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Alternative Field Representations and Integral Equations for Modeling Inhomogeneous Dielectrics

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Abstract

New volume and volume-surface integral equations are presented for modeling inhomogeneous dielectric regions. The presented integral equations result in more efficient numerical implementations and should therefore be useful in a variety of electromagnetic applications.

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1 Introduction

The modeling of inhomogeneous dielectrics via an integral equation approach is traditionally accomplished via the introduction of equivalent volume electric and magnetic currents [1] – [8]. For a dielectric with non-trivial permittivity and permeability this type of modeling implies six scalar unknowns at each volume location. As a result, the implementation of the resulting integral equation is computationally intensive and has excessive storage requirements.

In this paper it is demonstrated that any inhomogeneous dielectric material, regardless of its permittivity and permeability profile, can be modeled by a single electric or magnetic current density. Alternatively, either the electric or magnetic fields within the dielectric can be used as the unknown quantities. It appears though that one must pay a price for resorting to these reduced-unknown and/or kernal-singularity representations. Specifically, because they involve derivatives of the unknown quantities, a higher (at least linear) basis function is required for discretizing the resulting integral equations. However, it is possible to relax this requirement by resorting to a new volume-surface field representation. In this case, either the undifferentiated electric or magnetic field within the dielectric is the unknown quantity along with the corresponding tangential electric or magnetic fields on the outer boundary. Provided the dielectric volume is not composed of a single thin layer, this volume-surface integral equation still represents a nearly fifty percent reduction in the number of unknowns relative to traditional implementations.

2 Volume Representations

Let us consider the dielectric/ferrite volume V_d, shown in Fig. 1, having relative constitutive parameters ϵ_r and μ_r which are arbitrary functions of position. Assuming some exterior excitation, $(\mathbf{E}^i, \mathbf{H}^i)$, the total field can be written as

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \quad \mathbf{H} = \mathbf{H}^i + \mathbf{H}^s \tag{1}$$

where $(\mathbf{E}^s, \mathbf{H}^s)$ are the scattered fields caused by the presence of the dielectric. Traditionally [1] the scattered fields are formulated in terms of the equivalent currents

$$\mathbf{J}_{eq} = jk_oY_o(\epsilon_r - 1)\mathbf{E}, \quad \mathbf{M}_{eq} = jk_oZ_o(\mu_r - 1)\mathbf{H} \tag{2}$$

with k_o and $Z_o = 1/Y_o$ being the free space wavenumber and intrinsic impedance, respectively. In terms of these effective or equivalent current densities, the scattered field is given by

$$\mathbf{E}^s = \iiint_{V_d} \left[\nabla\times\overline{\mathbf{\Gamma}}_o(\mathbf{r},\mathbf{r}')\cdot\mathbf{M}_{eq}(\mathbf{r}') + jk_oZ_o\overline{\mathbf{\Gamma}}_o(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}_{eq}(\mathbf{r}')\right] dv' \tag{3}$$

in which $\mathbf{r}$ and $\mathbf{r}'$ denote the observation and integration points, respectivcly,

$$\overline{\mathbf{\Gamma}}_o(\mathbf{r},\mathbf{r}') = -\left[\overline{\mathbf{I}} + \frac{\nabla\nabla}{k_o^2}\right] G_o(\mathbf{r},\mathbf{r}'), \tag{4}$$

is the free space dyadic Green's function,

$$\nabla\times\overline{\mathbf{\Gamma}}_o(\mathbf{r},\mathbf{r}') = -\nabla G_o(\mathbf{r},\mathbf{r}')\times\overline{\mathbf{I}}, \tag{5}$$

$$G_o(\mathbf{r},\mathbf{r}') = \frac{e^{-jk_o|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

$$\overline{\mathbf{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

is the unit dyad and $\mathbf{H}^s$ is given by the dual of (3). By substituting (3) and its dual into (1) and then into (2), we obtain the coupled set of integral equations

$$\frac{\mathbf{J}_{eq}(\mathbf{r})}{jk_oY_o(\epsilon_r - 1)} - \mathbf{E}^s = \mathbf{E}^i \qquad \mathbf{r}\in V_d \tag{7a}$$

$$\frac{\mathbf{M}_{eq}(\mathbf{r})}{jk_oY_o(\mu_r - 1)} - \mathbf{H}^s = \mathbf{H}^i \qquad \mathbf{r}\in V_d \tag{7b}$$

for a solution of the equivalent currents $\mathbf{J}_{eq}$ and $\mathbf{M}_{eq}$.

The aforementioned formulation appears to be the only approach that has so far been utilized for three dimensional implementations. However, as noted in the literature [4, 5, 9], the singularity of the kernal in (3) presents numerical difficulties. Also, for non-trivial permeability, six scalar unknowns are involved in the solution of (7). The first of these difficulties can be alleviated by resorting to higher order basis functions and expressing, for example, $\mathbf{E}^s$ as

$$\begin{aligned}\mathbf{E}^s &= \int\int\int_{V_d}\Big\{\mathbf{M}_{eq}\times\nabla G_o(\mathbf{r},\mathbf{r}') - jk_oZ_o\mathbf{J}_{eq}G_o(\mathbf{r},\mathbf{r}') \\ &\quad - \frac{jZ_o}{k_o}\nabla'\cdot\mathbf{J}_{eq}(\mathbf{r}')\nabla G_o(\mathbf{r},\mathbf{r}')\Big\}dv' \qquad (8)\end{aligned}$$

which is the volume equivalent to the Stratton-Chu surface integral equation. Likewise, the scattered magnetic field to be substituted in (7b) can be expressed by the dual of (8).

Although this appears to be the most popular approach in modeling three-dimensional dielectrics, it can be shown that there are several other ways to formulate the problem. Most importantly, it can also be shown that (7) can be replaced with an equivalent system which involves only three (not six) scalar unknowns. Specifically, from Maxwell's equations [10] the radiation of $\mathbf{M}_{eq}$ is indistinguishable from the radiation of the electric current

$$\mathbf{J}'_{eq} = \frac{\nabla\times\mathbf{M}_{eq}}{jk_oZ_o} \qquad (9)$$

This can be combined with (2) giving a single equivalent electric current

$$\begin{aligned}\mathbf{J}''_{eq} &= jk_oY_o(\epsilon_r - 1)\mathbf{E} + \nabla\times[(\mu_r - 1)\mathbf{H}] \\ &= \frac{(\epsilon_r - 1)}{\epsilon_r}\nabla\times\mathbf{H} + \nabla\times[(\mu_r - 1)\mathbf{H}] \qquad (10)\end{aligned}$$

for representing the scattered fields $(\mathbf{E}^s, \mathbf{H}^s)$. From the dual of (3) we then obtain that the scattered magnetic field due to the current density (10) is

$$\begin{aligned}\mathbf{H}^s &= \int\int\int_{V_d}\left[\nabla G_o(\mathbf{r},\mathbf{r}')\times\bar{\mathbf{I}}\right]\cdot\Big\{\frac{\epsilon_r(\mathbf{r}') - 1}{\epsilon_r(\mathbf{r}')}\nabla'\times\mathbf{H}(\mathbf{r}') \\ &\quad + \nabla'\times[(\mu_r(\mathbf{r}') - 1)\mathbf{H}(\mathbf{r}')]\Big\}dv' \qquad (11)\end{aligned}$$

in which ∇' implies differentiation with respect to the primed/integration coordinates. When this is used in (1) we deduce the integral equation

$$\mathbf{H}^i(\mathbf{r}) = \mathbf{H}(\mathbf{r}) - \int\int\int_{V_d} \left[\nabla G_o(\mathbf{r},\mathbf{r}') \times \bar{\mathbf{I}}\right] \cdot \left\{ \frac{\epsilon_r(\mathbf{r}')-1}{\epsilon_r(\mathbf{r}')} \nabla' \times \mathbf{H}(\mathbf{r}') + \nabla' \times [(\mu_r(\mathbf{r}')-1)\mathbf{H}(\mathbf{r}')] \right\} dv' \qquad \mathbf{r} \in V_d \tag{12}$$

where the unknown quantity is now the magnetic field within V_d. Using a similar procedure it can be also shown that the scattered field may instead be represented by the radiation of a single magnetic current density

$$\mathbf{M}''_{eq} = -\frac{(\mu_r - 1)}{\mu_r} \nabla \times \mathbf{E} - \nabla \times [(\epsilon_r - 1)\mathbf{E}] \tag{13}$$

From the first of (1) and (3), we then deduce the integral equation

$$\mathbf{E}^i(\mathbf{r}) = \mathbf{E}(\mathbf{r}) - \int\int\int_{V_d} \left[\nabla G_o(\mathbf{r},\mathbf{r}') \times \bar{\mathbf{I}}\right] \cdot \left\{ \frac{\mu_r(\mathbf{r}')-1}{\mu_r(\mathbf{r}')} \nabla' \times \mathbf{E}(\mathbf{r}') + \nabla' \times [(\epsilon_r(\mathbf{r}')-1)\mathbf{E}(\mathbf{r}')] \right\} dv' \tag{14}$$

which as expected is the dual of (12). We observe that the kernel singularity associated with (12) and (14) is the same as that associated with (8). In addition, as in the case of the integral equation (7) in conjunction with (8), linear expansion functions such as those in [3] or [4] are required for the discretizing (12) and (14). Thus, even though the new integral equations (12) and (14) have half the unknowns, this was not achieved at the expense of increasing the kernel's singularity or the order of the expansion basis required in their implementation. It is remarked that special forms of these integral equations have already been successfully implemented for two dimensional applications [11, 12].

3 Volume-Surface Representation

The requirement to employ linear basis in connection with the implementation of (12) and (14) can be relaxed by resorting to a volume-surface integral equation (VSIE) such as that derived in [13] and [14] for two dimensional simulations. To do so we begin with (3) which in conjunction with (2) can be rewritten as

$$\mathbf{E}^s = \mathbf{E}^s_e + \mathbf{E}^s_m = -k_o^2 \int\int\int_{V_d} [\epsilon_r(\mathbf{r}') - 1]\, \mathbf{E}(\mathbf{r}') \cdot \overline{\boldsymbol{\Gamma}}_o(\mathbf{r}, \mathbf{r}')$$
$$-jk_o Z_o \nabla \times \int\int\int_{V_d} [\mu_r(\mathbf{r}') - 1]\, \mathbf{H}(\mathbf{r}') G_o(\mathbf{r}, \mathbf{r}') dv' \qquad (15)$$

where $\mathbf{E}^s_m$ is associated with the second integral and represents the field due to the magnetic equivalent current defined in (2). Setting $\mathbf{H} = \nabla \times \mathbf{E}/jk_o Z_o \mu_r$ in this integral, and invoking the identities

$$\nabla \times [\nabla' \times \phi \mathbf{E}] = \nabla \times [\nabla' \phi \times \mathbf{E}] + \nabla \times [\phi \nabla' \times \mathbf{E}]$$

$$\nabla(\phi\psi) = \psi \nabla \phi + \phi \nabla \psi$$

we obtain

$$\mathbf{E}^s_m = \mathbf{F}^{s1}_m + \mathbf{F}^{s2}_m + \mathbf{F}^{s3}_m \qquad (16)$$

with

$$\mathbf{F}^{s1}_m = \nabla \times \int\int\int_{V_d} \nabla' \times \left\{ \left(1 - \frac{1}{\mu_r(\mathbf{r}')}\right) G_o(\mathbf{r}, \mathbf{r}') \mathbf{E}(\mathbf{r}') \right\} dv' \qquad (17)$$

$$\mathbf{F}^{s2}_m = -\nabla \times \int\int\int_{V_d} \left\{ \left(1 - \frac{1}{\mu_r(\mathbf{r}')}\right) \nabla' G_o(\mathbf{r}, \mathbf{r}') \times \mathbf{E}(\mathbf{r}') \right\} dv' \qquad (18)$$

$$\mathbf{F}^{s3}_m = \nabla \times \int\int\int_{V_d} \left\{ G_o(\mathbf{r}, \mathbf{r}') \nabla' \left(\frac{1}{\mu_r(\mathbf{r}')}\right) \times \mathbf{E}(\mathbf{r}') \right\} dv' \qquad (19)$$

These integral expressions can be simplified through the use of various integral and differential identities.

The volume integral in (17) can be transformed to a surface integral by invoking Stoke's identity

$$\int\int\int_{V_d} (\nabla' \times \mathbf{A}) dv' = \oint\!\!\oint_{S_d} (\hat{n}' \times \mathbf{A}) ds' \qquad (20)$$

where S_d is the surface enclosing V_d and $\hat{n}' = \hat{n}(\mathbf{r}')$ denotes the outward unit normal to the surface S_d. We have

$$\begin{aligned}
\mathbf{F}_m^{s1} &= \nabla \times \oiint_{S_d} \left(1 - \frac{1}{\mu_r(\mathbf{r}')}\right) G_o(\mathbf{r}, \mathbf{r}') \left[\hat{n}' \times \mathbf{E}(\mathbf{r}')\right] ds' \\
&= -\oiint_{S_d} \left(1 - \frac{1}{\mu_r(\mathbf{r}')}\right) \left[\hat{n}' \times \mathbf{E}(\mathbf{r}')\right] \times \nabla G_o(\mathbf{r}, \mathbf{r}') ds' \qquad (21)
\end{aligned}$$

which is an integral involving the undifferentiated tangential electric field over the surface enclosing V_d. Turning now to the integral in (18) we first rewrite it as

$$\mathbf{F}_m^{s2} = -\int\int\int_{V_d} \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] \nabla \times \left[\nabla' G_o(\mathbf{r}, \mathbf{r}') \times \mathbf{E}(\mathbf{r}')\right] dv' \qquad (22)$$

and we note that [15, p. 487]

$$\nabla \times \left[\nabla' G_o \times \mathbf{E}(\mathbf{r}')\right] = \mathbf{E}(\mathbf{r}') \nabla^2 G_o - \mathbf{E}(\mathbf{r}') \cdot \nabla\nabla G_o \qquad (23)$$

Then, upon invoking the differential equation

$$\nabla^2 G_o(\mathbf{r}, \mathbf{r}') + k_o^2 G_o(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \qquad (24)$$

where $\delta(\mathbf{r}')$ denotes the Dirac delta function, it follows that

$$\mathbf{F}_m^{s2} = -k_o^2 \int\int\int_{V_d} \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] \mathbf{E}(\mathbf{r}') \cdot \overline{\mathbf{\Gamma}}_o(\mathbf{r}, \mathbf{r}') dv' + \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] \mathbf{E}(\mathbf{r}') \qquad (25)$$

Again, this involves only the undifferentiated electric field within the dielectric's volume. Finally, the last integral in (16) can be readily simplified and written as

$$\begin{aligned}
\mathbf{F}_m^{s3} &= \int\int\int_{V_d} \nabla \times \left\{ G_o(\mathbf{r}, \mathbf{r}') \nabla' \left[\frac{1}{\mu_r(\mathbf{r}')}\right] \times \mathbf{E}(\mathbf{r}') \right\} dv' \\
&= \int\int\int_{V_d} \nabla G_o(\mathbf{r}, \mathbf{r}') \times \left\{ \nabla' \left[\frac{1}{\mu_r(\mathbf{r}')}\right] \times \mathbf{E}(\mathbf{r}') \right\} dv' \qquad (26)
\end{aligned}$$

When (21), (25) and (26) are substituted into (16) and then into (15), we find that the total scattered field can be expressed as

$$\begin{aligned}
\mathbf{E}^s &= -k_o^2 \int\!\!\int\!\!\int_{V_d} \left[\epsilon_r(\mathbf{r}') - \frac{1}{\mu_r(\mathbf{r}')}\right] \mathbf{E}(\mathbf{r}') \cdot \overline{\Gamma}_o(\mathbf{r},\mathbf{r}')dv' \\
&+ \int\!\!\int\!\!\int_{V_d} \nabla G_o(\mathbf{r},\mathbf{r}') \times \left\{\nabla' \left[\frac{1}{\mu_r(\mathbf{r}')}\right] \times \mathbf{E}(\mathbf{r}')\right\} dv' \\
&- \oint\!\!\!\oint_{S_d} \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] [\hat{n}' \times \mathbf{E}(\mathbf{r}')] \times \nabla G_o(\mathbf{r},\mathbf{r}')ds' \\
&+ \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] \mathbf{E}(\mathbf{r})
\end{aligned} \tag{27}$$

For two dimensional simulations where the material parameters and the fields are invariant with respect to z, this expression can be readily shown to reduce to the VSIE given by Jin, etc. [13, equs. 28 and 31]. Expression (27) is also similar to the VSIE given by Tai [16]. However, Tai's expression was left in terms of differentiated field quantities and is only applicable for homogeneous dielectrics.

To obtain an integral equation on the basis of (27) we substitute this into the first of (1) and upon taking the principal value of the appropriate integrals we have

$$\begin{aligned}
&-k_o^2 \int\!\!\int\!\!\int_{V_d - V_o} \left[\epsilon_r(\mathbf{r}') - \frac{1}{\mu_r(\mathbf{r}')}\right] \mathbf{E}(\mathbf{r}') \cdot \overline{\Gamma}_o(\mathbf{r},\mathbf{r}')dv' \\
&+ \int\!\!\int\!\!\int_{V_d} \left\{\mathbf{E}(\mathbf{r}') \times \nabla' \left[\frac{1}{\mu_r(\mathbf{r}')}\right]\right\} \times \nabla G_o(\mathbf{r},\mathbf{r}')dv' \\
&- \oint\!\!\!\oint_{S_d - S_o} \left[1 - \frac{1}{\mu_r(\mathbf{r}')}\right] [\hat{n}' \times \mathbf{E}(\mathbf{r}')] \times \nabla G_o(\mathbf{r},\mathbf{r}')dv' \\
&+ \mathbf{E}^i = \begin{cases} \mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ not in } V_d \\ \frac{1}{2}\left[1 + \frac{1}{\mu_r(\mathbf{r})}\right] \mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ on } S_d \\ \frac{1}{3}\left[\epsilon_r(\mathbf{r}) + 2\mu_r(\mathbf{r})\right] \mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ in } V_d \end{cases}
\end{aligned} \tag{28}$$

In this, V_o is a vanishingly small spherical volume whereas S_o is a vanishingly small hemispherical surface both having their centers at $\mathbf{r}$. As given, (28) can be discretized via the moment method or some other technique for a solution of $\mathbf{E}(\mathbf{r})$ within the dielectric. Its kernal has, of course, the same singularity as (7a) but involves only a single unknown vector field in comparison with the two vector unknowns appearing in (7). If linear rather than pulse basis are employed for the solution of (28), then it may be desirable to rewrite the first integral of (28) in the form given by (8) with $\mathbf{M}_{eq} = 0$ and

$$\mathbf{J}_{eq} = \frac{jk_o}{Z_o}\left[\frac{\epsilon_r(\mathbf{r})\mu_r(\mathbf{r}) - 1}{\mu_r(\mathbf{r})}\right]\mathbf{E}(\mathbf{r}) \tag{29}$$

However, in this case one could also resort to the alternative integral equations (12) or (14). Of course, the dual of (28) is another integral equation. Further, linear combinations of (28) and its dual or (12) and (14) can be utilized if so desired.

In closing, we remark that if μ_r and/or ϵ_r are discontinuous within V_d, the surface integral in (27) and its dual must then be replaced by

$$\sum_i \oiint_{S_{d_i}} \left[u_+^i(\mathbf{r}') - u_-^i(\mathbf{r}')\right]\left[\hat{n}_i(\mathbf{r}') \times \mathbf{F}(\mathbf{r}')\right] \times \nabla G_o(\mathbf{r}, \mathbf{r}')ds'$$

where $\mathbf{F} = \mathbf{E}$ or $\mathbf{H}$. Here, S_{d_i} denotes the ith discontinuous surface within V_d, $\hat{n}_i(\mathbf{r})$ is the unit normal to S_i pointing from the $-$ side to the $+$ side (outermost side) and $u_\pm^i$ denotes the inverse relative dielectric constant at the $+$ or $-$ side of the surface S_{d_i}. In particular $u^i = 1/\mu_r^i$ for the E-field integral equation (27) and $u^i = 1/\epsilon_r^i$ for the H-field integral equation.

4 Conclusion

Some alternative formulations were proposed for modeling three-dimensional inhomogeneous dielectrics. These are summarized in figure 2 and the aim of the investigation was to generate integral equations for the fields within the dielectric scatterer utilizing the minimum number of unknowns and the least singular kernels. A purely volume integral equation was derived involving half the unknowns required with traditional equations for ferrite materials. The implementation of this reduced-unknown volume equation implies use of (at least) linear basis functions and to relax this requirement a volume-surface integral equation was derived. All of the integral equations presented here appear to be more efficient than the traditional ones without compromising the kernel's singularity. They should thus be found useful in a variety of radiation, scattering or SAR applications.

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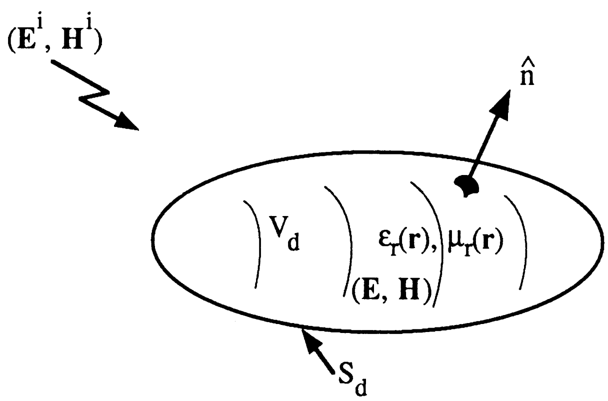

Fig. 1. Illustration of the inhomogeneous dielectric volume V_d enclosed by the surface S_d.

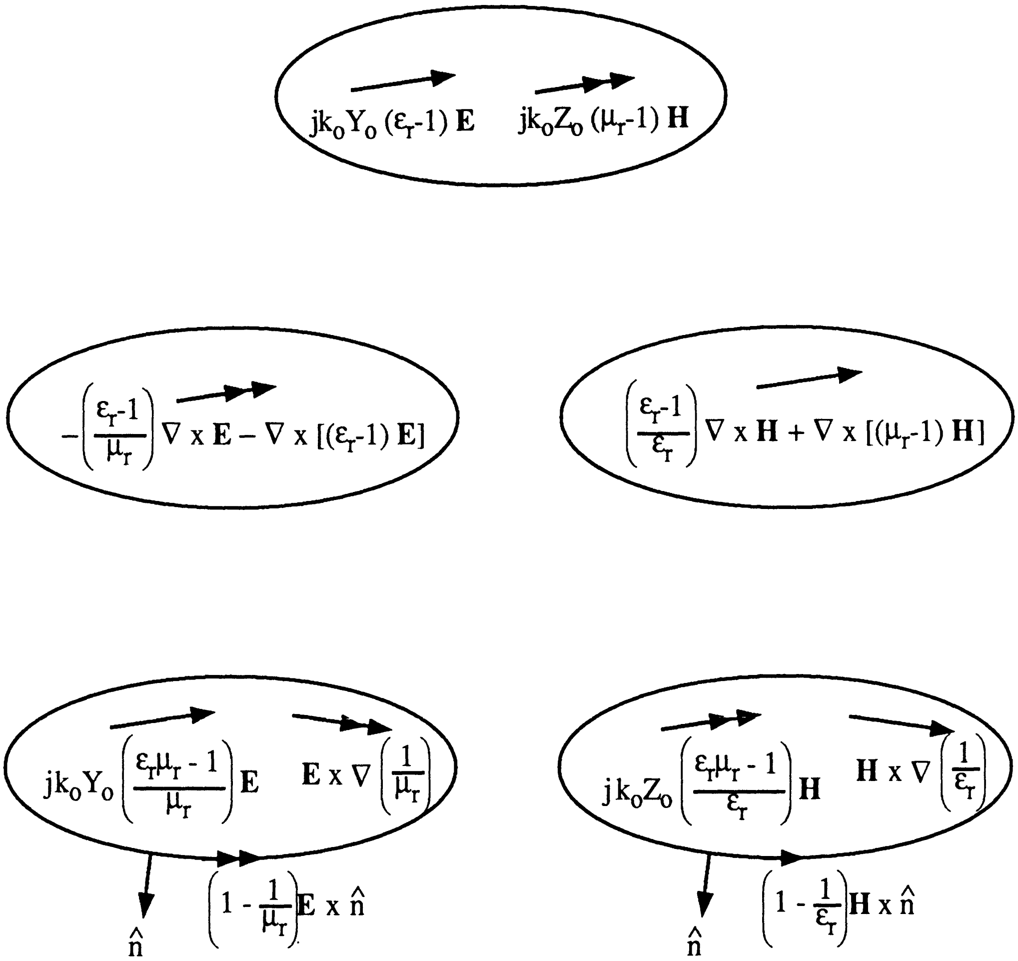

Fig. 2. Different volume equivalent currents for modeling the scattering by the inhomogeneous dielectric volume V_d in figure 1.