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SOME PROPERTIES OF CIRCULANTS

by

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ABSTRACT

A circulant is a square matrix which is completely determined by its first row in the following manner: The second row is obtained by shifting ("circulating") the first row one position to the right and placing the last entry of the first row in the first position of the second row. Each succeeding row is determined from the row above it in like manner. Circulants occur in various practical problems. This paper presents certain properties of circulants (Some already published and some new) which render such operations as matrix inversion, the determination of eigenvalues and eigenvectors, and even matrix multiplication considerably simpler than the corresponding operations with general matrices.

Basic to the simplified procedures is the fact that all circulants of a given size (N x N) have the same eigenvector matrix, which is easily determined from the N Nth roots of unity. Further simplifications result when the circulant is real or symmetric. In fact, if the circulant is both real and symmetric, the eigenvector matrix applicable to general circulants can be replaced by a real eigenvector matrix, and all the operations become even simpler. It then becomes possible to invert an arbitrary N x N real symmetric circulant on a digital computer using at most N+2 variable storage locations.

Singular-value decompositions of circulants are also discussed, as are many of the properties of "left-circulants"; i.e., matrices in which each succeeding row is obtained by circulating the row above to the left, rather than to the right.

I. INTRODUCTION

Square matrices A of the form

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-3} & a_{N-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{N-1} & a_0 \end{bmatrix} \qquad (1)$$

are called circulants. Note that each row is equal to the preceding row shifted one place to the right, with the last element moved to the beginning of the row. Circulants have a number of practical applications in diverse fields (see Refs. 1 and 2 and the references listed therein). Computations of eigenvalues, inverses, and singular values of circulants are much simpler than corresponding computations for general matrices of the same size, and for this reason circulants are profitably studied as a special class.

The aim of this report is to bring together in one place various useful properties of circulants. The results are presented as a series of lemmas and theorems. Some of these proofs repeat or parallel results given in the references, but are included here for the sake of continuity. The specialized results given here for real symmetric circulants, the simplified computational procedures, the singular value decompositions, and all of the material on left circulants are believed by the author to be original contributions.

II. GENERAL CIRCULANTS

We first study circulants in which the elements $a_0, a_1, \ldots, a_{N-1}$ of A are allowed to be complex numbers, and begin by making the following definitions.

Let $\underline{a}$ denote the N-component column vector which characterizes the circulant A; that is, let $\underline{a}$ be the column vector with elements equal to the elements of the first row of A:

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1} \end{bmatrix} \tag{2}$$

Let X be the N x N matrix whose columns are the column vectors

$$\underline{x}_k = \begin{bmatrix} 1 \\ r_k \\ r_k^2 \\ \cdot \\ \cdot \\ \cdot \\ r_k^{N-1} \end{bmatrix} \qquad k = 0, 1, \ldots, N-1 \tag{3}$$

where r_k is one of the k-th roots of unity, namely

$$r_k = \exp\left(jk\,\frac{2\pi}{N}\right) \tag{4}$$

Thus

$$X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & r_1 & \cdots & r_{N-1} \\ 1 & r_1^2 & \cdots & r_{N-1}^2 \\ \vdots & \vdots & & \vdots \\ 1 & r_1^{N-1} & \cdots & r_{N-1}^{N-1} \end{bmatrix} \tag{5}$$

Finally, let $\eta_0, \eta_1, \ldots, \eta_{N-1}$ denote the eigenvalues of A, and let

$$\underline{\eta} = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{N-1} \end{bmatrix} \quad \text{and } H = \begin{bmatrix} \eta_0 & & & 0 \\ & \eta_1 & & \\ & & \ddots & \\ 0 & & & \eta_{N-1} \end{bmatrix} \tag{6}$$

Lemma 1: X is symmetric; i.e.,

$$X^T = X$$

where the superscripted T denotes transposition.

Proof: This follows from the fact that x_{mk}, the element of X in the (m + 1)-th row and (k + 1)-th column, satisfies

$$x_{mk} = r_k^m = [\exp(jk\Theta)]^m = \exp(jkm\Theta) = [\exp(jm\Theta)]^k = r_m^k = x_{km}$$

for k,m = 0,1,...,N-1 where x_{km} is the element of X in the (k+1)-th row and (m+1)-th column, and where here, as henceforth, Θ denotes the quantity $2\pi/N$. Therefore X is symmetric. Q.E.D.

Because of the symmetry of X, no distinction will be made henceforth between X and X^T.

A fundamental property of circulants, which underlies all other results given below, can now be stated, as follows:

<u>Theorem 1</u>: The eigenvalues of A are given by

$$\eta_k = a_0 + a_1 r_k + a_2 r_k^2 + \ldots + a_{N-1} r_k^{N-1}, \quad k = 0,1,\ldots,N-1 \tag{7}$$

The vectors $\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{N-1}$ are the corresponding eigenvectors. In vector-matrix notation, these statements can be summarized as

$$\underline{\eta} = X \underline{a} \tag{8}$$

$$AX = XH \quad \text{or} \quad A = XHX^{-1} \tag{9}$$

Proof: For each $\underline{x}_k$, $k = 0,1,\ldots,N-1$,

$$A \underline{x}_k = \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ r_k \\ \vdots \\ r_k^{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + a_1 r_k + \cdots + a_{N-2} r_k^{N-2} + a_{N-1} r_k^{N-1} \\ a_{N-1} + a_0 r_k + \cdots + a_{N-3} r_k^{N-2} + a_{N-2} r_k^{N-1} \\ a_1 + a_2 r_k + \cdots + a_{N-1} r_k^{N-2} + a_0 r_k^{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + a_1 r_k + & \cdots & + a_{N-1} r_k^{N-1} \\ r_k(a_0 + a_1 r_k + & \cdots & + a_{N-1} r_k^{N-1}) \\ \vdots & & \\ r_k^{N-1}(a_0 + a_1 r_k + & \cdots & + a_{N-1} r_k^{N-1}) \end{bmatrix} = (a_0 + a_1 r_k + \ldots + a_{N-1} r_k^{N-1}) \begin{bmatrix} 1 \\ r_k \\ \vdots \\ r_k^{N-1} \end{bmatrix}$$

The next-to-last step above follows from the fact that $r_k^N = 1$ or, equivalently, $r_k^m = r_k^{m+N}$, k, m = $0, \pm 1, \pm 2, \ldots$. From the last step we see that $\underline{x}_k$ is an eigenvector of A and that the corresponding eigenvalue is $(a_0 + a_1 r_k + \ldots + a_{N-1} r_k^{N-1})$, as claimed. Equations 8 and 9 then follow at once from the definition of matrix multiplication, and from the symmetry of X. Q.E.D.

Next we note certain other properties of the eigenvector matrix X.

Lemma 2: The columns $\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{N-1}$ of X are orthogonal to each other and have norm $\sqrt{N}$ in terms of the Hermitian inner product*

$$(\underline{x}_h, \underline{x}_k) = \sum_{m=0}^{N-1} x_{mh} \bar{x}_{mk} \tag{10}$$

and the norm generated by this inner product, where x_{mh} is the (m+1)-th element of the vector $\underline{x}_h$.

Hence

$$X^{-1} = \frac{1}{N} \bar{X} \tag{11}$$

*A line over a scalar quantity denotes the complex conjugate of that quantity. A line over a vector or matrix quantity signifies that each element of the vector or matrix is to be replaced by its complex conjugate.

Proof: Consider any two of the vectors $\underline{x}_h$ and $\underline{x}_k$, and form their inner product:

$$(\underline{x}_h, \underline{x}_k) = \sum_{m=0}^{N-1} \exp(jmh\Theta) \exp(-jmk\Theta) = \sum_{m=0}^{N-1} \exp[jm(h-k)\Theta]$$

(Here, as before, the quantity $\frac{2\pi}{N}$ has been denoted by Θ.) We recognize this as a geometric series with common ratio $\exp[j(h-k)\Theta]$ and initial term unity. Therefore, for $h \neq k$

$$(\underline{x}_h, \underline{x}_k) = \frac{1 - [\exp j(h-k)\Theta]^N}{1 - \exp j(h-k)\Theta} = \frac{1 - \exp j(h-k)N\Theta}{1 - \exp j(h-k)\Theta} = \frac{1 - 1}{1 - \exp j(h-k)\Theta} = 0$$

For $h = k$, each term of the summation is equal to unity, so that

$$(\underline{x}_k, \underline{x}_k) = \|\underline{x}_k\|^2 = N \quad \text{and} \quad \|\underline{x}_k\| = \sqrt{N}$$

From this and from the symmetry of X, it follows that $X\overline{X} = \overline{X}X = NI$ or $X(\frac{1}{N}\overline{X}) = (\frac{1}{N}\overline{X})X = I$, where I is the identity matrix. Hence $X^{-1} = \frac{1}{N}\overline{X}$, as claimed. Q.E.D.

Substitution of this result into Eq. 9 shows that every circulant can be decomposed into a product of matrices of the form

$$A = \frac{1}{N} X H \overline{X} \tag{12}$$

Theorem 1 states that every circulant can be decomposed in a certain form. The following theorem states the converse.

Theorem 2: An N x N matrix A is a circulant if it can be decomposed as follows:

$$A = XHX^{-1} = \frac{1}{N} XH\overline{X} \tag{13}$$

where X is as defined above and H is a diagonal matrix with the eigenvalues of A as diagonal elements.

Proof: Assume that A can be decomposed in the form $\frac{1}{N} XH\overline{X}$, and let $\underline{\eta}$ be the N-component column vector of eigenvalues of A (i.e., $\underline{\eta}$ consists of the diagonal entries of H, as always). Let $\underline{b} = \frac{1}{N} \overline{X}\, \underline{\eta}$, and let B be the circulant characterized by $\underline{b}$ (i.e., the circulant having $\underline{b}^T$ as its first row). Then, from Eq. 8 and Lemma 2, the eigenvalues of B are given by

$$X\, \underline{b} = X(\tfrac{1}{N}\overline{X}\, \underline{\eta}) = \underline{\eta}$$

Since B is by definition a circulant, it follows from Eq. 12 that B can be written as

$$B = \frac{1}{N} XH\overline{X}$$

Therefore B = A, and therefore A is a circulant. The equivalence of the two expressions XHX^{-1} and $\frac{1}{N} XH\overline{X}$ given for the decomposition of A follows from Lemma 2, of course. Q.E.D.

Involved in the proof of Theorem 2 was the construction of a circulant having given eigenvalues. This is a useful result in its own right, and will therefore be presented as a lemma.

Lemma 3: A circulant B having specified eigenvalues* $\underline{\eta}$ can be constructed by choosing $\underline{b}^T$ as the first row of B, where

*Note, of course, that a different circulant B results if the same eigenvalues are arranged in different order in the vector $\underline{\eta}$.

$$\underline{b} = \frac{1}{N} \bar{X} \underline{\eta} \tag{14}$$

Proof: See proof of Theorem 2.

We have shown that every circulant of a given order N (i.e., every N x N circulant) has the same eigenvector matrix X (Theorem 1), and that every matrix having X as its eigenvector matrix is a circulant (Theorem 2). Many simple and convenient properties of circulants follow from these facts. Some of these are given below.

Lemma 4: Sums and differences of circulants of the same order are circulants, and the eigenvalues of the result are the corresponding sums or differences of the corresponding eigenvalues of the original circulants.

Proof: Let A and B be circulants of order N. Write $A = XH_AX^{-1}$ and $B = XH_BX^{-1}$. Then $A \pm B = [XH_AX^{-1}] \pm [XH_BX^{-1}] = X[H_A \pm H_B]X^{-1}$. H_A and H_B are diagonal matrices, and therefore $[H_A \pm H_B]$ is also. Therefore $A \pm B$ has the form required by Theorem 2, and consequently is a circulant. The generalization to more than two matrices is obvious. Q.E.D.

Lemma 5: Products of circulants of the same order are circulants. The resulting circulant is not affected by the order in which the factors are taken.

Proof: Let A and B be circulants of the same order N. Since A and B are circulants, they can be decomposed as $A = XH_AX^{-1}$ and $B = XH_BX^{-1}$. Thus $AB = XH_AH_BX^{-1}$ and $BA = XH_BH_AX^{-1}$. But since H_A and H_B are diagonal matrices, they commute, so that $AB = BA$. Furthermore, $H_A H_B = H_B H_A$ is also a diagonal matrix, so that $AB = BA$ is of the form required by Theorem 2, and hence is a circulant.

Note ease with which the inverse of a circulant can be obtained:

Lemma 6: If a circulant A has an inverse, then the inverse A^{-1} is a circulant, and is characterized by

$$\underline{b} = \frac{1}{N} \overline{X} \, [X \, \underline{a}]^{-1} \tag{15}$$

where $[X \, \underline{a}]^{-1}$ denotes the vector obtained by replacing each component of the vector $X \, \underline{a}$ by its reciprocal. This inverse exists if every component of $X \, \underline{a}$ is different from zero.

Proof: Since A is a circulant, it can be expressed in the form $A = XHX^{-1}$. Its inverse, if it exists, is thus given by

$$A^{-1} = [XHX^{-1}]^{-1} = XH^{-1} X^{-1} = \frac{1}{N} XH^{-1}\overline{X} \tag{16}$$

and is therefore a circulant, by Theorem 2. Also by Theorem 2, the diagonal elements $\eta_0, \eta_1, \ldots, \eta_{N-1}$ of H are given by $X \, \underline{a}$, so that H^{-1} exists if every component of $X \, \underline{a}$ is nonzero. Equation 15 then follows from Eqs. 8 and 14. Q.E.D.

Lemma 7: The pseudo-inverse* A^+ of a circulant A (obtained by replacing all nonzero eigenvalues of A by their reciprocals and leaving zero eigenvalues unchanged in the decomposition $A = XHX^{-1}$) is a circulant.

Proof: By the definition of the pseudo-inverse, A^+ is expressible in the form required by Theorem 2, and hence is a circulant. Q.E.D.

On the basis of these lemmas and of various properties of associativity, commutivity, distributivity, etc., inherited from matrix addition and multiplication, it is easily shown that circulants of a given order

*This pseudo-inverse is a special case of the Penrose pseudo-inverse applicable to square matrices. See Ref. 5.

1) form a commutative group with respect to matrix addition, with the zero matrix (itself a circulant) acting as the unit element;

2) form a commutative semi-group with respect to matrix multiplication having a unit element (the identity matrix, itself a circulant).

Circulants of a given order therefore form a commutative ring having a unit element with respect to multiplication. They fail to form a field because some circulants have no multiplicative inverse.

The following two lemmas allow the evaluation of $\|\underline{\eta}\|$ without actual computation of $\underline{\eta}$, and give an upper bound on the maximum modulus of any component of $\underline{\eta}$, which is also an upper bound* on $\|A\|$.

Lemma 8:

$$\|\underline{\eta}\| = \sqrt{N}\,\|\underline{a}\| \tag{17}$$

where the norm is as defined in Lemma 1.

Proof: From the definition of the norm and from Eq. 8,

$$\|\underline{\eta}\| = (\underline{\eta},\underline{\eta})^{\frac{1}{2}} = [\underline{\eta}^T\bar{\underline{\eta}}]^{\frac{1}{2}} = [\underline{a}^T X^T \bar{X}\,\bar{\underline{a}}]^{\frac{1}{2}}$$

From Lemmas 1 and 2,

$$X^T\bar{X} = X\bar{X} = X(NX^{-1}) = NI$$

Therefore $\|\underline{\eta}\| = [N\underline{a}^T\bar{\underline{a}}]^{\frac{1}{2}} = \sqrt{N}[\underline{a}^T\bar{\underline{a}}]^{\frac{1}{2}} = \sqrt{N}\,\|\underline{a}\|$ Q.E.D.

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*$\|A\|$ is here defined as $\max\limits_{\underline{x}\neq 0} \frac{\|A\underline{x}\|}{\|\underline{x}\|}$. It can be shown that for circulants $\|A\| = \max\limits_{i} |\eta_i|$.

Lemma 9: $|\eta_k| \leq \sum_{m=0}^{N-1} |a_m|$, k=0,1,...,N-1 (18)

Proof: From Eq. 7, $\eta_k = \sum_{m=0}^{N-1} r_k^m a_m$, so that

$$|\eta_k| = |\sum_{m=0}^{N-1} r_k^m a_m| \leq \sum_{m=0}^{N-1} |r_k^m a_m| = \sum_{m=0}^{N-1} |r_k^m| |a_m| = \sum_{m=0}^{N-1} |a_m|$$

since $|r_k^m| = 1$ for all k and m. Q.E.D.

Other properties of circulants can be derived just as easily. For instance, it can be shown that, if they exist, rational or irrational powers of circulants are circulants; that, if they exist, general functions of circulants are circulants; that polynomials of the form $Z^k + A_{k-1} Z^{k-1} + \ldots + A_1 Z + A_0$, where $A_{k-1}, \ldots, A_0$ are N x N circulants and Z is an unknown N x N matrix, have easily determined families of zeroes, each member of which is a circulant; etc.

Other convenient properties possessed by special classes of circulants are developed in the following sections. In discussing these special classes it will prove enlightening to investigate the number of degrees of freedom of circulants of various types; that is, the number of independent choices that must be made in order to specify the circulant completely. As a reference point for these later discussions, we note that the circulants so far considered have 2N degrees of freedom—the N real parts and N imaginary parts of the elements $a_0, a_1, \ldots a_{N-1}$ of $\underline{a}$. The circulant property then determines the remaining elements of A. Furthermore, because each choice of $\underline{a}$ uniquely determines an $\underline{\eta}$ (Theorem 1) and vice versa (Lemma 3), it is not surprising that there are 2N degrees of freedom involved in the choice of an $\underline{\eta}$—the N real parts and the N imaginary parts of the elements η_0, η_1, ..., η_{N-1}. There are no additional degrees of freedom, since X must be as specified by Eq. 5. Thus there is a 1:1 correspondence between N x N circulants and points in N-dimensional complex Euclidean space, so that circulants can be regarded as elements of an N-dimensional complex vector space.

III. REAL CIRCULANTS

If the elements $a_0, a_1, \ldots, a_{N-1}$ of A are required to be real, as they are in many of the practical applications of circulants, then clearly only N degrees of freedom remain in the choice of A. Similarly, if we wish to choose an $\underline{\eta}$ so as to generate a real circulant by means by Lemma 3, there can be only N degrees of freedom. We now investigate the restrictions on $\underline{\eta}$ which are implied by the realness of $\underline{a}$.

<u>Theorem 3</u>: The eigenvlaues of real circulants occur in complex conjugate pairs, with the exceptions of η_0 and, if N is even, $\eta_{N/2}$, which are not necessarily paired and must be real. Thus,

$$\eta_{N-k} = \overline{\eta}_k \qquad k=1,2,\ldots, \begin{cases} \frac{N-1}{2} & N \text{ odd} \\ \frac{N-2}{2} & N \text{ even} \end{cases} \tag{19}$$

Proof:

$$\eta_k = \sum_{m=0}^{N-1} a_m r_k^m \qquad \eta_{N-k} = \sum_{m=0}^{N-1} a_m r_{N-k}^m$$

But $r_{N-k}^m = \overline{r_k^m}$, since $r_{N-k}^m = \{\exp[j(N-k)\Theta]\}^m = \exp[jm(N-k)\Theta] =$ $[\exp(jmN\Theta)][\exp(-jmk\Theta)] = \exp(-jmk\Theta) = \overline{\exp(jmk\Theta)} = \overline{r_k^m}$. Thus $\eta_{N-k} = \sum_{m=0}^{N-1} a_m \overline{r_k^m}$, which, because of the realness of $\underline{a}$, is clearly the complex conjugate of the expression given above for η_k. Since $\eta_0 = \sum_{m=0}^{N-1} a_m$, η_0 is real.

For N even, $r^m_{N/2} = (-1)^m$, so that $\eta_{N/2} = \sum_{m=0}^{N-1} (-1)^m a_m$, which is also real. Q.E.D.

Lemma 10: The converse of Theorem 3 is true. That is, the circulant which corresponds to any $\underline{\eta}$ of the form given by Theorem 3 is real.

Proof: Let $\underline{\eta}$ have the stated form. Such an $\underline{\eta}$ corresponds to a circulant B characterized by $\underline{b} = \frac{1}{N} \overline{X}\, \underline{\eta}$, from Lemma 3. Assume that N is odd. Then, for $k = 0, 1, \ldots, N-1$,

$$b_k = \frac{1}{N} \sum_{m=0}^{N-1} \overline{r^k_m}\, \eta_m = \frac{1}{N} [\eta_0 + \sum_{m=1}^{(N-1)/2} (\overline{r^k_m}\, \eta_m + \overline{r^k_{N-m}}\, \eta_{N-m})]$$

$$= \frac{1}{N} [\eta_0 + \sum_{m=1}^{(N-1)/2} (\overline{r^k_m} + \eta_m + r^k_m\, \overline{\eta_m})]$$

which is always real. This last step follows from the assumed form of η and from the fact that

$$r^k_{N-m} = \exp\,[jk(N-m)\theta] = \exp(-jkm\theta) = \overline{r^k_m}$$

A similar argument, in which both η_0 and $(-1)^k\, \eta_{N/2}$ are taken outside the summation sign, leads to the stated conclusion for even N. Q.E.D.

There are therefore N degrees of freedom associated with $\underline{\eta}$—if N is odd, the real part of η_0 and the real and imaginary parts of $\eta_1, \eta_2, \ldots, \eta_{(N-1)/2}$; if N is even, the real parts of η_0 and $\eta_{N/2}$, plus the real and imaginary parts of $\eta_1, \eta_2, \ldots, \eta_{(N-2)/2}$. This reduces the amount of computation required to determine $\underline{\eta}$, since only $\frac{N+1}{2}$ or $\frac{N+2}{2}$ eigenvalues need be computed by Eq. 7, the remaining eigenvalues being obtained by taking complex conjugates, as indicated by Eq. 19.

The following result has some practical application in determining the maximum or minimum eigenvalue of A, and hence the norm of A, in special cases.

<u>Lemma 11</u>: If every element of $\underline{a}$ is real and has the same sign, i.e., if

$$\text{a)} \quad a_m \geq 0 \qquad m = 0,1,\ldots,N-1$$

or if

$$\text{b)} \quad a_m \leq 0 \qquad m = 0,1,\ldots,N-1$$

then an eigenvalue of A having maximum absolute value is η_0.

Proof: For case a), $|\eta_k| \leq \sum_{m=0}^{N-1} |a_m| = \sum_{m=0}^{N-1} a_m = \eta_0$, from Lemma 9 and Eq. 7. For case b), $|\eta_k| \leq \sum_{m=0}^{N-1} |a_m| = -\sum_{m=0}^{N-1} a_m = -\eta_0 = |\eta_0|$, by the same arguments. Q.E.D.

IV. SYMMETRIC CIRCULANTS

In this section we again allow the elements of A to be complex, but require that A be symmetric, i.e., that $A^T = A$, or, equivalently, that

$$a_k = a_{N-k}\,, \qquad k=1,2,\ldots,\begin{cases} \dfrac{N-1}{2} & N \text{ odd} \\ \dfrac{N-2}{2} & N \text{ even} \end{cases} \tag{20}$$

In this case N+1 degrees of freedom remain if N is odd—namely, the real and imaginary parts of a_0, a_1, $a_2,\ldots,a_{(N-1)/2}$. If N is even, there are N + 2 degrees of freedom—the real and imaginary parts of a_0, $a_1,\ldots,a_{N/2}$. These restrictions on A impose restrictions on $\underline{\eta}$ as follows:

<u>Theorem 4</u>: The eigenvalues of a symmetric (not necessarily real) circulant occur in pairs, with the exceptions of η_0 and, if N is even, $\eta_{N/2}$, which are not necessarily paired. Thus

$$\eta_{N-k} = \eta_k, \qquad k=1,2,\ldots,\begin{cases} \dfrac{N-1}{2} & N \text{ odd} \\ \dfrac{N-2}{2} & N \text{ even} \end{cases} \tag{21}$$

Proof: Assume that N is odd. Then for $k = 1,2,\ldots,\dfrac{N-1}{2}$,

$$\eta_k = \sum_{m=0}^{N-1} a_m r_k^m = a_0 + \sum_{m=1}^{(N-1)/2} a_m(r_k^m + r_k^{N-m}) = a_0 + \sum_{m=1}^{(N-1)/2} a_m(r_k^m + \overline{r_k^m})$$

Similarly, $\eta_{N-k} = a_0 + \sum_{m=1}^{(N-1)/2} a_m(r_{N-k}^m + \overline{r_{N-k}^m})$. But since

$r_{N-k}^{m} = \exp[jm(N-k)\theta] = \exp(-jmk\theta = \overline{r_k^{m}}$, it follows that $\eta_k = \eta_{N-k}, k=0,1,\ldots, \frac{N-1}{2}$, as claimed. A similar argument establishes the stated result for N even. Q.E.D.

Lemma 12: The converse of Theorem 4 is true. That is, the circulant which corresponds to any $\underline{\eta}$ of the form given by Theorem 4 is symmetric.

Proof: Let $\underline{\eta}$ have the stated form. Such an $\underline{\eta}$ corresponds to a circulant B characterized by $b = \frac{1}{N}\overline{X}\,\underline{\eta}$, from Lemma 3. Assume that N is odd. Then, for $k = 1, \ldots, \frac{N-1}{2}$

$$b_k = \frac{1}{N}\sum_{m=0}^{N-1} \overline{r_m^{k}}\,\eta_m = \frac{1}{N}\left[\eta_0 + \sum_{m=1}^{(N-1)/2} (\overline{r_m^{k}}\,\eta_m + \overline{r_{N-m}^{k}}\,\eta_{N-m})\right] =$$

$$\frac{1}{N}\left[\eta_0 + \sum_{m=1}^{(N-1)/2} \eta_m\,(\overline{r_m^{k}} + \overline{r_{N-m}^{k}})\right]$$

Similarly,

$$b_{N-k} = \frac{1}{N}\left[\eta_0 + \sum_{m=1}^{(N-1)/2} \eta_m(\overline{r_m^{N-k}} + \overline{r_{N-m}^{N-k}})\right]$$

But, as is easily shown, $\overline{r_m^{N-k}} = \overline{r_{N-m}^{k}}$ and $\overline{r_{N-m}^{N-k}} = \overline{r_m^{k}}$, so that $b_k = b_{N-k}$, as claimed. A similar argument holds for N even. Q.E.D.

There are thus the required number of degrees of freedom in $\underline{\eta}$: if N is odd, the N + 1 real and imaginary parts of the $\frac{N+1}{2}$ eigenvalues $\eta_0, \eta_1, \ldots, \eta_{(N-1)/2}$; if N is even, the N + 2 real and imaginary parts of the $\frac{N}{2} + 1$ eigenvalues $\eta_0, \eta_1, \ldots, \eta_{N/2}$. Therefore, in this special case too the use of circulants simplifies the computation of the eigenvalues of A.

V. REAL SYMMETRIC CIRCULANTS

A class of circulants having important practical applications (see Ref. 2, for example) is the class of real symmetric circulants. The conditions on $\underline{\eta}$ which correspond to imposing these conditions on A are easily obtained by combining the requirements for real circulants and symmetric circulants. We put this in the form of a theorem.

Theorem 5: The eigenvalues of a real symmetric circulant are real and satisfy

$$\eta_{N-k} = \eta_k \qquad k=1,2,\ldots, \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases}$$

Proof: As noted above, the proof follows at once from Theorems 3 and 4.

For real symmetric circulants, the procedure for computing the eigenvalues can be even further reduced, for the following reasons:

1) Because both $\underline{a}$ and $\underline{\eta}$ are known to be real, the imaginary part of the matrix X can be discarded in applying Eq. 8.

2) Because of the pairing of the eigenvalues η_k and η_{N-k}, only

$$\eta_0,\ \eta_1, \ldots, \begin{cases} \eta_{(N-1)/2} & N \text{ odd} \\ \eta_{N/2} & N \text{ even} \end{cases}$$

need be computed.

3) Symmetries in the real parts of the elements of X allow further reduction in the number of computations required.

To illustrate these points, let us define

$$Y = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \cos\Theta & \cos 2\Theta & \cdots & \cos M\Theta \\ 1 & \cos 2\Theta & \cos 4\Theta & \cdots & \cos 2M\Theta \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos M\Theta & \cos 2M\Theta & \cdots & \cos M^2\Theta \end{bmatrix} \qquad (22)$$

$$\text{where} \quad M = \begin{cases} (N-1)/2 & N \text{ odd} \\ N/2 & N \text{ even} \end{cases} \qquad (23)$$

and, as always, $\Theta = 2\pi/N$.

We can then prove the following theorem:

<u>Theorem 6</u>: If A is a real symmetric circulant, its eigenvalues are given by

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{(N-1)/2} \end{bmatrix} = Y \begin{bmatrix} a_0 \\ 2a_1 \\ \vdots \\ 2a_{(N-1)/2} \end{bmatrix} \qquad N \text{ odd}$$

$$(24)$$

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{(N-2)/2} \\ \eta_{N/2} \end{bmatrix} = Y \begin{bmatrix} a_0 \\ 2a_1 \\ \vdots \\ 2a_{(N-2)/2} \\ a_{N/2} \end{bmatrix} \qquad N \text{ even}$$

The remaining eigenvalues can be evaluated by using the pairing relationships of Theorem 5.

Proof: Write X and $X = X_R + jX_I$, where X_R and X_I are real. Then $\underline{\eta} = X\,\underline{a} = X_R\,\underline{a} + jX_I\,\underline{a}$. Because X_I and $\underline{a}$ are real, $jX_I\,\underline{a}$ is pure imaginary. But, from Theorem 5, $\underline{\eta}$ must be real, so that $X_I\,\underline{a}$ must be zero and $\underline{\eta} = X_R\,\underline{a}$.

Therefore, the first M + 1 eigenvalues are given by

$$\begin{bmatrix} \eta_0 \\ \\ \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_M \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \cos\Theta & \cos 2\Theta & \cdots & \cos(N-1)\Theta \\ 1 & \cos 2\Theta & \cos 4\Theta & \cdots & \cos 2(N-1)\Theta \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & \cos M\Theta & \cos 2M\Theta & \cdots & \cos(N-1)M\Theta \end{bmatrix} \begin{bmatrix} a_0 \\ \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1} \end{bmatrix}$$

since Re [exp(jkmΘ)] = cos kmΘ. Now consider the case where N is odd, and rewrite this equation as

$$\begin{bmatrix} \eta_0 \\ \\ \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_M \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cos\Theta & \cdots & \cos M\Theta & \cos(M+1)\Theta & \cdots & \cos(N-1)\Theta \\ 1 & \cos 2\Theta & \cdots & \cos 2M\Theta & \cos 2(M+1)\Theta & \cdots & \cos 2(N-1)\Theta \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 1 & \cos M\Theta & \cdots & \cos M^2\Theta & \cos M(M+1)\Theta & \cdots & \cos M(N-1)\Theta \end{bmatrix} \begin{bmatrix} a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_M \\ a_{M+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{N-1} \end{bmatrix}$$

Note(from the definition of the cosine) that $\cos k(N-m)\Theta = \cos km\Theta$ and (from the symmetry assumption) that $a_{N-m} = a_m$, for k, m = 1,2,...,M. Thus, the expression for η_k,

$$\eta_k = a_0 + a_1 \cos k\theta + a_2 \cos 2k\theta + \ldots + a_{N-1} \cos k(N-1)\theta$$

can be rearranged as

$$\eta_k = a_0 + [a_1 \cos k\theta + a_{N-1} \cos k(N-1)\theta] + \ldots + [a_M \cos kM\theta + a_{M+1} \cos k(M+1)\theta]$$

$$= a_0 + 2a_1 \cos k\theta + \ldots + 2a_M \cos kM\theta \ .$$

This establishes the validity of Eq. 24 for N odd. The proof for N even is similar, and is omitted here. Q.E.D.

The practical significance of this theorem becomes clear when we note that the already greatly simplified problem of finding the eigenvalues of a real circulant A by carrying out the multiplication of an N x N complex matrix and a real N-vector has been reduced to the multiplication of an (M+1)x(M+1) real matrix by a real (M+1) vector—a reduction from approximately 2N multiplications and 2(N-1) additions, for each of N eigenvalues, to M+1 multiplications and M additions for each of M+1 eigenvalues. This amounts to a reduction of almost 8 to 1 in the number of computations.

An even further reduction is possible. It turns out that every cosine appearing anywhere in the matrix Y also appears somewhere in the second row of Y, so that only M cosines must be computed in order to compute Y. The pattern is as follows: Let $y_k = \cos k\theta$, $k=0,1,\ldots,M$. Imagine these M+1 values arranged in a cyclic pattern of the form

$$y_0, y_1, \ldots, y_{M-1}, y_M, y_M, y_{M-1}, \ldots, y_1, y_0, y_1, \ldots, y_{M-1}, y_M, y_M, y_{M-1}, \ldots, y_1, y_0, y_1, \ldots$$

for N odd and

$y_0, y_1, \ldots, y_{M-1}, y_M, y_{M-1}, \ldots, y_1, y_0, y_1, \ldots, y_{M-1}, y_M, y_{M-1}, \ldots, y_1, y_0, y_1, \ldots$ for N even.

The first row of Y is then obtained by starting with y_0 and indexing <u>none</u> at a time (i.e., $y_0, y_0, y_0, y_0, \ldots$); the second row is obtained by starting with y_0 and indexing <u>one</u> at a time (i.e., $y_0, y_1, y_2, \ldots$); the third row is obtained by starting with y_0 and indexing <u>two</u> at a time (i.e., $y_0, y_2, y_4, y_6, \ldots$); and so on. Alternatively, the j-th element of the i-th row of Y is given by $Y_{ij} = y_p$ where $p = \begin{cases} m & m \leq M \\ N-m & m > M \end{cases}$ and where m is computed as the remainder, mod N, of the expression $(i-1)\cdot(j-1)$.

Taking advantage of these simplifications and symmetries, one can find the eigenvalues of or invert an N x N real symmetric circulant by using only $\left\{\begin{matrix} N+1 & N \text{ odd} \\ N+2 & N \text{ even} \end{matrix}\right\}$ memory locations* for storage of the matrix A, then its eigenvalues, and then its inverse. (Because A and its inverse are symmetric and circulant, we of course need store only part of the vectors which characterize A and its inverse).

Some of the results derived for general circulant matrices in the previous sections take on an even simpler form for real symmetric circulants. For instance,

<u>Lemma 13</u>: The real symmetric N x N circulant having the eigenvalues $\eta_0, \eta_1, \ldots, \eta_M$ is characterized by the vector $\underline{b} = [b_0, b_1, \ldots, b_{N-1}]$, where

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*Not including the accumulator, two storage locations for indices, one storage location for N, and storage for the program itself. If the information about the original matrix is not to be destroyed in the process, M additional locations are needed.

$$\begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_M \end{bmatrix} = \frac{1}{N} Y \begin{bmatrix} \eta_0 \\ 2\eta_1 \\ \cdot \\ \cdot \\ \cdot \\ 2\eta_M \end{bmatrix} \text{ (N odd)} \quad \text{or} \quad \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_M \end{bmatrix} = \frac{1}{N} Y \begin{bmatrix} \eta_0 \\ 2\eta_1 \\ \cdot \\ \cdot \\ \cdot \\ 2\eta_{M-1} \\ \eta_M \end{bmatrix} \text{(N even)}$$

and the remaining elements of $\underline{b}$ are obtained by use of the symmetry conditions.

Proof: Since for real symmetric circulants $\underline{a}$ and $\underline{\eta}$ possess the same symmetries, the arguments used in the proof of Theorem 6 can be used here to obtain the stated result from Lemma 3. Q.E.D.

Also of interest is the fact that Lemmas 4, 5, 6, and 7 can be proven with the word "circulant" replaced by "real symmetric circulant" throughout, and hence that real symmetric circulants of a given order also form a commutative ring having a unit element with respect to multiplication.

One other point is worthy of mention here. Whereas in the case of general real circulants, the eigenvectors occur in complex conjugate pairs, it is possible to write an eigenvector matrix for real symmetric circulants (call it W) which is real. This results from the fact that for real symmetric circulants the eigenvalues occur in pairs, so that the two (complex conjugate) eigenvectors $\underline{x}_k$ and $\underline{x}_{N-k}$ spanning the subspaces corresponding to a given pair of eigenvalues η_k and η_{N-k} can be replaced by any other orthogonal pair spanning the same subspace. It turns out that there is a convenient pair of <u>real</u> orthonormal eigenvectors $\underline{w}_k$ and $\underline{w}_{N-k}$, spanning each such two-dimensional subspace, namely,

$$\underline{w}_k = \frac{1-j}{2} \underline{x}_k + \frac{1+j}{2} \underline{x}_{N-k} = \operatorname{Re}(\underline{x}_k) + \operatorname{Im}(\underline{x}_k)$$

and

$$\underline{w}_{N-k} = \frac{1+j}{2}\,\underline{x}_k + \frac{1-j}{2}\,\underline{x}_{N-k} = \mathrm{Re}(\underline{x}_k) - \mathrm{Im}(\underline{x}_k)$$

The eigenvectors corresponding to η_0 and, if N is even, $\eta_{N/2}$, are real in any case, of course. Then, by using the definition of $\underline{x}_k$ and various trigonometric identities, W can be written as

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cos\theta + \sin\theta & \cdots & \cos(N-1)\theta + \sin(N-1)\theta \\ 1 & \cos 2\theta + \sin 2\theta & \cdots & \cos 2(N-1)\theta + \sin 2(N-1)\theta \\ \vdots & \vdots & & \vdots \\ 1 & \cos(N-1)\theta + \sin(N-1)\theta & \cdots & \cos(N-1)^2\theta + \sin(N-1)^2\theta \end{bmatrix}$$

Lemma 14: If, in all the above theorems and lemmas applicable to the matrix X and/or to general circulants, the word "circulant" is replaced by "real symmetric circulant" and the matrices X and $\overline{X}$ are replaced by W, and if, in Theorem 2 and Lemma 3 the restriction is added that $\underline{\eta}$ be of the form appropriate to real symmetric circulants (see Theorem 5), then all remain valid.

Proof: The proof is omitted here. It is straightforward, and depends on the fact that the eigenvalues of a real symmetric circulant occur in pairs, so that pairs of $\underline{x}$-vectors spanning two-dimensional subspaces can be replaced by pairs of $\underline{w}$-vectors spanning the same subspaces.

VI. SOME OTHER SPEICAL CLASSES

The main results of the previous three sections are summarized in Table I. along with the corresponding results for four additional special classes of circulants: "anti-symmetric," Hermitian (self-adjoint), skew-symmetric, and skew-Hermitian. Proofs of these results either follow from properties of matrices in general (e.g., those for skew-Hermitian circulants) or else are similar to proofs already given, and are omitted here.

TABLE I

Class	Condition on A	Conditions* on $\underline{a}$	Degrees of Freedom		Conditions* on $\underline{\eta}$
			N odd	N even	
Real	$A = \bar{A}$	$a_0, \ldots, a_{N-1}$ real	N	N	η_0, $\eta_{N/2}$ real; $\eta_k = \overline{\eta_{N-k}}$
Symmetric	$A = A^T$	$a_{N-k} = a_k$	N+1	N+2	$\eta_k = \eta_{N-k}$
Anti-Symmetric	$A + A^T$ = diagonal matrix	$a_{N-k} = -a_k$ $a_{N/2} = 0$	N+1	N	$\eta_{N/2} = \eta_0$ $\eta_{N-k} = 2\eta_0 - \eta_k$
Hermitian	$A = \bar{A}^T$	a_0, $a_{N/2}$ real $a_{N-k} = a_k$	N	N	$\eta_0, \ldots, \eta_{N-1}$ real
Skew-Symmetric	$A = -A^T$	$a_0 = a_{N/2} = 0$ $a_{N-k} = -a_k$	N-1	N-2	$\eta_0 = \eta_{N/2} = 0$ $\eta_{N-k} = -\eta_k$
Skew-Hermitian	$A = -\bar{A}^T$	a_0, $a_{N/2}$ pure imaginary $a_{N-k} = -\bar{a}_k$	N	N	$\eta_0, \ldots, \eta_{N-1}$ pure imaginary
Real Symmetric	$A = \bar{A} = A^T$	$a_0, \ldots, a_{N-1}$ real $a_{N-k} = a_k$	$\frac{N+1}{2}$	$\frac{N}{2} + 1$	$\eta_0, \ldots, \eta_{N-1}$ real $\eta_{N-k} = \eta_k$

*In this table, conditions given for $a_{N/2}$ and $\eta_{N/2}$ apply only when N is even, and the index k always runs from 1 to (N-1)/2 for N odd and from 1 to (N-2)/2 for N even.

VII. SIMPLIFIED COMPUTATIONS FOR GENERAL REAL CIRCULANTS

We have seen that the process of computing the eigenvalues of real symmetric circulants can be greatly simplified by taking advantage of the symmetries of X and $\underline{a}$. A significant reduction (approximately 4 to 1) can also be achieved for general real circulants, based on these results. The key idea here is the breaking up of the circulant into a symmetric part and a skew-symmetric part (a process that can always be carried out uniquely).

We begin by making some additional definitions: Let $\underline{b}$ be the symmetric part of $\underline{a}$, and let $\underline{c}$ be the skew-symmetric part; i.e.,

$$\begin{aligned} b_0 &= a_0 \; ; \quad b_k = \frac{1}{2}(a_k + a_{N-k}) \\ c_0 &= 0 \; ; \quad c_k = \frac{1}{2}(a_k - a_{N-k}) \end{aligned} \qquad k=1,2,\ldots,N-1 \qquad (25)$$

Let B and C denote the circulants characterized by $\underline{b}$ and $\underline{c}$, respectively. Note that, since $\underline{a}$ is real, $\underline{b}$ and $\underline{c}$ are also. Note also that B is symmetric and C is skew-symmetric (See Table I). Thus

$$A = B + C \qquad \underline{a} = \underline{b} + \underline{c}$$

Finally, let

$$Z = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \sin\theta & \sin 2\theta & \cdots & \sin M\theta \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \sin M\theta & \sin 2M\theta & \cdots & \sin M^2\theta \end{bmatrix} \qquad (26)$$

where, as always, $M = \frac{N-1}{2}$ for N odd and $\frac{N}{2}$ for N even.

We note that jZ is simply the imaginary part of the upper left-hand quarter of X, just as the matrix Y defined above was the real part of this same sub-matrix.

We can now state

<u>Theorem 7</u>: The eigenvalues $\eta_0, \eta_1, \ldots, \eta_M$ of the real circulant A are given by

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_M \end{bmatrix} = Y \begin{bmatrix} b_0 \\ 2b_1 \\ 2b_2 \\ \cdot \\ \cdot \\ \cdot \\ 2b_M \end{bmatrix} + jZ \begin{bmatrix} c_0 \\ 2c_1 \\ 2c_2 \\ \cdot \\ \cdot \\ \cdot \\ 2c_M \end{bmatrix} \qquad N \text{ odd}$$

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_M \end{bmatrix} = Y \begin{bmatrix} b_0 \\ 2b_1 \\ \cdot \\ \cdot \\ \cdot \\ 2b_{M-1} \\ b_M \end{bmatrix} + jZ \begin{bmatrix} c_0 \\ 2c_1 \\ \cdot \\ \cdot \\ \cdot \\ 2c_{M-1} \\ c_M \end{bmatrix} \qquad N \text{ even} \tag{27}$$

The remaining eigenvalues are given by

$$\eta_{N-k} = \bar{\eta}_k \qquad k = 1, 2, \ldots, \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases}$$

Also, η_0 and, if N is even, $\eta_{N/2}$, are real.

Proof: Write the equation $\underline{\eta} = X\,\underline{a}$ (from Theorem 1) as

$$\underline{\eta} = (X_R + jX_I)\ (\underline{b} + \underline{c})$$

where, as before, X_R is the real part of X and jX_I is the imaginary part. From Theorem 5, $X\,\underline{b}$ is real, since it is the eigenvalue vector corresponding to the real symmetric circulant B. Hence $jX_I\,\underline{b} = \underline{0}$. Now, note that C is real and skew-symmetric. From Table I, this means that the eigenvalues $X\,\underline{c}$ of C must satisfy the conditions given there for both cases. This implies that the eigenvalues of C, which are expressible as $X_R\underline{c} + jX_I\underline{c}$, are pure imaginary, and hence that

$$X_R\underline{c} = \underline{0}$$

Thus $\underline{\eta} = X_R\underline{b} + jX_I\underline{c}$; in other words, the symmetric part of A gives rise to the real part of $\underline{\eta}$ and the skew-symmetric part of A gives rise to the imaginary part of $\underline{\eta}$. The replacement of the term $X_I\underline{c}$ by the term

$$Z \begin{bmatrix} c_0 \\ 2c_1 \\ \cdot \\ \cdot \\ \cdot \\ 2c_M \end{bmatrix} \quad \text{or} \quad Z \begin{bmatrix} c_0 \\ 2c_1 \\ \cdot \\ \cdot \\ \cdot \\ 2c_{M-1} \\ c_M \end{bmatrix}$$

is then easily carried out along the same lines used in replacing X_R by Y in the proof of Theorem 6. The rest of this theorem is a repetition of parts of of Theorem 5. Q.E.D.

One other slight simplification might be noted here: For N even, the constant $c_{N/2}$ is always zero and hence need not be computed.

A simplified procedure for inverting a general real circulant based on these results, is also possible.

Theorem 8: Let A be a real circulant and let $\underline{\eta}$ be its eigenvalue vector. Let A be invertible (i.e., let $\underline{\eta}$ be such that $\eta_k \neq 0$, k = 0,1,...,N-1), and let the inverse be denoted by D. Let

$$\frac{1}{\eta_k} = \delta_k + j\psi_k \qquad k = 0,1,\ldots,N-1$$

where δ_k and ψ_k are real. Then D is a real circulant characterized by $\underline{d} = \underline{b} + \underline{c}$, where

$$\begin{bmatrix} b_o \\ b_1 \\ \vdots \\ b_M \end{bmatrix} = \frac{1}{N} Y \begin{bmatrix} \delta_o \\ 2\delta_1 \\ \vdots \\ 2\delta_M \end{bmatrix}; \quad \begin{bmatrix} c_o \\ c_1 \\ \vdots \\ c_M \end{bmatrix} = \frac{1}{N} Z \begin{bmatrix} \psi_o \\ 2\psi_1 \\ \vdots \\ 2\psi_M \end{bmatrix} \quad \text{N odd}$$

$$\begin{bmatrix} b_o \\ b_1 \\ \vdots \\ b_M \end{bmatrix} = \frac{1}{N} Y \begin{bmatrix} \delta_o \\ 2\delta_1 \\ \vdots \\ 2\delta_{M-1} \\ \delta_M \end{bmatrix} \quad \begin{bmatrix} c_o \\ c_1 \\ \vdots \\ c_M \end{bmatrix} = \frac{1}{N} Z \begin{bmatrix} \psi_o \\ 2\psi_1 \\ \vdots \\ 2\psi_{M-1} \\ \psi_M \end{bmatrix} \quad \text{N even} \tag{28}$$

$$b_{N-k} = b_k \quad c_{N-k} = -c_k \qquad k=1,2,\ldots, \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases}$$

and where, as always, $M = \frac{N-1}{2}$ for N odd and $\frac{N}{2}$ for N even.

Proof: Since A is a real circulant, its eigenvalues occur in complex conjugate pairs η_k and η_{N-k} (except for η_0 and $\eta_{N/2}$, which are unpaired and real). Thus the reciprocals of these η's satisfy the same conditions, so that D, the inverse of A, is also a real circulant, by Lemma 10. By Lemma 6,

$$\underline{d} = \frac{1}{N} \bar{X} [\underline{\eta}]^{-1} = \frac{1}{N} \bar{X} (\underline{\delta} + j\underline{\psi})$$

$$= \frac{1}{N} (X_R - jX_I) (\underline{\delta} + j\underline{\psi})$$

$$= \frac{1}{N} (X_R\underline{\delta} + X_I\underline{\psi}) + j \frac{1}{N} (X_R \underline{\psi} - X_I\underline{\delta})$$

Since $\underline{d}$ is real (as noted above), the imaginary part of this expression must be zero, so that

$$\underline{d} = \frac{1}{N} X_R\underline{\delta} + \frac{1}{N} X_I \underline{\psi}$$

The reduction of this expression to the form given in the statement of the theorem then follows by the same steps used in the proofs of Theorems 6 and 7, since $\underline{\delta}$ and $\underline{\psi}$ have the same symmetries as $\underline{b}$ and $\underline{c}$ of Theorem 7, respectively.

Q.E.D.

The following lemma is needed in the next section.

Lemma 15: Let A be a real circulant characterized by $\underline{a}$. Then the eigenvalue vector $X[a_0, a_{N-1}, a_{N-2}, \ldots, a_1]^T$ of A^T is the complex conjugate of the eigenvalue vector $X\underline{a}$ of A.

Proof: The vector $[a_0, a_{N-1}, \ldots, a_1]^T$ is of course the vector characterizing A^T, and A^T is a circulant, so that the above product does indeed give the eigenvalues of A^T. The symmetric part of A gives rise to the real part of $\underline{\eta} = X\,\underline{a}$, and the skew-symmetric part of A gives rise to the imaginary part of $\underline{\eta}$, as noted above. The symmetric parts of A and A^T are the same, and the skew-symmetric parts of A and A^T are negatives of each other. Thus the imaginary part of the eigenvalue vector of A^T is the negative of that of A, and the real parts are the same. Q.E.D.

VIII. SINGULAR VALUE DECOMPOSITIONS OF CIRCULANTS

Singular value decompositions (see Ref. 3) play a key role in Naylor's transform technique for time-varying systems (Ref. 3 and 4), and also offer considerable insight into the properties of matrices as transformations (see Ref. 3). The computation of the singular value decomposition of a real circulant is much simpler than that for a general matrix, and turns out to be closely related to the eigenvalue decomposition. These points are explored in the lemmas and theorems to follow.

Let us first mention two procedures which can be used to obtain singular value decompositions of a general real N x N matrix A: Form the real, symmetric, positive semi-definite matrix $A\,A^T$, and find its eigenvalues and any set of real orthogonal eigenvectors. (Such a set of real orthogonal eigenvectors always exists because $A\,A^T$ is a normal matrix. See Bellman, Ref. 1, p. 197.) The eigenvalues of $A\,A^T$ are real and non-negative, and the singular values of A are the positive square roots of these eigenvalues. Form a diagonal matrix Λ with the singular values as diagonal elements, and form a matrix U having the above-noted eigenvectors (normalized to $\sqrt{N}$ and taken in the same order as the corresponding singular values in Λ) as columns. Finally, let V be any orthonormal matrix (with columns having norm $\sqrt{N}$) for which

$$\frac{1}{N}\, U\, \Lambda\, V^T = A \tag{29}$$

This expression, $\frac{1}{N}\, U\, \Lambda\, V^T$, is called a singular value decomposition of A, and always exists under the conditions given here for A (and, in fact, for much more general conditions, also; see Ref. 3). It turns out that the matrix V can be obtained by replacing all the zero columns* of $A^T U \Lambda^+$ by any set of orthonormal vectors (norm $\sqrt{N}$) which are also orthogonal to all the nonzero columns of $A^T U \Lambda^+$. Of course, if Λ is invertible, so that $\Lambda^+ = \Lambda^{-1}$, then $A^T U \Lambda^+$ has no zero columns and V is completely determined by $A^T U \Lambda^+$. Even here, of course, the decomposition is not necessarily unique, since U may not be unique.** That is, when two or more of the eigenvalues of $A\,A^T$ are equal, there is some freedom in choosing the eigenvectors to span the corresponding subspace.

An alternative procedure for finding a singular value decomposition for A is to form the diagonal matrix*** Λ having the positive square roots of the eigenvalues of $A^T A$ as its diagonal elements, and the matrix V having any corresponding set of real orthonormal eigenvectors (norm $\sqrt{N}$) of $A^T A$ as its columns. Then U is obtained from $A\,V\,\Lambda^+$ by a procedure analogous to that given above for V, and the singular value decomposition is again given by Eq. 29.

These decompositions may differ from each other in more than order and sign changes on corresponding columns of U and V unless the singular values are distinct, in which case the two decompositions are equivalent.

<u>Lemma 16</u>: The singular values $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ of a real circulant A can be expressed in terms of the eigenvalues $\eta_0, \eta_1, \ldots, \eta_{N-1}$ as follows:

---

*Λ^+ is the pseudo-inverse of Λ. See Lemma 7 and Ref. 5.

**This lack of uniqueness may involve more than changes in the order of the singular values and eigenvectors, or changes in the signs of corresponding columns of U and V. Such order and sign variations are regarded here as giving rise to equivalent decomposition.

***It will be the same as the Λ defined above, except for possible differences in the order of the diagonal elements.

$$\lambda_0 = |\eta_0|$$

$$\lambda_k = |\eta_{N-k}| \qquad k = 1,2,\ldots,N-1$$

Since, from Theorem 3, the eigenvalues of a real circulant occur in complex conjugate pairs, (i.e., $\eta_{N-k} = \bar{\eta}_k$, k=1,2,...,N-1), this can be rewritten as

$$\lambda_0 = |\eta_0|$$

$$\lambda_k = \lambda_{N-k} = |\eta_k| = |\eta_{N-k}|$$

$$k=1,2,\ldots,\begin{cases}(N-1)/2 & N \text{ odd}\\ (N-2)/2 & N \text{ even}\end{cases}$$

$$\lambda_{N/2} = |\eta_{N/2}| \quad N \text{ even}$$

Proof: The singular values of a real circulant A characterized by $\underline{a}$ are the positive square roots of the elements of the column vector XA $\underline{a}$, since A $\underline{a}$ is the vector which characterizes the real symmetric circulant A A^T, and XA $\underline{a}$ is the eigenvalue vector of A A^T, by Theorem 1. Application of Eq. 12 gives

$$XA\,\underline{a} = X\left(\frac{1}{N}XH\bar{X}\right)\underline{a} = \frac{1}{N}X^2\,H\,(\bar{X}\underline{a})$$

where, as always, H is the eigenvalue matrix of A; i.e., its diagonal elements are the corresponding elements of $\underline{\eta} = X\underline{a}$. Now, since $\underline{a}$ is real, $\bar{X}\underline{a} = \overline{X\underline{a}} = \bar{\underline{\eta}}$, so that $H\bar{X}\underline{a}$ is simply $H\bar{\underline{\eta}} = [|\eta_0|^2, |\eta_1|^2, \ldots, |\eta_{N-1}|^2]^T$. From Lemma 2 and from the fact that $\underline{x}_{N-k} = \bar{\underline{x}}_k$, k=1,2,...,N-1, if follows that

$$\frac{1}{N} X^2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

It then follows that the singular values of A are $|\eta_0|, |\eta_{N-1}|, \ldots, |\eta_1|$. Q.E.D.

Lemma 17: Let A be a real N x N circulant, let Λ be the diagonal matrix with the singular values $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ of A (as defined in Lemma 16) as its diagonal elements, and let U and V (not to be identified as yet with the U and V matrices defined above) be defined as

$$U = AW\Lambda^+ + W(I - \Lambda\Lambda^+)$$
$$V = A^T W\Lambda^+ + W(I - \Lambda\Lambda^+)$$

where W is the eigenvector matrix for real symmetric circulants, defined in Section V. Then U and V are orthonormal matrices, and their columns have norm $\sqrt{N}$.

Proof: Consider V first. V will have been shown to be orthonormal with column norms $\sqrt{N}$ if it satisfies $V^{-1} = \frac{1}{N} V^T$, or $V V^T = NI$. We now show that V does indeed satisfy this equation:

$$V V^T = A^T W \Lambda^+ \Lambda^+ WA + A^T W\Lambda^+ (I - \Lambda \Lambda^+) W + W(I - \Lambda\Lambda^+)\Lambda^+ WA$$
$$+ W(I - \Lambda \Lambda^+) (I - \Lambda\Lambda^+)W$$
$$= A^T W(\Lambda^+)^2 WA + W(I - \Lambda \Lambda^+)^2 W$$

since $\Lambda^+(I - \Lambda\Lambda^+)$ is always zero for any Λ. (Since W is symmetric, we have written W instead of W^T here). Now, we note that $(\Lambda^+)^2$ and $(I - \Lambda\Lambda^+)^2$ satisfy the symmetry requirements of Theorem 5, so that $W(\Lambda^+)^2W$ and $W(I - \Lambda\Lambda^+)^2W$ are real symmetric circulants. Hence they can be rewritten as $X(\Lambda^+)^2\overline{X}$ and $X(I - \Lambda\Lambda^+)^2\overline{X}$. Noting also, from Eq. 12 and Lemma 15, that $A = \frac{1}{N}XH\overline{X}$ and $A^T = \frac{1}{N}X\overline{H}\overline{X}$, we have that

$$
\begin{aligned}
V V^T &= \left(\frac{1}{N}X\overline{H}\overline{X}\right) X(\Lambda^+)^2\overline{X}\left(\frac{1}{N}XH\overline{X}\right) + X(I - \Lambda\Lambda^+)^2\overline{X} \\
&= X\overline{H}(\Lambda^+)^2 H\overline{X} + X(I - \Lambda\Lambda^+)^2\overline{X} \\
&= X\overline{H}H(\Lambda^+)^2\overline{X} + X(I - \Lambda\Lambda^+)^2\overline{X} \\
&= X(\Lambda\Lambda^+)^2\overline{X} + X(I - \Lambda\Lambda^+)^2\overline{X}
\end{aligned}
$$

where the second line follows from the first by Lemma 2, the third follows from the second because diagonal matrices commute, and the fourth follows from the third because $H\overline{H} = \Lambda^2$, by Lemma 16. Now, $\Lambda\Lambda^+$ and $(I - \Lambda\Lambda^+)$ are diagonal matrices having only ones and zeroes on the main diagonal, so that $\Lambda\Lambda^+ = (\Lambda\Lambda^+)^2$ and $(I - \Lambda\Lambda^+) = (I - \Lambda\Lambda^+)^2$. Hence

$$
\begin{aligned}
V V^T &= X(\Lambda\Lambda^+)\overline{X} + X(I - \Lambda\Lambda^+)\overline{X} \\
&= X(I)\overline{X} \\
&= NI
\end{aligned}
$$

The proof for the orthonormality of U follows similar lines. Q.E.D.

Theorem 9: Let A be a real N x N circulant, let W be the orthonormal matrix defined in Section V, let Λ be the diagonal matrix with the singular values $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ of A (as defined in Lemma 16) as its diagonal elements, and let U and V be as defined in Lemma 17. Then

$$A = \frac{1}{N} W \Lambda V^T$$

and

$$A = \frac{1}{N} U \Lambda W$$

are singular value decompositions of A.

Proof: Note that AA^T and A^TA are real symmetric circulants (real because A is real; obviously symmetric; circulants by Lemma 5), so that, from Lemma 14, W may be used as the orthonormal eigenvector matrix called for in each of the above-mentioned procedures for finding singular value decompositions. The theorem then follows at once from these procedures and from Lemma 17. Q.E.D.

Let us now examine the significance of these results. The singular value decomposition procedures call for replacing the zero columns of $A\,W\,\Lambda^+$ and $A^T W\,\Lambda^+$ by any set of orthonormal column vectors normal to all nonzero column of $A\,W\,\Lambda^+$ and $A^T\,W\,\Lambda^+$, respectively. Lemma 17 shows that the columns of W corresponding to the zero columns of these matrices will always suffice, regardless of the other singular values. This somewhat surprising result follows from the fact (to be shown below) that the very same partitioning of the whole N-dimensional space into one- and two-dimensional subspaces observed in the eigenvalue decomposition of real symmetric circulants occurs here also. This result is expressed in the following lemmas, which also give simple expressions for finding the nonzero columns of $A\,W\,\Lambda^+$ and $A^T\,W\,\Lambda^+$ from W and H without matrix multiplication.

Lemma 18: The columns $\underline{v}_0, \underline{v}_1, \ldots, \underline{v}_{N-1}$ of the matrix V appearing in Lemma 17 and Theorem 9 can be obtained from the expressions

$$\underline{v}_0 = \operatorname{sgn} \eta_0 \, \underline{w}_0 \qquad \eta_0 \neq 0$$

$$\left.\begin{aligned} \underline{v}_k &= \frac{R_e \eta_k}{|\eta_k|} \underline{w}_k - \frac{I_m \eta_k}{|\eta_k|} \underline{w}_{N-k} \\ \underline{v}_{N-k} &= \frac{R_e \eta_k}{|\eta_k|} \underline{w}_{N-k} + \frac{I_m \eta_k}{|\eta_k|} \underline{w}_k \end{aligned}\right\} \left\{\begin{aligned} &\eta_k \neq 0 \\ &k=1,2,\ldots, \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases} \end{aligned}\right.$$

$$\underline{v}_{N/2} = \operatorname{sgn} \eta_{N/2} \, \underline{w}_{N/2} \qquad N \text{ even}, \ \eta_{N/2} \neq 0$$

When $\eta_0 = 0$, $\underline{v}_0 = \underline{w}_0$. When $\eta_k = \eta_{N-k} = 0, \underline{v}_k = \underline{w}_k$ and $\underline{v}_{N-k} = \underline{w}_{N-k}$. When $\eta_{N/2} = 0, \underline{v}_{N/2} = \underline{w}_{N/2}$. Here $\operatorname{sgn} \eta = \begin{cases} +1 & \eta > 0 \\ -1 & \eta < 0 \end{cases}$

Proof: Let us look at a particular column $\underline{v}_k$ of V $(0 < k < N/2)$. If $\eta_k = 0$, then the first term of the expression $V = A^T W \Lambda^+ + W(I - \Lambda \Lambda^+)$ makes no contribution to this column, since the corresponding column of Λ^+ is then zero. The second term contributes $\underline{w}_k$.

If $\eta_k = 0$, then the second term contributes nothing and the first term yields

$$\underline{v}_k = A^T W[0,\ldots, \frac{1}{\lambda_k}, 0,\ldots,0]^T = \frac{1}{|\eta_k|} A^T \underline{w}_k = \frac{1}{N|\eta_k|} XH\bar{X} \underline{w}_k$$

by Lemmas 16 and 15. From the definitions of $\underline{w}_k$ and $\underline{w}_{N-k}$ given in Section V, it follows that

$$\bar{X}\underline{w}_k = [0,\ldots,0, N \frac{1-j}{2}, 0,\ldots,0, N \frac{1+j}{2}, 0,\ldots,0]^T$$

where, as in the steps to follow, the two nonzero entries are in the $(k+1)$-th and $(N-k+1)$-th positions. Then

$$\overline{HX}\underline{w}_k = [0,\ldots,0,N\,\bar{\eta}_k\,\frac{1-j}{2},0,\ldots,0,\;N\,\eta_k\,\frac{1+j}{2},0,\ldots,0]^T$$

$$\underline{v}_k = \frac{1}{N|\eta_k|}\,X\overline{HX}\underline{w}_k = \frac{\bar{\eta}_k}{|\eta_k|}\,\frac{1-j}{2}\,\underline{x}_k + \frac{\eta_k}{|\eta_k|}\,\frac{1+j}{2}\,\underline{x}_{N-k}$$

But, by inverting the equations of Section V which define $\underline{w}_k$ and $\underline{w}_{N-k}$ in terms of $\underline{x}_k$ and $\underline{x}_{N-k}$, we obtain

$$\underline{x}_k = \frac{1+j}{2}\,\underline{w}_k + \frac{1-j}{2}\,\underline{w}_{N-k} \quad \text{and} \quad \underline{x}_{N-k} = \frac{1-j}{2}\,\underline{w}_k + \frac{1+j}{2}\,\underline{w}_{N-k}$$

which, upon substitution into the above expression for $\underline{v}_k$, yield the results stated in the lemma. The other results follow in similar fashion. Q.E.D.

Lemma 19: The columns $\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_{N-1}$ of the matrix U appearing in Lemma 17 and Theorem 9 can be obtained from the expressions

$$\underline{u}_0 = \operatorname{sgn} \eta_0\, \underline{w}_0 \qquad \eta_0 \neq 0$$

$$\left.\begin{aligned} \underline{u}_k &= \frac{R_e\eta_k}{|\eta_k|}\,\underline{w}_k + \frac{I_m\eta_k}{|\eta_k|}\,\underline{w}_{N-k} \\ \underline{u}_{N-k} &= \frac{R_e\eta_k}{|\eta_k|}\,\underline{w}_{N-k} - \frac{I_m\eta_k}{|\eta_k|}\,\underline{w}_k \end{aligned}\right\} \left\{\begin{aligned} &\eta_k \neq 0 \\ &k \text{ integer},\ 0 < k < \frac{N}{2} \end{aligned}\right.$$

$$\underline{u}_{N/2} = \operatorname{sgn} \eta_{N/2}\, \underline{w}_{N/2} \qquad N \text{ even},\ \eta_{N/2} \neq 0$$

When $\eta_0 = 0$, $\underline{u}_0 = \underline{w}_0$. When $\eta_k = \eta_{N-k} = 0$, $\underline{u}_k = \underline{w}_k$ and $\underline{u}_{N-k} = \underline{w}_{N-k}$. When $\eta_{N/2} = 0$, $\underline{u}_{N/2} = \underline{w}_{N/2}$.

Proof: The proof parallels that of Lemma 18 exactly except that the decomposition $\frac{1}{N} X H \bar{X}$ of A is used in place of the decomposition $\frac{1}{N} \overline{X H X}$ of A^T. Q.E.D.

We have seen that the matrix W can always be taken (at will) as either the input basis or the output basis for a singular value decomposition of a real circulant. It is easily shown that there are special cases where it is both the input basis and the output basis, or where one basis or the other differs from W only in the signs of some of its basis vectors.

Lemma 20: Let A be a real symmetric circulant. Then the matrices U and V appearing in the singular value decompositions $A = \frac{1}{N} W \Lambda V^T$ and $A = \frac{1}{N} U \Lambda W$ (as given by Lemma 17) are equal, and differ from W only in the signs of some of their columns, as follows:

$$U = V = W[\mathrm{sgn}\ H]$$

where [sgn H] is by definition a diagonal matrix with diagonal elements sgn η_0, sgn η_1, ..., sgn η_{N-1} and where, in this case, $\mathrm{sgn}\ \eta = \begin{Bmatrix} +1 & \eta \geq 0 \\ -1 & \eta < 0 \end{Bmatrix}$.

Proof: We note from Theorem 5 that for real symmetric circulants the eigenvalues are real, so that Lemmas 18 and 19 reduce to the stated result.

Q.E.D.

Lemma 21: Let A be a real, symmetric, positive semidefinite circulant. That is, let A be a real symmetric circulant with non-negative eigenvalues. Then the singular value decompositions given by Theorem 9 both reduce to

$$A = \frac{1}{N} W \Lambda W = \frac{1}{N} W H W$$

This is, the eigenvalue and singular value decompositions given above are the same.

Proof: The first part of the result follows from Lemma 20, since in this case [sgn H] = I. The second part follows from Lemma 16 and from the real non-negative nature of the eigenvalues $\eta_0, \eta_1, \ldots, \eta_{N-1}$.

The results of this section can easily be generalized to singular value decompositions of complex matrices, by starting from the eigenvalue decomposition. The singular values must still be real and nonnegative, but the input and output basis vectors may be complex. Let us define "phase" functions of the form.

$$\text{phase } \eta = \begin{cases} \eta / |\eta| & \eta \neq 0 \\ 1 & \eta = 0 \end{cases}$$

and

$$[\text{phase } H] = \begin{bmatrix} \text{phase } \eta_0 & & & 0 \\ & \text{phase } \eta_1 & & \\ & & \ddots & \\ 0 & & & \text{phase } \eta_{N-1} \end{bmatrix}$$

We can then state

<u>Theorem 10</u>: Let A be a general circulant with eigenvalue matrix H. Then

$$A = \frac{1}{N} X \Lambda (\overline{X} \, [\text{phase } H])^T$$

is a singular value decomposition, where the diagonal elements of the diagonal matrix Λ are $|\eta_0|, |\eta_1|, \ldots, |\eta_{N-1}|$.

Proof: The equation is valid, since Λ[phase H] = H, $\overline{X}^T = X$, and $A = \frac{1}{N} XH\overline{X}$, by Eq. 12. Also, the columns of ($\overline{X}$[phase H]) form a valid (i.e., orthonormal) input basis, since $\overline{X}$ is orthonormal, and multiplying a column of $\overline{X}$ by a constant of unit modulus changes neither the norm of the column nor the subspace spanned by it. Finally, Λ has the required form, so that this is indeed a singular value decomposition. Q.E.D.

IX. LEFT CIRCULANTS

The rather extensive results derived above for circulants might lead one to ask if there are other classes of matrices for which comparable results can be obtained. Bellman (Ref. 1, p. 235, Exercises 1 and 2) suggests that there are other such classes, based on the roots of general N-th order polynomials (rather than on the roots of the polynomial $r^N = 1$, from which the circulant eigenvector matrix X is derived). Another class of matrices which, to the knowledge of this author, has not been investigated in detail, is the class of <u>left circulants</u>; that is, square matrices of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-2} & a_{N-1} \\ a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_0 \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} & a_{N-2} \end{bmatrix} \qquad (29)$$

which can be seen to resemble circulants except that each row is equal to the row above, circulated to the <u>left</u> rather than to the right—thus the name. The name "circulant," without modifier, will be reserved for circulants as originally defined, but in the following discussion, to add emphasis and avoid confusion, circulants, as originally defined, will be referred to as "right circulants."

Like right circulants, left circulants can be profitably studied as a special class because of simplifications that can be made in the procedures for manipulating them—although the author is not aware of there being any applications in which they arise. They are discussed here because the results obtained above for right circulants offer an excellent basis for deriving and evaluating the sometimes similar and sometimes contrasting properties of left circulants.

Let us note first that left circulants of a given order (greater than two) do not all have the same set of eigenvectors. In fact, the eigenvectors depend on the elements of the matrix, in general. For example, consider the general third-order left circulant A characterized by $[a_0, a_1, a_2]^T$. Its eigenvalues are $(a_0 + a_1 + a_2)$, $\eta = + [a_0^2 + a_1^2 + a_2^2 - a_0a_1 - a_0a_2 + a_1a_2]^{1/2}$, and $-\eta$, and the corresponding eigenvectors are $[1,1,1]^T$, $[1,f(\eta), - 1 - f(\eta)]^T$, and $[1, f(-\eta), - 1 - f(-\eta)]^T$, where $f(\eta) = (\eta a_0 - a_0^2 + a_1 a_2)/(\eta a_2 - a_2^2 + a_0a_1)$. These eigenvectors clearly depend on the values of a_0, a_1, and a_2. Thus, the extensive results obtained above for right circulants (which followed from their all having the same eigenvector matrix X) should not be expected to have analogs in the case of left circulants.

Surprisingly, however, there is a decomposition for left circulants which, although not an eigenvalue-type decomposition, has many of the same advantages. Many of the results presented here, including eigenvalue and singular value decompositions will be based on or derived from this decomposition.

We begin with the following decomposition theorem. Note that here the quantities $\eta_0, \eta_1, \ldots, \eta_{N-1}$ appearing in this theorem are in general not the eigenvalues of the left circulant A. We reserve these symbols for the eigenvalues of the right circulant characterized by the same vector $\underline{a}$. That is, $\underline{\eta}$ is as defined by Eq. 7 or 8. In this section we shall often discuss the properties of left circulants in terms of these same η's for convenience.

Theorem 11: Let A be an N x N left circulant characterized by the vector $\underline{a}$. Then

$$A = \frac{1}{N} X G \bar{X} \tag{30}$$

where X is the same matrix defined above (Eq. 5) for right circulants, where G is a matrix of the form

$$G = \begin{bmatrix} \eta_0 & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & 0 \ldots 0 & \eta_{N-1} \\ 0 & 0 & 0 \ldots \eta_{N-2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \eta_2 \ldots 0 & 0 \\ 0 & \eta_1 & 0 \ldots 0 & 0 \end{bmatrix} \tag{31}$$

and where the components $\eta_0, \eta_1, \ldots, \eta_{N-1}$ are given by Eq. 7 or 8.

Proof: Let us operate on any one of the basis vectors $\underline{x}_k$ of X with the matrix $\frac{1}{N} X G \bar{X}$. Because these vectors are orthonormal with norm $\sqrt{N}$ (see Lemma 2), and because of the particular form of G, the result is

$$\frac{1}{N} X G \bar{X} \; \underline{x}_k = \begin{cases} \eta_0 \, \underline{x}_0 & k = 0 \\ \eta_k \, x_{N-k} & k = 1, 2, \ldots, N-1 \end{cases}$$

Now operate on this same $\underline{x}_k$ with the matrix A. The result is

$$A\,\underline{x}_k = \begin{bmatrix} a_0 + a_1 r_k + a_2 r_k^2 + \quad \cdots \quad + a_{N-1} r_k^{N-1} \\ a_0 r_k^{N-1} + a_1 + a_2 r_k + \quad \cdots \quad + a_{N-1} r_k^{N-2} \\ a_0 r_k^{N-2} + a_1 r_k^{N-1} + a_2 + \quad \cdots \quad + a_{N-1} r_k^{N-3} \\ \vdots \\ a_0 r_k + a_1 r_k^2 + a_2 r_k^3 + \quad \cdots \quad + a_{N-1} \end{bmatrix}$$

$$= (a_0 + a_1 r_k + \ldots + a_{N-1} r_k^{N-1}) \begin{bmatrix} 1 \\ \bar{r}_k \\ \bar{r}_k^2 \\ \vdots \\ \bar{r}_k^{N-1} \end{bmatrix} = \eta_k \bar{\underline{x}}_k$$

$$= \begin{cases} \eta_0 \; \underline{x}_0 & k = 0 \\ \eta_k \; \underline{x}_{N-k} & k = 1,2,\ldots,N-1 \end{cases}$$

This is true for each of the vectors $\underline{x}_k$, $k = 0,1,\ldots,N-1$. Since these vectors span the whole N-dimensional space, it then follows that $A = \frac{1}{N} XG\bar{X}$ over this whole space. Q.E.D.

Thus, as with right circulants, the partitioning of the N-dimensional space into the two-dimensional subspaces spanned by $(\underline{x}_1, \underline{x}_{N-1}),(\underline{x}_2, \underline{x}_{N-2})$, etc., plus the one-dimensional subspaces spanned by $\underline{x}_0$ (and, if N is even, $\underline{x}_{N/2}$), crops up again. Here, however, a "rotation" is involved, in that an input $\underline{x}$ colinear with one of the vectors $\underline{x}_k$ spanning a particular subspace produces an output $A\,\underline{x}$

colinear with the other vector spanning the subspace.

We now note various properties of left circulants, in a series of lemmas. The first lemma follows obviously from the definition of a left circulant, and the rest are easily established by using the decomposition given by Theorem 11 or by following steps similar to those used in establishing the corresponding lemma for right circulants.

Lemma 22: All left circulants are symmetric.

Lemma 23: Every matrix which can be decomposed in the form given by Theorem 11 is a left circulant.

Lemma 24: Sums and differences of left circulants are left circulants.

Lemma 25: The product of two left circulants is a right circulant.

Lemma 26: Let L be a left circulant and R a right circulant. Then LR and RL are left circulants, and LR = RL if and only if R is symmetric.

Lemma 27: Let L_1 and L_2 be left circulants. The $L_1\ L_2 = [L_2L_1]^T$. A sufficient (but not necessary) condition that $L_1\ L_2 = L_2\ L_1$ is that the vectors characterizing L_1 and L_2 both satisfy conditions of the form given by Eq. 20.

Lemma 28: Let L be a left circulant characterized by $\underline{a}$. Then L^+, the pseudo-inverse of L (which becomes the inverse if the inverse exists), is a left circulant characterized by $\underline{b} = \frac{1}{N}\overline{X}\ \underline{\psi}$, where the elements of the column vector $\underline{\psi}$ satisfy the conditions

$$\psi_k = 0 \quad \text{whenever} \quad \eta_k = 0$$

$$\psi_k = \begin{cases} 1/\eta_0 & k = 0 \\ 1/\eta_{N-k} & k = 1,2,\ldots,N-1 \end{cases} \quad \text{for } \eta_k \neq 0$$

and where, as always, $\underline{\eta} = X\,\underline{a}$.

Lemma 29: Let L and R be left and right circulants, respectively, characterized by the same vector $\underline{a}$, and let R be symmetric (i.e., $a_{N-k} = a_k$, $k = 1, 2, \ldots, N-1$). Then their pseudo-inverses L^+ and R^+ are left and right circulants, respectively, and are characterized by the same vector $\underline{b}$, as given by Lemma 28.

Lemma 30: Let L and R be left and right circulants, respectively, characterized by the same vector $\underline{a}$. Then $L^2 = L\,L^T = L^TL = R\,R^T = R^TR$.

Lemma 31: The only left circulants which are also right circulants are characterized by vectors of the form

$$N \text{ odd:} \quad \underline{a} = [a_0, a_0, a_0, \ldots, a_0]^T$$

$$N \text{ even:} \quad \underline{a} = [a_0, a_1, a_0, \ldots, a_0, a_1]^T$$

Thus, odd-order right-and-left circulants have at most one nonzero eigenvalue, $\eta_0 = N\,a_0$, and even-order right-and-left circulants have at most two nonzero eigenvalues, $\eta_0 = N(a_0 + a_1)/2$ and $\eta_{N/2} = N(a_0 - a_1)/2$. Furthermore, for $N = 2$ every right circulant is also a left circulant, and vice versa.

The decomposition given by Theorem 11, along with the definitions of the vectors $\underline{w}_k$ and $\underline{w}_{N-k}$ given in Section V, allow the direct verification of the following eigenvalue decomposition theorem. For simplicity, we limit the theorem to real left circulants.

Theorem 12: Let L be a real left circulant characterized by $\underline{a}$, let $\eta_0, \eta_1, \ldots, \eta_{N-1}$ be the eigenvalues of the right circulant characterized by the same vector $\underline{a}$ (see Eq. 7), and let the vectors $\underline{w}_0, \underline{w}_1, \ldots, \underline{w}_{N-1}$ be as defined in Section V. Then the eigenvalues $\gamma_0, \gamma_1, \ldots, \gamma_{N-1}$ of L are

$$\gamma_0 = \eta_0$$

$$\left.\begin{aligned} \gamma_k &= |\eta_k| \\ \gamma_{N-k} &= -|\eta_k| \end{aligned}\right\} \quad k = 1,2,\ldots, \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases}$$

$$\gamma_{N/2} = \eta_{N/2} \qquad N \text{ even}$$

and a corresponding orthonormal eigenvector matrix P with columns $\underline{p}_0, \underline{p}_1, \ldots, \underline{p}_{N-1}$ (having norm $\sqrt{N}$) is

$$\underline{p}_0 = \underline{w}_0$$

$$\left.\begin{aligned} \underline{p}_k &= \alpha\, \underline{w}_k + \beta\, \underline{w}_{N-k} \\ \underline{p}_{N-k} &= -\beta\, \underline{w}_k + \alpha\, \underline{w}_{N-k} \end{aligned}\right\} \quad k = 1,2,\ldots, \begin{cases} (N-1)/2 & \text{for } N \text{ odd} \\ (N-2)/2 & \text{for } N \text{ even} \end{cases}$$

$$\underline{p}_{N/2} = \underline{w}_{N/2} \qquad N \text{ even}$$

where $\alpha = 1$, $\beta = 0$ when $\eta_k = 0$ and

$$\left.\begin{aligned} \alpha &= \sqrt{\frac{|\eta_k| + \operatorname{Im} \eta_k}{2|\eta_k|}} \\ \beta &= \sqrt{\frac{|\eta_k| - \operatorname{Im} \eta_k}{2|\eta_k|}} \end{aligned}\right\} \quad \text{when } \eta_k \neq 0$$

Thus, L may be decomposed as

$$L = \frac{1}{N} P \Gamma P^T$$

where Γ is the diagonal matrix of eigenvalues $\gamma_0, \gamma_1, \ldots, \gamma_{N-1}$.

Proof: This theorem can be verified directly by showing that each of the vectors $\underline{p}_k$ is an eigenvector with the stated eigenvalue, using Theorem 11 to simplify the computations. The proof is omitted here.

Note that, in general, the eigenvectors depend on the values of $\eta_1, \ldots, \eta_{N-1}$, and hence are not independent of $\underline{a}$. However, as an aside we note also that if all the eigenvalues except η_0 and $\eta_{N/2}$ are pure imaginary, then the eigenvector matrix P differs from W by no more than the interchanging of certain pairs of columns $\underline{p}_k$ and $\underline{p}_{N-k}$. Thus, for these cases the eigenvalue decomposition can be written in terms of W simply by interchanging pairs of eigenvalues in Γ. We can therefore identify, if we wish, a special class of real left circulants which, in common with real symmetric right circulants, has the matrix W as its eigenvector matrix. This class turns out to be easily specified in terms of the elements of the characterizing vector $\underline{a}$.

Let us discuss this in terms of the right circulant A characterized by the same vector $\underline{a}$. From Table I, we see that a real skew-symmetric right circulant has eigenvalues meeting all these requirements, except that η_0 and $\eta_{N/2}$ are zero for such a circulant. Thus, if we can find a class of right circulants which have all zero eigenvalues except for these two, which are real, we can add any one of these to any real skew-symmetric right circulant and (by Lemma 4) the resulting right circulant will have the desired eigenvalue pattern.

Upon noting that all the real right-and-left circulants discussed in Lemma 31, have zero eigenvalues, except η_0 and $\eta_{N/2}$, which are real, the following lemma follows at once:

Lemma 32: Any real left circulant L for which the corresponding right circulant A has a symmetric part which is a right-and-left circulant—that is, which has a symmetric part characterized by a vector $\underline{b}$ of the form $[b_0, b_0, b_0, \ldots, b_0]$ for N odd or $[b_0, b_1, b_0, b_1, b_0, \ldots, b_1]$ for N even—has W as an eigenvector matrix. Such real left circulants have $(N + 1)/2$ degrees of freedom for N odd and $(N + 2)/2$ for N even.

If one wishes, one can use Theorem 12 to identify other special classes of left circulants sharing a common eigenvector matrix (other than W). For instance, the class of real left circulants for which $\text{Im}\ \eta_k = 0$ for all $k = 0, 1, \ldots, N-1$ share a common eigenvector matrix, and left circulants of this class can be identified as those for which the corresponding right circulant is symmetric. And so on.

We can also derive simplified procedures for finding singular value decompositions of left circulants. The proofs can be based either on Theorem 11 or Theorem 12, and are omitted here.

Theorem 13:

$$L = \frac{1}{N} W \Lambda V^T \text{ and } L = \frac{1}{N} V \Lambda W$$

are singular value decompositions of the real left circulant L, where Λ is a diagonal matrix with diagonal entries $|\eta_0|, |\eta_1|, \ldots, |\eta_{N-1}|$ and

$$V = L W \Lambda^+ + W(I - \Lambda\Lambda^+)$$

Lemma 33: The columns of the matrix V appearing in Lemma 31 can be obtained from $\underline{\eta}$ and W as follows:

$$\underline{v}_0 = \operatorname{sgn} \eta_0 \, \underline{w}_0 \qquad \eta_0 \neq 0$$

$$\left.\begin{aligned} \underline{v}_k &= -\frac{\operatorname{Im} \eta_k}{|\eta_k|} \underline{w}_k + \frac{\operatorname{Re} \eta_k}{|\eta_k|} \underline{w}_{N-k} \\ \underline{v}_{N-k} &= \frac{\operatorname{Im} \eta_k}{|\eta_k|} \underline{w}_{N-k} + \frac{\operatorname{Re} \eta_k}{|\eta_k|} \underline{w}_k \end{aligned}\right\} \quad \left\{\begin{aligned} &\eta_k \neq 0 \\ &k = 1, \ldots, \begin{cases} (N-1)/2 & \text{for N odd} \\ (N-2)/2 & \text{for N even} \end{cases} \end{aligned}\right.$$

$$\underline{v}_{N/2} = \operatorname{sgn} \eta_{N/2} \, \underline{w}_{N/2} \quad \text{for N even, } \eta_{N/2} \neq 0$$

When $\eta_0 = 0$, $\underline{v}_0 = \underline{w}_0$. When $\eta_k = \eta_{N-k} = 0$, $\underline{v}_k = \underline{w}_k$ and $\underline{v}_{N-k} = \underline{w}_{N-k}$. When $\eta_{N/2} = 0$, $\underline{v}_{N/2} = \underline{w}_{N/2}$.

The results derived in this section can readily be used in studying many other properties of real left circulants. For instance, one can easily show from Theorem 12 and Lemma 31 that there are no positive definite or negative definite left circulants for $N > 2$, and that a left circulant can be positive semidefinite or negative semidefinite only if it is a right-and-left circulant. Extentions of the eigenvalue and singular value decomposition to complex-valued left circulants is also straightforward, and the results exhibit many more parallels to the corresponding results for right circulants.

X. CONCLUSIONS

This report has derived and tabulated many properties of circulants. When circulants do arise in practical applications, the results presented here have allowed and will allow great simplifications in certain matrix operations such as inversion.

Although it has no known practical applications, the material on left circulants has been included both for its own sake and so that it will be available should such practical applications arise

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A circulant is a square matrix which is completely determined by its first row in the following manner: The second row is obtained by shifting ("circulating") the first row one position to the right and placing the last entry of the first row in the first position of the second row. Each succeeding row is determined from the row above it in like manner. Circulants occur in various practical problems. This paper presents certain properties of circulants (Some already published and some new) which render such operations as matrix inversion, the determination of eigenvalues and eigenvectors, and even matrix multiplication considerably simpler than the corresponding operations with general matrices.

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