THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

FORMULATIONS AND SOLUTIONS OF THE EQUATIONS OF FLUID FLOW

By

Robert H. Wasserman

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan, 1957.

August, 1957

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ACKNOWLEDGMENTS

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TABLE OF CONTENTS

Page

CHAPTER VII

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

Most of the work done in fluid flow theory has been carried out only after certain assumptions are made regarding the geometry of the flow. That is, considerations are usually restricted at the very outset of the investigation to a specified subclass of fluid flows defined by one or more geometrical conditions. The most prominent example is, of course, the class of ordinary plane flows defined by the geometrical conditions that there exists a family of parallel plane stream surfaces and that the direction normal to these surfaces be a direction of "symmetry" or "degeneracy". Another important example is the class of axially symmetrical flows. The prominence of these classes of flows is at least in part due to the way in which the equations of fluid flow are usually formulated.

In this work we will first circumscribe our problem by imposing certain conditions - essentially not geometrical - to be assumed throughout, namely, we will confine ourselves to steady state flows for which viscosity and external forces can be neglected. One of our principal objectives will be to study classes of flows defined by geometrical conditions without trying to satisfy boundary conditions; i.e., we will apply "inverse methods" in the sense of Nemenyi(1)* and Prim (2). However before we attempt to prescribe such conditions we will try to formulate the usual equations of fluid flow (for our circumscribed problem) in more appropriate ways. These will be our "intrinsic" formulations.

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Intrinsic descriptions are well-known (3, p. 105). They have been used implicitly by many authors (e.g., Chaplygin's hodograph equations are closely related to an intrinsic formulation), and explicitly by some (4,5). Recently they have been employed more extensively and systematically by Coburn (6).

Finally, our equations will also have invariant form so that we will have the freedom to choose appropriate coordinate systems, as well as ask for flows having various geometrical properties.

CHAPTER II

INTRINSIC FORMULATIONS OF THE EQUATIONS OF MOTION AND THE EQUATION OF CONTINUITY

We will first briefly orient our work in the general field of fluid mechanics, and state the relevant basic equations. Then we shall proceed to change the equations to a form more appropriate for an inverse approach.

1. General Assumptions on the Fluid Flows under Consideration. For a fluid in a region of space-time with space curvilinear coordinates x^i $(i = 1, 2, 3)$ one defines the "Eulerian" functions ρ, the density p, the pressure, and v^i $(i = 1, 2, 3)$, the velocity of the particle at each point of the region. The motion of a fluid is subject to the laws of conservation of momentum and conservation of mass. These laws are expressed in terms of these functions as differential equations; the equations of motion and the equation of continuity, which can be written either in terms of a specified coordinate system, usually rectangular cartesian coordinates (7, pp, 5, 577), or in an invariant, vector (3, pp. 68, 539), or tensor (8, p. 260) form.

We shall be concerned only with non-viscous steady state flows, which are subject to no external forces. Since we shall be interested in intrinsic or geometrical properties of such fluid flows, it will be desirable to write our equations in an invariant form. Thus in tensor form the equations of motion and the equation of continuity for the fluid flows with which we will be concerned are

$$\rho v^j \nabla_j v_i = -\nabla_i p \tag{1.1}$$

$$\nabla_i \rho v^i = 0 \tag{1.2}$$

respectively. In these equations the notation ∇_i stands for covariant differentiation.

A fluid is also subject to the law of conservation of energy, and an equation of state which describes the physical properties of the fluid. Again, in our work our fluids and flows will be restricted by certain assumptions applied to these laws. For our approach it is most convenient to defer the introduction of these concepts until later (Chapter III). For the present we will find some interest in working with eqs. (1.1) and (1.2) alone.

2. <u>First Intrinsic Formulation</u>. Now we first describe eq. (1.1) in geometrical terms. In particular, it involves a congruence of curves, the stream lines, and a family of surfaces, the constant pressure surfaces. The relation between these geometrical entities expressed by eq. (1.1) is brought out more clearly if we describe the stream lines by their unit tangent vectors t^i, and write $v^i = q t^i$ where q is the magnitude of v^i. Then the vector $v^j \nabla_j v_i$ on the left-hand side of eq. (1.1) can be written in terms of t^i, and the unit principal normal vectors, n^i of the stream lines as follows:

$$v^j \nabla_j v_i = q t^j \nabla_j (q t_i) = \left(t^j \nabla_j \frac{q^2}{2}\right) t_i + q^2 t^j \nabla_j t_i$$
$$= \left(t^j \nabla_j \frac{q^2}{2}\right) t_i + q^2 K n_i$$

by the Frenet formula $t^j \nabla_j t_i = K n_i$ where K is the curvature of the stream lines. With this expansion eq. (1.1) becomes

$$\rho\left(t^j \nabla_j \frac{q^2}{2}\right) t_i + \rho q^2 K n_i = - \nabla_i p \qquad (2.1)$$

that is, expressed geometrically, the pressure gradient lies in the osculating planes of the stream lines. Introducing the unit binormal vectors, b^i, of the stream lines, we can write eq. (2.1) alternatively in the form

$$t^i \nabla_i p = - \rho t^j \nabla_j \frac{q^2}{2} \qquad (2.2)$$

$$n^i \nabla_i p = - \rho q^2 K \qquad (2.3)$$

$$b^i \nabla_i p = 0 \qquad (2.4)$$

Equation (2.1), or eqs. (2.2) - (2.4) may be said to constitute an intrinsic representation of the equations of motion; that is, the equations of motion are referred to the unit trihedron formed by the tangent, normal, and binormal vectors associated with the stream lines.

In the same way, we could have chosen other geometrical entities of the problem, on which to base an intrinsic representation, for example, the congruence of curves determined by $\nabla_i p$.

Equation (1.2) can be described geometrically also. Putting $v^i = q t^i$ in this equation we get $\nabla_i \rho q t^i = 0$ or,

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The term $\nabla_i t^i$ may be described as the mean curvature of the surfaces orthogonal to the stream lines, when such surfaces exist (6), and so in this case, eq. (2.5) gives a simple geometrical description of the change of ρq along stream lines. Eq. (2.5) is an intrinsic form of the continuity equation.

3. Second Intrinsic Formulation. Now we can go a little further with our intrinsic formulations. If we have a flow, then we not only have a congruence, t^i, the stream lines, but we also have two independent families of surfaces, the stream surfaces, obtained from the solutions of $t^i \nabla_i \Psi = 0$. Let $\{\omega = \text{const.}\}$ denote one of these families, let N^i be the unit normal vector to the surface, and let m^i denote the unit normal vector to the stream line which lies in the $\omega =$ const. surface. See Figure 1. Now we shall refer our equations to these $\omega =$ const. stream surfaces.

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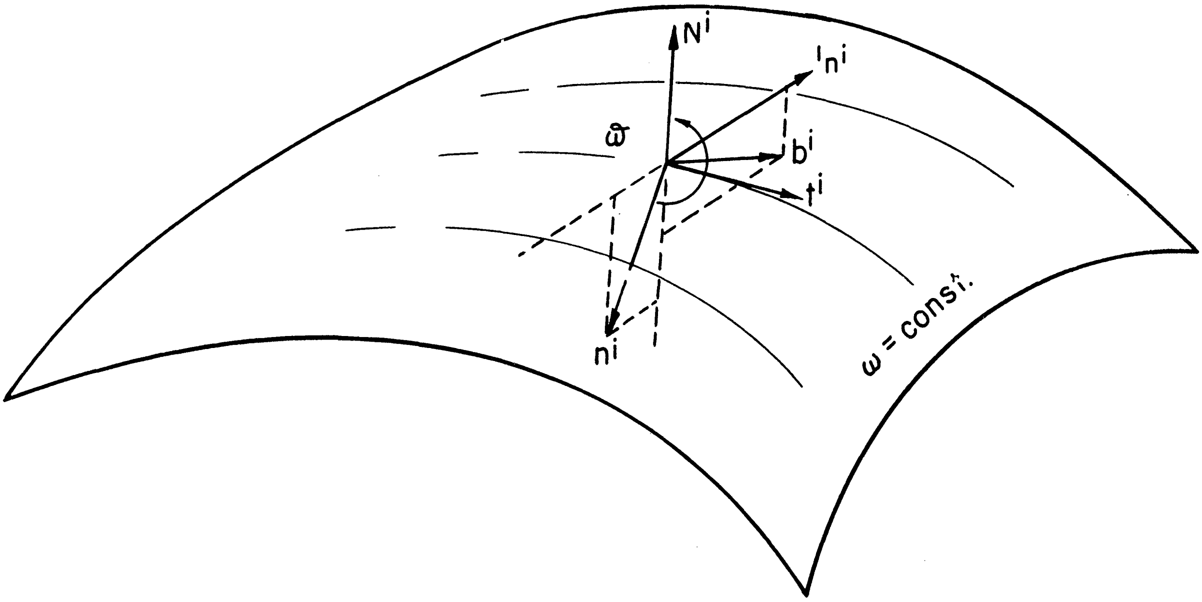

Figure 1. Vectors normal to the stream lines relative to a family of stream surfaces

a. <u>The Equations of Motion</u>. We can refer the equations of motion, eqs. (2.3) and (2.4), to the ω = const. surfaces by decomposing n^i and b^i into vectors in the $'n^i$ and N^i directions. To do this we introduce the angle ϖ between n^i and N^i. Then

$$n^i = \sin\varpi \; 'n^i + \cos\varpi \; N^i$$

But by Meusnier's theorem (9, p. 109)

$$K_N = K \cos \varpi \tag{3.1}$$

and

$$'K = K \sin \varpi \tag{3.2}$$

where $'K$ is the geodesic curvature and K_N is the normal curvature of the stream lines with respect to the ω = const. surfaces. Thus we obtain the decomposition

$$K n^i = 'K \, 'n^i + K_N N^i \tag{3.3}$$

Further, the cross product of eq. (3.3) and t^i yields (See Figure 1)

$$K b^i = - K_N \, 'n^i + 'K N^i \tag{3.4}$$

Substituting (3.3) and (3.4) into eqs. (2.3) and (2.4)respectively we get

$$'K\, 'n^i \nabla_i p + K_N N^i \nabla_i p = -\rho q^2 K^2$$

$$-K_N\, 'n^i \nabla_i p + 'K N^i \nabla_i p = 0$$

Then taking linear combinations of these two equations we get the intrinsic equations

$$'n^i \nabla_i p = -\rho q^2\, 'K \tag{3.5}$$

$$N^i \nabla_i p = -\rho q^2 K_N \tag{3.6}$$

b. <u>A Decomposition of $\nabla_i t^i$</u>. We can also refer the equation of continuity, eq. (2.5), to the ω = const. surfaces. This will be done by decomposing the divergence, $\nabla_i t^i$, appearing in eq. (2.5). The decomposition will also provide an additional geometrical interpretation of $\nabla_i t^i$.

Let $\{\psi = \text{const.}\}$ denote a second (independent) family of stream surfaces. On a surface ω = const. we have $\psi = \psi(u^1, u^2)$ where u^ξ $(\xi = 1, 2)$ are surface coordinates. If $'\nabla_\xi \psi$ denotes the surface gradient of ψ, $|'\nabla \psi|$ its magnitude and $|\nabla \omega|$ the magnitude of $\nabla_i \omega$ then we have the relation

$$\nabla_i t^i = -t^\xi\, '\nabla_\xi \ln|'\nabla\psi| - t^i \nabla_i \ln|\nabla\omega| \tag{3.7}$$

We prove (3.7) by introducing a particular coordinate system into our space, E^3. We choose the ω = const. surfaces as a family of coordinate surfaces X^3 = const., and we let the ψ = const. stream surfaces be the coordinate surfaces X^2 = const. The X^1 = const. coordinate surfaces may be any family such that a bona fide coordinate system is obtained. For such a coordinate system, with metric coefficients g_{ij},

$$t^i = \left(\frac{1}{\sqrt{g_{11}}} , 0 , 0 \right) \tag{3.8}$$

and

$$|\nabla \omega| = \sqrt{g^{ij} \frac{\partial \omega}{\partial X^i} \frac{\partial \omega}{\partial X^j}} = \sqrt{g^{33}} \tag{3.9}$$

On an $X^3 = \omega$ = const. surface we have

$$(ds)^2 = g_{11}(dx^1)^2 + 2 g_{12}\, dx^1 dx^2 + g_{22}(dx^2)^2$$

so that on such a surface with coordinate curves $u^1 = X^1$ = const. and $u^2 = X^2$ = const. the functions g_{11}, g_{12}, g_{22} in which the argument X^3 is constant are the metric coefficients $'g_{11}, 'g_{12}, 'g_{22}$ respectively of the surface. See Figure 2.

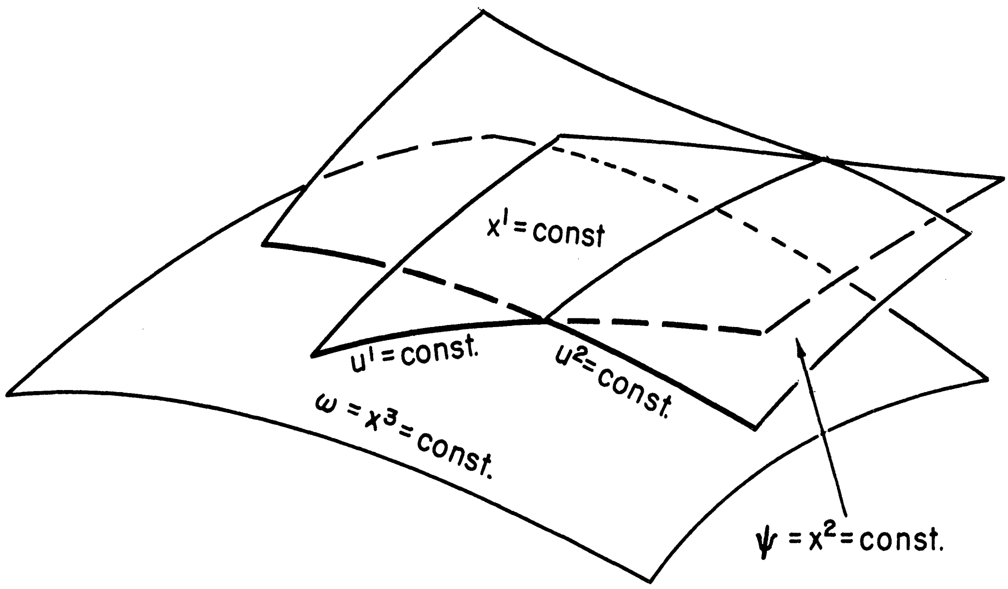

Figure 2. Space coordinates X^i and surface coordinates u^ξ on $X^3 = const.$

The quantities t^ξ and $|'\nabla\psi|$ defined on the $\omega = X^3 =$ const. surfaces are then given by

$$t^\xi = \left(\frac{1}{\sqrt{'g_{11}}}\ ,\ 0\right) \tag{3.10}$$

$$|'\nabla\psi| = \sqrt{g^{\eta\xi}\frac{\partial\psi}{\partial u^\eta}\frac{\partial\psi}{\partial u^\xi}} = \sqrt{'g^{22}} \tag{3.11}$$

Finally, we write the divergence $\nabla_i t^i$ as (8, p. 171)

$$\nabla_i t^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g}\, t^i \tag{3.12}$$

where $g = |g_{ij}|$, the determinant of the matrix (g_{ij}).

Now to prove (3.7) we substitute (3.8) - (3.12) into the expression

$$\nabla_i t^i + t^s\, '\nabla_s \ln |'\nabla\psi| + t^i \nabla_i \ln |\nabla\omega|$$

and we get

$$\nabla_i t^i + t^s\, '\nabla_s \ln |'\nabla\psi| + t^i \nabla_i \ln |\nabla\omega|$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial X^1} \sqrt{\frac{g}{g_{11}}} + \frac{1}{\sqrt{'g_{11}}} \frac{\partial}{\partial u^1} \ln \sqrt{'g^{22}} + \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial x^1} \ln \sqrt{g^{33}} \tag{3.13}$$

Since on each fixed ω = const. surface $g_{11} = {'g_{11}}$ and $X^1 = u^1$ eq. (3.13) may be written

$$\nabla_i t^i + t^s\, '\nabla_s \ln |'\nabla\psi| + t^i \nabla_i \ln |\nabla\omega|$$

$$= \left[\frac{\partial}{\partial u^1} \ln \sqrt{\frac{g}{'g_{11}}} + \frac{\partial}{\partial u^1} \ln \sqrt{'g^{22}} + \frac{\partial}{\partial u^1} \ln \sqrt{g^{33}} \right] \frac{1}{\sqrt{'g_{11}}}$$

$$= \frac{1}{\sqrt{'g_{11}}} \frac{\partial}{\partial u^1} \ln \sqrt{\frac{g\, g^{33}\, 'g^{22}}{'g_{11}}} \tag{3.14}$$

But $$\frac{g g^{33}}{'g_{11}}\, 'g^{22} = \frac{'g_{11}\,'g_{22} - 'g_{12}^2}{'g_{11}} \; \frac{'g_{11}}{'g_{11}\,'g_{22} - 'g_{12}^2} = 1$$

so that eq. (3.14) reduces to

$$\nabla_i t^i + t^\xi \,'\nabla_\xi \ln |'\nabla\psi| + t^i \nabla_i \ln |\nabla\omega| = 0$$

which is what we desired to prove.

In order to interpret eq. (3.7) geometrically, let us consider the two terms on the right-hand side separately.

The first term has the factor $t^\xi \,'\nabla_\xi |'\nabla\psi|$ which is a measure of the divergence of the stream lines on an ω = const. surface. In particular, consider the case when the stream lines are parallel on ω = const.; that is, the geodesics on the surface normal to any one stream line are also normal to all the others, or equivalently, the distance between any two stream lines on the surface measured along the geodesics normal to one of them is constant along the stream lines. See Appendix I(1) for a brief discussion of parallel curves on a surface. In this case, and in only this case we have the condition that $|'\nabla\psi|$ is a function of ψ alone. (See Appendix I1). However, the condition that $|'\nabla\psi|$ is a function of ψ alone is equivalent to the condition that $|'\nabla\psi|$ is constant when ψ is constant, that is, along stream lines. Thus the stream lines are parallel on an ω = const. surface if and only if $t^\xi \,'\nabla_\xi |'\nabla\psi| = 0$.

The second term on the right-hand side of eq. (3.7) has the factor $t^i \nabla_i \ln|\nabla\omega|$ which is a measure of the divergence of the stream surfaces ω = const. as we move along a stream line. In particular, consider the case when the surfaces ω = const. are parallel; that is, the straight lines normal to any one surface are also normal to all the others, or equivalently, the distance between any surfaces measured along the straight lines normal to one of them is constant on the surfaces. See Appendix I 2 for a brief discussion of parallel surfaces. In this case we have the condition that $|\nabla\omega|$ is a function of ω alone. (See Appendix I below eq. (I.7)) It follows that, in particular, $t^i \nabla_i \ln|\nabla\omega| = 0.$

Thus, in brief, we can interpret eq. (3.7) as decomposing $\nabla_i \cdot t^i$ into one part determined by the behavior of the stream lines on the ω = const. surfaces, and another part determined by the behavior of the family $\{\omega = \text{const.}\}$ itself.

Since for any scalar function f we have

$$t^{\xi}\,'\nabla_{\xi} f = t^{\xi} \frac{\partial x^i}{\partial u^{\xi}} \nabla_i f = t^i \nabla_i f$$

we see from eq. (3.7) that $\nabla_i \cdot t^i$ can also be written in the form

$$\nabla_i \cdot t^i = -t^i \nabla_i \ln|'\nabla\psi||\nabla\omega| \tag{3.15}$$

As pointed out to the author by Professor Coburn, $\nabla_i t^i$ can be written in still another form which will be useful in Chapter IV. To obtain this expression for $\nabla_i t^i$ we expand the identity

$$\nabla_i \nabla_j \omega = \nabla_j \nabla_i \omega$$

or,

$$\nabla_i |\nabla \omega| N_j = \nabla_j |\nabla \omega| N_i$$

and get

$$\nabla_i N_j + N_j \nabla_i \ln|\nabla \omega| = \nabla_j N_i + N_i \nabla_j \ln|\nabla \omega| \tag{3.16}$$

Multiplying (3.16) successively by N^i and t^j we get

$$t^j N^i \nabla_i N_j = t^j \nabla_j \ln|\nabla \omega| \tag{3.17}$$

Thus, the second term in the decomposition (3.7) of $\nabla_i t^i$ may be replaced by $t^j N^i \nabla_i N_j$, that is

$$\nabla_i t^i = -t^s \nabla_s \ln|\nabla \psi| - t^j N^i \nabla_i N_j \tag{3.18}$$

$N^i \nabla_i N_j$ is the curvature vector of the congruence defined by N^i, and so $t^j N^i \nabla_i N_j$ is the projection of this curvature vector along the stream lines.

c. The Equation of Continuity. Substituting the expression for $\nabla_i t^i$ given by eq. (3.7), or by eq. (3.15) into eq. (2.5) we obtain the equation of continuity in the intrinsic forms

$$t^i \nabla_i \ln \rho q = t^{\xi\prime} \nabla_{\xi} \ln |{}'\nabla\psi| + t^i \nabla_i \ln |\nabla\omega| \tag{3.19}$$

or,

$$t^i \nabla_i \frac{\rho q}{|{}'\nabla\psi|\,|\nabla\omega|} = 0 \tag{3.20}$$

respectively. Thus, we have the change of ρq along stream lines expressed geometrically in terms of the change of the magnitude of the gradient of ω and the change of the magnitude of the gradient of ψ in the ω = const. surfaces. A simple interpretation of eq. (3.20) is obtained by considering it to be a generalization of the "one dimensional" continuity equation (10, p. 102). The function $\frac{\rho q}{|{}'\nabla\psi|\,|\nabla\omega|}$ may be described as "the mass flow per unit cross-sectional area of a stream tube" at each point.

4. Intrinsic Formulations and Inverse Methods. In order to actually obtain solutions of our set of equations in intrinsic form we can proceed in one of two ways, just as we can when we start with the usual form,

eqs. (1.1) and (1.2) of the fluid flow equations. We can either choose a specific coordinate system in our space, E^3 , and try to solve the resulting differential equations, or else we can consider the g_{ij} appearing in the equations as unknowns, adjoin the integrability conditions $R_{ijkl} = 0$, where R_{ijkl} is the Riemann tensor of our space, and try to solve the resulting enlarged set of differential equations simultaneously. Our intrinsic formulation is probably no better than the usual formulation, eqs. (1.1) and (1.2), for proceeding in either of these two ways.

However, our intrinsic formulation does suggest another way of proceeding to which the usual formulation does not particularly lend itself. As already emphasized the intrinsic formulation brings into relief the geometrical representation of the functions and the relations among the functions describing the flow of a fluid. Thus we can classify fluid flows according to geometrical criteria, and, further, study classes of flows for which the equations can be simplified.

Also we can try to select coordinates with the object of finding general properties and/or simplifying our equations. Sometimes this can be done without loss of generality and sometimes not. For example, eqs. (2.2) - (2.4) can be simplified by choosing a coordinate system so that the congruence determined by t^i, n^i, and b^i are coordinate curves. Now this corresponds to imposing certain geometrical conditions. Thus the coordinate surfaces on which n^i and b^i lie are normal to t^i so there exist surfaces normal to the stream lines. Further the normal, n^i, of the stream lines is normal to the coordinate surfaces determined by t^i and b^i so there exist stream surfaces on which the stream lines are geodesics. Finally, the binormal, b^i, of the stream lines is normal to the coordinate surfaces determined by t^i and n^i so there exist stream surfaces on which the stream lines are asymptotic curves.

The class of flows having all these properties includes plane flows and axially symmetrical flows so it is not a particularly restrictive class. On the other hand, it is a subclass of the class of flows for which the stream lines can be taken as coordinate curves of an orthogonal coordinate system, and this class in turn is a subclass of the class of flows having a family of surfaces normal to the stream lines.

To get some idea of the scope of this last class of flows (those having a family of surfaces normal to the stream lines) we go back to ordinary plane flows and note that these are described by two conditions; namely, there exists a family of parallel plane stream surfaces, and there exists a direction (perpendicular to the planes) of "symmetry" or "degeneracy". Berker (11,5) calls flows satisfying only the first condition, pseudo-plane flows of the first kind, and he calls flows satisfying only the second condition, pseudo-plane flows of the second kind. Pseudo-plane flows of the first kind have been investigated recently by Kočina (12). By an example we will obtain later, we see that pseudo-plane flows of the second kind do not necessarily have surfaces normal to the stream lines.

It is perhaps worthwhile to note here that generalizations of pseudo-plane flows of the second kind, so called "degenerate flows", have been studied by several authors (13,14,15).

CHAPTER III

INTRINSIC EQUATIONS FOR INCOMPRESSIBLE FLUIDS AND FOR A CLASS OF COMPRESSIBLE FLUIDS

So far we have considered only the equations of motion and the equation of continuity. Before we proceed to apply geometrical conditions and look for classes of fluid flows we have yet, as remarked earlier, to adjoin a condition describing the physical properties of our fluid. We will want to study both incompressible and compressible fluids and so we will have to adjoin different conditions corresponding to these different kinds of fluids. Therefore, more specifically, at this point we will consider two separate cases corresponding to two important classes of fluids; homogeneous incompressible fluids, and a generalization of polytropic gases.

5. Homogeneous Incompressible Fluids. For a homogeneous incompressible fluid; i.e., ρ = const. we simplify our equations by putting

$$P = \frac{p}{\rho} \tag{5.1}$$

Then eq. (2.2) becomes

$$t^i \nabla_i \left(P + \frac{q^2}{2}\right) = 0 \tag{5.2}$$

or, defining

$$B = P + \frac{q^2}{2} \tag{5.3}$$

we have

$$t^i \nabla_i B = 0 \tag{5.4}$$

Eq. (5.3) is Bernoulli's equation for an incompressible fluid and B may be called the Bernoulli function.

Applying eq. (5.3) to eqs. (2.3) and (2.4) of the first intrinsic formulation we obtain

$$n^i \nabla_i P = -2(B-P)K \tag{5.5}$$

$$b^i \nabla_i P = 0 \tag{5.6}$$

and applying eq. (5.3) to eqs. (3.5) and (3.6) of the second intrisic formulation we obtain

$$'n^i \nabla_i P = -2(B-P)\,'K \tag{5.7}$$

$$N^i \nabla_i P = -2(B-P)K_N \tag{5.8}$$

Finally using eq. (5.3) in the equation of continuity, eq. (2.5), we get

$$t^i \nabla_i \ln(B-P) = -2\nabla_i t^i \tag{5.9}$$

or

$$t^i \nabla_i P = 2(B-P)\nabla_i t^i \tag{5.10}$$

or any one of several other forms obtained when the expressions for $\nabla_i t^i$ given by eqs. (3.7) or (3.15) are put into (5.9) or (5.10). In particular, when (3.15) is used in (5.9) we get

$$t^i \nabla_i \frac{B-P}{|\nabla\psi|^2|\nabla\omega|^2} = 0 \tag{5.11}$$

and when the stream lines are coordinate curves

$$\frac{B(\psi,\omega) - P}{|\nabla\psi|^2|\nabla\omega|^2} = \mu_1(\psi,\omega) \tag{5.12}$$

where μ_1 is an arbitrary function of the indicated variables. The equations (5.4) through (5.12) plus the indicated variations of these contain several complete sets of basic equations for our incompressible flows. Thus, for example, (5.4), (5.5), (5.6), and (5.10) constitute a complete set. Eqs. (5.4), (5.7), (5.8), and (5.12) constitute another complete set.

6. A Class of Compressible Fluids. Our generalization of polytropic gases is one considered by Prim (2) and by Hansen and Martin (16). Let us recall first that polytropic gases are perfect or ideal gases. For the latter the internal energy, E, of the gas is a function of temperature only (17, p. 8). For polytropic gases the internal energy is simply proportional to the temperature and in this case we have for an equation of state

$$\rho = \Xi\, p^{1/\gamma} e^{-S/c_p} \tag{6.1}$$

where S is the entropy of the gas, Ξ is a constant, $\gamma = c_p/c_v$ and c_p and c_v are the (constant) specific heats in constant pressure and volume changes, respectively.

We will consider now fluids with an equation of state

$$\rho = X(p)\, Y(S) \tag{6.2}$$

where X and Y are arbitrary functions having adequate differentiability for our future requirements. It should be noted that these fluids are generally not perfect gases. With this equation of state we have

$$\frac{\nabla_i p}{\rho} = \frac{\nabla_i p}{X(p)\, Y(S)} = \frac{1}{Y(S)} \nabla_i P(p) \tag{6.3}$$

where $\mathcal{P}$ is any solution of $\frac{d\mathcal{P}}{dp} = \frac{1}{X(p)}$. More specifically, we take

$$\mathcal{P}(p) = \int_0^p \frac{1}{X(p)}\, dp \tag{6.4}$$

Eq. (2.2) can now be written

$$t^i \nabla_i \mathcal{P} = -Y\, t^i \nabla_i \, q^2/2 \tag{6.5}$$

At this point we make the additional physical assumption that within the gas <u>the heat transfer by conduction may be neglected</u> (i.e., the gas has zero heat conductivity). Then there is no heat addition to a particle, and from the second law of thermodynamics (for a reversible process) the entropy of a particle must be constant. Thus we have (for steady flow)

$$t^i \nabla_i S = 0$$

That is, S is a function which is constant along stream lines but which may vary from stream line to stream line. (We can think of S as describing the initial values of the entropy over the set of stream lines.) Then in particular, $Y(S)$ can simply be considered to be a function defined over E^3 which is constant along stream lines, but which may vary from stream line to stream line. That is,

$$t^i \nabla_i Y = 0 \tag{6.6}$$

Now applying (6.6) to eq. (6.5) we have

$$t^i \nabla_i \left(P + \frac{q^2}{2} \Upsilon \right) = 0 \tag{6.7}$$

or, defining

$$\mathcal{B} = P + \frac{q^2}{2} \Upsilon \tag{6.8}$$

eq. (6.7) reduces to

$$t^i \nabla_i \mathcal{B} = 0 \tag{6.9}$$

Further, using eqs. (6.3) and (6.8), eqs. (2.3) and (2.4) of the first intrinsic formulation become

$$n^i \nabla_i P = -2(\mathcal{B} - P) K \tag{6.10}$$

$$b^i \nabla_i P = 0 \tag{6.11}$$

and eqs. (3.5) and (3.6) of the second intrinsic formulation become

$${}'n^i \nabla_i P = -2(\mathcal{B} - P)\, {}'K \tag{6.12}$$

$$N^i \nabla_i P = -2(\mathcal{B} - P) K_N \tag{6.13}$$

Finally, using (6.2), (6.6), and (6.8) in the equation of continuity, eq. (2.5), we get

$$t^i \nabla_i \ln X^2(\mathcal{B} - \mathcal{P}) = -2 \nabla_i t^i \tag{6.14}$$

or, since $t^i \nabla_i \ln X^2 = 2 \frac{dX}{dp} t^i \nabla_i \mathcal{P}$

$$\left[2(\mathcal{B} - \mathcal{P}) \frac{dX}{dp} - 1\right] t^i \nabla_i \mathcal{P} = -2(\mathcal{B} - \mathcal{P}) \nabla_i t^i \tag{6.15}$$

Note that in these equations X and $\frac{dX}{dp}$ can be considered to be functions of $\mathcal{P}$. Several other forms of the continuity equation are obtained when the expressions for $\nabla_i t^i$ given by eqs. (3.7) and (3.15) are put into (6.14) and (6.15). In particular, when (3.15) is used in (6.14) we get

$$t^i \nabla_i \frac{X^2(\mathcal{B} - \mathcal{P})}{|'\nabla\psi|^2 |\nabla\omega|^2} = 0 \tag{6.16}$$

and when the stream lines are coordinate curves

$$\frac{X^2(\mathcal{B}(\psi, \omega) - \mathcal{P})}{|'\nabla\psi|^2 |\nabla\omega|^2} = \mu_2(\psi, \omega) \tag{6.17}$$

where μ_2 is an arbitrary function of the indicated variables.

The eqs. (6.9) through (6.17) plus the indicated variations of these contain several complete sets of basic equations for our compressible fluids. Thus, for example, (6.9), (6.10), (6.11), and (6.15) constitute a complete set. Eqs. (6.9), (6.12), (6.13), and (6.17) constitute another complete set.

The new variables $\mathcal{P}$ and $\mathcal{B}$ appearing in our formulations are related to the more familiar functions enthalpy $h = \frac{p}{\rho} + E$ and stagnation enthalpy $H = h + \frac{q^2}{2}$. For, by the first law of thermodynamics we get

$$h = \frac{p}{\rho} + \int T\,dS - p\,d\frac{1}{\rho}$$

$$= \int \frac{1}{\rho}\,dp + T\,dS \tag{6.18}$$

where T is the temperature. If we integrate in the p, S plane from (0, 0) to (p, S) along the straight line segments $(0,0)$ to $(0,S)$ and $(0,S)$ to (p,S) , then eq. (6.18) may be written

$$h = \int_{(0,S)}^{(p,S)} \frac{1}{\rho}\,dp + \int_{(0,0)}^{(0,S)} T\,dS \tag{6.19}$$

Applying eqs. (6.2) and (6.4) to eq. (6.19) the first term on the right of (6.19) becomes $\frac{P}{\Upsilon}$. The second term on the right of (6.19) is a function of S alone. If we denote it by Z then eq. (6.19) becomes

$$h = \frac{P}{\Upsilon} + Z \tag{6.20}$$

Eq. (6.20) shows us the relation between P and h .

If we substitute P from eq. (6.20) into eq. (6.8) we get

$$\frac{B}{\Upsilon} = h - Z + \frac{q^2}{2} \tag{6.21}$$

so that

$$\frac{B}{\Upsilon} = H - Z \tag{6.22}$$

In terms of H eq. (6.21) becomes

$$H = h + \frac{q^2}{2} \tag{6.23}$$

the Bernoulli equation for compressible fluids. That is, with P expressed in terms of h by eq. (6.20) and with B expressed in terms of H by eq. (6.22), eq. (6.8) is transformed into Bernoulli's equation. Thus, we can consider (6.8) itself to be a form of Bernoulli's equation.

Another important compressible flow variable is the velocity of sound, defined by

$$c = \sqrt{\frac{\partial p}{\partial \rho}}$$

Then, by differentiating eq. (6.2) we get

$$c^2 = \frac{1}{\frac{dX}{dp} Y} \tag{6.24}$$

But differentiating eq. (6.4) we obtain

$$\frac{dX}{dp} = -\frac{1}{\left(\frac{d\rho}{dp}\right)^2} \frac{d^2\rho}{dp^2}$$

so that (6.24) may be written

$$c^2 = -\frac{\left(\frac{d\rho}{dp}\right)^2}{Y \frac{d^2\rho}{dp^2}} \tag{6.25}$$

Finally, we write some of the formulas above in the specific form they assume for the case of a polytropic gas. The basic relation is (comparing (6.1) and (6.2))

$$X(p) = p^{1/\gamma} \tag{6.26}$$

Then

$$\frac{dX}{dp} = \frac{1}{\gamma} p^{\frac{1-\gamma}{\gamma}} \tag{6.27}$$

Using (6.26) in the integrand eq. (6.4) becomes

$$\mathcal{P} = \frac{\gamma}{\gamma-1}\, p^{\frac{\gamma-1}{\gamma}} \tag{6.28}$$

Further,

$$\frac{d\mathcal{P}}{dp} = p^{-\frac{1}{\gamma}} \tag{6.29}$$

and

$$\frac{d^2\mathcal{P}}{dp^2} = -\frac{1}{\gamma}\, p^{-\frac{1+\gamma}{\gamma}} \tag{6.30}$$

The relation between h and p is obtained from (6.20) when $\mathcal{P}$ of eq. (6.28) is used, namely

$$h = \frac{1}{Y}\,\frac{\gamma}{\gamma-1}\, p^{\frac{\gamma-1}{\gamma}} + Z \tag{6.31}$$

Putting (6.29) and (6.30) into eq. (6.25) for the velocity of sound, we get

$$c^2 = \frac{\gamma}{Y}\, p^{\frac{\gamma-1}{\gamma}} \tag{6.32}$$

We indicated below eq. (6.15) that we can think of X and $\frac{dX}{dp}$ as functions of $\mathcal{P}$. In the polytropic case we eliminate p between (6.26) and (6.28) to get

$$\mathfrak{X} = \left(\frac{\gamma-1}{\gamma}\,\mathcal{P}\right)^{\frac{1}{\gamma-1}} \tag{6.33}$$

and we eliminate p between (6.27) and (6.28) to get

$$\frac{d\mathfrak{X}}{dp} = \frac{1}{\gamma-1}\,\frac{1}{\mathcal{P}} \tag{6.34}$$

7. Discussion and Applications of the Intrinsic Formulations. In this section we will say a little more about the function $\mathcal{B}$ of our formulations, and then obtain two important results in terms of these formulations.

a. The Functions B and $\mathcal{B}$. The Bernoulli function B is familiar, and we have discussed the function $\mathcal{B}$. In particular, $\mathcal{B}$ is described in terms of H in eq. (6.22). Also, it should be noted that $\mathcal{B}$ seems to play a role in the compressible case somewhat analogous to that of B in the incompressible case. We can pursue the comparison of B and $\mathcal{B}$ by introducing another important form of the equations of motion.

We go back to eq. (1.1) and we introduce the vorticity vector, $\Omega^k = \epsilon^{kij}\nabla_i v_j$ where ϵ^{kij} is the permutation tensor. Then the quantity $v^j\nabla_j v_i$ may be decomposed according to the identity (8, p. 261)

$$v^j \nabla_j v_i = -\epsilon_{ijk} v^j \Omega^k + \nabla_i \frac{q^2}{2}$$

Using this expression in eq. (1.1) we obtain the equations of motion in the form

$$-\epsilon_{ijk} v^j \Omega^k + \nabla_i \frac{q^2}{2} = -\frac{\nabla_i p}{\rho} \tag{7.1}$$

In the incompressible case replacing $\dfrac{\nabla_i p}{\rho}$ by $\nabla_i P$ and using eq. (5.3), eq. (7.1) can be written as

$$\epsilon_{ijk} v^j \Omega^k = \nabla_i B \tag{7.2}$$

In the compressible case, replacing $\dfrac{\nabla_i p}{\rho}$ by $\dfrac{\nabla_i \mathcal{P}}{\Upsilon}$ (eq.(6.3)) and using eq. (6.8), eq. (7.1) can be written as

$$\Upsilon \left(\epsilon_{ijk} v^j \Omega^k\right) = \nabla_i \mathcal{B} - \frac{q^2}{2} \nabla_i \Upsilon \tag{7.3}$$

When the compressible flow is isentropic, eq. (7.3) has the same form as eq. (7.2) with $\dfrac{\mathcal{B}}{\Upsilon}$ in place of **B**.

For incompressible flows, eq. (7.2) says that B is constant if and only if $\epsilon_{ijk} v^j \Omega^k = 0$. Flows having the property that $\epsilon_{ijk} v^j \Omega^k = 0$ are called Beltrami flows. Clearly, the class of Beltrami flows includes the flows with $\Omega^k = 0$, the irrotational flows.

For compressible flows eq. (7.3) says that if two of the quantities $\epsilon_{ijk} v^j \Omega^k$, $\nabla_i B$, $\nabla_i \Upsilon$ vanish then the third one vanishes also. This result as well as the general form of eq. (7.3) is reminiscent of Crocco's theorem (18, 19, 17, p. 22)

$$\epsilon_{ijk} v^j \Omega^k = \nabla_i H - T \nabla_i S$$

b. The Hodograph Equations of Chaplygin. For ordinary plane flows in the compressible case eq. (6.11) is identically satisfied so that our basic set of equations reduces to (6.9), (6.10), and (6.15).

Using (6.8) to eliminate ρ, and (6.24) to eliminate $\frac{d\Upsilon}{dp}$ eqs. (6.10) and (6.15) become

$$n^i \nabla_i \left(B - \frac{q^2}{2} \Upsilon \right) = -(q^2 \Upsilon) K \tag{7.5}$$

$$\left(\frac{q^2}{c^2} - 1 \right) t^i \nabla_i \left(B - \frac{q^2}{2} \Upsilon \right) = -(q^2 \Upsilon)(\nabla_i t^i) \tag{7.6}$$

In the isentropic, irrotational case $\mathfrak{B}$ = const. (See eq. (7.3)) and Υ can be taken to be unity so that eqs. (7.5) and (7.6) become

$$n^i \nabla_i q = q K \tag{7.7}$$

$$\left(\frac{q^2}{c^2} - 1\right) t^i \nabla_i q = q (\nabla_i t^i) \tag{7.8}$$

Eqs. (7.7) and (7.8) are special cases of equations obtained by Coburn (6). Professor Coburn has pointed out to the author that these intrinsic equations are closely related to the hodograph equations of Chaplygin.

To see this we first note that since in this case there are surfaces orthogonal to the stream lines, $\nabla_i t^i$ is the mean curvature of these surfaces (see discussion below, eq (2.5). Since a surface orthogonal to the stream lines is a cylinder its mean curvature is simply the negative of the curvature $\bar{K}$ of the curves orthogonal to the stream lines, the constant potential curves. Then

$$\nabla_i t^i = -\bar{K} \tag{7.9}$$

Introducing the stream lines and the constant potential curves as coordinates we have for the element of length

$$(ds)^2 = {}'g_{11}(d\phi)^2 + {}'g_{22}(d\psi)^2$$

where ϕ = const. are the constant potential curves and ψ = const. are the stream lines. If θ is the angle between the stream lines and the x-axis of a rectangular cartesian coordinate system in the plane, then

$$K = \frac{1}{\sqrt{'g_{11}}}\frac{\partial\theta}{\partial\phi} \qquad \bar{K} = -\frac{1}{\sqrt{'g_{22}}}\frac{\partial\theta}{\partial\psi} \tag{7.10}$$

In terms of the variables ϕ, and ψ and using eqs. (7.9) and (7.10) we can write eqs. (7.7) and (7.8) in the form

$$\frac{1}{\sqrt{'g_{22}}}\frac{\partial q}{\partial\psi} - \frac{q}{\sqrt{'g_{11}}}\frac{\partial\theta}{\partial\phi} = 0 \tag{7.11}$$

$$\left(\frac{q^2}{c^2} - 1\right)\frac{1}{\sqrt{'g_{11}}}\frac{\partial q}{\partial\phi} - \frac{q}{\sqrt{'g_{22}}}\frac{\partial\theta}{\partial\psi} = 0 \tag{7.12}$$

However, we have

$$\frac{1}{\sqrt{'g_{11}}} = q \tag{7.13}$$

since each side is equal to the derivative of ϕ with respect to length along the stream lines. Also, we can employ here eq. (6.17). In our special case this equation reduces to

$$\rho q \sqrt{'g_{22}} = \sqrt{K_2(\psi)}$$

Now if we parametrize the stream lines by ψ^* instead of ψ, where $\psi^* = \int \sqrt{K_2}\, d\psi$ then the new metric coefficient $'g^*_{22}$ is $'g^*_{22} = \frac{'g_{22}}{K_2}$ Thus, after making the change of parameter and then dropping the star notation we get

$$\rho q \sqrt{'g_{22}} = 1 \tag{7.14}$$

(cf. argument preceeding eq. (III.8) of Appendix III). By means of eqs. (7.13) and (7.14) we eliminate $\sqrt{'g_{11}}$ and $\sqrt{'g_{22}}$ from eqs. (7.11) and (7.12) to get (20)

$$\rho \frac{\partial q}{\partial \psi} - q \frac{\partial \theta}{\partial \phi} = 0 \tag{7.15}$$

$$\left(\frac{q^2}{c^2} - 1\right) \frac{\partial q}{\partial \phi} - \rho q \frac{\partial \theta}{\partial \psi} = 0 \tag{7.16}$$

Finally, we interchange the dependent and independent variables in (7.15) and (7.16), assuming the Jacobian does not vanish. (This assumption is not valid for simple wave flows (17, p. 39) and parallel flows.) This transformation results directly in the hodograph equations

$$\rho \frac{\partial \phi}{\partial \theta} - q \frac{\partial \psi}{\partial q} = 0$$

$$\left(\frac{q^2}{c^2} - 1\right) \frac{\partial \psi}{\partial \theta} - \rho q \frac{\partial \phi}{\partial q} = 0$$

c. Comparison of the Incompressible and the Compressible Cases. It will be noticed that we have been able to maintain a certain amount of parallelism of the developments for the incompressible and the compressible cases resulting in a certain correspondence between the equations of the resulting formulations. In particular, the compressible case is described by only four equations in four unknowns just as in the incompressible case. This is essentially an expression of the fact, referred to by Prim as the substitution principle (2), that there exists a decomposition of the class of compressible flows of 6b into equivalence classes in each of which all flows have the same stream line pattern and pressure distribution. Specifically, if one flow has $t^i = t^i_1$, $p = p_1$, $q = q_1$, $\rho = \rho_1$ and $\Upsilon = \Upsilon_1$, and another flow has $t^i = t^i_2$, $p = p_2$, $q = q_2$, $\rho = \rho_2$ and $\Upsilon = \Upsilon_2$, then the two flows are equivalent if $t^i_2 = t^i_1$, $p_2 = p_1$, $q_2^2 = \nu q_1^2$,

$\Upsilon_2 = \frac{1}{\nu}\Upsilon_1$ and $\rho_2 = \frac{1}{\nu}\rho_1$ where ν is an arbitrary function which is constant along stream lines.

Further, we notice that in any two corresponding sets of equations for the two cases, except for one pair (the continuity equation), the corresponding equations are actually formally the same. We shall look into the case later (Chapter VII) in which this exception submits to the rule; i.e., the case in which all the corresponding equations are formally the same.

CHAPTER IV

INTEGRABILITY CONDITIONS

Since we are using inverse methods, that is, we are interested in flows having certain geometrical properties, it is desirable to obtain any general conditions we can on the geometry of fluid flows. We will now obtain such conditions starting from the flow equations we derived in the previous chapter and obtaining integrability conditions for these equations. These integrability conditions will be examined and, in particular, it will be seen that results involving only the geometry of the stream lines can be derived from them.

First of all, we recall that among all the flow equations we obtained in the previous chapter we can pick out four equations in several different ways which together constitute a complete set of differential equations for the incompressible or the compressible flows. Starting with any such set of basic equations we can derive integrability conditions, and we can expect that, in general, different sets of basic equations will yield different conditions. In particular, we can either use a set of equations containing the continuity equation in the unintegrated form, (5.10) or (6.15), or we can use a set containing the continuity equation in the integrated form, (5.12) or (6.17). Also, we can either use the equations in the first intrinsic form or the equations in the second intrinsic form.

8. Integrability Conditions for the Equations in the First Intrinsic Form With the Unintegrated Continuity Equation. Here we shall consider primarily one particular set of flow equations; namely, the compressible flow equations

$$t^i \nabla_i \mathcal{B} = 0 \tag{6.9}$$

$$n^i \nabla_i \mathcal{P} = -2(\mathcal{B} - \mathcal{P}) K \tag{6.10}$$

$$b^i \nabla_i \mathcal{P} = 0 \tag{6.11}$$

$$\left[2(\mathcal{B} - \mathcal{P}) \frac{dX}{dp} - 1\right] t^i \nabla_i \mathcal{P} = -2(\mathcal{B} - \mathcal{P}) \nabla_i t^i \tag{6.15}$$

Note that this is a set of flow equations in what we have called the first intrinsic form.

We shall describe how we obtain the integrability conditions for these equations and will present the results. The calculations are relegated to Appendix II where they appear along with more general calculations valid for any complete set of flow equations containing the continuity equation in the unintegrated form. The results for the incompressible case will also be mentioned.

a. <u>Derivation of the Integrability Conditions of Equations (6.10), (6.11), and (6.15)</u>. If we think of eqs. (6.15), (6.10), and (6.11) as three differential equations for the function ρ , then they have conditions of integrability which are consequences of certain relations satisfied by the "intrinsic derivatives" $t^i \nabla_i \rho$, $n^i \nabla_i \rho$, and $b^i \nabla_i \rho$. Expressed in a compact form these relations are (21, p. 99)

$$\underset{v}{\lambda}^i \nabla_i (\underset{w}{\lambda}^j \nabla_j \rho) - \underset{w}{\lambda}^i \nabla_i (\underset{v}{\lambda}^j \nabla_j \rho) = \sum_{u=1}^{3} (\gamma_{uvw} - \gamma_{uwv}) \underset{u}{\lambda}^i \nabla_i \rho \tag{8.1}$$

where $\underset{1}{\lambda}^i = t^i$, $\underset{2}{\lambda}^i = n^i$, $\underset{3}{\lambda}^i = b^i$ and $\gamma_{uvw} = \underset{v}{\lambda}^i \underset{w}{\lambda}^j \nabla_j \underset{u}{\lambda}_i$

The quantities γ_{uvw} are defined for any system of triply orthogonal vector fields and are called <u>the coefficients of rotation</u> of the system. Two properties of these quantities which we shall use are the facts that (1) $\gamma_{uvw} = - \gamma_{vuw}$, and (2) $\gamma_{uvw} - \gamma_{uwv} = 0$ is the necessary and sufficient condition that these exists a family of surfaces orthogonal to the congruence determined by $\underset{u}{\lambda}^i$. Incidentally, we notice that if $\gamma_{uvw} - \gamma_{uwv} = 0$ for $u = 1, 2, 3$, and if we take the surfaces whose existence is guaranteed by the latter property as coordinate surfaces, then eq. (8.1) reduces to the ordinary conditions on the cross derivative of ρ .

Now if we substitute into eq. (8.1) the expressions for the intrinsic derivatives given by (6.15), (6.10), and (6.11), then expand the resulting equation, and finally eliminate the intrinsic derivatives of ρ again, we obtain <u>necessary conditions for the integrability of eqs. (6.15), (6.10), and (6.11)</u>. For the case of a polytropic gas our results are (see eqs. (11.9)-(11.11) of Appendix II).

$$\left[\tfrac{\gamma-1}{2}\rho(\beta-\rho)\right]\left[(\gamma_{123}-\gamma_{132})(\nabla_j t^j)\right]+\left[(\beta-\rho)\left(\beta-\tfrac{\gamma-1}{2}\rho\right)\right]\left[\ell^i\nabla_i\kappa-\gamma_{232}\kappa\right]$$
$$+\left[\left(\beta-\tfrac{\gamma+1}{2}\rho\right)(\ell^i\nabla_i\beta)\right]\kappa=0 \tag{8.2}$$

$$\left[\tfrac{\gamma-1}{2}\rho\left(\beta-\tfrac{\gamma+1}{2}\rho\right)(\beta-\rho)\right]\left[\ell^i\nabla_i(\nabla_j t^j)\right]-\left[(\beta-\rho)\left(\beta-\tfrac{\gamma+1}{2}\rho\right)^2\right]\left[(\gamma_{231}-\gamma_{213})\kappa\right]$$
$$-\left[\left(\tfrac{\gamma-1}{2}\rho\right)^2(\ell^i\nabla_i\beta)\right](\nabla_j t^j)=0 \tag{8.3}$$

$$\left[\tfrac{\gamma-1}{2}\rho\left(\beta-\tfrac{\gamma+1}{2}\rho\right)(\beta-\rho)\right]\left[n^i\nabla_i(\nabla_j t^j)\right]-\left[(\beta-\rho)\left(\beta-\tfrac{\gamma+1}{2}\rho\right)^2\right]\left[t^i\nabla_i\kappa-\gamma_{212}\kappa\right]$$
$$-\left[\tfrac{\gamma-1}{2}(\beta-\rho)\left(2\beta^2-\beta\rho-\tfrac{\gamma+1}{2}\rho^2\right)\right](\nabla_j t^j)\kappa-\left[\left(\tfrac{\gamma-1}{2}\rho\right)^2(n^i\nabla_i\beta)\right](\nabla_j t^j)=0 \tag{8.4}$$

In order to implement the further consideration of these rather complicated looking equations we have written each term as a product of two factors. The functions β and ρ appear only in the left-hand factor, and the right-hand factors contain only quantities dependent solely on the geometry of the stream line congruence. In particular, looking at the left-hand factors we see that eq. (8.2) is quadratic in ρ , and eqs. (8.3) and (8.4) are both cubic equations in ρ .

To see what these equations reduce to for the case of ordinary plane flows we look at the right-hand factors. Since for ordinary plane flows the binormal direction is normal to the planes of the stream lines and is a direction of symmetry, the derivative in the binormal direction of any quantity is zero. Thus,

$$b^i \nabla_i \kappa = b^i \nabla_i (\nabla_j t^j) = b^i \nabla_i \mathcal{B} = 0 \tag{8.5}$$

Moreover,

$$\underset{123}{\gamma} - \underset{132}{\gamma} = \underset{231}{\gamma} - \underset{213}{\gamma} = 0 \tag{8.6}$$

since there are surfaces normal to the stream lines and these are surfaces normal to the integral curves of the principal normals of the stream lines. Finally, from the definitions below eq. (8.1)

$$\underset{232}{\gamma} = b^i n^j \nabla_j n_i$$

So $\underset{232}{\gamma}$ is the projection in the binormal direction of the curvature vector of the principal normal congruence, and consequently

$$\underset{232}{\gamma} = 0 \tag{8.7}$$

Applying eqs. (8.5)-(8.7) to eqs. (8.2)-(8.4) we see that eqs. (8.2) and (8.3) are satisfied identically and we are left with only eq. (8.4).

b. Sufficient Conditions for the Integrability of Equations (6.9), (6.10), (6.11) and (6.15). In order to obtain sufficient conditions for integrability we can, on the one hand, ask that eqs. (8.2)-(8.4) be identities in ρ; that is, the condition of complete integrability. (22,p.15) Then the vanishing of the coefficients of the various powers of ρ constitute conditions on $\mathcal{B}$ and the geometry of the congruence of stream lines. As an example, it is seen that the condition of complete integrability is satisfied if $\nabla_i t^i = K = 0$. Flows with $\nabla_i t^i = K = 0$ have straight stream lines and (according to eqs. (6.10), (6.11), and (6.15)) $\nabla_i p = 0$.

On the other hand, if eqs. (8.2)-(8.4) are not identities in ρ then the question of sufficiency is more difficult than in the case of complete integrability.(22, pp. 16-18) Under the assumption $\nabla_i t^i = 0$, sufficient conditions can be obtained. These are derived in Chapter VII where we examine at some length the class of fluid flows having the property $\nabla_i t^i = 0$.

c. Conditions on the Geometry. Although eqs. (8.2)-(8.4) contain the functions $\mathcal{B}$, and ρ we will now see that it is possible to obtain conditions from these equations which involve only the geometry of the congruence of stream lines.

We have seen that when we employed both of the conditions $\nabla_i t^i = 0$ and $K = 0$, eqs. (8.2)-(8.4) were identically satisfied. Now let us apply each of these conditions alone.

When $K = 0$ only eq. (8.2), in general, yields a condition independent of β and ρ ; namely,

$$\underset{123}{\gamma} - \underset{132}{\gamma} = 0$$

This condition says that there exists a family of surfaces normal to the stream lines. This result may be obtained directly from eq. (2.1), from which we see that these surfaces are the constant pressure surfaces. We will use this result in Chapter VI.

When $\nabla_i t^i = 0$ eq. (8.3) gives

$$\underset{231}{\gamma} - \underset{213}{\gamma} = 0 \tag{8.9}$$

and eq. (8.4) gives

$$t^i \nabla_i K - \underset{212}{\gamma} K = 0 \tag{8.10}$$

Eq. (8.9) says that there exist surfaces normal to n^i, or there exists a family of stream surfaces on which the stream lines are geodesics. Since $\underset{212}{\gamma} = t^i n^j \nabla_j n_i$ and $n^j \nabla_j n_i$ is the curvature vector of the principal normal congruence of the stream lines, eq. (8.10) says that the change of $\ln K$ along stream lines is equal to the projection along the stream lines of the curvature vector of the principal normal congruence of the stream lines.

If when $\nabla_i t^i = 0$ we let the stream surfaces ω = const., which we introduced in Chapter II, be the surfaces on which the stream lines are geodesics, then $N_i = \pm m_i$ and $\underset{212}{\gamma} = t^i m^j \nabla_j m_i = t^i N^j \nabla_j N_i$. Thus, by eq. (3.17), $\underset{212}{\gamma} = t^j \nabla_j \ln |\nabla \omega|$ and with this eq. (8.10) becomes

$$t^i \nabla_i \frac{\kappa}{|\nabla \omega|} = 0 \tag{8.11}$$

that is, $\frac{\kappa}{|\nabla \omega|}$ <u>is constant along stream lines</u>. Also, from eq. (3.18) we have

$$-t^s {}'\nabla_s \ln |{}'\nabla \psi| - \underset{212}{\gamma} = 0$$

and with this, eq. (8.10) becomes

$$t^i \nabla_i \kappa |{}'\nabla \psi| = 0 \tag{8.12}$$

that is, $\kappa |{}'\nabla \psi|$ <u>is constant along stream lines</u>. These results for flows with $\nabla_i t^i = 0$ will be used in Chapter VII.

d. <u>Incompressible Flows</u>. We have thus far confined our attention to compressible flows as described by eqs. (6.9), (6.10), (6.11), and (6.15). A treatment analogous to that of 8a, 8b, and 8c can be carried out for incompressible flows. In fact, by using a more general notation much of the analysis can be performed for both types of flow simultaneously. This is done in Appendix II.

For the set of equations (5.4), (5.5), (5.6), and (5.10) for incompressible flows we can obtain equations analogous to eqs. (8.2)-(8.4) from the general equations (II.13)-(II.15) of Appendix II. Thus, the integrability conditions for eqs. (5.5), (5.6), and (5.10) are

$$(B-P)\left[b^i\nabla_i\kappa - \underset{232}{\gamma}\kappa\right] + (B-P)\left[\left(\underset{123}{\gamma} - \underset{132}{\gamma}\right)(\nabla_i t^i)\right] + (b^i\nabla_i B)\kappa = 0 \tag{8.13}$$

$$(B-P)\left[b^i\nabla_i(\nabla_j t^j)\right] - (B-P)\left[\left(\underset{231}{\gamma} - \underset{213}{\gamma}\right)\kappa\right] + (b^i\nabla_i B)(\nabla_i t^i) = 0 \tag{8.14}$$

$$(B-P)\left[t^i\nabla_i\kappa - \underset{212}{\gamma}\kappa\right] - (B-P)\left[n^i\nabla_i(\nabla_j t^j)\right] + (B-P)\left[\kappa(\nabla_i t^i)\right] - (n^i\nabla_i B)(\nabla_i t^i) = 0 \tag{8.15}$$

We see that the right-hand factors in eqs. (8.13)-(8.15) are the same as in eqs. (8.2)-(8.4). Further, looking at the left-hand factors we note that these equations are all linear in P. In particular, when $b^i\nabla_i B = n^i\nabla_i B = 0$ (i.e., when B is constant), the last term of each equation vanishes, $\mathcal{B}-\mathcal{P}$ may then be factored out of all the remaining terms, and the resulting equations no longer contain $\mathcal{B}$ or $\mathcal{P}$. Thus, in this case, eqs. (8.13)-(8.15) are also sufficient conditions for integrability.

Finally, we note that the results of 8c concerning the geometry of the stream lines are also valid for incompressible flows.

9. Other Integrability Conditions. For completeness it should be mentioned that if we use eqs. (6.12) and (6.13) instead of eqs. (6.10) and (6.11) in our set of flow equations (that is, using the second intrinsic formulation), then we obtain an apparently different set of integrability conditions. These conditions come directly out of the general results of Appendix II. However, they are more complicated looking than eqs. (8.2)-(8.4), and as yet nothing has been obtained from them which is not obtainable from eqs. (8.2)-(8.4).

Further, in the set of equations (6.9), (6.10), (6.11), and (6.15) used in section 8, we can replace (6.15) by eq. (6.16), or equivalently by

$$\underline{X}^2(\mathcal{B}-\mathcal{P}) = \mu_2 \, |'\nabla\psi|^2 |\nabla\omega|^2 \tag{9.1}$$

where μ_2 is any function such that $t^i \nabla_i \mu_2 = 0$. That is, the equations

$$t^i \nabla_i \mathcal{B} = 0 \tag{6.9}$$

$$n^i \nabla_i P = -2(\mathcal{B} - P)\kappa \tag{6.10}$$

$$b^i \nabla_i P = 0 \tag{6.11}$$

$$X^2(\mathcal{B} - P) = \mu_2 |'\nabla\psi|^2 |\nabla\omega|^2 \tag{9.1}$$

$$t^i \nabla_i \mu_2 = 0 \tag{9.2}$$

constitute a complete set of flow equations. For these equations we can also derive integrability conditions.

As mentioned at the beginning of the chapter, it would seem reasonable in an inverse method to try to obtain general conditions on the geometry of fluid flows. In Section 8 we have obtained some conditions which are relatively easily described. We have also indicated there and in this section how to obtain many others which were not explicitly presented because they are rather complicated and as yet not very enlightening geometrical interpretations of them have been found. Finally, it should be noted that many of our conditions contain the functions $\mathcal{B}$ and μ_2 (or B and μ_1) as well as the geometrical quantities. These functions are, in a sense, of secondary interest in our considerations since being constant along stream lines, they can be completely prescribed by boundary conditions. Thus, for example, one can often assume that far enough upstream the entropy, S, is constant and the vorticity is zero. It follows then, from eq. (7.3), that $\mathcal{B}$ = const.

CHAPTER V

FLOWS IN WHICH THE STREAM LINES ARE COORDINATE CURVES OF AN ORTHOGONAL COORDINATE SYSTEM

We will now obtain a characterization of a class of fluid flows which contains plane flows and axially symmetrical flows as special cases. This class, mentioned earlier (Section 4) is the class of flows for which the stream lines can be chosen as coordinate curves of an orthogonal coordinate system.

10. Some Stream Line Geometry. We suppose we have two mutually orthogonal families $\{\omega = \text{const.}\}$ and $\{\psi = \text{const.}\}$ of stream surfaces, and a family $\{\alpha = \text{const.}\}$ of surfaces perpendicular to the stream lines. If we choose these as our coordinate surfaces $\{X^3 = \text{const.}\}$, $\{X^2 = \text{const.}\}$, and $\{X^1 = \text{const.}\}$ respectively, then t^i, $'n^i$, and N^i are the unit tangent vectors to the coordinate curves and are simply expressed in terms of the metric coefficients;

$$t^i = \left(\frac{1}{\sqrt{g_{11}}},\ 0,\ 0\right)$$

$$'n^i = \left(0,\ \frac{1}{\sqrt{g_{22}}},\ 0\right) \tag{10.1}$$

$$N^i = \left(0,\ 0,\ \frac{1}{\sqrt{g_{33}}}\right)$$

where g_{ij} is the metric tensor of E^3.

$'K$ and K_N can also be simply expressed in terms of the metric coefficients. Thus using (3.3) in the Frenet formula $t^i \nabla_i t^j = K n^j$ we get

$$t^i \nabla_i t^j = 'K\, 'n^j + K_N N^j$$

which on writing $\nabla_i t^j = \dfrac{\partial t^j}{\partial x^i} + t^k \Gamma^j_{ki}$, and using (10.1) give, for j = 2 and 3

$$\frac{1}{g_{11}} \Gamma^2_{11} = 'K \frac{1}{\sqrt{g_{22}}}$$
$$\frac{1}{g_{11}} \Gamma^3_{11} = K_N \frac{1}{\sqrt{g_{33}}} \qquad (10.2)$$

Γ^j_{ki}, the Christoffel symbols of the second kind, are

$$\Gamma^j_{ki} = \frac{1}{2} g^{jj}\left(\frac{\partial}{\partial x^k} g_{ij} + \frac{\partial}{\partial x^i} g_{jk} - \frac{\partial}{\partial x^j} g_{ki}\right)$$

(no summation on j) for orthogonal coordinates. So

$$\Gamma^2_{11} = -\frac{1}{2} g^{22} \frac{\partial}{\partial x^2} g_{11}$$
$$\Gamma^3_{11} = -\frac{1}{2} g^{33} \frac{\partial}{\partial x^3} g_{11}$$

and eqs. (10.2) give

$$
\begin{aligned}
{}'\kappa &= -\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^2}\ln\sqrt{g_{11}} \\
\kappa_N &= -\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^3}\ln\sqrt{g_{11}}
\end{aligned}
\qquad (10.3)
$$

Further, using (3.3) and (3.4) in the Frenet formula $t^i\nabla_i b^j = -\tau n^j$ we get

$$
t^i\nabla_i\left(-\frac{\kappa_N}{\kappa}n^j + \frac{'\kappa}{\kappa}N^j\right) = -\tau\left(\frac{'\kappa}{\kappa}n^j + \frac{\kappa_N}{\kappa}N^j\right)
$$

when $\kappa \neq 0$. The scalar τ is the torsion of the stream lines. Expanding as before, and using (10.1) we get for $j = 2$ and 3

$$
\begin{aligned}
-\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\left(\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{22}}}\right) + \frac{1}{\sqrt{g_{11}}}\left[-\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{22}}}\Gamma_{21}^{2} + \frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{33}}}\Gamma_{31}^{2}\right] \\
= -\tau\frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{22}}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\left(\frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{33}}}\right) + \frac{1}{\sqrt{g_{11}}}\left[-\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{22}}}\Gamma_{21}^{3} + \frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{33}}}\Gamma_{21}^{3}\right] \\
= -\tau\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{33}}}
\end{aligned}
\qquad (10.4)
$$

But

$$
\Gamma_{31}^{2} = \Gamma_{21}^{3} = 0, \quad \Gamma_{21}^{2} = \frac{\partial}{\partial x^1}\ln\sqrt{g_{22}}, \quad \Gamma_{31}^{3} = \frac{\partial}{\partial x^1}\ln\sqrt{g_{33}}
$$

So eqs. (10.4) give

$$-\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\frac{1}{\sqrt{g_{22}}} - \frac{1}{\sqrt{g_{11}}}\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^1}\frac{\kappa_N}{\kappa}$$
$$+\frac{1}{\sqrt{g_{11}}}\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^1}\ln(\sqrt{g_{22}})^{-1} = -\tau\frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{22}}}$$

$$\frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\frac{1}{\sqrt{g_{33}}} + \frac{1}{\sqrt{g_{11}}}\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^1}\frac{'\kappa}{\kappa}$$
$$-\frac{1}{\sqrt{g_{11}}}\frac{'\kappa}{\kappa}\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^1}(\ln\sqrt{g_{33}})^{-1} = -\tau\frac{\kappa_N}{\kappa}\frac{1}{\sqrt{g_{33}}}$$

or

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\frac{\kappa_N}{\kappa} = \frac{'\kappa}{\kappa}\tau$$
$$\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\frac{'\kappa}{\kappa} = -\frac{\kappa_N}{\kappa}\tau \qquad (10.5)$$

By use of Meusnier's theorem, that is, eqs. (3.1) and (3.2), we can introduce the angle, ϖ , between the principal normal of the stream line and the normal of the $\omega = x^3$ = const. surface. Then eqs. (10.5) are

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\cos\varpi = \tau\sin\varpi$$
$$\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\sin\varpi = -\tau\cos\varpi$$

That is, when the stream lines are part of a triply orthogonal set, <u>a stream line is a plane curve if and only if the principal normal of the stream line makes a fixed angle with each of the orthogonal stream surfaces.</u>

11. <u>Flows with B = Const. or $\mathcal{B}$ = Const.</u> By means of (10.3), eqs. (5.7) and (5.8) for the incompressible case can be written

$$\frac{\partial P}{\partial x^2} = 2(B-P)\frac{\partial}{\partial x^2}\ln\sqrt{g_{11}} \tag{11.1}$$

$$\frac{\partial P}{\partial x^3} = 2(B-P)\frac{\partial}{\partial x^3}\ln\sqrt{g_{11}} \tag{11.2}$$

and in the compressible case eqs. (6.12) and (6.13) lead to the same equations with P replaced by $\mathcal{P}$ and B replaced by $\mathcal{B}$. Further, since in the present case $\frac{1}{|\nabla\psi|^2} = g_{22}$ and $\frac{1}{|\nabla\omega|^2} = g_{33}$ the continuity equation in the form (5.12) and (6.17) becomes respectively

$$\left(B(x^2, x^3) - P\right)g_{22}g_{33} = \mu_1(x^2, x^3) \tag{11.3}$$

$$\mathfrak{X}^2\left(\mathcal{B}(x^2, x^3) - \mathcal{P}\right)g_{22}g_{33} = \mu_2(x^2, x^3) \tag{11.4}$$

where μ_1 and μ_2 are arbitrary. With the additional conditions $\frac{\partial B}{\partial x^1} = \frac{\partial \mathcal{B}}{\partial x^1} = 0$ (eqs. (5.4) and (6.9)), we then have a complete set of flow equations for any flow for which the stream lines can be chosen as coordinate curves of an orthogonal coordinate system.

Now let us limit ourselves to incompressible flows for which $\frac{\partial B}{\partial x^2} = \frac{\partial B}{\partial x^3} = 0$, that is, B = const., and to compressible flows for which $\frac{\partial \mathbb{B}}{\partial x^2} = \frac{\partial \mathbb{B}}{\partial x^3} = 0$, that is, $\mathbb{B}$ = const.

In the case $\frac{\partial B}{\partial x^2} = \frac{\partial B}{\partial x^3} = 0$, eqs. (11.1) and (11.2) may be integrated to give

$$(B - P)\, g_{11} = \phi_1(x^1) \tag{11.5}$$

where ϕ_1 is an arbitrary function of x^1. Correspondingly, for the compressible case, when $\frac{\partial \mathbb{B}}{\partial x^2} = \frac{\partial \mathbb{B}}{\partial x^3} = 0$, we obtain

$$(\mathbb{B} - \mathbb{P})\, g_{11} = \phi_2(x^1) \tag{11.6}$$

in which ϕ_2 is an arbitrary function of x^1.

From (11.3) and (11.5) we get for the incompressible case

$$g_{22}\, g_{33}\, \phi_1(x^1) = g_{11}\, \mu_1(x^2, x^3) \tag{11.7}$$

From (11.4) and (11.6) for the compressible case

$$X^2 = \frac{g_{11}\,\mu_2(x^2, x^3)}{g_{22}\,g_{33}\,\phi_2(x')} \tag{11.8}$$

For a polytropic gas (11.8) becomes

$$\rho = \left[\frac{g_{11}\,\mu_2(x^2, x^3)}{g_{22}\,g_{33}\,\phi_2(x')}\right]^{\frac{\gamma-1}{2}} \tag{11.9}$$

substituting this into (11.6) we obtain

$$\left[B - \frac{\phi_2(x')}{g_{11}}\right]^{\frac{2}{\gamma-1}} g_{22}\,g_{33}\,\phi_2(x') = g_{11}\,\mu_2(x^2, x^3) \tag{11.10}$$

Thus, we can state the result that if $g_{11}, g_{22},$ and g_{33} are the metric coefficients of an orthogonal coordinate system in E^3, then there is an incompressible or a compressible flow with x'= variable curves as stream lines if and only if $g_{11}, g_{22},$ and g_{33} satisfy (11.7) or (11.10) respectively for arbitrary $\phi_1, \phi_2, \mu_1,$ and μ_2.

For example, for incompressible plane flows our condition is

$$g_{11}\,\phi_1(x^1) = g_{22}\,\mu_1(x^2)$$

But this is the condition that our orthogonal net be isometric (23, p. 163), so that our general criterion reduces to the well-known result (7, p. 68, or 10, p. 159) for this special case.

12. Application of Equations (11.7) and (11.10). One effective application of eqs. (11.7) and (11.10) is to use them to determine the existence of flows corresponding to given suitably restrictive classes of coordinate systems. To illustrate, we will consider two such classes.

a. Coordinate Systems for Which g_{ii} has a Product Form. Quantities g_{ii} of the form

$$\begin{aligned} g_{11} &= \mathcal{E}^{*2}(x^1)\,\mathcal{E}^2(x^2, x^3) \\ g_{22} &= \mathcal{F}^{*2}(x^1)\,\mathcal{F}^2(x^2, x^3) \\ g_{33} &= \mathcal{G}^{*2}(x^1)\,\mathcal{G}^2(x^2, x^3) \end{aligned} \qquad (12.1)$$

in which the functions on the right side are arbitrary functions of the indicated variables, satisfy eq. (11.7).

If these expressions are substituted into eq. (11.10) for the compressible case, then we get

$$\mathcal{E}^2\mathcal{B} - \frac{\phi_2}{\mathcal{E}^{*2}} = \mathcal{E}^2\left[\frac{\mathcal{E}^2\mu_2}{\mathcal{F}^2\mathcal{G}^2}\right]^{\frac{\gamma-1}{2}}\left[\frac{\mathcal{E}^{*2}}{\mathcal{F}^{*2}\mathcal{G}^{*2}\phi_2}\right]^{\frac{\gamma-1}{2}} \tag{12.2}$$

Now if either $\mathcal{E}$ = const., or $\mathcal{F}^*\mathcal{G}^*$ = const., then eq. (12.2), and therefore eq. (11.10), can be satisfied. Thus, if $\mathcal{E}$ = const., and if we choose $\mu_2 = \mathcal{F}^2\mathcal{G}^2$, then eq. (12.2) will contain only functions of X^1 (and constants). Then, in general, ϕ_2 can be chosen to satisfy eq. (12.2). If $\mathcal{F}^*\mathcal{G}^*$ = const., and if we choose $\phi_2 = \mathcal{E}^{*2}$, then eq. (12.2) will contain only functions of X^2 and X^3 (and constants). Then, in general, μ_2 can be chosen to satisfy eq. (12.2).

On the other hand, if we differentiate (12.2) successively with respect to $X^1 = \alpha$ and $X^2 = \psi$ we get

$$\frac{\partial}{\partial\psi}\left[\mathcal{E}^2\left(\frac{\mathcal{E}^2\mu_2}{\mathcal{F}^2\mathcal{G}^2}\right)^{\frac{\gamma-1}{2}}\right]\cdot\frac{\partial}{\partial\alpha}\left[\frac{\mathcal{E}^{*2}}{\mathcal{F}^{*2}\mathcal{G}^{*2}\phi_2}\right]^{\frac{\gamma-1}{2}} = 0$$

Differentiating (12.2) successively with respect to α and ψ yields a similar result, so that either

$$\frac{\mathcal{E}^{*2}}{\mathcal{F}^{*2}\mathcal{G}^{*2}\phi_2} = const. \tag{12.3}$$

or

$$\mathcal{E}^2\left[\frac{\mathcal{E}^2\mu_2}{\mathcal{F}^2\mathcal{G}^2}\right]^{\frac{\gamma-1}{2}} = const. \tag{12.4}$$

In the first case eq. (12.2) implies that $\dfrac{\phi_2}{\mathcal{E}^{*2}}$ = const., and therefore, from eq. (12.3), $\mathcal{F}^*\mathcal{G}^*$ = const. In the second case eq. (12.2) implies that $\mathcal{E}$ = const.

Thus, <u>quantities of the form (12.1) satisfy eq. (11.10) if and only if either $\mathcal{E}$ = const. or $\mathcal{F}^*\mathcal{G}^*$ = const.</u>

The problem now is to find orthogonal coordinate systems for which g_{ii} of the form (12.1) are the metric coefficients. In Appendix III we find that the requirement of the vanishing of the Riemann tensor implies that any such coordinate system has at least one of the following properties:

1. $g_{11} = 1$
2. $\mathcal{E}^* = \mathcal{F}^* = \mathcal{G}^* = 1$
3. it is a spherical coordinate system

By eq. (10.3) the first property implies that the X' variable curves, that is, the curves which are to be stream lines, are straight lines. The class of flows with straight stream lines will be described in Chapter VI.

In order to interpret the second property we first focus our attention on a fixed x^3 = const. surface. Taking the coordinate net u^1, u^2 on this surface as that determined by the curves of intersection of it and the x^1 = const. and x^2= const. surfaces, we get for its metric coefficients $'g_{11} = g_{11}$ and $'g_{22} = g_{22}$. (See Figure 2, and the discussion below eq. (3.9)). Since property 2 implies that $'g_{22}$ is not a function of u^1 , the u^1= const. curves are geodesics on x^3= const. (See Appendix I, eq. (I1) and below). By a similar argument, the intersections of x^1= const. and x^2= const. are geodesics on x^2= const. But since the curves of intersection of the x^1 = const. surfaces with the other two families are geodesics on the other two families, their normals lie in the x^1= const. surfaces and thus (8, p. 227 and p. 231) they are asympototics on the x^1 = const. surfaces. Since the asymptotics are orthogonal, the x^1 = const. surfaces are minimal surfaces, (23, p. 238) . The class of flows which have a family of minimal surfaces orthogonal to the stream lines will be described in Chapter VII.

With property 3 above we have

$$(ds)^2 = r^2(d\delta)^2 + (dr)^2 + r^2\sin^2\delta\,(d\theta)^2 \qquad (12.5)$$

where δ, r, θ are spherical coordinates given by

$$\delta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

with respect to the rectangular cartesian coordinates x, y, z. If we take $x^1 = r$, then $g_{11} = 1$ and the coordinate system has property 1. As mentioned above, the $x^1 = r$ = variable curves (the radial lines) are straight lines. If we take $x^1 = \theta$, then since the coefficients of eq. (12.5) do not contain θ, the coordinate system has property 2. As mentioned above, the $x^1 = \theta$ = const. surfaces (the coaxial planes) are minimal surfaces. Finally if we take $x^1 = \delta$, then from eq. (12.5) we see that $g_{11} = r^2 \neq 1$ and the last coefficient in eq. (12.5) is a function of x^1 so that the coordinate system has neither property 1 nor 2.

We have noted that for every possible coordinate system with g_{ii} of the form (12.1), eq. (11.7) is satisfied and then there is an incompressible flow with x^1 = variable curves as stream lines. Let us note now that for coordinate systems with the properties 1 or 2 we have the condition that either $\mathcal{E}$ = const. or $\mathcal{F}^*\mathcal{G}^*$ = const. which is required in order to satisfy eq. (11.10), and thus to have compressible flows with x^1 = variable curves as stream lines. That is, any coordinate system with property 1 or property 2 yields a compressible as well as an incompressible flow.

This is not the case for coordinate systems with property 3. We have seen above that there are three possibilities depending on whether we take $x^1 = r$, $x^1 = \theta$, or $x^1 = \delta$. The first possibility yields radial straight line (source-sink) incompressible and compressible flows since the coordinate system has property 1. The second possibility yields incompressible and compressible flows whose stream lines are the small circles on the spheres (vortex flows) since the coordinate system has property 2. The third possibility yields incompressible flows whose stream lines are the great circles on the spheres, but as we have seen from eq. (12.5), when $x' = \delta$ we have neither $\mathcal{E}$ = const. nor $\mathcal{F}^* \mathcal{G}^*$ = const., so there are no compressible flows (with $\mathcal{B}$ = const.) whose stream lines are the great circles on the spheres.

b. Ellipsoidal Coordinate Systems. Consider the family of central quadric surfaces

$$\frac{x^2}{a_3 - \lambda} + \frac{y^2}{a_2 - \lambda} + \frac{z^2}{a_1 - \lambda} = 1$$

where λ is a real parameter, and $a_1 < a_2 < a_3$. This family is composed of a family of ellipsoids for $\lambda < a_1$, a family of hyperboloids of one sheet for $a_1 < \lambda < a_2$, and a family of hyperboloids of two sheets for $a_2 < \lambda < a_3$. These three families form a triply orthogonal system of (confocal quadric) surfaces in E^3. Choosing these as coordinate surfaces we get (24, p. 26, or 25, p. 97).

$$g_{11} = \frac{1}{4}\,\frac{(x^1-x^2)(x^1-x^3)}{(a_3-x^1)(a_2-x^1)(a_1-x^1)}$$

$$g_{22} = \frac{1}{4}\,\frac{(x^2-x^3)(x^2-x^1)}{(a_3-x^2)(a_2-x^2)(a_1-x^2)} \tag{12.6}$$

$$g_{33} = \frac{1}{4}\,\frac{(x^3-x^1)(x^3-x^2)}{(a_3-x^3)(a_2-x^3)(a_1-x^3)}$$

where $x^1 = \lambda$ when λ is in one of the intervals $(-\infty, a_1)$, (a_1, a_2), or (a_2, a_3), $x^2 = \lambda$ when λ is in another of these intervals, and $x^3 = \lambda$ when λ is in the third interval.

Since $\dfrac{g_{11}}{g_{22}\,g_{33}}$ is the product of a function of x^1 and a function of x^2 and $x^{3\prime}$, these metric coefficients satisfy eq. (11.7). That is, any of the congruences of coordinate curves of an ellipsoidal coordinate system are stream lines of an incompressible fluid flow. More specifically, according to our assumption, these are flows with a constant Bernoulli function, B .

Now, an incompressible flow with B = const. is a Beltrami flow, by eq. (7.2). That is, the vorticity vector is parallel to the velocity vector or zero. But since in our case there are surfaces orthogonal to v^i, the integrability condition $v_i\, \epsilon^{ijk}\, \nabla_j v_k = 0$ is satisfied. This condition says that the vorticity vector is perpendicular to the velocity vector, and therefore not parallel to it. Hence the vorticity is zero.

Thus, the velocity has a potential ϕ , and the equipotential surfaces are the coordinate surfaces X^1 = const. These flows have been obtained by starting with Laplace's equation for the potential function. For, when this equation is written in ellipsiodal coordinates, then $\phi = X^1$ is a solution (7, p. 139 ff.).

Now we ask if the metric coefficients (12.6) satisfy eq. (11.10) for compressible fluid flows. We can write eq. (11.10) in the form

$$\frac{\mathcal{B}}{\phi_2} - \frac{1}{g_{11}} = \frac{1}{\phi_2}\left[\frac{g_{11}\mu_2}{g_{22}g_{33}\phi_2}\right]^{\frac{\gamma-1}{2}} \tag{12.7}$$

When the g_{ii} are given by (12.6) the right-hand side of e (12.7) is the product of a function of X^1 and a function of X^2 and X^3 so this equation can be written in the form

$$\phi_2^*(x^1) - \frac{1}{(x^1-x^2)(x^1-x^3)} = U(x^1)\, V(x^2, x^3)$$

where ϕ_2^*, U, and V are arbitrary functions of the indicated arguments, or

$$\frac{\partial^2}{\partial x^1 \partial x^2}\left(\ln\left[\phi_2^* - \frac{1}{(x^1-x^2)(x^1-x^3)}\right]\right) = 0$$

Carrying out the indicated differentiation we obtain

$$\left[2\,\phi_2^*(x^1-x^2)(x^1-x^3)-1\right]\left(2x^1-x^2-x^3\right)$$
$$-\left[\phi_2^*(x^1-x^2)(x^1-x^3)-1\right](x^1-x^2) = 0$$

But this must be an identity in X^1, X^2, X^3 , so that when it is written as a quadratic equation in X^3 the coefficients must be zero. The coefficient of $(X^3)^2$ is $2\phi_2^*(X^1 - X^2)$. Since ϕ_2^* cannot vanish, eq. (11.10) cannot be satisfied by the metric coefficients, (12.6), for an ellpsoidal coordinate system, and hence there are no corresponding compressible fluid flows.

CHAPTER VI

FLOWS WITH STRAIGHT STREAM LINES

In this section we will consider the class of fluid flows for which in each flow every stream line is a straight line. Going back to eq. (2.1) we see that (as long as $t^i \nabla_i q \neq 0$) this condition is equivalent to the statement that the constant pressure surfaces are everywhere perpendicular to the stream lines. (6 and 26).

13. The Basic Flow Equations. Flows with straight stream lines are a special case of flows in which the stream lines are coordinate curves of an orthogonal coordinate system. To see this we first note that the family $\{p = \text{const.}\}$ is a family of parallel surfaces, for there is a congruence of straight lines (the stream lines) which is orthogonal to all the p = const. surfaces (See Appendix I b for a brief discussion of parallel surfaces). Further, the straight lines through a line of curvature of one of the p = const. surfaces will intersect the other p = const. surfaces in lines of curvature. (See Appendix I below eq. (I.4)). These straight lines form a surface, and there are two families $\{\psi = \text{const.}\}$ and $\{\omega = \text{const.}\}$ of such surfaces. (See Figure 3 on page 67). Since the two lines of curvature through a point on a surface are orthogonal, these two families of surfaces, together with $\{p = \text{const.}\}$ constitute a triply orthogonal system of surfaces in E^3.

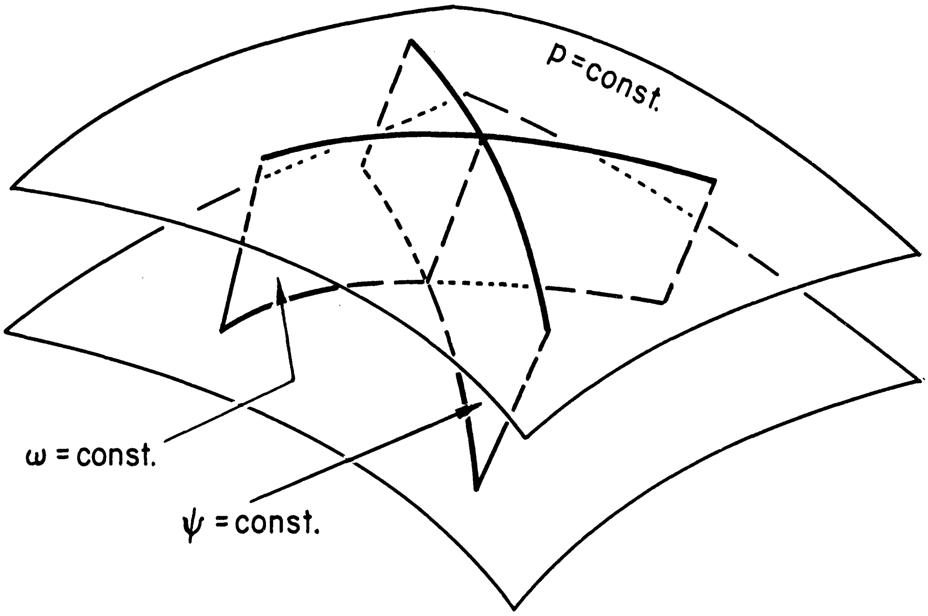

Figure 3. Orthogonal Coordinate System
with Parallel Surfaces

Now, let us parametrize the p = const. surfaces by the distance, α , along the stream lines, and let us take α, ψ, ω as our coordinates x^1, x^2, x^3 respectively. Our flow equations then have the form obtained at the beginning of section 11.

Equations (11.1) and (11.2) and the corresponding equations for compressible flow are satisfied identically for our special case of straight stream lines. Consequently, <u>in order to obtain flows with</u>

straight stream lines it is necessary and sufficient to satisfy eq. (11.3) in the incompressible case and eq. (11.4) in the compressible case.

In our $x^1 = \alpha$, $x^2 = \psi$, $x^3 = \omega$ coordinate system, the metric coefficients are (eqs. (I.7) of Appendix I)

$$g_{11} = 1 \tag{13.1}$$

$$g_{22} = \left[\alpha \mathbb{K}_2(\psi,\omega) - 1\right]^2 {}'\bar{g}_{22}(\psi,\omega) \tag{13.2}$$

$$g_{33} = \left[\alpha \mathbb{K}_3(\psi,\omega) - 1\right]^2 {}'\bar{g}_{33}(\psi,\omega) \tag{13.3}$$

in which $\mathbb{K}_2$ and $\mathbb{K}_3$ are the principal curvatures, and ${}'\bar{g}_{22}$ and ${}'\bar{g}_{33}$ are the metric coefficients of an arbitrary but fixed member of the family of parallel constant pressure surfaces. In terms of these expressions for g_{22} and g_{33} eqs. (11.3) and (11.4) become

$$\left(B(\psi,\omega) - P(\alpha)\right)\left(\alpha \mathbb{K}_2(\psi,\omega) - 1\right)^2\left(\alpha \mathbb{K}_3(\psi,\omega) - 1\right)^2 = \Psi'(\psi,\omega) \tag{13.4}$$

$$\bar{X}^2(\alpha)\left(\mathcal{B}(\psi,\omega)-P(\alpha)\right)\left(\alpha\bar{K}_2(\psi,\omega)-1\right)^2\left(\alpha\bar{K}_3(\psi,\omega)-1\right)^2 = \bar{\Psi}^2(\psi,\omega) \tag{13.5}$$

where $\bar{\Psi}^1(\psi,\omega) = \dfrac{\mu_1(\psi,\omega)}{'\bar{g}_{22}(\psi,\omega)\,'\bar{g}_{33}(\psi,\omega)}$ and $\bar{\Psi}^2(\psi,\omega) = \dfrac{\mu_2(\psi,\omega)}{'\bar{g}_{22}(\psi,\omega)\,'\bar{g}_{33}(\psi,\omega)}$

(Note that by eq. (5.3) $\bar{\Psi}^1$ vanishes if and only if $q = 0$, and by eq. (6.8) $\bar{\Psi}^2$ vanishes if and only if $q^2 \Upsilon = 0$. These exceptional cases will be excluded in the following discussion).

Eqs. (13.4) and (13.5) are to be identities in α, ψ, ω. We will see that the only way this can happen is if the principal curvatures K_2 and K_3 are both constant. The general idea of our method is to differentiate to obtain a polynominal in α and then draw our conclusions from the vanishing of the coefficients.

14. The Incompressible Case. For the incompressible case we write (13.4) as

$$B - P = \frac{\bar{\Psi}^1}{(\mathcal{K}\alpha^2 - \mathcal{M}\alpha + 1)^2} \tag{14.1}$$

where $\mathcal{K} = K_2 K_3$ = the total curvature of the surface and $\mathcal{M} = K_2 + K_3$ = the mean curvature of the surface. Now we differentiate (14.1) first with respect to α and then with respect to ψ to get

$$\frac{\partial}{\partial\psi}\,\frac{\Psi'(2\mathcal{K}\alpha-\mathcal{M})}{(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)^3} = 0$$

or, using subscript notation for derivatives with respect to ψ,

$$(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)\left[\Psi'(2\mathcal{K}_\psi\alpha-\mathcal{M}_\psi)+\Psi'_\psi(2\mathcal{K}\alpha-\mathcal{M})\right]$$
$$-3\Psi'(2\mathcal{K}\alpha-\mathcal{M})(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha) = 0$$

or

$$\left[-4\Psi'\mathcal{K}\mathcal{K}_\psi+2\Psi'_\psi\mathcal{K}^2\right]\alpha^3+\left[\Psi'\mathcal{M}\mathcal{K}_\psi+5\Psi'\mathcal{K}\mathcal{M}_\psi-3\mathcal{M}\mathcal{K}\Psi'_\psi\right]\alpha^2$$
$$+\left[2\Psi'\mathcal{K}_\psi-2\Psi'\mathcal{M}\mathcal{M}_\psi+\mathcal{M}^2\Psi'_\psi+2\mathcal{K}\Psi'_\psi\right]\alpha+\Psi'\mathcal{M}_\psi+\Psi'_\psi\mathcal{M} = 0 \tag{14.2}$$

If we differentiate (14.1) with respect to α and with respect to ω we get a similar equation with derivatives with respect to ω in place of derivatives with respect to ψ. The vanishing of the coefficients of α^3 and α^2, and of the constant term in eq. (14.2) and in the corresponding equation with derivatives with respect to ω in place of derivatives with respect to ψ implies that $\frac{\mathcal{K}^2}{\Psi'}$ = const., $\frac{\mathcal{K}\mathcal{M}^5}{(\Psi')^3}$ = const. and $\Psi'\mathcal{M}$ = const. Thus, for example, the vanishing of the coefficient of α^2 in eq. (14.2)

$$\Psi' \mathcal{M} \mathcal{K}_\psi + 5\Psi' \mathcal{K} \mathcal{M}_\psi - 3\mathcal{M}\mathcal{K}\Psi'_\psi = 0$$

implies that

$$\Psi'(\mathcal{M}^5\mathcal{K}_\psi + 5\mathcal{M}^4\mathcal{M}_\psi\mathcal{K}) - \mathcal{M}^5\mathcal{K}\Psi'_\psi - 2\mathcal{M}^5\mathcal{K}\Psi'_\psi = 0$$

or since $\Psi' \neq 0$

$$\Psi'(\mathcal{M}^5\mathcal{K})_\psi - \mathcal{M}^5\mathcal{K}\Psi'_\psi = \Psi'^2\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'}\right)_\psi = 2\mathcal{M}^5\mathcal{K}\Psi'_\psi$$

or

$$\Psi'\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'}\right)_\psi - 2\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'}\right)\Psi'_\psi = 0$$

Then

$$\Psi'^2\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'}\right)_\psi - \frac{\mathcal{M}^5\mathcal{K}}{\Psi'}(\Psi'^2)_\psi = 0$$

or

$$\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'^3}\right)_\psi = 0 \qquad (14.3)$$

Similarly, from

$$\Psi'\mathcal{M}\mathcal{K}_\omega + 5\Psi'\mathcal{K}\mathcal{M}_\omega - 3\mathcal{M}\mathcal{K}\Psi'_\omega = 0$$

we get

$$\left(\frac{\mathcal{M}^5\mathcal{K}}{\Psi'^3}\right)_\omega = 0 \qquad (14.4)$$

Eqs. (14.3) and (14.4) together give the result $\frac{\mathcal{K}\mathcal{M}^5}{\Psi'^3}$ = const.

If neither $\mathcal{K}$ nor $\mathcal{M}$ are zero then the three conditions = const., = const., and = const., imply that , , and are all constant, and thus, in particular, K_2 and K_3 are constant. If $\mathcal{K} = 0$ the vanishing of the coefficient of α gives $\frac{\mathcal{M}^2}{\Psi'}$ = const., and if $\mathcal{M} = 0$ this condition gives $\Psi'\mathcal{K}$ = const. and so in these cases we also get $\mathcal{K}$, $\mathcal{M}$, and Ψ', are const., and K_2 and K_3 are constant. Hence, in all cases <u>the principal curvatures K_2 and K_3 are constant.</u>

15. <u>The Compressible Case.</u> The compressible case requires more calculation. We first write (13.5) as

$$X^2(\mathcal{B} - \mathcal{P}) = \frac{\Psi^2}{(\mathcal{K}\alpha^2 - \mathcal{M}\alpha + 1)^2} \qquad (15.1)$$

Now we differentiate (15.1) twice with respect to ψ , obtaining successively (using subscript notation for derivatives with respect to ψ)

$$X^2 \mathcal{B}_\psi = \frac{(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)\,\Psi_\psi^2 - 2\,\Psi^2(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha)}{(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)^3} \tag{15.2}$$

$$\begin{aligned}&(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)^4 X^2 \mathcal{B}_{\psi\psi} + 3\left[(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)\,\Psi_\psi^2 - 2\,\Psi^2(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha)\right](\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha)\\ &=(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)\left[(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)\,\Psi_{\psi\psi}^2 - \Psi_\psi^2(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha) - 2\,\Psi^2(\mathcal{K}_{\psi\psi}\alpha^2-\mathcal{M}_{\psi\psi}\alpha)\right]\end{aligned} \tag{15.3}$$

Then eliminating X^2 between (15.2) and (15.3) we get

$$\begin{aligned}&(\mathcal{B}_{\psi\psi}\Psi_\psi^2 - \mathcal{B}_\psi \Psi_{\psi\psi}^2)(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)^2\\ &-2(\mathcal{B}_{\psi\psi}\Psi^2 - 2\,\mathcal{B}_\psi\Psi_\psi^2)(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha)\\ &= 2\,\Psi^2\mathcal{B}_\psi(\mathcal{K}\alpha^2-\mathcal{M}\alpha+1)(\mathcal{K}_{\psi\psi}\alpha^2-\mathcal{M}_{\psi\psi}\alpha)\\ &\quad + 6\,\Psi^2\mathcal{B}_\psi(\mathcal{K}_\psi\alpha^2-\mathcal{M}_\psi\alpha)^2\end{aligned} \tag{15.4}$$

For the case $\mathcal{B}_\psi = 0$ eq. (15.2) gives us

$$\left(\mathcal{K}\Psi_\psi^2 - 2\Psi^2\mathcal{K}_\psi\right)\alpha^2 + \left(2\Psi^2\mathcal{M}_\psi - \Psi_\psi^2\mathcal{M}\right)\alpha + \Psi_\psi^2 = 0$$

so that Ψ^2, $\mathcal{K}$ and $\mathcal{M}$ and thus K_2 and K_3 can be at most functions of ω.

Now we will assume that $\mathcal{B}_\psi \neq 0$. Eq. (15.4) is of the form

$$c_1\alpha^4 + c_2\alpha^3 + c_3\alpha^2 + c_4\alpha + c_5 = 0$$

The requirement that $c_i = 0$, $i = 1, \cdots, 5$ leads to the conditions

$$\left(\mathcal{B}_{\psi\psi}\Psi^2 - 2\mathcal{B}_\psi\Psi_\psi^2\right)\mathcal{K}\mathcal{K}_\psi + 3\mathcal{B}_\psi\Psi^2\mathcal{K}_\psi^2 - \mathcal{B}_\psi\Psi^2\mathcal{K}\mathcal{K}_{\psi\psi} = 0 \qquad (15.5)$$

$$\left(\mathcal{B}_{\psi\psi}\Psi^2 - 2\mathcal{B}_\psi\Psi_\psi^2\right)\left(\mathcal{K}\mathcal{M}_\psi + \mathcal{K}_\psi\mathcal{M}\right) + 6\mathcal{B}_\psi\Psi^2\mathcal{K}_\psi\mathcal{M}_\psi$$
$$- \mathcal{B}_\psi\Psi^2\left(\mathcal{K}\mathcal{M}_{\psi\psi} + \mathcal{K}_{\psi\psi}\mathcal{M}\right) = 0 \qquad (15.6)$$

$$\left(\mathcal{B}_{\psi\psi}\Psi^2 - 2\mathcal{B}_\psi\Psi_\psi^2\right)\left(\mathcal{K}_\psi + \mathcal{M}\mathcal{M}_\psi\right) + 3\mathcal{B}_\psi\Psi^2\mathcal{M}_\psi^2$$
$$- \mathcal{B}_\psi\Psi^2\left(\mathcal{K}_{\psi\psi} + \mathcal{M}\mathcal{M}_{\psi\psi}\right) = 0 \qquad (15.7)$$

$$\left(\mathcal{B}_{\psi\psi}\Psi^2 - 2\,\mathcal{B}_\psi \Psi_\psi^2\right)\mathcal{M}_\psi - \mathcal{B}_\psi \Psi^2 \mathcal{M}_{\psi\psi} \tag{15.8}$$

$$\mathcal{B}_\psi \Psi^2_{\psi\psi} - \mathcal{B}_{\psi\psi} \Psi^2_\psi = 0 \tag{15.9}$$

At this point we consider two cases.

(1) If $\mathcal{M}_\psi = 0$, then eq. (15.7) yields

$$\left(\mathcal{B}_{\psi\psi}\Psi^2 - 2\,\mathcal{B}_\psi \Psi_\psi^2\right)\mathcal{K}_\psi - \mathcal{B}_\psi \Psi^2 \mathcal{K}_{\psi\psi} = 0$$

and comparing this with eq. (15.5) we see that $\mathcal{K}_\psi = 0$ also, and again we have that $\mathbf{K}_2$ and $\mathbf{K}_3$ can be at most functions of ω .

(2) If $\mathcal{M}_\psi \neq 0$, we can solve (15.8) for $\mathcal{B}_{\psi\psi}\Psi^2 - 2\,\mathcal{B}_\psi \Psi_\psi^2$ and substitute it into eqs. (15.5) and (15.7) getting

$$\frac{\mathcal{M}_{\psi\psi}}{\mathcal{M}_\psi}\,\mathcal{K}\mathcal{K}_\psi + 3\,\mathcal{K}_\psi^{\,2} - \mathcal{K}\mathcal{K}_{\psi\psi} = 0 \tag{15.10}$$

$$\frac{\mathcal{M}_{\psi\psi}}{\mathcal{M}_\psi}\,\mathcal{K}_\psi + 3\,\mathcal{M}_\psi^{\,2} - \mathcal{K}_{\psi\psi} = 0 \tag{15.11}$$

Now $\mathcal{K}_\psi \neq 0$, for if $\mathcal{K}_\psi = 0$ then (15.11) implies $\mathcal{M}_\psi = 0$. Then from (15.10)

$$(\ln \mathcal{M}_\psi)_\psi + 3(\ln \mathcal{K})_\psi - (\ln \mathcal{K}_\psi)_\psi = 0$$

or

$$\mathcal{M}_\psi = \frac{\mathcal{K}_\psi \Lambda(\omega)}{\mathcal{K}^3}$$

where $\Lambda(\omega)$ is an arbitrary function of ω . Now we multiply (15.11) by $\mathcal{K}$ and subtract it from (15.10) and get

$$\mathcal{K}_\psi^{\,2} - \mathcal{K}\mathcal{M}_\psi^{\,2} = 0$$

Eliminating $\mathcal{M}_\psi$ from the last two equations we have

$$\mathcal{K}_\psi^{\,2} - \frac{\mathcal{K}_\psi^{\,2}\Lambda^2}{\mathcal{K}^5} = 0$$

or

$$\mathcal{K}_\psi^{\,2}(\mathcal{K}^5 - \Lambda^2) = 0$$

which contradicts the condition $\mathcal{K}_\psi \neq 0$. Thus, we have found that in all cases $\mathcal{K}$ and $\mathcal{M}$ and thus K_2 and K_3 are at most functions of ω .

Starting with eq. (15.1) and differentiating twice with respect to ω we can continue in an analogous fashion and arrive at the result that K_2 and K_3 are at most functions of ψ. Thus we have the result that the principal curvatures K_2 and K_3 of the fixed surface are constant. It follows (eqs. (I.10) of Appendix I) that <u>the principal curvatures of each surface of our family of parallel surfaces are constant</u>.

Now we ask the purely geometrical question, what kind of triply orthogonal systems of surfaces with the properties

(i) the system contains a family of parallel surfaces

(ii) each of the two principal curvatures is constant on each surface

can we have in E^3? The answer is that there are exactly those systems such that when they are taken as coordinate systems the Riemann tensor vanishes. The vanishing of the Riemann tensor for orthogonal coordinate systems constitutes six conditions, called the Lamé equations, on the three square roots $\sqrt{g_{11}}$, $\sqrt{g_{22}}$, $\sqrt{g_{33}}$ of the metric coefficients. See Appendix III.

In our case, since $\sqrt{g_{11}} = 1$, three of the Lamé equations, (eqs. (III.1), (III.5*) and (III.6) of Appendix III), reduce to

$$\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial\psi}\sqrt{g_{33}}\right)+\frac{\partial}{\partial\omega}\left(\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial\omega}\sqrt{g_{22}}\right)+\left(\frac{\partial}{\partial\alpha}\sqrt{g_{22}}\right)\left(\frac{\partial}{\partial\alpha}\sqrt{g_{33}}\right)=0 \qquad (15.12)$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{\sqrt{g_{33}}}\,\frac{\partial}{\partial\omega}\sqrt{g_{22}}\right) = 0 \tag{15.13}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{\sqrt{g_{22}}}\,\frac{\partial}{\partial\psi}\sqrt{g_{33}}\right) = 0 \tag{15.14}$$

But in our case we also have $g_{22} = (K_2\alpha - 1)^2\,'\bar{g}_{22}(\psi,\omega)$ and $g_{33} = (K_3\alpha - 1)^2\,'\bar{g}_{33}(\psi,\omega)$ (See eqs. (13.2) and (13.3)) from which we find by differentiation

$$\frac{1}{\sqrt{g_{33}}}\,\frac{\partial}{\partial\omega}\sqrt{g_{22}} = \frac{|K_2\alpha - 1|}{|K_3\alpha - 1|\sqrt{'\bar{g}_{33}}}\,\frac{\partial}{\partial\omega}\sqrt{'\bar{g}_{22}}$$

$$\frac{1}{\sqrt{g_{22}}}\,\frac{\partial}{\partial\psi}\sqrt{g_{33}} = \frac{|K_3\alpha - 1|}{|K_2\alpha - 1|\sqrt{'\bar{g}_{22}}}\,\frac{\partial}{\partial\psi}\sqrt{'\bar{g}_{33}} \tag{15.15}$$

Substituting (15.15) into our three conditions (15.12)-(15.14) we get

$$\frac{|K_3\alpha-1|}{|K_2\alpha-1|}\,\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{'\bar{g}_{22}}}\,\frac{\partial}{\partial\psi}\sqrt{'\bar{g}_{33}}\right) + \frac{|K_2\alpha-1|}{|K_3\alpha-1|}\,\frac{\partial}{\partial\omega}\left(\frac{1}{\sqrt{'\bar{g}_{33}}}\,\frac{\partial}{\partial\omega}\sqrt{'\bar{g}_{22}}\right)$$

$$\pm K_2K_3\sqrt{'\bar{g}_{22}}\,\sqrt{'\bar{g}_{33}} = 0 \tag{15.16}$$

$$\left(\frac{1}{\sqrt{'\bar{g}_{33}}}\frac{\partial}{\partial\omega}\sqrt{'\bar{g}_{22}}\right)\frac{\partial}{\partial\alpha}\frac{|K_2\alpha-1|}{|K_3\alpha-1|}=0 \qquad (15.17)$$

$$\left(\frac{1}{\sqrt{'\bar{g}_{22}}}\frac{\partial}{\partial\psi}\sqrt{'\bar{g}_{33}}\right)\frac{\partial}{\partial\alpha}\frac{|K_3\alpha-1|}{|K_2\alpha-1|}=0 \qquad (15.18)$$

Eqs. (15.16), (15.17) and (15.18), imply that either $K_2 = K_3$ or $K_2 \cdot K_3 = 0$. For if $K_2 \neq K_3$ then the left-hand factors of (15.17) and (15.18) must vanish. This, in turn, implies that the first two terms of (15.16) vanish so that, from the last term, $K_2 \cdot K_3 = 0$

If $K_2 = K_3 = 0$, then our parallel surfaces are planes. If $K_2 = K_3 \neq 0$, then every point of each parallel surface is an umbilical point, so each surface is a sphere (27, p. 97), and our parallel surfaces are concentric spheres.

If one of the principal curvatures of a surface is zero, then it is a developable surface; i.e., either a plane, a cylinder, a cone, or a tangent surface of a curve. It is evident geometrically that the non-zero principal curvature of cones and non-circular cylinders cannot be constant, so our parallel surfaces cannot be cones or non-circular cylinders. Thus it remains to consider only tangent surfaces of curves. The metric coefficients of a tangent surface of a curve referred to its lines of curvature are (23, p. 128), in our notation,

$$'g_{22} = \bar{K}^2(\psi)(\omega - \psi)^2 \quad , \quad 'g_{23} = 0 \quad , \quad 'g_{33} = 1 \qquad (15.19)$$

where $\bar{K}(\psi)$ is the curvature of the curve. But the Codazzi equations on our case imply $\frac{\partial}{\partial \omega} {}'g_{22} = 0$ (23, p. 230), which is incompatible with (15.19). Thus our parallel surfaces cannot be tangent surfaces either.

In summary, we have found that <u>our parallel surfaces can only be concentric spheres, concentric circular cylinders, or parallel planes</u>. Correspondingly, we have the result that <u>the only incompressible or compressible (of the class of section 6) flows with straight stream lines are point source flows, line source flows, and parallel flows</u>.

CHAPTER VII

FLOWS FOR WHICH $\nabla_i t^i = 0$

We noted in Section 2 that the pressure gradient lies in the osculating plane of the stream lines, and in Section 6 we examined the special case in which the pressure gradient lies along the tangent to the stream lines. Let us now consider the other special case in which the pressure gradient lies along the principal normal of the stream lines. By (2.1) this may also be described as the case in which $K \neq 0$ and $\underline{\mathfrak{z}}$ is constant along stream lines. Further, since $t^i \nabla_i p = 0$, eq. (5.10) shows that for incompressible fluid flows $\nabla_i t^i = 0$ is a necessary and sufficient condition for this case. For compressible fluid flows however, eq. (6.15) shows that the conditions $t^i \nabla_i p = 0$ and $\nabla_i t^i = 0$ are equivalent if and only if the coefficient $2(\mathcal{B} - \mathcal{P})\dfrac{d\mathfrak{X}}{dp} - 1$ does not vanish.

The case

$$2(\mathcal{B} - \mathcal{P})\frac{d\mathfrak{X}}{dp} - 1 = 0$$

corresponds to a flow in which $\mathfrak{z}^2 = c^2$. For by eq. (6.8) $2(\mathcal{B} - \mathcal{P}) = \mathfrak{z}^2 \Upsilon$ and by eq. (6.24) $\dfrac{d\mathfrak{X}}{dp} = \dfrac{1}{c^2 \Upsilon}$. Moreover if we solve $2(\mathcal{B} - \mathcal{P})\dfrac{d\mathfrak{X}}{dp} - 1 = 0$ for $\mathcal{B}$ and take the derivative in the t^i direction we get, since $t^i \nabla_i \mathcal{B} = 0$,

$$t^i \nabla_i \mathcal{P} + t^i \nabla_i \frac{1}{2\,\dfrac{d\mathfrak{X}}{dp}} = 0$$

or

$$X \frac{d^2 X}{dp^2} - 2\left(\frac{dX}{dp}\right)^2 = 0$$

which leads to

$$X = \frac{1}{\sigma_1 - \sigma_2 p}$$

where σ_1 and σ_2 are arbitrary constants. The equation of state, $\rho = X(p)Y(S)$, obtained with this X is a generalization of the form used by Chaplygin and by Karman and Tsien in their work with the hodograph method, (20). This special case in which $\nabla_i t^i = 0$ and $t^i \nabla_i p \neq 0$ will be excluded in the following considerations in this chapter. We shall return briefly to such flows in Chapter VIII.

16. <u>Related Incompressible and compressible Flows</u>. With $\nabla_i t^i = 0$ the equations

$$\nabla_i t^i = 0 \tag{16.1}$$

$$t^i \nabla_i P = 0 \tag{16.2}$$

$$t^i \nabla_i B = 0 \tag{5.4}$$

$$n^i \nabla_i P = -2(B-P)K \tag{5.5}$$

$$b^i \nabla_i P = 0 \tag{5.6}$$

constitute a complete set of conditions for an incompressible fluid flow, and in the compressible case we have the equations

$$\nabla_i t^i = 0 \tag{16.1}$$

$$t^i \nabla_i \mathcal{P} = 0 \tag{16.3}$$

$$t^i \nabla_i \mathcal{B} = 0 \tag{6.9}$$

$$n^i \nabla_i \mathcal{P} = -2(\mathcal{B} - \mathcal{P})K \tag{6.10}$$

$$b^i \nabla_i \mathcal{P} = 0 \tag{6.11}$$

Since these two sets of equations have the same form we have the following result: Corresponding to each incompressible (compressible) flow with $\nabla_i t^i = 0$ there is a compressible (incompressible) flow having the same stream lines and constant pressure surfaces.

The converse result is also true. That is, namely; if an incompressible and a compressible flow have the same (non-straight) stream lines and constant pressure surfaces, then they (both) have $\nabla_i t^i = 0$. For if an incompressible and a compressible flow have the same constant pressure surfaces, then $\nabla_i \mathcal{P} = F \nabla_i P$ where

F is a suitable scalar function. Replacing $\nabla_i \mathcal{P}$ by $F \nabla_i P$ in eqs. (6.11) and (6.15) we get

$$F n^i \nabla_i P = -2(\mathcal{B} - \mathcal{P}) K \tag{16.4}$$

$$\left[2(\mathcal{B}-\mathcal{P})\frac{d\mathfrak{X}}{dp} - 1\right] F t^i \nabla_i P = -2(\mathcal{B}-\mathcal{P}) \nabla_i t^i \tag{16.5}$$

respectively. Comparing (16.4) with the corresponding incompressible flow eq. (5.5) we see that, since n^i and K are the same for both flows and $K \neq 0$

$$F(B - P) = \mathcal{B} - \mathcal{P}$$

Similarly, comparing (16.5) with (5.10) we get

$$\left[2(\mathcal{B}-\mathcal{P})\frac{d\mathfrak{X}}{dp} - 1\right](\nabla_i t^i)(B - P) F = -(\nabla_i t^i)(\mathcal{B}-\mathcal{P})$$

Together these last two equations give $\frac{d\mathfrak{X}}{dp}(\mathcal{B}-\mathcal{P})^2 \nabla_i t^i = 0$ from which our conclusion follows.

These results describing related incompressible and compressible flows give a partial answer to questions such as posed by Gilbarg (28) as to the common stream line patterns of two different classes of flows. Such a result has also been obtained by Parsons (29), namely, if incompressible and compressible fluids have the same velocity fields, then the velocity magnitude must be constant on stream lines. Our second ("converse") result above is somewhat stronger than this in the sense that in this result the velocity fields and (by eq. (1.1)) the constant pressure surfaces are assumed to be the same for the two flows, whereas we only require that the stream lines and the constant pressure surfaces be the same for the two flows.

Since the equations for the incompressible and compressible cases are formally the same, we can proceed at this point considering both cases simultaneously. We will use the notation of the incompressible case.

17. <u>The Basic Equations.</u> We first introduce a coordinate system x^1, x^2, x^3 containing the family $\{\omega = \text{const.}\}$ of stream surfaces with $x^3 = \omega$. Then, the unit normal of ω = const., $N_i = \left(0, 0, \frac{1}{\sqrt{g^{33}}}\right)$, and

$$N^i = \frac{g^{i3}}{\sqrt{g^{33}}} \tag{17.1}$$

Further, taking the stream lines as coordinate curves

$$t^i = \left(\frac{1}{\sqrt{g_{11}}} , 0, 0 \right) \tag{17.2}$$

and $t_i = \dfrac{g_{i1}}{\sqrt{g_{11}}}$. Finally, expressing n^i as the cross product, $n^i = \dfrac{\epsilon^{ijk}}{\sqrt{g}} N_j t_k$ we get

$$n^i = \frac{1}{\sqrt{g_{11} g^{33} g}} \left(-g_{12} , g_{11} , 0 \right) \tag{17.3}$$

With t^i given by eq. (17.2), eqs. (16.2) and (5.4) become

$$\frac{\partial P}{\partial x^1} = \frac{\partial B}{\partial x^1} = 0 \tag{17.4}$$

Using eqs. (5.7) and (5.8) instead of (5.5) and (5.6) and using (17.4) and the expressions for n^i and N^i given by (17.3) and (17.1) we obtain

$$\sqrt{\frac{g_{11}}{g^{33}g}}\;\frac{\partial P}{\partial x^2} = -2(B-P)\,'K$$

$$\frac{g^{23}}{\sqrt{g^{33}}}\frac{\partial P}{\partial x^2} + \sqrt{g^{33}}\,\frac{\partial P}{\partial x^3} = -2(B-P)K_N$$

or

$$\frac{\partial P}{\partial x^2} = -2(B-P)\sqrt{\frac{g^{33}g}{g_{11}}}\;'K \qquad (17.5)$$

$$\frac{\partial P}{\partial x^3} = -2(B-P)\frac{-g^{23}\sqrt{\frac{g}{g_{11}}}\;'K + K_N}{\sqrt{g^{33}}} \qquad (17.6)$$

Now we bring in some of our results of Section 8 (Chapter IV). First of all, we found there (eq. (8.9)) that when $\nabla_i t^i = 0$ there exists a family of stream surfaces on which the stream lines are geodesics. We can take our surfaces $\{\omega = const.\}$ to be that family without loss of generality. Then $'K = 0$, and our flow eqs. (17.4), (17.5), and (17.6) reduce to

$$B = B(x^2, x^3)$$

$$P = P(x^3)$$

$$\frac{dP}{dx^3} = -2(B-P)\frac{K_N}{\sqrt{g^{33}}}$$

That is, P and B may be any (positive) functions of the indicated arguments which satisfy the ordinary differential equation, eq. (17.9).

In addition to eqs. (17.7), (17.8), and (17.9) we must remember that we have the condition $\nabla_{\alpha} \cdot t^{i} = 0$, eq. (16.1), to satisfy. By eqs. (3.15), (3.9), and (3.11) this condition may be written

$$\frac{\partial}{\partial x^1}\sqrt{g^{22}}\sqrt{g^{33}} = 0 \tag{17.10}$$

Let us examine our basic equations (17.7)-(17.10). One thing to notice is that eq. (17.8) says that our $\omega = x^3$ = const. surfaces are constant pressure surfaces. Another consequence of these equations is that

$$\frac{\partial}{\partial x^1} \frac{K_N}{\sqrt{g^{33}}} = 0 \tag{17.11}$$

This comes from (17.9) and the fact that neither P nor B are functions of X^1 . On the other hand, if eq. (17.11) is satisfied, then eqs. (17.7), (17.8), and (17.9) are satisfied with any $P(X^3) > 0$ and with B obtained from eq. (17.9). (To insure that $B > 0$, it may be necessary to add a positive constant to P). Thus, since eq. (17.11) is the necessary and sufficient condition to be able to solve eqs. (17.7), (17.8), and (17.9), then <u>eqs. (17.10) and (17.11) are necessary and sufficient conditions for the existence of fluid flows with $\nabla_i t^i = 0$.</u>

Finally, let us note that (since $K = \pm K_N$ and $|\nabla \omega| = \sqrt{g^{33}}$) eq. (17.11) is the same as

$$t^i \nabla_i \frac{K}{|\nabla \omega|} = 0 \tag{8.11}$$

which was obtained in Section 8 as a necessary condition for integrability of the flow equations when $\nabla_i t^i = 0$. Also, the relation

$$\frac{\partial}{\partial x^2} \ln \frac{K}{\sqrt{g^{33}}} + \frac{1}{B-P} \frac{\partial B}{\partial x^2} = 0 \qquad (17.12)$$

obtained by differentiating eq. (17.9), is the same as

$$b^i \nabla_i \ln K - \gamma_{232} + \frac{b^i \nabla_i B}{B-P} = 0 \qquad (17.13)$$

the integrability condition (8.13) when $\nabla_i \cdot t^i = 0$. To identify eqs. (17.12) and (17.13) we note that $\gamma_{232} = b^i \nabla_i \ln |\nabla \omega|$, since eq. (3.17) is valid with any vector in the surface ω = const. in place of t^i, and we put this expression for γ_{232} into eq. (17.13).

18. Two Subclasses of Flows. We have reduced our problem, in the case of flows for which $\nabla_i \cdot t^i = 0$, to that of finding quantities $\sqrt{g^{22}}$, K_N, and $\sqrt{g^{33}}$ which satisfy eqs. (17.10)

and (17.11). We will now obtain consequences of making certain general assumptions about the quantities $\sqrt{'g^{22}}$, K_N , and $\sqrt{g^{33}}$.

a. <u>The Case $\frac{\partial\sqrt{'g^{22}}}{\partial x^1} = \frac{\partial K_N}{\partial x^1} = \frac{\partial\sqrt{g^{33}}}{\partial x^1} = 0$</u>. From eqs. (17.10) and (17.11) we see that either each of the quantities $\sqrt{'g^{22}}$, K_N , and $\sqrt{g^{33}}$ are functions of x^1 , or else none of them are functions of x^1 . Let us consider the latter case.

The condition that $\sqrt{'g^{22}}$ is not a function of x^1 means that on the $\omega = x^3$ = const. surfaces the stream lines are parallel and their orthogonal trajectories are geodesics (see discussion above eq. (3.15), and Appendix I1). But we have chosen the $\omega = x^3$ =const. surfaces so that the stream lines are geodesics on them. Thus, taking the stream lines and their orthogonal trajectories as coordinate curves we have $'g_{11}$ is a function of only u^1 and $'g_{22}$ is a function of only u^2 (see Appendix I1). By changing the scale in the u^1 and u^2 directions (see discussion above (7.14)) we get $'g_{11} = 1$ and $'g_{22} = 1$, or

$$(ds)^2 = (du^1)^2 + (du^2)^2$$

for the first fundamental form of $\omega = x^3$ = const. That is, when

$$\frac{\partial\sqrt{'g^{22}}}{\partial x^{1}} = 0$$

(and hence also $\frac{\partial K_N}{\partial x^1} = \frac{\partial\sqrt{g^{33}}}{\partial x^1} = 0$), then

<u>the $\omega = x^3$ = const. surfaces must be developable surfaces.</u>

b. <u>$\nabla_i t^i = 0$ Flows Which are Geometrically Equivalent to the Beltrami Flows.</u> Differentiating eq. (17.9) with respect to x^2 we see that $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}} = 0$ if and only if $\frac{\partial B}{\partial x^2} = 0$. Now suppose we have a flow with $\frac{K_N}{\sqrt{g^{33}}}$ and B such that $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}} = \frac{\partial}{\partial x^2} B = 0$. Then our flow equations, eqs. (17.7), (17.8), and (17.9) can also be satisfied with the same $\frac{K_N}{\sqrt{g^{33}}}$ and <u>any</u> B such that $\frac{\partial B}{\partial x^2} = 0$. That is, if we have a flow with $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}} = 0$ and $\frac{\partial B}{\partial x^2} = 0$, then there are also flows with the same $\frac{K_N}{\sqrt{g^{33}}}$ and with any B such that $\frac{\partial B}{\partial x^2} = 0$ (and such that the solution P of eq. (17.9) is positive). In particular, there is also a flow with B = const., that is, a Beltrami flow

$$\epsilon_{ijk} v^j \Omega^k = 0 \qquad (18.1)$$

(see below eq. (7.3)) with the same stream lines. In other words, <u>the class of flows with $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}} = 0$ is geometrically equivalent to the class of Beltrami flows.</u>

We can get a little more information about the condition $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}} = 0$ if we express $\frac{K_N}{\sqrt{g^{33}}}$ in terms of the metric coefficients. Introducing the second fundamental tensor, h_{ij}, of the x^3 = const. surfaces we have the formulas (8, p. 228 and 227)

$$K_N = -h_{ij}t^i t^j = -t^i t^j \nabla_i N_j$$

Since in our case $t^i = \left(\frac{1}{\sqrt{g_{11}}}, 0, 0\right)$ and $N_i = \left(0, 0, \frac{1}{\sqrt{g^{33}}}\right)$, we get, using $\nabla_i N_j = \frac{\partial N_j}{\partial x^i} - N_k \Gamma^k_{ij}$

$$K_N = \frac{1}{g_{11}\sqrt{g^{33}}}\Gamma^3_{11}$$

or (8, p. 164),

$$K_N = \frac{g^{3k}}{g_{11}\sqrt{g^{33}}}\left[\frac{\partial g_{1k}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^k}\right]$$

and

$$\frac{K_N}{\sqrt{g^{33}}} = \frac{g^{3k}}{g_{11}g^{33}}\left[\frac{\partial g_{1k}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^k}\right] \qquad (18.2)$$

If we take the stream lines and their orthogonal trajectories as the coordinate curves on each $X^3 = \omega$ = const. surface and take a family of surfaces through these orthogonal trajectories as the X^1 = const. surfaces, then $g_{12} = 0$ and $\frac{\partial g_{11}}{\partial X^2} = 0$. (see Appendix I1). In these coordinates $\frac{g^{13}}{g^{33}} = -\frac{g_{13}}{g_{11}}$ and eq. (18.2) becomes

$$\frac{K_N}{\sqrt{g^{33}}} = -\frac{1}{2g_{11}^2}\frac{\partial g_{11}}{\partial X^1}g_{13} + \frac{1}{g_{11}}\frac{\partial g_{13}}{\partial X^1} - \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial X^3} \tag{18.3}$$

With $\frac{K_N}{\sqrt{g^{33}}}$ in this form we shall show that $\frac{\partial}{\partial X^2}\frac{K_N}{\sqrt{g^{33}}} = 0$ if and only if

$$g_{13} = \sqrt{g_{11}}\, A_2(X^2, X^3) + A_1(X^1, X^3) \tag{18.4}$$

where A_1 and A_2 are arbitrary functions. To see this, we first suppose $\frac{\partial}{\partial X^2}\frac{K_N}{\sqrt{g^{33}}} = 0$. Then from eq. (18.3)

$$-\frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{13}}{\partial x^2}+\frac{\partial}{\partial x^1}\left(\frac{\partial g_{13}}{\partial x^2}\right)=0$$

or,

$$\frac{\partial}{\partial x^1}\left(\ln\frac{1}{\sqrt{g_{11}}}\right)+\frac{\partial}{\partial x^1}\left(\ln\frac{\partial g_{13}}{\partial x^2}\right)=0 \qquad (18.5)$$

assuming $\frac{\partial}{\partial x^2}g_{13}\neq 0$. Then, integrating (18.5) we get

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial g_{13}}{\partial x^2}=A_0(x^2, x^3) \qquad (18.6)$$

where A_0 is an arbitrary function. Integrating (18.6) we get eq. (18.4). Conversely, substituting g_{13} in the form (18.4) into eq. (18.3) we obtain immediately the result $\frac{\partial}{\partial x^2}\frac{K_N}{\sqrt{g^{33}}}=0$.

In particular, if there are surfaces normal to the stream lines, that is,

$$v_i\,\Omega^i=0 \qquad (18.7)$$

then we can choose coordinates such that $g_{13} = 0$. g_{13} is then of the form (18.4) (put $A_1 = A_2 = 0$) and therefore $\frac{\partial}{\partial x^2} \frac{K_N}{\sqrt{g^{33}}} = 0$ It follows from the result below (18.1) that there is a Beltrami flow with the same stream lines . But by eqs. (18.1) and (18.7) a Beltrami flow with surfaces normal to the stream lines has $\Omega^i = 0$, that is, it is a potential flow. Thus, the class of $\nabla_i t^i = 0$ flows with surfaces normal to the stream lines is geometrically equivalent to the class of potential flows with $\nabla_i t^i = 0$.

Hamel (30) has shown that the only potential $\nabla_i t^i = 0$ flows are those obtained by the normal superposition of a plane potential vortex and a uniform rectilinear flow. Hence, <u>every $\nabla_i t^i = 0$ flow having surfaces normal to the stream lines has the stream lines of one of these flows</u>. This is a result which was obtained by Prim (2) by a different method. As an application of this result we note that plane and axially symmetrical flows are flows having surfaces normal to the stream lines so that the only plane or axially symmetrical flows with $\nabla_i t^i = 0$ are those whose stream lines are concentric circles or parallel straight lines (28, 2).

19. <u>Generalizations of Plane and Axially Symmetrical Flows</u>. We have seen that for $\nabla_i t^i = 0$ flows the additional requirement that there be a family of surfaces normal to the stream lines, and thus, in

particular, the requirement that the flow be either plane or axially symmetrical, leads to a rather restrictive class of solutions. Hence in order to try to obtain other $\nabla_i t^i = 0$ flows we forego this requirement and apply other geometrical conditions. The conditions we will apply can be obtained by abstracting certain general geometrical properties possessed by plane and axially symmetrical flows. Thus, plane flows have the property that (a) there is a family of cylindrical stream surfaces all having parallel generators and the stream lines are geodesics on these surfaces, and axially symmetrical flows have the property that (b) there is a family of stream surfaces which are coaxial surfaces of revolution and the stream lines are geodesics on these surfaces. Now we will look for flows in the classes (a) and (b).

a. Stream Lines are Geodesics on Cylindrical Stream Surfaces. Any one-parameter family of cylinders all having parallel generators can be described by

$$\begin{aligned} x &= x(u^1, \omega) \\ y &= y(u^1, \omega) \\ z &= u^2 \end{aligned} \tag{19.1}$$

where x, y, z are rectangular cartesian coordinates, ω is the parameter, and $x(u^1, \omega)$, $y(u^1, \omega)$ are arbitrary functions. This family of cylinders, $\{\omega = \text{const.}\}$, will be our family of stream surfaces.

The geodesics on these surfaces are helices each of which is given, for a fixed value of ω, by (23, p. 15)

$$u^2 = I \int^{u'} \sqrt{\left(\frac{\partial x}{\partial u'}\right)^2 + \left(\frac{\partial y}{\partial u'}\right)^2}\, du' + J \qquad (19.2)$$

where I and J are arbitrary constants. The integral

$$S(u', \omega) = \int^{u'} \sqrt{\left(\frac{\partial x}{\partial u'}\right)^2 + \left(\frac{\partial y}{\partial u'}\right)^2}\, du' \qquad (19.3)$$

is the length of the curve of intersection of the cylinder ω = const. and the x, y plane. The constant, I, is the cotangent of the angle β between the generator of the cylinder and the helix since $dz = \cot\beta\, dS$ (See Figure 4). But the helices are to be stream lines of a flow, so they must be the intersections of the stream surfaces ω = const. and a second family $\{\psi = \text{const.}\}$, and therefore the parameters I and J are, in general, functions

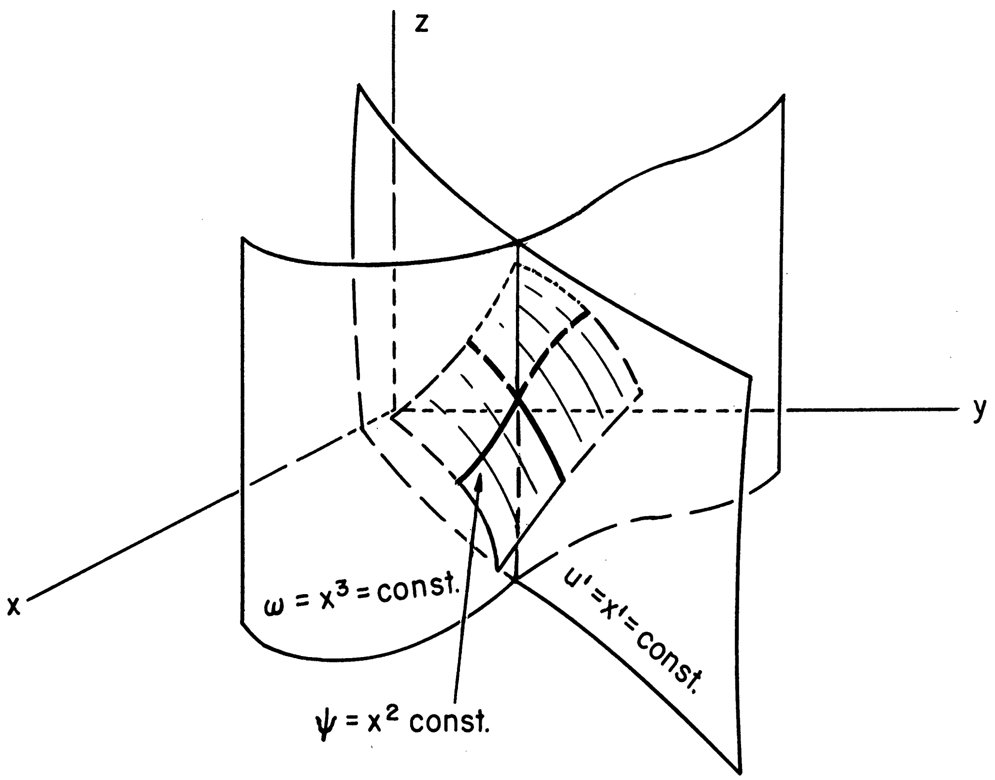

Figure 4. Stream Lines on Cylinders

of ψ and ω . Then with the notation introduced by eq. (19.3), eq. (19.2) reads

$$u^2 = I(\psi, \omega)\, S(u', \omega) + J(\psi, \omega) \tag{19.4}$$

Now taking the families $\{\psi = const.\}$ and $\{\omega = const.\}$ as coordinate families $\{X^2 = const.\}$ and $\{X^3 = const.\}$ we are in a position to try to satisfy eqs. (17.10) and (17.11). To this end we will first obtain the quantities $\sqrt{'g^{22}}$, K_N , and $\sqrt{g^{33}}$ appearing in eqs. (17.10) and (17.11) in terms of the arbitrary functions $X, \mathcal{Y}, I,$ and J . For convenience we will take for the coordinate family $\{X^1 = const.\}$ the cylinders with generators u^1 = const. and which are orthogonal to the $X^3 = \omega$ = const. surfaces. (See Figure 4). By suitably choosing u^1 in eq. (19.1) these surfaces can be described as u^1 = const. surfaces. Hence u^1, ψ, ω will be the independent variables in our subsequent analysis.

Let us first calculate the quantity $\sqrt{'g^{22}}$. When u^2 in eqs. (19.1) is replaced by its expression in (19.4) and ω is held constant we get a parametrization of the ω = const. surfaces by u^1 and ψ , and ψ = const. are the stream lines. Then a straight forward calculation for the metric coefficients $'g_{\varsigma\varkappa}$ of ω = const. gives

$$'g_{11} = (1 + I^2)\left(\frac{\partial s}{\partial u'}\right)^2$$

$$'g_{12} = I\frac{\partial s}{\partial u'}\left(\frac{\partial I}{\partial \psi} s + \frac{\partial J}{\partial \psi}\right)$$

$$'g_{22} = \left(\frac{\partial I}{\partial \psi} s + \frac{\partial J}{\partial \psi}\right)^2$$

and from these

$$\sqrt{'g^{22}} = \frac{\sqrt{1 + I^2}}{\left|\frac{\partial I}{\partial \psi} s + \frac{\partial J}{\partial \psi}\right|} \tag{19.5}$$

The normal curvature κ_N appearing in eq. (17.11) is expressed in terms of the principal curvatures K_1 and K_2 of ω = const. by the Euler formula (8, p. 230)

$$\kappa_N = K_1 \cos^2\beta + K_2 \sin^2\beta \tag{19.6}$$

But in our case $\mathrm{K}_1 = 0$ so

$$K_N = \mathrm{K}_2 \sin^2\beta \tag{19.7}$$

Now both K_2, and $\sqrt{g^{33}}$ can be considered to be quantities defined over the x, y plane; i.e., $|\mathrm{K}_2|$ is the curvature of the curve of intersection of the x, y plane and ω = const., and $\sqrt{g^{33}}$, the magnitude of the gradient of the ω = const. surfaces, is the same as the magnitude of the gradient of the ω = const. <u>curves</u> in the x, y plane. Consequently if a_{11} and a_{22} are the metric coefficients for the orthogonal u', ω coordinate net in the x, y plane, then by use of equations of type (10.3), we find

$$\mathrm{K}_2 = \pm \frac{1}{\sqrt{a_{11}}\sqrt{a_{22}}} \frac{\partial\sqrt{a_{11}}}{\partial \omega} \tag{19.8}$$

and since $a_{12} = 0$, we see that

$$\sqrt{g^{33}} = \sqrt{a^{22}} = \frac{1}{\sqrt{a_{22}}} \tag{19.9}$$

Substituting K_2 from (19.8) into (19.7) we get

$$K_N = \pm \frac{\sin^2\beta}{\sqrt{a_{11}}\sqrt{a_{22}}} \frac{\partial \sqrt{a_{11}}}{\partial \omega} \tag{19.10}$$

For a fixed value of ψ, $\sqrt{'g^{22}}$ can also be considered to be defined in the x, y plane. In particular, from (19.3) we have $s = \int \sqrt{a_{11}}\, du'$, and since $\dfrac{\partial \sqrt{a_{11}}}{\partial u'} = 0$

$$s = \sqrt{a_{11}}\, u' + \Omega_1 \tag{19.11}$$

where Ω_1 is an arbitrary function of ω. Substituting s from (19.11) into (19.5) we get

$$\sqrt{'g^{22}} = \frac{\sqrt{1 + I^2}}{\left| \frac{\partial I}{\partial \psi}\left(\sqrt{a_{11}}\, u' + \Omega_1\right) + \frac{\partial J}{\partial \psi} \right|} \tag{19.12}$$

In eqs. (19.9), (19.10) and (19.12) we have the quantities $\sqrt{g^{33}}$, K_N and $\sqrt{g^{22}}$ expressed in the form we desire. In this form we now put these quantities into eqs. (17.10) and (17.11).

Substituting (19.9) and (19.12) into eq. (17.10) with $X^1 = u^1$ and noting that I is not a function of u^1 , we get

$$\frac{\partial}{\partial u^1}\left[\frac{1}{\left|\frac{\partial I}{\partial \psi}(\sqrt{a_{11}}\, u^1 + \Omega_1) + \frac{\partial J}{\partial \psi}\right|} \cdot \frac{1}{\sqrt{a_{22}}}\right] = 0$$

so that

$$\sqrt{a_{22}} = \frac{1}{\Omega_2\left(\frac{\partial I}{\partial \psi}\sqrt{a_{11}}\, u^1 + \frac{\partial I}{\partial \psi}\Omega_1 + \frac{\partial J}{\partial \psi}\right)} \qquad (19.13)$$

in which ψ is held fixed and Ω_2 is an arbitrary function of ω .

Substituting (19.9) and (19.10) into eq. (17.11) with $X^1 = u^1$ and noting that the angle β is not a function of u^1 we get

$$\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{a_{11}}}\frac{\partial}{\partial \omega}\sqrt{a_{11}}\right) = 0$$

Thus $\sqrt{a_{11}}$ is a product of a function of u' and a function of ω, or, by a change of scale, $\sqrt{a_{11}}$ is a function of ω alone.

We have seen that eq. (17.10) yields (19.13) which is a relation between $\sqrt{a_{11}}$ and $\sqrt{a_{22}}$, and eq. (17.11) imposes a condition on $\sqrt{a_{11}}$. Now in order that $\sqrt{a_{11}}$ and $\sqrt{a_{22}}$ be metric coefficients corresponding to an orthogonal coordinate system in the plane it is necessary and sufficient that (see Appendix III)

$$\frac{\partial}{\partial \omega}\left(\frac{1}{\sqrt{a_{22}}}\frac{\partial}{\partial \omega}\sqrt{a_{11}}\right) + \frac{\partial}{\partial u'}\left(\frac{1}{\sqrt{a_{11}}}\frac{\partial}{\partial u'}\sqrt{a_{22}}\right) = 0$$

Substituting $\sqrt{a_{22}}$ from (19.13) into this condition we obtain a fourth degree equation in u'. The vanishing of the coefficients requires that the coefficient, $\frac{\partial I}{\partial \psi}\sqrt{a_{11}}$, of u' in eq. (19.13) be zero, that is

$$\frac{\partial I}{\partial \psi} = \frac{\partial \beta}{\partial \psi} = 0$$

This result means, geometrically, that on a fixed ω = const. surface all the helices are parallel. Our condition $\nabla_i \cdot t^i = 0$ then immediately tells us (see eq. (3.7)) that $t^i \nabla_i |\nabla \omega| = 0$, and consequently $|\nabla \omega|$ is constant on the cylinders ω = const. It follows that the ω = const. surfaces are parallel.

It is evident now, from geometrical considerations, that the ω = const. surfaces must be coaxial circular cylinders. More formally, we can note first from eq. (19.13) that $\sqrt{a_{22}}$ is independent of u^1. Then from (19.8) K_2 is independent of u^1; that is, the curvature of the cross-sections of the cylinders is constant.

We have arrived at the result that the only possible $\nabla_i t^i = 0$ flows whose stream lines are geodesics on cylinders are those whose stream lines are parallel helices on coaxial circular cylinders. On the other hand, as is seen from the geometrical interpretation of eqs. (17.10) and (17.11), any family of parallel helices on coaxial circular cylinders will correspond to a fluid flow. Such flows we shall call simply "helical flows" and we shall come back to them and describe them more completely in Chapter VIII.

b. Stream Lines are Geodesics on Surfaces of Revolution. Now we ask for flows in which there is a family of stream surfaces which are coaxial surfaces of revolution and the stream lines are geodesics on these surfaces. Any one-parameter family of coaxial surfaces of revolution can be described by

$$\begin{aligned} x &= u^1 \cos u^2 \\ y &= u^1 \sin u^2 \\ z &= f(u^1, \omega) \end{aligned} \tag{19.14}$$

where x, y, z are rectangular cartesian coordinates, u^1, u^2 are polar coordinates in the x, y planes, ω is the parameter, and $f(u^1, \omega)$ is an arbitrary function.

The geodesics on these surfaces are described by (23, p. 179)

$$u^2 = I \int^{u^1} \frac{\sqrt{1 + \left(\frac{\partial f}{\partial u^1}\right)^2}}{u^1 \sqrt{u^{1\,2} - I^2}}\, du^1 - J \tag{19.15}$$

where I and J are arbitrary constants; that is, we have a 2-parameter family on each surface. But if these curves are to be the stream lines of a flow, then they must be the intersections of the stream surfaces ψ = const. and ω = const., and thus the parameters I and J are, in general, functions of ψ and ω. Let

$$W(u^1, \psi, \omega) = I(\psi, \omega) \int^{u^1} \frac{\sqrt{1 + \left(\frac{\partial f}{\partial u^1}\right)^2}}{u^1 \sqrt{u^{1\,2} - I^2}}\, du^1 \tag{19.16}$$

Then eq. (19.15) reads

$$u^2 = W(u^1, \psi, \omega) - J(\psi, \omega) \tag{19.17}$$

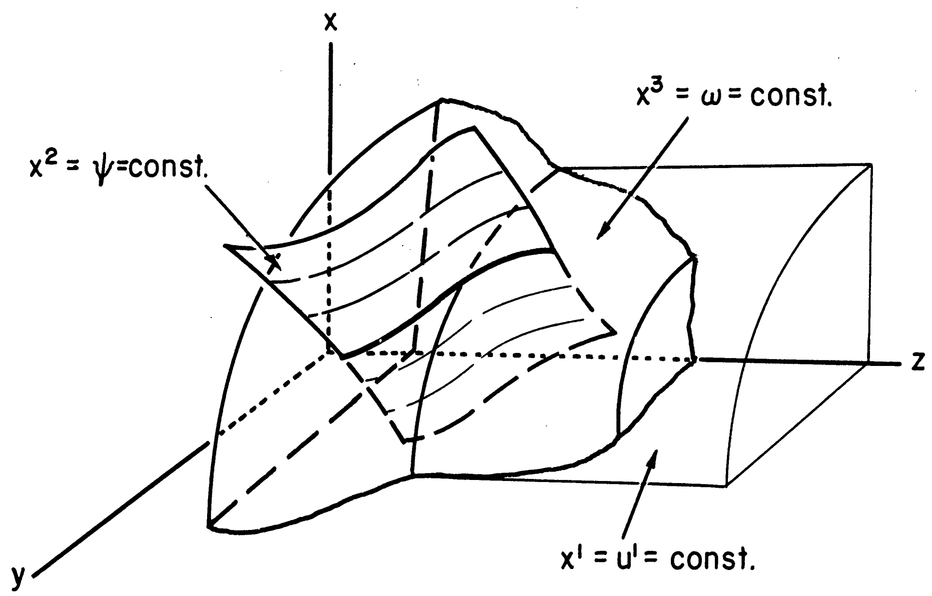

Figure 5. Stream Lines on Surfaces of Revolution

Now taking the families $\{\psi = const.\}$ and $\{\omega = const.\}$ as coordinate families $\{x^2 = const.\}$ and $\{x^3 = const.\}$ we are in a position to try to satisfy eqs. (17.10) and (17.11). To this end we will first obtain the quantities $\sqrt{g^{22}}$, K_N , and $\sqrt{g^{33}}$

appearing in eqs. (17.10) and (17.11) in terms of the arbitrary functions I , J , and f . For convenience we will take for the coordinate family $\{x' = const.\}$ the circular cylinders through the circles u' = const. and which have generators in the z direction. (See Figure 5). These surfaces can be described as u' = const. surfaces. Hence u' , ψ , ω will be the independent variables in our subsequent analysis.

The normal curvature K_N appearing in eq. (10.3) is expressed in terms of the principal curvatures of ω = const. by the Euler formula

$$K_N = K_1 \cos^2\beta + K_2 \sin^2\beta \tag{19.6}$$

as before. For surfaces of revolution we have (23, p. 227)

$$K_1 = \frac{\frac{\partial f}{\partial u'}}{u'\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^{1/2}} \quad , \quad K_2 = \frac{\frac{\partial^2 f}{\partial u'^2}}{\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^{3/2}} \tag{19.18}$$

Further, when u^2 in eqs. (19.14) is replaced by its expression (19.17) and ω is held constant we get a parametrization of the ω = const. surfaces by u' and ψ, and ψ = const. are the stream lines. Then $\cos^2\beta = \frac{'g_{12}^2}{'g_{11}\,'g_{22}}$. A straight forward calculation for the metric coefficients $'g_{\xi\varkappa}$ of ω = const. gives

$$'g_{11} = 1 + \left(\frac{\partial f}{\partial u'}\right)^2 + u'^2\left(\frac{\partial W}{\partial u'}\right)^2$$

$$'g_{12} = u'^2\frac{\partial W}{\partial u'}\left(\frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}\right)$$

$$'g_{22} = u'^2\left(\frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}\right)^2$$

Then

$$\cos^2\beta = \frac{u'^2\left(\frac{\partial W}{\partial u'}\right)^2}{1 + \left(\frac{\partial f}{\partial u'}\right)^2 + u'^2\left(\frac{\partial W}{\partial u'}\right)^2} \tag{19.19}$$

and

$$'g^{22} = \frac{1 + \left(\frac{\partial f}{\partial u'}\right)^2 + u'^2\left(\frac{\partial W}{\partial u'}\right)^2}{u'^2\left[1 + \left(\frac{\partial f}{\partial u'}\right)^2\right]\left(\frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}\right)^2} \tag{19.20}$$

Differentiating eq. (19.16) with respect to u' and substituting the result in (19.19) and (19.20) we get respectively

$$\cos^2\beta = \frac{I^2}{u'^2} \qquad\qquad \sin^2\beta = \frac{u'^2 - I^2}{u'^2} \qquad (19.21)$$

and

$$\sqrt{g^{22}} = \frac{1}{\sqrt{u'^2 - I^2}\left|\frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}\right|} \qquad (19.22)$$

Finally, substituting (19.18) and (19.21) into eq. (19.6) we obtain

$$K_N = \frac{I^2 \frac{\partial f}{\partial u'}}{u'^3\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^{1/2}} + \frac{(u'^2 - I^2)\frac{\partial^2 f}{\partial u'^2}}{u'^2\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^{3/2}} \qquad (19.23)$$

The metric coefficient g^{33} appearing in eq. (17.10) can be considered to be defined over the y, z meridional plane; i.e,

$\sqrt{g^{33}}$, the magnitude of the gradient of the ω = const. surfaces, is the same as the magnitude of the gradient of the ω = const. curves in the y, z plane. Consequently, if $a_{\xi\eta}$ are the metric coefficients of the y, z plane referred to the curves ω = const. and u' = const. then $\sqrt{g^{33}} = \sqrt{a^{22}}$. But, using

$$y = u'$$
$$z = f(u', \omega)$$

we get $\sqrt{a^{22}} = \dfrac{\sqrt{1 + \left(\frac{\partial f}{\partial u'}\right)^2}}{\left|\frac{\partial f}{\partial \omega}\right|}$ so

$$\sqrt{g^{33}} = \frac{\sqrt{1 + \left(\frac{\partial f}{\partial u'}\right)^2}}{\left|\frac{\partial f}{\partial \omega}\right|} \tag{19.24}$$

In eqs. (19.22), (19.23), and (19.24) we have the quantities $\sqrt{'g^{22}}$, K_N , and $\sqrt{g^{33}}$ expressed in the form we desire. In this form we now put these quantities into eqs. (17.10) and (17.11).

With the expressions given by eqs. (19.23) and (19.24) the condition (17.11) is obtained in the form

$$\frac{I^2 \frac{\partial f}{\partial u'} \frac{\partial f}{\partial \omega}}{u'^3\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]} + \frac{\left(u'^2 - I^2\right) \frac{\partial^2 f}{\partial u'^2} \frac{\partial f}{\partial \omega}}{u'^2\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^2} = \mathcal{N}(\psi, \omega) \tag{19.25}$$

where $\mathcal{N}(\psi, \omega)$ is an arbitrary function of ψ and ω. With the expressions given by eqs. (19.22) and (19.24) the condition (17.10) is obtained in the form

$$\frac{\sqrt{1+\left(\frac{\partial f}{\partial u'}\right)^2}}{\sqrt{u'^2 - I^2}\left|\frac{\partial f}{\partial \omega}\right|\left|\frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}\right|} = \varpi(\psi, \omega) \tag{19.26}$$

where $\varpi(\psi, \omega)$ is an arbitrary function of ψ and ω. Now the problem of finding fluid flows whose stream lines are geodesics on surfaces of revolution is reduced to solving eqs. (19.25) and (19.26). That is, corresponding to every set of functions $f(u', \omega)$, $I(\psi, \omega)$, $J(\psi, \omega)$, $\mathcal{N}(\psi, \omega)$, and $\varpi(\psi, \omega)$ which satisfy eqs. (19.25) and (19.26) we obtain such a fluid flow. We will now see that the possible solutions of eqs. (19.25) and (19.26) are limited to those for which I , J , $\mathcal{N}$, and $\mathcal{W}$ are independent of ψ .

Dividing both sides of eq. (19.25) by $u'^2 - I^2$ and differentiating with respect to ψ we get

$$\frac{\partial I^2}{\partial \psi} \frac{\frac{\partial f}{\partial u'} \frac{\partial f}{\partial \omega}}{u'\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]} = u'^2 \frac{\partial \mathcal{N}}{\partial \psi} - I^2 \frac{\partial \mathcal{N}}{\partial \psi} + \mathcal{N} \frac{\partial I^2}{\partial \psi}$$

Assuming $\frac{\partial I^2}{\partial \psi} \neq 0$, we can divide through by it and differentiate again with respect to ψ . Then the vanishing of the coefficients in the resulting equation in u' leads to $\mathcal{N} = \mathcal{N}_1 I^2 + \mathcal{N}_2$ where $\mathcal{N}_1$ and $\mathcal{N}_2$ are functions of ω alone. Putting this back into eq. (19.25) we obtain an equation linear in I^2 , and the vanishing of the coefficients leads to

$$\frac{\partial f}{\partial u'} \frac{\partial f}{\partial \omega} - u'(\mathcal{N}_1 u'^2 + \mathcal{N}_2)\left[1 + \left(\frac{\partial f}{\partial u'}\right)^2\right] = 0 \qquad (19.27)$$

$$\frac{\partial^2 f}{\partial u'^2} \frac{\partial f}{\partial \omega} - \mathcal{N}_2\left[1 + \left(\frac{\partial f}{\partial u'}\right)^2\right] = 0 \qquad (19.28)$$

Ignoring only the case of straight stream lines we eliminate $\frac{\partial f}{\partial \omega}$ between these two equations to get an equation containing $\frac{\partial f}{\partial u'}$ and $\frac{\partial^2 f}{\partial u'^2}$. The latter equation is then integrated twice with respect to u' so that f is expressed explicitly as a function of u', v_1, v_2, and two additional functions of ω. When the arbitrary functions of ω are suitably chosen f will also satisfy eqs. (19.27) and (19.28). In particular, these solutions will have

$$\frac{\partial f}{\partial \omega} = \pm \frac{\frac{1}{2}\frac{d v_3}{d\omega}}{\sqrt{v_3 - u'^2}} \tag{19.29}$$

where v_3 is an arbitrary function of ω.

From eq. (19.26) we can get another expression for $\frac{\partial f}{\partial \omega}$. We write (19.26) as

$$\pm \frac{\sqrt{1 + \left(\frac{\partial f}{\partial u'}\right)^2}}{\sqrt{u'^2 - I^2}\;\frac{\partial f}{\partial \omega}\; w} = \frac{\partial W}{\partial \psi} - \frac{\partial J}{\partial \psi}$$

and differentiate with respect to u'. The resulting equation is

$$\left[w_1 \frac{\partial I^2}{\partial \psi}\frac{1}{u'^2 - I^2}\right]\frac{\partial f}{\partial \omega} \pm \frac{1}{u'^2 - I^2} = w_2$$

where $\omega_1 = \frac{\omega}{2I}$ and ω_2 is a function of u' and ω. Differentiating this equation with respect to ψ we obtain

$$\frac{\partial f}{\partial \omega} = \pm \frac{\frac{\partial I^2}{\partial \psi}}{\omega_1\left(\frac{\partial I^2}{\partial \psi}\right)^2 - I^2\left[\frac{\partial}{\partial \psi}\left(\omega_1 \frac{\partial I^2}{\partial \psi}\right)\right] + u'^2\left[\frac{\partial}{\partial \psi}\left(\omega_1 \frac{\partial I^2}{\partial \psi}\right)\right]} \tag{19.30}$$

Now if we square eqs. (19.29) and (19.30) and equate the results, we find that we cannot obtain the vanishing of the coefficient of u'^2 as required. Since the foregoing was based on the assumption that $\frac{\partial I^2}{\partial \psi} \neq 0$, we can conclude that there are no flows with this condition. That is, the stream lines of a $\nabla_{\!\lambda} \cdot t^{\lambda} = 0$ flow whose stream lines are geodesics on surfaces of revolution are given by eq. (19.15) with I a function of only ω.

This result may be described geometrically by saying that the stream lines of such a flow are curves of rotation on each ω = const. surface. This is seen from eq. (19.21) by which, on the circles $\{u' = const., \omega = const.\}$ we have β = const. Also, from eqs. (19.23) and (19.24) we have $\frac{\partial}{\partial \psi} \frac{K_N}{\sqrt{g^{33}}} = 0$, so that (see Section 18b) these flows are geometrically equivalent to the class of Beltrami flows.

The equations to be satisfied, eqs. (19.25) and (19.26), now can be slightly simplified. Eq. (19.25) keeps the same form, but I and v are functions of only ω. That is, eq. (19.25) becomes

$$\frac{I^2 \frac{\partial f}{\partial u'} \frac{\partial f}{\partial \omega}}{u'^3\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]} + \frac{(u'^2 - I^2)\frac{\partial^2 f}{\partial u'^2}\frac{\partial f}{\partial \omega}}{u'^2\left[1+\left(\frac{\partial f}{\partial u'}\right)^2\right]^2} = v(\omega) \tag{19.31}$$

Eq. (19.26) can be written in the simpler form

$$\frac{1+\left(\frac{\partial f}{\partial u'}\right)^2}{(u'^2 - I^2)\left(\frac{\partial f}{\partial \omega}\right)^2} = w_3(\omega) \tag{19.32}$$

where $w_3(\omega) = \left|\frac{\partial J}{\partial \psi}\right| w(\psi, \omega)$

Thus we have found that the problem of obtaining $\nabla_x \cdot t' = 0$ flows whose stream lines are geodesics on surfaces of revolution reduces to that of obtaining solutions $f(u', \omega)$, $I(\omega)$, $v(\omega)$, $w_3(\omega)$ of eqs. (19.31) and (19.32).

In an attempt to obtain solutions of eqs. (19.31) and (19.32) we can try some of the simplest possible cases. For example, let us try the case in which f is a linear function of u' . That is, the ω = const. surfaces are circular cones. Applying this condition to eq. (19.32) we find that the result cannot be an identity in u' and ω . Thus there are no $\nabla_i t^i = 0$ flows in which the stream lines are geodesics on circular cones.

So far we have not obtained any solutions of eqs. (19.31) and (19.32). Consequently, we are led to the question of the existence of $\nabla_i t^i = 0$ flows (other than the very simple flows with straight or circular stream lines) whose stream lines are geodesics on surfaces of revolution, or equivalently, the question of consistency of eqs. (19.31) and (19.32)

CHAPTER VIII

HELICAL FLOWS

In Section 19 (Chapter VII) we found that a fluid flow with $\nabla_i t^i = 0$ and having a family of cylindrical stream surfaces on which the stream lines are geodesics (helices) must have a family of concentric circular stream surfaces on which the stream lines are parallel helices. We also saw, om the other hand, that every family of parallel helices on concentric circular cylinders is a possible stream line pattern of one or more fluid flows. Now we shall examine the class of all fluid flows with these parallel helical stream line patterns.

Some of the simpler helical flows are well-known. Parallel straight line flows and plane vortex flows are included in our class of helical flows as limiting cases. Potential helical flows have been completely characterized by Hamel (see Section 18, Chapter VII). These flows are used in the theory of compressors and turbines and there described as "free-vortex" flows (31, p. 201). Also, in this area of application a generalization called "forced-vortex flows" is sometimes considered. (31, p. 203). Nemenyi and Prim (32) have obtained some flows of the class of Beltrami helical flows. Such flows have also been obtained by Coburn (33) for the supersonic range by means of the characteristic equations.

20. <u>The Equation Governing Helical Flows</u>. We found, in Section 16 (Chapter VII) that when $\nabla_i \cdot t^i = 0$ the equations of fluid flow have the same form in both the incompressible and the compressible cases. Using the notation of the incompressible case these equations are

$$B = B(x^2, x^3) \tag{17.7}$$

$$P = P(x^3) \tag{17.8}$$

$$\frac{dP}{dx^3} = -2(B - P)\frac{K_N}{\sqrt{g^{33}}} \tag{17.9}$$

In the incompressible case

$$P = \frac{p}{\rho} \tag{5.1}$$

and

$$\frac{q^2}{2} + P = B \tag{5.3}$$

In the compressible case the corresponding relations are

$$\mathcal{P} = \int_0^p \frac{1}{\mathrm{X}(p)}\, dp \tag{6.4}$$

$$\Upsilon \frac{q^2}{2} + \mathcal{P} = \mathcal{B} \tag{6.9}$$

The $X^3 = \omega =$ const. surfaces are now the concentric circular cylinders and we can parametrize these surfaces by their cross-sectional radii. That is, we can put $X^3 = r$. Then

$$\sqrt{g^{33}} = |\nabla r| = 1 \tag{20.1}$$

Further, since the principal curvature, K_2, of $r =$ const. is $-\frac{1}{r}$, (19.7) yields

$$K_N = -\frac{1}{r} \sin^2 \beta(r) \tag{20.2}$$

where β is the angle between the helices and the generators of the circular cylinders, and this angle is constant on each circular cylinder.

Now eq. (17.8) is $P = P(r)$, and using eqs. (20.1) and (20.2), eq. (17.9) becomes

$$\frac{dP}{dr} = 2(B - P)\frac{\sin^2\beta(r)}{r} \tag{20.3}$$

Finally, from eq. (20.3) we see that B must be a function of r alone. That is, eq. (17.7) becomes $B = B(r)$. Thus, our problem reduces to satisfying eq. (20.3) with three functions, $\beta(r)$, $P(r)$, and $B(r)$.

The fact that we have only the one condition, eq. (20.3), on our fluid flows means that corresponding to any arbitrary assignment as a function of r of any two of the three quantities stream line slope, pressure, and Bernoulli function there is a fluid flow with the two prescribed properties and the third function given by eq. (20.3). For example, given a function β we can also select P (or B) and let eq. (20.3) determine B (or P). On the other hand, we can start with P and B and think of eq. (20.3) as determining β. These several procedures are subject to the conditions that P and B are positive and $0 \leq \sin^2\beta \leq 1$. Hence, we can think of the class of helical fluid flows as being described by three functions P(r), B(r), β(r) subject to the one condition, eq. (20.3), or as a class with two arbitrary functions.

21. <u>General Properties of Helical Flows</u>. Now we will obtain some general properties of our class of flows.

a. <u>The Pressure Variation</u>. From eq. (20.3) since $B - P \geq 0$ we see that $\frac{dP}{dr} \geq 0$; that is, in all cases, P is a non-decreasing function of r.

P may be either a bounded or an unbounded function. This may be easily shown by examples.

(i) Let $P = 1 - e^{-r}$. Then $\frac{dP}{dr} = e^{-r}$, and putting this into eq. (20.3) we see that if we then choose $B - P = r e^{-r}$ we get a flow with $\sin^2\beta = \frac{1}{2}$.

(ii) Let $P = r$. Then $\frac{dP}{dr} = 1$, and putting this into eq. (20.3) we see that if we then choose $B - P = \frac{r}{2} e^{r}$ we get a flow with $\sin^2\beta = e^{-r}$.

Since $\frac{\sin^2\beta}{r} \leqslant \frac{1}{r}$ for all functions β , the relations

$$0 \leqslant \frac{dP}{dr} \leqslant \frac{2}{r}(B - P) \tag{21.1}$$

always hold. For plane vortex flows eq. (20.3) becomes

$$\frac{dP}{dr} = \frac{2}{r}(B - P) \tag{21.2}$$

Hence, comparing (21.1) and (21.2) we see that $\frac{dP}{dr}$ is greater for a plane vortex flow than for any other helical flow with the same $B - P$. In terms of g, from eqs. (5.3) and (6.9), this can also be stated as follows: $\frac{dP}{dr}$ of any incompressible helical flow is bounded by $\frac{dP}{dr}$ of the incompressible plane vortex flow with the same g. If both flows have the same value of P at some radius, then P of the vortex flow is greater than P of the (non-plane vortex) helical flow at all subsequent radii. Also, $\frac{d\mathcal{P}}{dr}$ of any compressible helical flow is bounded by $\frac{d\mathcal{P}}{dr}$ of the compressible plane vortex flow with the same $g^2 \Upsilon$, and if both flows have the same value of $\mathcal{P}$ at some radius then $\mathcal{P}$ of the vortex flow is greater than $\mathcal{P}$ of the (non-plane vortex) helical flow at all subsequent radii.

If we have P, we get the pressure, $p_{(i)}$, in the incompressible case from

$$p_{(i)} = \rho_{(i)} P \tag{21.3}$$

and from (6.4) with $\mathcal{P}$ replaced by P we obtain the pressure of a compressible flow. For a polytropic gas we have from (6.28) with P in place of $\mathcal{P}$

$$p_{(c)} = \left(\frac{\gamma-1}{\gamma} P\right)^{\frac{\gamma}{\gamma-1}} \tag{21.4}$$

The relations expressed in eqs. (21.3) and (21.4) are shown in Figure 6.

In particular, from the relations between P and the pressure we see the fact that P is a non-decreasing function of r implies that the pressure is a non-decreasing function of r for all flows both incompressible and compressible (assuming $X > 0$). Further, the results stated below eq. (21.2) are valid for the pressure as well as for P and $\mathcal{P}$.

Further, eqs. (21.3) and (21.4) can be used to compare incompressible flows with flows of a polytropic gas. Thus, for example, suppose that we have a flow of each type both having the same q and β . If, moreover, $\mathcal{B} = B$ and $\Upsilon = 1$, then $P = \mathcal{P}$ but, from eqs. (21.3) and (21.4), $p_{(c)} = \left(\frac{\gamma-1}{\gamma} \frac{p_{(i)}}{\rho_{(i)}}\right)^{\frac{\gamma}{\gamma-1}}$. In the case $P = \mathcal{P} = r$ the pressure variations can be compared directly from Figure 6. Clearly, such comparisons also can be made between two incompressible or two compressible flows.

We can also compare the rate of change of pressure with r for an incompressible flow and a polytropic flow both having the same P. For then

$$\frac{dP}{dp_{(i)}} \frac{dp_{(i)}}{dr} = \frac{dP}{dp_{(c)}} \frac{dp_{(c)}}{dr}$$

or, from (21.3) and (21.4),

$$\frac{dp_{(c)}}{dr} = \frac{dp_{(i)}}{dr} \frac{\left(\frac{\gamma-1}{\gamma} P\right)^{\frac{1}{\gamma-1}}}{\rho_{(i)}} \tag{21.5}$$

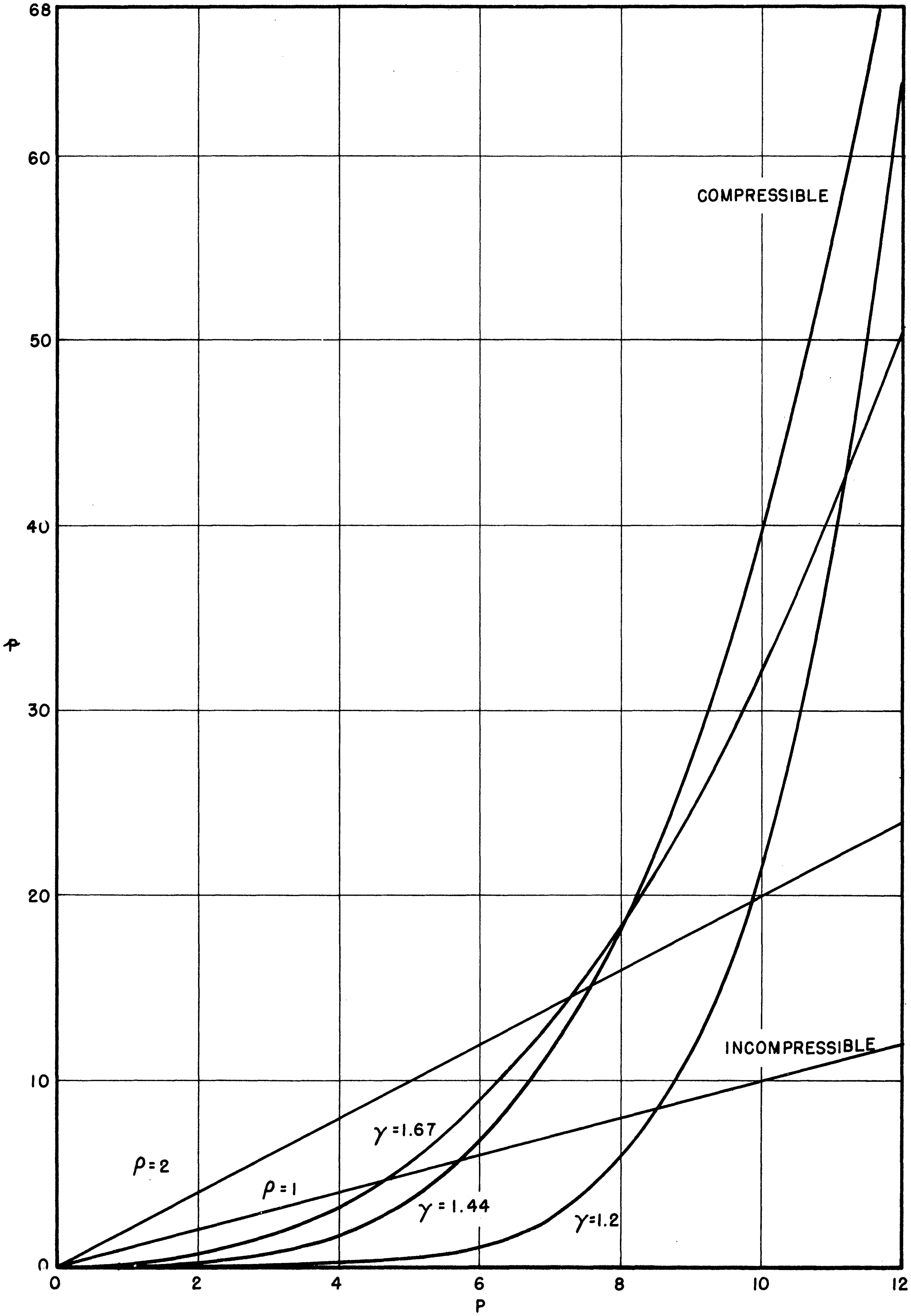

Figure 6. Pressure as a function of P for several values of γ and ρ

Since P is non-decreasing with r and $\rho_{(i)}$ is constant, the above result can be described in terms of the flow field by the statement that (for the same P), as the radius increases, the rate of change of pressure in the compressible case increases relative to the rate of change of pressure in the incompressible case.

Finally, flows in which B = const. ($\mathcal{B}$ = const. in the compressible case) are often of particular interest (see Section 7b, Chapter III). For such flows eq. (20.3) reduces to

$$\frac{d}{dr} \ln (B-P)^{-1} = \frac{2}{r} \sin^2 \beta \qquad (21.6)$$

Then

$$0 \leq \frac{d}{dr} \ln (B-P)^{-1} \leq \frac{2}{r} \qquad (21.7)$$

and for plane vortex flows eq. (21.6) becomes

$$\frac{d}{dr} \ln (B-P)^{-1} = \frac{2}{r} \qquad (21.8)$$

We can integrate eq. (21.7) from a point $r = r_0$ where $P = P_0$ to get

$$P \lessgtr B - \frac{r_0^2(B - P_0)}{r^2} \tag{21.9}$$

Since the equality sign holds for plane vortex flows we see that if an arbitrary helical flow and a plane vortex flow have the same constant B and the same P at some radius then <u>P (and the pressure) of the plane vortex flow is greater than P (and the pressure) of the (non-plane vortex) helical flow for all subsequent radii.</u>

b. <u>The Velocity Magnitude Variation</u>. To determine the variation of q we have to employ eqs. (5.3) and (6.9).

The first thing we notice from these equations is that while in the incompressible case q is only a function of r, in the compressible case q can also vary from stream line to stream line on r = const. if the entropy does. Further, unlike the pressure, q may either increase or decrease with increasing r depending on the behavior of B, or of $\mathcal{B}$ and Υ.

Instead of getting q from P or $\mathcal{P}$ by means of eq. (5.3) or eq. (6.9) we can use these equations to eliminate P (or $\mathcal{P}$) in eq. (20.3) and obtain differential equations for q. For the incompressible case we get

$$\frac{d}{dr}\frac{q^2}{2} + \frac{2\sin^2\beta}{r}\frac{q^2}{2} = \frac{dB}{dr} \tag{21.10}$$

and for the compressible case

$$\frac{d}{dr}\left(\frac{q^2}{2}\Upsilon\right) + \frac{2\sin^2\beta}{r}\left(\frac{q^2}{2}\Upsilon\right) = \frac{d\mathcal{B}}{dr} \tag{21.11}$$

or

$$\frac{\partial}{\partial r}\frac{q^2}{2} + \frac{2\sin^2\beta}{r}\frac{q^2}{2} = \frac{\partial}{\partial r}\frac{\mathcal{B}}{\Upsilon} + \frac{\mathcal{P}}{\Upsilon}\frac{\partial}{\partial r}\ln\Upsilon \tag{21.12}$$

The partial derivatives appear in (21.12) since, as we have noted, q and Υ may vary from stream line to stream line on r = const. as well as with r.

We can compare the incompressible and the polytropic flows from a somewhat different viewpoint than in 21a. Thus, for example, suppose we have a flow of each type both having the same p. If, moreover, $\mathcal{B} = B$ and $\Upsilon = 1$, then

$$\frac{q_{(i)}^2}{2} + P = \frac{q_{(c)}^2}{2} + \mathcal{P} \tag{21.13}$$

and from eqs. (21.3) and (21.4)

$$\frac{q_{(i)}^2}{2} + \frac{p}{\rho_{(i)}} = \frac{q_{(c)}^2}{2} + \frac{\gamma}{\gamma-1} p^{\frac{\gamma-1}{\gamma}} \tag{21.14}$$

In the special case p = r, then the difference $\frac{q_{(i)}^2}{2} - \frac{q_{(c)}^2}{2} = P_{(c)} - P_{(i)}$ is obtained directly from Figure 6. as the horizontal distance between the two appropriate curves.

We can also compare the rate of change of q with r for an incompressible flow and a polytropic flow both having the same p . From the identity

$$\frac{dP_{(c)}}{dp}\frac{dP_{(i)}}{dr} = \frac{dP_{(i)}}{dp}\frac{dP_{(c)}}{dr}$$

and by use of eqs. (21.3) and (21.4) we find

$$\frac{dP_{(i)}}{dr} = \frac{dP_{(c)}}{dr}\frac{p^{\frac{1}{\gamma}}}{\rho_{(i)}} \tag{21.15}$$

If we assume, moreover, that $\mathcal{B} = B$ and $\Upsilon = 1$, then we can use eq. (21.13). Differentiating eq. (21.13) and applying (21.15) we obtain the relations

$$\frac{d}{dr}\left(\frac{q_{(i)}^2}{2}\right) - \frac{d}{dr}\left(\frac{q_{(c)}^2}{2}\right) = p^{-\frac{1}{\gamma}}\frac{dp}{dr}\left[1 - \frac{p^{\frac{1}{\gamma}}}{\rho_{(i)}}\right]$$

$$= \frac{dp}{dr}\left[\frac{1}{p^{1/\gamma}} - \frac{1}{\rho_{(i)}}\right] \tag{21.16}$$

In the special case $p = r$ eq. (21.16) says that the difference $\frac{d}{dr} q_{(i)}^2 - \frac{d}{dr} q_{(c)}^2$ decreases with r. In particular, for small r, q^2 changes more rapidly in the incompressible case and for large r, q^2 changes more rapidly in the compressible case.

Finally, as in 21a we look at the case when B = const. In terms of q eq. (20.3) reduces to

$$\frac{d}{dr} \ln(\Upsilon q^2)^{-1} = \frac{2}{r}\sin^2\beta \tag{21.17}$$

for compressible flows, and now Υq^2 is a non-increasing function of r . Further,

$$0 \leq \frac{d}{dr} \ln(\Upsilon q^2)^{-1} \leq \frac{2}{r} \tag{21.18}$$

and for plane vortex flows eq. (21.17) becomes

$$\frac{d}{dr} \ln(\Upsilon q^2)^{-1} = \frac{2}{r} \tag{21.19}$$

We can integrate eq. (21.18) from a point $r = r_0$ where $\Upsilon q^2 = \Upsilon_0 q_0^2$ to get

$$\frac{r_0^2}{r^2}\left(\Upsilon_0 q_0^2\right) \leq \Upsilon q^2 \tag{21.20}$$

Since the equality sign holds for plane vortex flows we see that if an arbitrary helical flow and a plane vortex flow have the same Υq^2 at some radius then <u>Υq^2 of the (non-plane vortex) helical flow</u>

is greater than Υq^2 *of the plane vortex flow at all subsequent radii.* Also, if we replace Υq^2 by q^2 in this and the preceding paragraph we obtain corresponding results for incompressible flows.

c. *The Mach Number Variation.* In the compressible case another important quantity is the Mach number

$$M = \frac{q}{c}$$

where c is the velocity of sound (see Section 6). With q^2 from eq. (6.8) and c^2 from eq. (6.24) we get

$$M^2 = 2(\mathcal{B} - \mathcal{P})\frac{dX}{dp} \qquad (21.21)$$

Thus M, like p, is a function of r alone.

For a polytropic gas we can apply eq. (6.34), and eq. (21.21) becomes

$$M^2 = \frac{2}{\gamma - 1}\,\frac{\mathcal{B} - \mathcal{P}}{\mathcal{P}} \qquad (21.22)$$

Solving eq. (21.22) for $\mathcal{B}$ we see that M may vary in any manner whatsoever if $\mathcal{B}$ is not restricted. Solving eq. (21.22) for $\mathcal{P}$ and substituting the resulting expression into (20.3) in which P and B are replaced by $\mathcal{P}$ and $\mathcal{B}$ we get

$$\frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{d}{dr}\left(\frac{\gamma-1}{2}M^2\right)+\frac{2\sin^2\beta}{r}\left(\frac{\gamma-1}{2}M^2\right)=\frac{d}{dr}\ln\mathcal{B} \qquad (21.23)$$

As in 21a. and 21b. we consider the case $\mathcal{B}$ = const. Eq. (21.23) shows that in this case M is a non-increasing function of r. Further we can now write eq. (21.33) in the form

$$\frac{d}{dr}\ln\frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}+\frac{2\sin^2\beta}{r}=0 \qquad (21.24)$$

from which we find that, like the quantity Υq^2 (when $\mathcal{B}$ = const.), M is bounded below by the M of the plane vortex flow.

d. The Vorticity Variation. Another vector important for fluid flows in the vorticity, Ω^i. See Section 7a., Chapter III. This vector is the curl of the velocity,

$$\Omega^i = \epsilon^{ijk} \nabla_j v_k$$

In order to examine this vector it will perhaps be most instructive to decompose it according to the intrinsic formula (5 and 6)

$$\Omega^i = q\left(b^k n^j \nabla_j t_k - n^k b^j \nabla_j t_k\right) t^i + \left(b^j \nabla_j q\right) n^i + \left(q\kappa - n^j \nabla_j q\right) b^i \tag{21.25}$$

Now we will evaluate the coefficients of t^i, n^i and b^i in terms of the quantities q and β. To do this it will be convenient to introduce a specific coordinate system into our space.

Note that all that we have required of our coordinate system so far for our study of helical flows is that $X^3 = \lambda$, $X^2 =$ const. are surfaces through the helices, and no special conditions on the surfaces X^1 = const. We choose now the family of planes θ = const. through the axis of the coaxial cylinders for $\{X^1 = \text{const.}\}$. If $\{\psi = \text{const.}\}$ is any family of stream surfaces as described in Section 19a., Chapter VII, then we can select $\{J(\psi, \omega) = \text{const.}\}$ as our family $\{X^2 = \text{const.}\}$. Then with u^2 from eq. (19.2), eqs. (19.1) become

$$x = X^3 \cos X^1 = r\cos\theta$$

$$y = X^3 \sin X^1 = r\sin\theta$$

$$z = I X^1 X^3 + X^2 = I r\theta + X^2$$

where $I = \cot\beta$.

The metric coefficients g_{ij} of our X^1, X^2, X^3 coordinate system are

$$g_{ij} = \begin{pmatrix} (1+I^2)r^2 & Ir & \theta I r \frac{\partial}{\partial r}(Ir) \\ Ir & 1 & \theta \frac{\partial}{\partial r}(Ir) \\ \theta I r \frac{\partial}{\partial r}(Ir) & \theta\frac{\partial}{\partial r}(Ir) & 1 + \left[\theta \frac{\partial}{\partial r}(Ir)\right]^2 \end{pmatrix}$$

and

$$|g_{ij}| = r^2$$

Further, in this coordinate system we have (see Section 17, Chapter VII)

$$
\begin{aligned}
t_k &= \frac{g_{k1}}{\sqrt{g_{11}}} \\
n^j &= -g^{j3} \\
b^j &= \frac{1}{\sqrt{g_{11}g}}\left(-g_{12}, g_{11}, 0\right)
\end{aligned}
\tag{21.26}
$$

and from eq. (20.2)

$$K = \frac{\sin^2\beta}{r} \tag{21.27}$$

Now, substituting our values of g_{ij} and g^{ij} into eqs. (21.26) and then substituting the resulting expressions for t_k, n^j, and b^j, and K from eq. (25.27) into the coefficients of t^i, n^i, and b^i in eq. (21.25) we get

$$t_i \Omega^i = q\left(b^k n^j \nabla_j t_k - n^k b^j \nabla_j t_k\right)$$

$$= q\left[\frac{d\beta}{dr} + \frac{\sin\beta\cos\beta}{r}\right] \tag{21.28}$$

$$n_i \Omega^i = b^j \nabla_j q$$

$$= \frac{1}{\sin\beta}\ \frac{\partial q}{\partial x^2} \tag{21.29}$$

$$b_i \Omega^i = qK - n^j \nabla_j q \tag{21.30}$$

$$= q\,\frac{\sin^2\beta}{r} - \theta\left(\frac{d}{dr} r\cot\beta\right)\frac{\partial q}{\partial x^2} + \frac{\partial q}{\partial r}$$

It should be noticed, first of all, that $t_i \Omega^i$ is a function of only $x^3 = r$, $n_i \Omega^i$ may be a function of x^2 and x^3, and $b_i \Omega^i$ may vary with all three coordinates. $b_i \Omega^i$ is the first quantity appearing in our analysis of helical fluid flows which can vary along stream lines, and this can occur only when q varies from one stream line to another on $x^3 = r$ = const. surfaces.

Further, we should not expect to be able to assign a priori all three vorticity components, for then q and β would have to satisfy three differential equations.

Now let us consider the case $t_i \Omega^i = 0$. This is the case in which there exists a family of surfaces orthogonal to the stream lines (see Section 8, Chapter IV). From eq. (21.28) we get, by integration,

$$\frac{\cot\beta}{r} = \lambda_1 \tag{21.31}$$

where λ_1 is a constant.

On the other hand we can consider the case $n_i \Omega^i = b_i \Omega^i = 0$. This is the case in which the vorticity is parallel to the velocity, that is, these are Beltrami flows (see Section 7a., Chapter III). From eqs. (21.29) and (21.30) we get

$$\frac{\partial q}{\partial x^2} = 0 \tag{21.32}$$

$$\frac{dq}{dr} + q\,\frac{\sin^2\beta}{r} = 0 \tag{21.33}$$

Assuming both $t_i \omega^i = 0$ and $n_i \omega^i = b_i \omega^i = 0$ gives us potential flows. In this case eqs. (21.31), (21.32), and (21.33) are all valid. Applying (21.31) to (21.33) yields

$$q^2 = \left(\frac{1}{r^2} + \lambda_1^2\right)\lambda_2^2 \qquad (21.34)$$

where λ_2 is a constant. Introducing the orthogonal projections of v^i on the z axis and in the x, y plane, we find

$$v_{(z)} = q \cos\beta$$

$$v_{(\rho)} = q \sin\beta$$

and putting in β and q from eqs. (21.31) and (21.34) we get

$$v_{(z)} = \pm \lambda_1 \lambda_2$$

$$v_{(\rho)} = \pm \frac{\lambda_2}{r}$$

We have seen that the condition $t_i \omega^i = 0$ alone completely determines a stream line pattern - that of the potential flows. This is in accordance with our discussion in Section 18, Chapter VII where we found that corresponding to a $\nabla_i t^i = 0$ flow with $t_i \Omega^i = 0$ there is a potential flow with the same stream lines. This of course does not imply that all flows with this stream line pattern are potential flows. Non-potential flows with these stream lines can easily be constructed by means of eqs. (21.29) and (21.30).

The conditions $n_i \Omega^i = b_i \Omega^i = 0$ alone, i.e., Beltrami flow, give us eq. (21.33). If in addition Υ is constant, then eq. (21.33) is the same as eq. (21.17) which we saw was equivalent to the condition that $\mathcal{B}$ = const. On the other hand, if $\mathcal{B}$ = const., and Υ = const., then eq. (21.17) reduces to eq. (21.33) so that we have a Beltrami flow. These results are in accordance with what we noted from eq. (7.3).

Other conditions we can impose on the vorticity include the condition that one and only one of the components $n_i \Omega^i$ or $b_i \Omega^i$ vanish. As a final observation we note that the case $t_i \Omega^i = b_i \Omega^i = 0$ and $n_i \Omega^i \neq 0$ cannot occur. That is, the vorticity vector cannot lie in the radial direction. For eq. (21.30) with $n_i \Omega^i \neq 0$ and $b_i \Omega^i = 0$ implies that $r \cot \beta$ = const. But this is incompatible with eq. (21.31) which as we have seen comes from the condition $t_i \Omega^i = 0$.

22. Helical Flows of Fluids with the Chaplygin-Karman-Tsien Equation of State. At the beginning of Chapter VII we discussed a special case of $\nabla_i t^i = 0$ flows which can occur when the equation of state

$$\rho = \frac{\Upsilon(S)}{\sigma_1 - \sigma_2 p} \tag{22.1}$$

is valid. σ_1 and σ_2 are arbitrary constants. This case was subsequently excluded in our considerations in Chapter VII and thus far in this chapter. Now we shall investigate this case a little further.

Since now we do not necessarily have $t^i \nabla_i p = 0$ eqs. (6.12) and (6.13) do not simplify as they did in Section 17. (In Section 17 we referred to eqs. (5.7) and (5.8) instead of eqs. (6.12) and (6.13)). Thus instead of (17.5) and (17.6) we now have

$$-\frac{g_{12}}{\sqrt{g_{11} g^{33} g}} \frac{\partial P}{\partial x^1} + \sqrt{\frac{g_{11}}{g^{33} g}} \frac{\partial P}{\partial x^2} = -2(B - P)' K \tag{22.2}$$

$$\frac{g^{13}}{\sqrt{g^{33}}} \frac{\partial P}{\partial x^1} + \frac{g^{23}}{\sqrt{g^{33}}} \frac{\partial P}{\partial x^2} + \sqrt{g^{33}} \frac{\partial P}{\partial x^3} = -2(B - P) K_N \tag{22.3}$$

Again we can choose the surfaces $x^3 = \omega$ = const. to be those on which the stream lines are geodesics so that ${}'K = 0$. Further, we can choose the surfaces x^1 = const. to be surfaces through the orthogonal trajectories on $x^3 = \omega$ = const. of the stream lines, so that $g_{12} = 0$. Then from (22.2) we get

$$\frac{\partial P}{\partial x^2} = 0 \tag{22.4}$$

and (22.3) reduces to

$$\frac{g^{13}}{\sqrt{g^{33}}} \frac{\partial P}{\partial x^1} + \sqrt{g^{33}} \frac{\partial P}{\partial x^3} = -2(B - P) K_N \tag{22.5}$$

Now let us restrict ourselves to helical flows. We calculate the metric coefficients appearing in eq. (22.5) from the coordinate transformation

$$x^1 = z + \theta r \tan\beta$$

$$x^2 = z - \theta r \cot\beta$$

$$x^3 = r$$

and obtain (when we replace X^3 by r)

$$g^{13} = \theta \frac{d}{dr} r \tan\beta = (X^1 - X^2) \frac{\sin\beta\cos\beta}{r} \frac{d}{dr} r \tan\beta$$

$$g^{33} = 1$$

Eq. (20.2) for the normal curvature is valid. Using this, and the expressions above for g^{13} and g^{33} in eq. (22.5) we obtain

$$(X^1 - X^2)\sin\beta\cos\beta\left(\frac{d}{dr} r\tan\beta\right)\frac{\partial P}{\partial X^1} + r\frac{\partial P}{\partial r} = 2(B - P)\sin^2\beta \tag{22.6}$$

Note that as before $B = B(X^2, r)$.

Since $\frac{\partial P}{\partial X^2} = 0$, the left-hand side of eq. (22.6) is a linear function of X^2 so that $B = B_1(r)X^2 + B_2(r)$ and eq. (22.6) splits into

$$-\sin\beta\cos\beta\left(\frac{d}{dr} r\tan\beta\right)\frac{\partial P}{\partial X^1} = 2B_1\sin^2\beta \tag{22.7}$$

$$X^1\sin\beta\cos\beta\left(\frac{d}{dr} r\tan\beta\right)\frac{\partial P}{\partial X^1} + r\frac{\partial P}{\partial r} = 2(B_2 - P)\sin^2\beta \tag{22.8}$$

Eqs. (22.7) and (22.8) can be satisfied in several ways. For example, if we put

$$\frac{d}{dr} r \tan\beta = \mathcal{B}_1 = 0$$

$$\mathcal{B} = \mathcal{B}_2 = \text{const.}$$

then we get

$$\mathcal{B} - \mathcal{P} = \frac{\Delta^2 + r^2}{r^2}\,\Theta(\chi')$$

where $\Delta = r\tan\beta$ and Θ is an arbitrary positive function of χ'.

Further, since $\mathcal{B} - \mathcal{P} = \frac{q^2}{2}\Upsilon$

$$q = \frac{\sqrt{\Delta^2 + r^2}}{r}\,\frac{\Theta_1}{\sqrt{\Upsilon}} \qquad (22.9)$$

where Θ_1 is an arbitrary positive function of χ'. Finally, since for the flows of this section

$$q = \sqrt{\frac{\partial p}{\partial \rho}}$$

we get from (22.1)

$$q = \frac{\sigma_1 - \sigma_2 p}{\sqrt{\sigma_2 \Upsilon}} \qquad (22.10)$$

and thus, eliminating q between (22.9) and (22.10),

$$p = \frac{\sigma_1}{\sigma_2} - \frac{\sqrt{\Delta^2 + r^2}}{r} \frac{\Theta_1}{\sqrt{\sigma_2}}$$

23. Examples. To get an idea of the size of the class of helical flows, their geometrical and thermodynamic variety, we present the following examples.

a. Flows with Given P and $B-P$. We will examine a little further the examples mentioned in 21a. We can think of $B-P$ as essentially describing the velocity magnitude in the incompressible case, for in this case $q = \sqrt{2(B-P)}$. In the compressible case we have the relation $q\Upsilon = \sqrt{2(\mathcal{B}-\mathcal{P}}$.

In example (i) we had

$$P = 1 - e^{-r} \tag{23.1}$$

$$B - P = r e^{-r} \tag{23.2}$$

and from eq. (20.3)

$$\beta = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ$$

Further, adding eqs. (23.1) and (23.2) we get

$$B = 1 + (r-1)e^{-r}$$

and for a polytropic gas, from eqs. (21.22),(23.1), and (23.2) we have

$$M^2 = \frac{2}{\gamma-1} \; \frac{r}{e^r - 1}$$

The behavior of P, $B-P$ (and thus, qualitatively, of q in the incompressible case and of $q\sqrt{Y}$ in the compressible case), β B, and M is shown in Figure 7 for $\gamma = 1.4$.

In example (ii) we had

$$P = r \tag{23.3}$$

$$B - P = \frac{r}{2} e^r \tag{23.4}$$

and from eq. (20.3)

$$\sin\beta = e^{-\frac{r}{2}}$$

Further, adding eqs. (23.3) and (23.4) we get

$$B = r + \frac{r}{2} e^r$$

and for a polytropic gas, from eqs. (21.22), (23.3), and (23.4) we

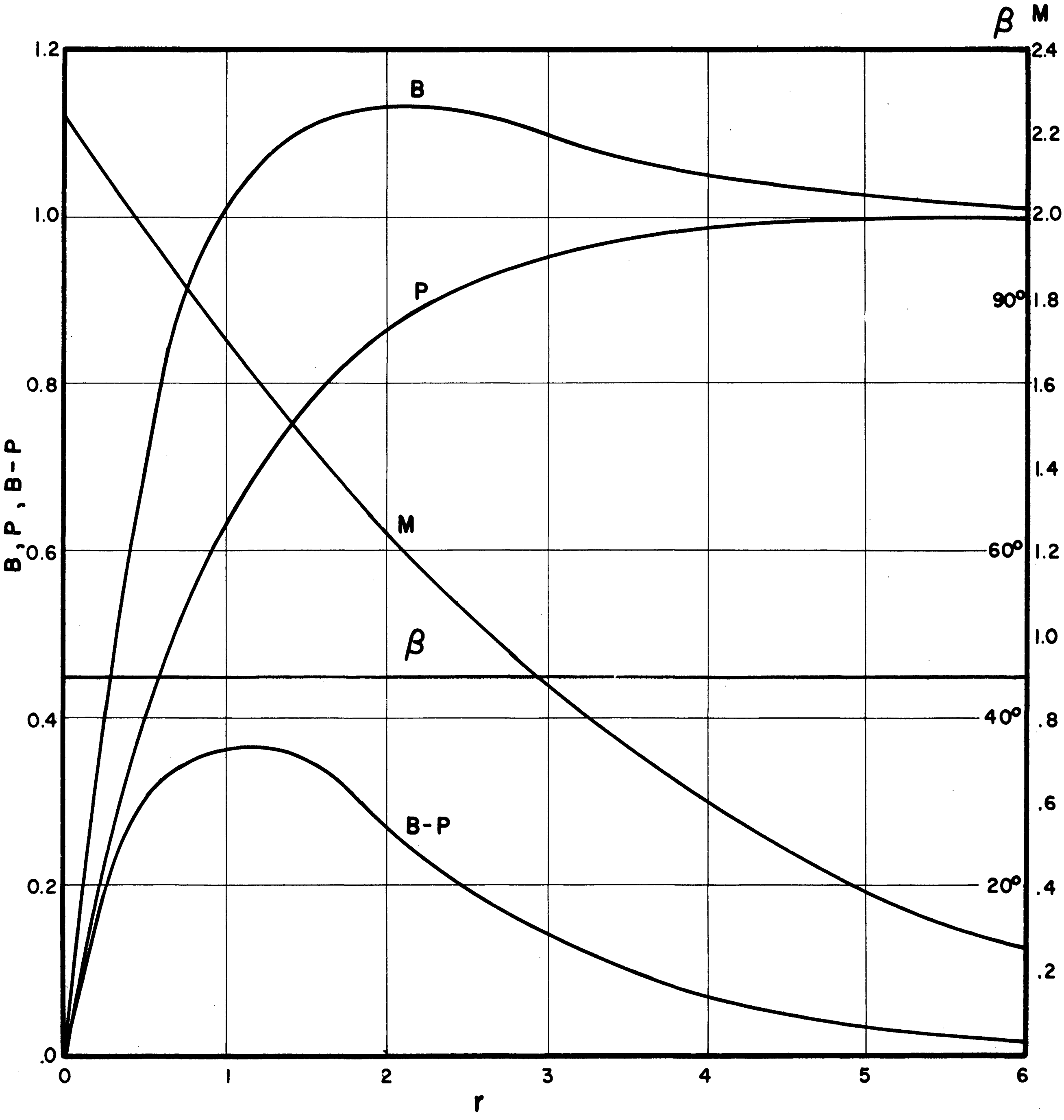

Figure 7. Example a. (Given $P = 1 - e^{-r}$, $B - P = re^{-r}$)

have

$$M^2 = \frac{1}{\gamma-1} e^r$$

In this case, each of the functions P, $B-P$, B, and M increase without bound monotonically with r while β decreases from $\pi/2$ and approaches 0 as r approaches infinity.

b. Flows with Given P and β. Let

$$P = r^n \qquad n > 0 \tag{23.5}$$

$$\tan\beta = r^m$$

Then $\frac{dP}{dr} = n r^{n-1}$ and $\sin^2\beta = \frac{r^{2m}}{1+r^{2m}}$, and therefore eq. (20.3) gives

$$B - P = \frac{n}{2} r^n \frac{1+r^{2m}}{r^{2m}} \tag{23.6}$$

Adding eqs. (23.5) and (23.6) we get

$$B = r^n\left[1 + \frac{n}{2}\frac{1+r^{2m}}{r^{2m}}\right]$$

Finally, for a polytropic gas, from eqs. (21.22), (23.5), and (23.6)

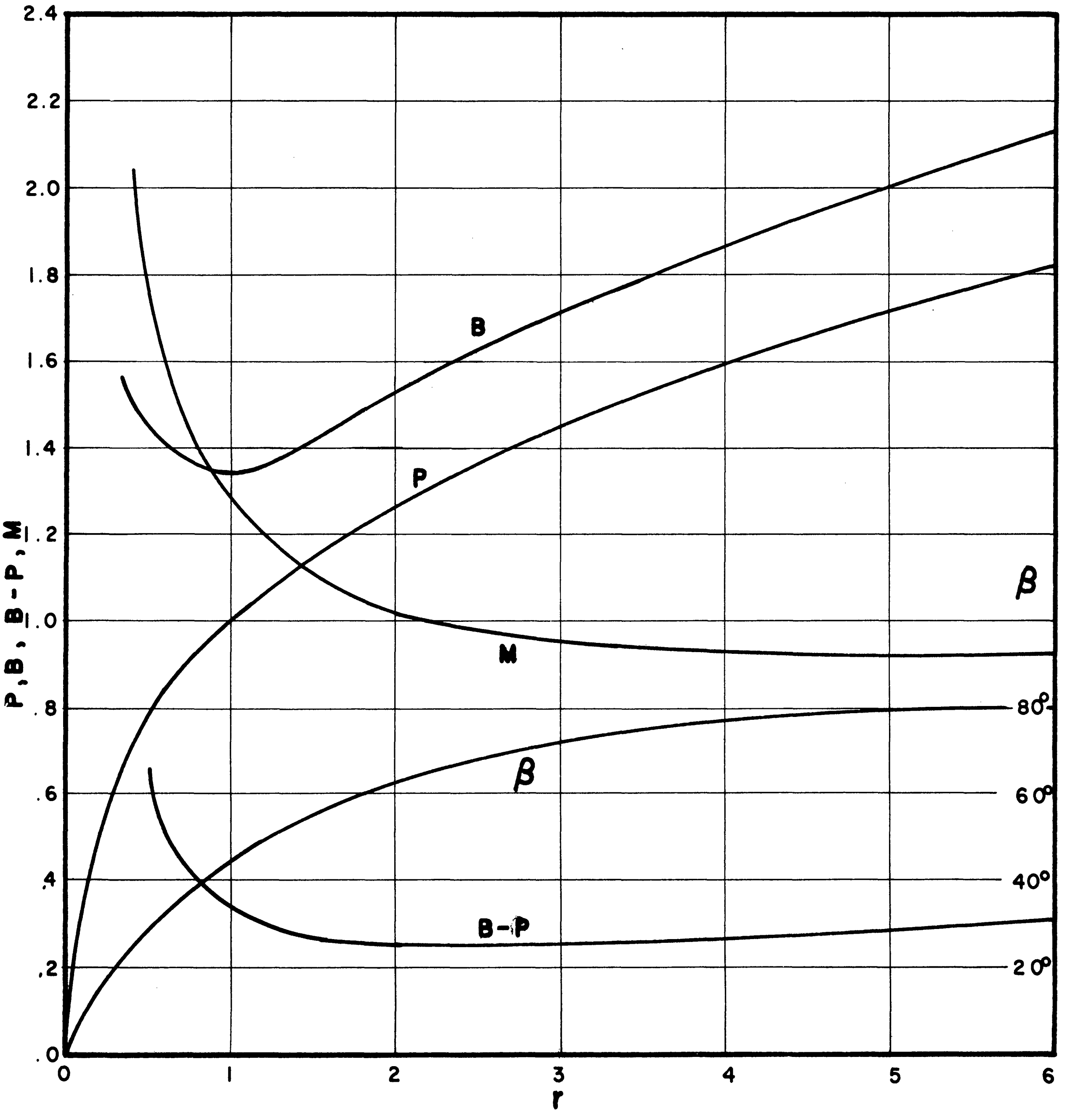

Figure 8. Example b. (Given $P = r^{1/3}$, $\tan\beta = r$)

we have

$$M^2 = \frac{n}{\gamma - 1}\left(\frac{1}{r^{2m}} + 1\right)$$

For these flows we can make the general statement that if β increases with r $(m>0)$ then M decreases with r, and if β decreases with r $(m<0)$ then M increases with r. An example of these flows in which $n = \frac{1}{3}$, $m = 1$ and $\gamma = 1.4$ is illustrated in Figure 8.

c. Flows with Given P and M. Let

$$P = r^n + l_1 \tag{23.7}$$

$$M^2 = \frac{2}{\gamma - 1}\,\frac{r}{e^{r - l_2} + l_3}$$

Then from eq. (21.22) (using the notation P and B in eq. (21.22) instead of $\mathcal{P}$ and $\mathcal{B}$)

$$B - P = \frac{r(r^n + l_1)}{e^{r - l_2} + l_3} \tag{23.8}$$

Substituting $\frac{dP}{dr} = n r^{n-1}$ and eq. (23.8) into eq. (20.3) we get

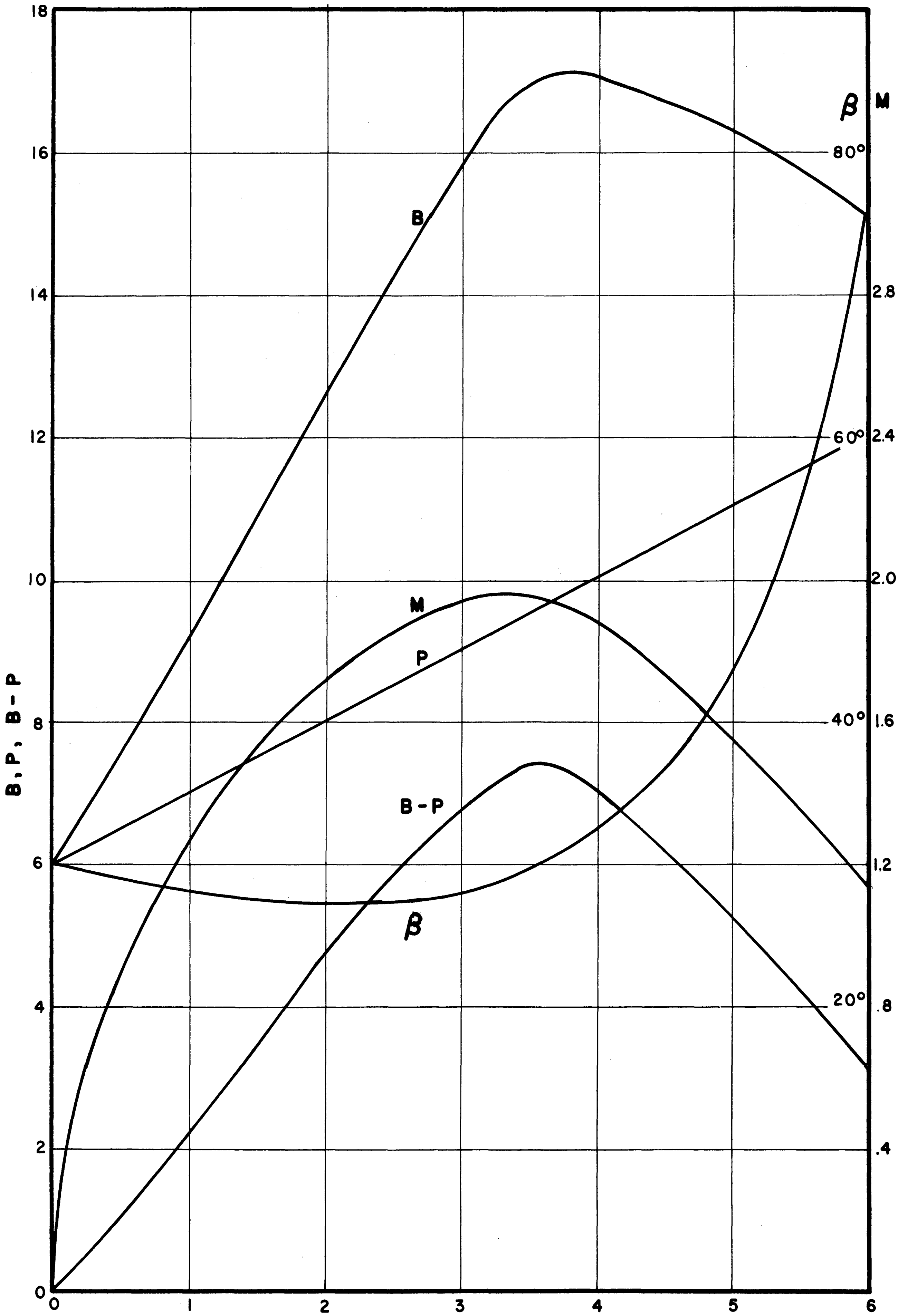

Figure 9. Example c. (Given $P = r + 6$, $M^2 = \frac{2}{\gamma - 1} \frac{r}{e^{r-3} + 3}$)

$$\sin^2\beta = \frac{\frac{n}{2} r^{n-1}(e^{r-l_2} + l_3)}{r^n + l_1}$$

Finally adding eqs. (23.7) and (23.8) we get

$$B = \frac{(r^n + l_1)(r + e^{r-l_2} + l_3)}{e^{r-l_2} + l_3}$$

Note that for these flows the range of the solutions is restricted to r less than a certain value determined by the condition that $\sin^2\beta \leqslant 1$. An example of these flows in which $n = 1$, $l_1 = 6$, $l_2 = l_3 = 3$, and $\gamma = 1.4$ is shown in Figure 9.

d. <u>Flows with Given β and $B-P$</u>. Let

$$\sin\beta = \frac{r}{e^{r-1}}$$

$$B - P = e^{r-1} \tag{23.9}$$

Substituting these expressions into eq. (20.3) we get

$$P = \bar{P} - \frac{2(r+1)}{e^{r-1}} \tag{23.10}$$

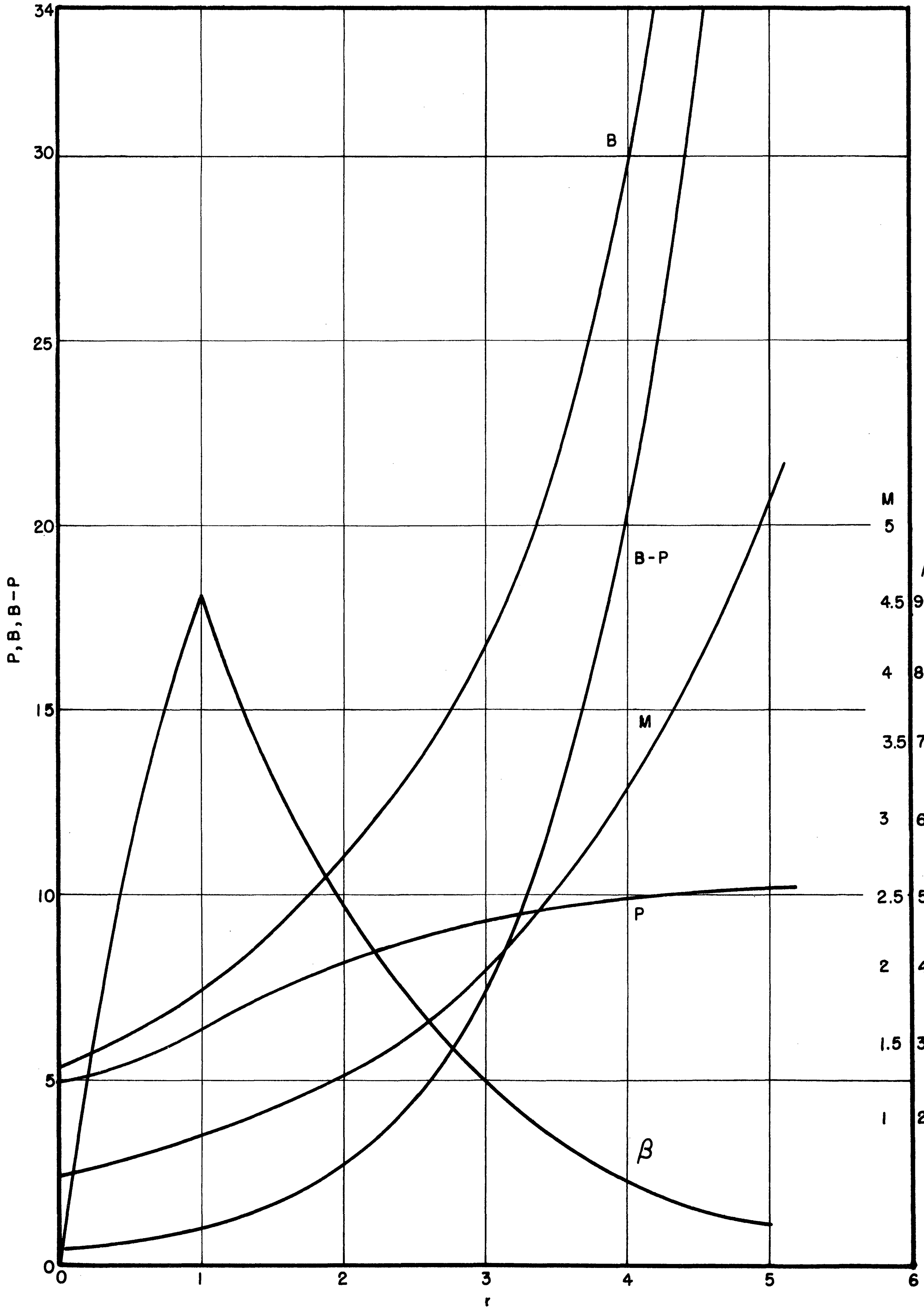

Figure 10. Example d. (Given $\sin\beta = \frac{r}{e^{r-1}}$, $B-P = e^{r-1}$)

where $\bar{P}$ is a constant of integration. Adding eqs. (23.9) and (23.10) we get

$$B = \bar{P} + e^{\lambda-1} - \frac{2(\lambda+1)}{e^{\lambda-1}}$$

and substituting (23.9) and (23.10) into (21.22) we obtain, for a polytropic gas

$$M^2 = \frac{2}{\gamma-1} \frac{e^{\lambda-1}}{\bar{P} - \frac{2(\lambda+1)}{e^{\lambda-1}}}$$

The flow with $\bar{P} = 2e + 5$ and $\gamma = 1.4$ is illustrated in Figure 10.

e. <u>Flows with Given M and β</u>. Let

$$M = l_1 \lambda^n$$

$$\sin\beta = l_2 \lambda^m \tag{23.11}$$

If in eq. (21.22) we use the notation P and B instead of $\mathcal{P}$ and $\mathcal{B}$ then we have

$$B - P = \frac{\gamma-1}{2} l_1^2 \lambda^{2n} P \tag{23.12}$$

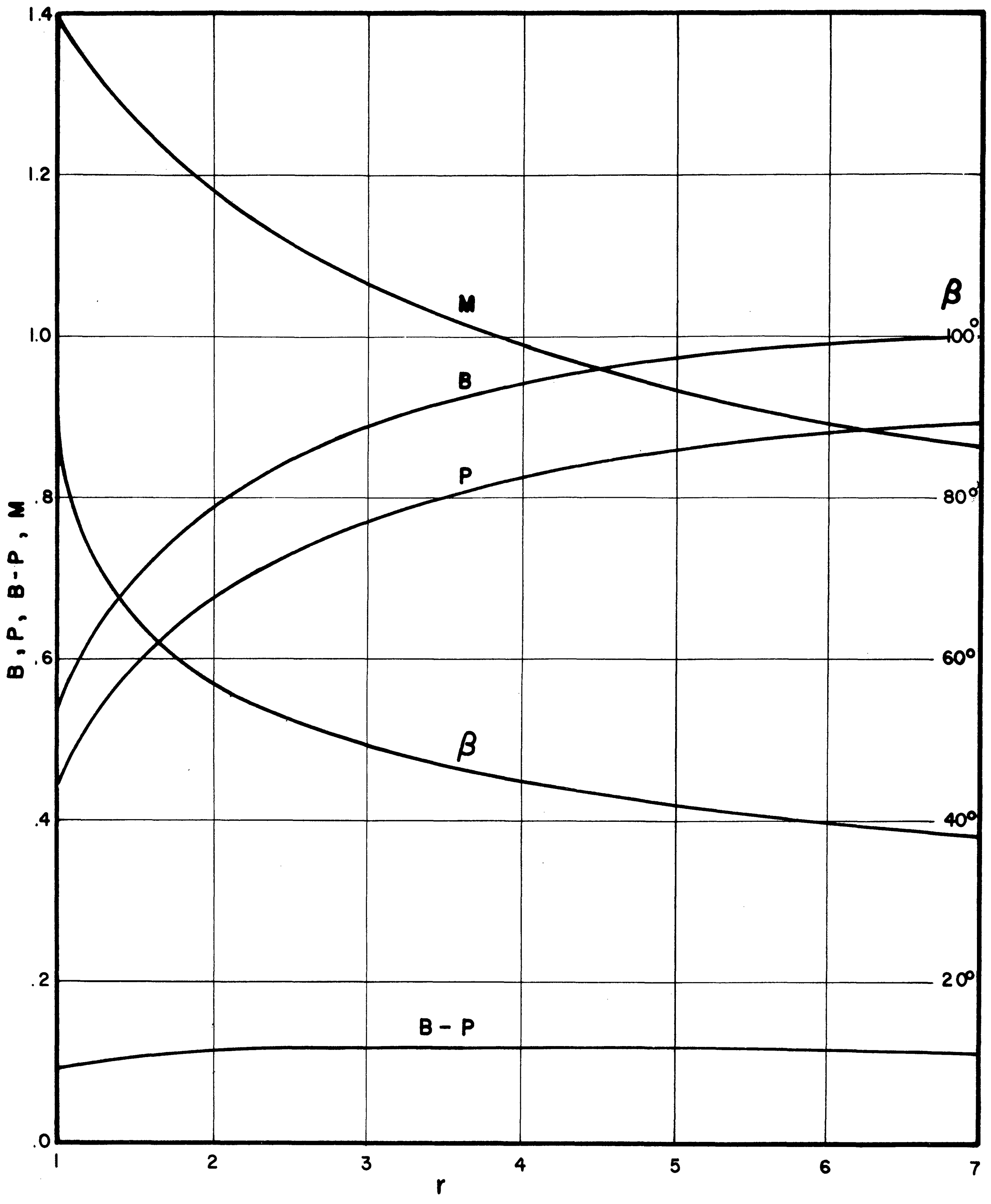

Figure 11. Example e. (Given $M = 1.4 r^{-\frac{1}{4}}$, $\sin\beta = r^{-\frac{1}{4}}$)

Substituting eqs. (23.11) and (23.12) into eq. (20.3) we obtain

$$P = \bar{P}\, e^{\frac{\gamma-1}{2(m+n)} l_1^2 l_2^2 r^{2(m+n)}} \tag{23.13}$$

where $\bar{P}$ is a constant of integration. Substituting eq. (23.13) into eq. (23.12) we get

$$B = \bar{P}\left(1 + \frac{\gamma-1}{2} r^n\right) e^{\frac{\gamma-1}{2(m+n)} l_1^2 l_2^2 r^{2(m+n)}}$$

and finally

$$B - P = \frac{\gamma-1}{2} \bar{P} r^n e^{\frac{\gamma-1}{2(m+n)} l_1^2 l_2^2 r^{2(m+n)}}$$

Note that by eq. (23.11) for these flows solutions are restricted to the range $0 \leq r \leq \left(\frac{1}{l_2}\right)^{\frac{1}{m}}$ when $m > 0$, and $\left(\frac{1}{l_2}\right)^{\frac{1}{m}} \leq r < \infty$ when $m < 0$. An example for which $m = n = -\frac{1}{4}$, $l_1 = 1.4$, $l_2 = 1$, $\bar{P} = 1$, and $\gamma = 1.4$ is shown in Figure 11.

APPENDIX I

PARALLEL CURVES ON A SURFACE AND PARALLEL SURFACES

Parallel curves on surfaces and parallel surfaces have appeared several times in our discussion. Therefore, for convenience, we append a short description of these concepts.

1. Parallel Curves on a Surface. Two curves on a surface are said to be parallel if there exists a family of geodesics (on the surface) orthogonal to both curves. Parallel curves on a surface may also be defined by the equivalent property: If we are given a curve C_1 and the family of geodesics orthogonal to C_1, then a curve C_2 is parallel to C_1 if C_2 is such that the distance measured along the geodesics between C_1 and C_2 is the same along each geodesic.

The equivalence of these two properties comes out of a result concerning geodesics on surfaces. If u^1, u^2 is an orthogonal coordinate net on the surface, then the geodesic curvature of the coordinate curves u^2 = const. is given by (23, p. 187)

$$'K = -\frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^2} \ln \sqrt{g_{11}} \tag{I.1}$$

Thus, the coordinate curves u^2 = const. are geodesics if and only if $\sqrt{g_{11}}$ is not a function of u^2. (Similarly, the coordinate curves u^1 = const. are geodesics if and only if $\sqrt{'g_{22}}$ is not a function of u^1.)

Now if C_1 and C_2 are orthogonal to a family of geodesics and this family is taken to be $\{u^2 = \text{const.}\}$ then C_1 and C_2 are the curves $u^1 = u^1_1$ = const. and $u^1 = u^1_2$ = const. The element of length is

$$(ds)^2 = {'g_{11}}(du^1)^2 + {'g_{22}}(du^2)^2$$

But since $'g_{11}$ is not a function of u^2, the distance between C_1 and C_2 along geodesics (u^2 = const.) is independent of u^2, that is, the distance is the same along each geodesic.

Conversely, given a curve C_1 and a family of geodesics orthogonal to C_1, if C_2 is not orthogonal to the geodesics it must cross the orthogonal trajectories of the geodesics. Since for the latter, the distance measured along geodesics to C_1 is the same along each geodesic, this cannot be true of C_2.

Another result which we use (Section 3 in Chapter II, and Section 17 in Chapter VII) is that $\{\psi(u^1, u^2) = \text{const.}\}$ is a family of parallel curves if and only if $|'\nabla\psi|$ is a function of only ψ. Let the curves $\psi = \text{const.}$ be chosen as the coordinate curves u^2 = const. Then $|'\nabla\psi| = \sqrt{'g^{22}}$ (eq.(3.11)). If $\{\psi = \text{const.}\}$ are parallel curves then $'g_{22}$ is a function of only $u^2 = \psi$, (see parenthetical remark below eq. (I.1)) and therefore $'g^{22}$ is only a function of $u^2 = \psi$. Conversely, suppose $|'\nabla\psi| = \Psi(\psi)$. Then, changing variables according to $\psi^* = \int \frac{d\psi}{\sqrt{\Psi(\psi)}}$ we get $|\nabla \psi^*| = 1$. Taking the $\psi^* =$ const. curves as coordinate curves $u^2 =$ const. we get $'g_{22} = 1$ (23, p. 157). Thus $\{\psi^* = \text{const.}\}$ are parallel curves, and hence so are $\{\psi = \text{const.}\}$.

2. Parallel Surfaces. Two surfaces in E^3 are said to be parallel if there exists a family of straight lines orthogonal to both surfaces. Parallel surfaces may also be defined by the equivalent property: If we are given a surface S_1 and the family of straight lines orthogonal to S_1, then a surface S_2 is parallel to S_1 if S_2 is such that the distance measured along the straight lines between S_1 and S_2 is the same along each straight line.

To show the equivalence of these two properties we introduce the rectangular cartesian coordinates y^i, $i = 1, 2, 3$. Let

$$S_1 : \quad y^i = \bar{y}^i(u^\xi)$$
$$S_2 : \quad y^i = \bar{y}^i(u^\xi) + \alpha(u^\xi) N^i(u^\xi) \qquad (I.2)$$

where $u^\xi = (u^2, u^3)$ are the surface coordinates of S_1, N^i is the unit normal vector to S_1, and thus $\alpha(u^\xi)$ is the distance between S_1 and S_2. Then

$$\sum_i N^i \frac{\partial y^i}{\partial u^\xi} = \sum_i N^i \left(\frac{\partial \bar{y}^i}{\partial u^\xi} + N^i \frac{\partial \alpha}{\partial u^\xi} + \alpha \frac{\partial N^i}{\partial u^\xi} \right) \qquad (I.3)$$

But N^i is the unit normal vector to S_1 implies that

$$\sum_i N^i \frac{\partial \bar{y}^i}{\partial u^\xi} = \sum_i N^i \frac{\partial N^i}{\partial u^\xi} = 0$$

so that eq. (I.3) becomes

$$\sum_i N^i \frac{\partial y^i}{\partial u^\xi} = \frac{\partial \alpha}{\partial u^\xi} \tag{I.4}$$

Eq. (I.4) says that S_2 is orthogonal to the normals of S_1 if and only if the distance, α , between S_1 and S_2 is constant.

At the beginning of Section 13 (Chapter VI) we use the result that any family of parallel surfaces can be imbedded in a triply orthogonal system of surfaces. To show this we first note that by (I.2) (with α = const.), the first fundamental tensor $'g_{\xi\eta}$ of S_2 is

$$'g_{\xi\eta} = \sum_i \frac{\partial y^i}{\partial u^\xi} \frac{\partial y^i}{\partial u^\eta} = \sum_i \left(\frac{\partial \bar{y}^i}{\partial u^\xi} + \alpha \frac{\partial N^i}{\partial u^\xi}\right)\left(\frac{\partial \bar{y}^i}{\partial u^\eta} + \alpha \frac{\partial N^i}{\partial u^\eta}\right) \tag{I.5}$$

Now we choose the coordinate curves u^ξ = const. on S_1 to be the lines of curvature of S_1 . Then (23, p. 230)

$$K_\xi \frac{\partial \bar{y}^i}{\partial u^\xi} + \frac{\partial N^i}{\partial u^\xi} = 0 \qquad \text{(Rodriguez' formula)} \tag{I.6}$$

where K_ξ $(\xi = 1, 2)$ are the principal curvatures on S_1.
Substituting (I.6) into (I.5) we get

$$'g_{\xi\eta} = (\alpha K_\xi - 1)(\alpha K_\eta - 1) \sum_i \frac{\partial \bar{y}^i}{\partial u^\xi} \frac{\partial \bar{y}^i}{\partial u^\eta}$$

So $'g_{\xi\eta}$ is proportional to the first fundamental tensor, $'\bar{g}_{\xi\eta}$, of S_1. Similarly, the second fundamental tensor, $'h_{\xi\eta}$, of S_2 is proportional to the second fundamental tensor $'\bar{h}_{\xi\eta}$ of S_1 (9, p. 159). Since u^ξ = const. are the lines of curvature of S_1 we have $'\bar{g}_{23} = '\bar{h}_{23} = 0$ (23, p. 230). Hence $'g_{23} = 'h_{23} = 0$ and u^ξ = const. are the lines of curvature of S_2 (23, p. 230). That is, the straight lines through the lines of curvature of one parallel surface will intersect the other parallel surface in lines of curvature.

The straight lines through a line of curvature of one of the parallel surfaces will form a surface. There are two families of such surfaces and clearly by the result above they are mutually orthogonal, and both families are orthogonal to the parallel surfaces.

The surfaces through the lines of curvature may be parametrized by u^2 and u^3 and the parallel surfaces may be parametrized by α. Let α, u^2, u^3 be a coordinate system x^1, x^2, x^3 in E^3. The transformation between this coordinate system and the system y^i $(i = 1, 2, 3)$ is given by the second part of eq. (I.2), with α now an independent variable and so

$$g_{11} = \sum_k \frac{\partial y^k}{\partial \alpha} \frac{\partial y^k}{\partial \alpha} = 1$$

$$g_{22} = \sum_k \frac{\partial y^k}{\partial u^2} \frac{\partial y^k}{\partial u^2} = (\alpha \bar{K}_2 - 1)^2 \, '\bar{g}_{22} \tag{I.7}$$

$$g_{33} = \sum_k \frac{\partial y^k}{\partial u^3} \frac{\partial y^k}{\partial u^3} = (\alpha \bar{K}_3 - 1)^2 \, '\bar{g}_{33}$$

There are two other properties of parallel surfaces which we have used. In Section 3 (Chapter II) we state that if $\{\omega = const.\}$ is a family of parallel surfaces then $|\nabla\omega|$ is a function of ω alone. If we take $\{\omega = const.\}$ as a family $\{x' = const.\}$ of a triply orthogonal system of coordinate surfaces, then the formulas (10.3) are valid, and since $'K$ and K_N vanish for straight lines they yield the result that g_{11} is a function of only $x' = \omega$. Since $|\nabla\omega| = \sqrt{g^{11}} = \frac{1}{\sqrt{g_{11}}}$, we see that $|\nabla\omega|$ is a function of ω alone.

In Section 15 (Chapter VI) we mention the fact that if the principal curvatures are constant on one surface of a parallel family, then they are constant on each surface of the family. This follows from the relations

$$\underline{K}^*_{\xi} = \frac{'h_{\xi\xi}}{'g_{\xi\xi}} \tag{I.8}$$

which are valid for each of the parallel surfaces, and (9, 159)

$$\frac{{}'h_{\xi\xi}}{{}'g_{\xi\xi}} = \pm \frac{1}{\overline{K}_{\xi}\alpha - 1} \frac{{}'\bar{h}_{\xi\xi}}{{}'\bar{g}_{\xi\xi}} \qquad (I.9)$$

Eqs. (I.8) and (I.9) together give

$$\overline{K}_{\xi}^{*} = \pm \frac{\overline{K}_{\xi}}{\overline{K}_{\xi}\alpha - 1} \qquad (I.10)$$

Thus, if $\overline{K}_{\xi} = const.$, then on each surface $\{\alpha = const.\}$ of the parallel family $\overline{K}_{\xi}^{*} = const.$

APPENDIX II

CALCULATION OF THE INTEGRABILITY CONDITIONS OF SECTION 8

1. Integrability Conditions for Equations (6.10), (6.11) and (6.15). In Section 8 we noted that the intrinsic derivatives $t^i \nabla_i \mathcal{P}$, $n^i \nabla_i \mathcal{P}$ and $b^i \nabla_i \mathcal{P}$ of $\mathcal{P}$ have to satisfy

$$\underset{v}{\lambda}^i \nabla_i (\underset{w}{\lambda}^j \nabla_j \mathcal{P}) - \underset{w}{\lambda}^i \nabla_i (\underset{v}{\lambda}^j \nabla_j \mathcal{P})$$
$$= \sum_{u=1}^{3} (\underset{uvw}{\gamma} - \underset{uwv}{\gamma}) \underset{u}{\lambda}^i \nabla_i \mathcal{P} \tag{8.1}$$

where $\underset{u}{\lambda}^i = (t^i, n^i, b^i)$ and $\underset{uvw}{\gamma} = \underset{v}{\lambda}^i \underset{w}{\lambda}^j \nabla_j \underset{u}{\lambda}_i$. If eqs. (6.10), (6.11), and (6.15) are thought of as three partial differential equations for the function $\mathcal{P}$, then the application of (8.1) leads to their integrability conditions.

In order to apply eq. (8.1) to eqs. (6.10), (6.11), and (6.15) we introduce the notation

$$\underset{u}{K} = (\nabla_i t^i, K, 0)$$

$$\underset{u}{\Gamma} = (\Gamma(\mathcal{B}, \mathcal{P}), 1, 0)$$

where $\Gamma(\mathcal{B}, \mathcal{P}) = \dfrac{1}{2(\mathcal{B}-\mathcal{P})\frac{dx}{dp} - 1}$. Then eqs. (6.15), (6.10), and (6.11) may be written

$$\underset{u}{\lambda}^i \nabla_i \mathcal{P} = -2(\mathcal{B}-\mathcal{P}) \underset{u}{\Gamma} \underset{u}{K} \tag{II.1}$$
$$u = 1, 2, 3$$

Now we substitute (II.1) into (8.1) obtaining

$$\lambda_{v}^{i}\nabla_{i}\left[(\mathcal{B}-\mathcal{P})\Gamma_{w}\kappa_{w}\right]-\lambda_{w}^{i}\nabla_{i}\left[(\mathcal{B}-\mathcal{P})\Gamma_{v}\kappa_{v}\right]$$

$$=\sum_{u=1}^{3}(\mathcal{B}-\mathcal{P})\Gamma_{u}\left(\gamma_{uvw}-\gamma_{uwv}\right)\kappa_{u} \qquad \text{(II.2)}$$

To simplify (II.2) we expand the first term on the left-hand side, thinking of the Γ_{u} as functions of $\mathcal{B}$ and $\mathcal{P}$;

$$\lambda_{v}^{i}\nabla_{i}\left[(\mathcal{B}-\mathcal{P})\Gamma_{w}\kappa_{w}\right]=(\mathcal{B}-\mathcal{P})\Gamma_{w}\lambda_{v}^{i}\nabla_{i}\kappa_{w}+\Gamma_{w}\kappa_{w}\left[\lambda_{v}^{i}\nabla_{i}\mathcal{B}-\lambda_{v}^{i}\nabla_{i}\mathcal{P}\right]$$

$$+\kappa_{w}(\mathcal{B}-\mathcal{P})\left[\left(\frac{\partial}{\partial\mathcal{B}}\Gamma_{w}\right)\lambda_{v}^{i}\nabla_{i}\mathcal{B}+\left(\frac{\partial}{\partial\mathcal{P}}\Gamma_{w}\right)\lambda_{v}^{i}\nabla_{i}\mathcal{P}\right]$$

$$=(\mathcal{B}-\mathcal{P})\Gamma_{w}\lambda_{v}^{i}\nabla_{i}\kappa_{w}+\kappa_{w}\left[(\mathcal{B}-\mathcal{P})\left(\frac{\partial}{\partial\mathcal{P}}\Gamma_{w}\right)-\Gamma_{w}\right]\lambda_{v}^{i}\nabla_{i}\mathcal{P}$$

$$+\kappa_{w}\left[(\mathcal{B}-\mathcal{P})\left(\frac{\partial}{\partial\mathcal{B}}\Gamma_{w}\right)+\Gamma_{w}\right]\lambda_{v}^{i}\nabla_{i}\mathcal{B}$$

In the right-hand side the quantity $\lambda_{v}^{i}\nabla_{i}\mathcal{P}$ may be replaced by its value according to eq. (II.1), so that we get

$$\lambda_{v}^{i}\nabla_{i}\left[(\mathcal{B}-\mathcal{P})\Gamma_{w}\kappa_{w}\right]=(\mathcal{B}-\mathcal{P})\Gamma_{w}\lambda_{v}^{i}\nabla_{i}\kappa$$

$$-2(\mathcal{B}-\mathcal{P})\Gamma_{v}\left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{P}}\Gamma_{w}-\Gamma_{w}\right]\kappa_{v}\kappa_{w}$$

$$+\left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\Gamma_{w}+\Gamma_{w}\right]\kappa_{w}\lambda_{v}^{i}\nabla_{i}\mathcal{B} \qquad \text{(II.3)}$$

for the first term on the left-hand side of eq. (II.2').

Interchanging v and w in (II.3) we obtain a similar expansion for the second term on the left-hand side of eq. (II.2);

$$\begin{aligned}\underset{w}{\lambda}^{i}\nabla_{i}(\mathcal{B}-\mathcal{P})\underset{v}{\Gamma}\,\underset{v}{\kappa} &= (\mathcal{B}-\mathcal{P})\underset{v}{\Gamma}\,\underset{w}{\lambda}^{i}\nabla_{i}\underset{v}{\kappa} \\ &\quad - 2(\mathcal{B}-\mathcal{P})\underset{w}{\Gamma}\left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{P}}\underset{v}{\Gamma} - \underset{v}{\Gamma}\right]\underset{v}{\kappa}\,\underset{w}{\kappa} \\ &\quad + \left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\underset{v}{\Gamma} + \underset{v}{\Gamma}\right]\underset{v}{\kappa}\,\underset{w}{\lambda}^{i}\nabla_{i}\mathcal{B} \qquad \text{(II.4)}\end{aligned}$$

Substituting eqs. (II.3) and (II.4) into eq. (II.2) we get

$$\begin{aligned}&(\mathcal{B}-\mathcal{P})\underset{w}{\Gamma}\,\underset{v}{\lambda}^{i}\nabla_{i}\underset{w}{\kappa} - (\mathcal{B}-\mathcal{P})\underset{v}{\Gamma}\,\underset{w}{\lambda}^{i}\nabla_{i}\underset{v}{\kappa} \\ &- 2(\mathcal{B}-\mathcal{P})^{2}\left[\underset{v}{\Gamma}\frac{\partial}{\partial\mathcal{P}}\underset{w}{\Gamma} - \underset{w}{\Gamma}\frac{\partial}{\partial\mathcal{P}}\underset{v}{\Gamma}\right]\underset{v}{\kappa}\,\underset{w}{\kappa} + \left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\underset{w}{\Gamma} + \underset{w}{\Gamma}\right]\underset{w}{\kappa}\,\underset{v}{\lambda}^{i}\nabla_{i}\mathcal{B} \\ &- \left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\underset{v}{\Gamma} + \underset{v}{\Gamma}\right]\underset{v}{\kappa}\,\underset{w}{\lambda}^{i}\nabla_{i}\mathcal{B} = \sum_{u=1}^{3}(\mathcal{B}-\mathcal{P})\underset{u}{\Gamma}\left(\gamma_{uvw} - \gamma_{uwv}\right)\underset{u}{\kappa} \qquad \text{(11.5)}\end{aligned}$$

If we write (II.5) as three separate equations, and then put in (t^{i}, n^{i}, b^{i}) for $\underset{u}{\lambda}^{i}$, $(\nabla_{i}t^{i}, \kappa, 0)$ for $\underset{u}{\kappa}$, and $(\Gamma(\mathcal{B},\mathcal{P}), 1, 0)$ for $\underset{u}{\Gamma}$ we obtain the desired integrability conditions for eqs. (6.10), (6.11) and (6.15). Thus, using the facts that $\gamma_{uuv} = 0$, $\gamma_{121} = \kappa$, and $\gamma_{131} = 0$ we get from (II.5)

$$-(\mathcal{B}-\mathcal{P})\, b^i \nabla_i \kappa - \kappa\left(b^i \nabla_i \mathcal{B}\right) = \frac{(\mathcal{B}-\mathcal{P})\left(\underset{123}{\gamma} - \underset{132}{\gamma}\right)\nabla_i t^i}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp} - 1} - (\mathcal{B}-\mathcal{P})\underset{232}{\gamma}\,\kappa \tag{II.6}$$

$$\left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\left(\frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1}\right) + \frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1}\right](\nabla_j t^j)\, b^i \nabla_i \mathcal{B}$$

$$+ \frac{(\mathcal{B}-\mathcal{P})\, b^i \nabla_i (\nabla_j t^j)}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1} = (\mathcal{B}-\mathcal{P})\left(\underset{231}{\gamma} - \underset{213}{\gamma}\right)\kappa \tag{II.7}$$

$$(\mathcal{B}-\mathcal{P})\, t^i \nabla_i \kappa + \left[2(\mathcal{B}-\mathcal{P})^2 \frac{\partial}{\partial\mathcal{P}}\left(\frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1}\right)\right](\nabla_i t^i)\,\kappa$$

$$-\left[(\mathcal{B}-\mathcal{P})\frac{\partial}{\partial\mathcal{B}}\left(\frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1}\right) + \frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1}\right](\nabla_i t^i)\, n^i \nabla_i \mathcal{B} \tag{II.8}$$

$$-\frac{(\mathcal{B}-\mathcal{P})\, n^i \nabla_i (\nabla_j t^j)}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1} = -\frac{(\mathcal{B}-\mathcal{P})\,\kappa(\nabla_i t^i)}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1} + (\mathcal{B}-\mathcal{P})\underset{212}{\gamma}\,\kappa$$

For a polytropic gas, $\frac{d\Sigma}{dp} = \frac{1}{\gamma-1}\,\frac{1}{\mathcal{P}}$ (eq.(6.34)) and so $\frac{1}{2(\mathcal{B}-\mathcal{P})\frac{d\Sigma}{dp}-1} = \frac{\frac{\gamma-1}{2}\mathcal{P}}{\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}}$. Putting this expression into eqs. (II.6), (II.7), and (II.8) we get

$$-(\mathcal{B}-\mathcal{P})\left[b^i \nabla_i \kappa - \underset{232}{\gamma}\,\kappa\right] - \left(b^i \nabla_i \mathcal{B}\right)\kappa$$

$$= \frac{\frac{\gamma-1}{2}\mathcal{P}(\mathcal{B}-\mathcal{P})}{\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}}\left(\underset{123}{\gamma} - \underset{132}{\gamma}\right)\nabla_i t^i \tag{II.9}$$

$$\frac{\frac{\gamma-1}{2}\,\mathcal{P}(\mathcal{B}-\mathcal{P})}{\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}}\, b^i\nabla_i(\nabla_j t^j) - \frac{\left(\frac{\gamma-1}{2}\mathcal{P}\right)^2}{\left(\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}\right)^2}(b^i\nabla_i\mathcal{B})(\nabla_j t^j)$$

$$= (\mathcal{B}-\mathcal{P})\left(\gamma_{231} - \gamma_{213}\right)K \qquad \text{(II.10)}$$

$$(\mathcal{B}-\mathcal{P})\left[t^i\nabla_i K - \gamma_{212} K\right] - \frac{\frac{\gamma-1}{2}\,\mathcal{P}(\mathcal{B}-\mathcal{P})}{\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}}\, n^i\nabla_i(\nabla_j t^j)$$

$$+ \frac{2(\mathcal{B}-\mathcal{P})^2\frac{\gamma-1}{2}\mathcal{B}}{\left(\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}\right)^2}(\nabla_j t^j)K + \frac{\left(\frac{\gamma-1}{2}\mathcal{P}\right)^2}{\left(\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}\right)^2}(n^i\nabla_i\mathcal{B})(\nabla_i t^i)$$

$$= -\frac{\frac{\gamma-1}{2}\,\mathcal{P}(\mathcal{B}-\mathcal{P})}{\mathcal{B}-\frac{\gamma+1}{2}\mathcal{P}}(\nabla_i t^i)K \qquad \text{(II.11)}$$

Multiplying eq. (II.9) by $\mathcal{B} - \frac{\gamma+1}{2}\mathcal{P}$, and multiplying eqs. (II.10) and (II.11) by $\left(\mathcal{B} - \frac{\gamma+1}{2}\mathcal{P}\right)^2$, and slightly rearranging terms in these equations we obtain eqs. (8.2) - (8.4)

2. Integrability Conditions for Other Basic Sets of Flow Equations. Now we would like to generalize the previous discussion. Up to now we have been concerned exclusively with the flow equations of the form (6.9), (6.10), (6.11) and (6.15). However, if eqs. (6.10) and (6.11) are replaced by eqs. (6.12) and (6.13) then we have another

acceptable set of flow equations. Also, if $\mathcal{B}$ and $\mathcal{P}$ are replaced by B and P and if the factor $\Gamma = 2(\mathcal{B}-\mathcal{P})\frac{dX}{dp} - 1$ in (6.15) is replaced by 1, then we obtain the eqs. (5.4), (5.5), (5.6) and (5.10) for incompressible flow. In all, we have four different sets of equations.

Rather than carry out separate calculations for each of these cases, we note that, by generalizing the notation, eq. (II.5) will be valid in all cases, and we can proceed to get the integrability conditions directly from it. If (B^*, P^*) can stand for (B , P) as well as ($\mathcal{B}$, $\mathcal{P}$) and if $\underset{u}{\Gamma}$ can stand for (1 , 1 , 0) as well as (Γ, 1 , 0), then eq. (II.1) with (B^* , P^*) in place of ($\mathcal{B}$, $\mathcal{P}$) is valid for incompressible as well as compressible flows, and consequently, eq. (II.5) with (B^*, P^*) in place of ($\mathcal{B}$, $\mathcal{P}$) , that is

$$
\begin{aligned}
&(B^*-P^*)\underset{w}{\Gamma}\,\underset{v}{\lambda}^i\nabla_i\underset{w}{K} - (B^*-P^*)\underset{v}{\Gamma}\,\underset{w}{\lambda}^i\nabla_i\underset{v}{K} \\
&\quad -2(B^*-P^*)^2\left[\underset{v}{\Gamma}\frac{\partial}{\partial P^*}\underset{w}{\Gamma} - \underset{w}{\Gamma}\frac{\partial}{\partial P^*}\underset{v}{\Gamma}\right]\underset{v}{K}\,\underset{w}{K} \qquad \text{(II.12)} \\
&+\left[(B^*-P^*)\frac{\partial}{\partial B^*}\underset{w}{\Gamma} + \underset{w}{\Gamma}\right]\underset{w}{K}\,\underset{v}{\lambda}^i\nabla_i B^* - \left[(B^*-P^*)\frac{\partial}{\partial B^*}\underset{v}{\Gamma} + \underset{v}{\Gamma}\right]\underset{v}{K}\,\underset{w}{\lambda}^i\nabla_i B^* \\
&\quad = \sum_{u=1}^{3}(B^*-P^*)\underset{u}{\Gamma}\left(\gamma_{uvw} - \gamma_{uwv}\right)\underset{u}{K}
\end{aligned}
$$

is also valid for both flows. Further, if $\underset{u}{\lambda}^i$ can stand for (t^i, n^i, N^i) as well as (t^i, n^i, b^i) , if $\underset{u}{K}$ can stand for $(\nabla_i t^i, K, K_N)$ as well as $(\nabla_i t^i, K, 0)$ and if $\underset{u}{\Gamma}$ can be $(\Gamma, 1, 1)$ or $(1, 1, 1)$

as well as the previous possibilities, then eq. (II.1) is valid for both intrinsic formulations, and consequently, so is (II.12).

For all cases we have $\lambda_1^i \nabla_i B^* = 0$, $\gamma_{121} = \kappa_2$, $\gamma_{131} = \kappa_3$, and Γ_2, Γ_3 are constant. If we write (II.12) as three separate equations and use these properties and the fact that $\gamma_{uuv} = 0$ we get

$$\begin{aligned}
&\left[(B^*-P^*)\Gamma_3\right]\left[\lambda_2^i \nabla_i \kappa_3 - \gamma_{323}\kappa_3\right] - \left[(B^*-P^*)\Gamma_2\right]\left[\lambda_3^i \nabla_i \kappa_2 - \gamma_{232}\kappa_2\right] \\
&- \left[(B^*-P^*)\Gamma_1\right]\left[(\gamma_{123}-\gamma_{132})\kappa_1\right] + \left(\Gamma_3 \lambda_2^i \nabla_i B^*\right)\kappa_3 - \left(\Gamma_2 \lambda_3^i \nabla_i B^*\right)\kappa_2 = 0
\end{aligned} \qquad \text{(II.13)}$$

$$\begin{aligned}
&\left[(B^*-P^*)\Gamma_1\right]\left[\lambda_3^i \nabla_i \kappa_1 - \kappa_3\kappa_1\right] - \left[(B^*-P^*)\Gamma_3\right]\left[\lambda_1^i \nabla_i \kappa_3 - \gamma_{313}\kappa_3\right] \\
&- \left[(B^*-P^*)\Gamma_2\right]\left[(\gamma_{231}-\gamma_{213})\kappa_2\right] - \left[2(B^*-P^*)^2\Gamma_3 \frac{\partial}{\partial P^*}\Gamma_1\right](\kappa_3\kappa_1) \\
&+ \left\{\left[(B^*-P^*)\frac{\partial}{\partial B^*}\Gamma_1 + \Gamma_1\right]\left(\lambda_3^i \nabla_i B^*\right)\right\}\kappa_1 = 0
\end{aligned} \qquad \text{(II.14)}$$

$$\begin{aligned}
&\left[(B^*-P^*)\Gamma_2\right]\left[\lambda_1^i \nabla_i \kappa_2 - \gamma_{212}\kappa_2\right] - \left[(B^*-P^*)\Gamma_1\right]\left[\lambda_2^i \nabla_i \kappa_1 - \kappa_2\kappa_1\right] \\
&- \left[(B^*-P^*)\Gamma_3\right]\left[(\gamma_{312}-\gamma_{321})\kappa_3\right] + \left[2(B^*-P^*)^2\Gamma_2 \frac{\partial}{\partial P^*}\Gamma_1\right](\kappa_1\kappa_2) \\
&- \left\{\left[(B^*-P^*)\frac{\partial}{\partial B^*}\Gamma_1 + \Gamma_1\right]\left(\lambda_2^i \nabla_i B^*\right)\right\}\kappa_1 = 0
\end{aligned} \qquad \text{(II.15)}$$

Eqs. (II.13), (II.14), and (II.15) contain the integrability conditons for all cases, and they reduce to a specific set when a specific choice is made for the functions involved. In particular, to compare the integrability conditions for the compressible case with those for the incompressible case, (as we do in section 8d) we consider eqs. (II.13) - (II.15) using the same intrinsic formulation in both cases. Then the right-hand factors of all the terms are the same in both cases. In the incompressible case the left-hand factors of the last terms of eqs. (II.13) - (II.15) reduce to $\overset{\cdot}{\underset{3}{\lambda}}{}^{i}\nabla_i B$, $\overset{\cdot}{\underset{3}{\lambda}}{}^{i}\nabla_i B$, and $\overset{\cdot}{\underset{2}{\lambda}}{}^{i}\nabla_i B$ respectively, and the left-hand factors of all the other terms reduce to $B-P$.

APPENDIX III

THE CONDITIONS ON g_{ij} OF EQUATION (12.1) WHICH ARE CONSEQUENCES OF $R_{ijkl} = 0$

For a triply orthogonal coordinate system X^1, X^2, X^3 in E^3 the condition that the Riemann tensor vanishes is expressed by the following conditions on g_{ii} ;

$$\frac{\partial}{\partial X^2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial X^2}\sqrt{g_{33}}\right)+\frac{\partial}{\partial X^3}\left(\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial X^3}\sqrt{g_{22}}\right)+\frac{1}{g_{11}}\left(\frac{\partial}{\partial X^1}\sqrt{g_{22}}\right)\frac{\partial}{\partial X^1}\sqrt{g_{33}}=0 \qquad \text{(III.1)}$$

$$\frac{\partial}{\partial X^3}\left(\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial X^3}\sqrt{g_{11}}\right)+\frac{\partial}{\partial X^1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial X^1}\sqrt{g_{33}}\right)+\frac{1}{g_{22}}\left(\frac{\partial}{\partial X^2}\sqrt{g_{33}}\right)\frac{\partial}{\partial X^2}\sqrt{g_{11}}=0 \qquad \text{(III.2)}$$

$$\frac{\partial}{\partial X^1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial X^1}\sqrt{g_{22}}\right)+\frac{\partial}{\partial X^2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial X^2}\sqrt{g_{11}}\right)+\frac{1}{g_{33}}\left(\frac{\partial}{\partial X^3}\sqrt{g_{11}}\right)\frac{\partial}{\partial X^3}\sqrt{g_{22}}=0 \qquad \text{(III.3)}$$

$$-\frac{\partial}{\partial X^2}\left(\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial X^3}\sqrt{g_{11}}\right)+\frac{1}{\sqrt{g_{22}g_{33}}}\left(\frac{\partial}{\partial X^2}\sqrt{g_{11}}\right)\frac{\partial}{\partial X^3}\sqrt{g_{22}}=0 \qquad \text{(III.4)}$$

$$-\frac{\partial}{\partial X^3}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial X^1}\sqrt{g_{22}}\right)+\frac{1}{\sqrt{g_{33}g_{11}}}\left(\frac{\partial}{\partial X^3}\sqrt{g_{22}}\right)\frac{\partial}{\partial X^1}\sqrt{g_{33}}=0 \qquad \text{(III.5)}$$

$$-\frac{\partial}{\partial X^1}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial X^2}\sqrt{g_{33}}\right)+\frac{1}{\sqrt{g_{11}g_{22}}}\left(\frac{\partial}{\partial X^1}\sqrt{g_{33}}\right)\frac{\partial}{\partial X^2}\sqrt{g_{11}}=0 \qquad \text{(III.6)}$$

These conditions are referred to as the Lamé equations. Eqs. (III.4), (III.5), and (III.6) may also be written

$$-\frac{\partial}{\partial x^3}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^2}\sqrt{g_{11}}\right)+\frac{1}{\sqrt{g_{22}g_{33}}}\left(\frac{\partial}{\partial x^2}\sqrt{g_{33}}\right)\frac{\partial}{\partial x^3}\sqrt{g_{11}} = 0 \qquad \text{(III.4*)}$$

$$-\frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^3}\sqrt{g_{22}}\right)+\frac{1}{\sqrt{g_{33}g_{11}}}\left(\frac{\partial}{\partial x^3}\sqrt{g_{11}}\right)\frac{\partial}{\partial x^1}\sqrt{g_{22}} = 0 \qquad \text{(III.5*)}$$

$$-\frac{\partial}{\partial x^2}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\sqrt{g_{33}}\right)+\frac{1}{\sqrt{g_{11}g_{22}}}\left(\frac{\partial}{\partial x^1}\sqrt{g_{22}}\right)\frac{\partial}{\partial x^2}\sqrt{g_{33}} = 0 \qquad \text{(III.6*)}$$

Letting $\alpha = x^1$, $\psi = x^2$, and $\omega = x^3$, and putting $g_{11} = 1$, we find that eqs. (III.1), (III.5*), and(III.6) reduce to eqs. (15.12), (15.13), and (15.14) in Chapter VI.

For an orthogonal coordinate system x^1, x^2 in E^2 the condition that the Riemann tensor vanishes is expressed by the following condition on the metric coefficients $g_{\xi\xi}$ $(\xi = 1, 2)$;

$$\frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\sqrt{g_{22}}\right)+\frac{\partial}{\partial x^2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^2}\sqrt{g_{11}}\right) = 0$$

This result may be obtained directly from the Lamé equations by noting that this is all that remains when all terms containing either g_{33}

or the derivative with respect to x^3 are deleted. Putting $x^1 = u^1$, $x^2 = \omega$, $g_{11} = a_{11}$, and $g_{22} = a_{22}$ we obtain the condition used in Section 19 below eq. (19.13).

In Section 12a we desire to find orthogonal coordinate systems for which the quantities

$$\begin{aligned} g_{11} &= \mathcal{E}^{*2}(x^1)\,\mathcal{E}^2(x^2, x^3) \\ g_{22} &= \mathcal{F}^{*2}(x^1)\,\mathcal{F}^2(x^2, x^3) \\ g_{33} &= \mathcal{G}^{*2}(x^1)\,\mathcal{G}^2(x^2, x^3) \end{aligned} \tag{12.1}$$

are the metric coefficients. To this end we proceed to determine what the application of the Lame conditions implies as to the structure of the first fundamental form. Since a certain amount of calculation will be required it will be expedient to first simplify the notation a bit.

First note that we can put $\mathcal{E}^* = 1$ in eq. (12.1) without loss of generality. For, if we have a coordinate system x^1, x^2, x^3 in which $\mathcal{E}^* \neq 1$ then the coordinate system $\bar{x}^1, \bar{x}^2, \bar{x}^3$ defined by

$$\bar{x}^1 = \int \mathcal{E}^* dx^1$$

$$\bar{x}^2 = x^2$$
$$\bar{x}^3 = x^3$$

has the same coordinate surfaces and coordinate curves as the system x^1, x^2, x^3 and $\bar{\mathcal{E}}^* = 1$ (cf. argument above eq. (7.14))

Now we again put $\alpha = X^1$, $\psi = X^2$, and $\omega = X^3$. Furthermore we will use subscript notation as well as the notation of the previous equations for partial derivatives. In this notation, and with $\mathcal{E}^* = 1$, substituting eq. (12.1) into eqs. (III.1) - (III.6) results in

$$\frac{\mathcal{G}^*}{\mathcal{F}^*}\frac{\partial}{\partial\psi}\left(\frac{\mathcal{G}_\psi}{\mathcal{F}}\right) + \frac{\mathcal{F}^*}{\mathcal{G}^*}\frac{\partial}{\partial\omega}\left(\frac{\mathcal{F}_\omega}{\mathcal{G}}\right) + \mathcal{F}^*_\alpha \mathcal{G}^*_\alpha \frac{\mathcal{F}\mathcal{G}}{\mathcal{E}^2} = 0 \qquad \text{(III.8)}$$

$$\frac{\mathcal{E}}{\mathcal{G}}\frac{\partial}{\partial\omega}\left(\frac{\mathcal{E}_\omega}{\mathcal{G}}\right) + \mathcal{G}^*\mathcal{G}^*_{\alpha\alpha} + \frac{\mathcal{G}^{*2}}{\mathcal{F}^{*2}}\frac{\mathcal{E}\,\mathcal{E}_\psi\,\mathcal{G}_\psi}{\mathcal{F}^2\mathcal{G}} = 0 \qquad \text{(III.9)}$$

$$\mathcal{F}^*\mathcal{F}^*_{\alpha\alpha} + \frac{\mathcal{E}}{\mathcal{F}}\frac{\partial}{\partial\psi}\left(\frac{\mathcal{E}_\psi}{\mathcal{F}}\right) + \frac{\mathcal{F}^{*2}}{\mathcal{G}^{*2}}\frac{\mathcal{E}\,\mathcal{E}_\omega\,\mathcal{F}_\omega}{\mathcal{G}^2\mathcal{F}} = 0 \qquad \text{(III.10)}$$

$$-\frac{\partial}{\partial\psi}\left(\frac{\mathcal{E}_\omega}{\mathcal{G}}\right) + \frac{\mathcal{E}_\psi\,\mathcal{F}_\omega}{\mathcal{F}\mathcal{G}} = 0 \qquad \text{(III.11)}$$

$$-\mathcal{F}^*_\alpha\frac{\partial}{\partial\omega}\left(\frac{\mathcal{F}}{\mathcal{E}}\right) + \frac{\mathcal{F}^*\mathcal{G}^*_\alpha}{\mathcal{G}^*}\frac{\mathcal{F}_\omega}{\mathcal{E}} = 0 \qquad \text{(III.12)}$$

$$-\mathcal{G}_\psi\frac{\partial}{\partial\alpha}\left(\frac{\mathcal{G}^*}{\mathcal{F}^*}\right) + \frac{\mathcal{G}^*_\alpha}{\mathcal{F}^*}\frac{\mathcal{E}_\psi\,\mathcal{G}}{\mathcal{E}} = 0 \qquad \text{(III.13)}$$

Further, eqs. (III.4*), (III.5*), and (III.6*) lead to

$$-\frac{\partial}{\partial\omega}\left(\frac{\mathcal{E}_\psi}{\mathcal{F}}\right) + \frac{\mathcal{G}_\psi \mathcal{E}_\omega}{\mathcal{F}\mathcal{G}} = 0 \qquad \text{(III.11*)}$$

$$-\mathcal{F}_\omega \frac{\partial}{\partial\alpha}\left(\frac{\mathcal{F}^*}{\mathcal{G}^*}\right) + \frac{\mathcal{F}^*_\alpha}{\mathcal{G}^*}\,\frac{\mathcal{E}_\omega \mathcal{F}}{\mathcal{E}} = 0 \qquad \text{(III.12*)}$$

$$-\mathcal{G}^*_\alpha \frac{\partial}{\partial\psi}\left(\frac{\mathcal{G}}{\mathcal{E}}\right) + \frac{\mathcal{G}^*\mathcal{F}^*_\alpha}{\mathcal{F}^*}\,\frac{\mathcal{G}_\psi}{\mathcal{E}} = 0 \qquad \text{(III.13*)}$$

We will now show, by means of eqs. (III.8) - (III.13) and eqs. (III.11*) - (III.13*), that for g_{ii} of the form (12.1), the first fundamental form has at least one of the properties:

(i) $\mathcal{E} = 1$

(ii) $\mathcal{F}^* = 1$

(iii) $\mathcal{G}^* = 1$

In the following proof we assume that none of the quantities $\mathcal{E}$, $\mathcal{F}^*$, or $\mathcal{G}^*$ are constant and we eliminate each of the possibilities which arise.

Let us first look at eqs. (III.13) and (III.12*). Since $\mathcal{E} \neq const.$ these equations tell us that either $\mathcal{G}_\psi$ or $\mathcal{F}_\omega$ is not zero. Suppose $\mathcal{G}_\psi \neq 0$. Then $\mathcal{E}_\psi \neq 0$, and from (III.13)

$$\mathcal{E} = \mathcal{G}^{c_1} f_1(\omega) \tag{III.14}$$

where c_1 = const. $\neq 0$, or

$$\mathcal{G} = \frac{\mathcal{E}^{1/c_1}}{f_1^{\;1/c_1}} \tag{III.15}$$

Thus, upon changing scale in the ω direction, the first fundamental form becomes

$$(ds)^2 = \mathcal{E}^2 (d\alpha)^2 + \mathcal{F}^{*2} \mathcal{F}^2 (d\psi)^2 + \mathcal{G}^{*2} \mathcal{E}^{2/c_1} (d\omega)^2 \tag{III.16}$$

Along with $\mathcal{G}_\psi \neq 0$ we can have either $\mathcal{F}_\omega \neq 0$ or $\mathcal{F}_\omega = 0$. If $\mathcal{F}_\omega \neq 0$, then $\mathcal{E}_\omega \neq 0$, and from (III.12*)

$$\mathcal{E} = \mathcal{F}^{c_2} f_2(\psi) \tag{III.17}$$

where c_2 = const. $\neq 0$, or

$$\mathcal{F} = \frac{\mathcal{E}^{1/c_2}}{f_2^{\;1/c_2}} \tag{III.18}$$

Eq. (III.16) is now

$$(ds)^2 = \mathcal{E}^2(d\alpha)^2 + \mathcal{F}^{*2}\mathcal{E}^{2/c_2}(d\psi)^2 + \mathcal{G}^{*2}\mathcal{E}^{2/c_1}(d\omega)^2 \qquad \text{(III.19)}$$

If $\mathcal{F}_\omega = 0$, then $\mathcal{E}_\omega = 0$ and eq. (III.16) becomes

$$(ds)^2 = \mathcal{E}^2(\psi)(d\alpha)^2 + \mathcal{F}^{*2}(d\psi)^2 + \mathcal{G}^{*2}\mathcal{E}^{2/c_1}(\psi)(d\omega)^2 \qquad \text{(III.20)}$$

If we start with $\mathcal{F}_\omega \neq 0$ instead of $\mathcal{G}_\psi \neq 0$ then we get eq. (III.18), and

$$(ds)^2 = \mathcal{E}^2(d\alpha)^2 + \mathcal{F}^{*2}\mathcal{E}^{2/c_2}(d\psi)^2 + \mathcal{G}^{*2}\mathcal{G}^2(d\omega)^2 \qquad \text{(III.21)}$$

If with $\mathcal{F}_\omega \neq 0$ we also have $\mathcal{G}_\psi \neq 0$ then we again get eq. (III.19). If with $\mathcal{F}_\omega \neq 0$ we also have $\mathcal{G}_\psi = 0$ then $\mathcal{E}_\psi = 0$ and from eq. (III.21)

$$(ds)^2 = \mathcal{E}^2(\omega)(d\alpha)^2 + \mathcal{F}^{*2}\mathcal{E}^{2/c_2}(\omega)(d\psi)^2 + \mathcal{G}^{*2}(d\omega)^2 \qquad \text{(III.22)}$$

In summary, eqs. (III.13) and (III.12*) imply that our first fundamental form is either (III.19), (III.20), or (III.22).

Now look at eqs. (III.12) and (III.13*). Since $\mathcal{F}_\omega$ and $\mathcal{G}_\psi$ cannot both vanish, one of these equations gives

$$\frac{\mathcal{F}^* \mathcal{G}^*_\alpha}{\mathcal{F}^*_\alpha \mathcal{G}} = c_3 \qquad \text{(III.23)}$$

where C_3 = const. $\neq 0$, or

$$\mathcal{G}^* = \mathcal{F}^{*c_3} \qquad \text{(III.24)}$$

(A constant factor which occurs in (III.24) in the derivation may be suppressed without loss of generality.)

Putting (III.23) and (III.17) into eq. (III.12) we get

$$-\frac{\partial}{\partial\omega}\left(\frac{\mathcal{F}}{\mathcal{F}^{c_2}}\right) + c_3 \frac{\mathcal{F}_\omega}{\mathcal{F}^{c_2}} = 0$$

or

$$\mathcal{F}_\omega \left(c_2 + c_3 - 1\right) = 0 \qquad \text{(III.25)}$$

Putting (III.23) and (III.14) into eq. (III.13*) we get

$$-\frac{\partial}{\partial\psi}\left(\frac{\mathcal{G}}{\mathcal{G}^{c_1}}\right) + \frac{1}{c_3} \frac{\mathcal{G}_\psi}{\mathcal{G}^{c_1}} = 0$$

or

$$\mathcal{G}_\psi \left(c_1 + \frac{1}{c_3} - 1 \right) = 0 \qquad \text{(III.26)}$$

We will make use of these relations presently.

Next, we go to the pair of eqs. (III.9) and (III.10). If we differentiate (III.9) and (III.10) with respect to ψ and ω we find two possibilities:

(1)
$$\mathcal{G}^* = \mathcal{F}^* \qquad \text{(III.27)}$$

and eqs. (III.9) and (III.10) split into

$$-\mathcal{G}^*\mathcal{G}^*_{\alpha\alpha} = \frac{\mathcal{E}}{\mathcal{G}}\frac{\partial}{\partial\omega}\left(\frac{\mathcal{E}_\omega}{\mathcal{G}}\right) + \frac{\mathcal{E}\mathcal{E}_\psi\mathcal{G}_\psi}{\mathcal{F}^2\mathcal{G}} = c_4 \qquad \text{(III.28)}$$

$$-\mathcal{F}^*\mathcal{F}^*_{\alpha\alpha} = \frac{\mathcal{E}}{\mathcal{F}}\frac{\partial}{\partial\psi}\left(\frac{\mathcal{E}_\psi}{\mathcal{F}}\right) + \frac{\mathcal{E}\mathcal{E}_\omega\mathcal{F}_\omega}{\mathcal{G}^2\mathcal{F}} = c_5 \qquad \text{(III.29)}$$

where c_4 and c_5 are constants. (A constant factor which occurs in (III.27) in the derivation may be suppressed without loss of generality).

$$\frac{\mathcal{E}\mathcal{E}_\psi\mathcal{G}_\psi}{\mathcal{F}^2\mathcal{G}} = c_6 \qquad \text{(III.30)}$$

(2)

$$\frac{\mathcal{E}\mathcal{E}_\omega\mathcal{F}_\omega}{\mathcal{G}^2\mathcal{F}} = c_7 \qquad \text{(III.31)}$$

where C_6 and C_7 are constants, and eqs. (III.9) and (III.10) split into

$$-\frac{\mathcal{E}}{\mathcal{G}}\frac{\partial}{\partial\omega}\left(\frac{\mathcal{E}_\omega}{\mathcal{G}}\right) = \mathcal{G}^*\mathcal{G}^*_{\alpha\alpha} + C_6\frac{\mathcal{G}^{*2}}{\mathcal{F}^{*2}} = C_8 \qquad \text{(III.32)}$$

$$-\frac{\mathcal{E}}{\mathcal{F}}\frac{\partial}{\partial\psi}\left(\frac{\mathcal{E}_\psi}{\mathcal{F}}\right) = \mathcal{F}^*\mathcal{F}^*_{\alpha\alpha} + C_7\frac{\mathcal{F}^{*2}}{\mathcal{G}^{*2}} = C_9 \qquad \text{(III.33)}$$

where C_8 and C_9 are constants.

We can eliminate the possibility (1), because comparing (III.27) and (III.24) we see that $C_3 = 1$. However, in eqs. (III.19) and (III.20) $\mathcal{G}_\psi \neq 0$ so (III.26) gives in these cases $C_1 = 0$ which is a contradiction. Similarly, for (III.22) $\mathcal{F}_\omega \neq 0$ and (III.25) gives $C_2 = 0$ which is again contrary to assumption.

Pursuing the possibility (2) we substitute the expressions for $\mathcal{G}$ and $\mathcal{F}$ of eqs. (III.15) and (III.18) (suppressing the functions f_1 and f_2) into (III.30) and (III.31). These conditions then become

$$\frac{\mathcal{E}_\psi}{\mathcal{F}} = const. \tag{III.34}$$

$$\frac{\mathcal{E}_\omega}{\mathcal{G}} = const. \tag{III.35}$$

Putting (III.34) into (III.11*) and putting (III.35) into (III.11) we get

$$\mathcal{E}_\psi \mathcal{F}_\omega = 0 \tag{III.36}$$

and

$$\mathcal{G}_\psi \mathcal{E}_\omega = 0 \tag{III.37}$$

Thus either $\mathcal{E}_\psi = 0$ or $\mathcal{E}_\omega = 0$.

Using the fact noted above that $\mathcal{G}^* = \mathcal{F}^*$, or $\frac{\partial}{\partial\alpha}\left(\frac{\mathcal{G}^*}{\mathcal{F}^*}\right) = 0$ is impossible, we see from (III.13) and (III.12*) that either $\mathcal{F}_\omega = 0$ or $\mathcal{F}_\psi = 0$. Consequently only the forms (III.20) and (III.22) with $\mathcal{G}^*$ given by (III.24) remain as possibilities, that is, either

$$(ds)^2 = \mathcal{E}^2(\psi)(d\alpha)^2 + \mathcal{F}^{*2}(d\psi)^2 + \mathcal{F}^{*2c_3}\mathcal{E}^{2/c_4}(\psi)(d\omega)^2 \tag{III.38}$$

or

$$(ds)^2 = \mathcal{E}^2(\omega)(d\alpha)^2 + \mathcal{F}^{*2}\mathcal{E}^{2/c_2}(\omega)(d\psi)^2 + \mathcal{F}^{*2c_3}(d\omega)^2 \tag{III.39}$$

In (III.38) we have $c_1 + \frac{1}{c_3} = 1$ and in (III.39) we have $c_2 + c_3 = 1$

Using eq. (III.30) and eq. (III.10) in (III.38) we get respectively

$$\mathcal{E}_{\psi\psi} = 0$$

$$\mathcal{F}^*_{\alpha\alpha} = 0$$

Similarly, from (III.31) and (III.9) we get for (III.39)

$$\mathcal{E}_{\omega\omega} = 0$$

$$\mathcal{G}^*_{\alpha\alpha} = 0$$

Finally, we use our results for (III.38) in eq. (III.8). Since $\mathcal{F} = 1$, it can be written

$$\frac{\mathcal{E}^2}{\mathcal{G}} \mathcal{G}_{\psi\psi} + \frac{\mathcal{F}^*}{\mathcal{G}^*} \mathcal{F}^*_\alpha \mathcal{G}^*_\alpha = 0$$

or

$$\mathcal{E}^{2-\frac{1}{c_1}} \frac{d^2}{d\psi^2} \mathcal{E}^{1/c_1} + \mathcal{F}^{*(1-c_3)} \left(\frac{d\mathcal{F}^*}{d\alpha}\right) \frac{d}{d\alpha} \mathcal{F}^{*\,c_3} = 0$$

or

$$\frac{1}{c_1}\left(\frac{d\mathcal{E}}{d\psi}\right)^2\left(\frac{1}{c_1} - 1\right) + c_3\left(\frac{d\mathcal{F}^*}{d\alpha}\right)^2 = 0$$

But $c_3 = \frac{1}{1-c_1}$

So

$$\frac{1}{c_1}\,\frac{1-c_1}{c_1}\left(\frac{d\mathcal{E}}{d\psi}\right)^2 + \frac{1}{1-c_1}\left(\frac{d\mathcal{F}^*}{d\alpha}\right)^2 = 0$$

and thus we are led to either $\frac{d\mathcal{E}}{d\psi} = 0$ or $\frac{d\mathcal{F}^*}{d\alpha} = 0$ which cases we have already excluded.

Similarly, applying (III.9) to (III.39) we find that either $\mathcal{E} = const.$ or $\mathcal{F}^* = const.$

This completes the proof of the statement below eq. (III.13*), that is, the first fundamental form must have at least one of the properties:

$$(i)\quad \mathcal{E} = 1$$
$$(ii)\quad \mathcal{F}^* = 1$$
$$(iii)\quad \mathcal{G}^* = 1$$

From eq. (III.12) we see that case (ii) can be split further into the two cases

$$(ii.1)\quad \mathcal{F}^* = \mathcal{G}^* = 1$$
$$(ii.2)\quad \mathcal{F}^* = \mathcal{F} = 1$$

From (III.13*) case (iii) splits into

$$(iii.1)\quad \mathcal{F}^* = \mathcal{G}^* = 1$$
$$(iii.2)\quad \mathcal{G}^* = \mathcal{G} = 1$$

Now we shall show that cases (ii.2) and (iii.2) lead to the fundamental form for spherical coordinates. With this result we can state our final conclusion that <u>our first fundamental form has at least one of the properties</u>

1. $\underline{\mathcal{E} = 1}$
2. $\underline{\mathcal{F}^* = \mathcal{G}^* = 1}$
3. <u>It is the first fundamental form for spherical coordinates</u>.

To obtain the implications of (ii.2) we can assume that $\mathcal{G}^*_\alpha \neq 0$. Then eq. (III.13*) gives us

$$\mathcal{G} = \mathcal{E} \qquad \text{(III.40)}$$

(changing scale in the ω direction). Further eq. (III.10) says that

$$\mathcal{E}_{\psi\psi} = 0 \qquad \text{(III.41)}$$

and eq. (III.11) (with (III.40)) says that

$$(\ln \mathcal{E})_{\psi\omega} = 0 \qquad \text{(III.42)}$$

Eqs. (III.41) and (III.42) together imply that

$$\mathcal{E} = f_3(\omega)\,\psi \qquad \text{(III.43)}$$

Using (III.40) and (III.43) in eq. (III.9) we get

$$\frac{d}{d\omega}\left(\frac{f_3'}{f_3}\right) + \mathcal{G}^*\mathcal{G}^*_{\alpha\alpha} + \mathcal{G}^{*2} f_3^2 = 0 \qquad \text{(III.44)}$$

Differentiating (III.44) with respect to ω we find that, since $\mathcal{G}^* \neq const.$ we must have $f_3 = const$. Then (III.44) reduces to

$$\mathcal{G}^*_{\alpha\alpha} + f_3^2 \mathcal{G}^* = 0$$

and this equation with (III.43) (in which f_3 = const.) gives the first fundamental form for spherical coordinates.

Because of symmetry we obtain the same result for case (iii.2) with ψ and ω interchanged.

APPENDIX IV

NOTATION

For convenience we list the symbols used in the text, along with a brief description of each and a reference to the section in which each is defined.

a_1, a_2, a_3	constants in equation of quadric surfaces, Sect. 12b
a_{11}, a_{22}	metric coefficients of planes in sect. 19a and 19b
A_0, A_1, A_2	arbitrary functions appearing in eq. (18.4) for g_{13} and in the derivation of eq. (18.4)
b^i	the unit binormal vector of the stream lines, sect. 2
B	the Bernoulli function, eq. (5.3)
B^*	used to denote either B or $\mathcal{B}$, app. II2
$\mathcal{B}$	the function corresponding to B for compressible flows, eq. (6.8)
$\mathcal{B}_1, \mathcal{B}_2$	functions of r in terms of which $\mathcal{B}$ is decomposed in sect. 22
c	the speed of sound, sect. 6
c_p, c_v	specific heats, sect. 6
$c_i \quad i=1,2,\dots$	denote arbitrary constants in several different contexts, sect. 15, sect. 19b, app. III
C_1, C_2	curves, app. I
E	internal energy, sect. 6
E^2, E^3	Euclidian spaces, sect. 3b, and app. III

$\mathcal{E}, \mathcal{E}^*$	factors of the metric coefficient g_{11}, eq. (12.1)
f	an arbitrary function in sect. 3b, and also used to express z as a function of u' and ω in eq. (19.14)
f_1, f_2, f_3	arbitrary functions in app. III
F	a scalar function in eqs. (16.4) and (16.5)
$\mathcal{F}, \mathcal{F}^*$	factors of the metric coefficient g_{22}, eq. (12.1)
g	the determinant of the metric coefficients, sect. 3b
g_{ij}	the first fundamental tensor of E^3, sect. 3b
$'g_{\xi\eta}, 'g^*_{\xi\eta}, '\bar{g}_{\xi\eta}$	the first fundamental tensor of surfaces in E^3, sect. 3b, sect. 7b, and eqs. (13.2) and (13.3)
$\mathcal{G}, \mathcal{G}^*$	factors of the metric coefficient g_{33} eq. (12.1)
h	enthalpy, sect. 6
h_{ij}	second fundamental tensor of a surface in E^3 referred to space coordinates, sect. 18b
$'h_{\xi\eta}, '\bar{h}_{\xi\eta}$	second fundamental tensor of a surface in E^3 referred to surface coordinates, app. I2
H	stagnation enthalpy, sect. 6
I	a function of ψ and ω, eq. (19.2) and eq. (19.15)
J	a function of ψ and ω, eq. (19.2) and eq. (19.15)
K_1, K_2	principal curvatures of ω = const. surfaces, eq. (19.6)
K_2, K_3, K_2^*, K_3^*	principal curvatures of α =const. surfaces, eq. (13.2) and (13.3), and app. I2.
$\mathcal{K}$	total (Gaussian) curvature of α = const. surfaces, sect. 14
l_1, l_2, l_3	arbitrary constants, sect. 23
m	exponent, sect. 23

M Mach number, sect. 21c

$\mathfrak{M}$ mean curvature of α = const. surfaces, sect. 14

n exponent, sect. 23

n^i unit principal normal vector of the stream lines, sect. 2

$'n^i$ unit normal vector of the stream lines which lies in the ω = const. surface, sect. 3

N^i unit normal vector of ω = const. surfaces, sect. 3

p, p_1, p_2 pressure, sect. 1 and sect. 7c

P $\frac{p}{\rho}$, eq. (5.1)

P^* used to denote either P or $\mathcal{P}$, app. II2

$\mathcal{P}$ $\int_0^p \frac{1}{X(p)}\, dp$, eq. (6.4)

q magnitude of the velocity, sect. 2

r used as a spherical coordinate, eq. (12.5), and as a cylindrical coordinate, sect. 20

R_{ijkl} the Riemann tensor, sect. 4

s length, sect. 3b

S, S_1, S_2 entropy, sect. 6, and sect. 7c

t^i, t^i_1, t^i_2 unit tangent vector of the stream lines, sect. 2 and sect. 7c

T temperature, sect. 6

u^ξ $(\xi = 1, 2)$ surface coordinates, sect. 3b

U an arbitrary function, sect. 12b

v, v_1, v_2, v_3 arbitrary functions, eq. (19.25) and below

v^i the velocity vector, sect. 1

V	an arbitrary function, sect. 12b
w, w_1, w_2, w_3	arbitrary functions, eq. (19.26) and below
W	a function defined by eq. (19.16)
x	a rectangular cartesian coordinate, sect. 12a
x^i	general coordinates in E^3, sect. 3b
X	a function of pressure, eq. (6.2)
y	a rectangular cartesian coordinate, sect. 12a
$y^i, \bar{y}^i$	rectangular cartesian coordinates, app. I1
Y, Y_1, Y_2	a function of entropy, eq. (6.2), and sect. 7c
z	a rectangular cartesian coordinate, sect. 12a
Z	a function of entropy, sect. 6
α	a coordinate, sect. 10
β	angle of helices, sect. 19a
γ	ratio of specific heats, sect. 6
γ_{uvw}	coefficients of rotation, sect. 8a
Γ	a coefficient in the continuity equation, app. II1
Γ_u	either $(1, 1, 0)$, $(1, 1, 1)$, $(\Gamma, 1, 0)$, or $(\Gamma, 1, 1)$, app. II2
Γ^j_{ki}	Christoffel symbols, sect. 10
δ	polar angle of spherical coordinates, eq. (12.5)
Δ	a constant, sect. 22
θ	used as a cylindrical coordinate, sect. 7b and as a polar coordinate, eq. (12.5)
Θ, Θ_1	arbitrary functions, sect. 22
κ	the curvature of the stream lines, sect. 2

$'\kappa$	geodesic curvature of the stream lines with respect to ω = const., eq. (3.2)
κ_N	normal curvature of the stream lines with respect to ω = const., eq. (3.1)
$\bar{\kappa}$	the curvature of the orthogonal trajectories of the stream lines in plane flows, sect. 7b
$\underset{u}{\kappa}$	either $(\nabla_i t^i, '\kappa, \kappa_N)$ or $(\nabla_i t^i, \kappa, 0)$, app. II2
λ	a parameter in equation of quadric surfaces, sect. 12b
λ_1, λ_2	constants, sect. 21d
$\underset{u}{\lambda^i}$	either (t^i, m^i, N^i) or (t^i, n^i, b^i) , app. II2
Λ	an arbitrary function of ω , sect. 15
μ_1, μ_2	arbitrary functions of ψ and ω , eqs. (5.12) and (6.17)
ν	an arbitrary function of ψ and ω , sect. 7c
Ξ	arbitrary constant, eq. (6.1)
ϖ	the angle between m^i and N^i , sect. 3a
ρ, ρ_1, ρ_2	density, sect. 1 and sect. 7c
σ_1, σ_2	arbitrary constants, eq. (22.1)
τ	torsion of the stream lines, sect. 10
ϕ	potential function, sect. 7b
ϕ_1, ϕ_2, ϕ_2^*	arbitrary function of χ' eqs. (11.5) and (11.6) and sect. 12b
ψ, ψ^*	stream function, sect. 3b and sect. 7b
Ψ	stream function, sect. 3
Ψ', Ψ^2	arbitrary functions of ψ and ω , eqs. (13.4) and (13.5)
ω	stream function, sect. 3
Ω_1, Ω_2	arbitrary functions of ω , eqs. (19.11) and (19.13)
Ω^k	the vorticity vector, sect. 7a

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