THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
Department of Meteorology and Oceanography

Technical Report No. 1

FORMULATION OF AN OBSERVATIONAL STUDY OF KINETIC ENERGY CONVERSIONS IN THE ATMOSPHERE

A. Wiin-Nielsen

ORA Project 06372

under contract with:

NATIONAL SCIENCE FOUNDATION
GRANT NO. GP-2561
WASHINGTON, D. C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

January 1966

TABLE OF CONTENTS

ABSTRACT

The total kinetic energy in the atmosphere has been subdivided into four energy reservoirs. The partition of the kinetic energy is accomplished by dividing the total flow into the vertical mean flow (the barotropic component) and the vertical shear flow (the baroclinic component). Each of these components is subdivided into the zonal components and the eddy components.

The complete energy exchange diagram is derived by dividing a given energy conversion in the contribution from the quasi-non-divergent flow and the contribution from the divergent flow. Such a subdivision of the energy conversion is advantageous because the calculations are based on geopotential data.

Calculations will be carried out for five months (January, April, July, October, 1962, and January, 1963) based on five isobaric surfaces (20, 30, 50, 70, and 85 cb). The complete energy diagrams will be presented for each month together with an averaged diagram representing the annual mean.

1. INTRODUCTION

The major components of the energetics of the atmosphere originally formulated by Lorenz [2], have been investigated in great detail during recent years. A comprehensive summary of calculations based on observations has been given by Oort [3] who also has included some results of studies of the general circulation based on long-term numerical integrations of theoretical models of the atmosphere.

The kinetic energy of the atmosphere has been subdivided into the kinetic energy of the zonally averaged flow and the kinetic energy of the remaining flow, the eddies. Several estimates have been made of the energy conversions between the eddy kinetic energy and the zonal kinetic energy. (Starr [8], Saltzman and Fleisher [4], Wiin-Nielsen, Brown and Drake [12, 13].) A different subdivision of the kinetic energy was introduced by Wiin-Nielsen [10] in close collaboration with Smagorinsky [7] who used this subdivision to describe the energetics of his basic general circulation experiment. The new subdivision consists of dividing the atmospheric flow into the vertically averaged flow (the so-called barotropic component) and the deviation from the vertical mean flow, the vertical shear flow (or the baroclinic component) of the atmospheric flow. The original pilot study by Wiin-Nielsen [10] was later extended (Wiin-Nielsen and Drake [14,15]) to cover much larger data samples and a greater vertical resolution in addition to an estimate of the contribution from the divergent part of the wind to the energy conversion between the vertical shear flow and the vertical mean flow.

Smagorinsky [7] extended the study of the energy conversions between the different forms of kinetic energy to cover all possible components of the energy transformation between the four forms of energy: (1) The kinetic energy of the zonally averaged vertical mean flow; (2) the kinetic energy of the eddy component of the vertical mean flow; (3) the kinetic energy of the zonally averaged vertical shear flow; and (4) the kinetic energy of the eddy component of the vertical shear flow. We shall in the following sections denote these components by the symbols: K_{MZ}, K_{ME}, K_{SZ}, K_{SE}. Numerical values of the many energy conversions between the four energy forms can be found in Smagorinsky's paper [7] for his model.

The same energy conversions have not to the knowledge of the authors been calculated from atmospheric data. It is the main purpose of this paper to present the formulation of such calculations based on atmospheric height data from five isobaric levels: 85, 70, 50, 30 and 20 cb. The height data will be the routine objective analysis carried out by the National Meteorological Center, U.S. Weather Bureau in connection with its short range numerical prediction program. The data which have been used in several studies of the energetics of the atmosphere (Wiin-Nielsen [11]) were originally made available to us by Dr. George P. Cressman.

There are significant differences between an observational study of atmospheric energy conversions and a study of the same quantities based on a numerical integration of a set of equations which simulates the thermohydrodynamic behavior of the atmosphere on a large time-scale including modeling of the atmospheric heat sources and the frictional dissipation. One of the main differences is the fact that atmospheric parameters such as the vertical velocity, the horizontal divergence, and the distribution of the atmospheric heat sources escape ordinary synoptic analysis and must be computed by more or less realistic indirect methods while they are readily available as by-products of the numerical integrations of the model equations with an accuracy as great as the electronic computer employed for the experiment will permit. Due to this fact it has been found advantageous to transform the original expressions for the kinetic energy conversions in such a way that we isolate easily computable quantities such as the horizontal wind and the vertical components of the vorticity in separate integrals while the quantities which must be computed by indirect methods (divergence etc.) are isolated in other integrals. This subdivision is made possible by using well known identities between terms in the hydrodynamic equations. As shown in the earlier paper (Wiin-Nielsen [10]), we can in this way distinguish between terms which will make contributions in a quasi-geostrophic formulation of the atmospheric dynamics and the terms which will contribute only in an atmospheric model based on the primitive equations.

Section 2 of this paper contains an outline of the basic framework and the formulas which will be used in the calculations.

2. BASIC EQUATIONS FOR THE ENERGY CONVERSIONS

The integrals which are used for the calculation of the energy conversions are naturally very similar to those derived by Smagorinsky [7], although his derivations apply to a two-level representation of the vertical structure of the atmosphere. We begin by defining the wind components used in this study. The vertically averaged wind $\vec{v}_M$ is defined by the expression

$$\vec{v}_M = \frac{1}{p_o} \int_o^{p_o} \vec{v} \, dp , \qquad (2.1)$$

in which $\vec{v}$ is the horizontal wind vector with components u and v, p is pressure and $p_o = 100$ cb.

The deviation of the wind from the vertical mean $\vec{v}_M$, the shear vector, is defined by the relation

$$\vec{v}_S = \vec{v} - \vec{v}_M . \qquad (2.2)$$

Each of the wind components will be subdivided in the zonal average defined by the relation

$$(\ )_Z = \frac{1}{2\pi} \int_o^{2\pi} (\ ) \, d\lambda , \qquad (2.3)$$

where λ is longitude, while the eddy component is defined as the following deviation:

$$(\ )_E = (\ ) - (\ )_Z . \qquad (2.4)$$

The amount of kinetic energy in the zonal part of the vertical mean flow can be evaluated from the formula:

$$K_{MZ} = \frac{p_o}{g} \int_S \frac{1}{2} (u_{MZ}^2 + v_{MZ}^2) \, dS \qquad (2.5)$$

in which $dS = a^2 \cos\varphi \, d\lambda \, d\varphi$ is the area element (a is the radius of the earth, and φ is latitude), while S is the total area of integration.

The corresponding expressions for the three additional energy forms are

$$K_{ME} = \frac{p_0}{g} \int_S \frac{1}{2} (u_{ME}^2 + v_{ME}^2) \, dS \qquad (2.6)$$

$$K_{SZ} = \frac{1}{g} \int_0^{p_0} \int_S \frac{1}{2} (u_{SZ}^2 + v_{SZ}^2) \, dS \, dp \qquad (2.7)$$

and

$$K_{SE} = \frac{1}{g} \int_0^{p_0} \int_S \frac{1}{2} (u_{SE}^2 + v_{SE}^2) \, dS \, dp \ . \qquad (2.8)$$

During the derivations it has been assumed that the boundary conditions for ω are $\omega = 0$ for $p = 0$ and $p = p_0$. It follows then from the general continuity equation in pressure coordinates that

$$\nabla \cdot \vec{v}_M = 0 \quad \text{and} \quad \nabla \cdot \vec{v} = \nabla \cdot \vec{v}_S \ . \qquad (2.9)$$

When we average $\nabla \cdot \vec{v}_M = 0$ in the zonal direction we obtain

$$\frac{1}{a \cos\varphi} \quad \frac{\partial \, v_{MZ} \cos\varphi}{\partial\varphi} = 0 \ ,$$

from which it follows that $v_{MZ} \cos\varphi$ is constant, but since $v_{MZ} \cos\varphi = 0$ at the poles we obtain $v_{MZ} = 0$ everywhere. The expression (2.5) reduces therefore to

$$K_{MZ} = \frac{p_0}{g} \int_S \frac{1}{2} u_{MZ}^2 \, dS \ . \qquad (2.10)$$

The complete equations for the energy transformations are obtained by deriving equations for the rate of change of K_{MZ}, K_{ME}, K_{SZ} and K_{SE}. From (2.10) and (2.6) we obtain

$$dK_{MZ}/dt = \frac{p_0}{g} \int_S u_{MZ} \frac{\partial u_{MZ}}{\partial t} \, dS \qquad (2.11)$$

and

$$dK_{ME}/dt = \frac{p_o}{g} \int_S [u_{ME} \frac{\partial u_{ME}}{\partial t} + v_{ME} \frac{\partial v_{ME}}{\partial t}] \, dS \qquad (2.12)$$

with expressions analogous to (2.12) obtained from (2.7) and (2.8). It is seen from (2.12) that we must go through the following steps in order to derive an equation for dK_{ME}/dt. First, we must derive the equations governing the local rate of change of u_{ME} and v_{ME}. The next step is to multiply the first of these equations by u_{ME} and the second by v_{ME}. The final step is to add the resulting equations and integrate over the domain S. A similar procedure must be followed in order to obtain equations for dK_{SZ}/dt and dK_{SE}/dt. The derivation of these equations is rather straightforward although laborious and cumbersome due to the many different subscripts and the two different averaging procedures which are being used. However, the equation for dK_{MZ}/dt is rather easy to easy to get according to (2.11). We shall therefore derive this equation in detail and also obtain the equation for dK_{ME}/dt while the remaining two equations will be given without derivation.

The first equation of motion will be used in the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{a \cos \varphi \, \partial \lambda} + \frac{v}{a} \frac{\partial u}{\partial \varphi} + \omega \frac{\partial u}{\partial p} = - \frac{1}{a \cos \varphi} \frac{\partial \Phi}{\partial \lambda} + fv$$

$$+ \frac{uv}{a} \tan \varphi + F_x , \qquad (2.13)$$

where $\omega = dp/dt$ is the vertical velocity in the pressure system, $\Phi = gz$ the geopotential of the isobaric surfaces, f the Coriolis parameter and F_x the frictional force per unit mass in the zonal direction.

We first perform an averaging process in the vertical direction by applying the operator (2.1) to (2.13). Writing

$$u = u_M + u_S, \; v = v_M + v_S \; , \; \text{etc.},$$

we obtain

$$\frac{\partial u_M}{\partial t} + \frac{u_M}{a \cos \varphi} \frac{\partial u_M}{\partial \lambda} + \frac{v_M}{a} \frac{\partial u_M}{\partial \varphi} + [\frac{u_S}{a \cos \varphi} \frac{\partial u_S}{\partial \lambda} + \frac{v_S}{a} \frac{\partial u_S}{\partial \varphi} + \omega_S \frac{\partial u_S}{\partial \varphi}]_M =$$

$$- \frac{1}{a \cos \varphi} \frac{\partial \Phi_M}{\partial \lambda} + f\ v_M + \frac{u_M v_M + (u_S v_S)_M}{a} \tan \varphi + F_{x,M} \qquad (2.14)$$

Our next step is to apply the operator (2.3) to (2.14). Writing $u_M = u_{MZ} + u_{ME}$ and analogous expressions for the other dependent variables we obtain

$$\frac{\partial u_{MZ}}{\partial t} + \left[\frac{v_{ME}}{a} \frac{\partial u_{ME}}{\partial \varphi} \right]_Z + \left[\frac{v_{SZ}}{a} \frac{\partial u_{SZ}}{\partial \varphi} \right]_M + \left[\frac{v_{SE}}{a} \frac{\partial u_{SE}}{\partial \varphi} \right]_{MZ} + \left[\omega_{SZ} \frac{\partial u_{SZ}}{\partial p} \right]_M +$$

$$\left[\omega_{SE} \frac{\partial u_{SE}}{\partial p} \right]_{MZ} = \qquad (2.15)$$

$$\frac{(u_{ME} v_{ME})_Z}{a} \tan \varphi + \frac{(u_{SZ} v_{SZ})_M}{a} \tan \varphi + \frac{(u_{SE} v_{SE})_{MZ}}{a} \tan \varphi + F_{x,MZ}$$

In order to obtain a more convenient and shorter form of (2.15) we make use of the continuity equations for the zonal and eddy parts of the vertical mean flow and the vertical shear flow, i.e.,

$$\frac{1}{a \cos \varphi} \left[\frac{\partial u_{ME}}{\partial \lambda} + \frac{\partial v_{ME} \cos \varphi}{\partial \varphi} \right] = 0 \qquad (2.16)$$

$$\frac{1}{a \cos \varphi} \left[\frac{\partial u_{SZ}}{\partial \lambda} + \frac{\partial v_{SZ} \cos \varphi}{\partial \varphi} \right] + \frac{\partial \omega_{SZ}}{\partial p} = 0 \qquad (2.17)$$

and

$$\frac{1}{a \cos \varphi} \left[\frac{\partial u_{SE}}{\partial \lambda} + \frac{\partial\ v_{SE} \cos \varphi}{\partial \varphi} \right] + \frac{\partial \omega_{SE}}{\partial p} = 0 \qquad (2.18)$$

We can then write (2.15) in the so-called momentum form and obtain

$$\frac{\partial u_{MZ}}{\partial t} + \frac{1}{a \cos^2 \varphi} \frac{\partial (u_{SZ} v_{SZ} \cos^2 \varphi)_M}{\partial \varphi} +$$

$$\qquad (2.19)$$

$$\frac{1}{a \cos^2 \varphi} \frac{\partial (u_{ME} v_{ME} \cos^2 \varphi)_Z}{\partial \varphi} = \frac{1}{a \cos^2 \varphi} \frac{\partial (u_{SE} v_{SE})_{MZ} \cos^2 \varphi}{\partial \varphi} = F_{x,MZ} .$$

The equation for dK_{MZ}/dt is now obtained by multiplying (2.19) by u_{MZ} and integrating over the region S which we assume to be bounded in such a way that $v \cos \varphi = 0$ at the boundary. This requirement will certainly be fulfilled if S is the region of the whole earth. We obtain

$$\frac{dK_{MZ}}{dt} = C(K_{ME}, K_{MZ}) + C(K_{SE}, K_{MZ}) + C(K_{SZ}, K_{MZ}) - D(K_{MZ}) \tag{2.20}$$

in which

$$C(K_{ME}, K_{MZ}) = \frac{2\pi\ a\ p_o}{g} \int_{\varphi_1}^{\varphi_2} (u_{ME} v_{ME})_Z \cos^2\varphi \frac{\partial}{\partial\varphi} \left(\frac{u_{MZ}}{\cos\varphi}\right) d\varphi \tag{2.21}$$

$$C(K_{SE}, K_{MZ}) = \frac{2\pi\ a\ p_o}{g} \int_{\varphi_1}^{\varphi_2} (u_{SE} v_{SE})_Z \cos^2\varphi \frac{\partial}{\partial\varphi} \left(\frac{u_{MZ}}{\cos\varphi}\right) d\varphi \tag{2.22}$$

$$C(K_{SZ}, K_{MZ}) = \frac{2\pi\ a\ p_o}{g} \int_{\varphi_1}^{\varphi_2} (u_{SZ} v_{SZ}) \cos^2\varphi \frac{\partial}{\partial\varphi} \left(\frac{u_{MZ}}{\cos\varphi}\right) d\varphi \tag{2.23}$$

and

$$D(K_{MZ}) = -\frac{2\pi\ a^2 p_o}{g} \int_{\varphi_1}^{\varphi_2} u_{MZ}\ F_{x,MZ} \cos\varphi\ d\varphi\ . \tag{2.24}$$

It will be noticed that the expressions (2.21) to (2.23) are obtained after an integration by parts using the lateral boundary condition.

The integral (2.21) can only be called the energy conversion from K_{ME} to K_{MZ} when it has been shown that the same integral with opposite sign appears in the equation for dK_{ME}/dt. Similar remarks apply to the integrals (2.22) and (2.23).

An inspection of the integrals (2.21) to (2.23) shows that they have the same form as the well-known energy conversion between the eddy motion and the zonal motion. The physical interpretation of energy conversions of this kind has been discussed by several authors (Kuo [1], Starr [8], Wiin-Nielsen et.al., [12,13]). The kinetic energy of the zonally averaged vertical mean flow will increase if we have a positive correlation between the proper momentum transport and the shear of the zonal, vertical mean current u_{MZ}. It is obvious that the three integrals represent physical processes of the same nature but while a numerical estimate of the first two integrals ((2.21) and (2.22)) can be obtained from the nondivergent assumption it is evident that the third integral depends entirely on a divergent wind component (v_{SZ}) because v_{SZ} vanishes when we substitute the assumption of a nondivergent wind. If we therefore want to calculate the integral (2.23) we must be able to calculate the vertical velocity, the horizontal divergence and thereby the divergent wind components. The procedure which is used for this purpose will be described in a later section.

We are next turning our attention to the derivation of the equation for dK_{ME}/dt. The first goal is to derive equations for the rate of change of u_{ME} and v_{ME} with respect to time. The equation for the rate of change of u_M has already been given as Eq. (2.14). The corresponding equation for v_M is derived in a similar manner from the equation

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{a \cos\varphi \, \partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\varphi} + \omega \frac{\partial v}{\partial p} = - \frac{1}{a}\frac{\partial\Phi}{\partial\varphi} - fu - \frac{u^2}{a}\tan\varphi + F_y \qquad (2.25)$$

and we obtain

$$\frac{\partial v_M}{\partial t} + \frac{u_M}{a \cos\varphi}\frac{\partial v_M}{\partial\lambda} + \frac{v_M}{a}\frac{\partial v_M}{\partial\varphi} + \left[\frac{u_S}{a \cos\varphi}\frac{\partial v_S}{\partial\lambda} + \frac{v_S}{a}\frac{\partial v_S}{\partial\varphi} + \omega_S \frac{\partial v_S}{\partial p}\right]_M =$$

$$- \frac{1}{2}\frac{\partial\Phi_M}{\partial\varphi} - f\, u_M - \frac{u_M^2 + (u_S^2)_M}{a}\tan\varphi + F_{y,M} \qquad (2.26)$$

In order to obtain the equation for $\partial u_{ME}/\partial t$ we must subtract Eq. (2.14) from Eq. (2.13). After some rearrangement we arrive at the equation

$$\frac{\partial u_{ME}}{\partial t} + \frac{u_{MZ}}{a \cos\varphi}\frac{\partial u_{ME}}{\partial\lambda} + \frac{u_{ME}}{a \cos\varphi}\frac{\partial u_{ME}}{\partial\lambda} + \frac{v_{ME}}{a}\frac{\partial u_{MZ}}{\partial\varphi} + \frac{v_{ME}}{a}\frac{\partial u_{ME}}{\partial\varphi} -$$

$$\left(\frac{v_{ME}}{a}\frac{\partial u_{ME}}{\partial\varphi}\right)_Z + \left[\frac{u_{SZ}}{a \cos\varphi}\frac{\partial u_{SE}}{\partial\lambda} + \frac{u_{SE}}{a \cos\varphi}\frac{\partial u_{SE}}{\partial\lambda} + \frac{v_{SZ}}{a}\frac{\partial u_{SE}}{\partial\varphi} + \frac{v_{SE}}{a}\frac{\partial u_{SZ}}{\partial\varphi} + \right.$$

$$\frac{v_{SE}}{a}\frac{\partial u_{SE}}{\partial\varphi} - \left(\frac{v_{SE}}{a}\frac{\partial u_{SE}}{\partial\varphi}\right)_Z + \omega_{SZ}\frac{\partial u_{SE}}{\partial p} + \omega_{SE}\frac{\partial u_{SZ}}{\partial p} + \omega_{SE}\frac{\partial u_{SE}}{\partial p} -$$

$$\left.\left(\omega_{SE}\frac{\partial u_{SE}}{\partial p}\right)_Z\right]_M = - \frac{1}{a \cos\varphi}\frac{\partial\Phi_{ME}}{\partial\varphi} + f\, v_{ME} + \frac{u_{MZ} v_{ME}}{a}\tan\varphi +$$

$$\frac{u_{ME}v_{ME}}{a}\tan\varphi - \frac{(u_{ME}v_{ME})_Z}{a}\tan\varphi + \frac{(u_{SZ}v_{SE})_M}{a}\tan\varphi + \frac{(u_{SE}v_{SE})_M}{a}$$

$$\tan\varphi - \frac{(u_{SE}v_{SE})_{MZ}}{a}\tan\varphi + \frac{(u_{SE}v_{SZ})_M}{a}\tan\varphi + F_{x,ME} \quad . \qquad (2.27)$$

We obtain the equation for $\partial v_{ME}/\partial t$ by subtracting (2.26) from (2.25). After a similar rearrangement we obtain

$$\frac{\partial v_{ME}}{\partial t} + \frac{u_{MZ}}{a\cos\varphi}\frac{\partial v_{ME}}{\partial\varphi} + \frac{u_{ME}}{a\cos\varphi}\frac{\partial v_{ME}}{\partial\lambda} - \left(\frac{u_{ME}}{a\cos\varphi}\frac{\partial v_{ME}}{\partial\lambda}\right)_Z + \frac{v_{ME}}{a}\frac{\partial v_{ME}}{\partial\varphi} -$$

$$\left(\frac{v_{ME}}{a}\frac{\partial v_{ME}}{\partial\varphi}\right)_Z + \left[\frac{u_{SZ}}{a\cos\varphi}\frac{\partial v_{SE}}{\partial\lambda} + \frac{u_{SE}}{a\cos\varphi}\frac{\partial v_{SE}}{\partial\lambda} - \left(\frac{u_{SE}}{a\cos\varphi}\frac{\partial v_{SE}}{\partial\lambda}\right)_Z +\right.$$

$$\frac{v_{SZ}}{a}\frac{\partial v_{SE}}{\partial\varphi} + \frac{v_{SE}}{a}\frac{\partial v_{SZ}}{\partial\varphi} + \frac{v_{SE}}{a}\frac{\partial v_{SE}}{\partial\varphi} - \left(\frac{v_{SE}}{a}\frac{\partial v_{SE}}{\partial\varphi}\right)_Z + \omega_{SZ}\frac{\partial v_{SE}}{\partial p} +$$

$$\left.\omega_{SE}\frac{\partial v_{SZ}}{\partial p} + \omega_{SE}\frac{\partial v_{SE}}{\partial p} - \left(\omega_{SE}\frac{\partial v_{SE}}{dp}\right)_Z\right]_M = -\frac{1}{a}\frac{\partial\Phi_{ME}}{\partial\varphi} - f\,u_{ME} -$$

$$\frac{u_{MZ}u_{ME}}{a}\tan\varphi - \frac{u_{ME}u_{MZ}}{a}\tan\varphi - \frac{u_{ME}^2}{a}\tan\varphi + \frac{(u_{ME}^2)_Z}{a}\tan\varphi - \frac{(u_{SZ}u_{SE})_N}{a}$$

$$\tan\varphi - \frac{(u_{SE}u_{SZ})_M}{a}\tan\varphi - \frac{(u_{SE}^2)_M}{a}\tan\varphi + \frac{(u_{SE}^2)_{MZ}}{a}\tan\varphi + F_{y,ME} \qquad (2.28)$$

The following step consists of multiplying (2.27) by u_{ME}, (2.28) by v_{ME} and adding the resulting equations. The resulting equation will express the local rate of change of the eddy kinetic energy: $k_{ME} = \frac{1}{2}(u_{ME}^2 + v_{ME}^2)$. By making use of the continuity equations we obtain the following equation:

$$\frac{\partial k_{ME}}{\partial t} + \frac{u_{MZ}}{a\cos\varphi}\frac{\partial k_{ME}}{\partial\lambda} + \frac{u_{ME}}{a\cos\varphi}\frac{\partial k_{ME}}{\partial\lambda} + \frac{v_{ME}}{a}\frac{\partial k_{ME}}{\partial\varphi} + \frac{u_{ME}v_{ME}}{a}\frac{\partial u_{MZ}}{\partial\varphi}$$

$$+ u_{ME} \left[\frac{1}{a \cos\varphi} \frac{\partial(u_{SZ}u_{SE})}{\partial\lambda} + \frac{1}{a} \frac{\partial(v_{SZ}u_{SE})}{\partial\varphi} - 2 \frac{\tan\varphi}{a} u_{SE}v_{SZ} \right.$$

$$+ \frac{1}{a \cos\varphi} \frac{\partial(u_{SE}u_{SE})}{\partial\lambda} + \frac{1}{a} \frac{\partial(u_{SE}v_{SE})}{\partial\varphi} -2 \frac{\tan\varphi}{a} u_{SE}v_{SE}$$

$$\left. + \frac{1}{a \cos\varphi} \frac{\partial(u_{SZ}u_{SE})}{\partial\lambda} + \frac{1}{a} \frac{\partial(u_{SZ}v_{SE})}{\partial\varphi} - 2 \frac{\tan\varphi}{a} u_{SZ}v_{SE} \right]_M$$

$$+ v_{ME} \left[\frac{1}{a \cos\varphi} \frac{\partial(v_{SE}u_{SZ})}{\partial\lambda} + \frac{1}{a} \frac{\partial(v_{SE}v_{SZ})}{\partial\varphi} = \frac{\tan\varphi}{a} v_{SE}v_{SZ} + \frac{\tan\varphi}{a} u_{SE}u_{SZ} \right.$$

$$+ \frac{1}{a \cos\varphi} \frac{\partial(v_{SE}u_{SE})}{\partial\lambda} + \frac{1}{a} \frac{\partial(v_{SE}v_{SE})}{\partial\varphi} - \frac{\tan\varphi}{a} u_{SE}u_{SZ} + \frac{\tan\varphi}{a} u_{SE}u_{SE}$$

$$\left. + \frac{1}{a \cos\varphi} \frac{\partial(v_{SZ}u_{SE})}{\partial\lambda} + \frac{1}{a} \frac{\partial(v_{SZ}v_{SE})}{\partial\varphi} - \frac{\tan\varphi}{a} v_{SE}v_{SZ} + \frac{\tan\varphi}{a} u_{SE}u_{SZ} \right]_M =$$

$$- \frac{1}{a \cos\varphi} u_{ME} \frac{\partial\Phi_{ME}}{\partial\lambda} - \frac{1}{a} v_{ME} \frac{\partial\Phi_{ME}}{\partial\varphi} - \frac{u_{ME}v_{ME}}{a} u_{MZ} \tan\varphi + u_{ME}F_{x,ME} +$$

$$v_{ME}F_{y,ME} \quad . \qquad (2.29)$$

The equation for dK_{ME}/dt is now obtained by integrating (2.29) over the region S employing the boundary conditions mentioned earlier. One could naturally be satisfied by simply listing the terms which would appear according to the form of (2.29). However, in order to distinguish between the terms depending essentially on the rotational and the irrotational part of the flow which in turn, according to our present experience, will be quantities of first and second order of magnitude, it is advantageous to rewrite Eq. (2.29) in terms of vorticity and divergence. This can be done by using a number of well known identities. Writing the terms in the new equations in the same order in which they appear in (2.29) we obtain:

$$\frac{\partial k_{ME}}{\partial t} + \frac{u_{MZ}}{a \cos\varphi} \frac{\partial k_{ME}}{\partial \lambda} + \frac{u_{ME}}{a \cos\varphi} \frac{\partial k_{ME}}{\partial \lambda} + \frac{v_{ME}}{a} \frac{\partial k_{ME}}{\partial \varphi} +$$

$$\frac{u_{ME} v_{ME}}{a} \cos\varphi \frac{\partial}{\partial \varphi} \left(\frac{u_{MZ}}{\cos\varphi}\right) + u_{ME} \left[\frac{1}{a \cos\varphi} \frac{\partial [u_{SZ} u_{SE} + v_{SZ} v_{SE}]}{\partial \lambda} -\right.$$

$$v_{SZ} \zeta_{SE} + u_{SE} \nabla \cdot \vec{v}_{SZ} + \frac{1}{a \cos\varphi} \frac{\partial [\frac{1}{2} (u_{SE}^2 + v_{SE}^2)]}{\partial \lambda} - v_{SE} \zeta_{SE} + u_{SE} \nabla \cdot \vec{v}_{SE}$$

$$\left. - v_{SE} \zeta_{SZ} + u_{SZ} \nabla \cdot \vec{v}_{SE}\right]_M + v_{ME} \left[\frac{1}{2} \frac{\partial [u_{SZ} u_{SE} + v_{SZ} v_{SE}]}{\partial \varphi} + u_{SZ} \zeta_{SE} + v_{SE} \nabla \cdot \vec{v}_{SZ}\right.$$

$$\left. + \frac{1}{a} \frac{\partial [\frac{1}{2} (u_{SE}^2 + v_{SE}^2)]}{\partial \varphi} + u_{SE} \zeta_{SE} + v_{SE} \nabla \cdot \vec{v}_{SE} + u_{SE} \zeta_{SE} + v_{SE} \nabla \cdot \vec{v}_{SE}\right]_M =$$

$$- \frac{1}{a \cos\varphi} u_{ME} \frac{\partial \Phi_{ME}}{\partial \lambda} - \frac{1}{a} v_{ME} \frac{\partial \Phi_{ME}}{\partial \varphi} + u_{ME} F_{x,ME} + v_{ME} F_{y,ME} \quad . \tag{2.30}$$

When Eq. (2.30) is integrated over the region S we make use of the fact that the windfield $\vec{v}_{ME}$ is nondivergent. The resulting equation can be written in the form

$$\frac{dK_{ME}}{dt} = C(K_{MZ}, K_{ME}) + C(K_{SZ}, K_{ME}) + C(K_{SE}, K_{ME}) +$$

$$C(K_{SE}, [K_{SZ}], K_{ME}) - D(K_{ME}) \ ,$$

where the symbols have their usual meaning except that the term $C(K_{SE}, [K_{SZ}], K_{ME})$ represent an energy conversion from K_{SE} to K_{ME} in which K_{SZ} participates although it remains unchanged. Such an energy conversion is called catalytic by Smagorinsky [7] in close analogy with the well known chemical reactions. The integral for $C(K_{MZ}, K_{ME}) = - C(K_{ME}, K_{MZ})$ has already been given in Eq. (2.21). The remaining integrals are:

$$C(K_{SZ},K_{ME}) = \frac{2\pi a^2}{g} \int_0^{p_0} \int_{\varphi_1}^{\varphi_2} [\{(u_{ME}\zeta_{SE})_Z v_{SZ} - (v_{ME}\zeta_{SE})_Z u_{SZ}\} -$$

$$\{(u_{ME}u_{SE})_Z + (v_{ME}v_{SE})_Z\} \nabla \cdot \vec{v}_{SZ}] \cos\varphi \, d\varphi \, dp \tag{2.32}$$

$$C(K_{SE},K_{ME}) = \frac{a^2}{g} \int_0^{p_0} \int_{\varphi_1}^{\varphi_2} \int_0^{2\pi} [(u_{ME}v_{SE}-v_{ME}u_{SE})\zeta_{SE} - (u_{ME}u_{SE} +$$

$$v_{ME}v_{SE}) \nabla \cdot \vec{v}_{SE}] \cos\varphi \, d\lambda \, d\varphi \, dp \tag{2.33}$$

$$C(K_{SE},[K_{SZ}],K_{ME}) = \frac{2\pi a^2}{g} \int_0^{p_0} \int_{\varphi_1}^{\varphi_2} [(u_{ME}v_{SE}-v_{ME}u_{SE})_Z \zeta_{SZ} -$$

$$\{(u_{ME} \nabla \cdot \vec{v}_{SE})_Z u_{SZ} + (v_{ME} \nabla \cdot \vec{v}_{SE})_Z v_{SZ}\}] \cos\varphi \, d\varphi \, dp \tag{2.34}$$

In order to complete all energy conversions it is necessary to derive the equations for $d\,K_{SZ}/dt$ and $d\,K_{SE}/dt$. This procedure is necessary because we can check the calculations leading to (2.20) and (2.31), and because we want to check that our interpretation of the catalytic energy conversion (2.34) is correct. It suffices to reproduce the main results

$$\frac{dK_{SZ}}{dt} = C(A_Z,K_{SZ}) + C(K_{SE},K_{SZ}) + C(K_{ME},K_{SZ}) + C(K_{MZ},K_{SZ}) - D(K_{SZ}) \tag{2.35}$$

and

$$\frac{dK_{SE}}{dt} = C(A_E,K_{SE}) - C(K_{SE},K_{SZ}) - C(K_{SE},K_{ME}) - C(K_{SE},K_{MZ})$$

$$- C(K_{SE},[K_{SZ}],K_{ME}) - D(K_{SE}) \; . \tag{2.36}$$

The first terms on the right-hand side are the conversions from the zonal and eddy available potential energy to the zonal and eddy shear flow kinetic energy, respectively. They have been calculated by Saltzman and Fleisher [5], [6] and Wiin-Nielsen [9]. The only other term which appears in (2.35) and (2.36) and which are not found in (2.20) or (2.31) is the energy conversion $C(K_{SE}, K_{SZ})$

which turns out to be

$$C(K_{SE},K_{SZ}) = \frac{2\pi a^2}{g} \int_0^{p_0} \int_{\varphi_1}^{\varphi_2} [(v_{SE}\zeta_{ME})_Z u_{SZ} - (u_{SE}\zeta_{ME})_Z v_{SZ} -$$

$$(\omega_{ME} \frac{\partial u_{SE}}{\partial p})_Z u_{SZ} - (\omega_{ME} \frac{\partial v_{SE}}{\partial p})_Z v_{SZ}] \cos\varphi \, d\varphi \, dp \quad . \qquad (2.37)$$

One would normally expect that an energy conversion like $C(K_{SE},K_{SZ})$ would contain quantities whith subscripts SE and SZ omly. In (2.37) we find, however, also the subscript ME. A similar remark can be made with respect to Eq. (2.32) expressing the energy conversion $C(K_{SZ},K_{ME})$. In addition to the subscripts SZ and ME we find also the subscript SE in (2.32). The reason for this special condition in (2.32) and (2.37) is that the zonal shear flow energy act as a catalyst for the energy conversion from K_{SE} to K_{ME}. The energy conversion, $C(K_{SE},K_{SZ})$ and $C(K_{SZ},K_{ME})$, given in our formulas (2.37) and (2.32) are in both cases the noncatalytic part of the total energy conversion. Hence the appearance of the additional subscripts. Smagorinksy [7] gives the necessary formulas to find the catalytic and noncatalytic parts of a given energy conversion.

Summarizing the results of the derivations of the energy conversions involving the four forms of kinetic energy we find a total of seven energy conversions given in Eq. (2.21), (2.22), (2.23), (2.32), (2.33), (2.34), and (2.37). The additional terms concerning a conversion from available potential energy or a frictional dissipation will not be considered in this paper. A schematic diagram given the seven energy conversions is shown in Figure 1, where the dashed line indicates the catalytic energy conversion. Only the integrand in the conversion integrals is shown in the diagram. The integration is carried out over the area S, where $dS = a^2 \cos\varphi \, d\lambda \, d\varphi$, and with respect to pressure, i.e., $\frac{1}{g}$ dp. The rectangular boxes in Figure 1 represent the amounts of energy while the hexagonal boxes show the energy conversions and the integrand which has to be used for the numerical evaluation of the integral.

It will be noticed from the formulas for the energy conversions and from Figure 1 that there are three different energy conversions. If any energy conversion requires any component of the mean meridional circulation, a divergence or a vertical velocity, it will be classified as a divergent component of the energy conversion and denoted by a subscript: D. The remaining parts of the energy conversions will be classified as nondivergent components and will have a subscript: ND. We find by inspection of the formulas that one energy conversion $C(K_{SZ},K_{MZ}) = C_D(K_{SZ},K_{MZ})$, while two conversions: $C(K_{ME},K_{MZ}) = C_{ND}(K_{ME},K_{MZ})$ and $C(K_{SE},K_{MZ}) = C_{ND}(K_{SE},K_{MZ})$ contain only nondivergent components. The remaining four kinetic energy conversions contain both nondivergent and divergent components. The nondivergent components will be present in a quasi-geostrophic model of the atmospheric motion, while we must use the primitive equations to incorporate the divergent components in a numerical integration.

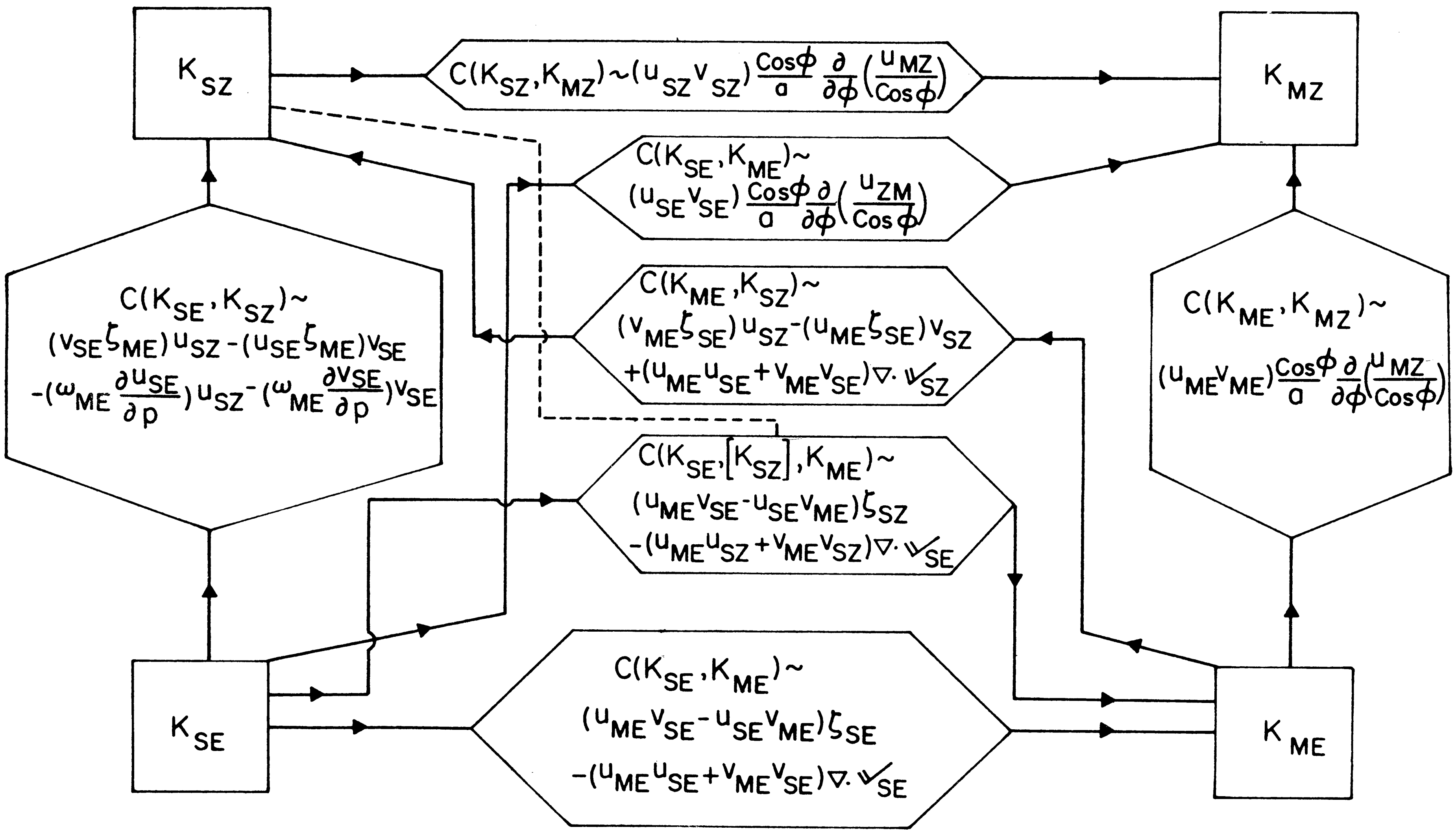

Figure 1. Schematic energy diagram showing all energy conversions between the four energy forms: K_{SZ}, K_{SE}, K_{MZ}, and K_{ME}. The hexagonal boxes show the energy conversion together with the integrand in the energy conversion integral.

The nondivergent energy components create no special problem in a calculation based on atmospheric data. Only height data are available for the calculations to be performed in this study. From the height data we will obtain a so-called geostrophic streamfunction ψ by solving the equation

$$\nabla^2\psi = \frac{1}{f}\nabla^2\Phi - \frac{1}{f^2}\nabla f \cdot \nabla\Phi \quad . \tag{2.38}$$

The vorticity will be computed as $\zeta = \nabla^2\psi$ while the wind components will be obtained from $u = -\frac{1}{a}\frac{\partial\psi}{\partial\varphi}$ and $v = \frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}$. The streamfunction ψ is first obtained by solving (2.38) at the five data levels mentioned in the introduction. The four components, ψ_{MZ}, ψ_{ME}, ψ_{SZ}, and ψ_{SE} are next computed using the definitions (2.1) to (2.4). The remaining calculations necessary to obtain all the components C_{ND} will be completed using a computational procedure as described by Wiin-Nielsen and Drake [14].

The energy conversions C_D require a knowledge of a vertical velocity, a divergence or the components v_{SZ} of the mean meridional circulation. In order to obtain a numerical estimate of the conversions C_D we will follow a procedure as outlined by Wiin-Nielsen and Drake [15] by computing the vertical velocity ω from the so-called ω-equation, obtain the horizontal divergence from the continuity equation and then calculate ω_{ME}, $\nabla\cdot\vec{v}_{SE}$ and $\nabla\cdot\vec{v}_{SZ}$ using the averaging procedures. Finally, the component v_{SZ} is obtained by integration of the continuity equation for the zonally averaged flow

$$\frac{1}{a\cos\varphi}\frac{\partial v_{SZ}\cos\varphi}{\partial\varphi} = \nabla\cdot\vec{v}_{SZ} \quad . \tag{2.39}$$

Equation (2.39) can be integrated starting from the North Pole where $v_{SZ}\cos\varphi = 0$.

REFERENCES

1. H. L. Kuo: "A Note on the Kinetic Energy Balance of the Zonal Wind Systems," Tellus, 3, pp. 205-207, 1951.

2. E. N. Lorenz: "Available Potential Energy and the Maintenance of the General Circulation," Tellus, 7, No. 2, pp. 157-167, 1955.

3. A. H. Oort: "On Estimates of the Atmospheric Energy Cycle," Monthly Weather Review, 92, No. 11, pp. 483-493, 1964.

4. B. Saltzman and A. Fleisher: "Spectrum of Kinetic Energy Transfer due to Large-Scale Horizontal Eddy Stresses" Tellus, 12, pp. 110-111, 1960.

5. B. Saltzman and A. Fleisher: "The Modes of Release of Available Potential Energy in the Atmosphere," Journal of Geophysical Research, 65, pp. 1215-1222, 1960.

6. B. Saltzman and A. Fleisher: "Further Statistics of the Modes of Release of Available Potential Energy," Journal of Geophysical Research, 66, pp. 2271-2273, 1961.

7. J. Smagorinsky: "General Circulation Experiments with the Primitive Equations, I-The Basic Experiment," Monthly Weather Review, 91, pp. 99-165, 1963.

8. V. P. Starr: "Note Concerning the Nature of the Large-Scale Eddies in the Atmosphere," Tellus, 5, pp. 494-498, 1953.

9. A. Wiin-Nielsen: "A Study of Energy Conversion and Meridional Circulations for the Large-Scale Motion in the Atmosphere," Monthly Weather Review, 87, pp. 319-332, 1959.

10. A. Wiin-Nielsen: "On Transformation of Kinetic Energy between the Vertical Shear Flow and the Vertical Mean Flow in the Atmosphere," Monthly Weather Review, 90, pp. 311-323, 1962.

11. A. Wiin-Nielsen: "Some New Observational Studies of Energy and Energy Transformations in the Atmosphere," Proceedings from WMO-IUGG Symposium on Research and Development Aspects of Longe Range Forecasting, Technical Note No. 66, 1964.

12. A. Wiin-Nielsen, J. A. Brown and M. Drake: "On Atmospheric Energy Conversions between the Zonal Flow and the Eddies," Tellus, 15, pp. 261-279, 1963.

13. A. Wiin-Nielsen, J. A. Brown, and M. Drake: "Further Studies of Energy Exchange between the Zonal Flow and the Eddies," Tellus, 16, pp. 168-180, 1964.

14. A. Wiin-Nielsen, and M. Drake: "On the Energy Exchange between the Baroclinic and Barotropic Components of Atmospheric Flow," Monthly Weather Review, 93, pp. 79-92, 1965.

15. A. Wiin-Nielsen and M. Drake: "The Contribution of Divergent Wind Components to the Energy Exchange between the Baroclinic and Barotropic Components," accepted for publication in Monthly Weather Review, 94, 1966.