ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

FINAL REPORT

RESEARCH IN GENERAL TOPOLOGY

By

R. L. WILDER

Project 2100

ORDNANCE CORPS, U. S. ARMY
CONTRACT DA-20-018-ORD-12738

October, 1953

FINAL REPORT

RESEARCH IN GENERAL TOPOLOGY

Introduction

The original motivation of the research that has been done on this project is to be found principally in the theory of optimum missile trajectories, although other possible applications that the investigators had in mind included the analysis of production and allocation, equilibrium of floating bodies and other phases of mechanics, as well as connections within mathematics, especially in analysis and geometry. As so frequently happens in such cases, the existing mathematical tools were in need of extensive building up, and it is not surprising that most of the time had to be spent on basic research of a mathematical nature. Work has centered particularly about certain aspects of algebraic topology and the Morse critical-point theory (as well as the relations between these); although a major part of the work on the project has been devoted to these, some application to the theory of optimum missile trajectories was begun.

In the more detailed summary given below, reference is made to the bibliography at the end of the report. There is also included an abstract of an unpublished paper which is referred to below and which is not yet on record elsewhere.

1. New Dualities for Manifolds

There is little doubt that the present state of critical-point theory is largely a result of the lack of the fullest and most efficient use of recently introduced topological methods. A specific case of this, worked on early in the course of the project, concerns certain inequalities relating the Betti numbers of manifolds and their subsets, announced by Morse in 1927[1] and published with proofs in 1952.[2] As remarked by Morse, such relations are "becoming increasingly important in questions of homology, homotopy and category" and "in studies related to the unsolved Schoenflies problem". Among the early results found during the course of the project was that these relations hold without the stringent analytic conditions imposed by Morse, as well as that, as a consequence of the methods used, the original inequalities form only a partial list of a complete set associated with certain exact sequences. An abstract concerning this work (which was presented at the New York meeting of the American Mathematical Society in April, 1953) may be found in the Bulletin of the American Mathematical Society.[3]

Further investigations along the same line revealed some new duality relations involving homology groups associated with a manifold with boundary. An abstract concerning these results is included at the end of this report.

2. The "Add" as a Topological Invariant

Another direction was taken in studying the relations between algebraic topology and critical-point theory. Recent work of Pitcher[4] employed critical-point theory to compute certain homotopy groups of spheres. It would seem plausible, therefore, to seek further applications of critical-point theory to general topology. One approach, which requires no analytic structure of the space S under consideration, would be to consider the class of all real-valued functions defined on S, defining an equivalence between functions in terms of their critical point or critical groups, and, following the model of homotopy and cohomotopy theory defining an addition between the related equivalence classes. The resulting structure turns out to be not a group but an add, as defined by R. Baer.[5] The add thus defined is a topological invariant. A more basic approach, however, is to deal with certain classes of filters of the space S, rather than classes of real-valued functions defined on S, and to define equivalence classes of filters and addition between equivalence classes.

This construction is described in detail in Reference 6. It has been shown that various add structures are capable of distinguishing certain manifolds undistinguished by their homology or homotopy groups. More significant results from the use of adds, e.g., in critical-point theories, must await further investigation, however.

3. Axiomatization of Critical-Point Theory

In order to apply Morse theory to as general a situation as possible, both in general topology and in variational problems, the foundations of the theory were investigated. The result was a new axiomatization of Morse theory. The point of view adopted was based on the consideration of a triple (S, f, H) where S is a space, f is a mapping of S into the reals, and H is a homology theory which is "admissible" on S and f. The axioms are stated in terms of properties of the homology theory H on the category of pairs arising from the function on S. This approach generalizes previous presentations and serves to clarify the relations between various levels of and approaches to Morse theory.

Several new theorems of interest were obtained. In particular, the "strong Morse relations" were stated in terms of isomorphisms of appropriate groups, and were derived without any assumption of finiteness of Betti numbers of the space or of the number of critical points of the function. Another theorem of particular interest in its application is the following:

> If S is a Hausdorff space, f an arbitrary mapping into the reals such that for each real number c the set of all points whose functional values do not exceed c is compact, and S is f-reducible, then with Cech homology theory, (1) the strong and weak Morse relations hold and (2) there exists a critical set at each critical value.

Applications of this theorem (cf Reference 6) were made to the theory of minimal surfaces as well as to optimum trajectory theory, described below.

The work described in this section will be presented at the November, 1953, meeting of the American Mathematical Society in Pasadena, California, and together with material in Section 2 was presented in the form of a doctoral dissertation by Mr. J. P. Roth this summer.

4. Applications to the Theory of Optimum Trajectories of Guided Missiles

The theorem above has been applied to the space of trajectories of a guided missile satisfying certain linearized equations of motion. In particular, it has been shown that under very weak conditions an optimum trajectory always exists. It has also been shown that all the Morse relations hold.

This result has been very recently extended to the nonlinearized case, but not all the details have been carried through yet. Unexpectedly, these results were obtainable without the development of a global theory for the problem of Bolza, due to the fact that the corresponding variational problem is of a very special type, which permitted the above described analysis to go through.

5. Uncompleted Work

The development of the global theory for the problem of Bolza remains to be carried out. Furthermore, the underlying algebraic structure of critical-point theory presages many additional results from the application to topological problems. Recent work of Serre, Leray, and Massey strongly suggest this.

ABSTRACT

DUALITIES FOR MANIFOLDS WITH BOUNDARY

By

R. L. Wilder and J. P. Roth

The following theorems on generalized n-dimensional manifolds with boundary are established:

I. Let T be the union of an open n-dimensional orientable generalized manifold S, uniformly locally connected in all dimensions up to and including n-1, and its boundary M. Let C be a closed subset of S. Then the kernel of the injection $H^r(C) \to H^r(S)$ is _link_-isomorphic to the kernel of the injection $h^{n-r-1}(S-C) \to h^{n-r-1}(S)$.

II. Under the same hypotheses as in I, the kernel of the injection $H^r(M) \to H^r(T)$ is link-isomorphic to a subgroup of $h^{n-r-1}(S)$.

By means of I and II we are able to establish as a sequel to the authors' previous paper[3] the following generalization of Corollary 7.4 of Morse.[2]

III. Let T be as in I. Let M_1 be an orientable (n-1)-generalized closed manifold contained in S. Let S_1 be the domain in S complementary to M_1 whose complete boundary is M_1. Let S_2 be the other domain in S complementary to M_1; its boundary is $M \cup M_1$. Let the closures of S, S_1, S_2, be denoted respectively by T, T_1, T_2; then

$$H^{n-r-1}(T) \oplus H^r(T_2) \;\tilde{\supset}\; H^{n-r-1}(T_1) \oplus H^r(M) ,$$

where $\tilde{\supset}$ means "contains a subgroup isomorphic to".

BIBLIOGRAPHY

1. M. Morse, "The Analysis and Analysis Situs of Regular n-Spreads in (n + r)-Space", Proc. Nat. Acad. Sci. 13, 813-817 (1927).

2. M. Morse, "Homology Relations on Regular Orientable Manifolds", Proc. Nat. Acad. Sci. 38, 247-258 (1952).

3. R. L. Wilder and J. P. Roth, "On Certain Inequalities Relating the Betti Numbers of a Manifold and its Subsets", Bull. Amer. Math. Soc., July, 1953, p. 371; abstract no. 439.

4. E. Pitcher, "Homotopy Groups of the Space of Curves with Applications to Spheres," Proc. Int'l Congress of Math'ns, 1, 528-529 (1950).

5. R. Baer, "Free Sums of Groups and Their Generalizations, II", Amer. J. of Math., 72 625-670 (1950).

6. J. P. Roth, "An Investigation in Algebraic Topology and Morse Theory", PhD thesis, University of Michigan, 1953.