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THE STABILITY OF A GAS CONFINED BETWEEN TWO ROTATING CYLINDERS

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TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	ii
LIST OF TABLES	iv
LIST OF FIGURES	v
NOMENCLATURE	vi
I INTRODUCTION	1
II MATHEMATICAL DESCRIPTION	15
III RESULTS AND CONCLUSIONS	43
IV POSSIBLE EXTENSIONS	68
APPENDICES	
I SOLUTION PROCEDURE	73
II COMMENTS ON THE CHARACTER OF THE EIGENVALUES	86
LIST OF REFERENCES	93

LIST OF TABLES

Table		Page
I	NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	45
II	NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING FOR $\Delta_\theta = 0$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	50
III	NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL ANGULAR VELOCITY FOR $\Delta_r = .1$, $\Delta_\theta = 0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	54
IV	NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity	56
V	NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = +.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	56
VI	NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = +.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity Included	57
VII	NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = .1$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity Included	60
VIII	NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	62
IX	NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 1$. With Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	65

LIST OF FIGURES

Figure		Page
1.	Physical Configuration	18
2.	Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	46
3.	Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	47
4.	Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	48
5.	Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	51
6.	Neutral Stability Reynolds Number vs Differential Angular Velocity for $\Delta_\theta = 0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity	55
7.	Neutral Stability Reynolds Number vs Differential Temperature for $\Delta_r = 10^{-4}$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms Included. No Variation of Viscosity and Thermal Conductivity	58
8.	Percent Variation of Neutral Stability Reynolds Number with Thermal Variation of Viscosity and Thermal Conductivity vs Differential Temperature for $\bar{N}_M = 10^{-5}$. No Dissipation Terms Included	61
9.	Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1$. No Variation of Viscosity and Thermal Conductivity	63

NOMENCLATURE

a	=	wave number
$\hat{A}_\mu$	=	$\frac{\mu(\hat{\theta})}{\mu_I}$
$\bar{A}_\mu$	=	$\hat{A}_\mu(\bar{\theta}) = \hat{A}_\mu$ evaluated for $\hat{\theta} = \bar{\theta}$
A'_μ	=	$\frac{d\hat{A}_\mu}{d\hat{\theta}}$ evaluated for $\hat{\theta} = \bar{\theta}$
$\hat{A}_k$	=	$\frac{k(\hat{\theta})}{k_I}$
$\bar{A}_k$	=	$\hat{A}_k(\theta) = \hat{A}_k$ evaluated for $\hat{\theta} = \bar{\theta}$
A'_k	=	$\frac{d\hat{A}_k}{d\theta}$ evaluated for $\hat{\theta} = \bar{\theta}$
b_1, b_2	=	integration constants
c_1, c_2	=	integration constants
c_p	=	specific heat at constant pressure per unit mass
c_v	=	specific heat at constant volume per unit mass
D_r	=	$\frac{\partial}{\partial r}$ and $\frac{d}{dr}$
$D_r^+ f$	=	$D_r f + \frac{f}{r} = \frac{1}{r} D_r(rf)$
$D_r^- f$	=	$D_r f - \frac{f}{r} = r D_r(\frac{f}{r})$
D_t	=	$\frac{\partial}{\partial t}$
D_x	=	$\frac{d}{dx}$
D_z	=	$\frac{\partial}{\partial z}$ and $\frac{d}{dz}$
D_ϕ	=	$\frac{1}{r}\frac{\partial}{\partial \phi}$ and $\frac{1}{r}\frac{d}{d\phi}$
E	=	$e^{\sigma t}$
I	=	indicates numerical integration operator
I	=	subscript representing inner cylinder
k	=	thermal conductivity
L	=	$D_r D_r^+ - a^2$ operator

$\bar{N}_R$ = $\dfrac{\tilde{\omega}_I \tilde{r}_I^{2} \tilde{\rho}_A}{\mu_I}$ a Reynolds number

N_R = $\bar{N}_R \Delta_r^{2}$ a Reynolds number

$\bar{N}_M$ = $\dfrac{\tilde{\omega}_I \tilde{r}_I^{2}}{(R\tilde{\theta}_I)^{1/2}}$ a Mach number

N_M = $\bar{N}_M \Delta_r$ a Mach number

N_{Pe} = $N_{Pr} \cdot N_R$ a Peclet number

N_{Pr} = $\dfrac{\mu c_p}{k}$ the Prandtl number

O = subscript representing outer cylinder

p = pressure

r = radial coordinate

s = entropy

R = gas constant

t = time

T = θ, temperature

u = radial component of velocity

U = $N_R u$

v = tangential component of velocity

w = axial component of velocity

x = $r - \Delta_r^{-1}$

y = $1 - x$

z = axial coordinate

α, β = special combining forms

γ = isentropic exponent

μ = viscosity

ϕ = angular coordinate

θ = temperature

ρ = density

σ = growth factor

Δ_r = spacing

Δ_ω = differential angular velocity

Δ_θ = differential temperature

$\tilde{v}$ wave over symbol represents dimensional form

$\hat{v}$ circumflex over symbol represents dimensionless form

$\bar{v}$ = bar over variable represents mean or stationary flow variable

v' prime on symbol represent perturbed flow variable

v^* star on symbol represents the axial part of perturbed variable

v = symbol without appendages represents the radial part of the perturbed variable

CHAPTER I

INTRODUCTION

In the following, the problem of a gas confined between two infinitely long concentric cylinders will be discussed. One or both of the cylinders are rotating. The flow of the gas is considered to be viscous. The purpose of this problem is to determine whether the initial mean laminar flow is stable or unstable with respect to the imposed conditions.

This is a variation on the problem discussed by Taylor[12] in his 1923 paper. There are, however, several distinct differences between the work of Taylor and the others who have followed him, and this work. First, the spacing between the cylinders is considered to be variable from very small to moderately wide, whereas most of the previous work has considered only very narrow spacing. Second, the fluid is considered to be a gas, that is, it is compressible and behaves according to the perfect gas law. The flow at Mach numbers up to one is considered. Third, the variation of the viscosity with temperature is included along with the variations of thermal conductivity with temperature. Last, the effect of viscous dissipation of mechanical energy into thermal energy is also included.

A method of solution differing from previous methods is used on this problem. This method is primarily a numerical method. However, it is based on a twenty-five term power series expansion of the variables rather than on a finite difference approximation. The detailed description of this method is given in Appendix I.

History

Previous to Taylor's time, the problem of the stability of a rotating fluid was considered by such people as Reynolds, Couette, and principally by Lord Rayleigh. These works may be considered as preliminary and inconclusive. Taylor's work, on the other hand, was quite complete. He considered a liquid between two rotating cylinders, the liquid to be viscous, and the spacing, in general, to be unspecified. Although he established a method of solving the problem under these conditions, he finally obtained a solution of the problem only in the case of small spacing between the cylinders. The large spacing problem was very formidable, involving determinants of high order with very complicated co-efficients. In addition to the analysis, Taylor performed a series of detailed experiments covering the range of variables discussed in his theory. He covered a moderate range of relative speeds between the two cylinders.

In the past 40 years, many people have attempted to clarify this problem, in particular, Low(7), and Jeffreys (5) showed that there is a direct analogy between this problem and the problem of a convecting fluid heated from below and confined between two flat plates of infinite extent. This analogy is only complete if in the rotating cylinder problem the spacing is small and both cylinders rotate in the same direction at almost the same speed. In 1940, Pellew and Southwell(8) published a paper treating these problems in a very simple way. The method they used was an integration method wherein they were able to show, in particular, that for these special cases, the imaginary part of the growth factor is zero whenever the real part of the growth factor is zero. This point is discussed further in Chapter II. In 1954 Chandrasekhar(1) discovered a particularly simple means of handling this type of problem when the spacing between the cylinders is small. His approach met with remarkable success, in that the determinental equation which results from his method could be solved to a high degree of accuracy with only a one by one determinant. However, Chandrasekhar did carry his calculation to as high as fourth order approximations. This approach was particularly suitable for finding the stability criteria for cases varying from no relative motion of the two cylinders up to quite large relative motion of the two cylinders. Many problems similar to this have been treated in the last few years by this method. It is a very powerful method.

In 1958 Chandrasekhar(2) further advanced the knowledge of this subject by covering the problem for a wide spacing between the cylinders instead of the narrow spacing. He approached this by a method similar to the one discussed above with the exception that he used Bessel functions in the place of the trigonometric functions used previously. At the same time Donelly and Fultz(3) performed a series of experiments which conclusively demonstrated the validity of Chandrasekhar's conclusions. Their work, however, was done only for the case for which the inner cylinder was half the radius of the outer.

Up until this time, nobody had considered the effects of temperature in the rotating cylinders problem. Clearly, the temperature plays an important role, in that temperature affects the density of the fluid and hence the propensity to become unstable. In 1961 Yih(13) discussed the role of temperature as it affected the stability of a liquid between rotating cylinders. Yih's work showed one additional factor; in some special cases viscosity could play a destabilizing role as well as a stabilizing role.

Concurrent with the developments discussed above was another series of studies, concerned with the stability of flow around blunt bodies, in particular, the stability of flows across concave and convex surfaces. In 1940, Görtler[4] showed that an instability pattern developed on concave surfaces very similar to that between two rotating cylinders. The principal difference between the flow conditions considered by Görtler and those treated by the others is the absence of the rotating cylinders, and in particular, only one solid surface was considered. The flows are similar, however, because of the curvature of the flow. In 1958, Lees[6] discussed variations on the Görtler flow with emphasis on convex surfaces. He discussed certain sufficient conditions for stability which were a variation upon those discussed by Rayleigh[9] and Synge[11]. Lees states that cooling the convex surface is never destabilizing.

Yih[13] showed in his 1961 paper that for a gas it is not sufficient to talk about the circulation gradient, but it is necessary to consider the entropy gradient as well. He stated that, in addition to the positive circulation gradient, a negative entropy gradient is necessary to assure stability of the flow. One of the main purposes of this research is to show the effect of the gaseous state, and hence the aforementioned entropy gradient, upon the stability of the flow.

Physical Description, General

Throughout this study, the walls of the two cylinders are chosen to be infinitely conducting; that is, the gas at the walls will follow exactly the temperature applied to the walls. The total mass of the gas remains constant for all variations of the various parameters; that is, no gas is allowed to enter or leave the region between the cylinders.

The usual measure of the stability of a fluid in a problem of this kind is the Taylor number, but because of the large variety of parameters studied herein, it is not suitable to represent all cases. Consequently, the Reynolds number will be used instead as a criterion of the relative stability. In all cases, the Reynolds number will be based on the velocity (angular velocity) of the inner cylinder, the radius of the inner cylinder, the viscosity of the gas at the inner cylinder and the mean density of the gas. The mean density of the gas corresponds to the density of the gas at the inner cylinder with no motion and no temperature gradient.

Of the several parameters to be used in the analysis, four are of primary importance to the preliminary discussion. The spacing denoted by Δ_r is the spacing between the outer and inner cylinder divided by the radius of the inner cylinder. The differential angular velocity symbolized by Δ_ω is the angular velocity of the outer cylinder minus that of the inner divided by the angular velocity of the inner cylinder. In a similar way, the differential temperature Δ_θ is defined to be the temperature of the outer cylinder minus the inner divided by the inner absolute temperature. The Mach number denoted by $\overline{N}_M$ is defined as the velocity of the inner cylinder divided by the square root of the gas constant multiplied by the absolute temperature of the inner cylinder.

A gas, rather than a liquid, being used presents several physical variations to be considered. The primary difference is that gases, in the presence of temperature or pressure variations, have rather large changes in relative density as compared to liquids. In addition, the pressure disturbances are transferred from one part of a liquid to another more rapidly than in a gas, because of the lower speed of sound in a gas. Thermal disturbances are assimilated rapidly in a gas because of the higher thermal diffusivity.

The viscosity usually is much lower in a gas than in a liquid. However, since a gas is much lighter than a liquid, the kinematic viscosity is larger for gas than for water. This implies that velocity disturbances are distributed more readily in a gas than in some liquids. However, the ratio of the momentum diffusivity to the thermal diffusivity, the Prandtl number, is less for a gas than for most liquids. The ultimate effects of these differences on the stability of a fluid is not clear from these considerations alone. A detailed description of the mechanism of stability, given subsequently, is necessary to evaluate the importance of these factors.

One other characteristic of a gas is important to this study. For a gas, the viscosity increases with temperature whereas viscosity decreases with temperature for a liquid. No effects of viscosity variations with temperature for a liquid will be discussed. The thermal conductivity of a gas also increases in approximately the same way as the viscosity with temperature. Gravity is neglected throughout the whole of this work.

Physical Description, Specific

The word stable is used to refer to the conditions of flow such that whenever an infinitesimal disturbance is applied to that flow, the disturbance will decay in time. The word unstable is used for a flow whose disturbed state grows with time. A flow that is stable no matter how large a Reynolds number is applied is referred to as absolutely stable (it is understood that only axisymmetric disturbances are considered), whereas flow that is only stable for a sufficiently low Reynolds number and unstable for a sufficiently high Reynolds number is called labile or conditionally stable. The numerical part of this work is concerned with the location of lines (or surfaces) of neutral stability which are lines (or surfaces) separating a region of stability from one of instability in a region of lability. In some cases, the locations of lines of lability, the lines between the regions of absolute stability and lability, are discussed.

To begin with, consider the flow of an inviscid liquid between the walls of two rotating cylinders. The physical reasoning of Rayleigh(9) proceeds along the following lines: If a ring of fluid initially at radius r_1 and having a tangential velocity v_1 is displaced to a radius r_2 having a local velocity v_2 , $r_2 > r_1$, will the disturbed ring continue in the same direction or will it tend to return to its initial position? The angular momentum of this ring will remain constant during its travels, since the absence of viscosity implies that no torques are applied to it. The fluid will, however, take a new velocity v_1' consistant with the angular momentum. Therefore, $\rho\, r_1\, v_1 = \rho\, r_2\, v_1'$ implying $v_1' = \frac{r_1}{r_2} v_1$. The centrifugal force field in the region of r_2 is $\rho \frac{v_2^2}{r_2}$ while the centrifugal force required to hold the displaced ring in place is $\rho \frac{(v_1')^2}{r_2}$. Then, the differential force on the ring is $\delta F = \frac{\rho}{r_2^3} (r_2^2 v_2^2 - r_1^2 v_1^2)$. If this force is positive the ring will tend to return to its original position, whereas, if it is negative an unstable condition will be present. This leads to the well known Rayleigh condition, that is, for an inviscid liquid the flow will be stable if and only if the gradient of the square of the circulation (rv) (or angular momentum) is positive.

Synge[11] has shown that for a viscous fluid a positive circulation gradient is also a sufficient condition for stability; in other words, viscous flow is absolutely stable in the presence of a positive circulation gradient. All fluids are viscous to some degree and only begin to look inviscid at high Reynolds numbers. The assumption of an inviscid fluid implies an infinite Reynolds number. What is the effect of the addition of viscosity to the fluid? The principal roles of viscosity are the transport of momentum in the presence of shearing flow and the transformation of mechanical energy into thermal energy. These tend to diffuse the momentum of the displaced ring, making it more nearly like that of its neighbors. In effect, this reduces the magnitude of force δF . For the case of a negative circulation gradient where the flow is unstable in the absence of viscosity this effect is stabilizing, and it has been demonstrated both experimentally and analytically (see authors quoted earlier) that for sufficiently low Reynolds numbers, the flow is indeed stable. The flow is unstable, however, for high Reynolds numbers. The region of negative circulation gradient is, therefore, a region of lability in which the viscosity is stabilizing. The region of lability for a viscous fluid, corresponds to the region of instability for an inviscid fluid.

In the region of positive circulation gradient, the magnitude of the force δF is again reduced by the addition of viscosity and the consequent assimilation of the displaced ring. However, the flow in this region is absolutely stable since δF is a restoring force; even the largest Reynolds number does not cause instability. Therefore, increasing the viscosity reduces the tendency to be absolutely stable, destabilizing the flow. The flow will never become unstable, since only an infinitely viscous fluid would reduce δF to zero. The region of absolute stability for a viscous fluid corresponds to the region of stability for an inviscid fluid.

Yih(13) showed that in the presence of other destabilizing influences, such as a positive temperature gradient, increasing the viscosity could be destabilizing. It is necessary to note here that this phenomenon is only realistic in the case of a liquid where the viscosity and thermal conductivity are relatively independent. In a gas, the thermal conductivity and the viscosity vary together yielding a Prandtl number of very nearly the same value for most gases, under widely varying conditions. This point will be discussed further in the next chapter.

Before going on to the temperature effects, a word should be mentioned about the effect of the change in differential angular velocity Δ_ω and spacing Δ_r. The Rayleigh criterion, in the case of a liquid, with no temperature effects included, reduces to the condition $(1 + \Delta_\omega)(1 + \Delta_r)^2 > 1$ for absolute stability. Thus, for infinitesimal spacing, the transition from the condition of absolute stability to lability occurs at $\Delta_\omega = 0$ as Δ_ω goes from positive to negative values.

As the value of Δ_ω becomes more negative, the circulation gradient becomes more negative, therefore, the force δF is more destabilizing. This means the Reynolds number, at the point of neutral stability, is decreased. This is verified by the work of Chandrasekhar(1), but is obscured by the fact that he discusses the Taylor number

$$N_T = -4\,\bar{N}_R^2\,\Delta_r^2\,\frac{(1+\Delta_\omega)(1+\Delta_r)^2 - 1}{\Delta_r(2+\Delta_r)}$$

which increases in this region.

At the value of $\Delta_\omega = -1$, the outer cylinder changes direction, as Δ_ω is reduced below minus one. The flow near the outer cylinder is now opposite to that near the inner cylinder, which gives rise to a ring of stagnation, zero velocity, somewhere between the two cylinders. This, of course, causes the flow to have a positive circulation gradient in the region between the stagnation ring and the outer cylinder. By analogy to the previous discussion, it is apparent that the flow is beset with both stabilizing and destabilizing influences. Therefore, the decreasing stability that is apparent in the region $0 \leq \Delta_\omega \leq -1$ is offset by a stabilizing influence in the region $-1 < \Delta_\omega < -\infty$. The work of Chandrasekhar shows that there is indeed a stabilizing effect in this region. A plot of this effect will be presented in Chapter III.

The form of the Rayleigh criterion previously given, $(1+\Delta_\omega)(1+\Delta_r)^2 > 1$, implies that as Δ_r becomes larger a more negative value of Δ_ω is allowable for absolute stability. The circulation gradient decreases even though the difference in angular velocity remains the same. The lability line, in the absence of gaseous and temperature effects, is a parabola defined by $(1+\Delta_\omega)(1+\Delta_r)^2 = 1$ in the Δ_r-Δ_ω plane, in a region of positive Δ_r (negative Δ_r is physically meaningless). The region of absolute stability extends into the area of negative Δ_ω as $\Delta_r \to \infty$ as far as $\Delta_\omega = -1$. Therefore, in the region $0 \geq \Delta_\omega \geq -1$ an increase in spacing can cause the flow to go from a labile condition to a condition of absolute stability.

In the region below $\Delta_\omega = -1$, the Rayleigh criterion does not have such a simple form since $\overline{v}$, the angular component of velocity, changes sign between the cylinder walls. However, an increase in spacing probably will not affect the character of the flow as described in a previous paragraph.

Another effect of spacing is apparent. The restraining influence of the walls on the disturbed flow, as is felt through the viscous effects, is lessened as Δ_r increases. Therefore, the wider spacing is destabilizing.

There are two influences then: the stabilizing effect as exemplified by the Rayleigh criterion which has its origins in inviscid analysis, and the destabilizing effect which comes from the relative viscous effects. For a given value of Δ_ω an increase in the spacing reduces the circulation gradient but it also reduces the damping. Therefore, the stability of the flow is affected by conflicting factors, and no conclusion can be reached here as to which will be dominant.

It was mentioned before that the presence of a positive temperature gradient would be destabilizing. It is easy to see that with a higher temperature at the outer wall than inner wall, the density would decrease with increasing radius. The cooler, denser fluid near the inner wall would tend to go outward in manner analogous to a fluid heated from below in the presence of a gravitational field. In this case, of course, the motivating force is the centrifugal force rather than gravity. A negative temperature gradient will have the opposite effect, that is, it will stabilize the flow.

The response to temperature is modified by the rate at which the thermal disturbance is assimilated to the local surroundings. If the fluid has a high thermal conductivity, or alternately, thermal diffusivity, the effect of a temperature disturbance will be small. In a gas, however, the thermal conductivity is relatively small; therefore, it is probable that a thermal disturbance will have a significant influence. An increase in thermal assimilation will be destabilizing whenever the temperature gradient is stabilizing and stabilizing whenever the temperature gradient is destabilizing. The effects of k, the thermal conductivity, changes are modified, of course, by similar changes in μ, the viscosity.

At this point it is necessary to enter a caveat. Although the mean density gradient is influential with respect to stability of the flow, it is not entirely responsible for the subsequent effects. The mean temperature gradient, which causes the density variation, is an important factor. This discussion will not be continued here, but will be pursued in Chapters II and III where reference can be made to equations and results of calculations.

Up to this point, the discussion has concerned a liquid. No changes in the preceding statements have to be made for a gas, but a few have to be added. The principal feature of a gas is its compressibility. As the cylinders rotate at higher rates, more of the gas will be concentrated at the outer cylinder increasing the density there. It would seem as if this has a stabilizing influence because it places the heaviest gas at the lowest part of the centrifugal field. The stabilizing influence is not a strong one, however, for two reasons. A large Mach number or spacing is required before a significant density difference is achieved, and a pressure disturbance is transmitted so readily, any resultant δF is quickly dispersed. The diffusion is so rapid, however, that it occurs approximately isentropically, which leaves a small residual density difference which has a small stabilizing effect. Since the transfer occurs almost isentropically, the entropy gradient, rather than the density gradient, determines whether or not δF is stabilizing or destabilizing.

The effects of the increase of viscosity with temperature are twofold. An increase in temperature causes an increase in the diffusive and the dissipative effects of viscosity, which increases the rate at which the disturbed ring assimilates with the local conditions. This, as noted before, stabilizes the flow in labile regions but destabilizes in regions of absolute stability. A decrease in temperature will have the opposite effect.

The dissipative effects of viscosity are manifested by an increase in the mean temperature and a change in disturbed temperature. The energy for this increase is obtained from the kinetic energy be means of the shearing action of the fluid. The kinetic energy is continually being replaced, of course, by a transfer of energy from the cylinder walls. (As will be noted in the next chapter, the inclusion of dissipation in this study affects only the energy equation not the momentum equations.) Since the walls hold the temperature of the fluid at the walls to the applied temperature, the temperature increase, arising from the dissipative effects, has a maximum somewhere between the two cylinders. This implies a positive temperature gradient near the inner cylinder which is destabilizing, and negative near the outer cylinder. Whichever of these effects is dominent depends on other factors in the flow.

Before going further with discussions of this kind, it is convenient to have at hand the equations governing the flow of the fluid and of temperature for reference and comparison to physical interpretations.

CHAPTER II

MATHEMATICAL DESCRIPTION

The basic equations used for describing the flow of fluid are:

the Navier Stokes equations,

the equation of continuity,

the constitutive relationship (perfect gas law),

the general energy equation.

The primary source for these equations has been Schlichting(10). Throughout the subsequent work, the volume viscosity is considered to be zero. The gas is considered to be dense, not rarified, but at normal pressures when no rotation is present (i.e., the gas is a continuum). Included in this derivation are the effects upon the flow caused by variations of viscosity and the thermal conductivity with temperature. However, variations of the specific heat and the perfect gas constant with temperature are neglected. This is justified by noting that the viscous and thermal conductivity variations are much stronger in ordinary gases, including air, than the variations in specific heat, and the gas constant. No effect of pressure on any of these coefficients is included. Explicity then, the following equations form the mathematical basis for this problem.

$$\begin{aligned}
\tilde{\rho}\left[\frac{\partial\tilde{u}}{\partial\tilde{t}} + \tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{r}} + \tilde{v}\frac{1}{r}\frac{\partial\tilde{u}}{\partial\tilde{\phi}} + \tilde{w}\frac{\partial\tilde{u}}{\partial\tilde{z}} - \frac{\tilde{v}^2}{\tilde{r}}\right] &= \\
-\frac{\partial\tilde{p}}{\partial\tilde{r}} &+ \frac{\partial}{\partial\tilde{r}}\left[\mu\left(2\frac{\partial\tilde{u}}{\partial\tilde{r}} - \frac{2}{3}\left(\frac{\partial\tilde{u}}{\partial\tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} + \frac{\partial\tilde{w}}{\partial\tilde{z}}\right)\right)\right] \\
&+ \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{\phi}}\left[\mu\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{\phi}} + \frac{\partial\tilde{v}}{\partial\tilde{r}} - \frac{\tilde{v}}{\tilde{r}}\right)\right] + \frac{\partial}{\partial\tilde{z}}\left[\mu\left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{r}}\right)\right] \\
&+ 2\frac{\mu}{r}\left[\frac{\partial\tilde{u}}{\partial\tilde{r}} - \frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} - \frac{\tilde{u}}{\tilde{r}}\right] .
\end{aligned} \tag{1}$$

$$\begin{aligned}
\tilde{\rho}\left[\frac{\partial\tilde{v}}{\partial\tilde{t}} + \tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{r}} + \frac{\tilde{v}}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} + \tilde{w}\frac{\partial\tilde{v}}{\partial\tilde{z}} + \frac{\tilde{v}\tilde{u}}{\tilde{r}}\right] &= \\
-\frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial\tilde{\phi}} &+ \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{\phi}}\left[\mu\left(\frac{2}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} - \frac{2}{3}\left(\frac{\partial\tilde{u}}{\partial\tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} + \frac{\partial\tilde{w}}{\partial\tilde{z}}\right)\right)\right] \\
&+ \frac{\partial}{\partial\tilde{z}}\left[\mu\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{\phi}} + \frac{\partial\tilde{v}}{\partial\tilde{z}}\right)\right] + \frac{\partial}{\partial\tilde{r}}\left[\mu\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{\phi}} + \frac{\partial\tilde{v}}{\partial\tilde{r}} - \frac{\tilde{v}}{\tilde{r}}\right)\right] \\
&+ \frac{2\mu}{\tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{\phi}} + \frac{\partial\tilde{v}}{\partial\tilde{r}} - \frac{\tilde{v}}{\tilde{r}}\right] .
\end{aligned} \tag{2}$$

$$\begin{aligned}
\tilde{\rho}\left[\frac{\partial\tilde{w}}{\partial\tilde{t}} + \tilde{u}\frac{\partial\tilde{w}}{\partial\tilde{r}} + \frac{\tilde{v}}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{\phi}} + \tilde{w}\frac{\partial\tilde{w}}{\partial\tilde{z}}\right] &= \\
-\frac{\partial\tilde{p}}{\partial\tilde{z}} &+ \frac{\partial}{\partial\tilde{z}}\left[\mu\left(2\frac{\partial\tilde{w}}{\partial\tilde{z}} - \frac{2}{3}\left(\frac{\partial\tilde{u}}{\partial\tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{\phi}} + \frac{\partial\tilde{w}}{\partial\tilde{z}}\right)\right)\right] \\
&+ \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left[\mu\,\tilde{r}\left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{r}}\right)\right] + \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{\phi}}\left[\mu\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{\phi}} + \frac{\partial\tilde{v}}{\partial\tilde{z}}\right)\right] .
\end{aligned} \tag{3}$$

$$\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\,\tilde{\rho}\,\tilde{u})}{\partial\tilde{r}} + \frac{1}{\tilde{r}}\frac{\partial(\tilde{\rho}\,\tilde{v})}{\partial\tilde{\phi}} + \frac{\partial(\tilde{\rho}\,\tilde{w})}{\partial\tilde{z}} = 0 . \tag{4}$$

$$\frac{\tilde{p}}{\tilde{\rho}} = R\tilde{\Theta} \; . \qquad (5)$$

$$C_p \, \tilde{\rho} \, \frac{D\tilde{\Theta}}{D\tilde{t}} = \frac{D\tilde{p}}{D\tilde{t}} + \nabla \cdot (k \, \nabla \tilde{\Theta}) + \mu \, \tilde{\Phi} \; , \qquad (6)$$

in which

$$\frac{D}{D\tilde{t}} = \frac{\partial}{\partial \tilde{t}} + u \frac{\partial}{\partial \tilde{r}} + \frac{\tilde{v}}{\tilde{r}} \frac{\partial}{\partial \tilde{\phi}} + \tilde{w} \frac{\partial}{\partial \tilde{z}} \; ,$$

$$\nabla \cdot (k \nabla \tilde{\Theta}) = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \, k \frac{\partial \tilde{\Theta}}{\partial \tilde{r}}\right) + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{\phi}} \left(\frac{k}{\tilde{r}} \frac{\partial \tilde{\Theta}}{\partial \tilde{\phi}}\right) + \frac{\partial}{\partial \tilde{z}} \left(k \frac{\partial \tilde{\Theta}}{\partial \tilde{z}}\right) \; ,$$

$$\tilde{\Phi} = 2 \left[\left(\frac{\partial \tilde{u}}{\partial \tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2} \left(\tilde{u} + \frac{\partial \tilde{v}}{\partial \tilde{\phi}}\right)^2 + \left(\frac{\partial \tilde{w}}{\partial \tilde{z}}\right)^2 \right]$$

$$+ \left[\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{1}{\tilde{r}} \frac{\partial \tilde{w}}{\partial \tilde{\phi}}\right]^2 + \left[\frac{\partial \tilde{w}}{\partial \tilde{r}} + \frac{\partial \tilde{u}}{\partial \tilde{z}}\right]^2 + \left[\frac{1}{\tilde{r}} \frac{\partial \tilde{u}}{\partial \tilde{\phi}} + \tilde{r} \frac{\partial}{\partial \tilde{r}} \left(\frac{\tilde{v}}{\tilde{r}}\right)\right]^2$$

$$- \frac{2}{3} \left[\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{1}{\tilde{r}} \left(\tilde{u} + \frac{\partial \tilde{v}}{\partial \tilde{\phi}}\right) + \frac{\partial \tilde{w}}{\partial \tilde{z}}\right]^2 \; .$$

$$C_p - C_v = R \; , \quad \frac{C_p}{C_v} = \gamma \; . \qquad (7)$$

The boundary conditions are:

$$\tilde{u} = 0 \; , \quad \tilde{w} = 0, \quad \tilde{v} = \tilde{\omega}_O \tilde{r}_O \; , \quad \tilde{\Theta} = \tilde{\Theta}_O \quad \text{at} \quad \tilde{r} = \tilde{r}_O \; ,$$

and

$$\tilde{u} = 0 \; , \quad \tilde{w} = 0, \quad \tilde{v} = \tilde{\omega}_I \tilde{r}_I \; , \quad \tilde{\Theta} = \tilde{\Theta}_I \quad \text{at} \quad \tilde{r} = \tilde{r}_I \; .$$

In the preceding equations, $\tilde{\rho}$ is the density of the gas; $\tilde{t}$ time, $\tilde{r}$, $\tilde{\phi}$ and $\tilde{z}$ the radial, tangential and axial coordinates, (see Figure 1);

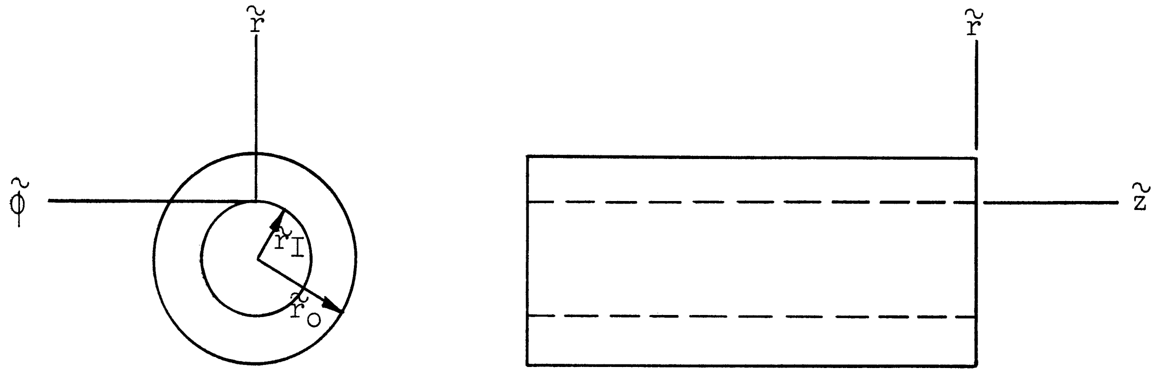


Figure 1. Physical configuration.

$\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ the radial, tangential and axial components of the velocity, μ the viscosity, $\tilde{p}$ the pressure; R the gas constant; k the thermal conductivity; $\tilde{\theta}$ the temperature, C_p and C_v are the specific heats per unit mass at constant pressure and temperature; and $\gamma = \frac{C_p}{C_v}$, the isentropic exponent. The above equations are dimensional equations; dimensionless forms of these equations are much easier to handle, so the following substitutions are made.

$$\tilde{u} = \hat{u}\tilde{\omega}_I\tilde{r}_I\Delta_r \ , \quad \tilde{v} = \hat{v}\tilde{\omega}_I\tilde{r}_I\Delta_r \ , \quad \tilde{w} = \hat{w}\tilde{\omega}_I\tilde{r}_I\Delta_r$$
$$\tilde{r} = r\tilde{r}_I\Delta_r \ , \quad \tilde{z} = zr_I\Delta_r \ , \quad \tilde{\phi} = \phi \ , \quad \tilde{t} = \frac{t}{\omega_I}$$
$$\tilde{p} = \hat{\theta}\,\tilde{\theta}_I \ , \quad \tilde{p} = \hat{p}\,\tilde{\rho}_A\tilde{\omega}_I{}^2\tilde{r}_I{}^2\Delta_r{}^2 \ , \quad \tilde{\rho} = \hat{\rho}\,\tilde{\rho}_A \ .$$

The subscript A on ρ represents the average density. The subscript I represents the inner cylinder and the subscript O represents the outer. Many choices of the reference length and reference velocity are possible. The particular ones chosen were chosen to make certain parts of the numerical solution have a convenient scale. The other scale factors are the usual ones.

Then:

$$\hat{\rho}\left[D_t + \hat{u}\,D_r + \hat{v}\,D_\phi + \hat{w}\,D_z\right]\hat{u} - \hat{\rho}\,\frac{\hat{v}^2}{r} =$$

$$- D_r\hat{p} + \frac{1}{N_R}\left\{D_r\left[\hat{A}_\mu\left(2D_r\hat{u} - \frac{2}{3}\,(D_r^+\,\hat{u} + D_\phi\hat{v} + D_z\hat{w})\right)\right]\right.$$

$$\left. + D_\phi\left[\hat{A}_\mu\,(D_\phi\hat{u} + D_r^-\hat{v})\right] + D_z\left[\hat{A}_\mu(D_z\hat{u} + D_r\hat{w})\right] + \frac{2\hat{A}_\mu}{r}\left[D_r^-\hat{u} - D_\phi\hat{v}\right]\right\}. \quad (8)$$

$$\hat{\rho}\left[D_t + \hat{u}\,D_r^+ + \hat{v}\,D_\phi + \hat{w}\,D_z\right]\hat{v} =$$

$$- D_\phi\hat{p} + \frac{1}{N_R}\left\{D_\phi\left[\hat{A}_\mu\left(2D_\phi\hat{v} - \frac{2}{3}\,(D_r^+\,\hat{u} + D_\phi\hat{v} + D_z\hat{w})\right)\right]\right.$$

$$\left. + D_z\left[\hat{A}_\mu\,(D_\phi\hat{w} + D_z\hat{v})\right] + D_r\left[\hat{A}_\mu(D_\phi\hat{u} + D_r^-\hat{v})\right] + \frac{2\hat{A}_\mu}{r}\left[D_\phi\hat{u} + D_r^-\hat{v}\right]\right\}. \quad (9)$$

$$\hat{\rho}\left[D_t + \hat{u}\,D_r + \hat{v}\,D_\phi + \hat{w}\,D_z\right]\hat{w} =$$

$$- D_z\hat{p} + \frac{1}{N_R}\left\{D_z\left[\hat{A}_\mu\left(2D_z\hat{w} - \frac{2}{3}\,(D_r\hat{u} + D_\phi\hat{v} + D_z\hat{w})\right)\right]\right.$$

$$\left. + D_r^+\left[\hat{A}_\mu\,(D_z\hat{u} + D_r\hat{w})\right] + D_\phi\left[\hat{A}_\mu(D_\phi\hat{w} + D_z\hat{v})\right]\right\}. \quad (10)$$

$$D_t\hat{\rho} + D_r^+(\hat{\rho}\hat{u}) + D_\phi(\hat{\rho}\hat{v}) + D_z(\hat{\rho}\hat{w}) = 0\,. \quad (11)$$

$$\frac{\hat{p}}{\hat{\rho}} = \frac{1}{N_M^2}\,\hat{\theta}\,. \quad (12)$$

$$\hat{\rho}\,\frac{D\hat{\theta}}{Dt} = (1 - \frac{1}{\gamma})\,N_M^2\,\frac{D\hat{p}}{Dt} + \frac{1}{N_{Pe}}\,\nabla\cdot\,(\hat{A}_k\nabla\hat{\theta}) + \hat{A}_\mu\,(1 - \frac{1}{\gamma})\,\frac{N_M^2}{N_R}\,\Phi\,, \quad (13)$$

in which

$$\frac{D}{Dt} = (D_t + \hat{u}\,D_r + \hat{v}\,D_\phi + \hat{w}\,D_z),$$

$$\nabla \cdot (\hat{A}_k \nabla \hat{\theta}) = \Big(D_r^{+}(\hat{A}_k D_r) + D_\phi(\hat{A}_k D_\phi) + D_z(\hat{A}_k D_z) \Big) \hat{\theta} \ ,$$

and

$$\hat{\Phi} = 2 \left[(D_r \hat{u})^2 + (\frac{\hat{u}}{r} + D_\phi \hat{v})^2 + (D_z \hat{w})^2 \right] + (D_z \hat{v} + D_\phi \hat{w})^2 + (D_r \hat{w} + D_z \hat{u})^2$$

$$+ (D_\phi \hat{u} + D_r^{-} \hat{v})^2 - \frac{2}{3} (D_r^{+} \hat{u} + D_\phi \hat{v} + D_z \hat{w})^2 \ .$$

The boundary conditions are:

$$\hat{u} = 0 \ , \quad \hat{w} = 0 \ , \quad \hat{v} = (1 + \Delta_r^{-1})(1 + \Delta_\omega) \ , \quad \hat{\theta} = 1 + \Delta_\theta \ \text{at} \quad r = 1 + \Delta_r^{-1} ,$$

and

$$\hat{u} = 0 \ , \quad \hat{w} = 0 \ , \quad \hat{v} = \Delta_r^{-1} \ , \quad \hat{\theta} = 1 \quad \text{at} \quad r = \Delta_r^{-1} \ .$$

In these equationsthe following are dimensionless parameters and operators.

$N_R = \dfrac{\tilde{\omega}_I \tilde{r}_I^2 \tilde{\rho}_A}{\mu_I} \Delta_r^2 = \bar{N}_R \Delta_r^2$, a Reynolds number.

$N_M^{\ 2} = \dfrac{\tilde{\omega}_I^{\ 2} \tilde{r}_I^{\ 2}}{R \ \tilde{\theta}_I} \Delta_r^2 = \bar{N}_M \Delta_r^2$, a Mach number squared.

$N_{Pe} = N_{Pr} \cdot N_R$, a Peclet number.

$N_{Pr} = \dfrac{\mu \ C_p}{k}$, the Prandtl number.

$\Delta_r = \dfrac{\tilde{r}_O - \tilde{r}_I}{\tilde{r}_I}$, the relative spacing.

$$\hat{A}_\mu = \frac{\mu}{\mu_I} \ , \quad \hat{A}_k = \frac{k}{k_I} \ , \quad \hat{A}_\mu = \hat{A}_\mu(\hat{\theta}) \ , \quad \hat{A}_k = \hat{A}_k(\hat{\theta}) .$$

$$D_r = \frac{\partial}{\partial r} \ , \quad D_r^{+} = \frac{\partial}{\partial r} + \frac{1}{r} \ , \quad D_r^{-} = \frac{\partial}{\partial r} - \frac{1}{r} \ ,$$

$$D_z = \frac{\partial}{\partial z} \ , \quad D_\phi = \frac{1}{r}\frac{\partial}{\partial \phi} \ , \quad D_t = \frac{\partial}{\partial t} \ .$$

The flow, in general, may be considered to consist of two parts; the stationary or mean flow which is the pure laminar flow having only an angular component of velocity, and the perturbed flow having components in each of the three directions.

Since the mean flow only has a component in the angular direction and only has variation in the radial direction,

$$D_t = D_z = D_\phi = 0 ,$$

$$\hat{u} = \bar{u} = 0 , \quad \hat{w} = \bar{w} = 0 ,$$

whereas,

$$D_r \neq 0 , \quad D_r^+ \neq 0 , \quad D_r^- \neq 0 , \quad \hat{\theta} = \bar{\theta} \neq 0 , \quad \hat{v} = \bar{v} \neq 0 .$$

Furthermore,

$$\bar{\rho}\,\frac{\bar{v}^2}{r} = D_r\,\bar{p} , \tag{14}$$

$$(D_r + \frac{2}{r})(\bar{A}_\mu D_r^- \bar{v}) = 0 , \tag{15}$$

$$\frac{\bar{p}}{\bar{\rho}} = \frac{1}{N_M^2}\,\bar{\theta} , \tag{16}$$

$$D_r^+ (\bar{A}_k\, D_r\, \bar{\theta}) + (1 - \frac{1}{\gamma})\, N_{Pr}\, N_M^2\, \bar{A}_\mu \,(D_r^- \bar{v})^2 = 0 , \tag{17}$$

in which $\bar{A}_\mu = \hat{A}_\mu$ and $\bar{A}_k = \tilde{A}_k$, evaluated for $\hat{\theta} = \bar{\theta}$. Since no variation in t , ϕ or z is present, Equations (14) to (17) are ordinary differential equations.

The barred variables are the mean flow variables. The boundary conditions are:

at $r = r_I$; $\bar{v} = 1$, $\bar{\theta} = 1$

and

at $r = r_o$; $\bar{v} = (1+\Delta_\omega)(1+\Delta_r^{-1})$, $\bar{\theta} = 1 + \Delta_\theta$.

The solution to the mean flow equations is discussed in Appendix I. For the remaining portion of the mathematical description of this problem, it will be assumed that a solution exists for the mean flow equations.

If the flow is perturbed, that is, disturbed in any way, this disturbance may either grow, decay, oscillate or remain the same. The next set of equations represents this disturbed flow. All products of disturbed quantities are assumed to be too small to be considered in these equations and are dropped. Furthermore, the mean flow terms that remain are subtracted away. The resulting equations govern an infinitesimal disturbance superimposed on the primary flow. Then,

$$\bar{\rho}\left[D_t + \bar{v}\, D_\phi\right] u' - \frac{2\bar{\rho}\bar{v}}{r}\, v' - \frac{\bar{v}^2}{r}\, \rho' =$$

$$- D_r p' + \frac{1}{N_R}\left\{D_r\left[\bar{A}_\mu\left(2D_r u' - \frac{2}{3}\left(D_r^+ u' + D_\phi v' + D_z w'\right)\right)\right]\right.$$

$$+ \bar{A}_\mu\, D_\phi\left[D_\phi u' + D_r^- v'\right] + A_\mu'\,(D_r^-\,\bar{v})\left[D_\phi\,\Theta'\right]$$

$$\left. + \bar{A}_\mu\, D_z\left[D_z u' + D_r w'\right] + \frac{2\bar{A}_\mu}{r}\left[D_r^-\, u' - D_\phi\, v'\right]\right\}, \tag{18}$$

$$\bar{\rho}\left[D_t + \bar{v}\, D_\phi\right] v' + (\bar{\rho}\, D_r^+\, \bar{v})\, u' =$$

$$- D_\phi\, p' + \frac{1}{N_R}\left\{\bar{A}_\mu\, D_\phi\left[2D_\phi v' - \frac{2}{3}\left(D_r^+ u' + D_\phi v' + D_z w'\right)\right]\right.$$

$$+ \bar{A}_\mu\, D_z\left[D_\phi w' + D_z v'\right] + \left[D_r + \frac{2}{r}\right]\left[\bar{A}_\mu\,(D_\phi\, u' + D_r^-\, v')\right]$$

$$\left. + A_\mu'\left[(D_r^+\,\bar{v})\, D_r\,\Theta' + \left(\left(D_r + \frac{2}{r}\right) D_r^-\bar{v}\right)\Theta'\right]\right\}, \tag{19}$$

$$\bar{\rho}\left[D_t + \bar{v}\,D_\phi\right] w' = -D_z p' + \frac{1}{N_R}\left\{\bar{A}_\mu D_z\left[2D_z w' - \frac{2}{3}\left(D_r^+ u' + D_\phi v' + D_z w'\right)\right]\right.$$

$$\left. + D_r^+\left[\bar{A}_\mu (D_z u' + D_r w')\right] + \bar{A}_\mu D_\phi\left[D_\phi w' + D_z v'\right]\right\} \quad , \quad (20)$$

$$D_t\rho' + D_r^+(\bar{\rho}u') + D_\phi(\bar{\rho}v' + \rho'\bar{v}) + D_z(\bar{\rho}w') = 0 \quad , \quad (21)$$

$$p' = \frac{1}{N_M^2}\left(\rho'\,\bar{\theta} + \bar{\rho}\,\theta'\right) \quad , \quad (22)$$

$$\bar{\rho}\left[(D_t + \bar{v}\,D_\phi)\,\theta' + (D_r\bar{\theta})u'\right] = \left(1 - \frac{1}{\gamma}\right) N_M^2\left[(D_t + \bar{v}D_\phi)p' + (D_r\bar{p})u'\right]$$

$$+ \frac{1}{N_{Pe}}\left[\left(D_r^+(\bar{A}_k D_r) + D_\phi(\bar{A}_k D_\phi) + D_z(\bar{A}_k D_z)\right)\theta'\right.$$

$$\left. + A_k'\left((D_r^+ D_r\bar{\theta})\theta' + (D_r\theta)D_r^+\theta'\right)\right]$$

$$+ \left(1 - \frac{1}{\gamma}\right)\frac{N_M^2}{N_R}\left[2\bar{A}_\mu\left((D_r^-\bar{v})D_r^- v' + (D_r^-\bar{v})D_\phi u'\right)\right.$$

$$\left. + A_\mu'\,(D_r^-\,\bar{v})^2\,\theta'\right] , \quad (23)$$

in which

$$\left.\begin{aligned} A_\mu' &= \frac{d}{d\hat{\theta}}\hat{A}_\mu \\ A_k' &= \frac{d}{d\hat{\theta}}\hat{A}_k \end{aligned}\right\} \quad \text{evaluated for } \theta = \bar{\theta} .$$

The primed variables are the perturbed functions. In this form the equations represent a set of partial differential equations in a rather involved form. To make the problem more amenable to solution, the variables are separated. That is, each of the individual dependent variables can be considered to be a product of three terms; the first term being a function of r only, the second term being a function of t only, and the third term a function of z only. Specifically, in addition, it is assumed that the rotationally symmetric mode of flow is the most unstable which is justified by all experiments that have ever been performed. By substituting

$$\rho' = \rho(r)\ E\ \rho^*(z)\ ,$$
$$p' = p(r)\ E\ p^*(z)\ ,$$
$$u' = u(r)\ E\ u^*(z)\ ,$$
$$v' = v(r)\ E\ v^*(z)\ ,$$
$$w' = w(r)\ E\ w^*(z)\ , \quad \text{in which} \quad E = e^{\sigma t}\ ,$$

the perturbed equations reduce to:

$$\bar{\rho}\ (\sigma u E u^*) - \frac{2\bar{\rho}\bar{v}}{r} vEv^* - \rho E \rho^* \frac{\bar{v}^2}{r} =$$
$$- (D_r p)Ep^* + \frac{1}{N_R}\left\{2D_r\left[\bar{A}_\mu(D_r u)\right] Eu^* - \frac{2}{3} D_r\left[\bar{A}_\mu\left((D_r^+ u)Eu^* + wE(D_z w^*)\right)\right]\right.$$
$$\left. + \bar{A}_\mu uE\ (D_z^2 u^*) + \bar{A}_\mu(D_r w)E(D_z w^*) + \frac{2\bar{A}_\mu}{r}\left[(D_r^- u)Eu^*\right]\right\} . \qquad (24)$$

$$\bar{\rho}\left[\sigma vEv^* + (D_r^+\bar{v})uEu^*\right] = \frac{1}{N_R}\left\{\bar{A}_\mu vE(D_z^2 v^*) + (D_r + \frac{2}{r})\left[\bar{A}_\mu(D_r^- v)\right]Ev^*\right.$$
$$\left. + A_\mu'\left[(D_r^{-}\bar{v})(D_r\Theta)E\Theta^* + \left((D_r + \frac{2}{r})D_r^-\bar{v}\right)\Theta E\Theta^*\right]\right\} . \qquad (25)$$

$$\bar{\rho}\,\sigma wEw^* = - pE(D_z p^*) + \frac{1}{N_R}\left\{2\bar{A}_\mu wE(D_z^2 w^*) - \frac{2}{3}\bar{A}_\mu\left[(D_r^+ u)E(D_z u^*) + wE(D_z^2 w^*)\right]\right.$$
$$\left. + D_r^+(\bar{A}_\mu u)\ E\ (D_z u^*) + D_r^+\left(\bar{A}_\mu(D_r w)\right) Ew^*\right\} . \qquad (26)$$

$$\sigma\rho E\rho^* + D_r^+(\bar{\rho}u)Eu^* + \bar{\rho}wE(D_z w^*) = 0 . \qquad (27)$$

$$pEp^* = \frac{1}{N_M^2}(\rho E\rho^*\bar{\theta} + \bar{\rho}\theta E\theta^*) \quad . \tag{28}$$

$$\begin{aligned}\bar{\rho}\left(\sigma\theta E\theta^* + uEu^*(D_r\bar{\theta})\right) = {} & (1 - \tfrac{1}{\gamma})\, N_M^2 \left(\sigma pEp^* + uEu^*(D_r\bar{p})\right) \\ & + \frac{1}{N_R} \cdot \frac{1}{N_{Pr}} \left\{ D_r^+\left(\bar{A}_k(D_r\bar{\theta})\right)E\theta^* + \bar{A}_k\theta E(D_z^2\theta^*) \right. \\ & \left. + A_k' \left[(D_r^+D_r\bar{\theta})\theta E\theta^* + (D_r\bar{\theta})(D_r^+\theta)E\theta^* \right] \right\} \\ & + (1 - \tfrac{1}{\gamma}) \frac{N_M^2}{N_R} \left\{ \bar{A}_\mu(2D_r^-\bar{v})(D_r^-v)Ev^* + A_\mu'(D_r^-\bar{v})^2\theta E\theta^* \right\} . \end{aligned} \tag{29}$$

The boundary conditions are:

$u' = 0, \quad v' = 0, \quad w' = 0, \quad \text{and} \quad \theta' = 0 \qquad \text{at} \quad r = \Delta_r^{-1}$

and $r = 1 + \Delta_r^{-1}$.

Since infinitely long cylinders are being considered, an arbitrary variation of the disturbance in the z direction, vanishing at $\pm\infty$, may be transformed to a periodic variation by means of the Fourier integral, the starred variables are assumed to be periodic and are assumed to have the same period. The phases of each of the variables must agree in any one equation, even though the absolute phase is of no importance. Therefore,

from (28), phase (p^*) = phase (ρ^*) = phase (θ^*) = cos (az) .

Likewise, from (29), phase (u^*) = phase (p^*) = cos (az).

Then, from (27), phase (Dw^*) = phase (u^*) = cos (az).

Therefore, phase (w^*) = sin (az),

but from (26), phase (Dp^*) = phase (w^*) = sin (az).

Therefore, phase p^* = cos (az), which agrees with earlier assumption.

From (25), phase (v^*) = phase (u^*) = cos (az), and (24) checks.

Therefore,

$$\rho^* = \cos(az),$$

$$p^* = \cos(az),$$

$$w^* = \sin(az),$$

$$u^* = \cos(az),$$

$$v^* = \cos(az),$$

$$\Theta^* = \cos(az).$$

Henceforth, neutral stability of the flow is considered. That is, the real part of σ equals zero. It is also assumed here that the imaginary part of σ equals zero. There is a question as to whether this is always true in these problems. The previously cited work of Pellew and Southwell(8) showed that, for the thermal convection problem and for the case of two cylinders rotating in the same direction, at the same speed, and with small spacing, the imaginary part of σ is zero if the real part is zero. However, their work was blessed with a particularly fortunate circumstance, that is, their equation was self-adjoint. The general equations are not self-adjoint, therefore, this principle, the "principle of exchange of stabilities" will be drawn on in this work without any attempt at proof. The perturbed equations as derived from the basic equations are then,

$$- \frac{2\bar{\rho}\bar{v}}{r} v - \frac{\bar{v}^2}{r} \rho = - D_r p + \frac{1}{N_R} \left\{ 2D_r \left(\bar{A}_\mu (D_r u) \right) - \frac{2}{3} D_r \left(\bar{A}_\mu (D_r^+ u + aw) \right) - \bar{A}_\mu a^2 u + \bar{A}_\mu a D_r w + \frac{2\bar{A}_\mu}{r} D_r^- u \right\}, \qquad (30)$$

$$(\bar{\rho}D_r^+\bar{v})u = \frac{1}{N_R}\left\{ -a^2\bar{A}_\mu v + (D_r + \frac{2}{r})\left(\bar{A}_\mu(D_r^- v)\right) + A'_\mu\left[(D_r^-\bar{v})D_r\theta + \left((D_r + \frac{2}{r})D_r^-\bar{v}\right)\theta\right]\right\} , \quad (31)$$

$$ap + \frac{1}{N_R}\left\{ -2a^2\bar{A}_\mu w + \frac{2}{3}\bar{A}_\mu(aD_r^+u + a^2w) - aD_r^+(\bar{A}_\mu u) + D_r^+\left(\bar{A}_\mu(D_r w)\right)\right\} = 0, \quad (32)$$

$$D_r^+(\bar{\rho}u) + \bar{\rho}aw = 0 , \quad (33)$$

$$p = \frac{1}{N_M^2}(\rho\bar{\theta} + \bar{\rho}\theta) ,$$

and

$$(\bar{\rho}D_r\bar{\theta})u = (1 - \frac{1}{\gamma})(D_r\bar{p})u + \frac{1}{N_R}\cdot\frac{1}{N_{Pr}}\left\{D_r^+\left(\bar{A}_k(D_r\theta)\right) - \bar{A}_k a^2\theta + A'_k\left((D_r^+D_r\bar{\theta})\theta + (D_r\bar{\theta})(D_r^+\theta)\right)\right\} + (1 - \frac{1}{\gamma})\frac{N_M^2}{N_R}\left[(2\bar{A}_\mu D_r^-\bar{v})(D_r^- v) + A'_\mu(D_r^-\bar{v})^2\theta\right] . \quad (35)$$

The boundary conditions are:

$u = 0$, $v = 0$, $w = 0$, and $\theta = 0$, at $r = \Delta_r^{-1}$ and $r = 1 + \Delta_r^{-1}$.

A change of the independent variable is convenient at this point, $r = \Delta_r^{-1} + x$ which results in $D_r = D_x$. By a substitution of variables $U = N_R u$, $V = v$ and $T = \theta$, and by solving (33) for w and substituting it into (30) and (32), then solving (32) for p and (34) for ρ and substituting these into (30), the next set of equations result.

$$\left\{D_x - N_M^2 \frac{\bar{v}^2}{r\bar{\theta}}\right\}\left\{a^2\left[-\frac{2\bar{A}_\mu}{\bar{\rho}} D_x^{\pm}(\bar{\rho}U) + D_x^{\pm}(\bar{A}_\mu U)\right] + D_x^{\pm}\left[\bar{A}_\mu D_x\left(\frac{1}{\bar{\rho}} D_x^{\pm}(\bar{\rho}U)\right)\right]\right\}$$

$$- a^2\left\{2 D_x\left(\bar{A}_\mu(D_x U)\right) + \frac{2}{3} N_M^2 \frac{\bar{v}^2}{r\bar{\theta}} \bar{A}_\mu \frac{D_x\bar{\rho}}{\bar{\rho}} U - \bar{A}_\mu D_x\left(\frac{1}{\bar{\rho}} D_x^{+}(\bar{\rho}U)\right) + \frac{2\bar{A}_\mu}{r}(D_x^{-}U)\right\}$$

$$+ a^4\bar{A}_\mu U = a^2 N_R^2\left[\frac{2\bar{\rho}\bar{v}}{r} V - \frac{\bar{\rho}\bar{v}^2}{r\bar{\theta}} T\right]. \qquad (36)$$

$$(D_x + \frac{2}{r})\left(\bar{A}_\mu(D_x^{-}V)\right) - a^2\bar{A}_\mu V = (\bar{\rho}D_x^{\pm}\bar{v})U - A_\mu'\left[(D_x^{-}\bar{v})D_x T + \left((D_x + \frac{2}{r})D_x^{-}\bar{v}\right)T\right]. \qquad (37)$$

$$D_x^{+}\left(\bar{A}_k(D_x T)\right) + A_k'\left((D_x^{+}D_x\bar{\theta})T + (D_x\bar{\theta})(D_x^{+}T)\right)$$

$$- \bar{A}_k a^2 T + (1 - \frac{1}{\gamma})N_{Pr}N_M^2 A_\mu'(D_x^{-}\bar{v})^2 T$$

$$= N_{Pr}\left[\bar{\rho}D_x\bar{\theta} - (1 - \frac{1}{\gamma})N_M^2 D_x\bar{p}\right]U - (1 - \frac{1}{\gamma})N_{Pr}N_M^2\bar{A}_\mu(2D_x^{-}\bar{v})(D_x^{-}V). \qquad (38)$$

The boundary conditions are:

$$u = 0\ , \quad v = 0\ , \quad \text{and } D_x u = 0 \quad \text{at} \quad x = 0 \text{ and } x = 1 .$$

For very small spacing the terms $\frac{1}{r}$ approach zero as Δ_r and D_x^{+} and D_x^{-} approach D_x as Δ_r. If the viscosity is not considered to vary with temperature as a first approximation, $\bar{A}_\mu = \bar{A}_k = 1$, and $A_\mu' = A_k' = 0$. Whenever the generation of thermal energy by viscous dissipation is not being considered, the second term in (17), the last term on the left-hand side of (38), and the second term on the right-hand side of (38) are dropped.

The Case of Small Mach Number and No Temperature Gradient

In the case of small Mach numbers and no temperature gradient, Equations (36) and (37) reduce to those for a liquid with constant density.

$$(D_x D_x^+ - a^2)(D_x D_x^+ - a^2)\ U \ = \ a^2 N_R^2 \frac{2\overline{v}}{r}\ V\ . \tag{39}$$

$$(D_x D_x^+ - a^2)\ V \ = \ (D_x^+ \overline{v})\ U\ . \tag{40}$$

The same boundary conditions apply here.

What is the mechanism from the point of view of the governing equations of flow? A small disturbance in the radial component of the velocity u (or alternately the radial momentum ρu) is transferred into the angular component of velocity v (or angular momentum ρv) by the convective action of u on the mean angular momentum gradient $D_x^+ \overline{v}$. The angular velocity is, in turn, diffused throughout the whole of the fluid by the action of the viscosity in the guise of $\frac{1}{N_R}$ in the equations. This disturbance in the angular velocity then disturbs the centrifugal force gradient $\frac{2\overline{\rho}\overline{v}}{r}$ which generates a disturbance in u , which is then also diffused by the viscosity into the entire fluid.

Clearly, there is a mechanism here for unstable action; that is, a u generates a v which, in turn, generates a u . If this new u is larger than the first, the flow is unstable. Of course, since all of these things happen together, there is no first and second u, but a single u which grows, decays, oscillates or remains constant with time. The thought of sequential events, however, is helpful for the purpose of understanding the mechanism involved.

Discussion of Effects in Variations in Δ_ω

The terms $\frac{2\bar{\rho}\bar{v}}{r}$ and $D_x^+\bar{v}$ are driving potentials for unstable flow, while the other terms represent the damping or diffusive effects. A close examination of these driving potentials is helpful in understanding some of the phenomena associated with this type of problem.

The term $\frac{2\bar{v}}{r}$ may be considered either positive or negative in the region of flow $-1 \leq \Delta_\omega < +\infty$, since the direction of positive rotation has no effect on the stability of the flow. In the range of Δ_ω between -1 and $-\infty$ the sign of $\frac{2\bar{v}}{r}$ is both positive and negative. Either the positive or negative region is near the inner cylinder depending again on the arbitrary choice of sign of rotation.

The Rayleigh criterion, as explained by Synge(11), states only that a positive $D_x(r\bar{v})^2$ is sufficient for absolute stability of a viscous fluid. It says nothing about a necessary condition for stability or, oppositely, it says nothing about a sufficient condition for lability. One necessary condition for lability is clearly a negative $D_x(r\bar{v})^2$. Is it also sufficient? From the Rayleigh criterion, absolute stability requires $D_x(r\bar{v})^2 > 0$, which means $2r^2\bar{v}\,D_x^+\bar{v}$ must be greater than zero. Therefore, $\bar{v}$ and $D_x^+\bar{v}$ must be of the same sign for absolute stability. In the range of Δ_ω between $+\infty$ and -1 the signs of $\frac{2\bar{v}}{r}$ and $D_x^+\bar{v}$ do not change between the two cylinders. In fact, for all values of Δ_ω, $D_x^+\bar{v}$ is a constant and only $\frac{2\bar{v}}{r}$ is variable. Experiments and analyses have shown that for small spacing the flow is labile between $\Delta_\omega = 0$ and $\Delta_\omega = -1$, and stable for $\Delta_\omega > 0$. Then in the region $+\infty > \Delta_\omega \geq -1$, it may be seen that the flow is always labile if $\frac{2\bar{v}}{r}$ and $D_x^+\bar{v}$ are of the opposite sign, and always absolutely stable if they are of the same sign.

The picture is not so clear for the region of Δ_ω between -1 and $-\infty$. The Rayleigh criterion for an inviscid fluid states that the flow is unstable for all $(1 + \Delta_\omega)(1 + \Delta_r)^2 < 1$. A small amount of viscosity added to the inviscid fluid probably amounts to a transition between stable and unstable flow at some finite Reynolds number. The region of instability for an inviscid fluid becomes a region of lability for a viscous fluid.

From another point of view, however, it would seem unlikely for the flow to be labile for large negative Δ_ω. The flow for large positive Δ_ω is absolutely stable. As the magnitude of Δ_ω increases the two flows become more nearly alike rather than unlike. The only difference between the two is the small region of opposite sign near the inner cylinder in the negative Δ_ω case. Since the flow has been shown in some instances to be labile it seems plausible that a small region where $\frac{2\bar{v}}{r}$ and $D_x^+\bar{v}$ are of opposite sign is sufficient for lability.

This judgment is supported by the work of Synge[11] for inviscid fluids. In effect, his results state that whenever $\frac{2\bar{v}}{r}$ and $D_x^+\bar{v}$ are of the same sign in a subinterval of the space between the cylinders, the flow has a stable set of eigenvalues. Whenever there is a subinterval that has opposite signs, the flow has an unstable set of eigenvalues. That is, whenever there is a change of sign in $\frac{2\bar{v}}{r}$, two sets of modes, a stable set and an unstable set, exist. The work in Appendix II shows that a similar situation exists for viscous fluids; that is, whenever there is a region of opposite sign and a region of the same sign, two modes, a labile mode and an absolutely stable mode, exist. Since a labile mode exists, the flow is labile for all Δ_ω sufficiently negative, even down to $\Delta_\omega = \infty$. The Reynolds number at the transition between stable and unstable flow (at the neutral stability line), of course, becomes larger as Δ_ω becomes a larger negative number.

The lability of flows with a large negative Δ_ω can be made more plausible by extending the physical interpretation given in Chapter I. Imagine the stagnation ring replaced by a middle cylinder at the same radius. The flow inside this middle cylinder would be unstable while the flow outside would be stable. The middle cylinder could not be rigid, since the perturbed flow is not necessarily zero at this interface. In fact, the perturbed flow feeds energy from the unstable part into the stable part through the interface. This causes the Reynolds number at neutral stability to be higher since the outer region acts as an energy sink while the inner region an energy source for the perturbed flow.

The terms on the left-hand sides of (39) and (40) represent the diffusion of the velocities into the fluid and provide the damping in the labile region.

Discussion of Spacing Variations

A little insight into the phenomenon of the stabilizing-destabilizing effect of increasing Δ_r may be gained by examining the equations. As Δ_r increases, $D_x^{+-}\bar{v}$ decreases in the labile region for $-1 \leq \Delta_\omega \leq 0$, while $\frac{2\bar{v}}{r}$ remains essentially constant, decreasing slightly. Then N_R^2 increases to maintain the balance between the size of U and the size of V in the two equations.

For $-1 < \Delta_\omega \leq 0$ and sufficiently large Δ_r, $D_X^+\bar{v}$ changes sign and the flow becomes absolutely stable. However, since $N_R = \bar{N}_R \Delta_r^2$, $\bar{N}_R$ decreases as Δ_r grows if N_R remains constant or does not increase too rapidly. These two effects tend to counteract each other; the net effect is not determinable until calculations are made.

The Case with Temperature Effects Included

When the effects of temperature, but not Mach number, viscous variations, or dissipation are considered, Equations (36) through (38) reduce to

$$D_X D_X^+ D_X \left(\frac{1}{\bar{\rho}} D_X^+ (\bar{\rho} U)\right) - a^2 D_X \left(\frac{1}{\bar{\rho}} D_X^+ (\bar{\rho} U) + D_X^+ U\right) + a^4 U$$
$$= a^2 N_R^2 \left(\frac{2\bar{\rho}\bar{v}}{r} V - \frac{\bar{\rho}\bar{v}^2}{r\bar{\Theta}} T\right) , \tag{41}$$

$$D_X D_X^+ V - a^2 V = \bar{\rho} D_X^+ \bar{v} U , \tag{42}$$

$$D_X^+ D_X T - a^2 T = N_{Pr} \bar{\rho} D_X \bar{\Theta} U . \tag{43}$$

The function $\bar{\rho}$ is a positive definite function varying at most $\frac{\Delta_\Theta}{2}$ from its mean value of 1. Equation (42) is essentially the same as Equation (40) with the exception of the $\bar{\rho}$ in the term $\bar{\rho} D_X^+ \bar{v}$, the angular momentum gradient. This effect is small except when Δ_Θ is large. Equation (39) is modified more than (40) since not only is the $\bar{\rho}$ included in $\frac{2\bar{\rho}\bar{v}}{r}$ but an additional term $-\frac{\bar{\rho}\bar{v}^2}{r\bar{\Theta}} T$ is added, plus variations in the diffusion term. The term $-\frac{\bar{\rho}\bar{v}^2}{r\bar{\Theta}} T$ is the modification of the centrifugal force by the density perturbation caused by the temperature disturbance.

Equation (43) is a new one to the system. A disturbance in u draws on the temperature gradient as modified by $\bar{\rho}$, generates a T disturbance, which is diffused by the action of the thermal diffusivity (represented by $\frac{1}{N_{Pr}N_R}$ in this equation).

There is a close resemblance between Equation (42) and (43); in fact, for small spacing and a small thermal gradient applied between the walls, $D_x\bar{\Theta}$ is the constant Δ_Θ, and $D_xD_x^+$ and $D_x^+D_x$ both reduce to D_x^2. Then the role played by the term $\bar{\rho}D_x\bar{\Theta}$ in (43) is the same as that played by $\bar{\rho}D_x^+\bar{v}$ in Equation (42), except, because of the minus sign in $-\frac{\bar{\rho}\bar{v}^2}{r\bar{\Theta}}T$ in Equation (41), a positive $\bar{\rho}D_x\bar{\Theta}$ is destabilizing and a negative sign is stabilizing. Whenever $D_x\bar{\Theta}$ may be considered a constant, the similarity of the V and T equations implies that the shape of V and T are similar. In fact, whenever $D_x\bar{\Theta} = \Delta_\Theta$

$$\frac{T}{V} = N_{Pr}\frac{\Delta_\Theta}{D_x^+\bar{v}} \quad .$$

Then, the three equations reduce to two, the same two that were previously discussed except now the driving potential term on the right hand side of (41) reduces to

$$\frac{2\bar{\rho}\bar{v}}{r}\left(1 - N_{Pr}\frac{\bar{v}\,\Delta_\Theta}{2\bar{\Theta}D_x^+\bar{v}}\right)V \tag{44}$$

By examining this relation it may be seen that the role played by $\bar{\rho}$ is not so strong as the role played by Δ_Θ itself in some cases. Whenever $D_x^+\bar{v} = 2[(1 + \Delta_\omega)(1 + \Delta_r)^2 - 1]/\Delta_r(2 + \Delta_r)$ is a small number, e.g., near $\Delta_\omega = 0$ for $\Delta_r \simeq 0$ (i.e. solid body rotation), small variations in Δ_Θ can be decisive.

If the observations of the paragraph on Δ_ω can be applied, a necessary and sufficient condition for lability is an interval in which (44) is of opposite sign to $D_x^+\bar{v}$. Alternately, the absence of a region of apposition is necessary and sufficient for absolute stability.

A sufficiently negative Δ_θ (in the presence of a negative $D_x^+\bar{v}$) causes the flow to become absolutely stable whenever $\bar{v}$ is sufficiently positive. However, as Δ_ω approaches -1. and goes beyond -1. it is not possible to achieve absolute stability. The reason for this is the size of Δ_θ has a lower bound of -1. since negative absolute temperatures are not attainable.

It may be seen from (44) that a positive circulation gradient is not sufficient for absolute stability any longer. A positive $D_x^+\bar{v}$ and a positive $\frac{2\bar{v}}{r}$ may be overcome by a sufficiently large positive Δ_θ. Therefore, labile flow, hence unstable flow, is possible even if $\frac{2\bar{v}}{r}$ and $D_x^+\bar{v}$ are of the same sign. Therefore, the comment of Lees(6) is not entirely correct, cooling a convex surface can be destabilizing if $D_x^+\bar{v}$ is not too large.

It is tempting here to say that the lability of the flow is caused by the mean density gradient, but this is not true. The mean density profile does not provide the necessary change of sign in expression (44), even though the density does change. The necessary effect results when the temperature gradient is disturbed by u, causing a density perturbation. If, by some stretch of the imagination, it were possible to have a fluid which did not change density with the mean temperature but did change with the perturbed temperature, it would be possible to have a labile condition with a constant mean density in the pressure of a positive $D_x^+\bar{v}$. It is worthwhile to notice that the origin of the term $D_x\bar{\theta}$, which is the driving potential in this case, is the convection term in the substantial derivative of temperature, not the density variation.

This argument suggests that it might be possible to construct an unusual physical situation. If some means were found to apply a positive density gradient in the presence of a positive $D_x^+\bar{v}$ and a positive $\frac{2\bar{v}}{r}$, it still would be possible to have labile flow. The subsequent paragraphs on compressibility effects will cover this point.

The effect of thermal diffusivity is apparent from (44). As k increases, the Prandtl number is reduced which reduces the thermal effects. Therefore, insofar as the flow is labile by virtue of positive thermal gradient, an increase in k is stabilizing. On the other hand, if the flow is stabilized by a negative temperature gradient an increase in k is destabilizing. In a gas, however, the relative constancy of N_{Pr} does not allow this effect to occur.

Examination of Equations (41), (42), and (43) and the expression (44) reveals that increasing the viscosity can never have a destabilizing effect for a gas. Viscosity appears in two places in these equations; in the Reynolds number and in the Prandtl number. As noted previously, the Prandtl number remains nearly the same for a wide range of gases, and a wide range of pressures and temperatures. Therefore, there is no change in the driving potential (44) as viscosity is increased.

In view of this, the use of the Reynolds numbers or Taylor number as a measure of stability eliminates any apparent change of the point of neutral stability when μ changes. If angular velocity is used as a stability measure, an increasing viscosity demands an increasing angular velocity; therefore, the flow is stabilized in a labile region.

It is possible in the case of a liquid to construct a situation whereby increasing the viscosity will cause the flow to go from absolutely stable to labile. The thermal diffusivity and the viscous diffusivity are relatively independent for liquids, therefore, the Prandtl number is increased with increasing viscosity. The second term in (44) can go from a value less than one to a value greater than one. This causes the driving potential near the inner cylinder to go from positive to negative, which is sufficient for lability in the presence of a positive $D_x^+\bar{v}$ and positive $D_x\bar{\Theta}$. As the viscosity is increased beyond this point, the driving potential becomes increasingly negative; the Reynolds number then decreases indicating a destabilizing influence of viscosity in a labile region. The angular velocity at neutral stability decreases with increasing μ until a certain value of μ is reached. From that point onward an increase in μ has a stabilizing influence.

The mean density terms on the left-hand side of the equations are a little harder to analyze. Since the source of most of these terms is the continuity equation and the axial momentum equation, the effects will probably be minor ones on the U and w profiles. It is not clear how they affect stability.

The effects of temperature at the larger spacings are undoubtably similar to those for small spacings. In general, the two Equations (42) and (43) cannot be reduced to one. This makes it more difficult to assess the effects of the driving potentials.

The Effects of Compressibility

Inclusion of the compressibilty effects principally adds two terms to the T equation and one to the U equation. The V equation remains the same as (42) and is not repeated. The boundary conditions do not change. Equations (36) and (38) now assume the forms,

$$\left\{ D_X - N_M^2 \frac{\bar{v}^2}{r\bar{\Theta}} \right\} \left\{ a^2 \left[-\frac{2}{\bar{\rho}} D_X^+(\bar{\rho}U) + D_X^+U \right] + D_X^+D_X\left(\frac{1}{\bar{\rho}} D_X^+(\bar{\rho}U)\right)\right\}$$

$$- a^2 \left\{ 2D_X^2 U + \frac{2}{3} N_M^2 \frac{\bar{v}^2}{r\bar{\Theta}} \frac{D_X\bar{\rho}}{\bar{\rho}} U - D_X \left(\frac{1}{\bar{\rho}}D_X(\bar{\rho}U)\right) + \frac{2}{r} (D_X^-U)\right\} + a^4U$$

$$= a^2N_R^2 \left[\frac{2\bar{\rho}\bar{v}}{r} V - \frac{\bar{\rho}\bar{v}^2}{r\bar{\Theta}} T \right], \qquad (45)$$

$$D_X^+D_XT - a^2T = N_{Pr} \left[\bar{\rho}D_X\bar{\Theta} - (1 - \frac{1}{\gamma})N_M^2 D_X\bar{p} \right] U . \qquad (46)$$

The modifications of the U equation are difficult to determine. The second term of the operator $D_X - \frac{N_M^2\bar{v}^2\bar{\rho}}{r\bar{\Theta}}$ in the U equation represents the assimilation of the pressure perturbation. As such, it probably acts to counteract the effects of $\bar{\rho}$ in the terms on the right-hand side of (45) and (46), and the $D_X\bar{p}$ term in (46). It also changes the U profile by modifying the viscous diffusion effects. The other term modifies the interchange of momentum between u and w , and probably has a minor effect on stability.

The principal effect of the compressibility is the pressure gradient term in the T Equation (46). Its influence is identical to that of the temperature except that a positive pressure gradient is stabilizing rather than destabilizing, as was the case with temperature. By the nature of the centrifugal effects, $D_X\bar{p}$ can only be positive in this problem.

Its effects grow with increasing Mach number. The comments applied to the temperature gradient, of a previous paragraph, with respect to the weak influence of the mean density, apply here as well.

The terms $D_x\bar{\theta}$ and $D_x\bar{p}$ do not necessarily have the same influence in the T equation. If Equation (14) is substituted on the right-hand side of (45), it reduces to:

$$\bar{\rho}N_{Pr}\left[D_x\bar{\theta} - (1 - \frac{1}{\gamma})\, N_M^2\, \frac{\bar{v}^2}{r}\right].$$

By noting $N_M = \bar{N}_M\Delta_r$ and that $\bar{v}$ and r may be approximated by $\bar{v} = \Delta_r^{-1}(1 + \Delta_\omega x)(1 + \Delta_r x)$ and $r = \Delta_r^{-1}(1 + \Delta_r x)$, the right-hand side becomes: $\bar{\rho}N_{Pr}[\Delta_\theta - \Delta_r(1 - \frac{1}{\gamma})\bar{N}_M^2(1 + \Delta_\omega x)^2(1 + \Delta_r x)]$. Then, the influence of the second term is of order Δ_r , and is small whenever Δ_r is small. The factors $(1 - \frac{1}{\gamma})$ and $\bar{N}_M^2$ weight this term as well. The same, of course, is true for the second term in the operator $D_x - \frac{N_M^2\bar{v}^2\bar{\rho}}{r\bar{\theta}}$ in Equation (45). Therefore, the spacing must be significant before the influence of compressiblity can be felt. The total mass of the gas between the cylinders increases when Δ_r increases. Hence, for a given rotational speed, the density increases with Δ_r .

The pressure gradient offers the means of applying a positive density gradient by means other than temperature: therefore, it may be possible to construct a flow with a positive temperature, density, and circulation gradient, that is labile. By differentiating and rearranging Equation (16), $D_x\bar{\rho} = \frac{\bar{\rho}}{\bar{\theta}}[N_M^2\frac{\bar{v}^2}{r} - \Delta_\theta]$; for small spacing, $D_x\bar{\rho}$ must be positive if this construction is valid. Therefore, $\frac{\bar{v}^2}{r}N_M^2 > \Delta_\theta$. If the right-hand side of (46) is approximately a constant, which it will be for small Δ_r, Δ_ω, and Δ_θ, then

$$\frac{\theta}{v} \simeq \frac{\Delta_\theta - (1 - \frac{1}{\gamma})N_M^2 \frac{\bar{v}^2}{r}}{D_x^+ \bar{v}} \quad .$$

Therefore, the right-hand side of (45) becomes

$$\frac{2\bar{\rho}\bar{v}}{r}\left\{1 - N_{Pr}\frac{\bar{v}}{2\bar{\theta}}\left[\frac{\Delta_\theta - (1 - \frac{1}{\gamma})N_M^2 \frac{\bar{v}^2}{r}}{D_x^+ \bar{v}}\right]\right\} \quad .$$

Since $D_x^+\bar{v}$ is positive, a necessary condition for lability here is that this term must be negative. Therefore,

$$1 < N_{Pr}\frac{\bar{v}}{2\bar{\theta}}\left[\frac{\Delta_\theta - (1 - \frac{1}{\gamma})N_M^2\frac{\bar{v}^2}{r}}{D_x^+\bar{v}}\right]$$

or

$$\frac{2\bar{\theta}D_x^+\bar{v}}{N_{Pr}\bar{v}} + (1 - \frac{1}{\gamma})\ N_M^2\ \frac{\bar{v}^2}{r} < \Delta_\theta \ .$$

If Δ_θ is chosen such that

$$\frac{2\bar{\theta}D_x^+\bar{v}}{N_{Pr}\bar{v}} + (1 - \frac{1}{\gamma})\ \frac{N_M^2\bar{v}^2}{r} < \Delta_\theta < \frac{N_M^2\bar{v}^2}{r} \quad ,$$

the prescribed conditions are possible, and the flow is labile. This is, of course, a very artificial situation, but it illustrates the idea that the density gradient and circulation gradient are not dominant in stability problems of this type. A positive entropy gradient is necessary for lability, however.

The right-hand side of Equation (46) is essentially the entropy gradient

$$D_x\bar{s} = \frac{1}{\bar{\Theta}}\left[D_x\bar{\Theta} - (1 - \frac{1}{\gamma})N_M^2 \frac{\bar{v}^2}{r}\right] .$$

The right-hand side of (46) then may be written

$$N_{Pr}\bar{\rho}\bar{\Theta}D_x\bar{s} \quad .$$

Then, everything that has been said about the temperature profile could be repeated for an entropy profile, and the entropy gradient plays the dominant role as a driving potential for gases. It is certainly true that a sufficient condition for absolute stability is a negative entropy gradient <u>and</u> a positive circulation gradient as stated by Yih[13]. It is not a necessary one, since a small positive entropy gradient will not assure lability.

The changes in entropy arising either from a pressure gradient or a temperature gradient have the effect of shifting the lability line. That is, the transition from labile flow to absolutely stable or vice versa occurs at differing values of Δ_ω and/or Δ_r than they would for the case with no entropy gradient. The rapid diffusion of the pressure disturbance probably modifies this further.

The Other Effects

The addition of the variation of viscosity and thermal conductivity with temperature to the equations complicates them a great deal. It is difficult to assess the effects from the equations. The principal effect is that it makes the solution of these equations very involved.

The dissipation term in the energy equation has two effects. It introduces a mean temperature profile in the absence of an applied temperature gradient. The temperature gradient is positive near the inner cylinder and negative near the outer cylinder. It also introduces a term into the perturbed equations which is a direct coupling between the angular velocity gradient and the temperature disturbance. Both of these effects cause the thermal driving potential to be positive, that is, destabilizing, near the inner cylinder for negative circulation gradients.

It should be noted that the second term in Equation (17), which is the origin of the mean temperature profile, is of the order Δ_r to the zeroth power. This means that the mean temperature profile is relatively independent of Δ_r. It is not, however, completely independent; the profile changes in detail, but not in general shape or size.

The second term on the right-hand side of Equation (38) probably is destabilizing since it represents the increase in the temperature because of the shear stresses raised by the disturbed flow. Likewise, since $D_{\bar{x}}\bar{v}$ is negative, its influence is like a positive $D_x\bar{\theta}$. The size of this term is of order Δ_r, so its influence is not felt strongly at small Δ_r.

Increasing the magnitude of Δ_ω and the Mach number amplifies the effect of dissipation since the higher the shear stresses are, the more significant the increase in temperature. The extraction of the mechanical energy from the velocity is implicit in the diffusion terms and requires no additional terms in the U and V equations.

The solution method for these equations is explained in Appendix I.

CHAPTER III

RESULTS AND CONCLUSIONS

All of the calculations were performed for values of the Prandtl number N_{Pr} equal to .72 and the isentropic exponent γ equal to 1.41. The mean flow equations were considered acceptable when c_2 (see Appendix I) and the function agreed at two successive iterations to a value of one part in a million. Most of the disturbed flow calculations were accepted when N_R^2 and the U, V and T functions agreed to one part in ten thousand. In some cases a less stringent criterion was used.

Whenever results for more than one set of parameters were obtained from a single run, the U, V and T functions of the previous calculation were used as the first estimate of the functions for the new parameters. For the first set in any run, however, a standard first guess was used. The U function was assumed to have only three non-zero coefficients, $U_2 = 8$, $U_3 = -16$ and $U_4 = 8$; V and T were assumed to be identically zero. The first guess for $\bar{\theta}$ was always $\bar{\theta} = 1 + \Delta_\theta x$. (See Appendix I for description of the processes.) Whenever the effects of viscosity and thermal conductivity are included in the calculations, it is assumed that they both vary as the square root of the temperature.

The results presented here are at the value of a for which the Reynolds number is minimum. In all cases, the Reynolds number was calculated for several values of a, with at least one value on each side of the minimum. The usual a interval was .02, but in some cases, where broader variations were detected, the interval was .04. Therefore, the results given may be considered valid within an a interval of .04.

Two forms of the Reynolds number are referred to in many places: the Reynolds number based on the angular velocity of the inner cylinder and the radius of the inner cylinder, $\bar{N}_R = \frac{\tilde{\omega}_I \tilde{\rho}_A \tilde{r}_I^2}{\mu_I}$; and $N_R = \bar{N}_R^2 \Delta_r^2$ which in effect bases the Reynolds number on the space between the two cylinders instead of the radius of the inner cylinder. N_R is presented in most of the graphs and tables and $\bar{N}_R$ is presented in several places where it is interesting. The principal reason for using N_R in most places is that the values of $\bar{N}_R$, at the point of neutral stability, vary through many orders of magnitude while N_R is confined to moderate changes. In a few interesting places N_T , the Taylor number, is also given.

The variations of the neutral stability Reynolds number with the spacing Δ_r are given in Table I and Figures 2, 3, 4, and 5, for $\Delta_\omega = -1.0$; and in Table II and Figure 5, for $\Delta_\omega = -.01$. These runs were made for $\Delta_\Theta = 0$, $\bar{N}_M = 10^{-5}$, and with no dissipation terms or effects of variation of viscosity and thermal conductivity included. The solid curves a. in Figures 2, 3, and 4 show the values calculated in this work. The small dashed curves b. indicate the more probable tendancy curves, determined only by fitting a smooth curve to the datum point at $\Delta_r = 1$ calculated by Chandrasekhar(2). The long dashed curves c. represent the results as were determined from small space theory; that is, determined by use of the differential equations with terms of the order Δ_r or less neglected. Figure 2 shows the variations of N_R over a wide range of Δ_r and Figure 3 over the narrower but most interesting range. Figure 4 indi-

TABLE I

NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING
FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$

No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

Δ_r	a	N_R	$\bar{N}_R$	N_T
10^{-4}	3.14	.41170	$.41170 \times 10^8$	3390.0
10^{-2}	3.14	4.1410	$.41410 \times 10^5$	3398.0
.1	3.14	13.785	$.13785 \times 10^4$	3620.0
.2	3.14	20.608	$.51525 \times 10^3$	3861.0
.3	3.14	26.644	$.29603 \times 10^3$	4115.0
.4	3.14	32.610	$.20380 \times 10^3$	4431.0
.5	3.10*	40.700*	$.16250 \times 10^3$*	5305.0
.6	3.08**	50.380**	$.13990 \times 10^3$**	6480.0
1.0	3.20***	68.250***	$.68250 \times 10^2$***	6200.0

* Valid to one part in a thousand.

** Valid to one part in a hundred.

*** From the data of Chandrasekhar[2].

Chandrasekhar used a different equation for Taylor number. The values of N_T , the Taylor number, are valid to about one part in a hundred.

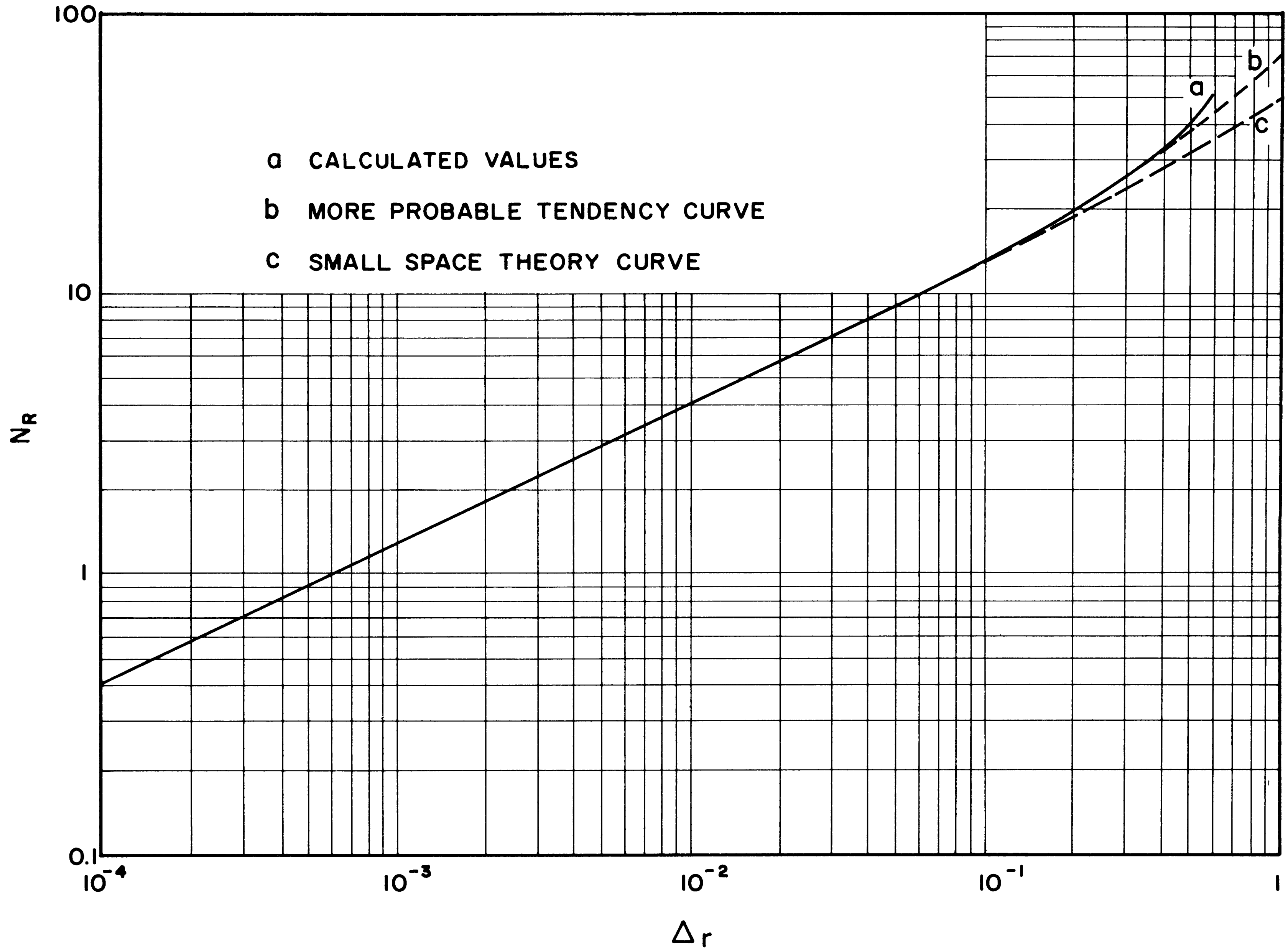


Figure 2. Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation terms. No Variation of Viscosity and Thermal Conductivity.

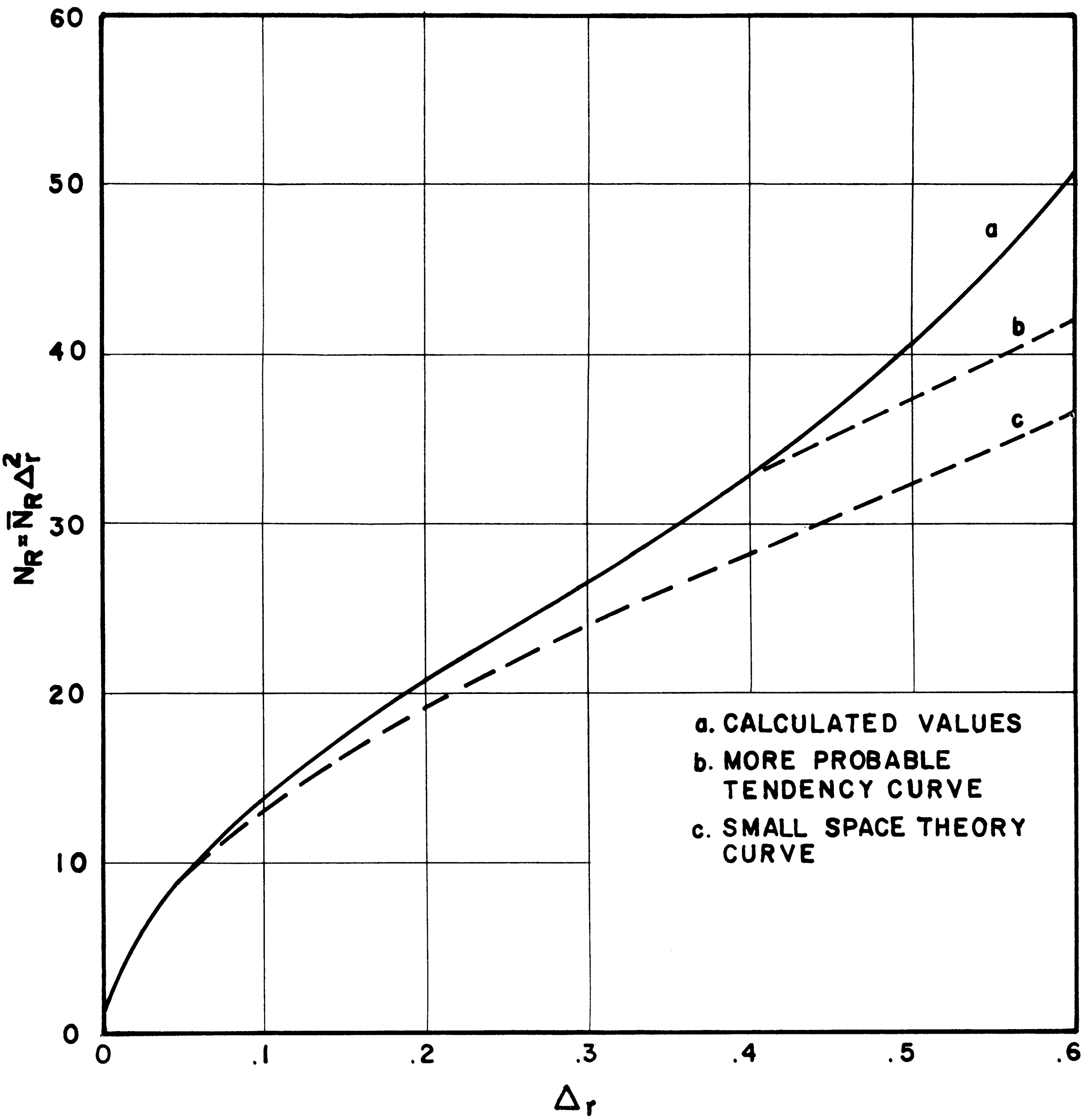


Figure 3. Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta=0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

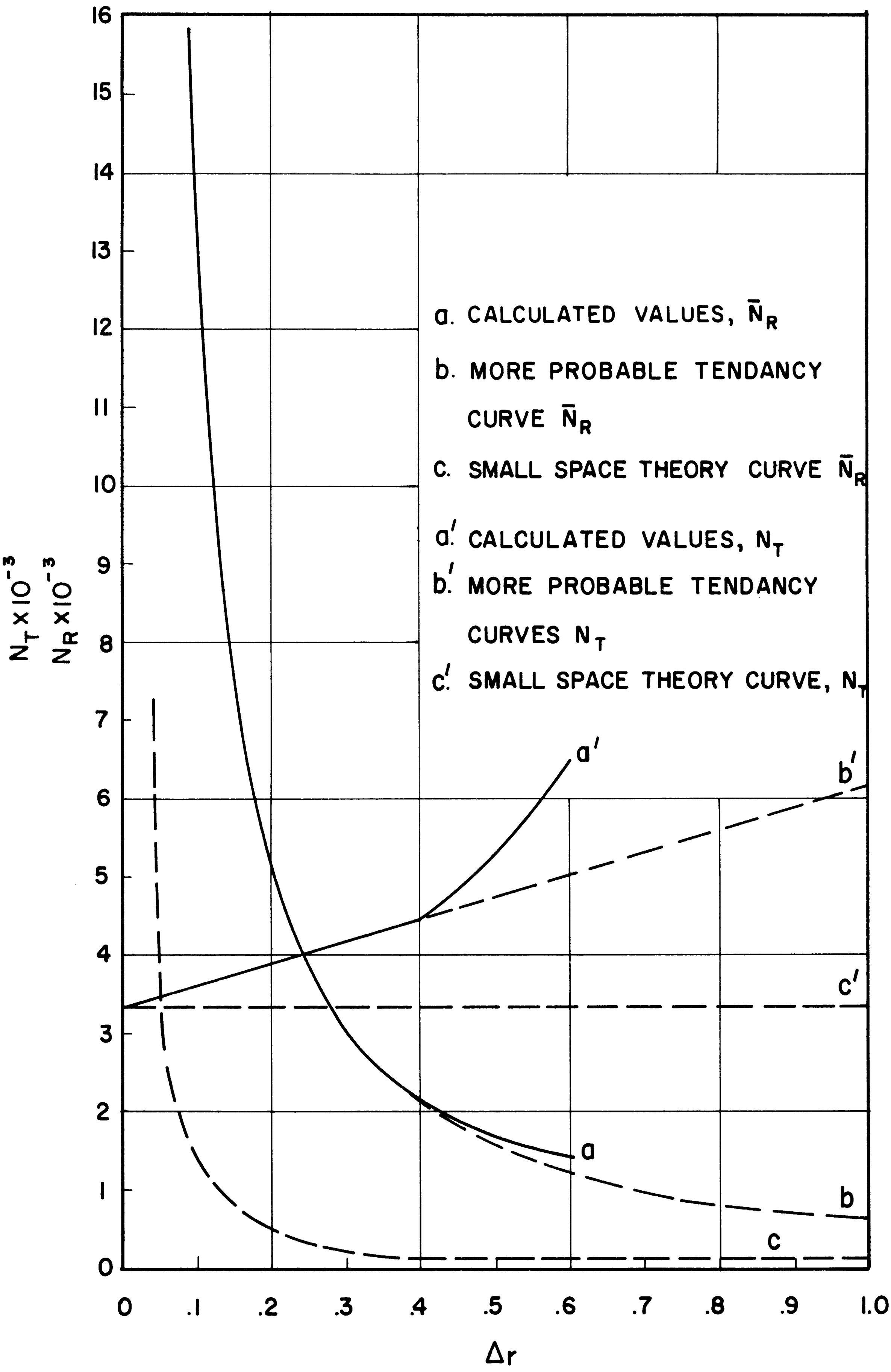


Figure 4. Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

cates how the values of $\bar{N}_R$ and N_T vary in the wide space interval. These figures are presented merely to show how the tendencies of the various measures of stability give different impressions of stability of the flow.

The N_T and N_R profiles give the impression that wider spacing in this range of Δ_r is stabilizing, whereas the $\bar{N}_R$ profile indicates that it is destabilizing. The data in Table II and in Figure 5 show the destabilizing effect of increasing Δ_r at small spacing for $\Delta_\omega = -1$ and the strong stabilizing effect as Δ_r aproaches the lability line. It should be noted that because of the small spacing the information presented in Table II and Figure 5 is essentially that of small space theory, which is indicated by the relative constancy of the Taylor number.

The expansions used for solution are, in general, valid up to values of Δ_r of about .5. The curves in Figures 2, 3, and 4 indicate that there may be an error due to the incomplete convergence of the various series, although examination of the U, V and T series indicates that at $\Delta_r = .5$, the results are satisfactory to about one part in a hundred. A less stringent convergence criterion was used here, because the number of interations tends to be excessive in this range of Δ_r if the criterion used is too strong. The information determined by Chandrasekhar(2) at $\Delta_r = 1$, was obtained from an expansion valid in this range and is more likely to be correct. The slowness of convergence probably accounts for the discrepancy in a in the upper range of Δ_r as well.

TABLE II

NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING
FOR $\Delta_\theta = 0$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$
No Dissipation Terms. No Variation of
Viscosity and Thermal Conductivity.

Δ_r	a	N_R	$\bar{N}_R$	N_T
10^{-4}	3.12	3.1688	316.88×10^6	1716.4
10^{-3}	3.12	10.347	10.347×10^6	1716.4
2.0×10^{-3}	3.12	16.871	4.2178×10^6	1716.4
3.0×10^{-3}	3.12	25.229	2.8032×10^6	1716.4
3.5×10^{-3}	3.12	31.369	2.5607×10^6	1716.5
3.7×10^{-3}	3.12	34.579	2.5259×10^6	1716.5
4.0×10^{-3}	3.12	40.822	2.5514×10^6	1716.5
5.0×10^{-3}	3.12	239.09	9.5638×10^6	1716.5

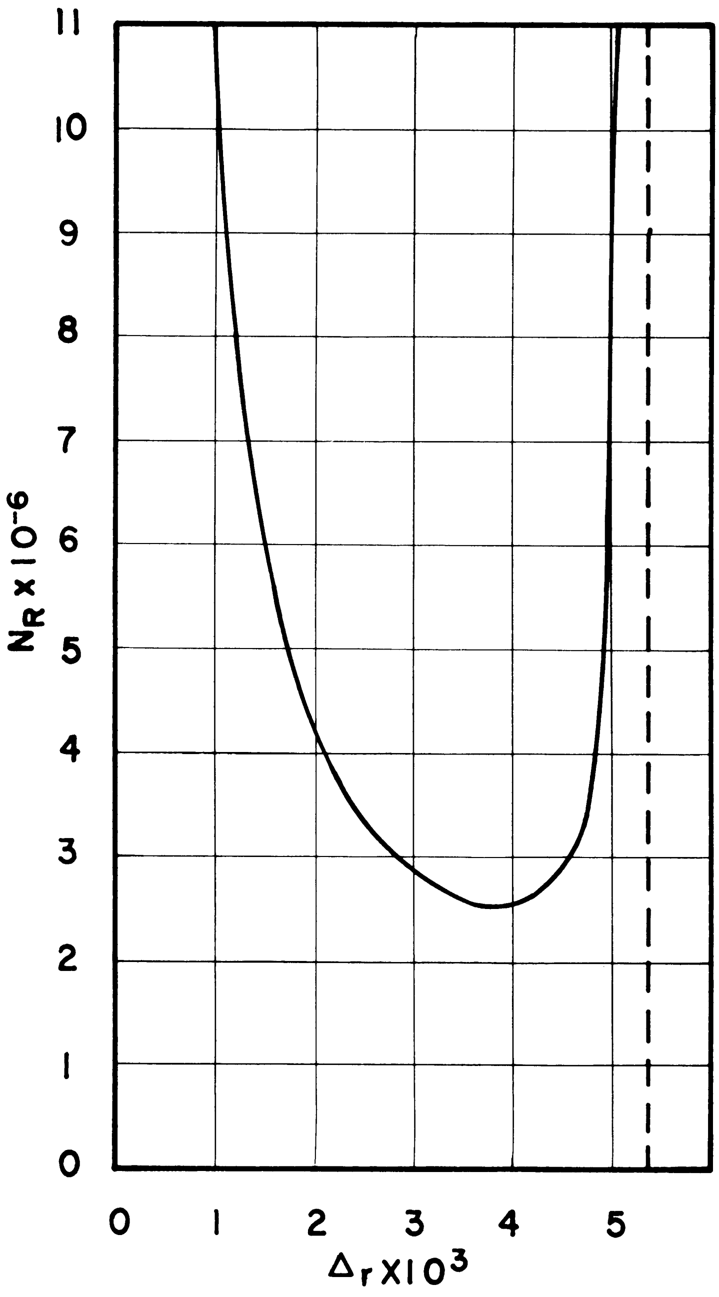


Figure 5. Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

The curves in Figures 4 and 5 clearly indicate that the flow for a given Δ_ω is destabilized as Δ_r grows larger except near the lability line. This means the decreasing influence of the walls by means of the viscosity is strong compared to the changing circulation gradient throughout most of the range of Δ_ω and Δ_r. However, near the point of Δ_r where the circulation gradient changes sign, that is, near the lability line, the effect of the circulation gradient is dominant.

The value $\Delta_\omega = -1$ as the main value of Δ_ω studied was chosen as being indicative of the general tendencies as well as being the value of Δ_ω for a minimum Reynolds number. At a given value of Δ_ω near $\Delta_\omega = 0$, the flow becomes absolutely stable as soon as Δ_r is increased to a finite value, which renders this region of flow relatively uninteresting for more detailed study. The lability line goes to $\Delta_r = \infty$ at $\Delta_\omega = -1$, which implies labile flow for all Δ_r.

A secondary consideration is important to this work; when the runs at higher Mach numbers and the runs that include dissipation are considered, a basis of comparison is needed. To get significant heat generation, a non-zero shear stress is required, and the shear stress requires that the magnitude of Δ_ω be finite. The expense of running a wide range of Δ_ω or any other parameter was prohibitive at this time.

It is tempting to try to apply the results of studies of this sort to the flows on single curved surfaces. The qualitative aspects of these flows are similar, but the quantitative aspects are undoubtably quite different. Even for the moderate range of Δ_r covered in this study, divergences are quite apparent; the values at $\Delta_r = \infty$ would be greatly different.

The decreasing values of a rather than the rising values found by Chandrasekhar, in the larger Δ_r range, is also probably due to the poor convergence of the series in this range.

Table II gives the values for the neutral stability Reynolds number for various values of Δ_ω at $\Delta_r = .1$. A curve is shown for comparison at $\Delta_r = 10^{-4}$ in Figure 4. These results are for $\Delta_\theta = 0$, $\bar{N}_M = 10^{-5}$ and no dissipation or variation of viscosity and thermal conductivity with temperature. Most of the data for the $\Delta_r = 10^{-4}$ curve is due to Chandrasekhar(1), but his values were verified at six points during the course of this work. The dashed curve for $\Delta_r = 0.1$ represents the value of N_R that would be obtained if the small-spacing theory were used instead of the wider-spacing theory. The transition from absolute stability to lability is clearly indicated here by the tendency for the curves to go to infinity at $\Delta_\omega \simeq 0$ for $\Delta_r = 10^{-4}$ and at $\Delta_\omega \simeq .2$ for $\Delta_r = 0.1$. The minimum value of N_R occurs at $\Delta_\omega = -1.0$ as expected in these curves. See Table III and Figure 6.

Tables IV, V, and VI give the values of N_R for various values of Δ_θ with and without the inclusion of the variation of viscosity and thermal conductivity with temperature. These values were calculated for $\Delta_\omega = +.01$, $\Delta_\omega = -.01$, and $\Delta_\omega = -1.0$ to illustrate the contrasting behavior in regions near and far from solid body rotation, and to show labile flow is possible for a positive circulation gradient. Figure 7 illustrates these values. Table VI gives corresponding values at $\Delta_r = .1$. All of the results here were obtained for values of $\bar{N}_M = 10^{-5}$ and $\Delta_r = 10^{-4}$ and no variation of viscosity and thermal conductivity with temperature or dissipation were included.

TABLE III

NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL ANGULAR VELOCITY FOR $\Delta_r = .1$, $\Delta_\theta = 0$, $\bar{N}_M = 10^{-5}$
No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

$\Delta\omega$	a	N_R	$\bar{N}_R$
- .2	3.14	84.100*	8410.0*
- .4	3.14	21.350**	2135.0**
- .5	3.14	17.585	1758.5
-1.0	3.14	13.785	1378.5
-1.5	3.16	15.841	1584.1

* Valid to one part in a hundred.

** Valid to one part in a thousand.

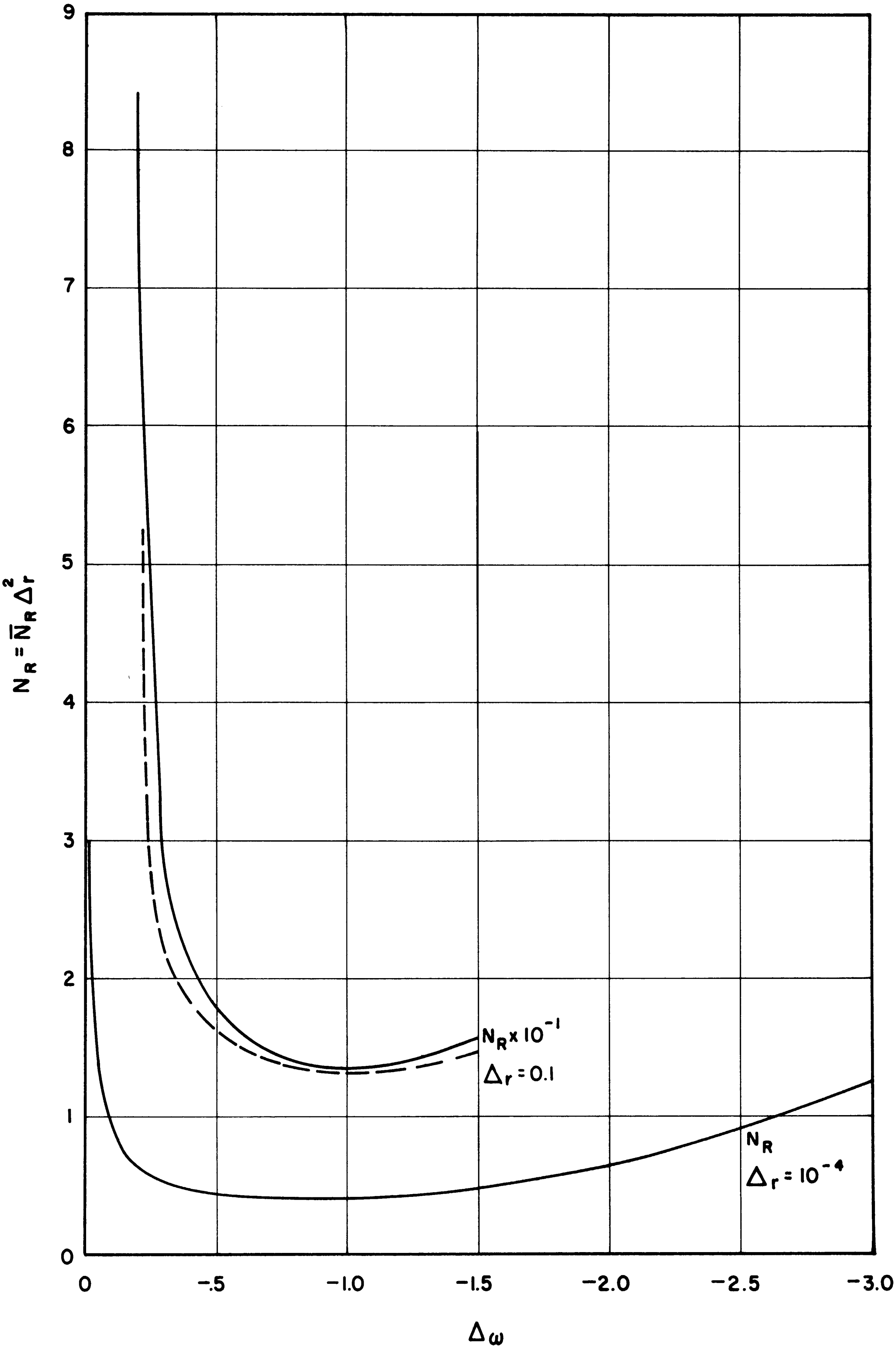


Figure 6. Neutral Stability Reynolds Number vs Differential Angular Velocity for $\Delta_\theta = 0$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

TABLE IV

NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = -.01$, $\bar{N}_M = 10^{-5}$

No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity.

Δ_θ	a	N_R without viscosity variation	N_R with viscosity variation	% Change
-.02	3.12	5.7847	5.7555	- .519
-.01	3.12	3.7200	3.7105	- .242
0	3.12	2.9590	2.9590	0
+.01	3.12	2.5340	2.5402	+ .237
+.02	3.12	2.2538	2.2650	+ .488
+.05	3.12	1.7738	1.7958	+1.24
+.10	3.12	1.3979	1.4323	+2.43
+.20	3.12	1.0702	1.2210	+4.86
+.50	3.12	.75259		

TABLE V

NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = +.01$, $\bar{N}_M = 10^{-5}$

No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

Δ_θ	a	N_R
+.02	3.12	5.6125
+.05	3.12	3.3877
+.1	3.12	2.3851

TABLE VI

NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = 10^{-4}$, $\Delta_\omega = +.01$, $\bar{N}_M = 10^{-5}$
No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity Included.

Δ_θ	a	N_R without viscosity variation	N_R with viscosity variation	% Change
-.10	3.14	.41350*		
-.02	3.14	.41250*	.41108	- .433
-.01	3.14	.41211	.41139	- .215
0	3.14	.41170	.41170	0
+.01	3.14	.41108	.41201	.219
+.02	3.14	.41049	.41233	.438
+.05	3.14	.40874	.41428	1.37
+.10	3.14	.40599	.41690	2.69
+.20	3.14	.40096	.4292	5.21
+.50	2.92	.39729		

* Valid to one part in a thousand.

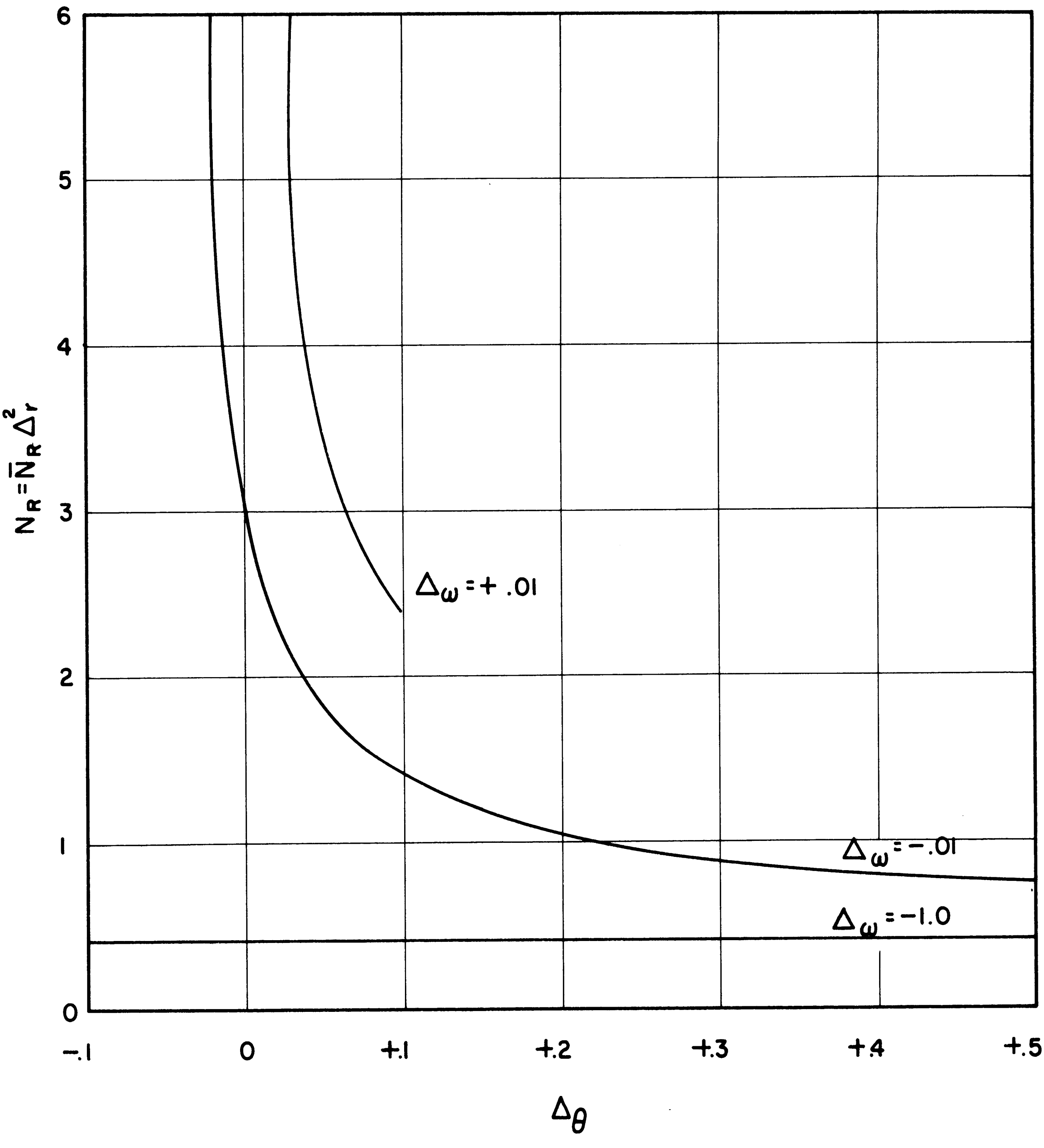


Figure 7. Neutral Stability Reynolds Number vs Differential Temperature for $\Delta_r = 10^{-4}$, $\bar{N}_M = 10^{-5}$. No Dissipation Terms Included. No Variation of Viscosity and Thermal Conductivity.

The results indicate the relative sensitivity of the flows near $\Delta_\omega = 0$ to variations in temperature because of the proximity of the lability line. The flow readily changes from absolutely stable to labile with small changes in differential temperature. A positive temperature gradient of about .025 is required for the transition when $\Delta_\omega = +.01$ and Δ_θ of about -.025 for $\Delta_\omega = -.01$. Compared to the results near $\Delta_\omega = 0$ the results near $\Delta_\omega = -1.0$ are quite striking by their lack of significant variation. This is indicative of the remoteness of the lability line and the small spacing. The work of Yih[13] shows this same phenomenon in a different form. Also see Table VII.

The effect of the variation of the viscosity and thermal conductivity with temperature, shown in Tables IV, V, and VI and Figure 8, is not spectacular in any way. As anticipated, the flow is stabilized slightly in the labile region. The effect is slightly larger for $\Delta_\omega = -1.0$ than for $\Delta_\omega = -.01$ probably due to the effect of the larger shear stresses. The variation with the larger spacing is nil. The small variation is, of course, partially due to the small size of the Prandtl number for a gas. The results would probably be more interesting for a liquid which has a larger Prandtl number and a wider variation of viscosity with temperature.

The results for the effects of compressibility are recorded in Table VIII and Figure 9, along with the effects of the dissipation factor in the energy equation. The results are for $\Delta_\theta = 0$ and include no variation of viscosity and thermal conductivity with temperature.

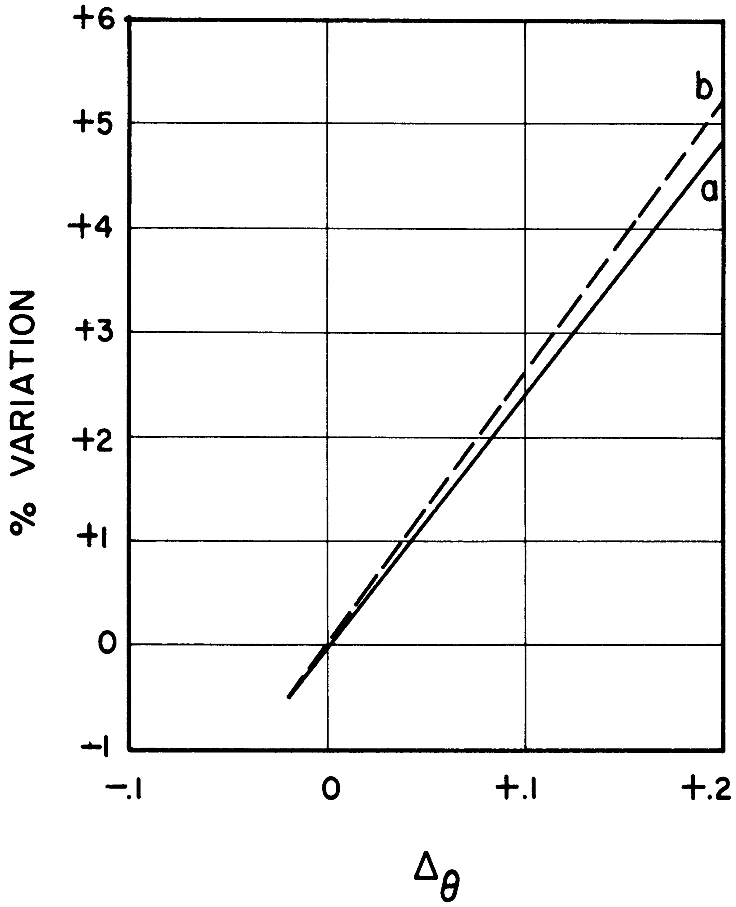
The curves for $\bar{N}_M = 1$ and $\bar{N}_M = 2$ are interesting only in view of the small amount of variation that is shown. An estimate of N_R made by

TABLE VII

NEUTRAL STABILITY REYNOLDS NUMBER VS DIFFERENTIAL TEMPERATURE FOR $\Delta_r = .1$, $\Delta_\omega = -1.0$, $\bar{N}_M = 10^{-5}$
No Dissipation Terms. With Variation of Viscosity and Thermal Conductivity Included.

Δ_θ	a	N_R without viscosity variation	N_R with viscosity variation	% Change
0	3.14	13.785	13.785	0
+.01	3.14	13.764	13.802	.291
+.02	3.14	13.743	13.819	.582
+.05	3.14	13.681	13.868	1.39
+.10	3.14	13.600*	13.940*	2.69

* Valid to one part in a thousand.



a. $\Delta_\omega = -.01$, $\Delta_r = 10^{-4}$

b. $\Delta_\omega = -1.0$, $\Delta_r = 10^{-4}$ and $\Delta_r = .1$

Figure 8. Percent Variation of Neutral Stability Reynolds Number with Thermal Variation of Viscosity and Thermal Conductivity vs Differential Temperature for $\bar{N}_M = 10^{-5}$. No Dissipation Terms Included.

TABLE VIII

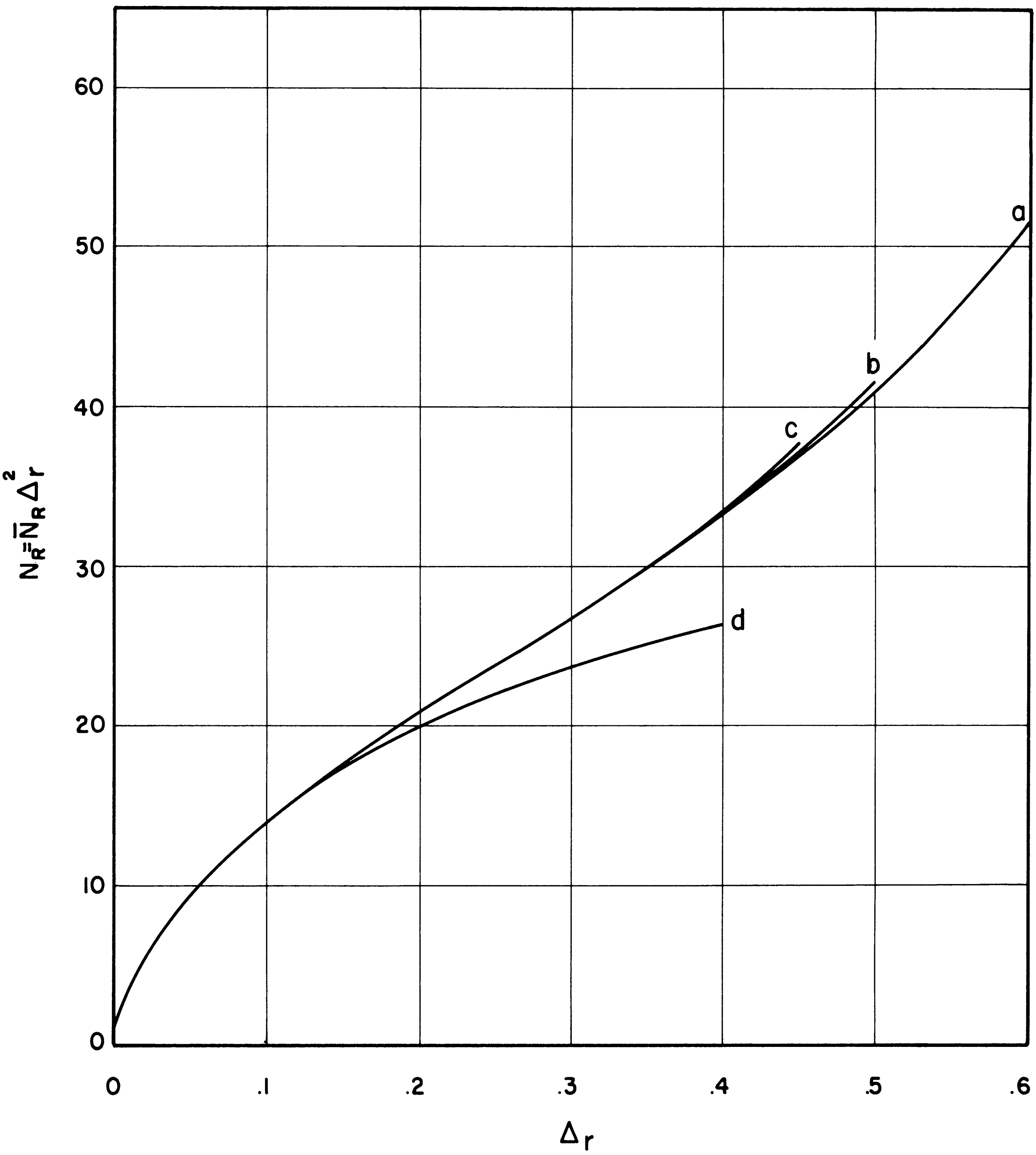
NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$

No Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

	$\bar{N}_M = 1$		$\bar{N}_M = 2$	
Δ_r	a	N_R	a	N_R
10^{-4}	3.14	.41173	3.14	.41172
.1	3.14	13.790	3.14	13.788
.2	3.20	20.694	3.24	20.681
.3	3.27	26.582	3.36	26.709
.35			3.45	29.920*
.40	3.35	33.270*	3.56	33.600**
.45	3.45	37.200**	3.68	37.700**
.50	3.57	41.800**		

* Valid to one part in a thousand.

** Valid to one part in a hundred.



a. $\bar{N}_M < 1$, no dissipation terms
b. $\bar{N}_M = 1$, no dissipation terms
c. $\bar{N}_M = 2$, no dissipation terms
d. $\bar{N}_M = 1$, with dissipation terms

Figure 9. Neutral Stability Reynolds Number vs Spacing for $\Delta_\theta = 0$, $\Delta_\omega = -1$. No Variation of Viscosity and Thermal Conductivity.

hand calculation and based only on the entropy gradient, which can only be rough at best, indicates that the variation should be in excess of 10% at $\Delta_r = .4$ and $\bar{N}_M = 2$. The calculated variation is only about 2%. This is probably due to the modification of the U profile by the assimila- of the pressure perturbation. Future work should clarify the roles of the various terms in the equations.

For values of Δ_ω nearer to zero the effects would have been more striking. In fact, at $\Delta_\omega = -.00005$ and $\Delta_r = 10^{-4}$ there is a transition to absolute stability at about $\bar{N}_M = 0.5$. The variations of neutral stability Reynolds number with $\bar{N}_M$ as they were with Δ_θ are only large in the vicinity of a lability line. Unfortunately, in this area which is most interesting, the solution process is very sensitive and no answers were obtainable.

The curve in Figure 9 which shows the effects of the inclusion of the dissipation term in the temperature equation is interesting because it indicates that this effect is quite strong compared to the effect of the compressibility. The destabilization is caused by the strength of the positive mean temperature gradient near the inner cylinder, as well as the positive temperature gradient contributed by the disturbed flow. The negative temperature gradient near the outer cylinder is de-emphasized by the low centrifugal effect there. Also see Table IX.

The minor variation of N_R with dissipation at low spacings is undoubtedly due to the strong damping influence of the viscosity at small spacing and the dominant influence of the circulation gradient at small spacings. The influence of the second term on the right-hand side of (38) is greater at wider spacings.

TABLE IX

NEUTRAL STABILITY REYNOLDS NUMBER VS SPACING
FOR $\Delta_\theta = 0$, $\Delta_\omega = -1.0$, $\bar{N}_M = 1$
With Dissipation Terms. No Variation of Viscosity and Thermal Conductivity.

Δ_r	a	N_R
10^{-4}	3.14	.41174
.1	3.14	13.791
.2	3.26	20.111
.3	3.54	23.743
.35	3.96	25.204*
.40	4.04	26.455**

* Valid to one part in a thousand.

** Valid to one part in a hundred.

A single run at a single value of a was made to illustrate the conjecture that labile flow is possible with a positive circulation gradient and a positive density gradient. The calculation was made for

$$\Delta_r = 10^{-4}, \quad \bar{N}_M = 6, \quad \Delta_\theta = .216 \times 10^{-2}, \quad \text{and} \quad \Delta_\omega = + 10^{-4}, \; a = 3.14,$$

which satisfies the conditions specified in the previous chapter. This gives a density gradient of about 14.4×10^{-4} and $D_x^+ \bar{v} \simeq 3.0$ with a positive $\frac{\bar{\rho}\bar{v}}{r}$. The entropy gradient is also positive, of course, since it provides the essential driving potential. The Reynolds number at neutral stability for these conditions is 8.3418, a real number, which indicates that the flow is indeed labile. More work needs to be done here to see the influence of the pattern of assimilation of pressure disturbance, which probably emphasizes this effect.

Conclusion

The conclusions of this study may be summarized as follows:

1. Small spacing theory is inadequate for studying the effects of compressibility.

2. In the region studied, the effects of spacing accounted for a significant variation in the neutral stability Reynolds number.

3. Increased spacing can be both stabilizing and destabilizing, depending on the proximity to the lability line.

4. Compressibility effects are small as compared to other effects, except at high Mach numbers and wide spacing, because of the speed at which a pressure pulse is assimilated. The effects are slightly stabilizing.

5. The effects of adding the dissipation terms to the energy equation are destabilizing to a degree greater than the compressibility effects are stabilizing.

6. The inclusion of the effects of the variation of viscosity and thermal conductivity stabilizes the flow slightly.

7. All effects are stronger near the lability line than they are away from it.

8. Increasing the viscosity never has a destabilizing effect for a gas, as it does in some cases for a liquid, in fact, by examing the neutral stability angular velocity it can be seen to be always stabilizing.

9. The entropy gradient acts as a driving potential for the flow, however, its effects are modified by the rate at which a pressure disturbance is assimilated. However, a sufficient condition for stability still is a positive circulation gradient and a negative entropy gradient.

10. A positive density gradient and a positive circulation gradient are not sufficient conditions for stability.

CHAPTER IV

POSSIBLE EXTENSIONS

The present program is capable of calculating the values of N_R between Δ_ω of about +3. to Δ_ω of about -3. up to a Mach number of about 10, up to a spacing Δ_r of about .5 , for values of Δ_θ between about +.5 and -.5 , with or without the effects of viscosity variation and with or without effects of dissipation. The expense, however, is very large; it is probably excessive for the value received. There are many ways of improving the situation.

Before covering these improvements, it is desirable to say that the general technique can be applied to a wide variety of ordinary differential equations, linear or non-linear, that have analytic solutions in a closed region. An experienced programmer can program, with the use of the existing subroutines, the solution of a set of equations of the size of the ones covered in this work in about four hours. Very little "debugging" is required.

The use of a polynomial in the powers of x is not the best expansion to use in problems of this sort. Expansions in several of the more common sets of orthogonal functions will give much more rapidly converging solutions. Instead of the twenty-five terms required for a satisfactory answer, as few as five terms in a trigonometric series gives the same results.

Expansions in terms of trigonometric functions and Tchebyshev polynomials have been tried on the simple Taylor problem with very satisfactory results. Some of the other Jacobi polynomials would probably give even better results. A natural choice of functions, because of the cylindrical nature of the problem, would be the Bessel functions. In general, the closer the generating differential equation of the orthogonal functions is to the equation to be solved, the better the results.

Much time is spent in the present program finding the value of a at which N_R is a minimum. The procedure merely calculates N_R at successive values of a at some predetermined interval between predetermined limits. A simple and accurate process can be included to rapidly locate the minimum point.

By including, with these improvements, some programming techniques which would speed up the processing, it is estimated that the results obtained in this work could be calculated for less than $100.00 on an IBM 7090. Once the changes in technique are incorporated into the calculation system, it is possible to generate as complete a set of curves for a wide range of the parameters as is desired, and to do it cheaply.

It would be interesting to see a set of neutral stability curves for, say, ten values of each of Δ_r, Δ_ω, Δ_θ and $\bar{N}_M$, and the location of the lability lines for each of these parameters. By including the complex forms of the equations, both the real and imaginary parts of σ at various values of N_R can be studied. It is desirable to know whether the imaginary part of σ is always zero near the point of neutral stability.

It would be particularly interesting to look at the solution near the lability line for the effects of compressibility. The rapid assimilation of a pressure pulse to its surroundings will probably have a significant effect on the location of the line and on the character of the results near it. These results require the better solution technique described in a previous paragraph to reduce the amount of machine time required to gain an answer.

A slightly different form of these equations will help get a solution near the lability line, since the lability line implies that the Reynolds number is infinite, an expansion in the inverse powers of the Reynolds number will aid the solution processes. The formal expansion of Equations(30) through (35) would lead to a degenerate set of equations as the zeroth approximation. But if the solution procedure itself is altered to incorporate an expansion, rapid convergence can be expected with only the second and third powers of the inverse Reynolds number. Solutions close to the lability line should be easily obtained this way.

An improvement that would aid in the analysis of the results would be the printing of the curves of all the terms in the differential equation or at least those of particular interest. This could be done for the cost of the time to print the answer. This would aid in analyzing such things as the role of certain parts of the equations such as the effect of the rapid assimilation of a pressure pulse.

The general techniques used here are applicable to the solution of the problem of the stability of flow over curved surfaces. The problem is to get satisfactory expansion valid throughout the whole region of flow or throughout the significant region of flow i.e., the boundary layer. Rather than reach conclusions from indirect approaches, such as interpreting the results of this work, it would be better to attack the curved surface problems directly. There would be no real difficulty involved even though the general case involves solution of partial, rather than ordinary, differential equations.

The solution of the boundary value problems of physics, whether they be exemplified by ordinary or partial differential equations, is only a matter of effort and money with the use of high speed digital computation. This does not mean that care should not be taken. The method, expansions and approximations, have to be fitted to the problems. Just any procedure will not work. As an example, the use of the usual numerical methods for the solution of problems of the sort discussed in this paper fail badly. But it may certainly be said that any problem of physical origin can be handled with careful application of suitable techniques. The important thing is the cost. The more independent variables or parameters the problem has, and the more complex the equation, the higher the cost. The most significant sections of a problem must be selected and filtered. Any analytic technique used to get answers to boundary problems can be used on a digital computer, usually with more speed, less cost and higher accuracy.

If it is desirable, the machine can give algebraic answers rather than numerical answers. It can perform any formal mathematical process and even do a limited amount of the selection of the techniques to be used.

Experimental verification of some of the results of this study would be desirable. This is particularly true for the small Mach effect and the comparatively large effect of dissipation. Experiments of this sort with gases would require extremely delicate and precise apparatus, in addition to highly controlled temperature conditions. The investment in material and labor would be large, and the results would probably not be conclusive. A more extensive set of numerical results would be necessary first as a guide to the best regions of the various variables to run the experiments.

APPENDIX I

SOLUTION PROCEDURE

All of the calculations were done on the IBM 709 electronic data processing machine at the University of Michigan. Most of the pertinent calculations were performed in the floating point mode which has a basic precision of eight significant decimal figures.

The procedure used by Chandrasekhar(1) is not satisfactory for handling a problem as general as this one. There is difficulty in getting expansions for the complementary solution of (36) which converges rapidly enough to give good numerical results. Furthermore, Equations (37) and (38) are sufficiently unlike each other in the general case that they cannot be reduced to one equation. Therefore, a new procedure was developed to handle this problem.

The basic calculations procedure is based on the assumption that any variable in the range $0 \leq x \leq 1$ may be approximated by a twenty-five term polynomial in the powers of x. In other words, any function of x within the scope of this problem is satisfactorily represented by a Taylor series truncated to twenty-five terms. Thus, the function f may be represented by $f = \sum_{n=0}^{24} f_n x^n$. All of the calculations deal with the twenty-five coefficients f_n as an ordered array, except, of course, where actual numerical values of the functions are required. The array of coefficients f_n may be thought of as a vector in a 25 space and is referred to as such in the subsequent text.

The usual numerical procedures divide the interval into small segments. The calculation proceeds by determining, by some algorithm, the values of the various functions being calculated at each segment in turn. In contrast to this, the calculation procedure used here calculates the value of the function for the whole of the x interval at one time. No segmenting of the interval is required.

Algorithms are necessary for the various operations on these vectors. The basic processes used are:

1. Addition of two vectors
2. Subtraction of two vectors
3. Multiplication of a vector by a scaler
4. Product of two vectors
5. Derivative of a vector
6. Integral of a vector
7. Reciprocal of a vector
8. Square root of a vector
9. Exponentiation of a vector.

After every operation, the resultant vector is surveyed and all terms less than 10^{-14} are set to zero to prevent underflow in future operations. This does not limit the accuracy of the calculations because the U vector, as will be seen later, is always set equal to one near the midpoint of the interval.

The first three of these do not require any special comment. The product of two vectors uses the equation $C_i = \sum_{j=0}^{i} a_{i-j} b_j$ to calculate the individual terms. Any coefficient of powers of x higher than 24 are disregarded. The derivative operation uses the equation

$$b_j = (j + 1)\, a_{j+1}; \quad j = 0, 1, \ldots 23$$

The 24th term is always set to zero. The integral, in a similar fashion, is calculated by $b_{j+1} = \frac{a_j}{j+1}$; $j = 0, 1, \ldots 23$. The 24th a_j is always discarded and the zeroth term is set to zero. Whenever these operations are referred to in the following text, the symbols D which represents the derivative operation and I the integral operation, are used. Successive integrations will be noted by powers of I; for example, $I^3 f$ represents the third integral of f. It is important to note that the first three terms of $I^3 f$ will be zero.

The reciprocal of a vector is calculated by noting that if the vector b is the reciprocal of the vector a, then $\sum_{j=0}^{24} a_{i-j} b_j = 1$. First b_o is determined by $b_o = \frac{1}{a_o}$, then b_1 from $a_o b_1 + a_1 b_o = 0$, etc. This process is only valid for sequences that converge fairly rapidly. This presents one of the principal limitations of this system. The division process is accomplished by successive use of the reciprocal operation and the multiplication operation.

The square root of a vector is accomplished in a manner similar to the reciprocal process. That is, if $b = \sqrt{a}$ then b^2 must equal a and $\sum_{j=0}^{24} b_{i-j} b_j = a_i$ then $b_o = \sqrt{a_o}$ and $b_1 = \frac{a_1}{2b_o}$ etc.

The exponentiation process is not a general one. It assumes that the first term in the exponentiated series is zero, which it is in use, and that the remaining terms represent a function which is less than one in magnitude throughout the interval. This is also true in the way it is used. The exponential is calculated then from the Taylor expansion $b = e^a = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots$, in which each of the powers of a is calculated by means of the algorithm for the product of vectors. Each of these procedures is coded in a subroutine so that they need not be recoded each time they are used.

The mean flow Equations (14) through (17) are handled in the following way.

From (15)

$$\frac{d}{dr} q + \frac{2q}{r} = 0$$

in which

$$q = \bar{A}_\mu D_r^- \bar{v} .$$

This has the solution

$$q = - 2c_2 r^{-1}$$

in which c_2 is the integration constant; therefore,

$$\bar{A}_\mu \frac{d}{dr} (\frac{\bar{v}}{r}) = - 2c_2 r^{-3}$$

Integrating

$$\frac{\bar{v}}{r} = - 2c_2 \int \frac{dr}{\bar{A}_\mu r^3} + c$$

or

$$\bar{v} = c_1 r - 2c_2 r \int \frac{dr}{\bar{A}_\mu r^3} .$$

In terms of the operational notation,

$$\bar{v} = c_1 r - 2c_2 r \, I \, (\frac{1}{\bar{A}_\mu r^3}) , \qquad \text{(I-1)}$$

c_1 and c_2 are integration constants. r is a two term vector $\Delta_r^{-1} + x$ and in general $\bar{A}_\mu$ is a vector. In particular, $\bar{A}_\mu$ is used either as $\bar{A}_\mu = 1$ or $\bar{A}_\mu = \sqrt{\bar{\Theta}}$ throughout the whole calculation, $\bar{A}_k = \bar{A}_\mu$ is used in all the work.

From (17),

$$D_r^+\left(\bar{A}_k\,(D_r\bar{\Theta})\right) + \bar{A}_\mu\,(1 - \tfrac{1}{\gamma})\,N_{Pr}N_M^2\,(D_r^-\bar{v})^2 = 0,$$

or by substituting q,

$$\frac{1}{r}\,\frac{d}{dr}\left(r\bar{A}_k\,\frac{d\bar{\Theta}}{dr}\right) + \frac{q^2(1 - \frac{1}{\gamma})N_{Pr}N_M^2}{\bar{A}_\mu} = 0,$$

or

$$\frac{d}{dr}\,(r\,\bar{A}_k\,\frac{d\bar{\Theta}}{dr}) = \frac{-4c_2^2(1 - \frac{1}{\gamma})\,N_{Pr}\,N_M^2}{\bar{A}_\mu r^3}\,.$$

Then by integrating,

$$r\bar{A}_k\,\frac{d\bar{\Theta}}{dr} = -\,4c_2^2\,(1 - \tfrac{1}{\gamma})\,N_{Pr}N_M^2\int\frac{dr}{\bar{A}_\mu r^3} + b_1$$

or

$$\frac{d\bar{\Theta}}{dr} = \frac{-4c_2^2\,(1 - \frac{1}{\gamma})N_{Pr}N_M^2}{\bar{A}_k r}\int\frac{dr}{\bar{A}_k r^3} + \frac{b_1}{r\,\bar{A}_k}\,.$$

By integrating again,

$$\bar{\Theta} = -\,4c_2^2\,(1 - \tfrac{1}{\gamma})\,N_{Pr}N_M^2\int\frac{dr}{r\bar{A}_k}\int\frac{dr}{r^3\bar{A}_\mu} + b_1\int\frac{dr}{r\bar{A}_k} + b_2,$$

in which b_1 and b_2 are integration constants, in terms of the operators.

Then

$$\bar{\Theta} = -\,4c_2^2\,(1 - \tfrac{1}{\gamma})N_{Pr}N_M^2\,I\left(\frac{1}{r\bar{A}_k}\;I\left(\frac{1}{r^3\bar{A}_\mu}\right)\right) + b_1 I\left(\frac{1}{r\bar{A}_k}\right) + b_2\,.$$

The boundary conditions are repeated here.

At $x = 0,\ r = \Delta_r^{-1},\ \bar{\Theta} = 1$ and $\bar{v} = \Delta_r^{-1}$

and

at $x = 1,\ r = 1 + \Delta_r^{-1},\ \bar{\Theta} = 1 + \Delta_\Theta$ and $\bar{v} = (1 + \Delta_\omega)(1 + \Delta_r^{-1})$.

Since none of the integral functions have a zeroth order term, the integrals are all zero at $x = 0$. Therefore, by applying the boundary conditions, the integration constants are:

$$c_1 = 1,\quad b_2 = 1,$$
$$c_2 = -\frac{\Delta_\omega}{2I_1\left(\frac{1}{r^3A_\mu}\right)}, \tag{I-3}$$
$$b_1 = \frac{\Delta_\Theta + 4c_2^{\,2}(1 - \frac{1}{\gamma})N_{Pr}N_M^{\,2}\left[I_1\left(\frac{1}{r\bar{A}_k}\, I\left(\frac{1}{r^3\bar{A}_\mu}\right)\right)\right]}{I_1\left(\frac{1}{r\bar{A}_k}\right)}.$$

The symbol I_1 is used to indicate that the integral function is evaluated at $x = 1$. From (14) and (16),

$$\frac{d}{dr}\bar{p} = N_M^{\,2}\,\frac{\bar{p}\bar{v}^2}{r\bar{\Theta}^2}$$

which has the solution

$$c\bar{p} = \exp\left(I\left(\frac{\bar{v}^2N_M^{\,2}}{r\bar{\Theta}}\right)\right). \tag{I-4}$$

c is the integration constant. Likewise,

$$\bar{\rho} = \frac{cN_M^{\,2}}{\bar{\Theta}}\exp\left(I\left(\frac{\bar{v}^2N_M^{\,2}}{r\bar{\Theta}}\right)\right). \tag{I-5}$$

The calculation of these functions proceeds in the following way:

1. Estimate c_2 and $\bar{\theta}$.
2. Calculate $\bar{A}_k$ and $\bar{A}_\mu$
3. Perform the calculations to get the integrals.
4. Evaluate the integrals.
5. Determine b_1 and c_2 .
6. Calculate $\bar{\theta}$.
7. Test to see if new c_2 is sufficiently close to the old c_2 . If the test is satisfactory, go to 8., if not, return to 2.
8. Test to see if components of the new $\bar{\theta}$ vector are sufficiently close to the old $\bar{\theta}$ vector. If the test is satisfactory, go to 9., if not, return to 2.
9. Calculate $\bar{v}$, $\bar{p}$ and $\bar{\rho}$.

The constant c in (I-4) is chosen such that $I_1(\bar{\rho}) = 1$. This guarantees that there is the same total mass of fluid between the cylinders for all runs.

Usually no more than five iterations are required to obtain the one part in a million accuracy demanded on c_2 and the $\bar{\theta}$ function. A few results were spot checked by hand and always found to be satisfactory.

The perturbed equations are solved in a slightly different way but the basic procedures are the same. The program for the solution is constructed directly from the Equations (36), (37), and (38) without any further reduction. The reduction process would be a long, tedious algebraic process resulting in a very clumsy and inefficient form for programming.

It is relatively easy to program directly from the above forms because the reduction algebra can be handled satisfactorily in the computer.

Each differential equation is considered to be an array or matrix consisting of five vectors. The zeroth column vector represents the function which is the coefficient of the zeroth order derivative term in the differential equation. The first column is the coefficient of the first order derivative; and so forth. Operational subroutines are used on these matrices to reduce the programmed equations to the equivalent canonical form.

$$(a_4D^4 + a_3D^3 + a_2D^2 + a_1D + a_0)f = a_5Dg + a_6g + a_7Dh + a_8h$$

in which the a_n's are the coefficient function vectors. The operational subroutines perform such operations as addition, subtraction, multiplication, taking a derivative, etc. on the differential equation matrix.

The differential system solved consists of two second order equations and a fourth order equation of the following form:

$$(D^4 + a_3D^3 + a_2D^2 + a_1D + a_o)\ U = N_R^2(a_5V + a_6T). \qquad \text{(I-6)}$$

$$(D^2 + b_1D + b_o)\ V = b_3U + b_4T + b_5DT\ . \qquad \text{(I-7)}$$

$$(D^2 + c_1D + c_o)\ T = c_3U + c_4V + c_5DV\ . \qquad \text{(I-8)}$$

The objective is to find the values of N_R^2 which satisfy this differential system subjected to the boundary conditions

$$U = DU = T = V = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1\ .$$

In general, the coefficients a_n, b_n and c_n are functions of the applied conditions, the mean flow variables, and the wave number a. Once the equations are in the canonical form, a special process is used on them.

The general procedure is to estimate a solution of the differential equations and the eigenvalue; place the terms with low order derivatives on the right hand sides of the equations; integrate the appropriate number of times; apply the boundary conditions; and start over again. To be a satisfactory process, the new solution should be closer to the true solution than the earlier one. The process must give a new and better estimate of the eigenvalue as well as the eigenfunctions, at each iteration. This is not an automatic procedure. In general, integration is an error reducing process, but it is not so strong as to override a poor re-estimating technique. There are literally hundreds of ways of strengthening (or weakening) this process. Several were tried until the simple and straightforward method described below was tried.

The fourth order Equation (I-6) is rearranged in the following form,

$$D^4U = g + N_R{}^2 h \tag{I-9}$$

in which g and h are functions of the last estimate of V, U and T and the coefficient functions a_n. Since, in general, U, V and T are not the true solutions at any stage of the iteration process, they contain errors, therefore, g and h are erroneous.

The form (I-9) is integrated four times. Then,

$$U = I^4 g + I^4 h\,(N_R{}^2) + U_o + U_1 x + U_2 x^2 + U_3 x^3 \tag{I-10}$$

the U_n are integration constants. U_0 and U_1 are zero since U and its first derivative are zero at $x = 0$. For the moment, the fact that the function must be zero at the upper boundary is ignored. The function U is calculated by applying the condition $DU = 0$ at $x = 1$, and an additional condition, U at ξ, equal to some predetermined value c. The value of ξ is somewhere between the limits $x = 0$ and $x = 1$, usually taken to be .5, and c is usually chosen to be 1. The integration constants may be then determined.

$$U_2 = \frac{U(\xi)}{\xi^2} - \frac{I_\xi{}^4 g}{\xi^2} - N_R{}^2 \frac{I_\xi{}^4 h}{\xi^2} - U_3 \xi \,, \tag{I-11}$$

and

$$U_3 = \frac{1}{\xi^2(-3+2\xi)} \left[\xi^2 I_1{}^2 g - 2I_\xi{}^4 g + 2c + N_R{}^2(\xi^2 I_1{}^3 h - 2I_\xi{}^4 h) \right], \tag{I-12}$$

in which the subscripts on the I's represent values of the integrals at the place where they are evaluated, i.e., at $x = 1$ and $x = \xi$.

If the "true value" of $N_R{}^2$ were known and substituted into (I-10) along with (I-11) and (I-12) and U evaluated at $x = 1$, the value $U(1)$ would probably not be zero, but some finite value. This is due to the fact that the functions U, V and T are imperfect estimates. Since the true value of $N_R{}^2$ is not known, neither is this non-zero value of $U(1)$. By rearranging (I-10) evaluated at $x = 1$ the following form results.

$$N_R{}^2 = \frac{U(1) - \alpha U(\xi) + \left[\alpha I_3{}^4 g + \beta I_1{}^3 g - I_1{}^4 g\right]}{-\left[\alpha I_3{}^4 h + \beta I_1{}^3 h - I_1{}^4 h\right]} \,, \tag{I-13}$$

in wich

$$\alpha = \frac{1}{\xi^2(3-2\xi)} \quad , \qquad \text{and} \quad \beta = \frac{1-\xi}{3-2\xi} \ .$$

The substitution of $U(1) = 0$ into (I-13) leads to an incorrect N_R^2 in the presence of incorrect U, V and T . This value of N_R^2 is, however, used in (I-10), (I-11), and (I-12) to give a new estimate of the U function. Then in turn V and T are calculated by double integration from (I-7) and (I-8) and applying the boundary conditions.

The question arises whether or not this process at any step gives an answer which is closer to the correct solution than the values at a previous step. When the results of calculations are compared to known answers, the eigenvalues N_R^2 appear to oscillate about the correct solution with an almost equal, but slightly decreasing, amplitude. If this process is continued it converges to the correct result. In a typical case, however, the process requires as many as forty or fifty steps to get an answer valid to one part in a thousand.

The oscillating character of the solution process permits it to be improved greatly. If, at any stage, the current estimate and the last estimate of U, V and T are averaged, and this average value is used instead of the new estimate, a result is obtained which is very close to the true answer. By using the averaging process, answers valid to one part in ten thousand are obtained from the calculations from a crude initial guess in as few as four steps. The more normal run requires seven or eight steps.

The programmed procedure proceeds like this:

1. Estimate U , V , T and N_R^2 .
2. Calculate V from (I-7).
3. Calculate T from (I-8).
4. Calculate g and h .
5. Calculate the integrals of g and h .
6. Calculate N_R^2 from (I-13) and U from (I-10).
7. Compare N_R^2 to previous value, if it is satisfactory, go to 8., otherwise go to 9.
8. Compare U , V and T to previous values, if satisfactory go to 10, if unsatisfactory go to 9.
9. Average U , V and T with their previous values and return to 2.
10. Print results and exit.

The actual process deviates slightly from the above description but it is correct in essence.

The question may be asked whether or notthere is any mathematical deductive reasoning which assures the result obtained is the correct result. Up to the time of this writing, the answer is a qualified no. By the use of a rather extensive set of preliminary proofs, and under some severely restricting assumptions, it can be shown that the basic process is oscillatory and the answer oscillates around the true solution. Also, it can be shown that the averaging process does indeed give a closer estimate to the true functions. The assumptions are so severe, however, as to make the "proof" weak beyond usefulness.

No such proof is necessary, since the resulting functions fit the differential equations so well there is no doubt of their validity.

Likewise, in every case calculated, where a previous result had been obtained by another method, the two answers agreed to within the stated accuracy of the results.

APPENDIX II

COMMENTS ON THE CHARACTER OF THE EIGENVALUES

The numerical procedure described in Appendix I, at times, yields negative values of the square of the Reynolds number; that is, the Reynolds number is imaginary. This is interpreted to mean that the flow is absolutely stable, which means there is no value of the Reynolds number for which the flow can be made unstable. Several arguments are given in the subsequent paragraphs to make this plausible. Only a very restricted set of equations are used as an example. The more general equations are mostly beyond the scope of these arguments.

The Equations (39) and (40) reduce to

$$(L - \sigma)\, LU = a^2 N_R^2 \phi V, \text{ and} \qquad \text{(II-1)}$$

$$(L - \sigma)\, V = \psi U \qquad \text{(II-2)}$$

if σ were not assumed to be zero, but assumed to be a real constant. The operator $L = D_x D_x^+ - a^2$ and the functions $\phi = 2\bar{v}/r$ and $\psi = D_x^+ \bar{v}$.

Two preliminary relations are required:

$$\int rfLgdx = - \int r \left[(D_x^+ f)(D_x^+ g) + a^2 fg \right] dx \qquad \text{(II-3)}$$

if f is zero at the limits of the integration, and

$$\int rfLgdx = \int rgLfdx \qquad \text{(II-4)}$$

if, in addition to f being zero at the limits, either g or Df is zero as well.

The limits on these and subsequent integrals will be between $x = 0$, $(r = \Delta_r^{-1})$, and $x = 1$, $(r = \Delta_r^{-1} + 1)$ (II-3) can be proven by integration by parts.

$$\int rfLgdr = \int rf \left[D_r D_r^+ g - a^2 g \right] dr$$

$$= rfD_r^+ g \Big]_{x=0}^{x=1} - \int \left[D_r(rf)\, D_r^+ g + a^2 rfg \right] dr$$

$$= - \int r \left[(D_r^+ f)(D_r^+ g) + a^2 fg \right] dr,$$

since $D_r^+ f = \frac{1}{r} D_r(rf)$. (II-4) can be deduced by noticing the symmetry of (II-3) with respect to f and g and by an additional integration by parts.

By assuming the existance of the solutions, multiplying (II-1) by rU and integrating,

$$\int rUL^2Udr - \sigma \int rULUdr = a^2 N_r^2 \int rU\phi Vdr. \qquad \text{(II-5)}$$

The first of these integrals is

$$\int rUL^2Udr = \int r\, (LU)^2\, dr,$$

by the use of (II-4). The second,

$$\int rULUdr = - \int r \left[(D_r^+ U)^2 + a^2 U^2 \right] dr,$$

by the use of (II-3).

(II-5) can then be expressed by

$$I_1 + \sigma I_2 = a^2 N_R^2 I_3 \quad , \tag{II-6}$$

in which

$$I_1 = \int r(LU)^2 \, dr$$

$$I_2 = \int r \left[(D_r^+ U)^2 + a^2 U^2 \right] dr$$

and

$$I_3 = \int rU\phi V dr.$$

I_1 and I_2 are positive definite, in a non-trivial case, since all the terms under the integral sign are positive in the range of integration. No such statement can be made with respect to I_3; in fact, I_3 can be positive, negative, or zero depending upon the character of the functions U, V and ϕ.

By multiplying (II-2) by rV and integrating, it reduces to

$$- I_4 - \sigma I_5 = I_6 \ , \tag{II-7}$$

in which

$$I_4 = \int r \left[(D_r^+ V)^2 + a^2 V^2 \right] dr$$

$$I_5 = \int rV^2 dr$$

and

$$I_6 = \int rV\psi U dr \quad .$$

I_4 and I_5 are positive definite, while again I_6 is dependent on the character of the integrand.

It is necessary here to limit severely the generality of the approach. Assume that both ψ and ϕ are non-zero constants. This is always true for ψ but only true for ϕ near solid body rotation, i.e. near $\Delta_\omega = 0$. Then,

$$\frac{I_6}{\psi} = \frac{I_3}{\phi},$$

or

$$I_3 = \frac{\phi}{\psi} I_6 = -\frac{\phi}{\psi}(I_4 + \sigma I_5) .$$

Substituting this into (II-6) yields:

$$N_R^2 = -\frac{\psi}{a^2\phi} \frac{I_1 + \sigma I_2}{I_4 + \sigma I_5} \qquad \text{(II-8)}$$

All of the integrals appearing here are positive definite. Thus, for $\sigma = 0$ the sign of N_R^2 is opposite to that of ψ/ϕ, or for a positive ψ/ϕ, N_R^2 is negative.

From the Rayleigh criterion, it is known that whenever ϕ and ψ are of the same sign, the flow is absolutely stable. Therefore, an absolutely stable condition yields an imaginary value of N_R.

From another point of view, the question may be asked: if a real positive N_R is imposed on the flow, what is the nature of the flow? Examination of (II-3) discloses that whenever ψ/ϕ is positive,

$$\frac{I_1 + \sigma I_2}{I_4 + \sigma I_5}$$

must be negative for a positive N_R^2. The integrals are all positive, thus only a negative σ, which implies decaying disturbances, satisfies these conditions.

A slightly different approach can be taken. If ψ is assumed to be positive and ϕ negative throughout the whole interval, a labile condition exists, which implies a real finite N_R at neutral stability. By making the transformation $M_R^2 = - N_R^2$ and $\phi = - \eta$, (II-1) reduces to

$$(L - \sigma)\ LU = a^2 M_R^2\ \eta V \qquad \text{(II-9)}$$

(II-9) is a differential equation of exactly the same form as (II-1) except that η is a positive function instead of the negative ϕ. The nature of the eigenvalue M_R^2 of (II-9) is clear from the transformation; that is, it is negative. Therefore, a negative eigenvalue results from a positive function η in the presence of a positive ψ. Again, recalling that a sufficient condition for stability is a positive η and ψ, an absolutely stable condition exists with a negative M_R^2.

Another question occurs if a negative N_R^2 is considered indicative of an absolutely stable condition. What is the nature of the flow when ϕ changes sign in the region between the two cylinders? For an inviscid fluid, Synge[11] has shown that two sets of eigenvalues exist; a stable and an unstable set. It is reasonable to believe that for a viscous fluid a similar condition exists, that is, an absolutely stable set of eigenvalues and a labile set exist.

This may be demonstrated by the application of two transformations. Consider a positive ψ and a ϕ that is negative near the inner cylinder and positive near the outer, which is known to generate a positive N_R^2. If the transformation previously used, $M_R^2 = - N_R^2$ and $\eta = - \phi$ is applied, the new driving potential η is then positive near the inner cylinder and negative near the outer, with a negative eigenvalue M_R^2. The resultant equations is identical to (II-9). A second transformation, $x = 1 - y$, does not change in any way the character of the differential equation; but, in effect, it reverses the functions U, V and η from left to right. When this transformation is applied to (II-9), it has no effect on M_R^2. The new η, however, is similar to the original ϕ, that is, it is negative near the inner cylinder and positive near the outer. In fact, whenever Δ_r is small and $\Delta_\omega = - 2$ the functions are identical. Therefore, two sets of eigenvalues exist, a positive and a negative set. If the negative value is interpreted as an absolutely stable mode and the positive as a labile mode, it is clear that both stable and unstable conditions exist when the driving potential changes sign.

The discussion of this section is not intended to imply that there are always two sets of roots when there is a sign change in the driving potential; in fact, there probably can be regions wherein ϕ changes sign, but only one set of roots exist. The shape of the ϕ curve would have to be other than nearly linear, however.

The transformation $x = 1 - y$ applied to (II-1) with a positive or negative definite ϕ has no effect on the character of the equation or the eigenvalue. Therefore, in this case there is only one set of eigenvalues.

The existance of two sets of eigenvalues, a positive and a negative set, causes some difficulty in the numerical solution. At times, the estimate of the eigenfunction is nearer the negative, absolutely stable, solution than it is to the positive, labile, solution. The iteration processes then converge to the wrong mode and indicate the stable solution instead of the labile one. In order to counteract this effect, it is necessary to select the first estimate carefully. This is done by calculating the eigenfunction at a neighboring condition and using it as the first estimate. As an example, the results are calculated for $\Delta_\omega = -1$ which has one set of roots, the positive ones, then its results are used for the first estimate of the eigenfunctions for $\Delta_\omega = -1.5$. Then these results are used for the first estimate of U, V and T for $\Delta_\omega = -1.75$ etc. As the process continues, smaller steps in Δ_ω are necessary to separate the roots. The presence of the negative root still causes difficulty by making the iteration process slow to converge.

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