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RESONANCE IN PRESSURIZED PIPING SYSTEMS

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NOMENCLATURE

Symbol	Meaning
A	Cross sectional area of pipe
A_1	Amplitude of sinusoidal head variation
C	Lumped capacitance, $C = \gamma \forall / K'$
C_D	Orifice discharge coefficient
D	Pipe diameter
E	Young's modulus of elasticity
F	Pressure wave traveling in the pipe
F	Resistance factor such that $H_F = FQ_F$
H	Amplitude of instantaneous piezometric head
H_o	Steady state or mean pressure head
H(x)	Non time-varying complex number for head
H_{sub}	Complex number notation for instantaneous head at a point. The subscript designates the location.
HA,HB,HC	Piezometric head at known computational points in x-t plane of characteristics solution.
HP	Piezometric head at unknown computational point in x-t plane of characteristics solution.
HP'	Total instantaneous head at valve in penstock
HR,HS	Piezometric head at interpolation points in x-t plane of characteristics solution
HVB	Total instantaneous head in valve body
I	Lumped inertance, $I = L/gA$
K	Bulk modulus of elasticity of liquid
K'	System modulus of elasticity, $a = \sqrt{K'/\rho}$

System	Meaning
L	Pipe length
L'	Effective length of choke or neck of resonator
PR	Pressure ratio of resonator
Q	Amplitude of instantaneous discharge
Q(x)	Non time-varying complex number for discharge
QSL	Instantaneous discharge at leaky seal
Q_{sub}	Complex number for instantaneous discharge at a point. The subscript designates the location
R	Subscript for receiving end of line
R_S	Real number denoting the real part of the complex impedance
S	Subscript for sending end of line
T	Theoretical period of system, $\sum_{k=1}^{n} 4\ L_k/a_k$
T_a	Apparent period of system
T_F	Period of forcing function
$\bar{V}$	Average oscillatory velocity over a half period
$\forall$	Volume
V_o	Steady state or mean velocity
VA,VB,VC	Velocity at known computational points in x-t plane of characteristics solution
VP	Velocity at unknown computational point in x-t plane of characteristic solution.
VR,VS	Velocity at interpolation points in x-t plane of characteristic solution.
VN	Total instantaneous velocity in neck of resonator
X_S	Real number denoting the imaginary part of the complex impedance

Symbol	Meaning
Z_o	Characteristic impedance, $Z_o = a/gA$
Z_{sub}	Hydraulic impedance at a point denoted by the subscript. Complex ratio of pressure head to discharge.
a	Speed of pressure pulse
e	Conduit wall thickness
f	Pressure wave traveling in the pipe
f	Darcy-Weisbach friction factor
g	Gravitational acceleration
h'	Dimensionless head, H/H_o
h	Instantaneous oscillatory pressure head
i	$\sqrt{-1}$
q	Instantaneous oscillatory discharge
t	time; as a subscript denotes partial differentiation
v'	Dimensionless velocity, V/V_o
x	Distance along pipe; as a subscript denotes partial differentiation
α	Reflection function, $\alpha_n = -f_n/F_n$
γ	Phase constant, $i\omega/a$
γ	Specific weight of fluid
Γ_R	Reflection coefficient, complex ratio of the reflected wave to the incident wave
δ	Amplitude of oscillatory motion of valve
ρ	Mass density
ρ	Pipe line constant or Allievi parameter, $aV_o/2gH_o$
τ	Dimensionless number describing discharge coefficient and area of opening at a valve, $(C_DA)/(C_DA)_o$

Symbol	Meaning
ψ	Phase angle
ω	Angular frequency, $2\pi/T$
ω_F	Frequency of forcing function, $2\pi/T_F$
ω_n	Natural frequency of resonator
$\mathcal{C}$	Capacitance per unit length, gA/a^2
$\mathcal{I}$	Inertance per unit length, $1/gA$

I. INTRODUCTION

Periodic oscillations of pressure and discharge in fluid systems have been observed by all of the many investigators in the field of fluid transients. Occasionally the characteristics of a system are such that when a periodic disturbance is introduced to the system, an amplification of the motion occurs resulting in a resonating condition. This is the area of the present investigation.

Numerous descriptions of incidents of resonance in different system types can be found in the literature. Over the years there have been a number of interesting resonating conditions in power conduits, many of which have resulted in severe and spectacular accidents. Out of twenty hydro plant accidents reported by Jaeger[27] in 1948, a total of twelve could be explained by various forms of resonance or self-induced oscillations. Self-excited vibrations of control valves in hydraulic control systems are also often observed.[2,17,29,37] The frequency of vibration in the latter type of system is generally considerably higher than in the former but the principles of mechanics involved in both are identical.

In order for a resonating situation to develop on a system, a mechanism must exist which produces a net positive influx of energy during each cycle. The main energy input can occur at the source of the excitation or at another terminal condition as a result of fluctuations established by the exciter. A reciprocating pump commonly produces a pulsating outflow which, if the pump is connected to a system of particular characteristics, can be amplified into a severe vibrating condition. Rhythmic motion of a valve can allow a variable outflow accompanied by pressure fluctuations which may be amplified if the connecting system responds to the same period of excitation. Resonance builds up when the net influx of energy to the system in one complete cycle exceeds the energy dissipated within the system during the cycle and continues to do so for successive periods. This accumulation of energy manifests itself in the form of increased amplitudes of the pressure head and discharge oscillations.

The primary objective of this study is to present a method of determining the resonant characteristics, frequencies, amplitudes, and phase relationships, of a number of practical fluid systems. Particular attention is given to those system types which are likely to exist in hydro power developments and water or oil supply lines. However, the same basic equations apply to the smaller scaled hydraulic control systems. The system types include pipes of single characteristics; lines of multiple characteristics having abrupt geometric changes; systems with branch connections as dead-end lines, open surge tanks, and free outflow lines; parallel lines; and numerous practical combinations of these situations. These are continuous systems in which distributed parameters are considered. Some appurtenances to a fluid system, such as resonators, can be satisfactorily represented by lumped parameters. These are also included in this study.

The wave equations describing unsteady flow in a frictionless system are solved in a manner analogous to the concepts of electrical engineering as related to transmission lines.[20,31,49] The concept of hydraulic impedance is used as the basis for the determination of the resonating frequencies in a particular system. Typical examples solved by the impedance approach are verified using the characteristics solution of the partial differential equations on the digital computer.

For laboratory and prototype verification of the method, results of experiments conducted in France and reported by Camichel, Eydoux, and Gariel,[10] are compared to those obtained by methods presented herein. The most significant experimental verification of the theory results from an analysis of an actual resonance condition which developed at Bersimis No. 2 Power Plant, Quebec, Canada.[1] The reasons the plant vibrated at particular frequencies and with definite pressure and discharge amplitudes are shown. The complete Bersimis system is also programmed for solution on the digital computer and the field resonating conditions are reproduced as they actually occurred on the site.

The present theory of resonance treats simple lines completely, and partially treats a few special cases of complex systems. This thesis extends the resonance theory to cover many different types of complex systems. Such systems that have not previously been analyzed include those that are made up of parallel lines and those having branch lines fitted with an orifice or valve at the branch terminal. As a consequence of this investigation, it will be possible to determine the resonant characteristics of many complex systems without introducing approximations which previously had to be made.

Impedance methods have been used extensively in steady state vibration problems.[11] The specific application of these concepts to resonance in fluid systems is accomplished herein.

II. LITERATURE

Although a thorough background in the treatment of basic water hammer is a prerequisite to the study of resonant conditions in fluid systems, a complete review of all the literature on this topic would be redundant. Rhythmic variations of pressure surges have been calculated by all investigators of water hammer and the essential properties of the transformation of the pressure wave into periodic oscillations following total valve closure on a line are well known. An analytical, graphical or numerical approach can be used to solve most practical problems.

The phenomenon of resonance in a simple line has also been the subject of intensive investigations with experimental results providing confirmation of the theoretical approaches. Experimental results have indicated the nature of resonance in systems of multiple characteristics, however, the theory is incomplete in covering these more complex situations.

It should be noted that the term resonance in pipe lines is often used to describe surging conditions in which inertial effects predominate. This investigation, and therefore this literature survey, is primarily interested in the higher resonating frequencies of the fundamental and higher harmonics, cases of fluid transients in which both inertial and elastic effects must be included.

Allievi[3,4] analyzed the waves developed in a simple pipe due to partial, continuous, rhythmical, alternating movements of opening and closing a valve. By considering the line connected to a reservoir at one end, and provided with a device at the free end which is governed by the gate equation,

$$v' = \tau \sqrt{h'} \; ,$$

he successfully demonstrated the limits on the amplification. v' and h' represent the dimensionless velocity and head as a function of time at the valve, V/V_o and H/H_o, respectively, and τ represents the dimensionless number describing the discharge coefficient and area of opening at the valve as a function of time, $(C_D A)/(C_D A)_o$.

Allievi began with the fundamental partial differential equations of water hammer,

$$H_x = \frac{1}{g} V_t$$

$$H_t = \frac{a^2}{g} V_x \; .$$

Using the general solution of these equations,

$$H - H_o = F_{t-L/a} + f_{t+L/a}$$

$$V - V_o = -\frac{g}{a} (F_{t-L/a} - f_{t+L/a}) \; ,$$

he obtained the following:

$$(\sqrt{h'_t} + \rho\tau_t)^2 + (\sqrt{h_{t-T/2}} - \rho\tau_{t-T/2})^2 = 2 + \rho^2\tau_t^2 + \rho^2\tau_{t-T/2}^2$$

This is the equation of a circle in coordinates $\sqrt{h'_t}$ and $\sqrt{h'_{t-T/2}}$.

The symbol ρ stands for the dimensionless pipe line or Allievi constant, $aV_o/2gH_o$. T is the theoretical period of the pipe, $4L/a$. The equation is quite general and is valid for any continuous periodic valve movement on a simple pipe line.

The valve is now assumed to be in a rhythmic motion such that

$$\tau_o = \tau_{2T/2} = \tau_{4T/2} = \ldots = C'$$

$$\tau_{T/2} = \tau_{3T/2} = \tau_{5T/2} = \ldots = 1,$$

so that a steady oscillatory pressure fluctuation is obtained. C' is a constant between 0 and 1 . With this relationship a pair of circles is defined, one representing conditions at even time increments 0, 2T/2, 4T/2, ... , and the other representing conditions at odd time increments T/2 , 3T/2, 5T/2, ... The coordinates of the intersection of these two circles, $\sqrt{h'_2}$, $\sqrt{h'_1}$, represent the solution of the two equations. From the geometry the following equation is obtained

$$\sqrt{h'_2}^2 + \sqrt{h'_1}^2 = 2 .$$

As the amplitude of the valve movement is increased, that is, as C' approaches 0, the intersection point moves toward the $\sqrt{h'_t}$ axis and thus $\sqrt{h'_2}$ approaches $\sqrt{2}$ and $\sqrt{h'_1}$ approaches 0 . Thus, in the limit, the pressure oscillation varies between 0 and $2H_o$.

This periodic valve motion of period T produces the fundamental mode on the simple system. A periodic τ variation of

$T_F = T/3$ can be shown to satisfy the above boundary condition and produces the third harmonic. Similarly the fifth, seventh and other odd harmonics can be produced on the simple pipe.

Camichel, Eydoux, and Gariel[10] were the first to report laboratory and full scale experiments on resonating conditions. They found that every compound pipe line possesses two different primary periods of oscillation. If an abrupt individual pressure disturbance is followed along a pipe system with variable diameter, the period of the whole system is found to be the sum of the individual periods of each sector of constant section. Following Camichel, this period is the theoretical period of the complex system.

$$T = \frac{4L_1}{a_1} + \frac{4L_2}{a_2} + \frac{4L_3}{a_3} + \cdots + \frac{4L_n}{a_n} = 4 \sum_{k=1}^{n} \frac{L_k}{a_k}$$

On the contrary, most periodic motions or pressure surges have a different period, which is generally shorter, called the apparent period of the system, T_a. Thus, the fundamental mode of oscillation on a complex system is at the apparent period whereas on a simple line it is at the theoretical period. The existence of the apparent period results from partial reflections of the pressure waves from each change in line characteristics as well as from the terminal points.

Camichel's work further demonstrated resonating conditions on certain higher frequencies, namely the odd harmonics of the system. These were produced both on simple lines and on complex systems.

On a simple pipe line the harmonics were found to be related to the theoretical period by integral multiples, that is, the period of the third harmonic multiplied by three equals the theoretical period, T . This can also be the case on complex lines according to the experiments but the generality of this conclusion is uncertain.

In most of the experiments reported by Camichel, resonance was produced by means of a rotating cock driven by a motor which opened and closed the valve rhythmically. Besides the demonstration of the existence of the apparent period and higher harmonics in a complex pipe system, the phenomenon of doubling of the static pressure in the case of resonance was also observed. It was further established that through flow was a minimum during conditions of severe resonance. The results of these experiments were quite completely reported but were presented without extensive theoretical explanation. More recent investigators have on many occasions referred to these data as experimental verification of their theories.

The earliest reporting of an accidental condition of severe pressure oscillation combined with machine vibration was made by Wilkins(48) in 1923. Conditions of a similar nature were discussed by Den Hartog(13) in 1929. These were vibrations in the penstock and power house resulting from pulsations created by runner buckets passing the guide vanes in Francis turbine installations. A change in the number of runner blades removed the source of the impulses so the vibration in the penstock disappeared. The frequency demonstrated the validity of the phenomenon of doubling the static pressure in a simple line activated by a rhythmical valve motion. By considering cases with both small and large values of the Allievi parameter, he was led to the observation that for large values it is possible to have cavitation at the valve and to reach superpressures in excess of the ordinarily considered maximum value of double the initial pressure. This work is summarized more recently(8) and broadened in scope to include graphical examples of the development of pressure oscillations in a simple system connected to a positive displacement pump. In some cases, depending upon the speed of the pump relative to the natural period of the system, severe pressure fluctuations can be shown to exist at the pump.

Schnyder(38) first presented the graphical development of resonance in a simple line resulting from the elastic behavior of a closed but elastic valve. If the valve geometry is such that a small leakage occurs in the closed position and the area of leakage gap decreases with increased pressure, then the graph shows there will be a pressure amplification at the valve.

Angus(5) discussed the possibility of a condition of resonance due to "chattering" of valves. Valve vibration may arise from one of two sources, by mechanical defects of the valve such as loose gates, loose stems or non-rigid stems, or by hydraulic defects. The latter causes the formation of vortices at the valve orifices which are periodically detached or swept away by the water flowing through the orifice. The effect of these vortices in changing the flow velocity, and consequently the pressure conditions, may be considered as that of a change in the coefficient of contraction of the valve orifice. Using the graphical procedure it is shown that when the period of oscillation of the valve is the same as that of the pipe, resonance occurs.

Probably Charles Jaeger[23,24,26] was the greatest single contributor to a basic understanding of the mechanics involved in resonance in fluid conduit systems. The practical nature of his discussions presents a clear mental picture of the conditions of resonance. Discussing Camichel's experimental work, Jaeger[27] says: "This opening-closing movement produces resonance in the shape of stationary waves in the pipe or system of pipes. At the end near the rotating cock, there is a loop of the oscillating system, while at the upper open end there is a node.... With the resonance of odd harmonics these pressures... are invariably repeated at each loop of the system, making the resonance of the harmonics much more dangerous than that of the fundamental."

A convenient visualization of these conditions can be obtained by considering a vibrating string of finite length having one end fixed. The fixed point is comparable to the constant head reservoir or upper open end, and displacement of the string from a horizontal position comparable to pressure head variation. If a sinusoidal displacement is pictured the fundamental would be represented by a 1/4 wave length, the second harmonic by a 1/2 wave length, the third harmonic by a 3/4 wave length, the fourth harmonic by a full wave length, etc. The free end undergoes an oscillatory movement only in the odd harmonic cases. The same condition exists in the fluid system. That is, pressure fluctuations occur at one terminal, the other being connected to a constant head source, only for the fundamental and odd harmonics. Nodes of the fluid system, at points of zero pressure flucation, correspond to nodes of zero displacement on the vibrating string. Similarly loops have comparable meanings at points of maximum fluctuations.

Jaeger(23) was the first to develop an analytical expression for the calculation of the apparent period of a system. Using the general solution of the water hammer equations, with complete consideration given to partial reflections created at points of discontinuity in the system geometry, he arrives at the following equation for a system of two pipes connected in series. The lines are terminated with a valve at one end and constant head reservoir at the other. A sinusoidal pressure variation is assumed on the system.

$$\tan \frac{\omega L_1}{a_1} = \frac{A_1 a_2}{a_1 A_2} \cot \frac{\omega L_2}{a_2}$$

In this expression ω is the angular frequency of the assumed sinusoidal pressure variation, and the subscripts refer to the particular pipes. The solution of this expression yields the frequency of the fundamental vibration in the series system. By using the same approach, a similar expression was developed for a branch system with open end boundary conditions on two of the branches and the valve on the third branch.

$$\frac{A_1}{a_1} \tan \frac{\omega L_1}{a_1} = \frac{A_2}{a_2} \cot \frac{\omega L_2}{a_2} + \frac{A_3}{a_3} \cot \frac{\omega L_3}{a_3}$$

The degree of complexity of the development was much higher and although the statement is made that this approach could be used for much more complex systems, this seems to be the practical end limit of its application.

With this theory for determination of the apparent period in a few practical systems, Jaeger[26] then turned his attention to a theory for resonance of harmonics. In this development it is assumed that a permanent regime of resonance is already established. The system is then examined to see what is required of it to support the resonating condition. The main key in the theory revolves around a function α defined at a point A in the uniform pipe leading to the valve. It relates the reflected pressure wave to the primary wave in a form expressed by the equation

$$f_n = -\alpha_n F_n \quad .$$

The function α depends only on the system geometry and the movement of the valve or gate creating the resonant regime. The success in the application of the theory depends upon the evaluation of α for a particular system geometry and boundary conditions. Since α is a reflection coefficient which summarizes a time history of events at the point A, it is necessary to have a knowledge of reflection and transmission coefficients wherever the pressure waves are subject to change. To succeed in this evaluation of α a restrictive assumption is necessary which severely limits the scope of application. This is that the half period in each pipe in the system must be an integral multiple of the half period in the pipe from the valve to point A . With this limitation the theory is able to predict the periods of some of the harmonics of the system.

In a system in which the function α has the value of unity at every instant, that is, when $f = -F$ at the point A , a condition of total reflection of the ascending pressure wave exists. This is tantamount to saying that a nodal point of the oscillating periodic system exists at point A with the corresponding loop at the valve. The characteristics necessary for resonance have been provided when this condition is satisfied.

The limitation imposed on the development was necessary in order to be able to evaluate the function α . As a result of the restriction the method predicts only those resonating periods which are related to the theoretical period of the system by integers. This led Jaeger to the conclusion that the harmonics in complex systems are related to the theoretical period while the fundamental is associated with the apparent period. A further conclusion drawn is that the theory of the fundamental must be quite different from the theory of the harmonics.

Even though this harmonic theory and theory of the fundamental can handle systems of relatively simple geometry, and then only predicts some of the resonating periods, it is still the most comprehensive and complete treatment on the subject. The results of the two theories are in close agreement with some of the earlier experimental work.

Recently Jaeger[25] had occasion to apply his methods to the resonant condition at Bersimis No. 2 Power Plant. He found it necessary to replace the actual complicated manifold and penstock system with an "equivalent" penstock to handle it in his theory. With this assumption he concludes that particular high harmonics existed in the pressure tunnel and penstock during the resonating condition. His analysis does not include the characteristics of the exciter nor is the complete system represented.

An additional valuable contribution of Jaeger[25,27] was a compilation of data on accidents which have occurred in pumping and hydro-power systems throughout the world. The earliest summary,[27] which includes accidents resulting from a variety of causes, emphasizes the importance of a basic understanding in this area of resonance since over fifty per cent of the reported conditions were the result of resonance or self-induced oscillations.

The severe accident at the Lac Blanc- Lac Noir pumped storage scheme in Europe in 1934, in which the power station was destroyed, was attributed to a resonance caused by the vibration of the guide vanes of the pump. Shortly afterward, Rocard[36] published his theory of auto-oscillations according to which the elastic mechanical parts of a system may start to oscillate at their natural frequency. If this frequency happens to correspond to the natural frequency of the mass of water in the pipe a dangerous condition can develop. In this particular case the moveable guide vanes of the centrifugal pump gave the first impulse leading to the rupture of the pipe.

Rocard introduces the concept of hydraulic impedance into his analysis and arrives at equations which theoretically are not difficult to handle but practically are almost impossible to use with numerical values. To avoid this practical limitation, he suggests replacing the true pipe system with an "equivalent" pipe. A stability analysis with numerical values, including the pump vane and the system, is admitted to be such a large amount of work that it could hardly be undertaken. Unfortunately digital computers were not at his disposal.

Evangelisti[15] uses the equivalent of electrical admittance to arrive at equations to predict the natural resonating frequencies of three system types, a simple line, two pipes in series, and two branch pipes. The resulting equation for the two series pipes connected to a reservoir is

$$Y = -\frac{ig}{a_1} \frac{\frac{A_2 a_1}{A_1 a_2} \cot \frac{\omega L_1}{a_1} \cot \frac{\omega L_2}{a_2} - 1}{\cot \frac{\omega L_1}{a_1} + \frac{A_2 a_1}{A_1 a_2} \cot \frac{\omega L_2}{a_2}} .$$

where Y is the admittance at the free end. The relationship for the admittance at the junction of two dead end branch pipes is shown to be

$$Y = -ig \left(\frac{A_1}{a_1} \cot \frac{\omega L_1}{a_1} + \frac{A_2}{a_2} \cot \frac{\omega L_2}{a_2}\right) .$$

The behavior of the system is not discussed nor is the theory compared to experimental work. Numerical calculations of examples are also not presented so the theory has apparently never been applied to practical situations.

Favre[18] developed a theory for the condition of resonance in a conical pipe line. He defines a quantity σ which is a "characteristic" of the conduit

$$\sigma = (1 + \frac{\gamma}{2}) [\mu(1 + \frac{\gamma}{2}) + \quad]$$

where

$$= \frac{a_o - a_m}{a_m}$$

and

$$\mu = \frac{D_A - D_o}{D_o}$$

The subscripts o, m, and A refer to conditions at the valve end, mid-point, and reservoir end of the conduit, respectively. Starting with the fundamental concepts of water hammer and assuming a sine wave disturbance, the following equation is developed which defines the fundamental and harmonic resonating frequencies of the conical pipe.

$$\tan \frac{\omega L}{a_m} = - \frac{\omega L}{a_m} \cdot \frac{1}{\sigma}$$

As the conical pipe approaches a cylinder, σ approaches 0, and the equation predicts the fundamental and odd harmonics of the simple line.

Favre also examines the question of the period a rhythmic valve movement must have in order to develop a resonance condition. He decides that for a continuous periodic valve movement the apparent period is the one which would create the most severe pressure oscillations. However, if a valve is opened and closed very rapidly at regularly spaced intervals it is the theoretical period which would be the most dangerous.

H. Y. Paynter[34,35] has discussed and used the electrical-hydraulic analogy as it pertains to water hammer and surge conditions. His use of the concepts of surge and hydraulic impedance appears to be the earliest in American literature concentrating in the area of unsteady hydraulic problems. As an example of the use of these concepts he examines cases of rhythmic motion of a valve at the end of a pipe line.[34] His analysis is confined to cases wherein the exciting period is greater than the theoretical period of the pipe. A graphical summary is presented relating period of gate movement, magnitude of gate movement, theoretical pipe period, Allievi parameter, phase angle and amplitude of the pressure fluctuations. The results are valid for small changes in the valve position. The same presentation was first made by D. Gaden[19] in 1945 but was derived numerically using the basic equations of water hammer.

Using an analogy to electrical transmission lines, Waller[28,45,46,47] carried out an extensive investigation of surge pressures in oil pipe lines supplied from reciprocating pumps. By performing a harmonic analysis of the pump discharge curve and calculating impedance values from the connecting pipe lines, he was able to predict the characteristics of the resulting pressure oscillations. This enabled him to design corrective devices for the system. Conditions of resonance are not specifically analyzed by Waller although a number of the elements for such an investigation are present in his work.

Taylor[44] uses the electrical transmission line theory including frictional losses to provide an analysis of the pulses in the arterial system. The periodic input flow can be resolved by a Fourier analysis into a set of oscillations which can then be considered with impedance concepts. The work is applied to simple lines and excellent experimental confirmation is provided on a long, water-filled tube supplied by a pump delivering sinusoidal flow. Hardung[22] more recently treats the same problem and provides good experimental confirmation for an infinite line condition. Experiments performed on a series line fail to confirm his approach to the theory.

The existence of self-excited vibrations of hydraulic control-valve pipe lines is discussed by Lee and Blackburn,[29] Ainsworth,[2] Ezekiel,[17] and Saito.[37] One common type of system in which serious pressure oscillations can occur is a system which includes a spool-type control valve. If the natural frequency of the connected piping is near the natural frequency of the valve mechanism a resonant condition can develop. In this type of system the frequency of vibration is usually quite high making it necessary to consider the connected piping as a distributed medium even with relatively short lines involved. Again these experiments and analyses were conducted with the connected piping being represented by simple lines.

The effect of vibration of the fluid line in the longitudinal direction on the dynamic response of a simple hydraulic control line is studied by D'Souza and Oldenburger.[14] Viscous effects are taken into account with only laminar flow being analyzed. Experiment and theory show the very marked effect of the natural frequency of vibration of the pipe in the longitudinal direction when the magnitude of the applied test frequency approximates that of the natural frequency.

The report of the malfunction and subsequent controlled testing of the resonant case at Bersimis No. 2 by McCaig, Gibson, and Abbott(1) is of particular interest in this thesis. These are the data which are analyzed as part of the experimental evidence in this investigation.

These authors describe the plant, the conditions under which the original malfunctions occurred, and the extensive controlled testing which followed. They present a graphical analysis of the same type of disturbance mechanism attached to a simple line and correctly exhibit the conditions under which the resonance phenomenon developed. They did not, however, examine the system or excitation device in detail in an effort to explain the particular characteristics of the resonance. System details and resonant conditions will be presented later in this thesis.

In summarizing the literature on the topic of resonance in fluid pipe lines, it is apparent that a large number of incidents have occurred in full scale as well as laboratory situations. The theory is complete for simple line conditions and has been partially extended into the domain of the more complex systems.

Undoubtedly Allievi, Camichel and Jaeger have done the most to promote a broad understanding of the topic. Rocard and Evangelisti seed the idea of the use of impedance methods in the theory but it has never been directly applied. Paynter, in his analysis of the slower surge conditions, and Waller, in his analysis of pipe lines connected to reciprocating pumps, use the impedance concepts to great advantage.

It remains to develop the concepts and to apply them explicitly to resonating cases of the fundamental and higher harmonics in complex systems, to show that higher harmonics can occur without the necessity of pressure nodal or loop points existing at a geometry change, to present a method to avoid the use of the "equivalent pipe", and to present a method of determining the resonating periods which generally will be the most serious in a given system.

III. BASIC CONCEPTS AND EQUATIONS OF WATER HAMMER

The basic partial differential equations for water hammer in a pipe have been derived in a number of references.[33,42,43] Also, the solution of these nonlinear partial differential equations by the method of characteristics has been well established and the broad extent of usefulness of this method, combined with the use of a high speed digital computer, has been reported and adequately confirmed by experimental methods.[12,40,41,43] The characteristic equations are expressed as finite difference equations, and interpolations are used to apply the method of specified time intervals. These finite difference equations and interpolation equations together with the original basic partial differential equations will be summarized and used herein without further reference to the details of development.

Evangelisti[16] suggested a slight change in the point of application of the frictional term in the finite difference equations which yields more accurate results. This modification is incorporated into this thesis.

The assumptions involved in the derivation of the basic water hammer equations are summarized so their limitations may be realized. A discussion is also presented on the usefulness and limitations of the characteristics solution when applied to the transient flow phenomenon between an initial steady state and a final condition of steady oscillatory motion.

A. Fundamental Equations of Water Hammer: Equations of Motion and Continuity

The water hammer equations are commonly developed using a one dimensional approach. Pressure is assumed to vary only in the axial direction and the pressure across a transverse section is taken equal to that at the center line. Velocity is considered in the same manner with V representing the mean velocity at a section. The pipe is assumed full at all times with minimum pressure always being greater than vapor pressure. Perfect elasticity is assumed in the pipe walls and liquid. Friction is evaluated as a distributed parameter with the assumption being made that the friction loss at any instant is equal to the loss which would occur for the same condition of steady flow.

Listed below are the equations of continuity and motion developed for a segment of an elastic tube, including a term which takes into account the friction loss in the pipe. The equation of motion is

$$gH_x + \frac{fV^2}{2D} + VV_x + V_t = 0 \quad . \tag{1}$$

The subscripts on H and V denote partial differentiation with respect to the independent variables, distance and time. The constant f represents the Darcy-Weisbach friction factor. The first and second terms of the equation represent the pressure gradient and friction loss terms, respectively, while the last two are the fluid acceleration terms.

The equation of continuity for horizontal pipes is

$$V_x + \frac{g}{a^2} (VH_x + H_t) = 0 . \qquad (2)$$

The velocity gradient appears as the first term while the elastic effects of the pipe and liquid are represented by the second term. The velocity of the pressure wave, a, is expressed as

$$a^2 = \frac{K/\rho}{1 + C_1 KD/Ee} \qquad (3)$$

where the constant C_1 is defined[21] as a modifying expression depending upon the Poisson ratio of the pipe wall material and the manner in which the longitudinal movement of the pipe is restricted. It is pertinent to note that the velocity of the pressure pulse through the pipe, as it is represented, is dependent upon the geometric properties of the pipe, the physical properties of the pipe material and the fluid, and not upon the level of frictional resistance in the pipe nor upon the manner in which the pulse is created.

The quantity VV_x in Equation (1) may be shown[33] to be small compared with V_t for pipes and may be neglected. Similarly the term VH_x in Equation (2) is small compared to H_t. The final form of the equations is

$$gH_x + \frac{fV|V|}{2D} + V_t = 0 , \qquad (4)$$

$$H_t + \frac{a^2}{g} V_x = 0 . \qquad (5)$$

The absolute value sign is introduced into the frictional term so that it has the proper sign for flow reversal in the pipe.

B. Computer Solution Using the Method of Characteristics

The nonlinear partial differential equations are transformed by the methods of the theory of characteristics into a pair of total differential equations, each subject to a restriction imposed by a companion secondary equation. The secondary equation defines a "characteristic line" on the $x-t$ plane along which the original equation is valid.

The equations written in finite difference form are

$$VP - VR + \frac{g}{a}(HP - HR) + \frac{f(VR)|VR|\Delta t}{2D} = 0 \qquad (6)$$

$$x_P - x_R - a(t_P - t_R) = 0 \qquad (6a)$$

$$VP - VS - \frac{g}{a}(HP - HS) + \frac{f(VS)|VS|\Delta t}{2D} = 0 \qquad (7)$$

$$x_P - x_S + a(t_P - t_S) = 0 \, . \qquad (7a)$$

Referring to Figure 1, conditions at point P are unknown. Values of V and H are considered known at points A, B, and C at time t_o . VR, HR, VS, and HS can be obtained by linear interpolation between A, C, and B using the equations:

$$\left.\begin{aligned} VR &= VC - \theta a(VC - VA) \\ HR &= HC - \theta a(HC - HA) \\ VS &= VC - \theta a(VC - VB) \\ HS &= HC - \theta a(HC - HB) \end{aligned}\right\} \qquad (8)$$

In these equations $\theta = \Delta t/\Delta x$. In a particular problem Δx is usually selected by dividing the pipe into a convenient number of

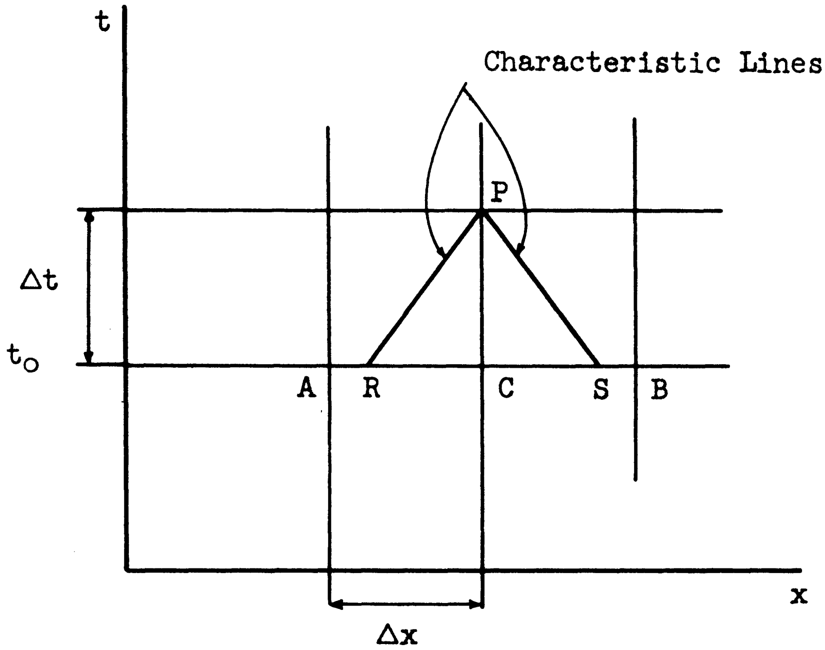

Figure 1. Characteristic Lines on the x - t Plane.

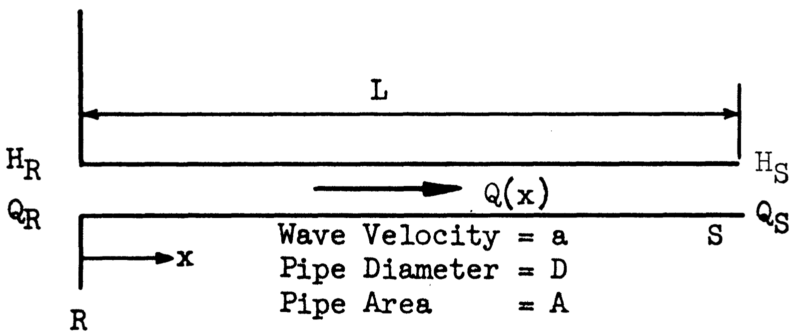

Figure 2. Simple Line.

sections. The magnitude of Δt should be no greater than $\Delta x/a$ for the solution to yield meaningful results.

For the evaluation of interior points on the pipe line, Equations (6) and (7) are solved simultaneously yielding

$$\left.\begin{aligned} VP &= \frac{VR + VS}{2} + \frac{g}{2a}\left[(HR - HS) - \frac{f\Delta t}{2D}\left(VR|VR| + VS|VS|\right)\right] \\ HP &= \frac{HR + HS}{2} + \frac{a}{2g}\left[(VR - VS) - \frac{f\Delta t}{2D}\left(VR|VR| - VS|VS|\right)\right] \end{aligned}\right\} \quad (9)$$

At the pipe terminals, either Equation (6) or (7) is used, with the boundary condition supplying either a relationship between VP and HP , or a defined value of one of these unknowns. Computations progress systematically from point to point along the pipe at a particular time; then they are repeated at time Δt later. The process is continued until the required information is obtained.

The computer solution as outlined in this chapter forms an integral part of this investigation, however, a problem often arises in the interpretation of the results provided by the computer. Many conditions of resonance develop quickly to high amplitude vibrations which are easily recognizable and conclusively real. On the other hand, many cases exhibit severe transient oscillations which may or may not end in a severe steady oscillatory motion.

In this latter category, a periodic excitation applied to some steady state condition often results in what appears to be a steady beat on the system. In reality, in most cases, the beat proves to be the result of steady oscillatory motion with a transient motion superimposed upon it, the transient condition being a function of the initial conditions of the system. If the computer program is allowed to run long enough the transient eventually disappears, leaving the condition of steady oscillatory motion which would exist physically. This is sometimes a very slow process in terms of computer time, although in a physical situation it would happen quickly. Inasmuch as the objectives of this study are primarily concerned with the non-transient condition, results of an analysis by the method of characteristics on the computer must be viewed carefully to be sure that a transient does not remain in what appears to be steady oscillatory motion. It is often necessary to make a detailed comparison of the output from two consecutive periods to make this decision.

A further discussion of the topic of beats is included in the next chapter where a complete analytical solution is provided for a typical example.

IV. IMPEDANCE APPROACH TO THE THEORY OF RESONANCE

It is advantageous to separate the total instantaneous pressure head, H, into two parts, the steady state or mean pressure head, H_o, and the oscillatory pressure head, h.

$$H = H_o + h \tag{10}$$

Similarly the discharge can be expressed

$$Q = Q_o + q , \tag{11}$$

where Q is the total instantaneous discharge at a point, Q_o is the steady uniform or mean flow rate, and q is the oscillating flow.

Using this notation on a frictionless system and at the same time replacing the flow velocity with the discharge, the equations of motion and continuity (Equations (4 and 5)) become

$$h_x + \frac{1}{Ag} q_t = 0 \tag{4a}$$

$$h_t + \frac{a^2}{Ag} q_x = 0 . \tag{5a}$$

The assumed flow direction and direction in which x is measured are shown on Figure 2. By taking the partial derivative of the first equation with respect to x and combining it with the partial derivative of the second with respect to t, one of the classical one-dimensional wave equations is obtained.

$$h_{tt} = a^2 h_{xx} \tag{12}$$

In a similar manner, a second wave equation identical in form is obtained.

$$q_{tt} = a^2 q_{xx} \tag{13}$$

A. Analytical Solution

The general solution of these equations is presented in a number of engineering and mathematics books.[39]

$$h = F(t + \frac{x}{a}) + f(t - \frac{x}{a}) \tag{14}$$

$$q = \frac{-gA}{a} \left[F(t + \frac{x}{a}) - f(t - \frac{x}{a}) \right] \tag{15}$$

The functions F and f represent pressure waves of arbitrary shape traveling with velocity a in the negative and positive directions of x, respectively. Both functions depend on the boundary conditions for the steady state solution and upon both the boundary and initial conditions for the complete solution. With specific boundary and initial conditions the shape of the wave is specified.

Unfortunately when a complete solution is given to a specific problem the results can sometimes be a little misleading, particularly when the desired end result is a steady state solution of oscillatory motion. This is because the partial differential equations and the indicated general solution are written for a frictionless system. Any terms introduced to the functions by the initial conditions are transient in nature and, although they would be retained in this solution, they eventually will disappear in a physical situation. The complete solution is of little value, therefore, when the desired result is a steady oscillatory motion.

An example follows to illustrate this situation as well as to illustrate an analytical approach to a particular problem. The mathematical shorthand, common in vibration problems, involving the use of complex numbers is used. The instantaneous value of a sinusoidal pressure head variation can be expressed $A_1 \cos \omega t = \mathrm{Re}\, A_1 e^{i\omega t}$ where Re stands for "the real part of". The letters Re are commonly dropped but are understood to exist. The real part of the final result represents the solution.

Consider a system consisting of a reservoir with sinusiodal waves on its surface and a simple pipe line leading from it with the downstream end closed. The objective is to find the final oscillatory motion which is superimposed upon the static conditions.

The partial differential equations, boundary conditions, and initial conditions are summarized.

$$q_x = \frac{-Ag}{a^2} h_t$$

$$q_t = -Ag\, h_x$$

$$q(L,t) = 0$$

$$h(0,t) = A_1 \sin \frac{2\pi}{T_F} t = -A_1 i\, e^{i\frac{2\pi}{T_F} t}$$

$$q(x,0) = 0$$

$$h(x,0) = 0$$

A solution of steady oscillatory motion of the form of Equations (14) and (15) can be assumed, where C_1 and C_2 are constants.

$$h(x,t) = C_1 e^{i\omega(t+\frac{x}{a})} + C_2 e^{i\omega(t-\frac{x}{a})}$$

$$= e^{i\omega t}(C_1 e^{i\omega\frac{x}{a}} + C_2 e^{-i\omega\frac{x}{a}})$$

$$q(x,t) = -C_1 \frac{gA}{a} e^{i\omega(t+\frac{x}{a})} + C_2 \frac{gA}{a} e^{i\omega(t-\frac{x}{a})}$$

$$= \frac{gA}{a} e^{i\omega t} (-C_1 e^{i\omega\frac{x}{a}} + C_2 e^{-i\omega\frac{x}{a}})$$

This assumed solution satisfies the controlling partial differential equations. Using the boundary condition at the dead end, $x = L$, $t > 0$,

$$C_2 = C_1 e^{2i\omega\frac{L}{a}} ,$$

Using the boundary condition at the reservoir, $x = 0$, $t > 0$,

$$\omega = 2\pi/T_F$$

$$C_1 + C_2 = - A_1 i .$$

Combining these conditions and using the notation $T = 4L/a$,

$$C_1 = \frac{-A_1 i}{1+e^{i\pi T/T_F}} ,$$

and

$$C_2 = \frac{-A_1 i}{1+e^{-i\pi T/T_F}} .$$

Placing these results in the assumed solution yields the solution representing steady oscillatory motion.

$$h(x,t) = A_1 \sin \frac{2\pi t}{T_F} \cos\frac{\pi}{2} \frac{T}{T_F} (1 - \frac{x}{L}) \sec \frac{\pi}{2} \frac{T}{T_F}$$

$$q(x,t) = \frac{A_1 g A}{a} \cos \frac{2\pi t}{T_F} \sin \frac{\pi}{2} \frac{T}{T_F} (1 - \frac{x}{L}) \sec \frac{\pi}{2} \frac{T}{T_F}$$

Since the equations describing this system are linear, additional terms can be superposed on the solution and, as long as the boundary conditions are satisfied, the solution is still valid. To match the initial conditions as specified, a number of odd harmonics of the natural period of the pipe can be introduced with the boundary conditions remaining satisfied. Using Fourier methods these can be made to satisfy the initial conditions. The form of this part of the solution can be expressed

$$h(x,t) = \sum_{0}^{M} A_{2m-1} \sin(2m-1) \frac{\pi}{2} \frac{x}{L} \sin \frac{2\pi}{T} (2m-1)(t-t_{2m-1})$$

$$q(x,t) = \sum_{0}^{M} A_{2m-1} \frac{gA}{a} \cos(2m-1)\frac{\pi}{2} \frac{x}{L} \cos \frac{2\pi}{T} (2m-1)(t-t_{2m-1}).$$

The pressure at a point from the total solution will appear as a beat having a period related to the forcing function period and the natural period of the pipe. For another set of initial conditions, a different complete solution would be obtained. The important conclusion is that the given steady state solution of oscillatory motion describes the condition which would exist in the physical situation and is, therefore, the desired solution in this resonance study.

B. Impedance Method for Steady Oscillatory Motion

The foregoing analytical approach can be used for many problems although it becomes somewhat cumbersome for complex boundary conditions. A more general approach is desirable wherein a solution of steady oscillatory motion can be determined more directly knowing the specific boundary conditions. This is provided by the impedance concepts.

1. Line Parameters

The general solution given as Equations (14) and (15) can be combined to eliminate the function, f, leaving an equation involving the oscillatory head, discharge, and a pressure pulse moving in the negative x direction.

$$h - \frac{a}{gA} q = 2F(t + \frac{x}{a})$$

If this wave is observed at a specific location x_1 at time t_1, and then again at a small increment of time later, the following relationships are obtained.

$$h_{x_1} - \frac{a}{gA} q_{x_1} = 2F(t_1 + \frac{x_1}{a})$$

$$h_{x_1 - a\Delta t} - \frac{a}{gA} q_{x_1 - a\Delta t} = 2F(t_1 + \Delta t + \frac{x_1 - a\Delta t}{a})$$

Combining, we get

$$\Delta h - \frac{a}{gA} \Delta q = 0 .$$

This equation is written,

$$\Delta h = Z_o \, \Delta q \ , \tag{16}$$

where Z_o is defined as the characteristic impedance of the line with the magnitude

$$Z_o = \frac{a}{gA} \ . \tag{17}$$

The characteristic impedance is a characteristic of the pipe line and is a constant for any particular line of constant geometric properties. It can be considered as a change in pressure head per unit change in volumetric flow rate. It is applicable in the form of Equation (16), at any section x , to the individual components of oscillatory head and flow; it does not apply to the total pressure head and discharge.

Equations (4a) and (5a) can be written in a different form in terms of two line parameters, inertance and capacitance. Each is a distributed characteristic of the pipe line, inertance arising out of inertial behavior and capacitance out of elastic behavior. The per unit length value of inertance is defined

$$\mathcal{I} = \frac{1}{gA} \ , \tag{18}$$

while the per unit length value of capacitance is defined

$$\mathcal{C} = \frac{gA}{a^2} \ . \tag{19}$$

With this notation the water hammer equations become

$$h_x + \mathcal{I}\, q_t = 0 , \tag{4aa}$$

$$h_t + \frac{1}{\mathcal{C}}\, q_x = 0 . \tag{5aa}$$

Also, the wave equations and the characteristic impedance can be expressed in terms of inertance and capacitance.

$$Z_o = \sqrt{\frac{\mathcal{I}}{\mathcal{C}}} \tag{17a}$$

$$h_{xx} = \mathcal{I}\mathcal{C}\, h_{tt} \tag{12a}$$

$$q_{xx} = \mathcal{I}\mathcal{C}\, q_{tt} \tag{13a}$$

The parallel between these equations and those describing a lossless electric transmission line is now evident. The analogies are voltage-pressure head, current-discharge, inductance-inertance, capacitance-capacitance, and characteristic impedance-characteristic impedance.

2. Solution in Terms of Terminal Conditions

The desired solution of steady oscillatory motion is one for which both pressure head and discharge in the pipe are sinusoidal, with angular frequency ω. The pressure head and discharge can be given by

$$h = h(x,t) = H(x)e^{i\omega t} \tag{20}$$

$$q = q(x,t) = Q(x)e^{i\omega t} \quad . \tag{21}$$

Performing the indicated differentiation in wave Equation (12a) and factoring the common exponential, we obtain the following total differential equation,

$$\frac{d^2H(x)}{dx^2} = \gamma^2 H(x) \quad , \tag{22}$$

where

$$\gamma = i\frac{\omega}{a} \quad . \tag{23}$$

The solution of Equation (22) is

$$H(x) = A'e^{-\gamma x} + B'e^{\gamma x} \quad , \tag{24}$$

where A' and B' are integration constants. Following the same procedure using the second wave equation, $Q(x)$ is found.

$$Q(x) = C'e^{-\gamma x} + D'e^{\gamma x} \tag{25}$$

The expression for the instantaneous values of head and discharge can now be written,

$$h(x,t) = (A'e^{-\gamma x} + B'e^{\gamma x})\, e^{i\omega t} \tag{20a}$$

$$q(x,t) = (C'e^{-\gamma x} + D'e^{\gamma x})\, e^{i\omega t} \quad , \tag{21a}$$

the "real part of" being understood. Using the continuity Equation (5a), two of the integration constants can be eliminated leaving

$$h(x,t) = (A'e^{-\gamma x} + B'e^{\gamma x})e^{i\omega t} \tag{20b}$$

$$q(x,t) = (\frac{A'}{Z_o} e^{-\gamma x} - \frac{B'}{Z_o} e^{\gamma x}) e^{i\omega t} . \tag{21b}$$

Further evaluation of the constants requires introduction of specific boundary conditions.

Referring to Figure 2, the boundary conditions at $x = 0$ are

$$h(0,t) = H_R = H(0)e^{i\omega t} \tag{26}$$

$$q(0,t) = Q_R = Q(0)e^{i\omega t} . \tag{27}$$

R refers to the receiving end of the line and S stands for the sending end. The constants can now be evaluated as

$$A' = \frac{H(0) + Q(0)Z_o}{2}$$

$$B' = \frac{H(0) + Q(0)Z_o}{2} .$$

Substituting these into the solution and at the same time substituting the value of γ from Equation (23) yields

$$H(x) = H(0) \cos \frac{\omega x}{a} - iQ(0)Z_o \sin\frac{\omega x}{a}$$

or

$$H(x) = \left[H_R \cos \frac{\omega x}{a} - i\, Q_R\, Z_o \sin \frac{\omega x}{a} \right] e^{-i\omega t}, \tag{28}$$

and

$$Q(x) = Q(0) \cos \frac{\omega x}{a} - i \frac{H(0)}{Z_o} \sin \frac{\omega x}{a}$$

or

$$Q(x) = \left[Q_R \cos \frac{\omega x}{a} - i \frac{H_R}{Z_o} \sin \frac{\omega x}{a} \right] e^{-i\omega t} . \qquad (29)$$

The head and discharge can now be evaluated at any point on the line.

$$h(x,t) = H_x = H_R \cos \frac{\omega x}{a} - i Q_R Z_o \sin \frac{\omega x}{a} \qquad (30)$$

$$q(x,t) = Q_x = Q_R \cos \frac{\omega x}{a} - i \frac{H_R}{Z_o} \sin \frac{\omega x}{a} \qquad (31)$$

In these expressions $H(x)$ and $Q(x)$ are complex heads and discharges but are functions of x alone; H_R, H_x, Q_R, Q_x are complex heads and discharges which contain the operator $e^{i\omega t}$.

A pair of values of particular interest are those at the sending end where $x = L$.

$$H_S = H_R \cos \frac{\omega L}{a} - i Q_R Z_o \sin \frac{\omega L}{a} \qquad (32)$$

$$Q_S = Q_R \cos \frac{\omega L}{a} - i \frac{H_R}{Z_o} \sin \frac{\omega L}{a} \qquad (33)$$

Solving these equations for H_R and Q_R yields the receiving end oscillatory head and discharge expressed in terms of conditions at the sending end.

$$H_R = H_S \cos \frac{\omega L}{a} + i Q_S Z_o \sin \frac{\omega L}{a} \qquad (34)$$

$$Q_R = Q_S \cos \frac{\omega L}{a} + i \frac{H_S}{Z_o} \sin \frac{\omega L}{a} \qquad (35)$$

3. Hydraulic Impedance

The complex ratio of the oscillatory pressure head to the oscillatory flow at any point x is defined as the hydraulic impedance or just "impedance" at that point, Z_x.

$$Z_x = \frac{h(x,t)}{q(x,t)} = \frac{H(x)e^{i\omega t}}{Q(x)e^{i\omega t}} = \frac{H_x}{Q_x} \tag{36}$$

At $x = 0$, or the receiving end,

$$Z_R = \frac{H_R}{Q_R} . \tag{37}$$

At $x = L$, or the sending end,

$$Z_S = \frac{H_S}{Q_S} . \tag{38}$$

The impedance is seen to be a complex number calculated from physical and dynamical flow properties of the system. In a particular pipe it varies with x, boundary conditions, and frequency of oscillation.

The distinction between this impedance and the characteristic impedance is important to recognize. Characteristic impedance of the line is constant along a line of constant geometric properties. It applies to each component of head and flow and does not apply to the total oscillatory head and flow.

In applying the concepts of hydraulic impedance to a fluid system, the impedance at one point is expressed in terms of conditions at another location, usually a terminal condition.

$$Z_x = \frac{H_x}{Q_x} = \frac{H_R \cos \omega x/a - i\, Q_R Z_o \sin \omega x/a}{Q_R \cos \omega x/a - i\, \frac{H_R}{Z_o} \sin \omega x/a}$$

or

$$Z_x = \frac{Z_R - i\, Z_o \tan \omega x/a}{1 - i\, \frac{Z_R}{Z_o} \tan \omega x/a} \quad . \tag{39}$$

Expressions relating impedance at the sending end in terms of impedance at the receiving end, or vice versa, are particularly useful.

$$Z_S = \frac{Z_R - i\, Z_o \tan \omega L/a}{1 - i\, \frac{Z_R}{Z_o} \tan \omega L/a} \tag{40}$$

$$Z_R = \frac{Z_S + i\, Z_o \tan \omega L/a}{1 + i\, \frac{Z_S}{Z_o} \tan \omega L/a} \tag{41}$$

C. <u>Terminal Conditions</u>

In order to apply the solution representing steady oscillatory motion as presented, it is necessary to be able to represent the boundary conditions as terminal impedances. Some of the common situations are presented.

1. <u>Constant Head Reservoir or Open End</u>

$$H_R = 0$$

$$Z_R = \frac{H_R}{Q_R} = 0$$

2. Dead End or Closed End

$$Q_R = 0$$

$$Z_R = \frac{H_R}{Q_R} = \infty$$

3. Matched Line

A line terminated in its characteristic impedance, Z_o , is called a matched line.

$$Z_R = Z_o$$

No reflections are returned from the receiving end in this case.

4. Fixed Orifice

The terminal impedance of a fixed orifice having an oscillatory flow superimposed on a mean flow can be analyzed in the following manner.

Beginning with the total head and discharge related by the gate equation,

$$Q = C_D A \sqrt{2gH} ,$$

and writing the differential,

$$dQ = \frac{C_D A \sqrt{2g}}{2 \sqrt{H}} dH ,$$

we obtain

$$q = \frac{Q}{2H} h ,$$

or

$$\frac{h}{q} = \frac{2H}{Q} .$$

This last expression defines the impedance at the orifice. For small changes in H and Q with respect to H_o and Q_o, this can be written as an approximation as

$$Z_R = \frac{2H_o}{Q_o} \quad . \tag{42}$$

This is a real number indicating no phase difference between head and discharge.

5. Moveable Valve

An oscillating valve producing small pressure variations can be analyzed as follows.

The gate equation can be linearized by assuming $h \ll 2\,H_o$ and making the following approximation.

$$H_o + h = H_o \left(1 + \frac{h}{H_o}\right)$$

$$\approx H_o \left(1 + \frac{h}{2H_o}\right)^2$$

Substituting this into the gate equation gives

$$Q_o + q = C_D(A_o + \Delta A)\sqrt{2gH_o}\left(1 + \frac{h}{2H_o}\right) .$$

Expanding, neglecting the higher order term, and subtracting the steady state condition leaves

$$q = Q_o \left[\frac{C_D \Delta A}{C_D A_o} + \frac{h}{2H_o}\right] . \tag{43}$$

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The dimensionless number τ is defined as the ratio of the product of the discharge coefficient and area of opening at any time to the same product when the valve is in the steady open position.

$$\tau = \frac{C_D(A_o + \Delta A)}{(C_D A_o)_{max}}$$

If the valve position is assumed to oscillate about the mean flow condition of $\tau = .5$, between the limits 0 and 1 , the relationships describing the valve opening in Equation (43) can be simplified.

$$\Delta\tau = \frac{C_D \Delta A}{(C_D A_o)_{max}} = \frac{C_D \Delta A}{2(C_D A_o)}$$

$$q = Q_o \left[2\Delta\tau + \frac{h}{2H_o} \right] \tag{44}$$

By assuming $\Delta\tau$ to vary as a sine wave,

$$\Delta\tau = \delta \sin \omega t = -i\ \delta\ e^{i\omega t} ,$$

and introducing the valve impedance,

$$Z_R = \frac{H_R}{Q_R} = \frac{H(0)e^{i\omega t}}{Q(0)e^{i\omega t}} = \frac{h}{q} ,$$

expressions can be presented for the valve impedance.

$$Q_R = Q_o \left[-2i\ \delta e^{i\omega t} + \frac{H_R}{2H_o} \right]$$

By eliminating Q_R , we obtain

$$Z_R = \frac{2\ H_o\ H_R}{Q_o\left[-4\ H_o i\ \delta e^{i\omega t} + H_R\right]} \quad ,$$

or

$$Z_R = \frac{2\ H_o\ H(0)}{Q_o\left[-4\ H_o\ \delta i + H(0)\right]}$$

$$= \frac{2H_o}{Q_o}\left[\frac{H(0)^2 + 4H_oH(0)\delta i}{H(0)^2 + 16H_o^2\delta^2}\right] . \qquad (44a)$$

By eliminating H_R , we obtain

$$Z_R = \frac{2H_o}{Q_o} + \frac{4H_oQ_o\delta i e^{i\omega t}}{Q_R} \quad ,$$

or

$$Z_R = \frac{2H_o}{Q_o} + \frac{4H_oQ_o\delta i}{Q(0)} . \qquad (44b)$$

Thus, if a reasonable estimate can be made for either the pressure head or discharge oscillations, the terminal impedance can be determined.

D. Application of Concepts

The general applicability of the method can best be indicated by discussing particular situations. First, conditions where distributed parameters must be used are presented, followed by cases where the assumption of lumped parameters is acceptable. A case which combines a number of lumped parameters including a friction head loss is included. In these examples the receiving end impedance of the system is assumed known or definable, the problem being to determine impedance at other locations in the system.

1. Distributed

a. Simple line connected to a constant head reservoir

$$Z_R = 0$$

By Equation (39),

$$Z_x = -i\, Z_o \tan \frac{\omega x}{a} \; . \qquad (45)$$

b. Simple line with dead end or closed end

$$Z_R = \infty$$

By Equation (39),

$$Z_x = \frac{1}{-\dfrac{i}{Z_o} \tan \dfrac{\omega x}{a}} \; ,$$

or

$$Z_x = i\, Z_o \cot \frac{\omega x}{a} \; . \qquad (46)$$

c. Infinite line or matched line

This is a case where reflections are not returned from the receiving end. Only the sending end component of head and discharge will exist on the system and therefore the impedance at any point will be the characteristic impedance of the line.

$$Z_R = Z_o$$

$$Z_x = Z_o \qquad (47)$$

d. Series connection

A common pressure exists at a junction of two pipes and continuity of the discharge also must be satisfied. Referring to Figure 3,

$$H_{R_1} = H_{S_2} = H_{J_2} ,$$

$$Q_{R_1} = Q_{S_2} ,$$

therefore,

$$Z_{R_1} = Z_{S_2} .$$

The sending end impedance of one pipe becomes the receiving end impedance of the other pipe. The impedance at any location in pipe 1 becomes,

$$Z_{x_1} = \frac{Z_{R_1} - i\, Z_{o_1} \tan \omega x_1/a_1}{1 - i \frac{Z_{R_1}}{Z_{o_1}} \tan \omega x_1/a_1} ,$$

or

$$Z_{x_1} = \frac{Z_{S_2} - i\, Z_{o_1} \tan \omega x_1/a_1}{1 - i \frac{Z_{S_2}}{Z_{o_1}} \tan \omega x_1/a_1} . \qquad (48)$$

e. Branch connections

At a junction of three or more pipes, a common pressure exists at all times. Also continuity of the volumetric flow rate must be satisfied at all times. Referring to Figure 4,

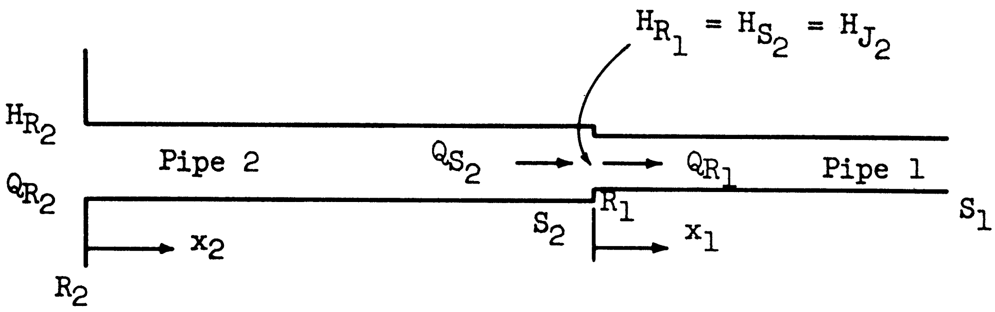


Figure 3. Series System.

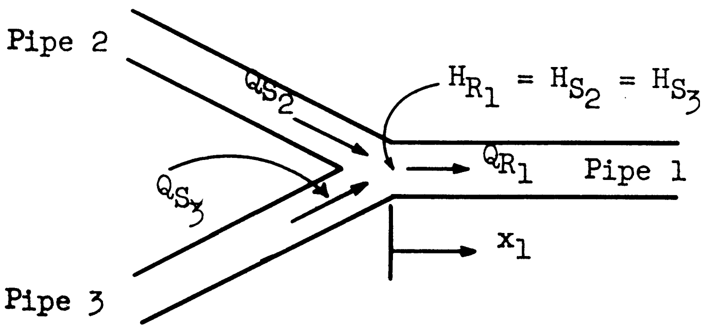


Figure 4. Branch Connection.

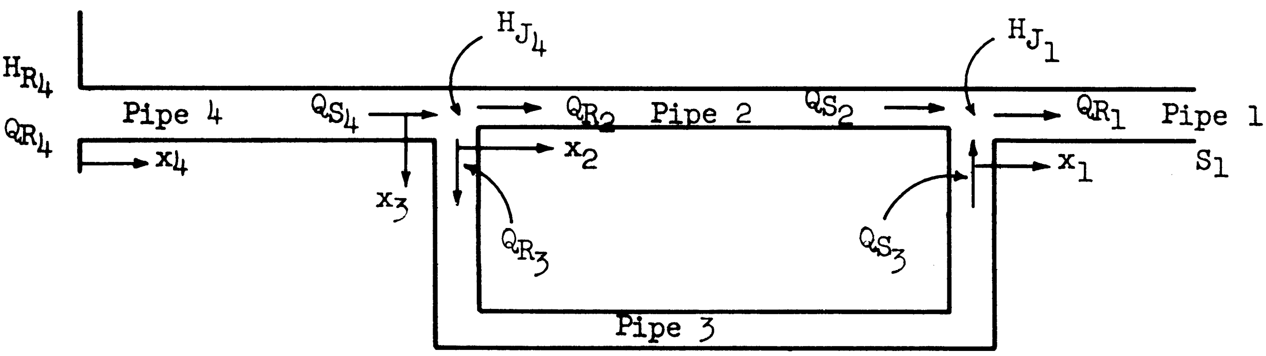


Figure 5. Parallel System.

$$H_{S_2} = H_{S_3} = H_{R_1} \quad ,$$

$$Q_{S_2} + Q_{S_3} = Q_{R_1} \quad ,$$

therefore,

$$\frac{1}{Z_{S_2}} + \frac{1}{Z_{S_3}} = \frac{1}{Z_{R_1}} \quad ,$$

or

$$Z_{R_1} = \frac{Z_{S_2} Z_{S_3}}{Z_{S_2} + Z_{S_3}} \quad . \tag{49}$$

With Z_{R_1} known, the impedance at any point in pipe 1 can be calculated using Equation (39).

f. Parallel system

To evaluate the impedance at any point in this system, it is necessary to look at the head and discharge relationships in the entire system, determining the unknowns by solving a set of simultaneous equations. Referring to Figure 5 and considering H_{R_4} and Q_{R_4} known, the following equations are available.

$$H_{J_4} = H_{R_4} \cos \frac{\omega L_4}{a_4} - i\, Q_{R_4} Z_{o_4} \sin \frac{\omega L_4}{a_4}$$

$$Q_{S_4} = Q_{R_4} \cos \frac{\omega L_4}{a_4} - i \frac{H_{R_4}}{Z_{o_4}} \sin \frac{\omega L_4}{a_4}$$

$$Q_{S_4} = Q_{R_3} + Q_{R_2}$$

$$H_{J_1} = H_{J_4} \cos \frac{\omega L_2}{a_2} - i\, Z_{o_2} Q_{R_2} \sin \frac{\omega L_2}{a_2}$$

$$H_{J_1} = H_{J_4} \cos \frac{\omega L_3}{a_3} - i\, Z_{o_3} Q_{R_3} \sin \frac{\omega L_3}{a_3}$$

$$Q_{S_2} = Q_{R_2} \cos \frac{\omega L_2}{a_2} - i \frac{H_{J_4}}{Z_{o_2}} \sin \frac{\omega L_2}{a_2}$$

$$Q_{S_3} = Q_{R_3} \cos \frac{\omega L_3}{a_3} - i \frac{H_{J_4}}{Z_{o_3}} \sin \frac{\omega L_3}{a_3}$$

$$Q_{S_2} + Q_{S_3} = Q_{R_1}$$

The unknowns involved in a given system are H_{J_4}, H_{J_1}, Q_{S_4}, Q_{R_3}, Q_{R_2}, Q_{R_1}, Q_{S_3}, and Q_{S_2} . The eight linear equations are sufficient to solve for each of these unknowns. With Q_{R_1} and H_{J_1} determined, Equation (39) can be used to evaluate the impedance at any point in pipe 1.

2. Lumped

Although all physical systems are continuous and therefore should be represented by distributed parameters, on some occasions it is possible to use a lumped parameter approximation. This happens when $\frac{\omega x}{a}$ becomes small so that $\frac{\omega x}{a} \approx \sin \frac{\omega x}{a}$. Values of $\frac{\omega x}{a}$ less than $\frac{\pi}{12}$ radians satisfy this requirement.

a. Open end

$$Z_R = 0$$

Using Equation (45), we obtain

$$Z_x = - i Z_o \frac{\omega x}{a} = - i \frac{\omega x}{gA} ,$$

or

$$Z_x = - i \omega x \mathcal{I} ,$$

where $\mathcal{I}$ is the local inertance, per foot. When the full length of the short section is considered,

$$Z_S = - i \omega I . \tag{50}$$

The last equation includes a lumped inertance defined,

$$I = \frac{L}{gA} . \tag{51}$$

b. <u>Closed or dead end</u>

$$Z_R = \infty$$

Using Equation (46), we obtain

$$Z_x = i Z_o \frac{a}{\omega x} = i \frac{a^2}{gAx\omega} ,$$

or

$$Z_x = \frac{i}{\omega x \mathcal{C}} ,$$

where $\mathcal{C}$ is the local capacitance, per foot. When the total volume of the enclosure is considered,

$$Z_S = \frac{i}{\omega C} . \tag{52}$$

The last equation includes a lumped capacitance defined,

$$C = \frac{g\forall}{a^2} = \frac{\gamma\forall}{K'} \quad (53)$$

c. General

In the general case where Z_R lies between the limiting cases defined above, and the lumped approximation is still valid, Equation (39) can be used to define the impedance.

$$Z_x = \frac{Z_R - i\, Z_o \frac{\omega x}{a}}{1 - i \frac{Z_R}{Z_o} \frac{\omega x}{a}}$$

$$Z_x = \frac{Z_R - i\, \omega x\, \mathcal{I}}{1 - i\, Z_R\, \omega x\, \mathcal{C}} \quad (54)$$

d. Example of a lumped system including a friction loss

A concentrated friction loss can also be included in the analysis of a lumped parameter system. An example is a Helmholtz resonator shown in Figure 6. In the short tube or neck, a mass of fluid oscillates back and forth. In keeping with the lumped parameter model it is assumed that the fluid acts as a solid plug. The volume of fluid in the enclosure is relatively large and acts essentially as a spring.

If high velocities occur in the small tube, a high frictional loss will be associated with it. In many cases a linearized assumption on the frictional loss resulting from the oscillatory flow is adequate.

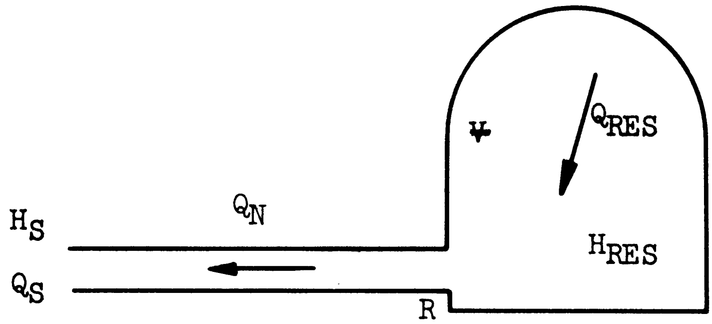


Figure 6. Helmholtz Resonator.

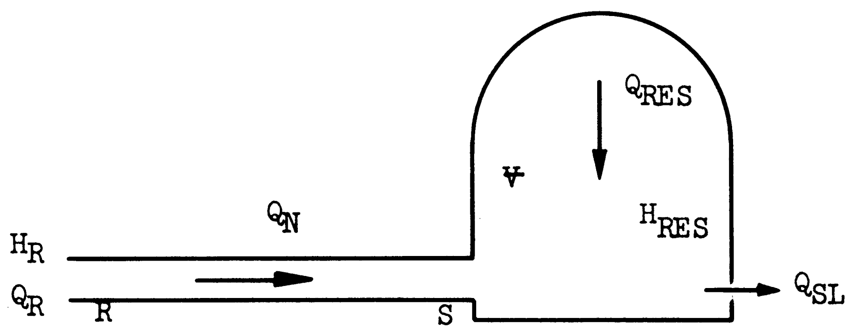


Figure 7. Helmholtz Resonator with Through Flow.

Thus, consider a resistance such that the frictional force opposing motion has a magnitude equal to a constant F times the discharge.

Referring to Figure 6, we have

$$Q_S = Q_N = Q_R = Q_{RES} ,$$

and

$$H_S + H_F + H_N = H_R = H_{RES} .$$

Using the complex impedance notation, we obtain

$$Z_S + Z_F + Z_N = Z_R = Z_{RES} ,$$

or

$$Z_S = Z_{RES} - Z_N - Z_F ,$$

where

$$Z_{RES} = \frac{i}{\omega C} ,$$

$$Z_N = i \omega I ,$$

and

$$Z_F = F .$$

Therefore,

$$Z_S = - F + i(-\omega I + \frac{1}{\omega C}) . \qquad (55)$$

Another example of a similar situation is a Helmholtz resonator with a small through flow. This is a case which is used later in the thesis. Referring to Figure 7, the following relationships can be expressed.

Examining the short tube, we have

$$Q_S = Q_N = Q_R ,$$

and

$$H_S + H_N + H_F = H_R .$$

Therefore,

$$Z_S = Z_R - Z_F - Z_N, \qquad (56)$$

where

$$Z_N = i \omega I ,$$

and

$$Z_F = F .$$

Examining the resonator volume, we have

$$H_{SL} = H_{RES} = H_S ,$$

and

$$Q_{SL} = Q_{RES} + Q_S .$$

Therefore,

$$\frac{1}{Z_{SL}} = \frac{1}{Z_{RES}} + \frac{1}{Z_S} ,$$

or

$$Z_{SL} = \frac{Z_S Z_{RES}}{Z_S + Z_{RES}} , \qquad (57)$$

where

$$Z_{RES} = \frac{i}{\omega C} ,$$

and Z_S is defined by Equation (56).

The lumped parameter systems can be connected to a distributed system or to other lumped systems without causing any particular difficulty in the analysis.

E. Reflection Coefficients, Standing Waves and Traveling Waves

Resonance would never occur on distributed parameter systems if it were not for the phenomenon of reflection. A reflection coefficient which can be defined in terms of the terminal impedance is often of value in analyzing problems.

In both the general solution of the wave equations, Equations (14) and (15), and the solution representing steady oscillatory flow, Equations (24) and (25), there are two terms, one representing a wave traveling in the increasing x direction and the other representing a wave traveling in the decreasing x direction. If the functions F and f are used to represent the complex pressure waves traveling in the negative and positive directions, respectively, the steady oscillatory head and discharge at the receiving terminal can be represented

$$H_R = F + f$$

and

$$Q = -\frac{F}{Z_o} + \frac{f}{Z_o} .$$

The reflection coefficient is commonly defined as the ratio of the reflected pressure head to the incident pressure head, which in this case can be written

$$\Gamma_R = \frac{F}{f} . \qquad (58)$$

Then

$$Z_R = \frac{H_R}{Q_R} = \frac{F + f}{\frac{-F + f}{Z_o}} = Z_o \frac{1 + \frac{F}{f}}{1 - \frac{F}{f}}$$

or

$$Z_R = Z_o \frac{1 + \Gamma_R}{1 - \Gamma_R} \quad . \tag{59}$$

Solving this equation for Γ_R gives

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o} \quad , \tag{60}$$

or

$$\Gamma_R = |\Gamma_R| \angle\psi ,$$

where ψ is the phase angle.

For a dead end pipe, $Z_R = \infty$, and

$$\Gamma_R = 1 = 1 \angle 0° \quad .$$

For an open end condition, $Z_R = 0$, and

$$\Gamma_R = -1 = 1 \angle 180° \quad .$$

For an infinite line, $Z_R = Z_o$, and

$$\Gamma_R = 0 \; .$$

For a purely real terminal impedance other than the above limiting cases the reflection coefficient is a real constant with a phase angle of 0° or 180° . This might occur at an orifice or valve located on the end of a line.

The more general reflection coefficients occurring at series and branch junctions will display magnitudes or moduli between 0 and 1 with phase angles varying from 0° to 360° .

If a pressure pulse is set up on a uniform infinite line, no reflection occurs since there are no changes in line conditions to bring about a reflection. When reflections are introduced into the system, the solution of steady oscillatory motion shows a combination of forward and reflected waves which result in a standing wave. Such a wave represents energy storage in the system and, in the case of a pure standing wave, not energy transmission. In contrast to these waves, traveling waves indicate a transmission of energy through the system.

When the receiving terminal condition is either open or closed, the reflected wave is the same size as the incident wave. In the idealized frictionless system, no energy is being transmitted through the system as a result of the oscillatory flow and only standing waves are present. If the terminal condition is such that a real part is introduced to the sending impedance, as in the case of an orifice, a traveling wave exists on the system and there is a net transmission of energy. However, even in this case, there is a reflection from the orifice which gives rise to a standing wave. The net result is a combination of a traveling wave with energy transmission and a standing wave with energy storage. The extreme limit for the terminal impedance in this latter direction is a purely real impedance equal to the characteristic impedance. This represents either a matched line or an infinite line.

F. Impedance Diagram

In general the terminal impedance is a complex number and can be expressed

$$Z_S = R_S + i X_S = |Z_S| \angle \psi , \tag{61}$$

On a particular system it varies with frequency and is maximum for the infinite number of resonating frequencies. These are the natural frequencies of the system and, in the case of a system originating in a constant head reservoir, they correspond to the fundamental and odd harmonics.

In a simple pipe line these natural frequencies are harmonically related like the overtones of a vibrating string. In a series system, in general, they are not harmonically related, however, they can be in particular cases. In a complex system as a branch or parallel system, there is a possibility of resonance in oscillatory modes in each branch. In a very complicated system, it is possible for these natural frequencies to crowd next to one another. Each natural frequency or resonance, however, is separated on the frequency scale by an anti-resonance, or frequency at which Z_S becomes small or equals zero.

An impedance diagram provides a very useful aid in assessing a system's frequency response. The most useful plot is that of the modulus of the impedance versus angular frequency. In some special instances it may be desirable to make plots of the real part, the imaginary part, or the phase angle of the terminal impedance versus frequency. Typical diagrams for complex systems are shown in Figures 13 to 16. The resonances and anti-resonances are visible as points of high impedance and low impedance, respectively.

G. General Discussion

1. Assumptions and Limitations

The assumptions outlined earlier which apply to the development of the water hammer equations are still relevant. In addition the limitations and assumptions of this particular analysis are summarized and discussed.

In the development of the impedance approach herein, friction is considered negligible. The validity of this assumption is certainly open to question in some cases as, for example, in small diameter pipes with high velocities. However, many cases of resonance are associated with small velocities and correspondingly low frictional losses.

The wave speed is of paramount importance in arriving at the resonating frequencies in a particular system. Since the wave speed, as commonly represented, is not a function of friction, the resonating frequencies of a system would not change appreciably with the inclusion of friction.

In the next chapter on computer verification of the theory, the nonlinear friction term is added on a number of occasions. The results are virtually identical whether friction is included or not. In the main experimental work in this thesis, Chapter VI, velocities are extremely low, generally less than one foot per second, so the assumption of negligible friction is valid.

The generality of the assumption of a sine wave on the system may also be questioned. One important justification for the assumption is that sinusoidal type pressure fluctuations have commonly been observed in experimental situations. Even more important, however, is the fact that with a linear system, a particular forcing function can be separated into a set of harmonics using a Fourier analysis. The method outlined herein can then analyze each harmonic at its own particular frequency. The results can then be superimposed yielding a complete steady oscillatory solution. Reference (45) provides a complete treatment of this approach on a system supplied by a reciprocating pump.

Two nonlinear conditions which are quite common in hydraulic systems are the moveable valve and the fixed orifice. The gate or orifice equation relates head to discharge. With small pressure oscillations this terminal condition can be linearized, as shown earlier, so that it can be used in this theory. Even in some cases of large amplitude pressure oscillations the velocity at the valve becomes very small thereby removing the effects of the nonlinearity. This will be shown in the next chapter.

Providing only the analysis of the condition of steady oscillatory flow disregards the transient conditions which may occur initially in the system. It is possible that particular initial and boundary conditions might produce transients which would be much more severe than the conditions represented by the solution of steady oscillatory motion. The prediction of the worst possible cases of water hammer, however, is not the objective in this resonance study. Resonant conditions can develop slowly from an apparently small oscillation or disturbance, conditions where the initial transients would be insignificant. When the transient solution is needed, methods of analysis are readily available to provide the solution. In a practical design situation, both should be considered.

Finally, it must be restated that this analysis deals only with the oscillatory part of the flow. It must be superimposed upon the mean flow conditions to obtain actual flow conditions. This is not considered to be a detriment to the method, nor a limitation of it.

2. Comparison

The limitations of the earlier approach to steady oscillatory motion on complex systems are now evident. The general solution of the water hammer equations was used and reflection coefficients were determined which expressed only amplitude changes at a discontinuity, saying nothing about the phase shift which also takes place.

The only cases which can be solved with this earlier approach are those wherein the phase shift is either $0°$ or $180°$, with a node or loop of the pressure diagram occurring at the junction. These are cases where the reflection coefficient, Γ_R , is a purely real number, either +1 or -1 . The results given by this approach are entirely correct but are quite incomplete. A theory of resonance was thus developed which included only resonating frequencies whose periods were integrally related to the theoretical period of the system. This limitation does not exist in the theory of this chapter.

V. EXPERIMENTAL AND COMPUTER VERIFICATION OF IMPEDANCE THEORY

Two methods of confirmation of the impedance theory are reported in this chapter. First, the theory is compared with experimental results from the literature. Systems on which experiments were performed, as reported by Camichel, Eydoux and Gariel,[10] are analyzed using the impedance relationships of the previous chapter. The calculated results are then compared with the reported results. Second, the theory is compared to results obtained from the solution of typical problems on the digital computer. For this study, systems were designed which represent examples of a number of practical situations. Impedance concepts were applied to determine the frequency response at the excitation point. Each complete system was then programmed for solution on the computer using the methods outlined in Chapter III. Various excitation devices were used to broaden the scope of the results. Impedance diagrams were constructed and computer results were plotted for easy comparison with impedance calculations. The chapter concludes with a general discussion on the adequacy of the confirmation.

A. Experimental

Results of tests conducted under laboratory conditions as well as on full scale conduit systems were reported by Camichel, Eydoux and Gariel.[10] Each system was a series pipe with a constant head reservoir as the upstream boundary condition. Excitation for these experiments was by a rotating cock located at the lower end of the line.

The laboratory experiments at "l'Institut électro-technique de Toulouse" were conducted on a series line made up of two sections whose characteristics are as follows:

$L_1 = 347.3'$, $L_2 = 661.5'$

$D_1 = .1312'$, $D_2 = .2625'$

$a_1 = 4449'/\text{sec.}$, $a_2 = 4265'/\text{sec.}$

$T = 0.932$ sec.

Figure 3 shows a schematic diagram of such a system, the reservoir being at R_2 and the rotating cock at S_1 .

The impedance at S_2 can be determined using Equation (45). With Z_{S_2} known, the impedance at any point in Pipe 1 is defined by Equation (48). In particular, the impedance at the rotating cock is defined by

$$Z_{S_1} = \frac{Z_{S_2} - i\, Z_{o_1} \tan \omega L_1/a_1}{1 - iZ_{S_2}/Z_{o_1} \tan \omega L_1/a_1} \quad , \qquad (48a)$$

where

$$Z_{S_2} = -\, i\, Z_{o_2} \tan \omega L_2/a_2 \quad . \qquad (45a)$$

By evaluating Z_{S_1} , at various values of ω , the natural frequencies of the system can be determined.

In this particular case the first natural period or fundamental was calculated to be .709 seconds. This is the apparent period of the system. It was determined experimentally to be .69 seconds. The second and third natural periods, corresponding to the third and fifth harmonics, were determined using Equations (45a and 48a) as .311 and .198 seconds while experimentally the measured values were .31 and .19 seconds.

Tests performed on "la conduite de Fully" by M. Boucher were also reported by Camichel. This was a full size series system defined as follows:

$L_1 = 7701.7'$, $L_2 = 7473.7'$

$D_1 = 1.64'$, $D_2 = 1.97'$

$a_1 = 4118'/\text{sec.}$, $a_2 = 3525'/\text{sec.}$

$T = 15.96$ seconds.

Using the same equations as in the previous case, the apparent period was calculated to be 13.72 seconds. The experimental value was 13.5 seconds.

A large number of tests were conducted at "l'usine hydro-electrique de Soulom" on two different penstocks. Each was a constant diameter penstock with a number of thickness changes in the pipe wall. Since each thickness change alters the wave speed slightly, the system can be analyzed as a number of pipe sections connected in series. Determination of the terminal impedance was carried out in a manner similar to the previous cases but with a larger number of series connections. Since the calculations become quite lengthy a program was written for solution on the computer.

The pipe identified as C_4 is 1759.7 feet long, 2.66 feet in diameter and consists of 15 sections of pipe of different thickness with the wave speed varying from 3156 to 3986 feet per second. The theoretical period is 2.007 seconds. The calculated apparent period is 1.888 seconds whereas it was measured at 1.882 seconds.

The pipe identified as P_3 is 1136.8 feet long, 3.937 feet in diameter and is made up of nine sections of different thickness with the wave speed varying from 2789 to 3576 feet per second. The theoretical period is 1.465 seconds. The odd harmonics through eleven were established experimentally on this pipe. Experimental data and calculated results, using Equations (45a and 48a), are listed for each harmonic.

Period	Calculated	Experimental
Apparent	1.379 sec.	1.368 sec.
Third Harmonic	.504	.505
Fifth Harmonic	.296	.310
Seventh Harmonic	.2073	.2150
Ninth Harmonic	.1627	.1667
Eleventh Harmonic	.1360	.1420

The agreement of the calculated periods, using the theory of Chapter IV, with the experimental results is seen to be satisfactory. A maximum difference of 4.5 percent exists with the largest differences occurring at the lower periods. With resonating periods greater than about .3 seconds the agreement is quite favorable.

B. Computer

Several system types with varying boundary conditions were used in this study. Single simple lines were not included as the theory of resonance is quite well established on simple pipes. Each system has a reservoir at the upstream source of supply with the excitation occurring at the downstream terminal. In the branch systems, various boundary conditions were provided on the extra leg of the system.

A number of possible excitation devices are available and can be mathematically represented as a boundary condition for the computer study. The periodic movement of a valve has been referred to in the literature and has been commonly used. This was used most generally in this study with the dimensionless number τ varying between the limits 0 and 1 .

The manner in which τ varies with time between these limits can take on many and various forms. The one most commonly used was the periodic variation of τ as a sine wave. This was done to permit easy application of the valve Equation (44a) to provide the opportunity for a quantitative comparison of results. For one particular series system in the study, other forms of τ movements were used. The periodic motion of τ as a square wave was applied as well as an intermittent periodic valve movement. All of these forms are shown in Figure 8.

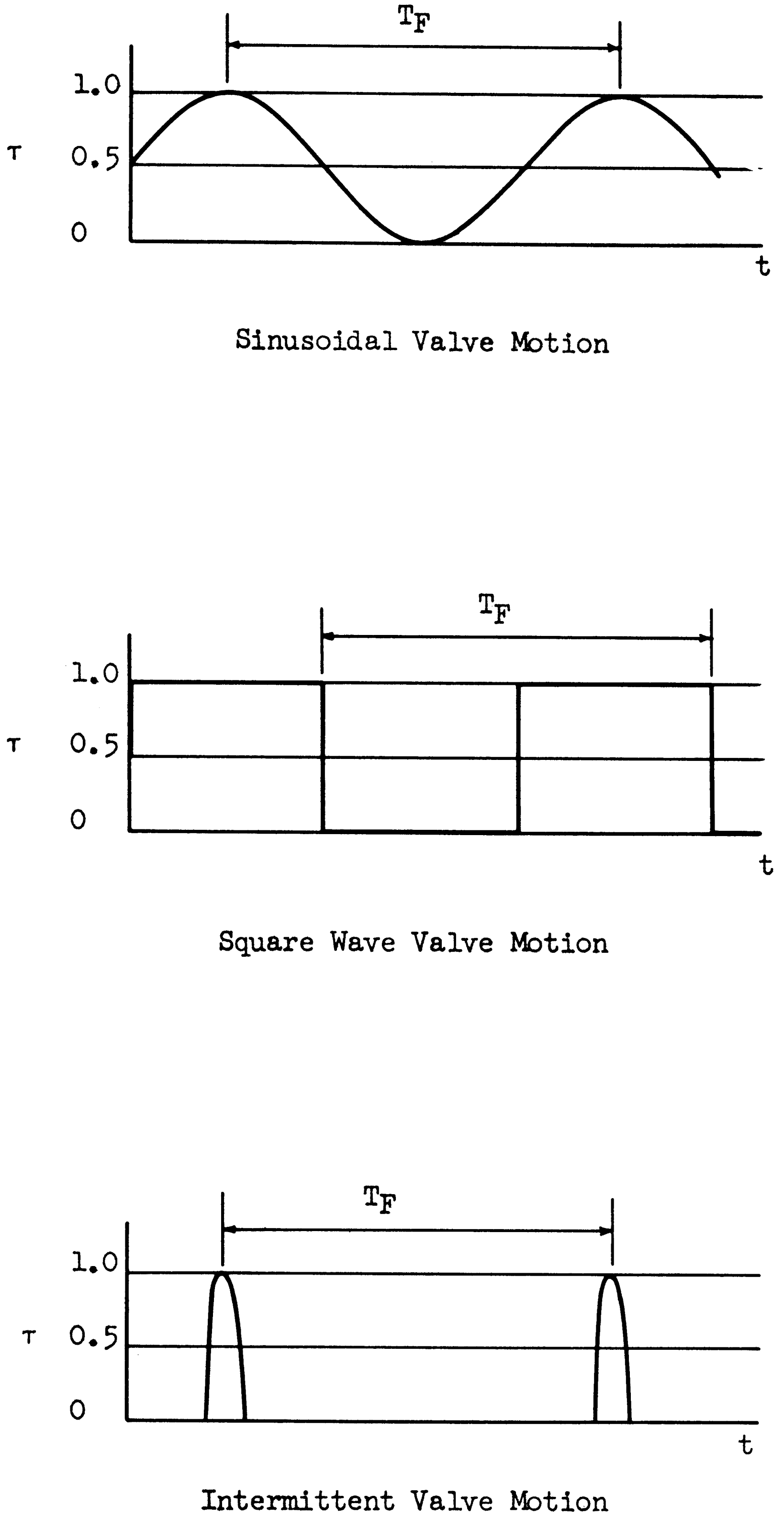
For the particular case of the sinusoidal valve movement, the following equations can be developed.

Equation (44a) gives the impedance at a valve in terms of the motion of the valve, mean flow conditions, and the amplitude of the pressure oscillations. The equation is valid for small pressure variations.

$$Z_R = \frac{2H_o\, H(0)}{Q_o(-4H_o\delta i + H(0))} \tag{44a}$$

Rearranging the equation to solve for the oscillatory pressure head at the sending end of the system, we have

$$H(L) = \frac{-2\, Q_o\, \delta i}{1/Z_S - Q_o/2H_o} \quad . \tag{62}$$



Sinusoidal Valve Motion

Square Wave Valve Motion

Intermittent Valve Motion

Figure 8. Periodic Variation of Tau: Sine Wave, Square Wave, Intermittent.

Since Z_S is a function of the frequency and the system properties, it can be evaluated.

$$Z_S = R_S + i\,X_S \tag{61}$$

Substituting into Equation (62),

$$H(L) = \frac{-2Q_o\,\delta i}{\left(\dfrac{R_S}{|Z_S|^2} - \dfrac{Q_o}{2H_o}\right) - \dfrac{i\,X_S}{|Z_S|^2}} ,$$

or

$$H(L) = \frac{2Q_o\delta\left[\dfrac{X_S}{|Z_S|^2} - i\left(\dfrac{R_S}{|Z_S|^2} - \dfrac{Q_o}{2H_o}\right)\right]}{\left(\dfrac{X_S}{|Z_S|^2}\right)^2 + \left(\dfrac{R_S}{|Z_S|^2} - \dfrac{Q_o}{2H_o}\right)^2} . \tag{63}$$

The modulus of $H(L)$ is the amplitude of the pressure head variation at the valve.

$$|H(L)| = |H_S| = \frac{2Q_o\delta}{\sqrt{\left(\dfrac{X_S}{|Z_S|^2}\right)^2 + \left(\dfrac{R_S}{|Z_S|^2} - \dfrac{Q_o}{2H_o}\right)^2}}$$

or

$$|H_S| = \frac{4H_oQ_o\delta\;|Z_S|}{\sqrt{4H_o^2 - 4H_oQ_oR_S + Q_o^2\;|Z_S|^2}} \tag{64}$$

If $R_S = 0$, the equation is simplified.

$$|H_S| = \frac{4H_o Q_o \delta |X_S|}{\sqrt{4H_o^2 + (Q_o X_S)^2}} \tag{65}$$

Values presented on the impedance diagrams were placed in dimensionless form so they could be more easily correlated. Impedance was nondimensionalized by dividing the hydraulic impedance by the characteristic impedance at the point under consideration. The dimensionless number representing the frequency is the forcing frequency divided by the frequency corresponding to the theoretical period of the system, that is, ω_F/ω or T/T_F. The amplitudes produced by a given excitation device can be expressed as impedances. However, the pressure head amplitude can also be expressed as a dimensionless value when it is compared to the mean or static head, that is, as the ratio of the total pressure head fluctuation to the static or mean head, $2h_{max}/H_o$. In this form the results can easily be compared to those of earlier investigators who reported the phenomenon of doubling of the static pressure during resonance on a system. As dimensionless values for the impedance diagram, Equations (64 and 65) become

$$\frac{2|H_S|}{H_o} = \frac{2h_{max}}{H_o} = \frac{8Q_o \delta \; |Z_S|}{\sqrt{4H_o^2 - 4H_o Q_o R_S + Q_o^2 |Z_S|^2}} \;, \tag{64a}$$

and

$$\frac{2h_{max}}{H_o} = \frac{8Q_o \delta |X_S|}{\sqrt{4H_o^2 + (Q_o X_S)^2}} \tag{65a}$$

Thus, on the impedance diagrams two different dimensionless parameters are plotted against the frequency ratio. Calculated impedance is plotted as a dimensionless number, $|Z_S|/Z_o$, versus frequency. Computer results and calculated pressure head amplifications are plotted as a dimensionless number, $2h_{max}/H_o$, versus frequency.

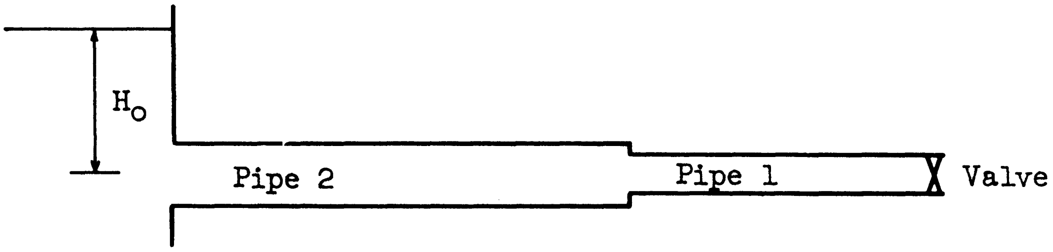
The system types, impedance diagrams and results follow.

1. Series

The details of the two series systems investigated are shown in Figure 9. Impedance values for the impedance diagram were obtained using Equations (45a and 48a). Computer results are shown on the impedance diagram, Figure 10 for system S1 , and Figure 11 for system S2 . Lines representing the linearized assumption of Equation (65a) are also shown. Since this approximation is valid only for small head variations, the lines are not extended above the value 2 on the dimensionless scale.

The computer results for the condition of the valve oscillating as a sine wave agree with the calculated line representing Equation (65a), where it is valid. For the other valve movements in system S2 , Figure 11, there is a qualitative agreement between the pressure head amplification and impedance. For a different mean flow condition these latter points would be shifted up or down but the same qualitative agreement would exist, that is, the largest pressure fluctuations would occur at frequencies of high impedance.

The computer results represented by points of highest amplification, above $2H_o$, require a negative flow through the valve in the



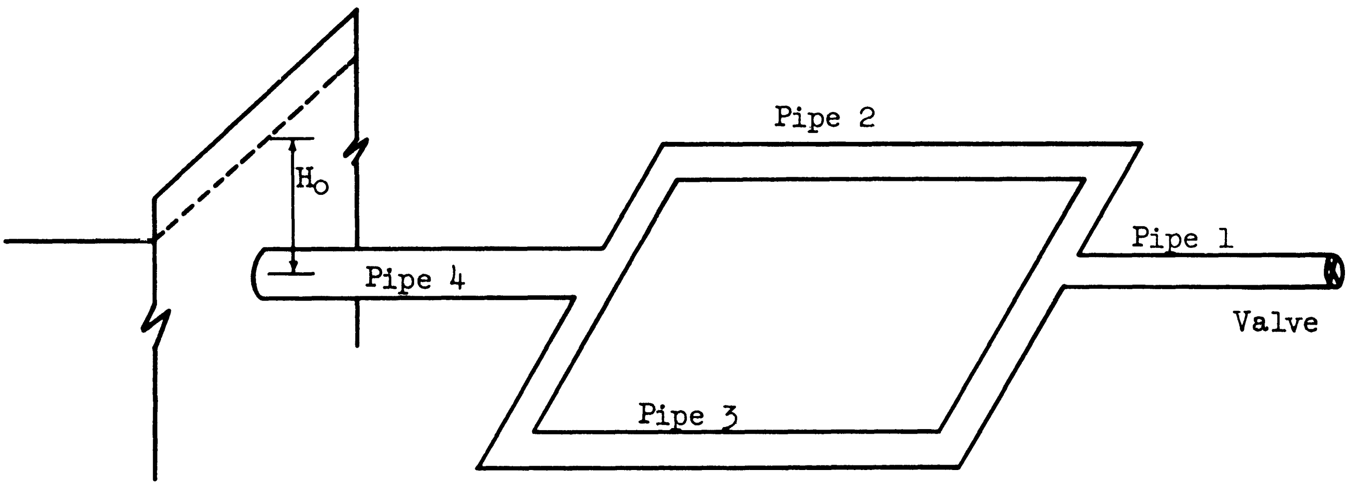
Series System

S1

L_1 = 3600' , L_2 = 1900'
D_1 = 1' , D_2 = 2'
a_1 = 3600'/sec, a_2 = 3800'/sec
T = 6 sec

S2

L_1 = 1800' , L_2 = 1900'
D_1 = 1' , D_2 = 2'
a_1 = 3600'/sec, a_2 = 3800'/sec
T = 4 sec



Parallel System

L_1 = 2500' , L_2 = 1100' , L_3 = 2200' , L_4 = 1100'
D_1 = 2' , D_2 = 2' , D_3 = 1' , D_4 = 2'
a_1 = 3600'/sec, a_2 = 4250'/sec, a_3 = 4250'/sec, a_4 = 4250'/sec
T = 5.87 sec.

Figure 9. Series Systems S1 and S2. Parallel System.

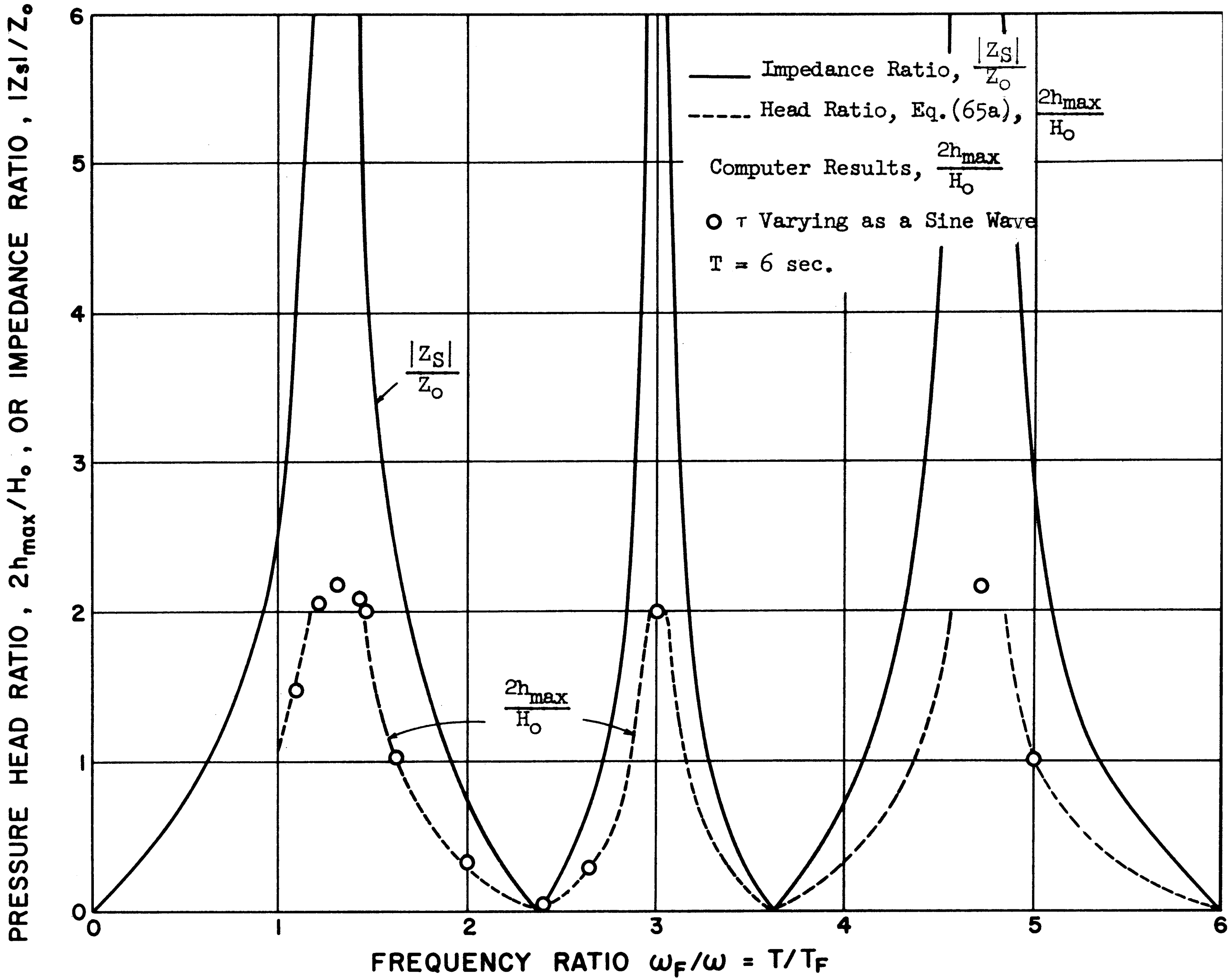


Figure 10. Impedance Diagram. Series System S1.

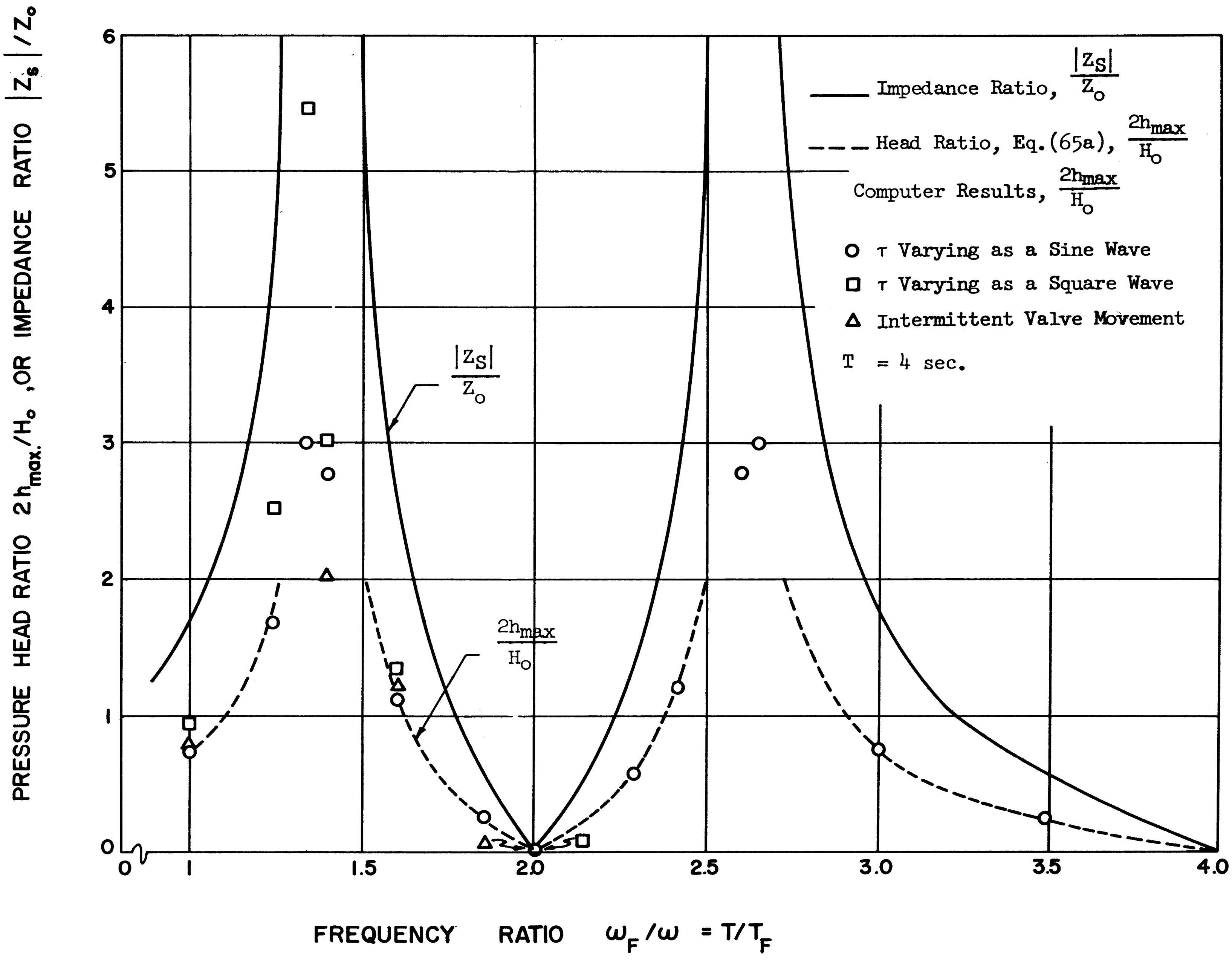


Figure 11. Impedance Diagram. Series System S2.

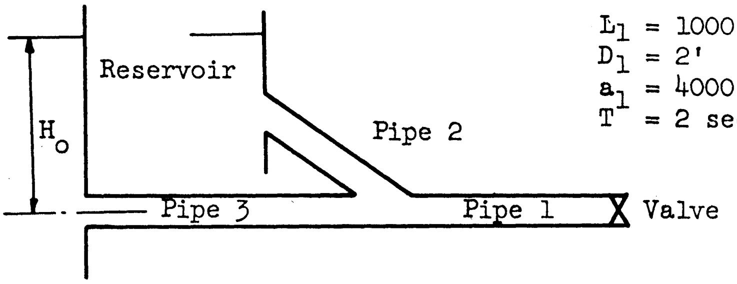
steady oscillatory condition. Therefore, for these amplitudes to exist, the valve would have to be submerged. If it were not submerged the same amplitudes probably would not be reached but a severe vibrating condition would prevail. The highest point on series S2 , resulting from τ varying as a square wave, exhibits vapor pressure on the system. Amplitudes of this magnitude would not exist physically.

2. Branch

Three branch system types, illustrated in Figure 12, were analyzed. The impedance determination was accomplished with the computer using a series of computations involving Equations (45a, 48a, and 49) for system B1; (45a, 46, 48a, and 49) for system B2; and (45a, 42, 48a, and 49) for systems B3a and B3b. The only excitation used in these cases was the rhythmical valve movement with τ varying as a sine wave.

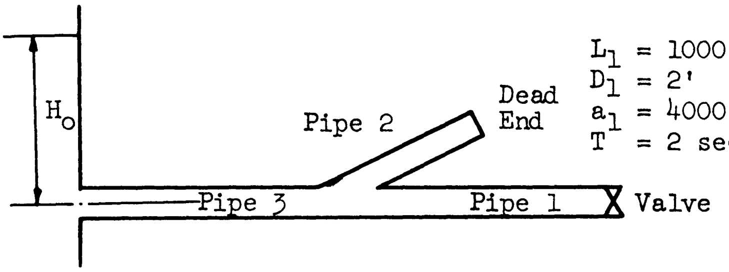
The results given in Figures 13, 14, 15, and 16 show that at the lower pressure amplifications there is agreement between computer results and calculated data from Equations (64a and 65a). At the higher amplifications there is a qualitative agreement between impedance and pressure head amplitude, as in the series cases. Again the results with pressure head magnitudes over $2H_O$ exhibit negative flows at the valve, meaning a submerged condition would be required to physically obtain this situation.

These impedance diagrams exhibit a few points of interest regarding standing and traveling waves. Systems B1 and B2 have boundary conditions such that total reflection of the incident waves occurs



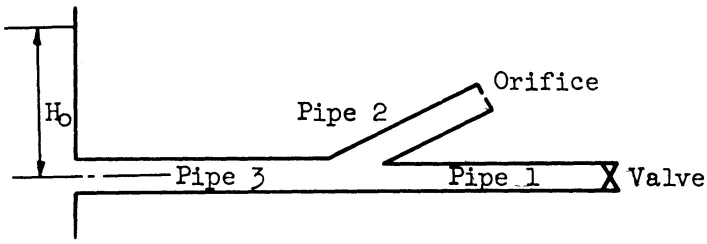
L_1 = 1000' , L_2 = 250' , L_3 = 1000'
D_1 = 2' , D_2 = 3' , D_3 = 3'
a_1 = 4000'/sec, a_2 = 4000'/sec, a_3 = 4000'/sec
T = 2 sec.

Branch System B1



L_1 = 1000' , L_2 = 250' , L_3 = 1000'
D_1 = 2' , D_2 = 3' , D_3 = 3'
a_1 = 4000'/sec, a_2 = 4000'/sec, a_3 = 4000'/sec
T = 2 sec.

Branch System B2



Branch System B3a

L_1 = 1000' , L_2 = 250' , L_3 = 2000'
D_1 = 2' , D_2 = 3' , D_3 = 3'
a_1 = 4000'/sec, a_2 = 4000'/sec, a_3 = 4000'/sec
T = 3 sec.

Branch System B3b

L_1 = 1000' , L_2 = 250' , L_3 = 1000
D_1 = 2' , D_2 = 3' , D_3 = 3'
a_1 = 4000'/sec, a_2 = 4000'/sec, a_3 = 4000'/sec
T = 2 sec.

Figure 12. Branch Systems B1, B2, and B3.

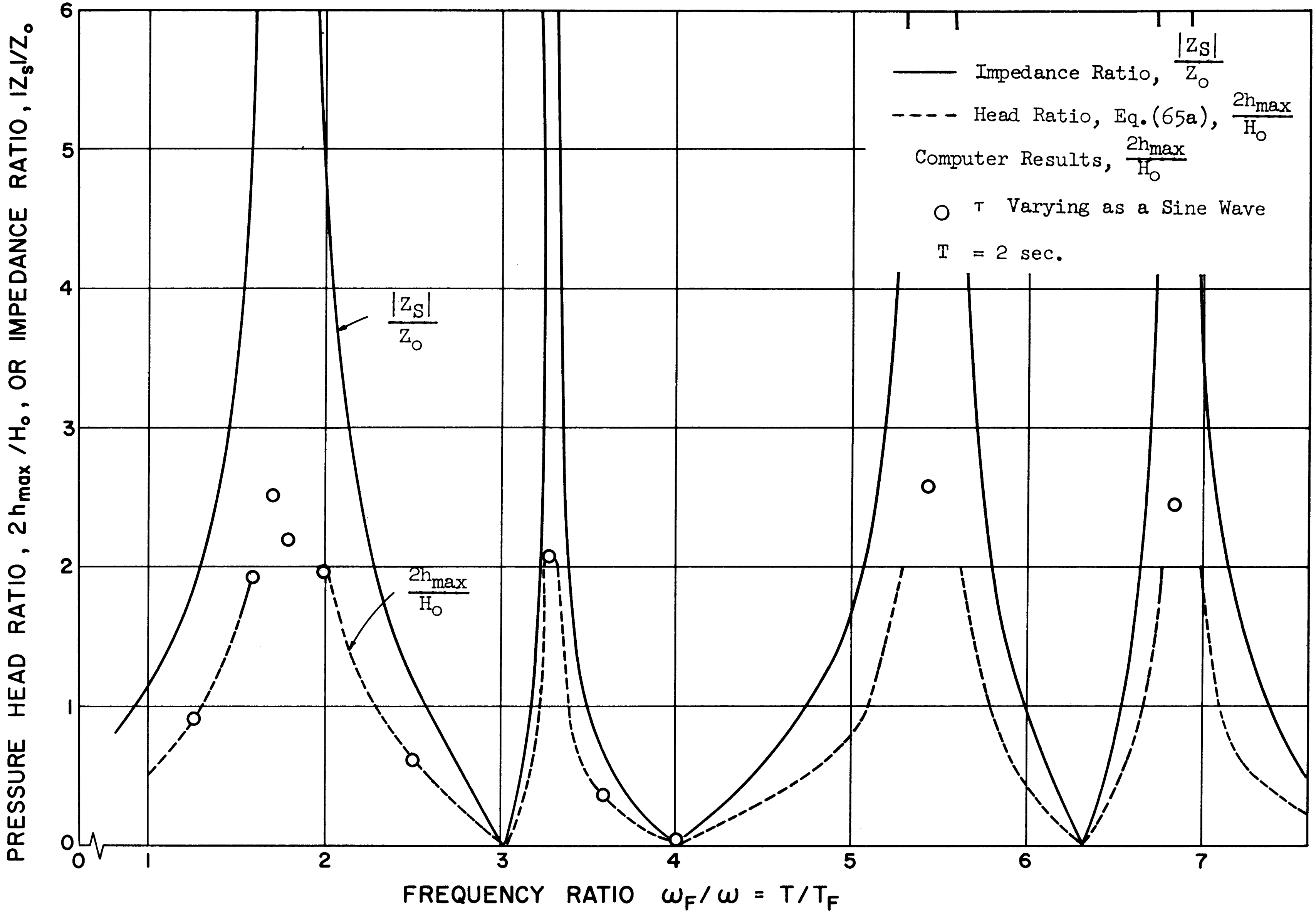


Figure 13. Impedance Diagram. Branch System Bl.

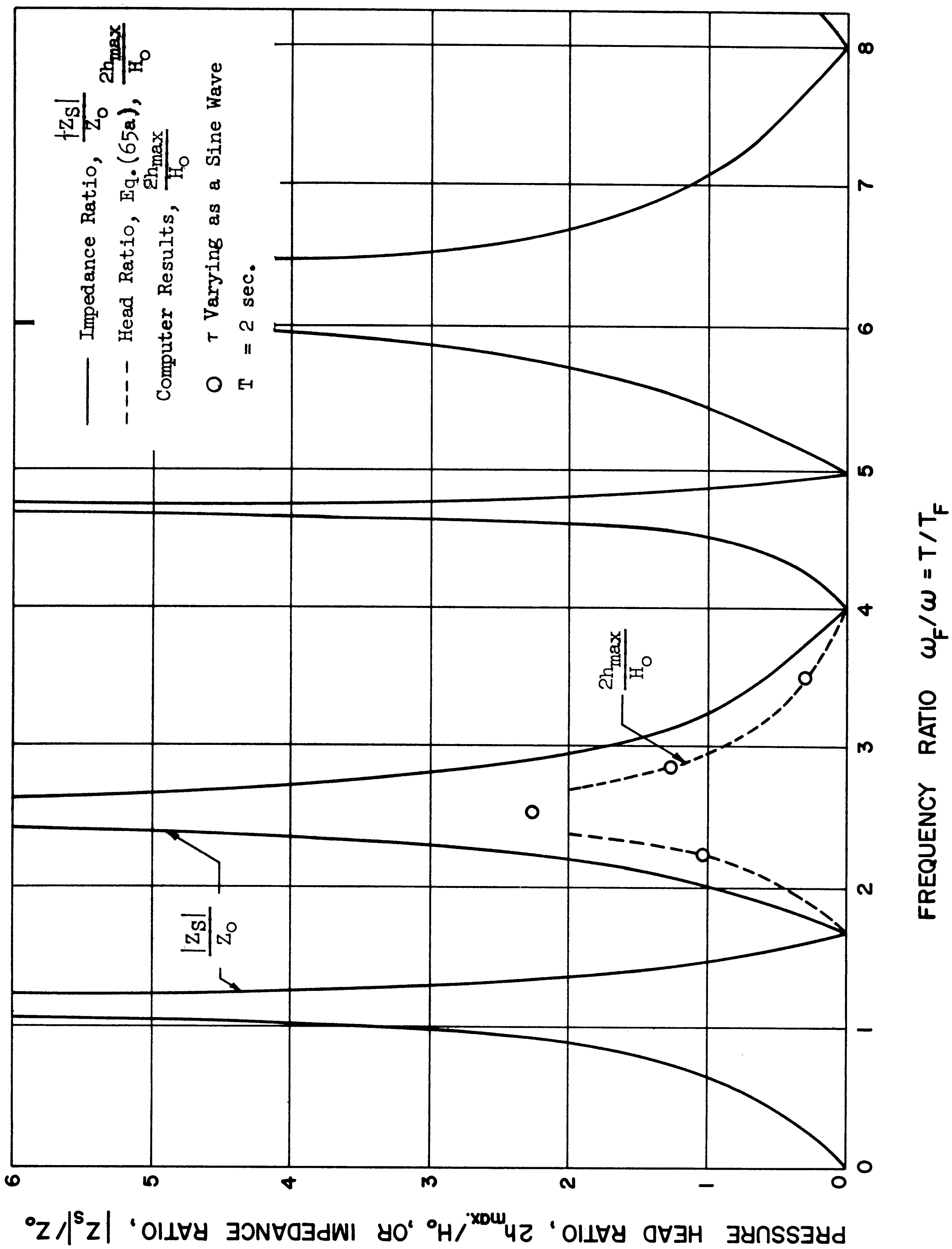


Figure 14. Impedance Diagram. Branch System B2.

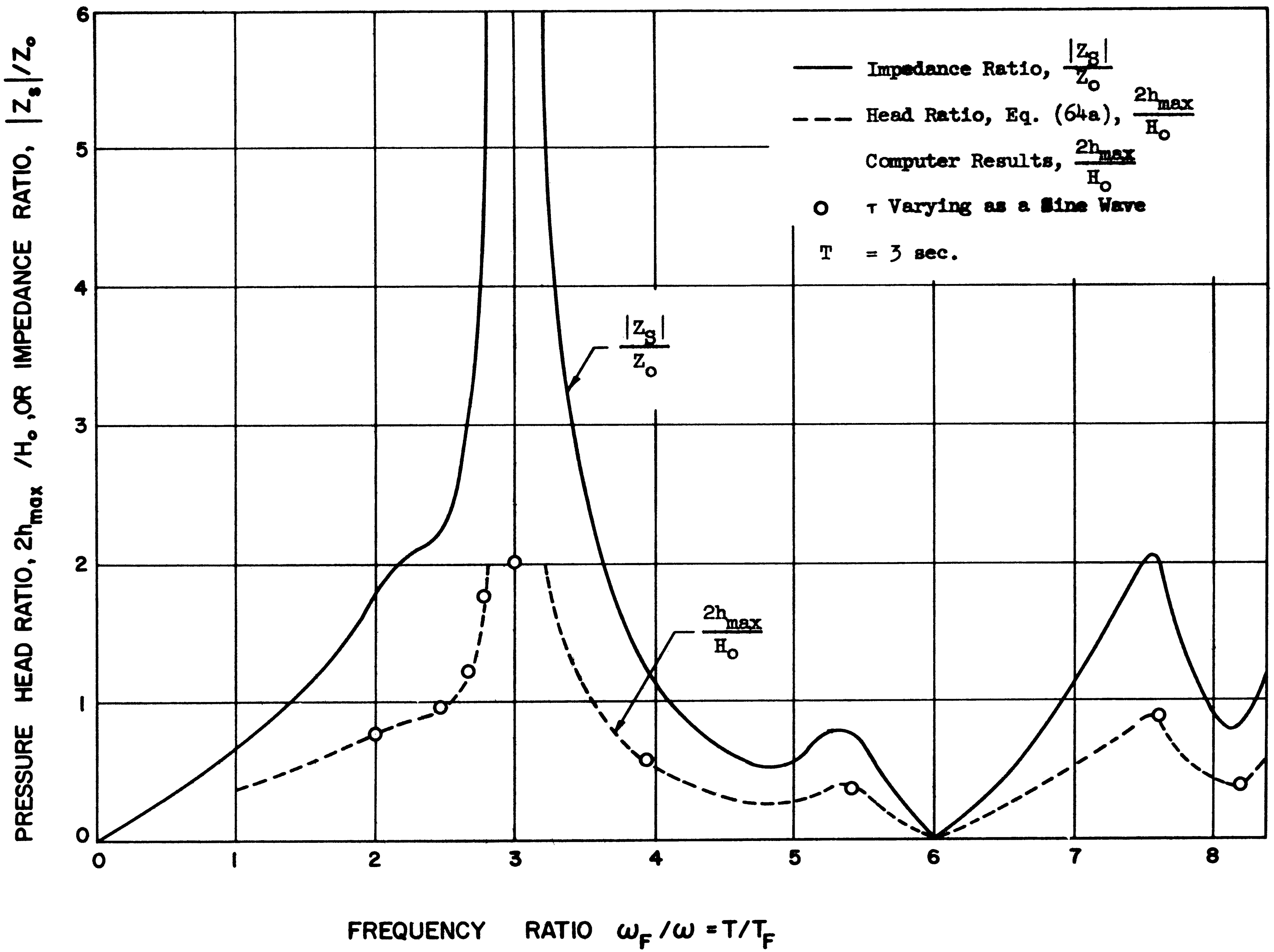


Figure 15. Impedance Diagram. Branch System B3a.

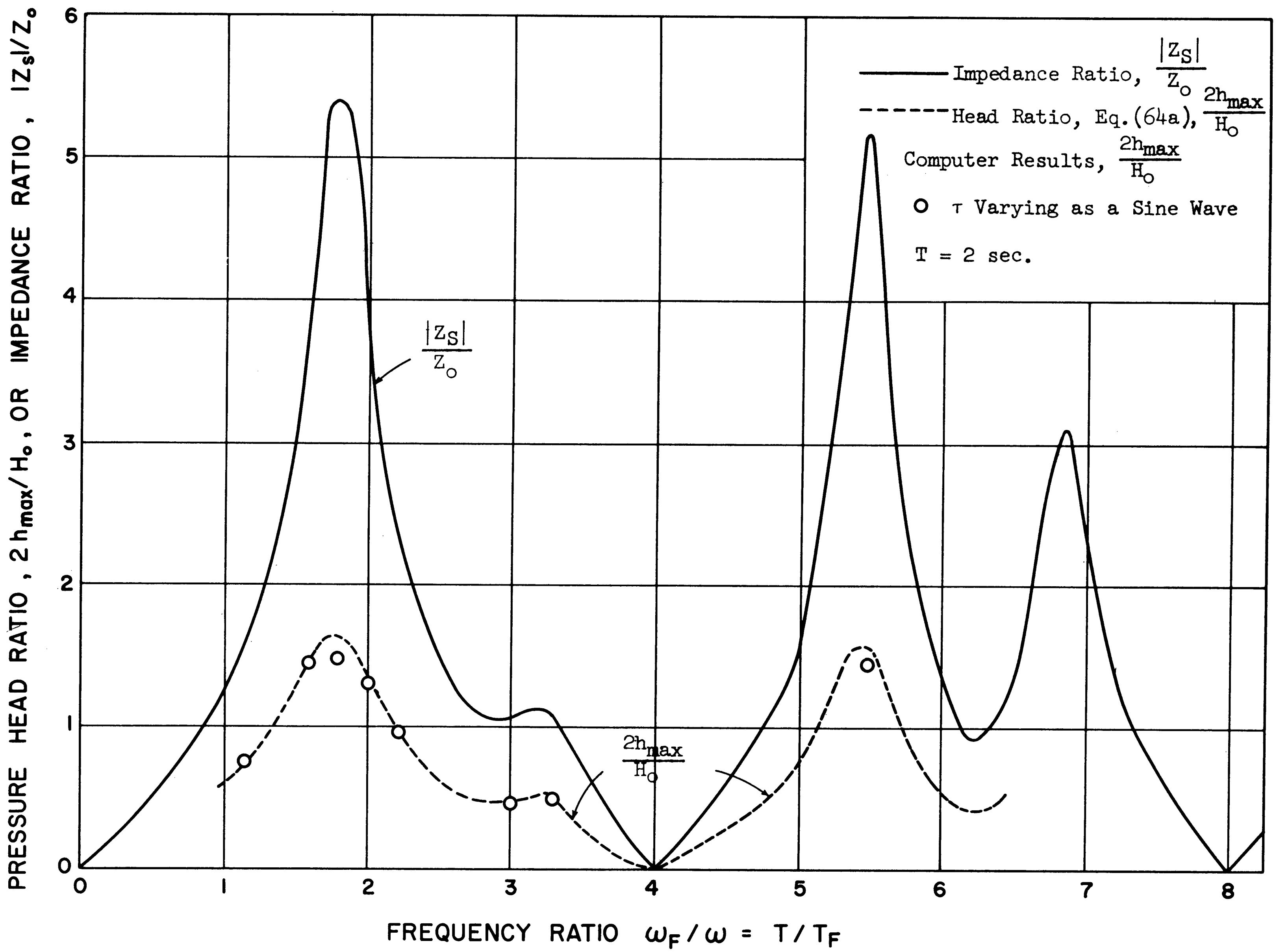


Figure 16. Impedance Diagram. Branch System B3b.

in both branches of the system. These are conditions which yield purely imaginary impedances at the excitation valve. At the resonating frequencies, $Z_S \rightarrow \infty$; at the antiresonances, $Z_S = 0$. If pure sine waves are established on these systems pure standing waves would exist with only energy storage and no energy transmission as a result of the oscillatory flow. Systems B3a and B3b have complete reflection at the reservoir but have only partial reflection at the orifice. The terminal impedance at the orifice is a real number giving rise to a terminal impedance at the sending end which includes both real and imaginary parts. The terminal impedance thus has finite limits at the resonances. This is tantamount to saying the oscillatory discharge at the terminal never goes to zero; therefore, there is energy transmission. With pure sine waves these last two examples would represent a condition of traveling waves superimposed upon standing waves.

3. Parallel

The parallel system is illustrated in Figure 9. The sending end impedance is calculated using Equations (45a, 48a), and the set of simultaneous equations presented for this purpose in Chapter IV. The impedance variation is presented in Figure 17 together with the computer results and the calculated values from the linearized Equation (65a).

Again there is agreement between calculated and computer results for small pressure variation. The one point above the head amplification of $2H_o$ requires a small negative flow at the valve.

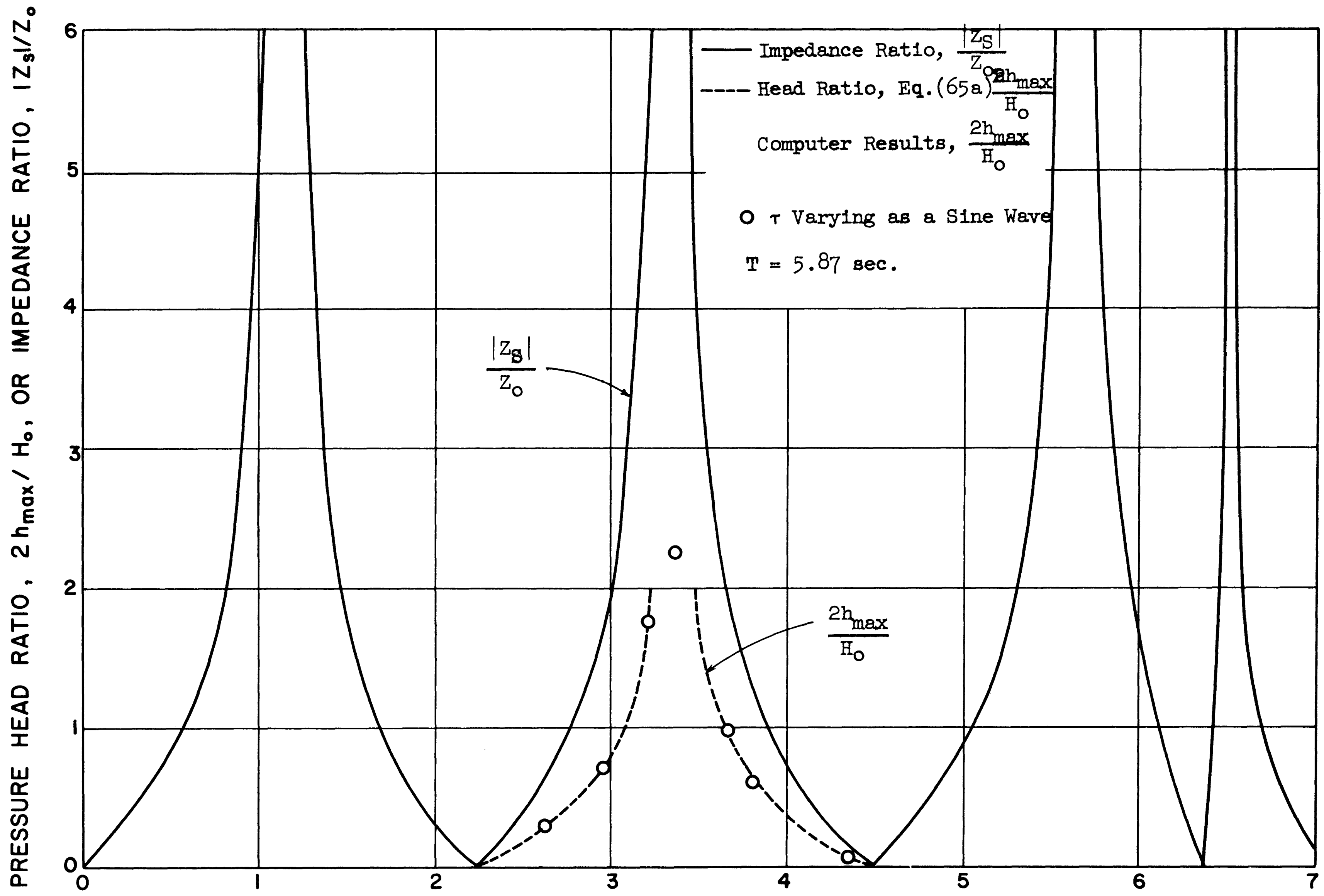


Figure 17. Impedance Diagram. Parallel System.

Results from the computer study, at the frequencies indicated by the two top points on Figure 17, show cavitation pressures at some locations on the system. Therefore, in the true physical situation, the indicated amplitudes would be unlikely at the valve at these frequencies.

C. Discussion

In general, the impedance theory is confirmed by both experimental and computer results. Although a rhythmical valve movement is not the only possible type of excitation, with the various shaped τ movements, it represents quite a broad coverage of possible physical excitations.

The results lead to the conclusion that if the forcing function has a time frequency equal to or approximately equal to one of the natural frequencies of the system, as defined herein, a severe vibrating condition will develop. A qualitative agreement is exhibited for all excitations attempted and a quantitative agreement exists in the special cases where amplitudes can be calculated. Thus in all cases the degree of amplification is directly related to the terminal impedance at the forced frequency.

The question arises as to why the pressure head fluctuation does not continue to grow during the forced valve movements. The answer lies in the fact that although the excitation remains, its effect becomes very small since the discharge is reduced to a value near zero in the resonating conditions. The mode of vibration is one in which a pressure loop exists at the valve together with a discharge node.

If the excitation were being forced by a pulsatile flow input, theoretically an infinite pressure could develop. In this case the discharge node can not exist at the exciter since the boundary condition is still requiring the pulsatile input. With the boundary condition remaining active the amplitude of the pressure head oscillation will continue to grow until cavitation sets in or a failure occurs.

A pure sine wave was established on some of the systems by using a mathematical model of a piston moving in a sinusoidal manner. These results are not reported in detail herein as the results are merely a numerical representation of the theory. The exact confirmation provided from these cases does, however, strengthen the justification for using the characteristics solution on the computer as a means of verification in this thesis.

The intermittent valve movement is of particular interest when it is compared to the theory proposed by Favre.[18] His work on a conical pipe line led him to the conclusion that for an intermittent valve movement the most serious oscillatory pressures would develop with a periodic motion of period T. This study, using the same excitation, indicates that the most serious oscillations occur at the same periods as when a continuous rhythmical valve movement is used. Favre did not have experimental data to confirm his theory.

In the computations, friction was included on a few occasions even though the theory of Chapter IV neglects it. The characteristic method of solution on the computer can include or exclude nonlinear friction depending on the value given to f in the program. Only extremely small differences in pressure amplifications could be detected in the results when friction was included. This fact affirms the conclusion that friction is of small importance in the resonance study of these types of systems.

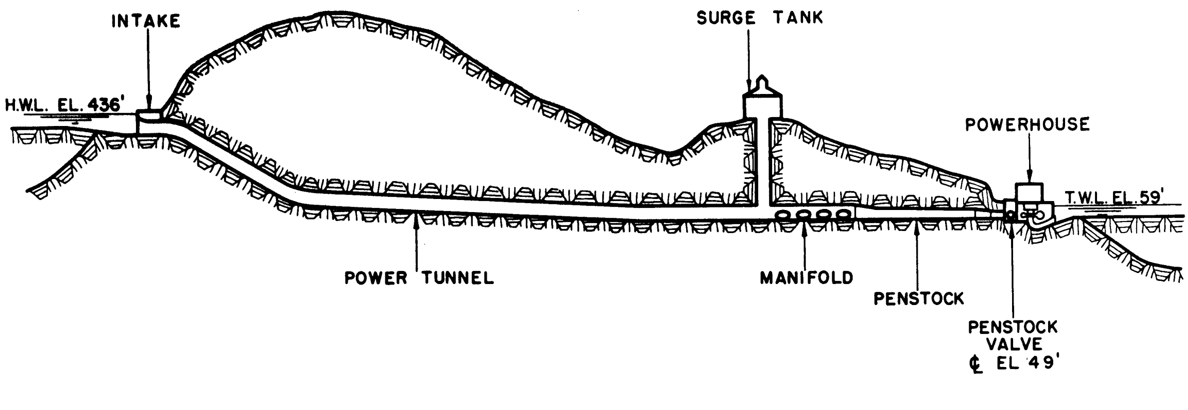
VI. THE BERSIMIS NO. 2 SYSTEM

The field data analyzed in this chapter were obtained at Bersimis No. 2 Hydro-Electric Development of the Quebec Hydro-Electric Commission, Canada. Information for analysis was obtained from References 1, 25 and 30 with additional details received by personal communication with the engineering consultants for the development, H. G. Acres and Company Limited, Niagara Falls, Canada.

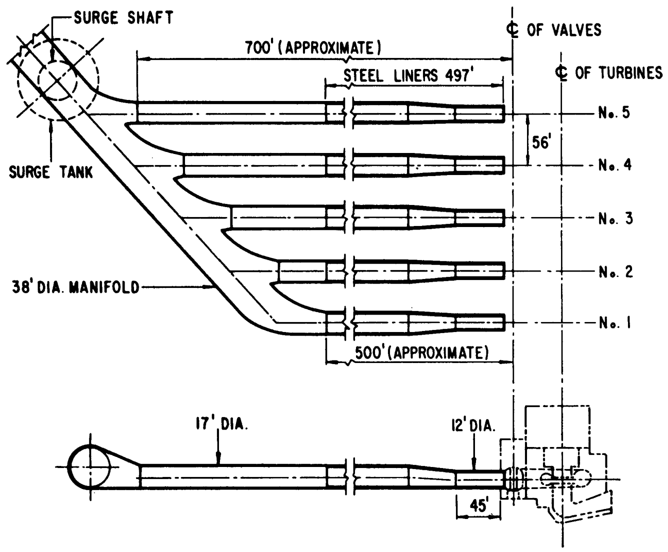
A. Description of Plant

Figure 18 shows a schematic diagram of the power development system. Basically it consists of a 38 foot tunnel, 2437 feet long supplying a 38 foot diameter manifold which feeds five penstocks. An orifice type surge tank is located at the junction of the tunnel and manifold. Each penstock is 17 feet in diameter reducing to 12 feet in diameter before reaching the control valve above the turbine. Normal operating pressure head on the plant is 377 feet.

The straight-flow valve located at the end of each penstock is shown in Figure 19. As illustrated, the valve door is in the closed position. When open the door rests in the space above the pipe, within the valve body. The valve body is spherical in shape with the 12 foot diameter pipe passing through it. When the valve is closed but not sealed a leakage path exists from the upstream pipe to the valve body and out into the downstream pipe. Final closure of the valve is achieved by inflating a flexible service seal which extends around the pipe on the downstream side of the door. A separate



Profile of Development



Plan of Manifold and Penstock

Figure 18. Schematic Diagram of Bersimis No. 2 Power Development.(1)

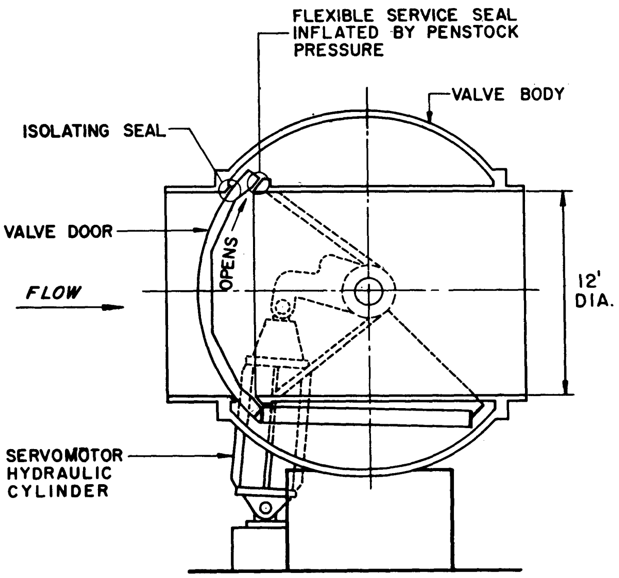


Figure 19. General Arrangement of Penstock Valve.(1)

isolating seal can be manually applied on the upstream side of the door, however, this seal is not normally used. A small by-pass valve is located in the valve body. When it is open it permits flow to pass from the penstock into the valve body and out into the downstream pipe.

The flexible service seal is inflated automatically with water at penstock pressure when the door is closed. If the service seal pressure drops below normal, a small leakage occurs. The volumetric flow varies inversely with the seal pressure under static conditions. It was this leakage together with the elastic properties of the valve that provided the excitation for the vibrating condition.

B. Accidental Conditions of Resonance and Controlled Testing

Valve vibration occurred accidentally on two occasions. These were followed by an extensive controlled testing program during which the valve was fully instrumented. For convenience in presentation herein, these occurrences will be identified as Cases a, b, and c beginning with the controlled testing. The most extensive information is available for this fully instrumented case. Other incidents pertaining to the investigation will be referred to as Case d.

Case a in Penstock No. 2

During this test all penstock valves were in the closed position. The by-pass valves in Penstock Nos. 1, 3, 4, and 5 were open but No. 2 was closed. The pipes below the valve were empty.

The service seal pressure, initially at normal operating pressure, was lowered gradually in steps until vibration began. With pressure reductions of 35 to 45 psi some vibration was observed on the instruments. It was not otherwise detectable. The seal pressure was restored to normal, then it was gradually lowered by about 70 psi. Vibration commenced a few seconds after the low pressure level was reached and gradually increased to a steady amplitude. This vibration was described by witnesses as being quite violent. An adequate record was obtained before the seal pressure was restored to stop the vibration.

The results of this test are summarized in Reference 1 in the following manner.

"(a) Penstock-valve vibration and pressure oscillations in the hydraulic system occurred if the seal pressure was lowered below the penstock pressure by 35 psi or more.

"(b) The vibration amplitude and the pressure oscillation amplitudes stabilized at some finite value for each seal-pressure value in the range tested.

"(c) The amplitudes increased approximately linearly as the seal pressure was lowered. The pressure oscillation amplitudes were not the same in the valve body, the seal, and the penstock, but were approximately in the ratio 2.4: 1.5: 1.

"(d) The frequency of all the vibrations and oscillations was 5.5 cycles per second,...."

An oscillograph record taken during the test is reproduced in Reference 1 and shows the penstock, the seal, and the valve body pressures as well as the valve feet displacement to be approximately in phase. The amplitudes on the penstock and valve body pressure heads at a seal pressure reduction of 70 psi were about 100 and 240 feet, respectively. The valve base movement had a total range of about .042 inch. All records exhibited a sinusoidal wave form. Discharge records showing the flow fluctuations were not recorded during the test, however, the leakage under static conditions with various lowered seal pressures was obtained.

An additional run was made which showed that opening the by-pass valve would also reduce and eliminate the vibration even with the seal pressure at a subnormal value.

Some pressure fluctuations were observed at the valves in other penstocks during the vibration in penstock No. 2. The by-pass valves were purposely opened in the other penstocks to reduce the possibility of such vibration.

Case b in Penstock No. 1

The first awareness of any malfunction at the Bersimis plant occurred June 11, 1961. The pressure in the service seal of penstock No. 1 accidentally dropped to about 30 percent of normal operating pressure. The vibration which developed continued for about seven minutes and was stopped by opening the by-pass valve.

At the time of this occurrence, only No. 5 penstock valve was open. There was no flow at penstock valves 2, 3, and 4 and only leakage flow at penstock valve No. 1. Other than the vibrations being described as severe, no specific amplitudes of the oscillations are known. The frequency of the vibration was observed on records of valve-base displacement. These indicated a vibration frequency of 1.8 cycles per second.

Case c in Penstock No. 3

The second accidental occurrence of vibration happened on June 29, 1961. Routine maintenance work on the control system was being performed at the time. The penstock valve was in the closed position and the service seal pressure accidentally dropped to zero. The penstock valve started to vibrate and was stopped by restoring the seal pressure to penstock pressure. No records of the oscillation were made but plant personnel present on the two accidental occurrences estimated the vibration frequencies to be about the same. At the time of this vibration, turbines 2 and 5 were operating and turbines 1 and 4 were standing.

Case d

On occasions prior to the accidental malfunctions, seal pressures had been lowered to zero on penstock valves for periods of fifteen minutes or more without causing vibration. The station was not operating on these occasions.

C. Analysis

The theories of the earlier chapters of this thesis were applied to the Bersimis system to explain the vibration phenomena.

In order to obtain values for comparison with the experimental results, the physical characteristics of the plant had to be clearly defined. This investigation is presented in this section, followed by a discussion of the impedance calculations and ending with the information on the characteristics solution on the digital computer. The comparison of the theory and experimental results is presented in the next section.

The geometric properties of the tunnel-penstock system are clearly defined and fixed as shown in Figure 18. The wave speed , a , in each pipe had to be estimated. Following Jaeger's discussion[25] the wave velocity may vary within the range 3950 to 4600 feet per second. A value of 4375 feet per second was used and was assumed to be constant throughout the system. Pipe diameter affects the wave speed very little on a system of this nature. The modulus of rigidity of the rock and tunnel lining material is the predominant control.

The geometric characteristics and elastic properties of the valve also had to be determined. The isolating seal forms a natural constriction between the penstock and the valve body. The area of the opening is equal to the product of the clearance and the circumference of the penstock. With the isolating seal completely withdrawn, which is its normal position, this area is 1.88 square feet. The effective length of this choke or neck includes the actual length of the constriction plus an allowance at each end since a mass of fluid at these locations must be accelerated and decelerated with the fluid in the neck.[32] An effective length, L' , of 5.45 feet was used. Since high velocities are likely to exist in the neck an appreciable head loss is likely to occur. A constant F , as defined for use in Equation (55), was determined by estimating the head loss to be one velocity head. Therefore, using the linearized approach,

$$F = \frac{H_F}{Q_N} = \frac{\bar{V}}{2gA} \quad , \tag{66}$$

where $\bar{V}$ is the average velocity in one direction during a half period.

The volume of the valve body cavity, $\forall$, was calculated to be 1900 cubic feet. The combined elasticity of the fluid and the valve body, K' , was estimated as 21,500,000 pounds per square foot.

The natural frequency of the resonator can be determined by referring to Figure 6 and Equation (55). This frequency, considered as the frequency at which maximum pressure fluctuations occur within the volume, exists when the discharge oscillations are largest in the neck.

$$Q_S \to \infty$$

$$Z_S \to 0$$

$$Z_S = - F + i(-\omega I + \frac{1}{\omega C}) \tag{55}$$

Z_S will be zero when both its real and imaginary parts are zero. The natural frequency is determined to be

$$\omega_n = \sqrt{\frac{1}{IC}} \tag{67}$$

In this particular case,

$$\omega_n = \sqrt{\frac{1}{IC}} = \sqrt{\frac{K'A}{\rho L'\forall}} = 44.9 \text{ radians/second.}$$

The amplification in the resonator of Figure 6 can be calculated. Denoting the amplification factor or pressure ratio by PR , we have

$$PR = \left|\frac{H_R}{H_S}\right| = \frac{|Z_R|}{|Z_S|} ,$$

$$PR = \sqrt{\frac{1}{\frac{F^2C}{I}\left(\frac{\omega}{\omega_n}\right)^2 + \left(-\left(\frac{\omega}{\omega_n}\right)^2 + 1\right)^2}} \qquad (68)$$

If the resistance factor, F is small, the equation reduces to

$$PR = \frac{\omega_n^{\,2}}{\omega_n^{\,2} - \omega^2} . \qquad (69)$$

For Case a , at the resonating frequency of 5.5 cycles per second or 34.5 radians per second, PR is 2.44. For Case b, at the frequency of 1.8 cycles per second or 11.3 radians per second, PR is 1.07. These can only be considered rough estimates as the through flow has not been included.

With the system completely defined, the impedance at the flexible service seal can be evaluated using the equations defined in Chapter IV. Inasmuch as all of the parameters of the system are fixed for any particular case, the sending end impedance is a function of frequency only. For a system of this complexity the calculations are very lengthy and would be prohibitive without the use of the computer. Program details for these impedance computations are presented in the Appendix. Results of the impedance calculations are presented as impedance diagrams in Figure 20 for Case a, and Figure 21 for Case b.

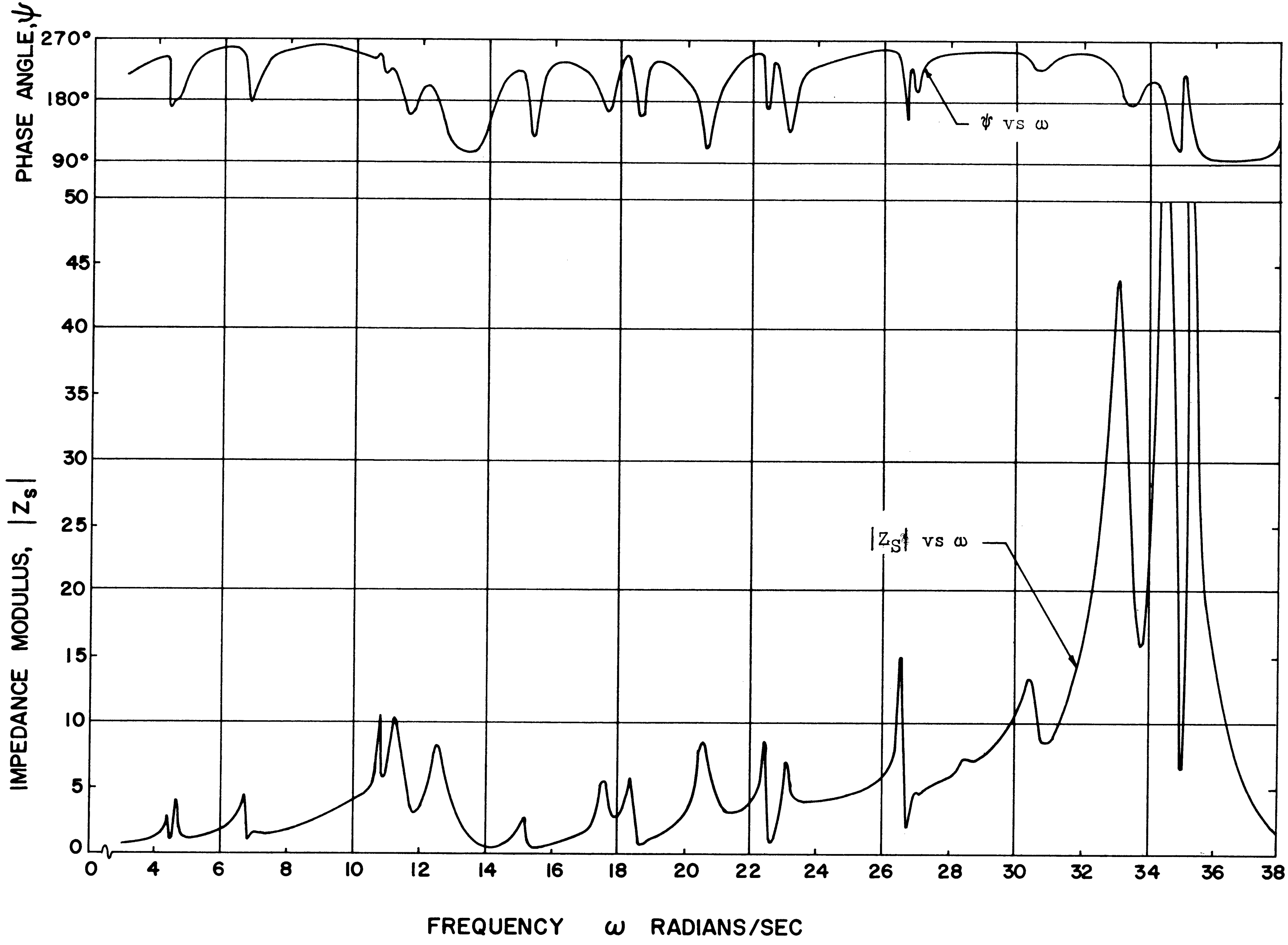


Figure 20. Impedance Diagram. Penstock No. 2.

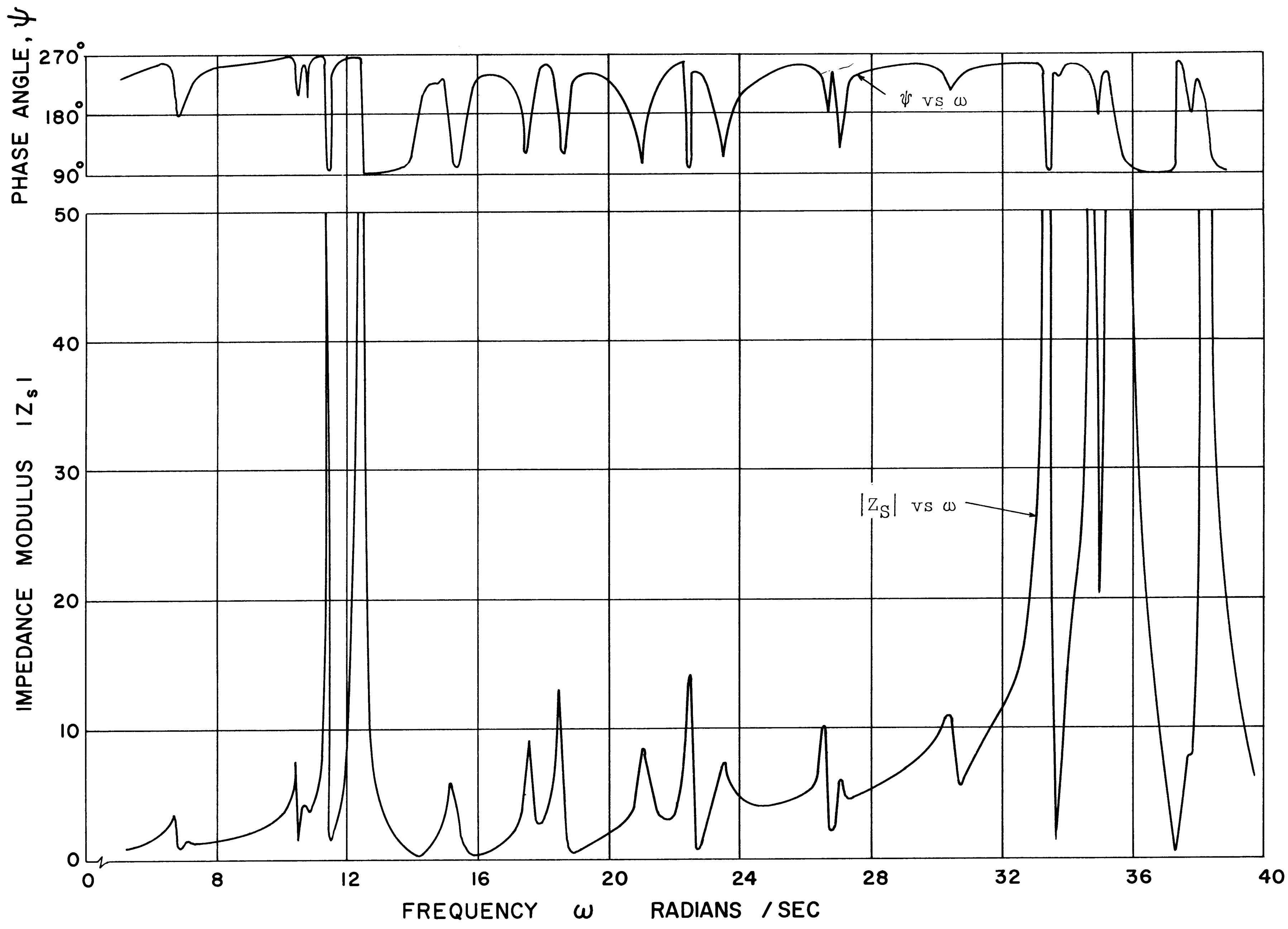


Figure 21. Impedance Diagram. Penstock No. 1.

The specific details of the system are included in each analysis. For example, the by-pass valves are open in the inactive penstocks for Case a, and only No. 5 penstock has any through flow in Case b.

The impedance diagram for Case a shows that the lowest frequency at which a small flow oscillation will produce a resonating condition is at 33.1 radians per second. There are other possible resonating frequencies immediately following and some at about 45 radians per second. The latter correspond to the natural frequency of the fluid in the valve body. The particular frequency at which the system resonated is dependent upon the leakage at the inflatable seal. For pure sine waves, a matched condition must exist between the impedance of the system and the impedance of the mechanism connected to it.

The impedance diagram for Case b shows the lowest frequency at which a small flow oscillation will produce a resonating condition is at 11.4 radians per second, the second at 12.4 radians per second. Others exist around 35 and around 45 radians per second. Again the one at which the system resonated is dependent upon the leakage at the seal.

To evaluate the characteristics of the leakage at the seal, the elastic behavior of the entire valve must be considered. A discussion of the valve and seal action is presented, followed by the equation used to describe the discharge.

When the seal pressure is lowered to some subnormal value a small steady state leakage occurs. If a disturbance is introduced to some point of the system sending a high pressure wave through the penstock to the valve, the valve gate is moved closing the leakage gap slightly. This area reduction and simultaneous flow reduction intiates a high pressure wave which returns to the valve one-half period later at a lower pressure. The lower pressure allows the valve gate to move, opening the seal leakage gap slightly, increasing the flow, and amplifying the low pressure wave. The low pressure wave returns one-half period later as a higher pressure, closing the leakage gap, decreasing the flow, and amplifying the high pressure wave. At the same time as this series of events is taking place the valve body cavity is acting as a resonator amplifying the pressure changes which in turn exaggerates the seal leakage changes. At a sufficiently low pressure in the valve body, the seal pressure eventually exceeds the valve body pressure and the flexible seal expands to form a complete closure, thus limiting further development of the pressure fluctuations. The seal leakage is also cut off on the high pressure side of the oscillation since the high penstock pressure moves the door to form a tight closure with the slightly deflated flexible seal.

The natural frequency of the valve door mechanism becomes of interest since in the above discussion it is assumed to respond almost immediately. Reference 25 gives the natural frequency of the Bersimis valves as 125 cycles per second. A mechanical device having such a high natural frequency would have little effect on the comparatively low resonant frequencies which actually occurred.

An accurate mathematical representation of the seal leakage would apparently be quite complicated since its description should include an area of opening which is a function of the penstock, valve body, and seal pressure heads while the head difference causing the flow is the instantaneous valve body pressure. One apparent simplification results from the fact that the experimental records showed the three pressure oscillations approximately in phase, at least in Case a.

An equation of the form of the orifice equation is desired but it must include the special properties mentioned above. That is, it must shut off the flow at a penstock pressure above the static condition; it must permit maximum flow somewhat below the static condition; and it must again shut off the flow before the head difference causing the flow becomes zero. An equation of the form

$$QSL = C_1(C_2 - HP') \sqrt{HVB} - C_3 \qquad (70)$$

was used with the subscripted constants being evaluated for each subnormal seal pressure. QSL , HP' , and HVB represent instantaneous values of seal discharge, total penstock pressure at the valve, and total valve body pressure, respectively. In effect, the area of opening is controlled by the penstock pressure; the head drop at the orifice is correctly represented by the valve body pressure, and the constant C_3 controls the lower pressure shut-off. All negative flows from the equation are set to zero. The resulting leakage curves for the two subnormal seal pressures of Cases a and b , are shown in Figure 22.

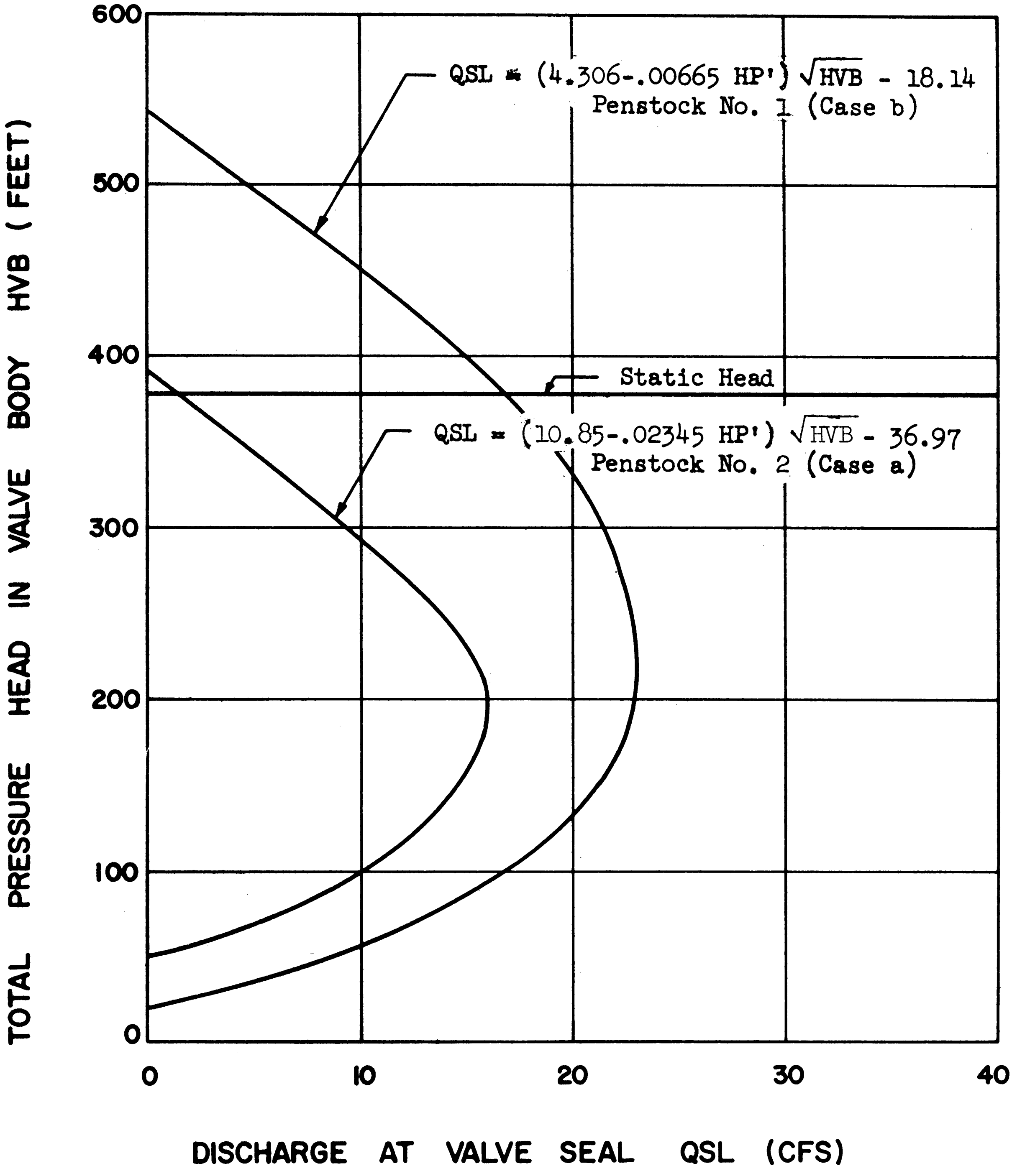


Figure 22. Leakage Curves of Penstock Valves.

For Case a, there is essentially no flow when the valve body pressure is above static conditions, and peak flow occurs at a pressure of about 200 feet. If the pressure drops as low as 50 feet the flow again shuts off. Assuming the pressure does not drop this far, there will be continuous flow at low pressures and no flow at high pressures. This is a condition which can be considered as a 180° phase shift, although we are not dealing with pure sine waves. Looking at the impedance diagram, Figure 20, the phase angle required at a frequency of 33.35 radians per second is approximately 180°. The impedance magnitude is about 31. With the seal discharge varying between 0 and 16.1 cfs, the pressure head should fluctuate over a total range of about 500 feet. The minimum pressure would be 127 feet so the above assumption regarding continuous flow is reasonable. Perfect sine waves are not going to exist so the numbers are approximate.

The predicted resonating frequency for Case a on the basis of this analysis is therefore 33.35 radians per second. The equivalent period is .188 seconds.

Referring to Figure 22 for the seal characteristics of Case b, the flow will be shut off when the pressure head is above 530 feet and will be maximum at about 235 feet. Should the pressure drop below 20 feet the flow will again shut off. For the first two resonating frequencies in Figure 21, the impedance magnitude is greater than 50. This impedance, with the seal discharge varying between zero and 23 cfs, would indicate a pressure fluctuation greater than twice the static head, assuming the existence of pure sine waves. With this amplitude of pressure oscillation, the flow would shut off at the low pressure as well as the high pressure and would exist only in the intermediate to low pressure ranges. In the steady oscillatory situation there would be two individual jets of fluid being discharged during each period. This type of flow variation cannot be described or even approximated by a simple sine wave of the full vibration period, so the direct application of the impedance concepts, including phase and amplitude, breaks down. The most that can be said is that the system will resonate at one of the two frequencies around 12 radians per second if resonance is going to occur as a result of relatively low discharge oscillations.

The predicted resonating frequency for Case b is either 11.7 radians per second or 12.5 radians per second. The equivalent periods are .55 seconds and .51 seconds.

Energy considerations pertaining to the action of the seal will be discussed at the end of this chapter. From an energy viewpoint, it can be shown that amplification of oscillations may occur whenever the seal discharges flow at a mean pressure below normal static conditions.

The complete Bersimis system was programmed for solution on the computer using the method of characteristics as outlined in Chapter III. The main problem in this part of the investigation involved the proper representation of the valve and leaky seal. The differential equations describing the flow at the valve were written and solved as part of the computer program by third order Runge-Kutta Methods.

The equation of motion written for the neck with the total penstock head, HP', on one end and the total valve body pressure head, HVB, on the other end is written as follows:

$$(HP' - HVB)\cdot\gamma\cdot A - VN\cdot|VN|\cdot\frac{\rho\cdot A}{2} = \rho\cdot A\cdot L'\cdot\frac{d(VN)}{dt} ,$$

or

$$\frac{d(VN)}{dt} = \frac{g}{L'}\cdot(HP' - HVB) - \frac{1}{2\cdot L'}\, VN\cdot|VN| \quad . \qquad (71)$$

The continuity expression takes the following form.

$$(A\cdot VN - QSL)\cdot dt = \frac{\forall\cdot\gamma}{K'}\cdot d(HVB)$$

By introducing the capacitance, $C = \gamma\cdot\forall/K'$, the equation becomes

$$\frac{d(HVB)}{dt} = (A\cdot VN - QSL)/C \quad . \qquad (72)$$

VN, A, and L' represent the total velocity, area, and effective length at the neck. QSL represents the seal discharge as defined by Equation (70).

At the initial time in the computer program HP', HVB, QSL, and VN are known. Solving the differential equations by the Runge-Kutta method yields HVB and VN for the next time increment. With VN known the next HP' can be determined from the characteristic equations; then QSL can be evaluated using Equation (70). These computations are repeated at the same time increments as the characteristics solution.

To initiate any oscillations with the computer calculations an initial disturbance had to be introduced. This was done by setting the initial conditions in the active penstock with too high a discharge. At the first numerical calculation the valve was closed to the proper seal leakage causing an abrupt pressure rise. This initial shock and the subsequent reflections produced ample excitation to start the system oscillating.

A point of interest in connection with this program is that it took approximately one minute of computer time to cover the calculations of one second real time. In order to reach a steady oscillatory condition without using an excessive amount of computer time, a severe shock was introduced initially. This produced high amplitude oscillations which fell into steady oscillatory motion quickly. On one occasion a very light disturbance was given since this would be more likely to occur physically. After fifteen minutes of execution time the development was only part way to a steady oscillatory condition, but the gradual build up to such a condition was evident.

Partial results of the computer study are presented graphically in Figure 23 for Case a, and Figure 24 for Case b.

D. Comparison and Discussion

The results of the analysis can now be summarized and compared to the results from the field tests. Each case will be discussed separately.

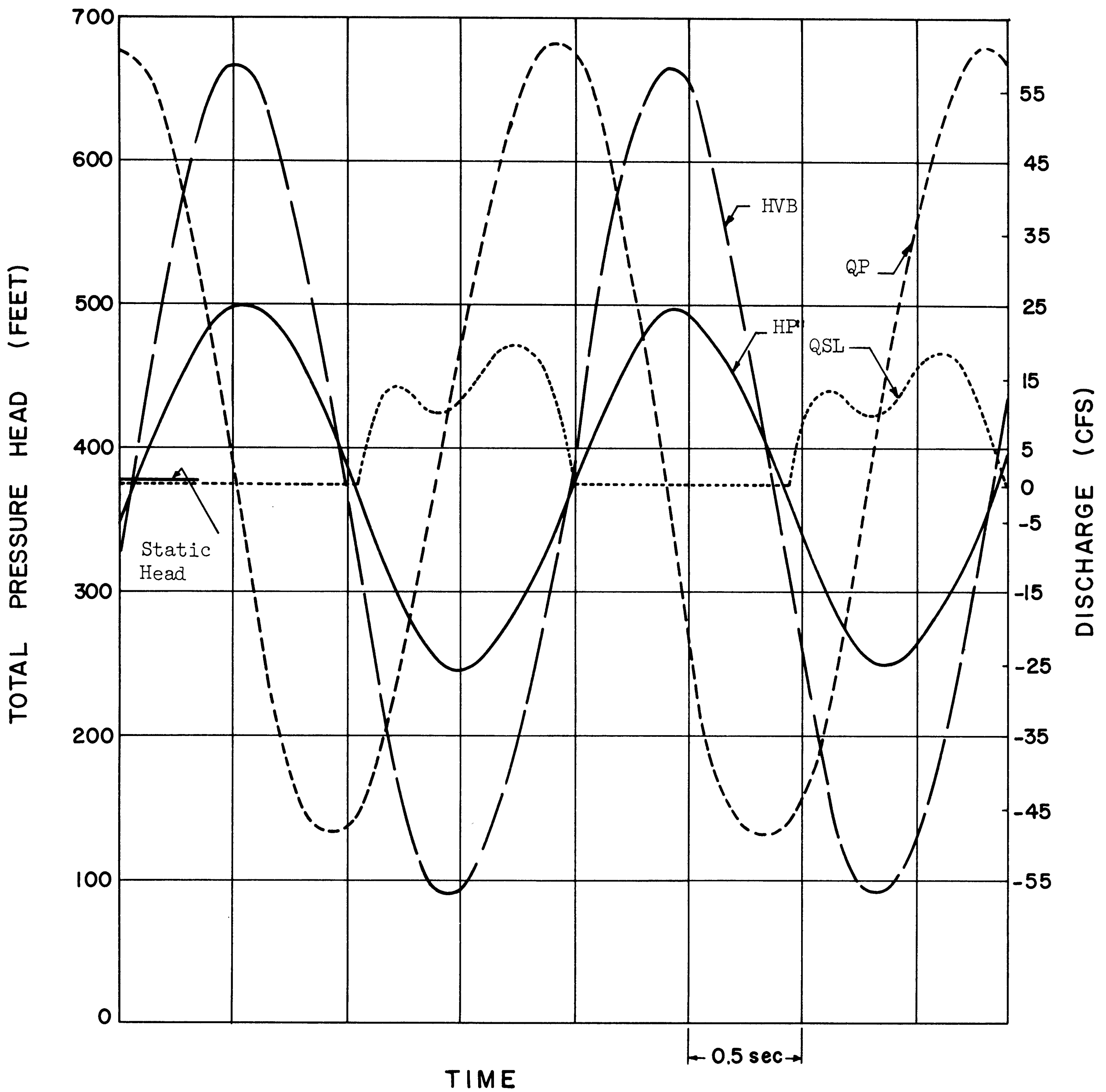


Figure 23. Pressure and Discharge Oscillations at Valve, Penstock No. 2.

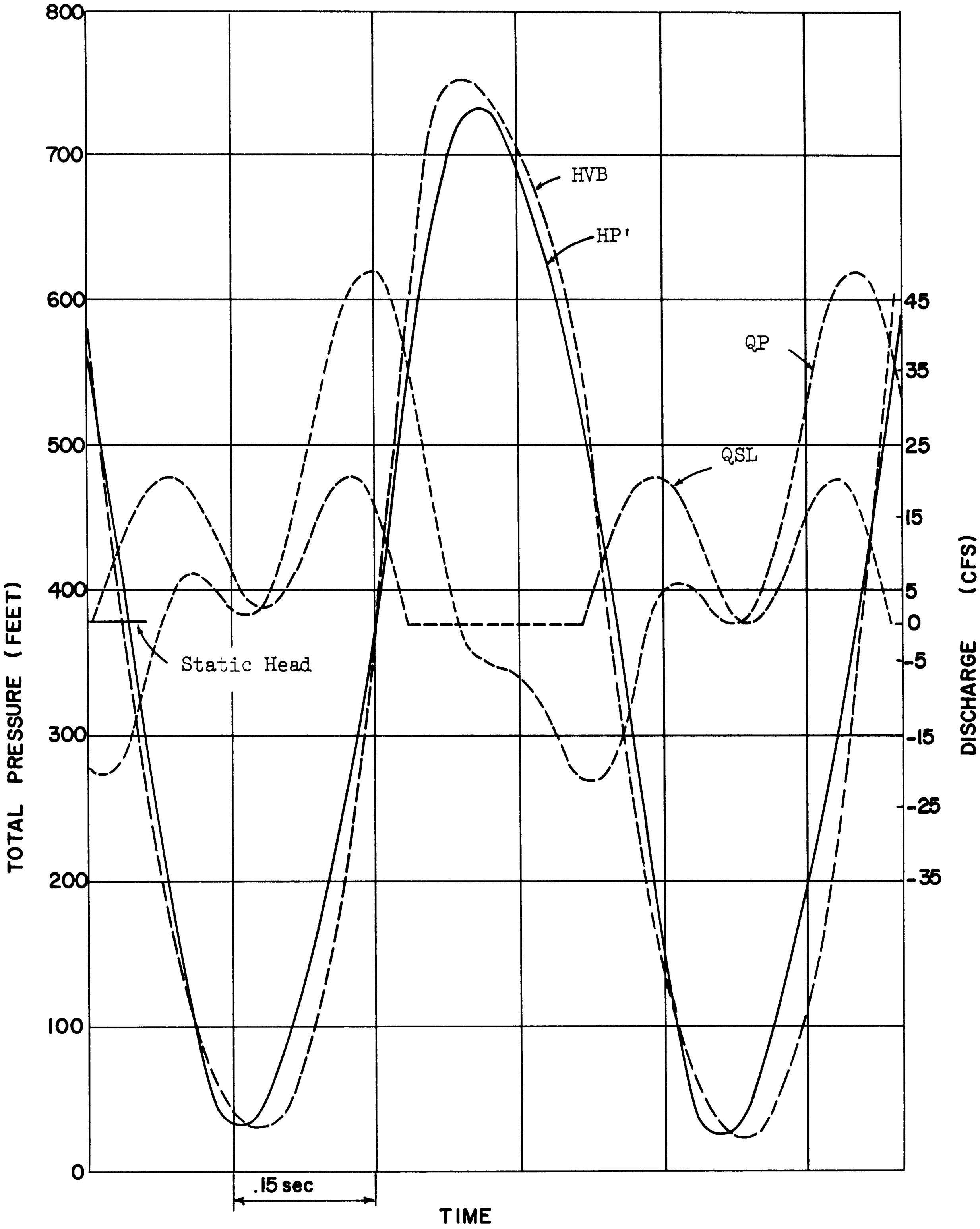


Figure 24. Pressure and Discharge Oscillations at Valve, Penstock No. 1.

Case a in Penstock No. 2

The reported frequency of vibration at the test site was 5.5 cycles per second or a period of .182 seconds. Impedance methods predicted a period of .188 seconds and the computer solution yielded a steady oscillatory condition at .188 seconds.

The actual amplitudes of the pressure head oscillations in the penstock and the valve body were approximately 100 and 240 feet, respectively, having a pressure ratio of 2.4. The comparable figures from the computer were 125 and 290 feet yielding a pressure ratio of 2.32. The estimated amplitude at the valve body was about 250 feet based on the impedance values. The predicted pressure ratio, based on the assumption of no through flow, was 2.44. The test records showed the pressure oscillations approximately in phase. This was confirmed by the computer results.

Discharge varied between 0 and 18 cfs with a mean value of about 5 cfs in the computer results.

The oscillations along the length of Penstock No. 2 appear as the third harmonic as shown on Figure 25. During this motion the large pipes of the system are not undergoing any appreciable pressure oscillations. However, according to the computer results, the valve at Penstock No. 3 is subjected to quite severe pressure oscillations, even with the by-pass valve open.

Case b in Penstock No. 1

The frequency of vibration measured at Bersimis was 1.8 cycles per second or a period of .555 seconds. Impedance methods

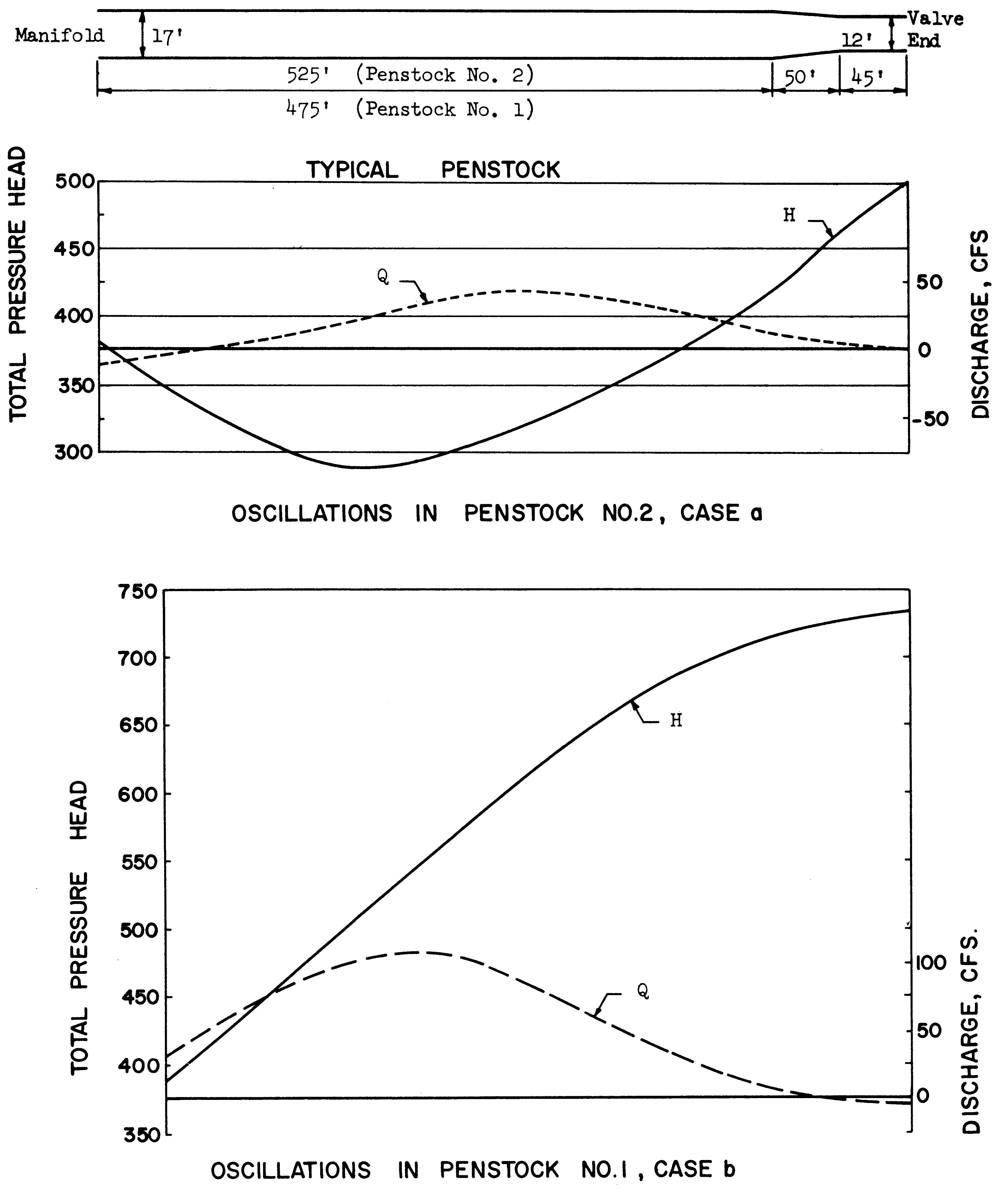


Figure 25. Pressure and Discharge Oscillations in Penstocks, Cases a and b.

predicted two probable periods of oscillation, either .55 or .51 seconds. The computer solution yielded a steady oscillatory condition at .505 seconds.

Amplitudes are not available from the test site. The computer yielded amplitudes of 360 and 370 feet for the penstock and valve body heads, respectively, giving a pressure ratio of 1.03. The pressure oscillations were approximately in phase on the computer data.

Computer results showed the discharge varying between 0 and 22 cfs with a mean value of about 8 cfs.

The pressure oscillations along the length of Penstock No. 1 appear as the fundamental as shown in Figure 25. Again, there is no appreciable pressure movement throughout most of the system except in Penstock No. 2 where the oscillations are almost as severe as at the excited valve. In this case Penstock No. 2 is completely closed. The oscillations at the two valves are exactly 180° out of phase indicating the existence of the second harmonic in this portion of the system.

Case c in Penstock No. 3

No actual records were obtained on this accidental condition, just the comment that the period of oscillation was about the same as for Case b. It seems likely that with the seal pressure at, or near, zero in a penstock similar to No. 1, the oscillations would be quite similar to those of Case b.

Case d

Inasmuch as the plant was not operating on these occasions when the seal pressure was lowered to zero, a likely explanation for the system not vibrating is that no initial pressure pulse was received at the valve to start the vibration. The steady state condition of leakage through the seal would remain stable if some disturbance did not initiate pressure waves in the system. When the plant is operating, various forms of disturbances might be possible such as those arising out of the governor control of the turbine, or from non-perfect flow conditions at bends or branches with the higher velocity flow.

The theory of this thesis agrees quite well with the experimental results when consideration is given to the variables in the system. Probably the one variable that can most affect the results of the calculations is the wave speed, a . Although the value used is reasonable it could vary as much as $\pm$ 7 percent and could also vary throughout the system. A wave speed of 4550 feet per second in Penstock No. 2 changes the resonating period to .182 seconds, the value measured in the field. The values used for the geometric and elastic properties of the valve are probably not perfect, nor is the frictional estimate in the neck. The particular properties of the seal equation greatly affect the final steady oscillatory amplitudes. These uncertain variables are the only explanation for the lack of perfect agreement in the frequency results of Case b in Penstock No. 1.

The development and perpetuation of this oscillatory condition is of interest from an energy viewpoint. During the development and eventual steady oscillatory condition, there is an energy output from the system at the seal. Due to the action of the seal and elastic behavior of the valve, the mean pressure at which the fluid leaves is below the static pressure. At the same time energy is being added at the reservoir in the amount of the product of the net positive inflow for a period and the static pressure. To simplify the discussion we can assume all valves closed so the only through flow is occurring at the leakage. When steady oscillatory motion is reached the input flow must equal the discharge leaving in each cycle. With the fixed static head at the reservoir and a fixed net positive flow input to the system, a definite amount of energy is being added to the system during each cycle. Since the same discharge must leave the system during each cycle, but at a lower discharge pressure because of the leakage characteristics, some of the added energy remains in the system. This must be dissipated within the system in each cycle or higher pressure and velocity oscillations will develop. Thus, for a given seal discharge relationship, pressure and velocity oscillations will develop until an energy balance is reached.

Considering again the particular pecularities of the seal, it can be seen that the most efficient exciter would be one that ejects fluid from the system at the lowest pressure head. If pure sine waves were involved, a phase angle of 180° would be the most efficient in the sense of developing the highest amplitudes. Case a matches these conditions very nicely. Case b has no through flow at the minimum pressure head but during each period the two fluid jets occur at a mean outflow pressure of about 235 feet and therefore provide ample excitation. On the other extreme, if a case existed where fluid could only be discharged at high pressure it can be seen that no energy would be available in the system to produce a resonating condition.

The type of oscillation discussed in this chapter has a distinctive characteristic when compared to those discussed earlier in the thesis. In this case the system is forcing the oscillation and controlling the operation of the boundary condition. This can be distinguished from a forcing function which forces the system at the period of the forcing function. The term auto-oscillation is sometimes used to describe the motion as discussed in this chapter.

VII. SUMMARY AND CONCLUSIONS

The primary objective of this thesis was to present a clearly defined method of evaluating the resonating characteristics of complex fluid systems. The impedance approach has been used.

The impedance method provides a powerful tool for the analysis of steady state fluid oscillations in systems where distributed parameters must be considered. Following the initial determination of the algebraic equations, it is no longer necessary to deal with the differential equations. The use of the algebraic equations and boundary conditions make it unnecessary to give consideration to reflection coefficients in complex systems. The algebraic equation of one pipe can be written in terms of the parameters of that pipe and in terms of its terminating impedance. These terminal impedances become the input impedance of the next pipe if a series connection exists. If one pipe joins two others, the output of the first becomes the input of the other two so that these two are in parallel. By starting at the very end of the system and systematically working back toward the input, the overall system equations can be obtained. The impedance diagram at the sending end then gives a useful aid in evaluating the frequency response of the system.

Two types of excitation have been used to present an evaluation of the impedance method.

One type consists of an external forcing function forcing the system to respond to a particular excitation frequency. The types of external forcing functions fall into two categories. The first is one in which the forcing function is either a pressure head or discharge which is a clearly defined periodic time function. In this case, the total system response can be determined by using a harmonic analysis of the forcing function, applying the impedance concepts to each harmonic, and superposing the results. The second category is one in which the forcing function requires a specific relationship between discharge and head. This is the more difficult of the two to analyze as the required relationship is most often nonlinear. With the nonlinear boundary condition the frequency response will depend upon the type of nonlinearity introduced. The principle of superposition is no longer valid. However, one of the most common nonlinear boundary conditions involves various forms of the gate or orifice equation. This equation can often be linearized for pressure amplitudes of modest size, and impedance methods can be successfully used. At the high impedance locations on the frequency scale, high pressure oscillations also exist making the quantitative prediction of amplitudes impossible. However, the impedance method has at least designated the zones on the frequency scale which are to be avoided with this type of excitation.

The second type of excitation is one that responds to the system, as did the leaky seal in Chapter VI. In this case the impedance diagram clearly identifies the possible severe resonating frequencies. The development of a resonating condition of this type is dependent upon the existence of an exciter which will permit a net positive influx of energy to the system on each cycle.

The specific conclusions which can be presented as a result of this investigation, and which therefore further the understanding in the area of resonance in pressurized piping systems, are summarized.

a. The impedance method is shown to be a workable, practical, and valid method of analysis in this particular field. "Hardware items" peculiar to fluid systems have been analyzed so they can be handled in impedance calculations. Complex systems which previously could not be analyzed have been included in this study.

b. The resonating frequencies of a complex system can be predicted.

c. An integral relationship does not necessarily exist between the theoretical period of a system and the periods of the higher harmonics.

d. With standing waves, or a combination of traveling and standing waves on a system, pressure loop and nodal points are not necessarily required to be located at geometric changes.

e. The use of the "equivalent pipe" is no longer necessary in a resonance study of a complex system.

f. Energy considerations have been discussed and the concept of an energy balance is shown to be a useful aid in the analysis of a resonance condition. However, a complete study of this aspect of resonance has not been attempted.

VIII. RECOMMENDATIONS FOR FURTHER STUDY

The most natural investigation following this thesis is the presentation of the theory for the design of corrective devices for the systems. Dead end lines, open end lines, relief valves, accumulators, chokes, or various combinations of these elements can probably be designed to alter the frequency response curve of a system resulting in a smooth operation over the frequency range of the attached exciter.

The inclusion of a distributed friction loss would further enhance the value of the method as it could then be applied to a broader range of problems. The linearization of the friction loss applied to the oscillatory flow may be quite acceptable. This would be particularly true when corrective devices are being used since the end result is ideally a small oscillation.

The effects of friction on oscillating flows in long pipe lines has not been extensively covered in the literature. A few incidents of practical significance have been reported but a strong theoretical examination has not been accomplished. A study in this area relating the energy input to the factors controlling attenuation of the pressure waves would certainly be of value.

An approach to the topic of resonance in fluid systems may be possible based entirely on energy concepts. If so, an investigation into this aspect could lead to an extremely valuable contribution to the level of knowledge in this field.

APPENDIX

COMPUTER PROGRAM DETAILS FOR SOLUTION OF THE COMPLEX IMPEDANCE IN MULTIPLE BRANCHING SYSTEMS

The solution of the complex terminal impedance in systems containing a number of branches necessitates many calculations For a clearly defined system, the digital computer can perform these calculations for a number of different frequencies yielding the desired information about the system.

To perform the operations of complex algebra on the computer it is necessary to divide the complex numbers into real and imaginary parts. Using the subscripts A and B to indicate the real and imaginary parts, and numerals to indicate pipe numbers, the following expressions describe the sending and receiving end impedances of pipe 1, as an example.

$$Z_{S1} = Z_{SA1} + iZ_{SB1}$$

$$Z_{R1} = Z_{RA1} + iZ_{RB1}$$

Impedance calculations in a branching system involve the use of the following equation in each pipe.

$$Z_S = \frac{Z_R - iZ_o \tan \omega L/a}{1 - i \frac{Z_R}{Z_o} \tan \frac{\omega L}{a}}$$

or

$$Z_{S2} = \frac{Z_{RA2} + i(Z_{RB2} - Z_{o2} \tan \frac{\omega L_2}{a_2})}{(1 + \frac{Z_{RB2}}{Z_{o2}} \tan \frac{\omega L_2}{a_2}) - i \frac{Z_{RA2}}{Z_{o2}} \tan \frac{\omega L_2}{a_2}}$$

In the computer program an internal function, IF1 , can be defined to perform this calculation. Two new complex numbers, representing the numerator and denominator of Z_{S2} can be established in the internal function.

$$Z_{S2} = \frac{A' + iB'}{C' + iD'}$$

Then

$$Z_{S2} = \left(\frac{A'C' + B'D'}{(C')^2+(D')^2}\right) + i\left(\frac{B'C' - A'D'}{(C')^2+(D')^2}\right)$$

so

$$Z_{SA2} = \frac{A'C' + B'D'}{(C')^2+(D')^2}$$

and

$$Z_{SB2} = \frac{B'C' - A'D'}{(C')^2+(D')^2}$$

At each branch of the system the following equation must be solved.

$$Z_{R1} = \frac{Z_{S2}Z_{S3}}{Z_{S2} + Z_{S3}}$$

A second internal function, IF2 , can be defined to evaluate this equation at each branch.

In any given system the receiving end impedances at the extreme end of each branch must be known. With this information the sending end impedance can be calculated using internal function IF1. This becomes the receiving end impedance for a series connected pipe. Re-using IF1 yields the sending end impedance of that pipe. This process is continued until a branch connection is reached. With the sending end impedances known in two of the branches, the receiving end impedance can be calculated for the third branch using IF2. Then IF1 is re-used to determine the sending end impedance in that branch. The process is continued through the complete system until the sending end of the system is reached.

In the computer program, an iteration statement can be written which will yield real and imaginary parts of the sending end impedance of the system for various frequencies. The moduli and phase angles can be determined permitting an evaluation of the frequency response of the entire system.

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