LOT SIZING IN ASSEMBLY SYSTEMS WITH RANDOM COMPONENT YIELDS

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Abstract

We investigate the problem of choosing optimal lot sizes in assembly systems when component manufacturing or procurement yields, and possibly assembly yields, are random. For a single-period setting, we analyze two different models. The first has multiple components with identical yield distributions and costs, random demand and random yield in assembly. The second has two components with non-identical yield distributions and costs, and possibly initial stock of one component. We provide complete analyses of both models, as well as comparative statics for the first.

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1 Part of this work was done while the first author was visiting the Department of Industrial and Operations Engineering, University of Michigan.

1. Introduction

This work analytically investigates some of the implications of yield randomness on component lot sizing decisions in assembly systems. These problems arise frequently in electronic and mechanical applications where, because of the nature of the manufacturing processes of the components, the yield (i.e., fraction of units that are usable) of many of the components may be random. Although a considerable amount of research has been done on lot-sizing when yields are random (e.g. Sepheri et al. 1986, Mazzola et al. 1987, Gerchak et al. 1988, Lee and Yano 1988, Henig and Gerchak 1989, Yano and Lee 1989 and references therein), relatively little has been done on assembly systems.

Yao (1988) analyzes a single period model in which the objective is to minimize the cost of producing components subject to a constraint on the probability of meeting a known demand for the finished product. Singh et al. (1988) consider a related problem in which there is a constraint on the total number of components processed (because of limitations on processing capacity) and the objective is to maximize the probability of meeting demand for a set of products. Yano and Chan (1989) provide a heuristic for the problem of minimizing the sum of expected holding costs for excess components and expected holding and shortage costs for the finished product, assuming that demand and assembly yields are deterministic.

We analyze two single-period expected profit maximization models which trade off the cost of production or procurement against the potential revenue from selling units of the assembled product. Like Yao (1988) and Yano and Chan (1989), we assume that yields are stochastically proportional to the lot size; that is, the distribution of the fraction good is assumed insensitive to the lot size. We also assume a 100% inspection of components by a perfect inspection process. While the model can take into account salvage values (or holding costs) of excess assembled units, it does not take into account the salvage values of unassembled components.

The first model we consider deals with products assembled of n components with identical yield distributions and costs. There are three reasons for our interest in such symmetric model. First, many electronic and mechanical devices as well as furniture, contain several non-substitutable components of identical level of complexity and cost (e.g. 'left' and 'right' parts). Second, as we are interested in the effect of the *number* of components on the optimal component lot size and associated costs, we are naturally led to consider symmetric systems. Third, we are able to obtain analytical, interpretable results for a fairly complex symmetric model.

The second model we analyze has two components with nonidentical yield distributions and costs, and possibly initial component stocks.

2. Multiple Components with Identical Yield Distributions and Costs

Suppose that for each component i, the random yield associated with a lot of size Q is $Y_Q = QP_i$, where P_i, $i = 1, \ldots, n$ are i.i.d. with common distribution H; let $\overline{H} = 1 - H$. Each component costs c to produce and inspect, and there are no initial component inventories. Since optimal component lot sizes will clearly be equal, the number of good sets available for assembly will be $Q\min_i P_i$, where the density of the random variable $P \equiv \min_i P_i$ is

$$f(p) = nh(p)\{\overline{H}(p)\}^{n-1}. \tag{1}$$

Demand X for the product is random with distribution G. The cost of assembling a unit of the product is a, each unit sells for r, and has a salvage value of v. The assembly stage is imperfect, and the yield associated with a lot of size S is $Y_S = SZ$, where Z is independent of the P_i's, and has distribution E, with mean α. We assume that $r\alpha > a + nc/E(\min_i P_i)$, so that some positive production level will be worthwhile.

This is a two-stage decision problem. First, one selects the lot size for the components. Then, given the number of resulting good sets, one decides how many units to assemble. Let us first consider the latter decision.

Had there been an abundant supply of component sets, the problem of how many sets S to assemble would be equivalent to a single-period single-item production problem with random yield (Gerchak et al. 1988). The solution $\bar{S}$ to this problem, using our notation, is given by

$$\int_0^{\infty} ze(z)G(\bar{S}z)dz = (r-a)\alpha/(r-v). \tag{2}$$

Since the objective function of that problem is concave (Gerchak et al. 1988), if Qp sets are available the optimal solution to the assembly problem is

$$S^* = \min(\bar{S}, Qp). \tag{3}$$

Given Q and p, the corresponding expected profit excluding component manufacturing costs, π_0, can be shown to equal

$$E^*(\pi_0 \mid Q,p)=\begin{cases} (r-v)\int_{z=0}^{\infty} e(z)\int_{x=0}^{\tilde{S}z} xg(x)dxdz \ \ if\ \tilde{S}\le Qp \\ (r-a)\alpha Qp-(r-v)Qp\int_0^{\infty} ze(z)G(Qpz)dz+(r-v)\int_{z=0}^{\infty} e(z)\int_{x=0}^{Qpz} xg(x)dxdz \ \ if\ \tilde{S}>Qp \end{cases}.$$

Taking the expectation of (4) with respect to P, we have

$$E^*(\pi_0 \mid Q)=(r-v)\{\overline{H}(\tilde{S}/Q)\}^n\int_{z=0}^{\infty} e(z)\int_{x=0}^{\tilde{S}z} xg(x)dxdz+n(r-a)\alpha Q\int_{p=0}^{\tilde{S}/Q} ph(p)\{\overline{H}(p)\}^{n-1}dp$$

$$-n(r-v)Q\int_{p=0}^{\tilde{S}/Q} ph(p)\{\overline{H}(p)\}^{n-1}\int_{z=0}^{\infty} ze(z)G(Qpz)dzdp$$

$$+n(r-v)\int_{p=0}^{\tilde{S}/Q} h(p)\{\overline{H}(p)\}^{n-1}\int_{z=0}^{\infty} e(z)\int_{x=0}^{Qpz} xg(x)dxdzdp. \quad (5)$$

In the component lot sizing decision, we therefore seek the value of Q which maximizes the overall expected profit

$$E(\pi)=-ncQ+E^*(\pi_0 \mid Q). \quad (6)$$

It can be shown that

$$\partial E(\pi)/\partial Q=-nc+n(r-a)\alpha\int_0^{\tilde{S}/Q} ph(p)\{\overline{H}(p)\}^{n-1}dp$$

$$-n(r-v)\int_{p=0}^{\tilde{S}/Q} ph(p)\{\overline{H}(p)\}^{n-1}\int_{z=0}^{\infty} ze(z)G(Qpz)dzdp, \quad (7)$$

and that $d^2E(\pi)/dQ^2<0$. The optimal Q^* is thus obtained by equating (7) to zero, and the associated expected costs by substituting that Q^* into (5)-(6). The result is summarized in the following.

Proposition 1

$E(\pi)$ is concave, and the optimal Q^* is the solution of

$$\int_0^{\tilde{S}/Q^*} ph(p)\{\overline{H}(p)\}^{n-1}\left\{(r-a)\alpha-(r-v)\int_{z=0}^{\infty} ze(z)G(Q^*pz)dz\right\}dp = c, \tag{8}$$

where $\tilde{S}$ is given by (2). The corresponding value of $E(\pi)$ is

$$E^*(\pi) = (r-v)\{\{\overline{H}(\tilde{S}/Q^*)\}^n \int_{z=0}^{\infty} e(z)\int_{x=0}^{\tilde{S}z} xg(x)dxdz$$
$$+n\int_{p=0}^{\tilde{S}/Q^*} h(p)\{\overline{H}(p)\}^{n-1}\int_{z=0}^{\infty} e(z)\int_{x=0}^{Q^*pz} xg(x)dxdzdp. \tag{9}$$ ||

How does Q^* react to a change in n? Assuming that the unit revenue, salvage value and assembly costs do not change, then it follows from (8) that since $\{\overline{H}(p)\}^{n-1}$ is decreasing in *n*, $Q^*(n)$ must be such that $G\{Q^*(n)pz\}$ be decreasing and $\tilde{S}/Q^*(n)$ increasing in *n*. Since G is an increasing function, and $\tilde{S}$ does not depend on *n*, it follows that

Corollary

$Q^*(n)$ is decreasing in *n*.

||

Since our assumptions imply that the cost of producing a set of components is increasing in their number, the profit margin becomes smaller, which makes the above result intuitive.

Some special cases of this model might be of interest. If assembly is perfect and free, and the product has no salvage value, we have a single stage decision problem, and (8) and (9) reduce to

$$\int_0^{\infty} ph(p)\{\overline{H}(p)\}^{n-1}\overline{G}(Q^*p)dp = c/r \qquad (10)$$

and

$$E^*(\pi) = rn\int_{p=0}^{\infty} h(p)\{\overline{H}(p)\}^{n-1}\int_{x=0}^{Q^*p} xg(x)dxdp \qquad (11).$$

respectively. If $n = 1$ these reduce to the results in Gerchak et al. (1988).

3. Two Components with Non-Identical Yield Distributions and Costs

Suppose now that $n = 2$, and that $Y_{Q_i}^{(i)} = Q_i P_i$, $i = 1, 2$, where $P_1 \sim H_1$ and $P_2 \sim H_2$ are independent. The unit component costs are c_1 and c_2 respectively. For simplicity, suppose that demand D is given, and assembly is perfect and free. If we order Q_i units of component type i, the density of $P(Q_1, Q_2) \equiv \min(Q_1P_1, Q_2P_2)$ is

$$l(p) = h_1(p/Q_1)\overline{H}_2(p/Q_2)/Q_1 + h_2(p/Q_2)\overline{H}_1(p/Q_1)/Q_2. \qquad (12)$$

Thus

$$E(\pi) = -c_1Q_1 - c_2Q_2 + r\{D\overline{H}_1(D/Q_1)\overline{H}_2(D/Q_2) + \int_0^D p\{h_1(p/Q_1)\overline{H}_2(p/Q_2)/Q_1 + h_2(p/Q_2)\overline{H}_1(p/Q_1)/Q_2\}dp\}. \qquad (13)$$

Some calculus establishes that

$$\partial E(\pi)/\partial Q_1 = -c_1 + r\int_0^{D/Q_1} p h_1(p)\overline{H}_2(Q_1 p/Q_2)dp,$$

$$\partial E(\pi)/\partial Q_2 = -c_2 + r\int_0^{D/Q_2} p h_2(p)\overline{H}_1(Q_2 p/Q_1)dp,$$

$$\partial^2 E(\pi)/\partial Q_1^2 = -r\left\{\int_0^{D/Q_1} p^2 h_1(p)h_2(Q_1 p/Q_2)dp/Q_2 + D^2 h_1(D/Q_1)\overline{H}_2(D/Q_2)/Q_1^3\right\} < 0,$$

$$\partial^2 E(\pi)/\partial Q_2^2 = -r\left\{\int_0^{D/Q_2} p^2 h_2(p)h_1(Q_2 p/Q_1)dp/Q_1 + D^2 h_2(D/Q_2)\overline{H}_1(D/Q_1)/Q_2^3\right\} < 0, \qquad (14)$$

and that

$$\partial^2 E(\pi)/\partial Q_1\partial Q_2 = rQ_1\int_0^{D/Q_1} p^2 h_1(p)h_2(Q_1 p/Q_2)dp/Q_2^2 = rQ_2\int_0^{D/Q_2} p^2 h_2(p)h_1(Q_2 p/Q_1)dp/Q_1^2, \qquad (15)$$

$$\{\partial^2 E(\pi)/\partial Q_1^2\}\{\partial^2 E(\pi)/\partial Q_2^2\} - \{\partial^2 E(\pi)/\partial Q_1\partial Q_2\}^2 = r^2 D^2\{D^2 h_1(D/Q_1)h_2(D/Q_2)\overline{H}_1(D/Q_1)\overline{H}_2(D/Q_2)/Q_1^3 Q_2^3$$
$$+ h_1(D/Q_1)\overline{H}_2(D/Q_2)\int_0^{D/Q_2} p^2 h_2(p)h_1(Q_2 p/Q_1)dp/Q_1^4$$
$$+ h_2(D/Q_2)\overline{H}_1(D/Q_1)\int_0^{D/Q_1} p^2 h_1(p)h_2(Q_1 p/Q_2)dp/Q_2^4, \qquad (16)$$

which is clearly positive. So we have

Proposition 2

$E(\pi)$ is concave in (Q_1, Q_2) and the optimal lot sizes solve

$$\int_0^{D/Q_1^*} p h_1(p)\overline{H}_2(Q_1^* p/Q_2^*)dp = c_1/r, \qquad (17)$$

$$\int_0^{D/Q_2^*} p h_2(p)\overline{H}_1(Q_2^* p/Q_1^*)dp = c_2/r. \qquad (18) \quad ||$$

Suppose now that there is an initial stock of good units of one of the components (if there are stocks of both, their minimum can be simply deducted from the demand). Without loss of generality assume that initially $I_1 = 0$ and $0 \le I_2 < D$. Then

$$E(\pi) = -c_1Q_1 - c_2Q_2 + r\{D\overline{H}_1(D/Q_1)\overline{H}_2\{(D-I_2)/Q_2\} + \int_0^D p h_1(p/Q_1)\overline{H}_2\{(p-I_2)/Q_2\}\,dp/Q_1$$

$$+\int_{I_2}^D p h_2\{(p-I_2)/Q_2\}\overline{H}_1(p/Q_1)dp/Q_2\}, \qquad (19)$$

and the optimality conditions are

$$\int_0^{D/Q_1} p h_1(p)\overline{H}_2\{(Q_1 p - I_2)/Q_2\}\,dp - \int_0^{I_2/Q_1} p(1+p)h_1(p)h_2\{(Q_1p - I_2)/Q_2\}\,dp = c_1/r \qquad (20)$$

$$\int_0^{(D-I_2)/Q_2} p h_2(p)\overline{H}_1\{(Q_1 p + I_2)/Q_1\}\,dp = c_2/r. \qquad (21)$$

4. Concluding Remarks

Our goal here was to pioneer the modeling of assembly systems with random component manufacturing yields and random assembly yields in an unconstrained profit maximization setting. We have modeled a symmetrical yet otherwise quite complex system, derived the optimality conditions, and showed that the component order quantity decreases in the number of components. We also analyzed a non-symmetrical two-component system with initial component stock.

It is interesting to note that what makes the lot sizing decisions for components here non-separable (and thus difficult), is the fact that product shortages are permitted. Had the (given) demand been rigid, and repeated lots of each component would have to be produced until there are enough units to satisfy demand, possibly incurring component type-specific setup costs with each lot, the problem would be entirely separable. Of course, each component lot sizing decisions then constitutes a dynamic program (e.g. Klein 1966).

The exact approach used here will probably not be practical for nonsymmetrical systems consisting of more than two or three components. Also, extensions to multi-period situations will require explicit modeling of unmated components and their associated holding costs, a difficult task in light of possible non-concavity of the objective function (Yano and Chan 1989). We nevertheless believe that some basic design tradeoffs in assembly systems with random yields can be made clearer even by idealized models like the ones discussed here.

References

1. Gerchak Y., R.G. Vickson and M. Parlar (1988) "Periodic Review Production Models with Variable Yield and Uncertain Demand" IIE Transactions, 20, 144-150.

2. Henig M. and Y. Gerchak (1989) "The Structure of Periodic Review Policies in the Presence of Random Yield" to appear in Operations Research.

3. Klein M. (1966) "Markovian Decision Models for Reject Allowence Problems" Management Science, 12, 349-358.

4. Lee H.L. and C.A. Yano (1988) "Production Control in Multi-Stage Systems with Variable Yield Losses" Operations Research, 36, 269-278.

5. Sepheri M., E.A. Silver and C. New (1986) "A Heuristic for Multiple Lot Sizing for an Order Under Yield Variability" IIE Transactions, 18, 63-69.

6. Singh M.R., C.T. Abraham and R. Akella (1988) "Planning for Production of a Set of Components when Yield is Random", Graduate School of Industrial Administration, Carnegie Mellon University.

7. Yano C.A. and T.J. Chan (1989) "Production and Procurement Policies for an Assembly System with Variable Component Yields", Dept. of Industrial and Operations Engineering, University of Michigan.

8. Yano C.A. and H.L. Lee (1989) "Lot Sizing with Random Yields: A Review" T.R. 89-16, Dept. of Industrial and Operations Engineering, University of Michigan.

8. Yao D.D. (1988) "Optimal Run Quantities for an Assembly System with Random Yields" IIE Transactions, 20, 399-403.

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Abstract

We investigate the problem of choosing optimal lot sizes in assembly systems when component manufacturing or procurement yields, and possibly assembly yields, are random. For a single-period scenario in which component salvage values of unmated components are zero, we analyze several different situations: n components with identical yield distributions and costs, under several different assumptions about demands, assembly yields, and assembly costs; and two components with non-identical yield distributions and costs.

Introduction

This work analytically investigates some of the implications of yield randomness on component lot-sizing decisions in assembly systems. These problems arise frequently in electronic assembly applications where, because of the nature of the manufacturing processes for the components, the yield (i.e., fraction of units that are usable) of many of the components may be random. Although a considerable amount of research has been done on lot-sizing when yields are random (e.g., see references in Sepehri et al. 1986, Mazzola et al. 1987, Gerchak et al. 1988, Lee and Yano 1988), relatively little has been done on assembly systems.

Yao (1988) analyzes a model in which the objective is to minimize the cost of producing components subject to a constraint on the probability of meeting a known demand for the finished product. Singh et al. (1988) consider a related problem in which there is a constraint on the total number of components processed (because of limitations on processing capacity) and the objective is to maximize the probability of meeting demand for a set of products. Yano and Chan (1989) provide a heuristic for the problem of minimizing the sum of expected holding costs for excess components and expected holding and shortage costs for the finished product. They assume that demand and assembly yields are deterministic. All three papers treat only the single-period case.

We analyze several single-period expected profit maximization models which trade off the cost of production or procurement against the potential revenue from selling units of the assembled product. More specifically, we analyze several different situations: n components with identical yield distributions and costs, under various assumptions about demand, assembly yields, and assembly costs; and two components with non-identical yield distributions and costs.

Our primary (and critical) simplification is to ignore the salvage values (or holding costs) of unassembled components. This simplification makes the resulting objective functions concave, thereby permitting us to obtain simpler, interpretable expressions as well as optimal solutions. However, the models can take into account salvage values of excess assembled units without sacrificing concavity of the objective function. Like Yao (1988) and Yano and Chan (1989), we assume that yields are stochastically proportional to the lot size (i.e., the distribution of the fraction good is insensitive to the lot size). We also assume that there is 100% inspection by a perfect inspection process. Any cost associated with the inspection can be included in the unit production or assembly cost.

In the next section we formulate and analyze models with n non-substitutable components with identical yield distributions and costs. We start with a simple scenario and gradually generalize it. The simplest scenario assumes known demand, and a costless and perfect assembly stage. The second model permits demand to be random. The third model adds positive assembly costs. Finally, we permit the assembly stage to have a random yield.

In the following section we analyze a simple model for two components with non-identical yield distributions and costs. We show that the bivariate objective is concave, and derive two equations for finding the optimal lot sizes. We also consider the effect of initial stocks on the solution to this problem.

We conclude by proposing further research directions in this area.

Multiple Components with Identical Yield Distributions and Costs

For each component i, the random yield associated with a lot of size Q is $Y_Q = Q \cdot P_i$, where $P_i, i = 1, \ldots, n$ are i.i.d. with common distribution H. Let $\bar{H} = 1 - H$. Each component costs c to produce, and each unit of the finished product sells for r, where in order for the process to be profitable, one needs to assume that $r > nc/E(\min_i P_i)$.

To aid in exposition, we shall start by modeling the simplest scenario and then gradually add random demand, positive assembly costs and an assembly lot size decision, and an imperfect assembly process to the model.

Simplest Scenario

Suppose that demand D is known, and that both the cost of assembly and all salvage values are zero. Since component lot sizes clearly will be equal, the profit will be

$$\pi = -ncQ + \begin{cases} rD & \text{if } Q \min_i P_i \geq D \\ rQ \min_i P_i & \text{if } Q \min_i P_i < D. \end{cases} \tag{1}$$

We note that

$$Pr[\min_i P_i > p] = [\bar{H}(p)]^n, \tag{2}$$

and the density of the minimum is

$$f(p) = nh(p)[\bar{H}(p)]^{n-1}. \tag{3}$$

Thus it is easy to show that

$$\begin{aligned} E(\pi) = & \quad -ncQ + rD[\bar{H}(D/Q)]^n \\ & + rnQ \int_0^{D/Q} \quad ph(p)[\bar{H}(p)]^{n-1}\, dp. \end{aligned} \tag{4}$$

Differentiating, we obtain

$$\partial E(\pi)/\partial Q = -nc + rn \int_0^{D/Q} \quad ph(p)[\bar{H}(p)]^{n-1} dp, \tag{5}$$

$$\partial^2 E(\pi)/\partial Q^2 = -nrD^2 h(D/Q)[\bar{H}(D/Q)]^{n-1}/Q^3 < 0. \tag{6}$$

Thus we have

Proposition 1.1

$E(\pi)$ is concave, the optimal lot size, Q^*, satisfies

$$\int_0^{D/Q^*} \quad ph(p)[\bar{H}(p)]^{n-1} dp = c/r, \tag{7}$$

and the corresponding optimal profit is

$$E^*(\pi) = rD[\bar{H}(D/Q^*)]^n. \quad || \tag{8}$$

This result agrees with results on the single component problem in Gerchak et al. (1986) in the special case where $n = 1$ and there is no initial inventory.

How does Q^* react to a change in n? Assuming that the unit revenue r does not change, then since $[\bar{H}(u)]^{n-1}$ is decreasing in n, it follows from (7) that $Q^*(n)$ is decreasing in n. This is not surprising since if r remains unchanged, an increase in n makes the product less profitable. If r is increasing in n, reflecting higher prices for more complex products, Q might not decrease in n.

Random Demand

Suppose now that demand X is random with distribution G and $\bar{G} \equiv 1 - G$. All other assumptions remain the same. Then it is easy to show that

$$\begin{aligned} E(\pi) = & -ncQ + rn\{\int_{p=0}^{\infty} h(p)[\bar{H}(p)]^{n-1}[\int_{x=0}^{Qp} xg(x)dx]dp \\ & + Q\int_{p=0}^{\infty} ph(p)[\bar{H}(p)]^{n-1}\bar{G}(Qp)dp\}. \end{aligned} \tag{9}$$

Differentiating, we obtain

$$\partial E(\pi)/\partial Q = n\{-c + r\int_{0}^{\infty} ph(p)[\bar{H}(p)]^{n-1}\bar{G}(Qp)dp\} \tag{10}$$

$$\partial^2 E(\pi)/\partial Q^2 = -nrQ\int_{0}^{\infty} ph(p)[\bar{H}(p)]^{n-1}g(Qp)dp < 0. \tag{11}$$

So we have

<u>Proposition 1.2</u>

$E(\pi)$ is concave, the optimal lot size, Q^*, satisfies

$$\int_{0}^{\infty} ph(p)[\bar{H}(p)]^{n-1}\bar{G}(Q^*p)dp = c/r, \tag{12}$$

and at the optimum

$$E^*(\pi) = rn\int_{u=0}^{\infty} h(p)[\bar{H}(p)]^{n-1}[\int_{x=0}^{Q^*p} xg(x)dx]dp. \;|| \tag{13}$$

This result also agrees with results in Gerchak et al. (1988) on the single-component problem in the special case where $n = 1$ and there is no initial inventory.

If unit revenue does not change with n, then since $[\bar{H}(p)]^{n-1}$ is decreasing in n, $Q^*(n)$ must be such that $\bar{G}[Q^*(n)p]$ is increasing in n. Since $\bar{G}$ is a decreasing function, it follows from (12) that $Q^*(n)$ is again decreasing in n.

Positive Assembly Costs and Product Salvage Values

Suppose now that the cost of assembling a unit of the finished product is a, and $r > a + nc/E(\min_i P_i)$ so it is profitable to make the product. Demand is random, and the salvage value of an excess assembled unit is v. In this case, there are two decisions: (1) the lot size for the components and (2) given the outcome of good units, how many units of the finished product to assemble. Let us first consider the latter decision.

There are $Q \min_i P_i \equiv Qy$ sets of good components, and we wish to decide how many, S, to assemble. As in an ordinary newsboy problem, the expected profit (ignoring component manufacturing costs) is

$$E(\pi_a) = (r-a)S + (v-r)SG(S) + (r-v)\int_0^S xg(x)dx. \tag{14}$$

The unconstrained solution to this problem is

$$G(\tilde{S}) = (r-a)/(r-v). \tag{15}$$

Since (14) is concave in S, the optimal solution to the assembly problem is

$$S^* = \min(\tilde{S}, Qy). \tag{16}$$

It can be shown that the corresponding expected profit is to equal

$$E^*(\pi_a|Q,y) = \begin{cases} (r-v)\int_0^{\tilde{S}} xg(x)dx & \text{if } \tilde{S} \le Qy \\ (r-a)Qy + (v-r)QyG(Qy) & \\ \quad +(r-v)\int_0^{Qy} xg(x)dx & \text{if } \tilde{S} > Qy. \end{cases} \tag{17}$$

In the component production decision, we therefore seek the value of Q which maximizes

$$E(\pi) = -ncQ + E^*(\pi_a|Q), \tag{18}$$

where $E^*(\pi_a|Q) = E_Y[E^*(\pi_a|Q,Y)]$. It can be shown that

$$\partial E(\pi)/\partial Q = -nc + \int_0^{\tilde{S}/Q} y[(r-a)+(v-r)G(Qy)]f(y)dy, \tag{19}$$

from which it also follows that $\partial^2 E(\pi)/\partial Q^2 < 0$. Using (3), we thus have,

Proposition 1.3

$E(\pi)$ is concave, the optimal lot size, Q^*, satisfies

$$(r-a)\int_0^{\tilde{S}/Q^*} ph(p)[\bar{H}(p)]^{n-1}dp - (r-v)\int_0^{\tilde{S}/Q^*} ph(p)[\bar{H}(p)]^{n-1}G(Q^*p)dp = c, \tag{20}$$

where $\tilde{S}$ is given by (15), and at the optimum,

$$E^*(\pi) = (r-v)\{\bar{H}(\tilde{S}/Q^*)\int_0^{\tilde{S}} xg(x)dx + n\int_{p=0}^{\tilde{S}/Q^*} h(p)[\bar{H}(p)]^{n-1}\int_{x=0}^{Q^*p} xg(x)dxdp\}.|| \tag{21}$$

We note that when $a = v = 0$, the result of Proposition 1.3 reduces to that of Proposition 1.2, since in that case $\tilde{S} = \infty$.

Again, it can be shown that $Q^*(n)$ is decreasing in n, although the argument is more complex.

Imperfect Assembly Process

Finally, we allow the assembly stage in the previous model to be imperfect. That is $Y_s = S \cdot Z$, where Z has distribution E and mean α. We assume that $r\alpha > a + nc/E(\min_i P_i)$ (i.e., it is profitable to produce).

Then the solution to the unconstrained assembly-stage decision is

$$\int_0^{\infty} ze(z)G(\tilde{S}z)dz = (r-a)\alpha/(r-v) \tag{22}$$

and

$$S^* = \min(\tilde{S}, Qy). \tag{23}$$

The corresponding expected profit is

$$E^*(\pi_a|Q,y) = \begin{cases} (r-v)\int_{z=0}^{\infty} e(z)\int_{x=0}^{\tilde{S}z} xg(x)dxdz & \text{if } \tilde{S} \le Qy \\ (r-a)\alpha Qy - (r-v)Qy\int_0^{\infty} ze(z)G(Qyz)dz & \\ \quad +(r-v)\int_{z=0}^{\infty} e(z)\int_{x=0}^{Qyz} xg(x)dxdz & \text{if } \tilde{S} > Qy. \end{cases} \quad (24)$$

By substitution of (24) into (18), it can be shown that

$$\begin{aligned} \partial E(\pi)/\partial Q &= -nc + (r-a)\alpha\int_0^{\tilde{S}/Q} yf(y)dy \\ &\quad -(r-v)\int_{y=0}^{\tilde{S}/Q} yf(y)\int_{z=0}^{\infty} ze(z)G(Qyz)dzdy, \end{aligned}$$

and that $\partial^2 E(\pi)/\partial Q^2 < 0$. Using (3), we have

Proposition 1.4

$E(\pi)$ is concave, and the optimal Q^* is the solution of

$$(r-a)\alpha\int_0^{\tilde{S}/Q^*} ph(p)[\bar{H}(p)]^{n-1}dp - (r-v)\int_{p=0}^{\tilde{S}/Q^*} ph(p)[\bar{H}(p)]^{n-1}\int_{z=0}^{\infty} ze(z)G(Q^*pz)dzdp = c, \quad (25)$$

where $\tilde{S}$ is given by (22). ||

Two Components with Non-Identical Yield Distributions and Costs

We shall now use the simplest scenario (deterministic demand and assembly yields, zero assembly cost) to analyze the case of two components with non-identical yield distributions and costs. Let $Y_{Q_i} = Q \cdot P_i$, $i = 1,2,$ where $P_1 \sim H_1$ and $P_2 \sim H_2$ are independent. Also suppose that demand D is known, and unit component costs are c_1 and c_2 respectively.

The density of min (Q_1P_1, Q_2P_2) is

$$l(p) = h_1(p/Q_1)\bar{H}_2(p/Q_2)/Q_1 + h_2(p/Q_2)\bar{H}_1(p/Q_1)/Q_2$$

Thus

$$\begin{aligned} E(\pi) &= -c_1Q_1 - c_2Q_2 + r\{D\bar{H}_1(D/Q_1)\bar{H}_2(D/Q_2) \\ &+ \int_0^D p[h_1(p/Q_1)\bar{H}_2(p/Q_2)/Q_1 + h_2(p/Q_2)\bar{H}_1(p/Q_1)/Q_2]dp\}. \end{aligned}$$

Some calculus establishes that

$$
\begin{aligned}
\partial E(\pi)/\partial Q_1 &= -c_1 + r\int_0^{D/Q_1} ph_1(p)\bar{H}_2(Q_1p/Q_2)dp, \\
\partial E(\pi)/\partial Q_2 &= -c_2 + r\int_0^{D/Q_2} ph_2(p)\bar{H}_1(Q_2p/Q_1)dp, \\
\partial^2 E(\pi)/\partial Q_1^2 &= -r\{\int_0^{D/Q_1} p^2h_1(p)h_2(Q_1p/Q_2)dp/Q_2 \\
&+ D^2h_1(D/Q_1)\bar{H}_2(D/Q_2)/Q_1^3\} < 0, \\
\partial^2 E(\pi)/\partial Q_2^2 &= -r\{\int_0^{D/Q_2} p^2h_2(p)h_1(Q_2p/Q_1)dp/Q_1 \\
&+ D^2h_2(D/Q_2)\bar{H}_1(D/Q_1)/Q_2^3\} < 0, \\
\partial^2 E(\pi)/\partial Q_1\partial Q_2 &= rQ_1\int_0^{D/Q_1} p^2h_1(p)h_2(Q_1p/Q_2)dp/Q_2^2 \\
&= rQ_2\int_0^{D/Q_2} p^2h_2(p)h_1(Q_2p/Q_1)dp/Q_1^2.
\end{aligned}
$$

Some additional algebra establishes that the Hessian is positive definite, so we have:

Proposition 2.1

$E(\pi)$ is concave in (Q_1, Q_2) and the optimal lot sizes solve

$$
\int_0^{D/Q_1^*} ph_1(p)\bar{H}_2(Q_1^*p/Q_2^*)dp = c_1/r,
$$

$$
\int_0^{D/Q_2^*} ph_2(p)\bar{H}_1(Q_2^*p/Q_1^*)dp = c_2/r. \qquad ||
$$

We shall now analyze the effect of initial component stocks, I_i, $i = 1, 2$, in this model. Suppose that $I_1 \leq I_2 < D$. Then

$$
\begin{aligned}
E(\pi) &= -c_1Q_1 - c_2Q_2 + r\{D\bar{H}_1[(D-I_1)/Q_1]\bar{H}_2[(D-I_2)/Q_2] \\
&+ \int_{I_1}^{D} ph_1[(p-I_1)/Q_1]\bar{H}_2[(p-I_2)/Q_2]dp/Q_1 \\
&+ \int_{I_2}^{D} ph_2[(p-I_2)/Q_2]\bar{H}_1[(p-I_1)/Q_1]dp/Q_2.
\end{aligned}
$$

After some tedious calculus, we obtain

$$
\partial E(\pi)/\partial Q_2 = -c_2 + r\int_0^{(D-I_2)/Q_2} ph_2(p)\bar{H}_1[(Q_2p + I_2 - I_1)/Q_1]dp,
$$

and since $I_1 \leq I_2$ the other first partial derivative is more complex:

$$
\partial E(\pi)/\partial Q_1 = -c_1 + r\{\int_0^{(D-I_1)/Q_1} ph_1(p)\bar{H}_2[(Q_1p - I_2 + I_1)/Q_2]dp
$$

$$- \int_0^{(I_2-I_1)/Q_1} p^2 h_1(p) h_2[(Q_1 p - I_2 + I_1)/Q_2] dp$$
$$- \int_0^{(I_2-I_1)/Q_1} p h_1(p) h_2[(Q_1 p - I_2 + I_1)/Q_2] dp \}.$$

Note that if $I_1 = I_2 \equiv I$, the situation reduces to one with demand $= (D - I)$ and no initial stocks.

If $0 = I_1 < I_2$, the simpler necessary condition becomes

$$\int_0^{(D-I_2)/Q_2^*} p h_2(p) \bar{H}_1[(Q_2^* p + I_2)/Q_1^*] dp = c_2/r.$$

But since Q_1^* itself depends on I_2, not much can be said about the relation between Q_2^* and I_2.

Concluding Remarks

One simplifying assumption that we made is zero salvage value (holding costs) of unmated components. It was shown by Yano and Chan (1989) that inclusion of such values renders the resulting models impractical for exact analytic treatment, even in single-period models. Extensions to multi-period situations will require explicit modeling of unmated components and their associated costs. Moreover, real-life problems involving assembly systems with random component yields are likely to involve more than two components, nonidentical yield distributions and costs. Undoubtedly, because of the complexity of these problems, practical approaches will require heuristic procedures. Further research along these lines is needed. Nevertheless, insights from simple models such as those analyzed here may provide the basis for good heuristics.

Finally, we have examined only pure make–to–plan (known demand) and make–to–stock (random demand) situations. It would be of interest to study assemble–to–order situations, where only a forecast of demand is known when component lot sizes are selected, but final assembly is based upon actual orders.

References

- Gerchak Y., M. Parlar and R.G. Vickson,(1986), "A Single Period Production Model with Uncertain Output and Demand", Proceedings of the 25th IEEE Conference on Decision and Control, 1733–1736.

- Gerchak Y., R.G. Vickson and M. Parlar, (1988), "Periodic Review Production Models with Variable Yield and Uncertain Demand", IIE Transactions, 20, 144-150.

- Lee H.L. and C.A. Yano, (1988), "Production Control in Multi- Stage Systems with Variable Yield Losses", Operations Research, 36, 269-278.

- Sepehri M., E.A. Silver and C. New, (1986), "A Heuristic for Multiple Lot Sizing for an Order Under Yield Variability", IIE Transactions, 18, 63- 69.

- Singh M.R., C.T. Abraham and R. Akella, (1988), "Planning for Production of a Set of Components when Yield is Random," Graduate School of Industrial Administration, Carnegie Mellon University

- Yano, C.A. and T.J. Chan, (1989), "Production and Procurement Policies for an Assembly System with Variable Component Yields", Dept. of Industrial and Operations Engineering, University of Michigan.

- Yao D.D., (1988), "Optimal Run Quantities for an Assembly System with Random Yields", IIE Transactions, 20, 399-403.