SINGLE-PERIOD MODELS FOR SINGLE-STAGE PRODUCTION/INVENTORY SYSTEMS WITH UNCERTAIN YIELDS

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ABSTRACT

We investigate the effects of yield uncertainty in single-stage, single-period inventory systems with both zero and positive setup costs, as well as deterministic and stochastic demand. For each of the systems, we determine the form of the optimal policy or present results which suggest what the form of the policy may be. We also provide numerical results which indicate that increasing uncertainty in systems with less-than-perfect yields may actually lead to optimal policies that are more conservative. That is, fewer items are purchased, and generally only in much more restrictive circumstances. This differs significantly from rules of thumb which suggest that increased uncertainty leads to optimal inventory levels which are larger as well.

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1. Introduction

Much of the research on production systems with yield losses has focused on inspection policies (e.g. Klein (1966), White (1967), and more recently Lee and Rosenblatt (1984) and Porteus (1984)) but little work has been done on production control for such systems. Two notable exceptions are papers by Giffler (1960) and Levitan (1960) where the objective is to find a reject allowance which optimizes the tradeoff between shortages and overages. Yet, variable yield losses are pervasive in microelectronic fabrication and many chemical processes, and optimal control of production is difficult but important in such situations.

We present four single-period models of single-stage production systems in which yields are uncertain. The four variants deal with (1) zero or positive setup costs and (2) deterministic or stochastic demand. Our models differ from that of Giffler and Levitan in that we assess a shortage cost per unit whereas they use a penalty per stockout occasion.

The primary motivation for this model development and analysis is to build a foundation for investigation of multi-period and multi-stage production scenarios in which yields (and possibly) demand are uncertain. Yet, despite this primary motivation, we recognize that there are existing production scenarios in which many similar characteristics exist. For instance, production of low-volume repair parts may be done infrequently, and lead times for input materials may be long, so that each production run is effectively independent. Changeovers on the various pieces of equipment required to produce a repair part may be expensive because of disruptions to the normal schedule and because long-forgotten calibrations must be done. Finally, demand for these parts may be highly uncertain and shortages may necessitate expediting or expensive alternate repair procedures. Many other direct applications exist in procurement of seasonal goods, for which only one order will be placed (e.g. Halloween costumes). The vendor may use a rationing policy so that orders are partially filled. The percentage of orders actually received depends upon total supply and demand at the vendor, making the actual "yield" rate uncertain at the time the retailer places the order.

In the next section we discuss the assumptions common to the four models. In each of the four subsequent sections we present a model along with related results and solution procedures. We then provide some simple examples to illustrate how the solutions change when yield uncertainty is introduced into well-established inventory models. We conclude with a discussion of implications of the models and results for analyses of more complex systems. Throughout the paper, we use the notation listed in Table 1.

TABLE 1

2. Assumptions

Similar assumptions are made in the four models which follow. We assume that the yield rate distributions, and where applicable, the demand distributions, are assumed to be known, continuous, and twice-differentiable. We assume that the actual yield rate may take on values between zero and one, inclusive, but the models can be generalized quite easily to other yield rate distributions with finite upper limits and to discrete situations.

Linear penalty costs are assessed for unused inventory at the end of the period and for unfilled demand. A production cost is charged on each unit of input, reflecting cost of materials and processing which is normally a function of the input, not of the output. Defective units are assumed to be unusable and to be disposed of at no additional cost.

Table 1

Notation

Parameter	Definition
L	Inventory holding cost per unit remaining at end of period
π	Shortage cost per unit of satisfied demand
A	Setup cost
w	Production cost per unit of input or purchase quantity
D	Demand for the product (may be random
p	Actual yield rate (as fraction of input quantity
$F_p(\cdot)$	Distribution of the yield rate*
$f_p(\cdot)$	Density of the yield rate*
$\bar{p}$	Average yield rate = $\int_0^1 pf(p)dp$
$F_D(\cdot)$	Distribution of demand*
$f_D(\cdot)$	Density of demand*
I	Initial inventory
u	Input quantity (decision variable)

* Note: For simplicity, we drop the subscripts except where necessary for clarity.

3. Model with Deterministic Demand and No Setup Costs

We begin with the simplest of the four models. One example of a relevant scenario is the newspaper carrier who has a subscription-based route in an apartment complex. The newspaper company delivers papers to an unsecure location near the apartment complex and some newspapers may be stolen before the carrier has a chance to distribute them. If a subscriber does not receive a paper because of a shortage, the subscriber calls the newspaper to request a paper and the carrier receives a demerit (and perhaps a financial penalty). (This is a real scenario which the author observed as a subscriber!).

The objective in this problem is simply to choose the order quantity which minimizes expected inventory holding costs, shortage costs, and purchase costs. The problem thus is

$$\text{minimize } h \int_{p=D/u}^{1} (pu-D)f(p)\,dp + \pi \int_{p=0}^{D/u} (D-pu)f(p)\,dp + wu \equiv G(u)$$

Taking the first derivative, we get:

$$\frac{\partial G(u)}{\partial u} = h \int_{p=D/u}^{1} pf(p)dp - \pi \int_{p=0}^{D/u} pf(p)\,dp + w$$

which, after simplification, gives:

$$\frac{\partial G(u)}{\partial u} = h\bar{p} + w - (\pi + h) \int_{p=0}^{D/u} pf(p)dp. \qquad (1)$$

To demonstrate that the objective function is convex, we note that:

$$\frac{\partial^2 G(u)}{\partial u^2} = (\pi+h)(D^2/u^3)f(D/u) > 0 \quad \text{for } D \leq u < \infty$$

Thus, we can optimize by setting (1) equal to zero. Doing so gives us:

$$\int_{p=0}^{D/u^*} pf(p)\, dp = (h\bar{p} + w)/(\pi + h)$$

Observe that u is finite for all positive values of h and strictly greater than D for all positive values of π provided $h > 0$. The form of the optimal solution has the flavor of the newsboy model but is considerably more difficult to solve because it involves numerical integration, whereas the newsboy model only requires finding a fractile of the demand distribution.

4. Model with Deterministic Demand and a Positive Setup Cost

An example of a situation with deterministic demand and positive setup costs might be the sale of Girl Scout cookies. The Girl Scouts take orders for cookies but the troop must arrange for delivery from the cookie factory and pay a delivery cost which is insensitive to the quantity delivered. Unfortunately, some of the cookies may get damaged (or eaten) in transit, but it is uneconomical to return them for replacement.

If there were no setup cost, the troop should order u* as indicated in the previous section. However, if the setup cost is large enough, it would be better to suffer the loss of profit. If the order is placed, the cost is

$$A + G(u^*)$$

If the order is not placed, the cost is

$$\pi(D-I) \qquad \text{for } I < D$$

where, in the case of the Girl Scouts, I=0. (If $I \geq D$, there is no need to order). Setting these costs to be equal permits us to determine the largest value of D-I at which we would place an order. So we have a policy in which we order u^* if

$$(D-I)^+ > [A + G(u^*)]/\pi$$

Observe that the ordering policy has a very different form than in an (s,S) inventory model. While there is still an "order-up-to-point," the trigger quantity is net requirements.

5. Model with Stochastic Demand and No Setup Costs

Introducing stochastic demand into the first model leads to a problem of the form.

$$\text{minimize} \quad h \int_{D=0}^{u+I} \int_{p=(D-I)/u}^{1} (pu+I-D)f(p)f(D)\,dp\,dD$$
$$+ \pi \int_{D=I}^{\infty} \int_{p=0}^{(D-I)/u} (D-pu-I)f(p)f(D)\,dp\,dD$$
$$+ wu \equiv H(u)$$

Again, we take the first derivative and obtain

$$\frac{\partial H(u)}{\partial u} = h \int_{D=0}^{u+I} \int_{p=(D-I)/u}^{1} pf(p)f(D)\,dp\,dD$$
$$- \pi \int_{D=I}^{\infty} \int_{p=0}^{(D-I)/u} pf(p)f(D)\,dp\,dD + w$$
$$= (\pi+h)\ \bar{p}\ F_D(u+I) - (\pi+h) \int_{D=I}^{u+I} \int_{p=0}^{(D-I)/u} pf(p)f(D)\,dp\,dD + w - \pi\bar{p} \qquad (2)$$

To ascertain whether the objective function is convex, we observe that the second derivative is

$$\frac{\partial^2 H(u)}{\partial u^2} = [(D-I)^2/u^3]\ [h \int_{D=I}^{u+I} f_p\left(\frac{D-I}{u}\right)f(D)\,dD] + \pi\bar{p}f_D(u+I) > 0.$$

The optimal solution therefore can be found by solving for the u that equates (2) to zero. Observe that since the first derivative is strictly increasing in u (since the absolute value of the second term ($\bar{p}\ F_D(u+1)$ is strictly greater than the double integral expression), for u to be positive, we must have

$$w - \pi\bar{p} < 0 \quad \text{or}$$
$$w < \pi\bar{p}$$

One can interpret this as a requirement that the cost of the input must be less than the expected contribution of that input to reducing shortage costs.

Since the second term in (2) vanishes as $u \to \infty$, to ensure that u^* is finite, we must have

$$F_D(u+I) > (\pi - w)/(\pi + h)\bar{p}$$

for some $u < \infty$. But the right hand side is less than 1 for any $w \geq 0$ and $h \geq 0$, or $|h|\bar{p} < w$ if $h < 0$, so under these conditions, there exists some $u < \infty$ which satisfies (2).

It also can be shown that

$$\frac{\partial H(u)}{\partial u \ I} = [(\pi+h)/u^2] \int_{D=I}^{u+I} (D-I)f(\frac{D-I}{u}) f(D)dD > 0$$

so as I increases, u^* decreases, as intuition would indicate.

6. Model with Stochastic Demand and Positive Setup Costs

Generalizing the model to incorporate setup costs gives us the problem of deciding whether to procure the product. If we place an order, the cost is

$$A + H(u^*|I) \tag{3}$$

on the other hand, if an order is not placed, we incur an expected cost of

$$\pi \int_{D=I}^{\infty} (D-I)f(D)dD + h \int_{D=0}^{I} (I-D)f(D)dD \tag{4}$$

where $E(\cdot)$ denotes expectation and $(x)^+$ denotes max $(0,x)$.

Since demand is stochastic, we cannot simply use net demand as we did in section 4. In addition, comparing $\partial H(u)/\partial I|_{u^*}$ and the derivative of (4) with respect to I, does not provide any additional insight about the form of the optimal policy. Thus, we cannot even conclude that the decision about whether or not to order is monotonic in I, although intuitively we would expect it to be so.

7. Numerical Examples

In the numerical examples, we use binomial yield and demand distributions to provide a relatively realistic view of the qualitative differences in the solutions and costs, resulting from uncertain yields. We use the following parameter values:

$h = 1$

$\pi = 4$

$A = 10$ (where appropriate)

$w = 2 < \pi\bar{p} = 3.2$

$\bar{D} = 10$ (= D when demand is deterministic)

$\bar{p}_D = .5$ (for demand; thus n=20 and $\sigma_D^2 = 5$)

$\bar{p}_y = .80$ (for yields)

$I = 0$ except where it is a decision variable

Example 1. Deterministic Demand, No Setup Costs

If yields were deterministic, we would simply order $D/\bar{p}$=12.5 units at w=2 per unit for a total cost of \$25. With uncertain yields, we would choose u=12, giving a total cost of \$27.32. Observe that it is optimal to order slightly less, but the cost is greater due to the yield variability. Therefore, optimal order quantities are not always monotonically increasing in the yield variance, contrary to popular belief. Of course, the results depend upon the actual yield distribution, but this example (which was selected arbitrarily) demonstrates that "intuitive" solutions may not be optimal when yields are uncertain.

Example 2. Deterministic Demand with Setup Cost

With deterministic yields, we would have the choice of ordering for units at a total cost of $10 + 2(10-I)/.8 = 32.5 - 2.5I$ or of incurring a shortage cost of $4(10-I) = 40 - 4I$. Thus, for $I \geq 5$, we would not order. Otherwise, we would order $12.5 - 1.25I$ units.

When yields are random, the optimal order quantity depends upon I. The values of u* and G(u*) are listed for $I = 0,\dots,9$ in Table 2, For each value of I, we must compare the cost of placing an order, or

$$A + G(u^*|I) = 10 + G(u^*|I)$$

with the cost incurred if an order is not placed, which is

$$\pi(D - I) = 4(10 - I).$$

It is evident that for $I > 2$, it is optimal not to order. Observe that the initial inventory which triggers an order is considerably less than the trigger value, when yields are deterministic.

TABLE 2

Example 3. Stochastic Demand with No Setup Cost

The single-period model with stochastic demand and no setup costs is commonly referred to as the newsboy model. With deterministic yields we would have an adjusted purchase or production cost $c = w/\bar{p} = 2.5$. We would therefore find an order-up-to-point, S*, which satisfies

$$F(S^*) = (\pi - c)/(\pi + h) = .5$$

This gives $S^* = 10$ and we must actually order $S^*/\bar{p} = 12.5$ to account for the yield loss. For each available unit of inventory, the quantity ordered would be reduced by 1.25. The cost of the system would be \$29.40 for I=0.

Using the model in section 5, we find that for $I = 0$, $u^* = 12$ and the total expected cost is \$30.84. Observe the change in the actual cost of the system because of the yield uncertainty.

TABLE 2

Results for Example 2

I	u*	G(u*)
0	12	27.32
1	11	24.84
2	10	22.42
3	9	20.05
4	8	17.76
5	6	14.11
6	5	11.64
7	4	9.25
8	3	6.96
9	2	4.80

Example 4. Stochastic Demand with Positive Setup Costs

We determined the optimal value of u and associated cost for several values of I as well as the expected system cost if no order is placed. These values are listed in Table 3. With A = 10, the optimal policy is to use u = 12 if I = 0 and not to order otherwise. In this particular case, the optimal policy is of the (s,S) type as we conjectured in the previous section.

TABLE 3

The results are notably different from those in Example 2. As we might expect, the costs are higher (by 10% or more). The values of u*, however, are smaller than in example 2. Thus (as in example 1), introduction of uncertain demand has led to more conservative policies, whereas typical practice is to increase safety stock in response to greater uncertainty.

8. CONCLUSIONS

We have presented models of and suggested solution approaches for four single-product, single-period inventory models with stochastic yields. For the cases with no setup costs, we found that a single-critical-number policy is optimal. When there are setup costs, the optimal policy need not have a two-critical number (e.g., (s,S)) type of policy. The order quantity is not necessarily linear in the initial inventory so a constant order-up-to-value does not necessarily exist.

There were two other interesting results. First, the "reorder" point in models with positive setup costs appear to decrease when the yield variance increases even with the same average yield rate. (This was evident in example 2). Second, introduction of demand uncertainty into a model with yield uncertainty (with the average demand held constant) may actually decrease order quantities, as we observed in several of the examples. Thus, increasing

TABLE 3

Results for Example 4

I	u*	H(u*)	Expected Cost with u = 0
0	12	29.85	40.00
1	10	26.94	36.00
2	9	24.45	32.00
4	7	19.54	24.00
6	4	14.14	16.14
8	2	9.21	9.09
10	0	4.40	4.40

uncertainty in a system need not increase optimal order quantities. This implies that ad hoc implementation of simple and "logical" policies which are based upon our current understanding of safety stock in the context of demand uncertainty may provide both incorrect results and incorrect intuition about the form of optimal policies when yield uncertainty is present.

Much more research needs to be done to develop an understanding of the effects of yield uncertainty in multi-period and multi-echelon systems.

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