The Position of the Student in Geometry Instruction:
A Study from Three Perspectives

by

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Abstract

This study aims to understand the student's position in instruction. I conceptualize instruction as interactions between the teacher, students, and mathematics, in educational environments (Cohen, Raudenbush & Ball, 2003; Lampert, 2001). In the three manuscripts contained in this dissertation, I look at the *position* (Harré & van Langenhove, 1999) of student from the perspective of the teacher, the student, and the mathematics.

"Mathematical Arguments in a Virtual High School Geometry Classroom" looks at the position of the student from the perspective of mathematics. It examines the mathematical arguments that could be made by learners in response to a virtual classroom discussion by comparing arguments made by a learner who had taken a geometry class to arguments made by a learner who had not. It shows the virtue of the two-column proof in its affordance to support chains of implications in arguments. However it also shows the drawback of the two-column proof in its lack of flexibility to support backings and rebuttals in arguments.

"Teachers' Perceptions of Geometry Students" looks at the position of the student from the perspective of the teacher. It examines teachers' perceptions of students that are instrumental in the work of teaching. It shows that while 'making conjectures' teachers perceive students in terms of engagement, ignoring the mathematical value of students' work. While 'doing proofs' teachers perceive students in terms of the mathematical

content at stake. These different perceptions of students crucially influence how students are supported in their mathematical work.

"The Work of 'Studenting' in High School Geometry Classrooms" looks at the position of the student from the perspective of the student. It examines the work that students do in instruction and the tacit knowledge that could guide this work. A theoretical model that describes 'studenting' is developed as well as a model for the rationality that supports 'studenting.'

Each group of participants involved in this study responded to the same scenario of geometry instruction, depicting a geometry class working on an open ended mathematical problem. These data sets provide three points of view on instruction. Together they serve to inform the instructional position of students.

Chapter 1

Introduction

This dissertation is a study of instruction with the aim of better understanding the students' role in this complex system. I conceptualize instruction as interactions between the teacher, students, and mathematics, in educational environments (Cohen, Raudenbush & Ball, 2003; Lampert, 2001). While instances of instruction are enacted by specific teachers, and specific students, learning specific mathematical ideas, in specific environments, the two actors, teacher, and student, can be thought of as positions (Harré & van Langenhove, 1999), that are taken up by individuals when they enter into instructional contexts. These positions guide individuals' notions of what is appropriate to do in a given instructional moment, they establish the relationship between individuals in these positions, and they give meaning to individuals' actions.

The students' position in instruction is an interesting topic of study because the position of student is taken up by a large number of individuals who, unlike the individuals who take up the position of teacher, have no official training for how to enact their position. In many ways it is remarkable that instruction in school is possible. One could imagine that if each student came to school and acted in the way that they deemed most fit for expressing who they are as individuals, instruction might be impossible. Mathematical work might or might not be done, but the phenomenon that we call instruction could not take place without a position of student for individuals to take up when they entered the classroom.

This is not simply a matter of classroom management—of getting students to sit in their seats and respond to the teacher's instructions. This is a deeper issue of students *knowing how* to "do their job" as a student. By the time students are in high school, they know more than *that they should* follow the teacher's instructions, they know *how* to follow the teacher's instructions. When the teacher gives them a task they know more than that they *should* complete the task, they know what actions to perform, using what (mental or physical) tools, to complete the task. They know the unspoken rules for how to act in the classroom and they know when it is acceptable to bend these rules to meet their goals (Mehan, 1979).

The notion of position, and positioning theory (Harré & van Langenhove, 1999), says that individuals continually construct their position through the ways that they participate in communities, and how they make meaning from that participation. Therefore, a student's position in instruction is built up through the students' actions within the environment of instruction.

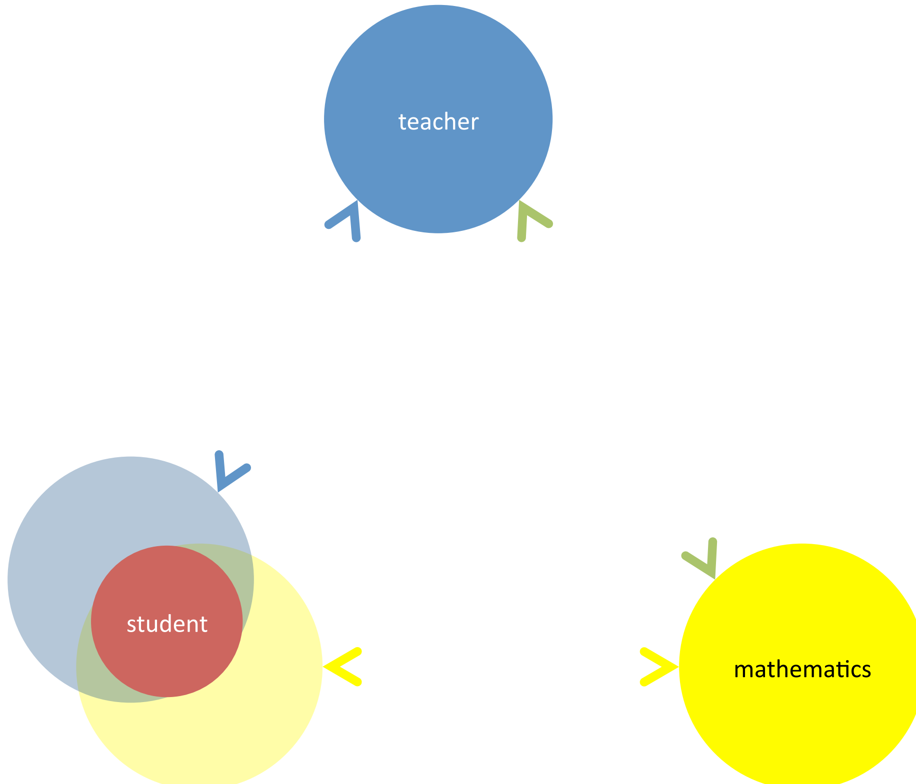

Figure 1: The students' position in the instructional triangle

From the instructional triangle (Cohen, Raudenbush & Ball, 2003; Lampert, 2001) one sees that central to the model are the relationships between teacher and student, student and content, and teacher and content. These relationships are represented by arrows in Figure 1. Thinking about the position of student as constructed by students' participation in instruction one sees that this participation exists within these arrows, and this interaction with the teacher and the content shapes the possibility for further interaction and the meaning that students make of this interaction. So, one can think of the position of student as it is constructed by the student's interactions with the teacher (represented in Figure 1 as a blue circle surrounding "student") and of the position of student as it is constructed by the student's interactions with the mathematical content that is to be learned in the geometry course (represented in Figure 1 as a yellow circle surrounding "student").

In the intersection of these two spheres of interaction is the space that is available to students in instruction. Within this space, students further respond to their own obligations and tacit understanding about what is appropriate for students to do in instruction (represented in Figure 1 as a red circle labeled "student"). This space, shaped by students' interactions with both the teacher and the content to be studied, and ultimately determined by the actions that students deem appropriate, is what I consider to be the position of the student in instruction.

The three perspectives

Through the three papers that comprise this dissertation I seek to understand better how it is that geometry students enact their position in instruction. From the instructional triangle one sees that the position of the student is partially defined by its relationship to the teacher's position and to the mathematics to be learned. So, each paper takes the point of view of one of the elements of instruction, the mathematics, the teacher, or the student, and attempts to explain how the students' position is constructed from this point of view. Each point of view adds information about the resources and constraints that are available to students as they enact their position.

The first paper, "Mathematical Arguments in a Virtual High School Geometry Classroom," looks at the position of the student from the point of view of mathematics. In this paper I examine the mathematical arguments made by learners of geometry in response to a virtual classroom discussion. This study asks the research questions,

- How does the mathematics being studied in high school geometry classrooms shape the position of the student in geometry instruction?
    - What are the modes of mathematical argumentation that learners can employ in response to classroom discussions?
    - What mathematical territory can learners cover while elaborating mathematical arguments in response to classroom discussions?
    - How can the structure of the two-column proof affect the mathematical arguments that learners make and the mathematical territory that they cover?

By studying these questions I am able to better understand how the mathematics that students engage with can shape the students' actions, and therefore the position of the student.

The second paper, "Teachers' Perceptions of Geometry Students," looks at the position of the student from the point of view of the teacher. In this paper I examine how teachers perceive the position of the student. I ask the research question,

- What perceptions of students are instrumental in the work of teaching geometry?

By studying this question I am able to better understand how the work that the teacher does, using her students as resources, can shape the position of the student.

The final paper, "The Work of 'Studenting' in High School Geometry Classrooms," looks at the position of the student from the point of view of students. In this paper I examine the work that students see themselves doing in instruction and the tacit knowledge that guides this work. I ask the research questions,

- How can hypothesized norms of instruction be used to justify student actions?

- What other justifications for instructional actions do students provide for their actions, when the action supported by the norm is deemed inappropriate?
- What is a model of 'studenting' that takes into account instructional norms for student actions as well as other research on students' actions?

By examining these research questions I am better able to understand the actions that students see as viable in instruction, and therefore shape the position of the student.

Together these points of view show the affordances and constraints that students work with when they enact their position in instruction. One sees the mathematical space that a learner might occupy as she learns geometry. One sees the space that is created for student actions by the perception of the teacher, who conducts instruction. And one sees the space that students feel entitled to occupy in instruction, guided by instructional norms.

The participants

The data used in this study was collected from three settings, each representing a different element of the instructional triangle. Each group of participants was asked to respond to the same animated scenario of geometry instruction, depicting a teacher and her class working on an open ended mathematical problem. The three data sets collected in relation to this animated scenario provide three different points of view on instruction. Together they serve to inform the instructional position of students.

The first group of participants, representing the position of mathematics, was composed of two mathematically successful adolescent learners of geometry. These participants engaged with the animated scenario from a mathematical viewpoint, attempting to understand the mathematics discussed in the scenario. Their responses map the mathematical territory that a student might encounter when she is learning geometry.

The second group of participants, representing the position of the teacher, was study groups of experienced geometry teachers. These participants engaged with the animated scenario from the position of the teacher, attempting to make sense of the teaching displayed in the scenario. Their responses contain evidence for how the teachers perceive their students and how this perception of students is instrumental to the work of teaching.

The third group of participants, representing the position of the student, was focus groups with high school geometry classes. These participants engaged with the animated scenario from the position of the student, attempting to understand the actions of the animated students shown in the scenario. Their responses provide evidence for the hypothetical norms that support students' instructional actions.

The data collection tool

Below I describe the animated scenario that each of these groups of participants responded to. I describe the animated scenario in two different ways. First I discuss the features that make this, and similar animated scenarios, useful for collecting data about participants' views of instruction. Second I describe the story of instruction that is presented in this particular animated scenario and why it was chosen to use as a data collection tool in this dissertation.

Description of animated scenarios

The animations of scenarios of instruction used in this study were developed in the context of project ThEMaT (Thought Experiments in Mathematics Teaching), directed by

Patricio Herbst (Herbst & Chazan, 2003). The instructional scenarios are animated using a set of cartoon characters that are sparse in terms of their human features (Herbst, Chazan, Chen, Chieu, and Weiss, 2010). The characters are two-dimensional with large round heads, trapezoidal bodies, and stick legs. Their hands are circles and they don't have arms. The characters show emotions through stylistic eyebrows and the changing shape of their mouths. Despite their simplicity the characters have proven to be effective in representing the instructional scenarios that they enact so as to compel people to discuss these (Chazan & Herbst, in press; Herbst & Chazan, 2006; Herbst, Nachlieli, & Chazan, in press). One reason I hypothesize that the characters are so effective is due to the fact that they are clearly not human, they encourage the suspension of disbelief, and viewers imagine themselves in the story depicted in the animated scenario.

In addition to their visual characteristics, these animated scenarios present particular instances of instruction that can be used as conversation prompts. By presenting a particular instance, the animated scenario allows for participants to discuss specific moves made by the teacher or students, or specific mathematical ideas that are at play, instead of talking in generalities about instruction. Therefore, the animated scenario supports a discussion about the instructional particulars (meaning the particular details of the instructional story, or the interactions between teachers, students, and the mathematics), which yields information about the decisions and judgments that teachers and students make during instruction.

Because the animated scenarios are constructed as opposed to recorded (as is the case with videos of instruction) they can embody targeted norms of instruction. In the design of animated scenarios, viewers can perceive these norms to be complied with or breached by the characters, hence allowing for the testing of the existence of hypothesized norms (Aaron & Herbst, 2007; Herbst, Nachlieli, & Chazan, in press). Inside the description of each study in this dissertation I describe the designed aspects of the animated scenario that are related to the research questions of that study and the expected responses of participants that correspond with that design.

The animated scenarios allow different viewers to notice different dimensions of the scenario (Ball & Lampert, 1999). For instance, a participant might view an animated scenario as if she were the teacher in the scenario, or as if she were a student in the class, or from the point of view of an observer. As an example, the three groups of participants in the studies presented here discussed different sets of issues in response to the scenario. The learners who participated in the case studies reported in the first paper focused on the mathematics in the scenario and were drawn into explorations of the number of possible intersection points of the angle bisectors of a quadrilateral. The teachers in the study group sessions reported in the second paper discussed the value of students' work in terms of students' progress towards learning to prove. The students in the focus group sessions reported in the third paper were concerned with the way that a student could disagree with another student's conjecture. Each of these three groups commented on issues visible from their point of view, but those issues were not necessarily present in the comments of the other groups.

These animated scenarios immerse participants in the work of instruction so that when participants answer questions about the prompt, I hypothesize, they answer as if they were actors in the story, not detached observers. Asking participants to respond to an animated scenario depicting a geometry class engaged in the angle bisectors problem[1] is very different than asking students to work on the angle bisectors problem, or asking teachers how they would attend to their students as they engage them in the angle bisectors problem, or asking students what instructional moves they would make in response to the angle bisectors problem. The answers to these questions often rely on tacit knowledge that would not be activated without some immersion in instruction. The animated scenarios are a tool for tapping into this tacit knowledge about instruction.

The animated scenarios are, in one way, very particular, in that they show a particular instantiation of instruction. In another way, however, they represent a very general scenario. There is no information in the animated scenario that describes the school in which the scenario takes place, the race, ethnicity, or social class of the characters (arguably, some of this is contained in the voices of the characters), or what events precede or follow the scenario. Connelly & Clandinin (2006) refer to these details as time, place, and sociality. These details are essential parts of a complete narrative but are purposely excluded from the animated scenarios. This lack of particulars in terms of the narrative, but dependence on particulars in terms of instruction (how the teacher and student interact around the mathematics), allow for the gathering of data that, I hypothesize, informs actions that may be common across time, place, and people,

[1] The angle bisectors problem asks students to consider what can be said about the angle bisectors of a quadrilateral.

providing specific information about the instructional activities that are shown in the animated scenario.

Description of The Square

I now move to describing the particular animated scenario, The Square[2], which was used for data collection in this dissertation. The Square depicts a high school geometry class working on the angle bisectors problem, or "what can one say about the angle bisectors of a quadrilateral?" The class begins by making conjectures in response to the problem and then they work on proving one of those conjectures. Below I provide details of the story and highlight some possible alternatives to the story that is presented in The Square. These possible alternatives are meant to showcase the designed nature of the story and prompt the reader to engage with the story by thinking about how the story could have gone differently.

The Square begins with the teacher reminding the class that when they studied triangles they learned that the angle bisectors of a triangle meet at a point. The teacher then poses an open-ended question, "what can one say about the angle bisectors of a quadrilateral?" She asks the class to make conjectures with the idea that they will later try to prove those conjectures. The problem that the teacher assigns to the class is abnormal[3] because it

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[2] The Square and the rest of ThEMaT's animated classroom scenarios can be viewed at http://grip.umich.edu/themat.

[3] Throughout this description of the animated episode, The Square, I refer to actions that would be "normal" in a high school geometry class. These views are based on norms for the situations of 'making conjectures' and 'doing proofs,' which are underdevelopment in the GRIP research project at the University of Michigan, under the direction of Patricio Herbst. These norms and instructional situations are described in more detail in Chapters 3 and 4 of this dissertation. The

combines the activities of making conjectures and proving. Normally, students would be asked to either make conjectures or do proofs, but not to prove conjectures that they make. The teacher does not specify any particular resources or operations that the students are expected to use while working on this task. The only resources that are specified are the mathematical objects, quadrilaterals and angle bisectors, as these are included in the problem statement. However, by bringing up the angle bisectors of triangles the teacher implicitly invokes points of intersection as a resource that could be used in the task.

The problem is open-ended in the sense that the teacher does not specify the types of claims that students should make about the angle bisectors of quadrilateral or how students should go about looking for those claims. From the story that follows, it appears that no student worked on the question about a general quadrilateral. The ensuing discussion revolved around particular quadrilaterals. This work on particular quadrilaterals reflects some of the open-endedness of the problem.

After the teacher gives instructions, students turn to their partners and begin to talk about the task. Students work on the task for an unknown period of time before the teacher calls the class to order. As the students are ending their conversations one student can be overheard saying that she drew a kite and the angle bisectors made a point. When the students are quiet the teacher asks a different student, named Alpha, to share his conjecture with the class; in spite of the fact that the teacher had not asked for volunteers,

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adjective "normal" does not mean desirable or healthy, but rather perceived as appropriate by teachers and students of the usual geometry course (see Herbst, Nachlieli, and Chazan, in press).

and that Alpha had not volunteered to share his conjecture. In light of that, one would expect the teacher to appreciate Alpha's contribution and find a way to make it a useful part of the conversation. One can notice that the teacher did not call on the student that drew a kite, a mathematically more complicated quadrilateral that could support a more difficult proof.

From his seat Alpha shares “about the square, they would have to bisect each other.” The teacher asks him to come to the board to draw his diagram. Alpha comes to the board and draws his diagram, a square with opposite vertexes connected, and describes it as “a square and the diagonals, they bisect each other.” Alpha’s conjecture is reasonable in some ways and problematic in others. On one hand, Alpha is correct; the diagonals of a square bisect each other. Also, in the case of a square, the angle bisectors and the diagonals overlap so it is understandable to use the terms interchangeably. On the other hand, Alpha does not seem to be answering the question about angle bisectors. There is a large conceptual gap between a statement about diagonals and a question about angle bisectors.

The teacher responds to Alpha’s conjecture by telling Alpha “the problem is about angle bisectors, not about diagonals.” Here the teacher could have responded differently to Alpha. Because the angle bisectors and the diagonals are the same in a square, and the problem the teacher asked is about angle bisectors, the teacher could have interpreted Alpha’s conjecture as saying that in a square the angle bisectors bisect each other. While this is not a formally correct conjecture (angle bisectors are rays not segments) it could be seen as a useful starting point for a discussion about what one can say about the angle bisectors of a quadrilateral. Also, the teacher began the lesson by reminding the class that in triangles the angle bisectors meet at a point. Alpha has brought up the case of the square where the angle bisectors also meet at a point. The teacher could highlight this connection and Alpha's diagram and overlook the exact statement that Alpha made.

The teacher's evaluation of his conjecture appears to lead Alpha to hang his head and seem dejected. The emotional power of this statement can be read two ways. First, this is upsetting for Alpha because the teacher does not appear to see the mathematical contribution that he is making to the progress on the problem. In this case the teacher is not fulfilling her obligation to understand the mathematics at least as well as the students and to weave the students' contributions into a complete mathematical solution. The second way of reading the teacher's reaction to Alpha is that the teacher is mean. In this reading the teacher does appear to understand how Alpha's conjecture could be a reasonable response to the angle bisectors problem but instead she focuses on the fact that Alpha's conjecture is not explicitly about angle bisectors and she uses this as grounds for dismissal of his conjecture. In the first reading the teacher's actions are flawed because she does not act as if she has the mathematical expertise that is required of a teacher. In the second reading the teacher's actions are also flawed because she willfully dismisses Alpha's conjecture despite her ability to see its usefulness to the discussion.

After the teacher reminds Alpha that the problem is about angle bisectors, another student, later addressed as Beta, is seen telling her neighbor that she thought that the diagonals and angle bisectors are the same thing. The teacher and Alpha do not seem to hear this remark. Beta's comment could be understood in at least two ways. The first is to see Beta as helping Alpha by pointing out that in the case of a square the angle bisectors and the diagonals are the same objects. In this case Alpha would have reason to be grateful for Beta's comment. In the second reading, Beta is talking out of turn and could be seen as increasing Alpha's discomfort by drawing more attention to the fact that the teacher has told him he is wrong.

After Beta's whisper the teacher addresses the class and reframes the question to be one about only squares instead of all quadrilaterals. Alpha returns to his seat, saying as he walks, "I just thought that the diagonal cuts the square in half." The teacher seems to like this conjecture. She writes "Alpha: The diagonal cuts the square in half" on the board and asks the class to elaborate on the statement. Beta says, "Alpha means that the diagonals are also the angle bisectors." As with her earlier comment, Beta's rewording of Alpha's conjecture can be seen as either helping Alpha clarify his idea or as highlighting the fact that the teacher is not sanctioning his idea.

It is interesting that the teacher wrote Alpha's second statement on the board because she did not write Alpha's earlier statement on the board, that the diagonals bisect each other. This could be because the second statement could arguably be more useful in the proof that the diagonals of a square are also its angle bisectors. The teacher asks for elaboration of Alpha's statement and Beta replies that the angle bisectors are the same as the diagonals.

The teacher asks the class if they agree with Beta's statement, that the diagonals are also the angle bisectors. A student named Gamma speaks up and says, "Obviously they are not." She goes on to talk about the case of a rectangle where angle bisectors are not the same as diagonals. Alpha protests that he was only talking about squares so Gamma's counter-example does not apply; but the teacher asks Gamma to elaborate anyway.

Gamma comes to the board and extends Alpha's diagram of a square into a diagram of a rectangle. She does not seem to take responsibility for showing how her ideas fit with Alpha's conjecture or into the bigger context of the discussion about angle bisectors of quadrilaterals. Another student defends Alpha by saying that when he made his conjecture he was not talking about rectangles, only squares.

One could imagine that at this point the teacher could take up the discussion about angle bisectors of rectangles and ask if other students had made conjectures about the angle bisectors of rectangles. However, the teacher returns to the case of the square and asks if Gamma's counterexample disproves Alpha's conjecture, that the diagonals cut the square in half.

At this point Beta, Gamma, and Alpha discuss Alpha's conjecture. The exchange ends with Alpha restating his conjecture as "in a square the angle bisectors meet at a point because they are the diagonals." The teacher paraphrases Alpha's conjecture on the board by writing, "in a □ the ang bis ⊠" and calls for volunteers to do a proof. The teacher calls on Lambda. By calling on Lambda to produce a proof the teacher shifts the activity from making conjectures to doing proofs. There is a shift in the goal of the activity that students are working on as well as in the participation structure of the class. Whereas in the first half of the animated scenario several students participated in the discussion, in the second half of the animated scenario, while Lambda presents a proof, there is much less participation from the other students. Also, the teacher can be seen to take on a more directive role.

Lambda begins his proof by saying that the diagonals cut the square into equal pieces so they are the same as the angle bisectors. The class presses him to prove this claim. The teacher ignores the students' call for a proof and asks Lambda which triangles he is talking about. Without waiting for a response from Lambda, the teacher interprets Lambda's statement as saying that the four small triangles formed by both diagonals are all congruent. Beta confirms that these are congruent but Lambda responds by asking the teacher to remove one diagonal from the diagram so that the square is only cut into two triangles. This is an abnormal request from a student because in the situation of doing proofs students do not alter the diagram available for use on the task (in particular they do not add or take away mathematical objects; see Herbst et al., 2009).

Despite Lambda's insistence that the proof only requires one diagonal the teacher leaves the diagonal on the diagram, citing the fact that squares have two diagonals. Lambda continues with his proof, saying that the diagonal forms two triangles that are congruent and isosceles. At this point the teacher asks Lambda what it is that he is trying to prove. This is understandable because the statement that the teacher wrote on the board before asking for a proof, although it was ambiguous, was a statement about at least two segments or rays. Lambda seems to have segued into a proof about congruent isosceles triangles.

Lambda tells the teacher that he is trying to prove that "the base angles on both triangles have to be equal to each other." The teacher marks the base angles of one of the small triangles as angles 1 and 2 and asks Lambda if these are the angles that he is trying to prove congruent. If the teacher had erased one of the diagonals, as Lambda had requested, then this triangle would not exist. The teacher is making the conversation easier to follow by marking angles and giving them labels but she is also complicating the conversation by marking angles that are not the ones that Lambda is talking about.

Instead of telling the teacher what it is that he is trying to prove Lambda again asks the teacher to erase one diagonal. Beta supports the teacher's decision to not erase one diagonal by expressing confusion with the idea of not showing both diagonals. The teacher again refuses to erase the diagonal, this time saying that since Lambda is proving something about the intersection of the diagonals he must have both diagonals in the diagram. The teacher says this despite the fact that Lambda has not told her what it is that he is trying to prove. The teacher's assertion about what Lambda is trying to prove does not match the statement that she asked Lambda to prove, or the statement that Lambda is attempting to prove. Lambda tells the teacher that he is trying to prove that "the triangle is isosceles" so he only needs one diagonal of the square. The teacher acquiesces to Lambda's request and erases one diagonal.

Even though Lambda is acting abnormally by asking the teacher to remove one diagonal, he seems aggravated by the fact that she will not comply with his request. Like the teacher's response to Alpha, the teacher's response to Lambda could be read as either meaning that the teacher does not understand the mathematical argument that Lambda is advancing or that she is purposefully being obtuse.

Lambda continues his argument, saying that the base angles of both triangles are congruent. Instead of giving an argument for why this is true he simply asserts that it could be proven for one triangle and then the proof could apply to the other triangle. The teacher accepts this reasoning and asks how this is connected to Alpha's conjecture.

Lambda responds by saying that he's showing that the diagonal is also an angle bisector and that the same proof could be used to show that the other diagonal is also an angle bisector. The teacher does not sanction or reject Lambda's line of reasoning. Beta appears surprised by the fact that Lambda has brought the second diagonal back into the argument and another student justifies this move by saying, "the teacher said he could do it!"

The teacher takes control of the conversation by writing the given and prove on the board. First she writes, "given: ABCD is a square." Then for the proof statement Alpha volunteers that the angle bisectors meet at one point. The teacher writes, "prove: the angle bisectors meet at a point." Beta expresses confusion at this proof statement. The animated scenario ends with the teacher calling for a proof of this statement and Lambda being indignant because he believes that he just did that proof. By asking for the proof again and writing the given and prove statements on the board the teacher is devaluing Lambda's contribution as not an acceptable proof. This move of the teacher could also be interpreted as an attempt to see if the rest of the class was following Lambda's argument.

Features of The Square

This animated scenario has several characteristics that make it useful for gathering information about the position of the student in geometry instruction. First is that it depicts two instructional situations, 'making conjectures' (Herbst et al, 2010) and 'doing proofs.' (Herbst & Brach, 2006) The inclusion of both of these situations allows for comparisons between these two situations. Seeing differences between the situations highlights the nature of each situation. Also, each of these two situations contains activities that are important in the discipline of mathematics. Their importance in the discipline makes them important activities to study in the classroom.

Another characteristic of this animated scenario that makes it useful for data collection in this dissertation is that it shows students engaging in a substantial discussion of mathematical ideas. This type of student-centered discussion is something that is not often seen in geometry classrooms but that is a goal of educational reform (NCTM, 2001). Gathering responses to the animated scenario can inform what it is about this type of instruction that makes it uncommon.

In addition to showing student-centered discussion, the animated scenario shows students supporting contradictory mathematical ideas. In the beginning of the animated scenario Alpha puts forward a conjecture and then Gamma presents a counter-example. Because students identify so closely with their ideas, students rarely share ideas that oppose the ideas of other students (Johnson, 1979). Again, by using this particular animated scenario as a data collection tool one can gather information about how the position of the student in geometry instruction mingles mathematical ideas with student identities and results in students being unwilling to support opposing mathematical ideas.

This animated scenario also showcases two student actions that are unusual and worthy of study. The first is Lambda's presentation of his proof. Lambda's presentation focuses on the key ideas of his proof without stating the details or producing statements and reasons in alternating order, like one would when constructing a two-column proof. Lambda is sharing a very smart idea, but it seems to have no traction in the classroom because of the form in which he shares his idea.

The other unusual student action is Lambda's request that the teacher remove one of the diagonals from the diagram on the board. It is unusual for a student to modify the resources that they are given to use in a task, like modifying the diagram. The teacher resists this change and yet Lambda persists in requesting the change until the teacher finally acquiesces. By removing the diagonal Lambda was able to focus his attention on one diagonal at a time and show that each diagonal was also an angle bisector. Removing the diagonal was instrumental in Lambda's proof but it was not something that the teacher seemed prepared to do.

This discussion of the animated scenario used for data collection in this dissertation highlights several features that recommend it as a useful tool for gathering information pertaining to the position of the student. There are features of the animated scenario as a genre, like its immersive quality, and features of this animated scenario in particular, like the presence of student-centered discussion, that prompt the participant to respond to the animated scenario in ways that are tied to instruction and to the specific instructional activities that are of interest.

The papers presented in the body of the dissertation are in the following order; "Mathematical Arguments in a Virtual High School Geometry Classroom," "Teachers' Perceptions of Geometry Students," and "The Work of 'Studenting' in High School Geometry Classrooms." Each paper is self contained, however, the second and third papers build on a shared theoretical framework, and therefore reference each other for aspects of their theoretical development. The dissertation concludes with a brief discussion of the understanding that these studies have provided of the position of student in instruction.

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Chapter 2

Mathematical Arguments in a Virtual High School Geometry Classroom

Since the time of the new math, and in particular, apropos of Bruner's famous principle that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (Bruner, 1960, p. 33), mathematics educators have been interested in the relationship between mathematics done by students in school and mathematics done by mathematicians working in the discipline. Understanding this relationship is especially important in high school geometry classrooms where there is a particular emphasis on the mathematical activity of proving, the preeminent tool of knowledge production and verification in the discipline (Lakatos, 1976; Thurston, 1994). According to prior research in high school geometry classrooms, students do proofs in a two-column format, proving statements that are already known to be true and using a predetermined set of resources (Herbst, 2002, 2006; Herbst & Brach, 2006). In the discipline of mathematics, however, mathematicians do not follow a particular form; more importantly they do not sit down to do a proof knowing what they will end up proving or what tools they will employ in the process (Lakatos, 1976).

This paper is part of a larger project to understand the position (Harré & van Langenhove, 1999) of the student in geometry instruction. In particular, this study is aimed at understanding how engagement with mathematical knowledge and practices shapes the nature of the work of being a geometry student. I hypothesize that being a student in a geometry class requires work on the part of the student that can be traced to the nature of the content under study and is not dependent on the individual student doing that work.

I make use of a metaphor that likens a classroom to an orchestra (Herbst & Balacheff, 2009). In this metaphor the students are the musicians and the teacher is the conductor. The mathematics that is under study in the classroom is analogous to the musical piece that is performed by the orchestra. It takes the class as a whole, not an individual student, to enact this mathematics and this enactment is under the direction of the teacher. Extending this metaphor, I think of each student's contribution in the classroom as consisting of both the argumentation style employed by the student and the mathematical content of the arguments that they elaborate. This is analogous to the musical style and musical part that a musician plays.

This metaphor helps to make visible the impact that the mathematical content has on the position of student. Just like the work of being a musician is shaped by the music the musician plays, the work of the student is likewise shaped by the mathematical content they are studying, in terms of both the argumentation style and mathematical territory covered. In most high school geometry classrooms one particular style of argumentation, the two-column proof, is favored above all others and the content is predetermined by the textbook (Herbst, 2002). This arrangement of content in the high school geometry class has both advantages and drawbacks and both need to be explored in more detail.

This paper reports on a case study wherein the data represent cases of mathematically successful adolescent learners encountering novel mathematical problems in the context of classroom discussions. From these particular cases I aim to better understand how the position of the student is shaped by the mathematical arguments and mathematical territory that could be elaborated by learners in these circumstances. Stake (1978) makes the point that case studies are useful when the desired outcome of the study is "a full and thorough knowledge of the particular, recognizing it also in new and foreign contexts" (p. 6). My goal in this study is to move closer to a full and thorough knowledge of the work involved in elaborating arguments in geometry instruction. These case studies provided understanding of two different ways (represented by two different participants) in which that work could proceed.

In this study I am using two cases of students elaborating mathematical arguments as a means for better understanding the process of elaborating mathematical arguments within geometry instruction. I am not interested in making general claims about these two students as individuals; rather, I am interested in making claims about the nature of the work that these students are engaged with. Also, I claim that the nature of this work is somewhat separate from the individuals conducting that work. The two students chosen to participate in this study represent two different possible ways to doing the work of elaborating arguments, insofar as the arguments that they elaborate reflect different modes of argumentation and cover different mathematical territory.

While one of the participants provides the opportunity to study arguments that reflect the structure of the two-column proof format that is common in American high school geometry classrooms, the other provides the opportunity to study arguments that were not influenced by this proof format. I ask what affordances and constraints two column proofs may provide to students, as can be seen in their capacity to make authentic mathematical arguments.

I explore the following research questions:

- How can the mathematics being studied in high school geometry classrooms shape the position of the student in geometry instruction?
    - What are the modes of mathematical argumentation that learners can employ in response to classroom discussions?
    - What mathematical territory can learners cover while elaborating mathematical arguments in response to classroom discussions?
    - How can the structure of the two-column proof affect the mathematical arguments that learners make and the mathematical territory that they cover?

The overarching research question that I am exploring looks at the position of the geometry student in instruction and asks how the mathematics that is being studied shapes this position. To answer this question I conceptualize the mathematics being studied as being comprised of both the style of argumentation being used and the mathematical territory being covered. The first sub question looks at the mathematical arguments that learners build and the modes of argumentation that are reflected in these arguments. The second sub question looks at the mathematical territory that learners cover while they are building these arguments. The third sub question looks at the differences uncovered in the previous two sub questions and asks how the structure of the two-column proof can explain these differences.

In this study I compare mathematical arguments elaborated by two middle school students, one who had taken a geometry course at a local high school and one who had not. I do not make general claims about students who have and have not taken this course, but rather, I look at the characteristics of students' arguments, some that reflect the structure of the two-column proof and some that do not. I compare the data gathered from these two students as a way to see the different ways that learners can engage with the mathematical content being studied and in particular, see how this engagement is shaped by the mathematical arguments and territory that they elaborate.

In the following sections I describe research on students' use and understanding of proof in mathematics classrooms and writings that address the generation of mathematical knowledge in the discipline of mathematics. I then describe the participants that took part in this study and the method of data collection that was used, including a description of the animated classroom scenario that was used as a conversation prompt in discussions with the participants, in terms of the mathematical arguments that it contains and the mathematical work that it might elicit. The method of analysis used in this study builds on Toulmin's scheme for analyzing arguments. I describe Toulmin's scheme and how I applied it to the data. The second half of the paper is devoted to showcasing the mathematical territory covered by the participants and the arguments elaborated by the participants in response to the animated scenario. The discussion returns to the research questions, making comparisons between the characteristics of the arguments made by the two participants. In making these comparisons I make use of the notion of 'disciplinary agency' described in the literature section, and argue that the arguments elaborated by one participant reflect the disciplinary agency of the high school geometry classroom, while the arguments elaborated by the other participant reflect a disciplinary agency that is closer to that found in the discipline of mathematics.

Relevant literature

To frame this study I draw from theoretical literature from two broad areas. The first is literature on students' work in classrooms (e.g., Hanna (1983, 1989, 1990, 1993); de Villiers (1990); Balacheff (1987, 1988); Chazan (1993); Schoenfeld (1985); Fosnot & Jacob (2009); Maher & Martino (1996); and Herbst (2002, 2004, 2006, Herbst & Brach, 2006)) and the second is literature on the creation of mathematical knowledge (e.g., Lakatos (1976); Wilder (1981); Kitcher (1984); and Pickering (1995)). These areas are related in that it is within the high school geometry classroom that teachers attempt to teach students about the creation of mathematical knowledge through proof.

Mathematical *ideas* can come from many sources. They can come from empirical exploration, intuition, guessing, teachers, textbooks, etc. However, mathematical *knowledge*, that is, ideas that are verified to be true, can only come through proof. Proof is the only accepted source of mathematical knowledge. In this study I consider the activity of doing proofs in a high school mathematics classroom as being comparable the activity of generating mathematical knowledge in the discipline of mathematics. On the surface this may seem absurd; one obvious difference between these activities is that the statements that are proven by students are already known to be true, while the statements that are proven by mathematicians have not yet been verified. However, the activity of proof in classrooms does resemble the generation of mathematical knowledge in the discipline on a much smaller timescale. Students may be convinced of the truth of the statement but they are still required to prove it, they are compelled to deploy their resources in a way that they cannot fully anticipate at the outset, and at the end of each proof they expand their knowledge of the mathematical territory that they are exploring. When the process of proving in high school classrooms is viewed in this light, I argue, it becomes reasonable to compare the activities of learners and the activities of mathematicians.

Another reason that this comparison is worthwhile connects to the quote from Bruner. I'd like to raise the question, is the teaching of the two-column proof an "intellectually honest form" of teaching students about the generation of mathematical knowledge? I argue here that the two-column proof provides both affordances and constraints in the quest to teach mathematics in an intellectually honest form.

Literature on proof in classrooms

Proof holds a contested but prominent place in American schools, and in geometry classrooms in particular. Below are perspectives on the teaching and learning of proof in classrooms. I divide the literature into three groups. The first group contains papers that discuss what proof in schools should look like from a disciplinary perspective, the second group contains papers that discuss the learning of proof, and the third group contains papers that discuss the teaching of proof.

Proof in schools from a disciplinary perspective

The following writings give a vision of what proof in schools should look like if it were to resemble closely proof in mathematics. These writings look at proof in mathematics and then make an effort to think about how these proofs can have an educative role in classrooms. I look in particular at the work of Hanna (1983, 1989, 1990, 1993) and de

Villiers (1990). Both of these authors look at the function that proof fulfills in the discipline and suggest that proof in schools could and should fulfill the same functions.

Unfounded assumptions about proof in the discipline

Hanna (1983, 1989, 1993) argues that the current place of proof in schools is based on unfounded assumptions. She comes to the conclusion that the place of proof in schools should rest on its explanatory power as well as various ways that proofs can be applied to new ideas both in mathematics and related fields.

Hanna asks the question, what does proof look like in the discipline of mathematics and what implications does this have for what proof should look like in school? She finds that three common assumptions about proof in mathematics are false, leading to the conclusion that the ways proof is taught in schools are inconsistent with proof in modern and historical mathematical practice.

Three assumptions; that proof is used as a way to convey understanding, that proof is the defining characteristic of mathematical practice, and that there exists a stable, agreed upon method of accepting new theorems based on their proof among practicing mathematicians, have lead to a misappropriation of rigorous proof in the mathematics classroom (Hanna, 1983). According to Hanna rigorous proof is one in which every definition, postulate, and theorem, as well as the logical connections, are explicitly stated. It is this rigorous proof that Hanna claims has been wrongly incorporated into classrooms, and that is not the way that proof is conceptualized in the discipline of mathematics. Hanna goes on to argue that proof should have a central role in the classroom, but it should be a role of communicating mathematical understanding, and in their communicative role, not all proofs are made equally.

The first assumption about proof is that it is a good way to convey mathematical understanding. Hanna claims that some proofs, “proofs that prove,” do not necessarily have any explanatory power (Hanna, 1983, 1990). She cites specifically proofs by induction and proofs by contradiction. In Hanna’s opinion, these proofs do not provide any illumination of the underlying mathematical structures in question, and therefore they do not have the ability to explain that can be found in other proofs. The proofs that have a place in the classroom are “proofs that explain.” Examples of types of proofs that explain are proofs by construction or visual proofs. Hanna gives the example of the proof that the sum of the first n whole numbers is equal to n (n+1)/2. This can be proved inductively without giving the reader any indication of why it is that the statement is true, or it can proved using a visual proof showing that the sum of an increasing series of triangular numbers and can be seen to be half of the rectangle of dimensions n by n+1 (see Figure 2).

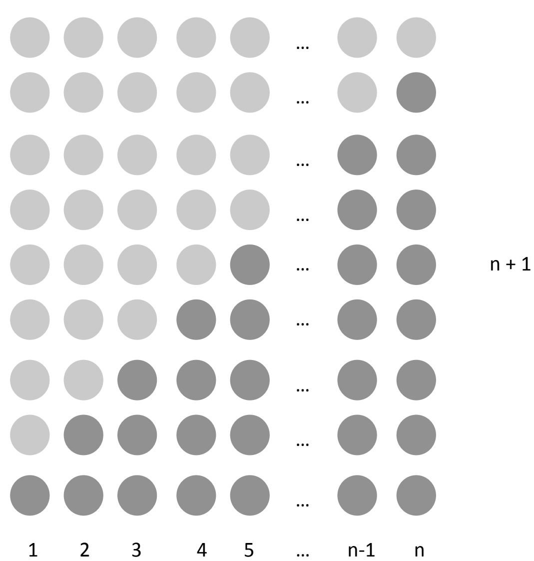

Figure 2: Visual proof of the claim that the sum of the first n whole numbers is equal to n(n+1)/2

According to Hanna, the second unfounded assumption about proof in mathematics is that rigorous proof is a key aspect of current mathematical practice (Hanna, 1983). A common reason for teaching rigorous proof to high school students is that rigorous proof is the cornerstone of modern mathematics practice. According to Hanna's research, "[m]athematicians accept a new theorem when some combination of the following holds:

1. They understand the theorem (that is, the concepts embodied in it, its logical antecedents, and its implications) and there is nothing to suggest it is not true;
2. The theorem is significant enough to have implications in one or more branches of mathematics, and thus to warrant detailed study and analysis;
3. The theorem is consistent with the body of accepted results;
4. There is a convincing mathematical argument for it, rigorous or otherwise, of a type they have encountered before." (Hanna, 1983, p. 70)

None of these reasons entail the existence of a rigorous proof. What is much more important is that the theorem is understood, significant in its domain, consistent with current thinking on the topic, and there exists a convincing argument. This convincing argument can take many forms, with a rigorous proof not necessarily being preferred. In fact, even when a formal proof exists its rigor and focus on detail can make it difficult for the reader to see the argument behind the lines of rigorous proof (Leron, 1983). According to Hanna, "it would therefore appear that what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinized and revised" (Hanna, 1989, p. 23).

The third and final assumption is that within mathematics there is a consensus on what constitutes a proof, and therefore on what grounds a theorem should be accepted as true (Hanna, 1983). Hanna points to the paradoxes that result from axiomatic set theory as proof that mathematics has carried on, even after this 'crisis of the foundations.' Three schools of thought emerged from the crisis: logicist, formalist, and intuitionist, each with their own understandings of the basic mathematical objects and logical system. In addition to the different schools of thought, history shows that what counts as rigor is changing. That is, what was rigorous to the Greeks was not rigorous to Euler was not rigorous to Cauchy is not rigorous to current logicians (Hanna, 1983).

If the negation of these three assumptions tell us how proof in mathematics is not best reflected in schools, then what are ways of understanding the place that proof holds in mathematics and therefore by extension in classrooms? Hanna places the meaning of mathematics in its application. This is not application only in the sense of its utility in other disciplines like physics, but application within the discipline of mathematics. Proof can reveal "new dimensions and new aspects of the theorems proved" (Hanna, 1993, p. 428). These applications give meaning to possibly otherwise trivial theorems or observations. However, these applications are not always within the scope of what the student understands so, in many cases, the student must take on faith a teacher's claim that a particular solution is better than another based on the teacher's greater knowledge of mathematics (i.e., what will come later, the generalizability of an argument). Of course it is preferable that students have understandings of the areas of application so that the applications will carry meaning (Hanna, 1993).

Starting from the assumption that proof in schools should reflect proof in mathematics, Hanna provides a vision for what position proof could hold in schools. She argues that the explanatory nature of proofs should be highlighted along with the realization that proofs gain importance from the contexts in which they can be applied to expand understandings.

According to Hanna, schools should abandon the unfounded assumptions that proof needs to be rigorous and that there is an agreed upon criteria for accepting proofs and the statements that they prove.

Functions of proof in the discipline of mathematics

de Villiers, like Hanna, has a vision of proof in schools reflecting proof in the discipline of mathematics. He claims that proof in the discipline of mathematics fulfills five functions: verification, explanation, systematization, discovery, and communication (de Villiers, 1990). de Villiers argues that through understanding these functions of proof and bringing them into the classroom, teachers will be able to satisfactorily answer students' question, "Why do we have to prove this?"

The first function, verification, is the starting place for de Villiers. He argues that many teachers and students are aware of this function and use this to justify their work with proofs. This function of proof is aimed at verifying that a particular mathematical proposition is true. de Villiers expands the notion of verification to include conviction, which places the emphasis of the justification not on mathematical and logical rules, but the ability to convince another person that the proposition is true. This move from verification to conviction is precipitated by the belief that there is no such thing as absolute certainty in mathematics.

The second function of proof is explanation. Proofs that explain provide understanding of why a proposition is true with respect to a particular set of assumptions. These proofs are different from proofs that convince because proofs that explain provide an understanding of *why* a proposition is true, beyond an understanding that the proposition *is* true. de Villiers claims that proofs that explain are more preferable to many mathematicians than proofs that convince. In an educational context, one can see the clear advantage of providing students with proofs that explain rather than proofs that only convince. Most likely a proof that explains will also convince, in addition to explaining.

Systematization, the third function of proof, shows how mathematical concepts involved in a proof are related to other mathematical concepts. Just as explanation is a function of proof that provides understanding, so is systematization. While explanation provides understanding of why a proposition is true with respect to a particular set of assumptions, systematization provides understanding of why a proposition is true with respect to a set of mathematical concepts. Systematization allows for axiomatization and defining, two important activities of metamathematics.

Discovery is the fourth function of proof put forward by de Villiers. He argues that throughout history mathematicians have discovered theorems in the process of proving other theorems. In particular, during a proof a mathematician comes upon the essential characteristics of the assumptions that allow for the conclusion, and therefore finds a more general proposition than she was originally proving.

The final function of proof discussed by de Villiers is communication. Proof allows for the discussion of mathematical ideas. Through this discussion mathematicians share their discoveries with each other, agree and disagree with each other, and learn from each other. Without this function of proof the mathematical community would cease to have any mathematically meaningful interactions with each other.

These five functions of proof are important to education because they help to expand the role of proof in the classroom beyond verification. Other functions of proof; explanation, systematization, discovery, and communication, could each provide students with another avenue for appreciating the value of proof in their mathematical work. These functions of proof are similarly useful for the current case study because they give ways to think about what the participants are doing when they create arguments, besides verifying the truth of a mathematical proposition.

The literature reviewed here, from Hanna and de Villiers, attempts to bring proofs in schools closer to those done by mathematicians by focusing on how proofs can lead to an explanation of the statement being proven and the usefulness of statements in terms of being applied to other mathematical domains. I focus on this set of writings because they provide a vision for what proof in classrooms might look like if school's primary obligation was to the discipline, as Bruner suggested that it might be, when he said that any topic could be taught in an intellectually honest form. However, as I will show in the following section, schools have other obligations, for instance, obligations to the learner of proof and to the work of teaching.

Learning proof in schools

In the following section I look at the literature on students' learning how to prove, and I review different ways that students view proofs and the various cognitive obstacles that students encounter when learning how to prove. This research on students learning how to prove will be beneficial in examining and understanding the arguments that study participants built.

Conceptions of proofs

In the following section I describe literature on students learning to prove. To frame this literature I begin with a quote from Balacheff (1987) that defines explanation, proof, and mathematical proof. Balacheff says,

> We call *explanation* a certain type of discourse that attempts to make understandable the truth character of a proposition or result acquired by the speaker. The reasons that he or she provides can be discussed, refuted, or accepted. We call *proof* (Fr. preuve) an explanation accepted by a given community at a given moment of time. The decision to accept it can be the object of a debate whose principal objective is to determine a common system of validation for the speakers. Within the mathematical community, only those explanations that adopt a particular form can be accepted as proof. They are sequences of statements organized according to determined rules: A statement is either known to be true or deduced from those that precede it using a rule of deduction from a set of well-defined rules. We call these sorts of proofs *mathematical proofs* (Fr. demonstration) (Balacheff, 1987, pp. 147-148, translated by Herbst).

This quote from Balacheff defines mathematical proof based on explanation and the acceptance of the mathematical community. This definition highlights the expectation that a mathematical proof would both explain why a statement is true and that members of the community employing mathematical proof would understand how that proof relates to the truth-value of the statement being proven. However, from the literature reviewed below, one sees that students have difficulty understanding how to interpret the results or existence of a proof. That is, students often will not understand the significance of a proof and they will not have the tools to tell when a proof is valid or not. This lack of understanding prevents students from seeing the value in mathematical proof. In addition, students do not feel compelled to prove a claim when they can *see* that the claim is true. These difficulties that students encounter when learning how to prove are discussed below.

Students view evidence as proof

Some students believe that claims can be proven through the presentation of empirical evidence (Chazan, 1993). These students are convinced by a series of examples that show the desired conclusion to be true. Balacheff gives two views of proof that fit into this "evidence is proof" category. These views are named "naïve empiricism" and "the crucial experiment." Both of these are described below.

Naïve empiricism

In this vision of proof, students take statements to be true if they can observe them in the world (Balacheff, 1988; Schoenfeld, 1985). This view of proof has the benefit of being quick. Students do not have to spend time considering the relationship between objects represented in the problem, but can make an observation and then report their finding. Also, this view of proof reflects a belief that a particular observable instance is the same as the phenomenon under examination.

Schoenfeld (1985) provides axioms for empiricism that he suggests can be used to predict student actions when students are acting in line with an empirical view of proof. These axioms are (paraphrased here):

1. Insight and intuition come from drawings.
2. Good hypotheses are visible in the diagram along with a path for testing them, and the hypotheses involve objects that are prominent in the diagram.
3. The most likely hypothesis is tested first.
4. Mathematical proof is irrelevant.

These axioms of empiricism, and especially axiom 4, can be seen to be direct barriers to students learning how to prove. As long as students believe that mathematical proof is irrelevant they will not move from an empirical view of proof to a more advanced view of proof.

The crucial experiment

In this vision of proof, students are more aware of the problem of generalization, than in the view of naïve empiricism, where the students are oblivious to concerns of generalization. Instead of taking any case of a phenomenon as the phenomenon itself, students engage in a crucial experiment for which they pick a useful case that they treat as a representative of a phenomenon that they wish to examine. The purpose of the experiment is to choose between two specific hypotheses (Balacheff, 1988). This choosing usually does not result in the confirmation of a hypothesis, but the rejection of one of the hypotheses.

Students view proof as evidence

Some students believe that claims that are proven through deductive means only hold for that particular case that is used in the proof (Chazan, 1993). For example, if a geometric proof is done using a diagram then students will believe that the proof holds for that diagram only and not for a general class of objects.

Generic example and thought experiment

In this vision of proof, students are able to construct an example that preserves the essential characteristics of the phenomenon while eliminating superficial characteristics (Balacheff 1988). Even students who have this more advanced view of proofs do not necessarily realize that this generic example generalizes to all cases in some infinite set. They might see that the claim has been proven for this example, but not for all cases.

Obstacles that students face when learning to prove

The obstacles discussed below are ways of thinking mathematically that are useful in developing an understanding of proofs. While they are not necessarily tied directly to the work of proving, in some cases they have been shown to be prerequisites for learning how to prove.

An obstacle that students face when they are learning how to prove is the distinction between objects and processes (Fosnot & Jacob, 2009). In mathematics this distinction is important because some objects can be understood as the result of an operation. For example, the arithmetic object *sum* can be understood as the result of the process of *addition*. So, the number sentence "3 + 2" could be seen as either an object or a process. An example from geometry is the object *angle bisector*, which is defined as a ray that *bisects an angle*. Therefore the process of bisecting an angle is reified in the object angle bisector.

According to Fosnot and Jacob there are certain mathematical ideas, such as the distinction between objects and process, which are prerequisites for more advanced deductive reasoning. They studied this in the context of students' conceptual reasoning with a number line. In their work the combination of the context of the problem situation (frogs jumping) and the representation (intervals on a number line) confronted students with the distinction between process (jumping) and objects (intervals). They concluded, "while examining students' early attempts to use deductive reasoning in the context of equivalence, we found that their construction of certain mathematical ideas was prerequisite" (Fosnot & Jacob, 2009). The object/process distinction is not a fundamental aspect of proving, however the understanding of this distinction facilitated students' learning of proof.

Maher and Martino (1996) noticed some obstacles in the development of proof through a case study that traced one student, Stephanie, from first to fifth grade. Since the researchers followed Stephanie for an extended length of time they were able to see how her thinking about proof and deductive reasoning developed. Of particular importance in Stephanie's learning of proof was the systemization of her reasoning. In attempting to construct an argument that would become a "proof by cases" she was able to able to list the cases using patterns that allowed her to reason about the exhaustiveness of her cases. She was able to use local patterns to generate cases from existing cases and she was able to use global patterns to check for repeated cases and verify the inclusion of all cases.

The literature reviewed here provides some guidance for other considerations that schools and teachers might have beyond their obligation to the discipline of mathematics. This literature shows that students need to learn how to value mathematical proofs as well as learn some mathematical ways of thinking that will facilitate their understanding of mathematical proof. In the next section I look at the learning of proof from the perspective of the classroom teacher.

Teaching

The literature below supports the claim that proof in schools responds to constraints that stem from the work of teaching. That is, the work that is entailed in teaching proofs shapes how students learn about proofs. One sees that proof in classrooms is taught inside the situation of 'doing proofs' (Herbst & Brach, 2006), in which students are taught how to create proofs in two-column form (Herbst, 2002). Below I discuss how the situation of ‘doing proofs’ and how the two-column proof impacts the proofs that students learn how to do.

The evolution of the two-column proof mode

In American geometry classrooms the most prevalent of style is the two-column proof (Herbst, 2002). The two-column proof requires that proofs are presented in successive lines, each line consisting of a statement and a reason why that statement is justified based on previous lines and logical connectives. Herbst argues that this format for the two-column proof supports teachers in assigning proof exercises that are of a suitable (short) length, and therefore doable by students, while still keeping structural similarities with proofs that are done by the book or teacher that are longer in length and more complicated in their content.

There is a tension between teaching students how to prove and wanting all students to succeed. Historically, it was thought that students should be able to prove new claims, but not all students should be expected to generate new knowledge (Herbst, 2002). After the establishment of the two-column proof in geometry classrooms, students rely on the teacher or text to provide more powerful and complicated theorems. Students are not expected to prove these statements but they need to use these more complicated statements as reasons in their own proofs. The proofs of these more complex theorems give students a chance to see the argumentation that leads to the creation of new knowledge, without students being expected to form these arguments themselves. By presenting these proofs teachers can justify that they are teaching the art of proof.

The high school geometry class has evolved to rest on the assumption that the proofs that generate new knowledge and the proofs that students produce should be similar, but to make the class function, and to have all students succeed, the proofs that students are expected to produce have to be separate from the need to generate new knowledge. According to D. E. Smith, who published a geometry textbook in 1895, as quoted by Herbst (2002, p. 300), "whereas it [is] not reasonable to expect that all students would 'discover new truths,' proving truths stated by somebody else [is] something that all students should be able to do." The difference in goals of mathematicians (to generate new mathematical knowledge and understanding) and that of schools (to show that students have gained the ability to do proofs) is reflected in the role that proofs hold in these different environments.

The instructional situation of 'doing proofs'

The environment of the high school geometry class is very different than the environment that mathematicians work in. The school environment can be studied through the lens of the didactical contract (Brousseau, 1997; Herbst, 2006), along with embedded situations and tasks. The contract provides a framework in which the work that students do on mathematical tasks is valued as fulfillment of mutual responsibility that the teacher and the students have to teach and learn mathematical content, respectively (the task, situation, and contract are described in more detail in "Teachers' Perceptions of Geometry Students," this volume). As I describe below, inherent tensions in this contract around the subject of proof can be seen.

Herbst explains relations between contract, situation, and task as follows:

> The didactical contract constitutes a classroom as a space for trade of work for knowledge. *Situation* and *task* are two constructs that point to things that matter in *how* this trade over knowledge takes place. What thinking a problem can elicit (what kind of task) and which frame actors can use to interpret and value such work (what kind of situation) are objects of negotiation: Participants interact as if

> they were constructing agreed-upon responses to those questions, responses that would let them preserve and fulfill the contract. (Herbst, 2006, p. 319)

The classroom, within which proof and reasoning occur, places constraints on the possibilities of action and interpretation that are available to teachers and students. Work done in the classroom trades for knowledge taught or learned (Herbst, 2004; Herbst & Brach, 2006). Negotiations of what will trade for what claims are often implicit, but they exert strong pressure on what students will do and how teachers will interpret students' work. According to this theory, when two-column proofs are done in high school geometry classrooms within the situation of 'doing proofs' the knowledge at stake is not only (or even at all) the goal of the task, it is to display teacher's and students' ability to teach and learn how to do proofs.

In the geometry classroom, the instructional situation of "doing proofs" is an elaboration of the didactical contract that frames activities of proving geometric claims. This is not an explicit agreement between an individual teacher and her students, but an implicit historical agreement that has been shaped over decades in the geometry classroom (Herbst, 2002). The situation of 'doing proofs' is the marketplace in which work on some tasks is exchanged for claims on learning or teaching 'proof.' Herbst and Brach (2006) lay out the accountability structure for the situation of 'doing proofs.' This accountability structure is an example of how teachers and students shape the activities of teaching and learning 'proof.' The teacher acts as if she is accountable for posing problems with clear statements of what shall be taken as given and what is the statement that is to be proved, as well as providing an accompanying diagram with all of the relevant geometric objects available for inspection. The student acts as if she is responsible for marking known statements on the diagram through various markings and for laying out a sequence of “statements” and “reasons” in the form of a two-column proof.

This description of proof in high school geometry classrooms shows how proofs exist in a web of constraints and affordances based on the interactions between the teacher, the students, the content, and the institutional constraints of schools. These affordances and constraints provide a possible explanation of how and why proof in classrooms is different from proof in the discipline of mathematics.

From this literature on proof in classrooms one sees proof described in three different ways. The first way of describing proofs is the way that proof in classrooms relates to proof in the discipline of mathematics. One sees that proof in classrooms in based on unfounded assumptions about proof in the discipline and that proof in the discipline fulfills functions beyond those that are capitalized on in classrooms. The second way of describing proofs in classrooms is the way students learn to prove, and in particular, difficulties that students face. One sees that students have difficultly understanding the value of proof, and that the persuasiveness of empirical reasoning blocks students learning of how to prove. Finally, the third way of describing proofs is the way that teachers teach proof. One sees the prevalence of the two-column proof and the central role that it plays in allowing teachers and students to show that they have taught and learned ‘proof,’ respectively. Below I look at how proof and the generation of mathematical knowledge are conceptualized in the discipline of mathematics.

The generation of mathematical knowledge: A review of relevant literature

In the following section I describe ways that researchers and philosophers of mathematics have conceptualized the generation of new mathematical knowledge in the discipline of mathematics. These writings attempt to explain how novel mathematical concepts and theories arise from known mathematical concepts and theories. I am interested in this because I would like to situate the current case studies as instances in which the participants are creating new mathematical knowledge. Even though the claims that are arrived at by the participants are not new to the mathematical community, they are new to the participants. The new mathematical knowledge created by the participants in this study also grew out of conversations, in response to authentic questions, so it was not delivered to the participants through an authority figure, but it was generated through their own work.

However, it is important that the participants did not generate this knowledge in isolation. They worked in reaction to animated thought experiments (as described in the methods section) and the researcher guided them. So, I've chosen to focus here on writings about the generation of new mathematical knowledge that pay careful attention to the interplay between the individual and the community in which the individual is working. In this literature review I will not look at writings, like Poincaré's "Mathematical Creation," which focus on the individual aspects of the creation of mathematical knowledge. Poincaré identifies mathematical creation as "the activity in which the human mind seems to take least from the outside world, in which it acts or seems to act only of itself and on itself" (Poincaré, 2000, p. 85). This view largely (if not totally) discounts the role of community in the creation of mathematical knowledge. While this view is useful in some instances, in the current study I am interested in how mathematical knowledge is created in the intersection of individual and social activity. Therefore, I have chosen to focus on writings that acknowledge the importance of "communities of interaction" on knowledge creation.

According to Nonaka (1994), "Although ideas are formed in the minds of individuals, interaction between individuals typically plays a critical role in developing these ideas. That is to say, 'communities of interaction' contribute to the amplification and development of new knowledge" (p. 15). I will follow his use of the term "communities of interaction" because it highlights both the fact that individuals create knowledge in communities that share knowledge and practices, and that within these communities individuals are interacting, or doing things together. This emphasis on interaction, and not only action, places the focus on the dynamics between individuals, not solely the action of each individual.

A similar concept to "communities of interaction" that is widely used is "communities of practice" (Wenger, 1998). "Practice" has a much more specific meaning than "interaction" (Cook & Brown, 1999) and I do not want to argue that the activities that the participants in this study engage in are practices in that sense. I will, however, argue that the activities that the participants in this study engage in can be seen to represent different ways of interacting mathematically that exist in the mathematics community and high school geometry classrooms.

The writings that I am focusing on are *Proof and Refutations* (Lakatos, 1976), which highlights the importance of the dialectic between proofs and refutations in the generation of new mathematical knowledge; *Mathematics as a Cultural System* (Wilder, 1981), which focuses on the role that the mathematical community plays in the generation of new mathematical knowledge; *The Nature of Mathematical Knowledge* (Kitcher, 1984), which argues for the empirical nature of mathematical knowledge; and *The Mangle of Practice* (Pickering, 1995), which discusses disciplinary agency and how it interacts with human agency to generate new mathematical knowledge. Below I give a brief summary of each author's thesis and relate their conceptualization of the generation of mathematical knowledge to the work that the participants do in the current study.

The importance of the dialectic between proofs and refutations

Lakatos' *Proofs and Refutations* presents the reader with a dialogue between a teacher and his students in which they develop some new (to them) mathematical knowledge. This dialogue is crafted to showcase the process through which mathematical knowledge is formed. The students in the class begin by stating a conjecture and collectively they alternate between refuting and refining that conjecture as they simultaneously develop a proof for the conjecture.

The dialogue begins with the teacher asking his students if they have a proof for the claim that "for all polyhedra V - E + F = 2, where V is the number if vertices, E the number of edges and F the number of faces" (Lakatos, 1976, p. 7), known as Euler's formula. Through the course of the dialogue the class develops a definition for polyhedra and a proof for the claim. A significant feature of the conversation that develops is the interplay between students' conjectures and the accompanying refutations of these conjectures. The different characters in the dialogue are able to take up different positions (which are based on the historical development of Euler's formula) and these differences push the argument forward.

In the introduction to the book Lakatos sets up his dialogue as a response to mathematical formalism and the rise of metamathematics. He argues, “in formalist philosophy of mathematics there is no proper place for methodology qua logic of discovery” (Lakatos, 1976, p. 3). Mathematical formalism, which attempts to remove the mathematical objects from mathematics and reduce it to a system of logical deduction using a few axioms, erases the creative process of mathematics leaving only the statements of theorems and the formal proofs used to verify them. The dialogue in *Proofs and Refutations* provides a counterexample to the claim that mathematics is formal mathematics. Lakatos highlights the importance of informal mathematics in mathematical discovery.

Lakatos helps us see the value in the informal mathematics that is done by the participants in the case studies. His dialogue places the importance of mathematical work on discovery and uncertainty. One sees that arguments that do not end in the statement of a correct mathematical claim can still be valuable if the arguer gains insight into the objects or propositions that are being explored.

The role of the mathematical community

Wilder’s primary project is to trace the development of mathematics through the movement of the component subfields of mathematics over the course of history (Wilder, 1981, p. 16). He stresses that he is not engaged in describing a history of mathematical culture, instead he is engaged in describing an evolution of mathematical culture, or “a *process of change*—a process by which various forms and structures change into (‘improved’) forms or structures, and is generally motivated by certain forces whose nature is dependent upon the types of forms or structures involved” (Wilder, 1981, p. 18).

He takes the view that evolution of mathematical subfields is propelled by the generation of new knowledge in these subfields.

The role of the culture is essential to Wilder's vision of the creation of mathematical knowledge. While the actions of an individual may be the catalyst for the discovery, the resources that made that discovery possible are present in the culture. "In each of these cases [of mathematical discovery], the so-called 'inventor' took a critical step in a series of steps leading to his invention; he was totally dependent not only upon ideas he gleaned from others, but, more important, for the *push* to invent which already existed in his culture." (Wilder, 1981, p. 10) That is, each individual contributes ideas to a culture, and even though a single individual who contributes the final idea, or has the thought to tie the other ideas together, is credited with the discovery, this discovery would not have been possible without the support and "push" of the culture.

Both Wilder and Kitcher (discussed below) see the importance of culture in maintaining the continuity of mathematical knowledge. While Kitcher focuses on the role of authorities in transmitting knowledge from one generation to another, Wilder focuses on the growth that occurs between generations. "A new generation does not have to re-do or re-invent concepts which were created by the older generation. Rather, the new generation takes up from where the older left off, and goes on from there. In this way, a culture grows, or *evolves*" (Wilder, 1981, p. 14). It is not only that mathematical knowledge is created, but also it is created *from something* and this something is held by the culture.

Wilder's focus on culture is useful for thinking about the current case studies. The animated scenarios that are used as discussion prompts in this study provide the participants with ideas they can react to and build on. I hypothesize that by having these extra cultural resources the participants are able to do mathematical work and build mathematical arguments that they might not have been able to do in isolation. In the classroom, where students are evaluated on their individual merits, one might interpret this support of cultural resources as "cheating." But in the discipline of mathematics, where colleagues are indispensable resources of the work, this is actually more authentic than working alone. Also, Wilder helps us see the "push" of the culture as an essential part of mathematical discovery. In the case studies the culture that is embedded in the animated scenario provides the participants with the impetus to solve problems and explore questions that are meaningful in the culture.

The "empirical nature" of mathematics

In *The Nature of Mathematical Knowledge* (1984) Kitcher advances the argument that Mathematics is fundamentally an empirical discipline. He argues that Mathematics grew out of humans' perceptual interactions with the world and was developed over time by transmission through authorities and changes to mathematical practice. In Kitcher's view mathematics began with arithmetic. Further, the claims of arithmetic are "true in virtue not of what *we can do* to the world but rather of what *the world* will let us do *to it*" (Kitcher, 1984, p. 108). He goes on to say that it is not necessarily what we can do but what an ideal subject could do while executing ideal operations. The result is an idealized empiricism that combines our interactions with, and observations of, the world with a vision of what these interactions and observations in an ideal world would admit.

In the case studies presented here, it is helpful to value the empirical aspects of mathematical work. Work done in geometry throughout history has been supported by visual aids (Netz, 1998). The participants in this study make extensive use of diagrams and move fluidly between thinking about concepts and thinking about objects represented through diagrams. Kitcher helps to highlight the value of this empirical work and place it in an historical tradition.

The role of disciplinary agency

In relation to the creation of mathematical knowledge, Pickering (1995) focuses on the confrontation of human agency with disciplinary agency. Human agency is understood as the combination of the ability to make choices and then to impose those choices on the world. Pickering proposes disciplinary agency as the answer to the question, "How can the workings of the mind lead itself into problems?" (Weil quoted by Pickering, p. 113) Disciplinary agency takes over when the actor follows the path that has been established by their discipline. In mathematics, disciplinary agency takes hold when mathematicians apply definitions or procedures. A mathematician may decide that it is useful to look at the product of a • (b+c), but it is disciplinary agency that dictates the result is ab + ac.

Mathematical knowledge is created in the intermingling, or mangle, of human agency and disciplinary agency. That is, when humans choose to apply disciplinary agency to new contexts. The result is not the outcome of the will of the mathematician, nor could it have existed without the push of the mathematician. Pickering labels the application of known procedures to new contexts *bridging* and the application of disciplinary agency in these contexts as *transcribing*. The final step of this process is *filling*, or the interpretation of the new results in ways that are meaningful with respect to what is already known.

The current study gives two examples of ways in which human agency can interact with disciplinary agency. In the first case, the participant works in ways that mirror work in the high school geometry classroom (which I assume consists of its own specialized discipline, in particular, the two-column proof format, as described earlier) and in the second case the participant works in ways that are less constrained by this discipline, and closer to authentic mathematical work that might be seen in the discipline of mathematics. As I will show in the results section, the first participant, Maria, who has had exposure to the two-column proof format, constructs arguments that are more efficient and predictable than those of the second participant, Sonia, because Maria acts as if she is led more reliably by the disciplinary agency of the two-column proof.

This review of literature on the generation of mathematical knowledge in the discipline provides a lens with which to look at the participants' arguments and a frame of reference for saying how mathematically authentic (or intellectually honest, à la Bruner) the participants' work is. For the discussion of the results that follow I focus on the writings of Lakatos and Pickering, because these two authors describe important aspects of argument creation that could be useful for understanding the arguments of the participants that are presented in this study. From Lakatos one sees the integral role that the dialectic between proofs and refutations plays in the development of a mathematical idea. Even when the participants create arguments that are not valid, one can still value the process through which these arguments develop. From Pickering one is introduced to disciplinary agency. This idea helps to explain the process through which mathematical processes and human decisions about the application of these mathematical processes combine to result in new mathematical knowledge. Disciplinary agency can be useful for understanding how the participants create their arguments. In particular, one can see what forms of disciplinary agency the participants deploy in their arguments.

Definition of proving

From literature describing proving in classrooms and proving in the discipline one sees two very different definitions of proving. In the classroom, proving is something that is done by individual students in a particular form (the two-column proof), where the statement being proven is provided by the teacher and known to be true. The process of proving in classrooms is reduced to producing a series of statements and reasons that follow from the givens of the proof problem and rules of deduction and reaches a conclusion that is always known in advance.

In the discipline of mathematics, one sees that the statement to be proven is developed simultaneously with the proof. The process of proving is a dialectic between justifying, refuting, and refining claims that is done in a culture that provides support and impetus for the process of proving. One sees that proving has an empirical basis that connects the work of the mathematician to the physical world, and the operations that mathematicians are able to perform in the world. The structure of mathematical work can be seen in the form of disciplinary agency. This agency is not under the control of the mathematician, but is under the control of the discipline of mathematics (determined by the properties that the community of mathematicians ascribes to mathematical objects) and dictates the outcome of the mathematician's actions.

Data

Description of data

The data used in this study were collected in individual meetings with two middle school students, Maria and Sonia, during ten and fifteen one-hour sessions, respectively. In each session the participant watched animated scenarios (see description below) with the researcher, and both participant and researcher paused the animated scenario to ask questions or give reactions. Often the animated scenario would be paused for several minutes while the participant worked on exploring mathematical questions inspired by the animated scenario.

Both participants in this study were successful mathematics students and about to enter ninth grade. The first participant, Maria had taken a geometry course at a local high school during eighth grade, where she had been introduced to two-column proofs. The previous year Maria had taken Algebra 1 in seventh grade, and the year before that she had tested out of her sixth grade mathematics class. The second participant, Sonia would be taking geometry for the first time in ninth grade and had taken Algebra 1 in eighth grade. Both participants were members of their school's MATHCOUNTS® team. These participants were chosen because they enjoyed mathematics and were inclined to spend time outside of school thinking about mathematics. The fact that Maria had taken a high school geometry class and Sonia had not taken a high school geometry class allowed for the possibility to see the influence of their knowledge of the two-column proof format in the arguments they made.

These two participants were chosen because it was expected that they could represent two different ways of doing mathematical work. Maria's experience in taking a high school geometry class meant that she might produce mathematical arguments that reflect a style of argumentation that resembles the two-column proof. Since Sonia had not formally been exposed to the two-column proof or a high school geometry class, but had engaged in a significant amount of mathematical work both in school and in out-of-school programs, it was expected that she might produce mathematical arguments that reflect a style of argumentation that resembles more authentic mathematical work that is informed by experience contemplating mathematical ideas and problems but that is not structured by a particular form or method of doing proofs. This "authentic mathematical work" is also closer to the style of argumentation that is found in the discipline of mathematics (this is similar to AM_D from Weiss et al (2009)).

The participants watched several animated scenarios over the course of the study. In this study I showcase arguments that the participants made in response to the animated scenario, The Square[4], in which an animated teacher and her students consider the question, what can one say about the angle bisectors of a quadrilateral? In particular, they discuss the relationship between diagonals and angle bisectors of a square (for more information about The Square and its use, see the Introduction to this volume). This question was chosen because it provides learners with an opportunity to explore an open-ended question.

Both participants viewed The Square twice. Maria viewed it during her first and last sessions. Sonia viewed it over the course of her first three sessions and during a session

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[4] The Square and the rest of ThEMaT's animated classroom scenarios can be viewed at http://grip.umich.edu/themat.

near the end of the study. Since Sonia was not as familiar with the content of geometry as Maria, it took her longer to watch the animated scenario. Sonia would often stop the animated scenario to work through the mathematics that was being discussed. Maria would watch longer sections of the animated scenario without pausing. As a result Sonia created more arguments than Maria, as will be seen in the next section.

The animated scenarios used in these case studies were developed for a different purpose; for uncovering the practical rationality of experienced high school geometry teachers (Herbst & Chazan, 2003). Although these animated scenarios were not designed for the purpose of supporting students in making mathematical arguments, I argue that the same features that allow teachers to become immersed in the scenarios (Aaron & Herbst, 2007) also support students in work of making arguments as if they were engaged in an instructional scenario. The animated scenarios invite immersion through the graphics that compose the animated scenarios and through the story that is presented in the animated scenario. In terms of the graphics, the characters in the story are represented by 2D-characters, which schematically represent human features (for example, they have hands but their hands are not attached to their bodies and their faces only display mouths when they are speaking). By using non-realistic looking characters in the representations, I hypothesize that the participants are prevented from rejecting the idea that they could be the student in the scenario. Since the characters clearly do not represent any real person, anyone could fill their position.

In terms of the story depicted in the animated scenario, important aspects of the narrative have been omitted, like who the characters are beyond their performance in this scenario, what comes before or after this story in time, and in what school, in what city, in what environment this story took place. All of these missing aspects of the narrative invite the participant to project their own experiences onto the context of the story (Chazan & Herbst, in press; Herbst & Chazan, 2006; Herbst et al., in press). The simplicity of the character set and narrative are balanced by the instructional complexity of the story being told.

Another benefit of the use of the animated scenarios is that they allow the participants access to a mathematical discussion that they can pause, fast-forward, or rewind. This control allows them to examine the discussion in detail, reviewing moments of interest and moving quickly through moments that did not catch their attention. Because the mathematical ideas and concepts displayed in the animated scenario were constructed through the voices of several characters, the participants could more easily follow the positions and opposing views that went into building the discussion.

This data collection prompt and the interview protocol of viewing an animated classroom scenario and asking participants to work through the mathematics presented therein, was employed because it allows access to the arguments that learners might produce in the context of a classroom discussion. This is true for two reasons. First, the data were collected from learners of mathematics who are encountering this mathematical content, as it exists on the edge of their mathematical horizon. Like individuals in the position of student, these participants cannot yet be expected to see the connections between the ideas that they are encountering or connections to other areas of mathematics. Second, the data were collected in the context of virtual classroom discussions. The participants were able to build on the ideas of their virtual classmates; that is, they were able to make sense of the mathematical ideas of others in addition to being accountable for creating those mathematical ideas on their own. If the data had been collected from mathematicians, or from learners in response to a mathematical problem (not embedded in action), the analysis would not be as useful in answering questions about students in instruction.

Sonia viewed The Square on July 8, July 9, July 10, and August 21. Maria viewed The Square on July 7 and August 1. Excerpts from these transcripts in the text are labeled, SA070808, SA070908, SA071008, SA082108, MC070708, and MC080108, respectively. Lines of transcript are numbered according to the turns of talk.

Below I describe the mathematical arguments that are elaborated in The Square and the mathematical work that could be prompted by engaging with The Square. I offer this as a backdrop against which to view the mathematical territory covered and the mathematical argument elaborated by the participants. In particular, the a priori analysis of the arguments elaborated in The Square helps us see the additional mathematical resources that the participants had to work with in addition to the statement of the angle bisectors problem. The analysis of the mathematical work that could be prompted by the square helps us see the ways that the participants do (and do not) engage in mathematical practices that might advance their exploration of the problem.

Map of the argument in the animated scenario

Below I describe the mathematical arguments that the animated students built in The Square. I provide these as a backdrop to the arguments that were elaborated by the participants, which will be described in the results section.

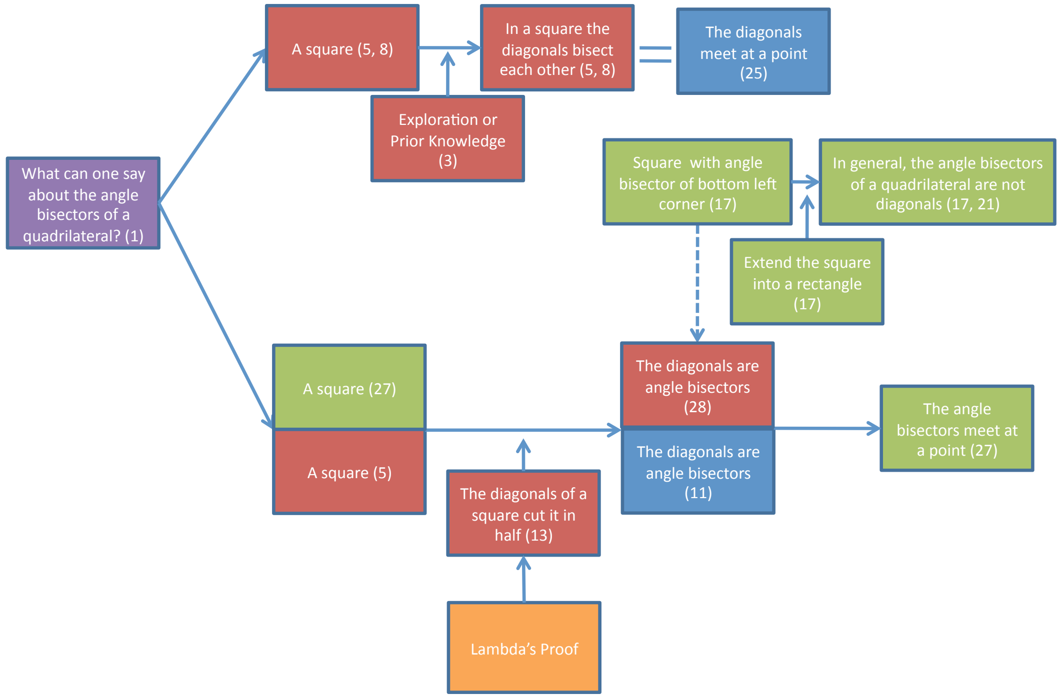

Figure 3: Arguments in the first half of The Square

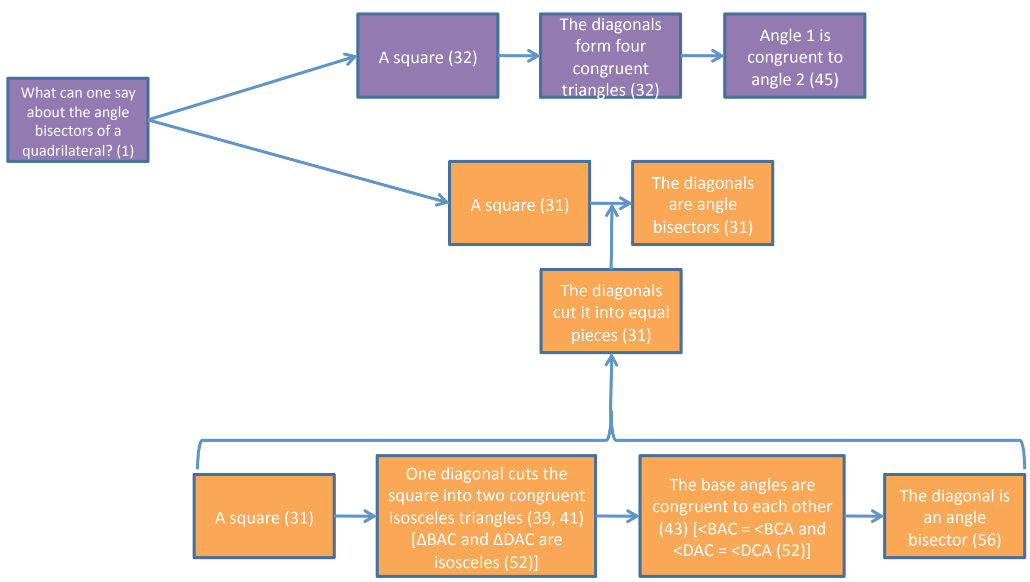

Figure 4: Arguments in the second half of The Square

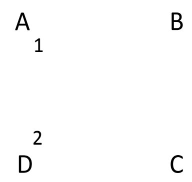

Figure 5: The diagram used by the animated teacher

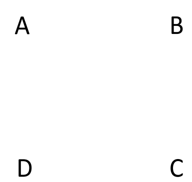

Figure 6: The diagram used by Lambda

In the animated scenario, The Square, there are four distinct arguments. Figures 3 and 4 show these arguments. Each node is color coded to represent the animated student who contributed the node. Nodes are stacked on top of each other when they were contributed by more than one animated student during the course of the animated scenario. Inside each node is a number in parenthesis that represents the turn number from the script of the animated scenario in which the node was contributed. Purple nodes were contributed by the animated teacher; red nodes were contributed by Alpha; blue nodes were contributed by Beta, green nodes were contributed by Gamma, and yellow nodes were contributed by Lambda. In the first half of the animated scenario there are two arguments, collectively elaborated by Alpha, Beta, and Gamma, and Lambda. In the second half of the animated scenario there are two more arguments, elaborated by the teacher and Lambda, respectively.

The first argument, which is created by Alpha and Beta, and can be seen in the top of Figure 3, states that in a square the diagonals bisect each other. In the animated scenario Alpha warrants this by his exploration (or prior knowledge) of a square. Beta rephrased Alpha's conclusion by saying that it is equivalent to the claim that in a square the diagonals meet at a point.

The second argument, which was created by Alpha, Beta, Gamma, and Lambda can also be seen in Figure 3. This argument consists of one main line of argument along with one set of warrants/backings and a rebuttal. The main line of argument begins with the claim that in a square the diagonals are angle bisectors. The node that the angle bisectors are the diagonals is then used to support the conclusion that the angle bisectors meet at a point. Alpha provides a warrant for this first implication by saying that the diagonals of a square cut it in half. Lambda further supports this by providing backing in the form of the proof seen in Figure 4. In response to the node that claims that the diagonals are angle bisectors, Gamma forms a rebuttal that this is not true in general, in particular, it is not true in a rectangle. She bases her claim on the data of a square and its angle bisector from the bottom left corner and based on a transformation of this diagram into a rectangle, concludes that the angle bisector in the bottom left corner of the diagram will not be a diagonal if the diagram were a rectangle.

There are two arguments formed in the second half of the animated scenario. In the second half of the story the animated teacher and Lambda have difficulty communicating with each other and this results in them producing two parallel and disjoint arguments simultaneously (see Figure 4).

The animated teacher's argument claims that in a square the diagonals form four congruent triangles and therefore angle 1 is congruent to angle 2 (see Figure 5). Lambda does not dispute this argument, but builds a different argument, using a different diagram (see Figure 6).

Lambda's argument contains one embedded sub argument. The main argument is that in a square the diagonals are angle bisectors because the diagonals cut the square into equal pieces. Lambda then goes on to support this warrant with an embedded sub argument. He begins by asserting that in a square one diagonal cuts the square into two congruent isosceles triangles ($\Delta BAC \cong \Delta DAC$ in Figure 6). Since these triangles are congruent and isosceles, their base angles are congruent ($\angle BAC \cong \angle BCA$ and $\angle DAC \cong \angle DCA$ in Figure 6). Then, Lambda claims, these angles being congruent implies that the diagonal is an angle bisector. This conclusion, however, does not directly follow because he does not provide the additional claim that the base angles of the two triangles are congruent ($\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$ in Figure 6), which would imply that the diagonals are angle bisectors.

In the sessions with participants, they were asked to view this animated scenario, and to untangle these arguments from the unfolding conversation. As is shown in the results section, the participants reacted to these arguments in different ways and with differing levels of success. The arguments made by the participants all respond to the animated teachers' initial question, "What can one say about the angle bisectors of a quadrilateral?" All sessions with participants were audio recorded and then transcribed and indexed for analysis.

Possible mathematical work done in response to the angle bisectors problem

The mathematical work showcased in The Square is one example of how mathematical work in response to the angle bisectors problem could proceed. Figure 7 shows a map, developed by Herbst (P. Herbst, personal communication, September 28, 2010), which describes the general mathematical moves that one could make in response to the angle bisectors problem. This map provides a backdrop against which to understand the mathematical significance of the work done in The Square, and the work done by the participants in response to The Square.

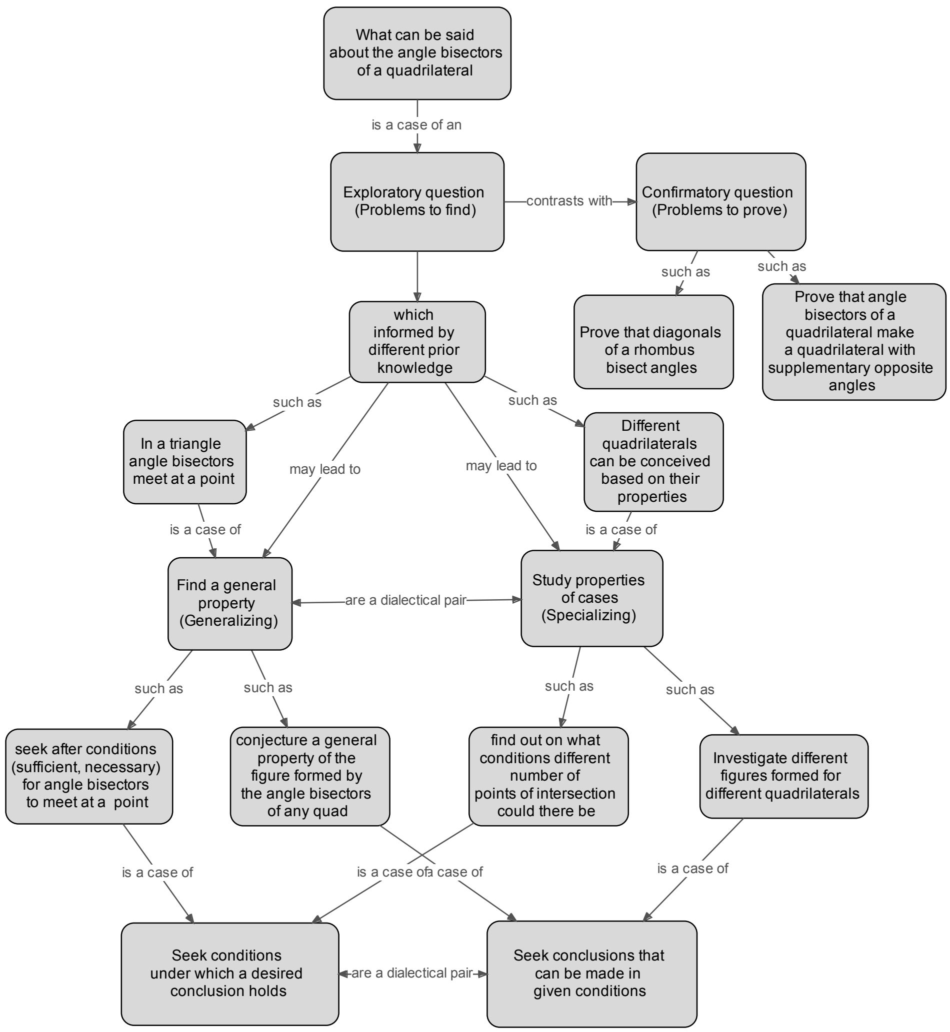

Figure 7: Map of mathematical moves in response to the angle bisectors problem (P. Herbst, personal communication, September 28, 2010)

This map begins with the problem, what can be said about the angle bisectors of a quadrilateral? This problem is classified as an exploratory question; that is, one that prompts the solver to discover some information. Exploratory problems are contrasted with confirmatory questions; that is, problems that prompt the solver to prove that a particular statement is true. From the map, one sees that mathematical work on exploratory problems can be supported by different sets of prior knowledge. In the case of the angle bisectors problem, two examples of useful prior knowledge are the taxonomy of quadrilaterals, or the theorem that the angle bisectors of a triangle meet at a point.

Beyond the use of specific prior knowledge, the problem could be either generalized or specialized. Generalization is the practice of finding a general property that could be used to answer the mathematical problem, while specialization is the practice of studying the properties of special cases. Generalizing in response to the angle bisectors problem could be done in at least two ways; looking for sufficient and/or necessary conditions for the angle bisectors to meet at a point, or conjecturing a general property of the figure formed by the angle bisectors of any quadrilateral. Specializing in response to the angle bisectors problem could also be done in at least two ways; finding out the conditions necessary for the angle bisectors to meet at a specific number of points, or investigating the different figures formed by the angle bisectors of different quadrilaterals.

Within the practices of generalizing and specializing, the solver can either seek conditions under which a desired conclusion holds or seek conditions that could be made in given conditions. Because answering the angle bisectors problem requires the solver to make some statement that consists of conditions and a conclusion, the solver could fix either of these and adjust the other to find interesting statements.

This map of mathematical moves that could be made in response to The Square provides a description of some of the work that a solver could do while working on the angle bisectors problem, and in particular, that the participants might do in response to engaging with The Square. Two important types of moves are the decision to generalize or specialize and the decision to fix either the conditions or the conclusion and to vary the other. The results section will show that Sonia, who had not taken a high school geometry course, and therefore had not been exposed to the two-column format for proofs, was more inclined to make these types of mathematical moves than Maria, who had taken a high school geometry course, and therefore was familiar with the two-column proof format. Maria was more inclined to work as if the angle bisectors problem was a confirmatory problem and attempt to prove that a given statement was true.

Available resources

An important factor in the mathematical work that a learner might do in response to the angle bisectors problem is the physical resources that they have available to them. In this study the participants worked with a researcher who structured the sessions, in terms of setting up the activity of watching the animated scenario, and occasionally pausing the animated scenario to ask questions of the participant. The participants also had plain white paper and pencil available to them. The participants used these to sketch geometric diagrams and write mathematical notes (mathematical questions, proofs, and ideas for proofs). The biggest physical resource that participants interacted with was the computer that played the animated scenario. The participants used the computer to watch the animated scenario but did not use the computer for any other purpose.

A physical resource that is common in many high school geometry classes is dynamic geometry software. This software allows students to use computers or hand-held calculators to quickly construct geometric objects that students can use for exploration and the development of conjectures and proofs. The most powerful function of this software is the ability to "drag" aspects of the construction to instantly visualize an infinite number of related shapes. For instance, using dynamic geometry software, a student could construct a quadrilateral with one right angle (a rectangle) and its angle bisectors, and then "drag" one edge or vertex of the quadrilateral until the quadrilateral was a square. As the student transformed the rectangle into a square they could observe the corresponding change in the angle bisectors (in this case, seeing them transform from four rays whose intersection form a square into two pairs of opposite overlapping rays which intersect in the center of the square). This mode of interacting with geometric objects significantly changes the nature of the mathematical work that students engage in (Hölzl, 1996). Since the participants in this study did not have dynamic geometry software available to them, I do not consider the effect that these resources could have on the mathematical work done by the participants in this study.

Methods

The current case study focuses on two mathematically successful adolescent learners' interactions with an animated scenario from a high school geometry classroom. Each of the learners worked on the same geometric task that was being worked on in the animated scenario. The learners alternated working the mathematics and watching the animated scenario.

The analysis in this case study examines the data to uncover the mathematical arguments that the participants create. These arguments were made in response to mathematical tasks and I am interested in examining the characteristics of these arguments. In particular I'm interested in seeing how the arguments that are made by the participants correspond to the arguments that could be found in a high school geometry class or in the discipline of mathematics.

Figure 8 shows a representation of the data collection methodology. The left side of the figure depicts the thought experiment that the participants engaged in. This thought experiment consisted of an animated instructional scenario that depicted a teacher and her class working on building an argument in response to a mathematical problem. The right side of the figure shows the participant interacting with this thought experiment. The participant and the researcher engaged with the thought experiment by viewing the animated scenario and discussing the mathematics done by the teacher and her class. During a viewing of the animated scenario the participant created new mathematical work in response to the mathematical work done in the animated scenario.

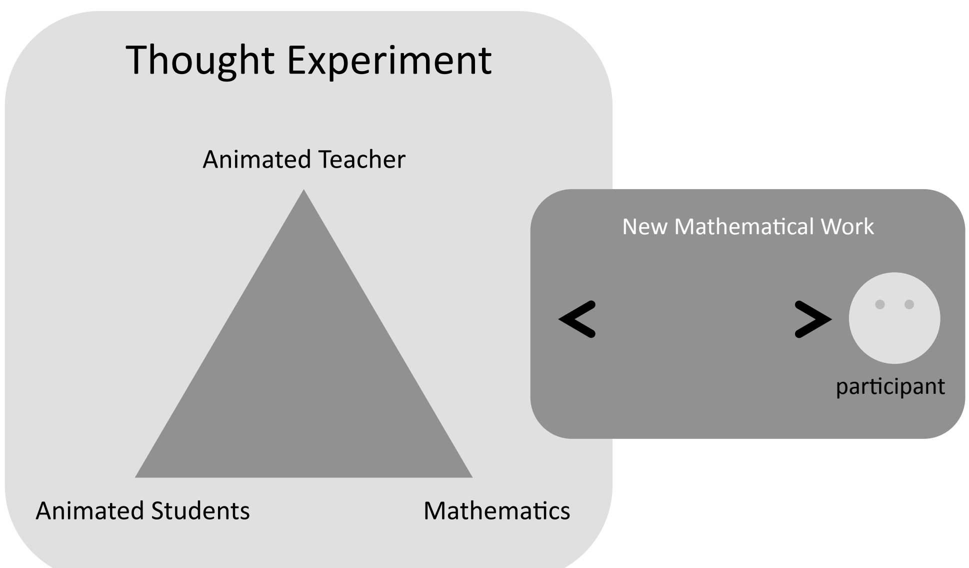

Figure 8: Data collection methodology

Analysis of mathematical arguments

The conversations with participants were analyzed using Toulmin's (1958) argument model (Figure 9). Arguments made by the participants were compared in terms of the characteristics of arguments found in the literature on the creation of new mathematical knowledge in classrooms. Toulmin's method of modeling arguments is a tool for describing the connections made between a set of data and a conclusion. Importantly, connecting these two are a warrant, or reason to believe that the conclusion follows from the data, a backing, or further support for the warrant, a qualifier that conveys the arguer's confidence in the argument, and a rebuttal, or a counter argument that the arguer acknowledges. Not every argument will make use of all these components, but Toulmin claims that all arguments can fit into this form.

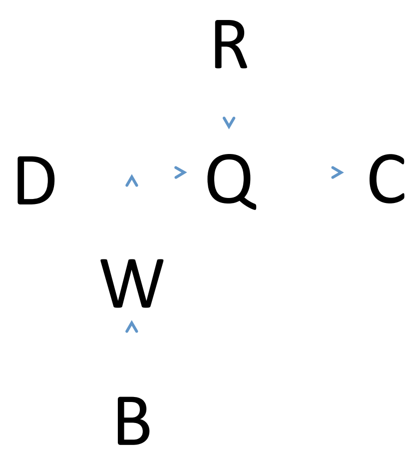

Figure 9: Toulmin's model of arguments, which can be read as "D implies C with probability Q, on the basis of W, supported by B, unless R"

When the case study participants make mathematical arguments, both separately and in tandem with the students in the animated scenario, they are constructing an argument connecting the hypotheses to the conclusion. To analyze the data I search the transcript for components of mathematical arguments. I use Toulmin's argument model as a template and therefore I code utterances as data, conclusions, warrants, backings, qualifiers, or rebuttals. In the results section I provide maps for each of the arguments made by the participants. When needed, a geometric diagram related to the argument is also provided.

Toulmin's argument model is useful for mapping arguments that are rational, in the sense that there is reasoning done in their creation, but that are not strictly deductive. Since the warrants and backings for arguments could come from any field, the argumentation could be based on any warrant that is appropriate for the arguer. In particular, it allows for the justification of geometric claims on the basis of intuition, exploration, or empirical observation.

Another feature of Toulmin's argument model is that it does not display the temporal sequence or duration of the process of creating the argument. Once the argument has been recorded its nodes can be read in any order. This lack of temporality takes the actions out of temporal sequence and allows the examination of their substantive relationships. As with any simplification, this highlights some aspects and hides others. Arguments recorded using Toulmin's model can be easily compared to each other in terms of overall structure and content of nodes. This model for recording arguments hides information about how long it took to build the argument or the process through which the nodes were created. In this study this is useful because it emphasizes the mathematical characteristics of the argument.

Earlier uses of Toulmin's argument model

Toulmin's model has been used before in studies that look at students' argumentation in mathematics classes (Krummheuer, 1995, 1998, 2000; Pedemonte, 2003, 2007; Wood, 1999; Yackel, 2001; Yackel & Rasmussen, 2002). These applications of Toulmin's model have been used to study whole group or small group discussions, not arguments produced by a single student, as is done in the current study. However, the participants in this study are able to use statements from the virtual students in the animated scenario as resources in their arguments. Instead of the argument being elaborated by a group of students, led by a teacher, as in collective argumentation (Krummheuer, 1992; Miller, 1986, 1987), the argument is elaborated by a single participant in interactions with the animated scenario. The knowledge that is embedded in the animated scenario, in the form of utterances of the animated students, is available to the participants, and allows them to build virtually collective arguments. These virtually collective contexts provide learners with collective resources, but result in arguments that more clearly reflect the thinking of one individual.

Coding of the arguments

The role of discourse in mediating activity cannot be underestimated. Moreover, some activities are not just mediated by discourse, but discourse constitutes the activity. The data presented in this study is an example of this type of discourse. The participants deploy other recourses, like gestures and diagrams, to communicate, but the activity of elaborating arguments is primarily a discursively constituted activity. That is, the operations performed by the participants, in concert with the animated scenario and the researcher, are performed through language.

So, then, to study the operations that the participants perform it is necessary to look at the linguistic resources that they use to construct those operations. The examination of these linguistic resources is supported by the claim of Bakhtin (1986), that "even in the most free, the most unconstrained conversation, we cast our speech in definite generic forms" (p. 78). Following this, I claim that the participants use language in predictable ways to construct arguments. Below I describe the linguistic resources that are used as markers to find the arguments in the transcripts of the participants' discussion of The Square. This analysis is based on Systemic Functional Linguistics, a theory for looking at language a primarily a tool for making meaning (Halliday, 1994).

The transcripts were coded for linguistic markers that pointed to various parts of Toulmin's argument model according to the following scheme that I developed. In particular, conjunction and process analysis (Martin & Rose, 2003) were useful in uncovering the structure of the participants' arguments in the transcripts. Conjunction analysis is concerned with the connections between clauses that describe how the clauses are related to each other. For example, the conjunction "because" often points to a causal relationship between two clauses. The preceding clause is likely a conclusion and the following clause is likely data. Process analysis is concerned with the actions that are visible in the transcripts and can describe how objects or actors relate to each other. For example, the process "proves" often points to the fact that the related mathematical statement will be the conclusion of an argument.

Sections of transcript were designated as arguments if they contained a mathematical implication (see Table 1). These are often marked by conjunctions like "because," "so," "since," "if," "then," or "when" (Martin & Rose, 2003). These conjunctions convey a causal relationship between clauses. In the case of “because” the clause before the conjunction is the conclusion of the implication and the clause after the conjunction is the data. For the conjunction “so” the position of the clauses is reversed.

Once it has been determined that a portion of transcript contains an implication the pieces of the argument need to be found. The markers for data and conclusion, the end points of an implication, include all of the markers of implications. In addition to the markers mentioned above, phrases like “I know,” “from here,” and “prove” are also markers. “I know” could indicate either a data or conclusion, while “from here” and “prove,” indicate data and conclusion, respectively.

The warrant and backing are found with the same markers because they have very similar functions in the argument model. They each provide a justification for the data implying the conclusion. In Toulmin’s model the backing is a more general rule than the warrant. The markers for the warrant and the backing could be similar to the markers for data, and conclusion, but the form of the warrant and backing is theoretically different from the form of the data and conclusion. That is, theoretically, while the data and conclusion are both simple statements, the warrant and backing are both composite propositions. However, in practice, arguers do not always spell out both the hypotheses and conclusion of the warrant and backing; these are often elided in speech.

Qualifiers are words or phrases that increase or decrease the modality of a statement. Common markers for qualifiers are “maybe,” “might,” and “could.” Qualifiers are different than the other nodes in an argument because they are not statements or propositions. Instead they are words or phrases that modify the force of the implication.

In recording the arguments of the participants I was not concerned with recording the qualifiers of their arguments because these are non-essential parts of the argument and they do not provide any mathematical content.

Rebuttals are marked with conjunctions like “unless,” “but” or processes that contain marks for negation, like “didn’t cross,” “don’t make a square,” “wouldn’t be true.” The conjunctions “unless” and “but” connect a conclusion to a rebuttal. These conjectures signal that whatever follows it is contrary to what came before. The rebuttal of an argument is in opposition to the conclusion. Words like “didn’t,” “don’t,” and “wouldn’t” could also mark phrases that are in opposition to the conclusion.

Table 1: Markers for parts of an argument

Part of argument	**Linguistic Markers**
Implication	*Because, so, since, if, then, when, and…*
Data	*Because, so, since, if, then, when, I/we/you know, from there, from here, and…*
Conclusion	*Because, so, since, if, then, when, and, I/we/you know, proved, discovered…*
Warrant	*Because, so, since…*
Backing	*Because, so, since…*
Qualifier	*Maybe, might, could be…*

Rebuttal	*Unless, but, not, can't, didn't, doesn't, wouldn't…*

The circumstances of clauses, or the information about the setting for the rest of the information to be conveyed by the sentence (Martin & Rose, 2003), also contain markers about which parts of the argument the sentence contains. If the circumstance sets the sentence temporally in the past it marks the following clause as data or warrant because it is implying that that information was known before the argument was formed. Similarly, if the circumstance places the sentence in the future that implies that the following clause contains a conclusion because it was not known until after the argument had been made.

Another method for coding parts of the arguments provided by the participants was to notice the question of the researcher. During the conversation the researcher would prompt the participant to fill in a particular part of the argument. For instance the researcher might ask, "what would you draw?" referring to the quadrilateral that the participant was basing her argument on. If the participant responded, "square" then this would be coded as data. Similarly, if the researcher asked, "why do you think [the conclusion]?" the reply from the participant would be coded as warrant.

For arguments where the transcript is especially complicated I include a table that shows the coding for each piece of the argument. In these tables, implications are labeled with "←" or "→" depending on the direction of the implication and in arguments that contain multiple implications I include a number to show where the implication falls in a series of implications. Each part of the argument is labeled by its name, as well as a number that corresponds to which implication this part of the argument is related to. This is because a statement may be used as a conclusion in one implication and as data in the following implication.

Analysis of mathematical territory

To examine the mathematical territory covered by the arguments elaborated by each participant, I map the mathematical territory covered by her arguments. This analysis highlights the mathematical content of the nodes of the participants' arguments and allows for a comparison between the mathematical content explored by each participant.

To perform this analysis, for each participant, I recorded the data and conclusion of every argument that she made and then arranged them in an array that collected similar data and conclusions, linking the conclusions to the data from which the participant deduced them. Valid implications are recorded with solid arrows and invalid implications are recorded with dashed arrows. The center of each array is the question, "what can one say about the angle bisectors of a quadrilateral?", surrounded by all the cases that participants used to examine this question, represented by shaded shapes. The outer ring of the array shows the conclusion that each participant arrived at based on each case. These conclusions are represented by white ovals. If the participant arrived at the same conclusion using two different sets of conditions, the conclusion was only recorded once, with an arrow connecting it to each set of conditions.

The methods described here are used to code the transcripts of the discussion between the participants and the researcher. These codes yield maps of arguments that allow for analysis of the participants' mathematical work and comparison of their work to the work of proving, as it exists in high school geometry classrooms and in the discipline of mathematics. It also allows for a mapping of the mathematical territory covered by these arguments. Below I describe the arguments that were elaborated by each participant, including a discussion of the validity of those arguments, and I describe the mathematical territory that was covered by the arguments elaborate by each participant.

Results

In the following section I describe the results of this analysis. The results are divided into two sections. The first section describes the arguments that were elaborated by the participants in response to the animated scenario and the second section describes the mathematical territory that was coved by the arguments elaborated by each participant. From the first set of results one can see that the style of argumentation employed by Maria, who had taken a high school geometry class, reflect the structure of the two-column proof, while the style of argumentation employed by Sonia, who had not taken a high school geometry class, reflects features of argumentation that might be found in the discipline of mathematics. From the second set of results one sees that Maria covered less mathematical territory than Sonia, although the arguments that Sonia used to cover this territory were not necessarily valid.

Maps of arguments

In the following section I describe the arguments that were made by each of the participants. The arguments created by the two participants are substantially different. From the descriptions of the creation of mathematical knowledge as it exists in schools, and especially the process of proof employing the two-column proof form, one can see that the participants' arguments are different from each other. These differences support the claim that Maria's arguments would fit well into the system of proving using two-column proofs that exists in classrooms, while Sonia's arguments would not be supported by this form of proof. In addition, one sees that Maria's arguments are able to connect data and conclusions using strings of implications and Maria's arguments are, in general, more likely to be valid than Sonia's.

In the argument maps below the solid rectangles correspond to pieces of the argument that were made by animated students and then referenced by the participant. Rectangles that are outlined are pieces of the argument that were produced by Maria or Sonia. Nodes that are crossed out are claims that the participants made and then later rejected. Nodes are labeled with the turn number from the transcript that they come from. In the description of the arguments quotes from transcripts are referenced as (II, DDDDD, TN) where II are the initials of the participant's pseudonym, DDDDDD is the date on which the conversation took place, and TN is the turn number of the transcript from which the quote was taken. Sonia's initials are SA and Maria's initials are MC. In the quotes from the transcript markers for pieces of the argument are recorded in italics.

Maria's arguments

The following section describes maps of arguments that were elaborated by Maria. Maria elaborated six arguments, over two viewings of the square. For each argument I describe the elaboration of the argument as stated by Maria and discuss the validity of each argument.

Maria's conjectures in response to the angle bisectors problem

Figure 10 shows maps of two arguments elaborated by Maria, who had taken a high school geometry course. These arguments were made during the first session with Maria. After seeing the animated teacher pose the problem, "What can one say about the angle bisectors of a quadrilateral?" the animated scenario was paused and Maria spent some time working on the angle bisectors problem. The following argument map (see Figure 10) displays the conjectures she made and the ways that she justified those conjectures.

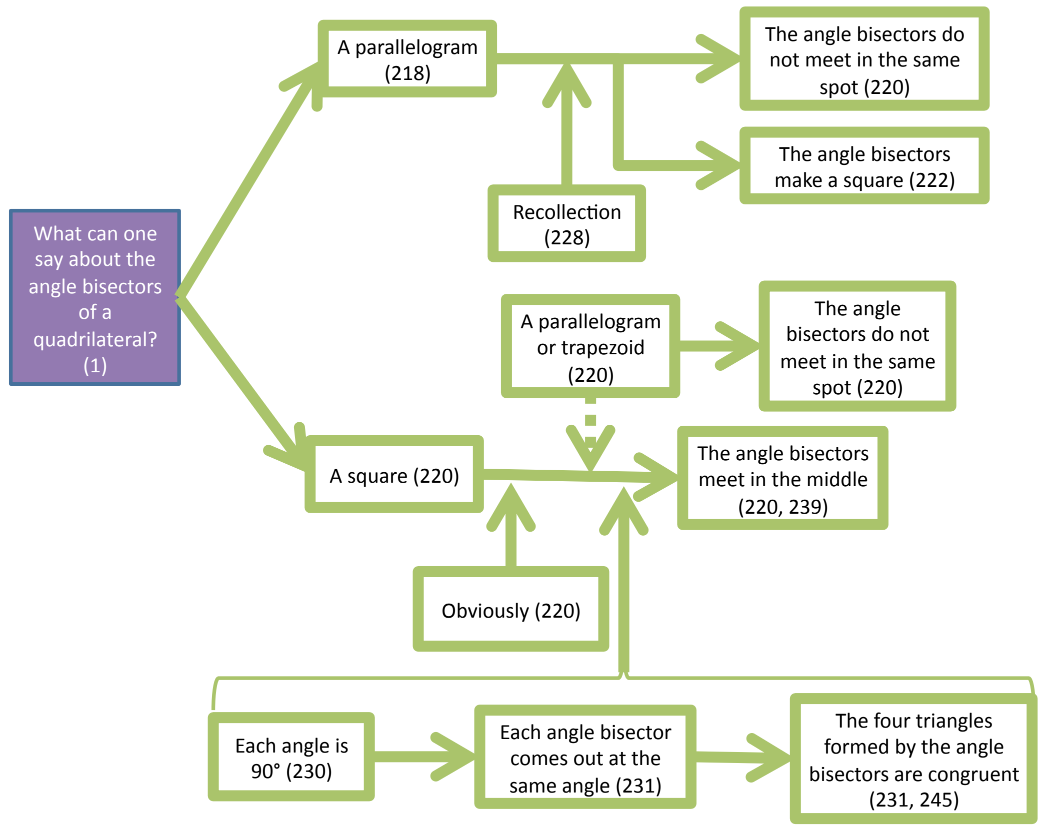

Figure 10: Maria's conjectures in response to the angle bisectors problem

The first conjecture looks at the case of a parallelogram. Maria made the claim that in a parallelogram the angle bisectors do not meet at a point and that the angle bisectors make a square. Maria said, "I *would, like, draw* a quadrilateral, not necessarily a square but…maybe a parallelogram" (MC070708, 216-218). The process "would, like, draw" marks "a quadrilateral" as the data for the argument. Maria further refined this by saying that in particular, she would draw a parallelogram. Maria went on to say, "*with a parallelogram…* [the intersection of the angle bisectors] wouldn't be in the same spot" (MC070708, 220) and "well, *the parallelogram*, the angle bisectors would form, I think, a square" (MC070708, 222). In the first sentence "with a parallelogram" sets up the circumstances of the sentence, which marks the data of the argument. The clause that follows marks the conclusion. In the second sentence, the circumstances, again, mark "the parallelogram" as the data and what follows is the conclusion. These comments from Maria set up to parallel arguments both beginning with the data "parallelogram." When asked to justify these claims she responded, "I think there's been a proof about it, but I can't exactly remember. But they won't meet in the same point" (MC070708, 228). This warrant is marked by the researcher's question that preceded it, "why do you think that? You can just imagine that in your head?"

One sees that the first of these conjectures, that the angle bisectors of a parallelogram do not meet at a point, is true, however, the second, that the angle bisectors of a parallelogram make a square, is false. The second conjecture is only true if the parallelogram is a rectangle.

Maria's next conjecture is supported by a more elaborate argument. The basic conjecture is that in a square the angle bisectors meet in the middle. Maria said, "Because, well, the square, it [the intersection of the angle bisectors] would obviously be in the middle" (MC070708, 220). "The square" is the circumstance of this sentence and this marks it as the data of the argument. The following phrase "it would obviously be in the middle" is marked as the conclusion by the fact that this is the assertion of the sentence.

Even though Maria's phrasing is very informal, this basic implication, that the angle bisectors of a square "meet in the middle" is true if one interprets "meet in the middle" to mean that the angle bisectors meet at the center of the square. Maria's justification for this implication is also very informal, but it seems to point toward the symmetry of the square, which supports the claim that the angle bisectors of a square meet at the center of the square's rotational symmetry.

In response to the researcher's prompt, "how do you think you would prove that?" Maria provided the following chain of reasoning; each angle in the square is 90°, so each angle bisector comes from the vertex at the same angle, so the four triangles formed by the angle bisectors are congruent. Maria said, "*I'd start off* with 'in a square *there'd* be 90° angles'…*and then*… every line is coming out of every angle at the, like, same angle, which would eventually, like, after many theorems, you *could prove* that the four triangles that the lines make are congruent" (MC070708, 231). There are several markers that allow us to parse the argument from this sentence. First, the phrase "I'd start off with" marks "in a square" as the data because in time it is prior to the rest of the argument. Next "there'd be" and "and then" mark "ninety degree angles" and "every line is coming out of every angle at, like, the same angles," respectively, as the conclusions of implications. The "and then" also marks "every line is coming out of every angle at the, like, the same angle" as the data for the second implication. The process "could prove" marks the following clause, "the four triangles that the lines make are congruent" as the conclusion to the argument. It is unclear what Maria meant by "after many theorems." Theorems are usually used as warrants, but Maria is not specific about which implications she would warrant with which theorems (See Table 2 below).

Table 2: Maria's elaboration (1)

Well *I'd start off* with	Marks data

'in a square *there'd be* ninety degree angles'…	Data (1)
and then…	→ (1)
every line is coming out of every angle at the, like, the same angle,	Conclusion (1) Data (2)
which would eventually, like, after many theorems,	
you *could prove* that	→ (2)
four triangles that the lines make are congruent.	Conclusion (2)

This argument that Maria embedded in the warrant of her main implication begins with a true premise, that each vertex of the square is 90°, and ends with a true conclusion, that the four triangles formed by the angle bisectors are congruent. However, the middle node, that each angle bisector comes out at the same angle is unclear. Also, since this node only discusses angles, it would not be sufficient to make the claim that the triangles are congruent. This claim would require some information about the sides of the congruent triangles. It is true that the sides of the triangles are congruent because they share sides with the square, but Maria did not include this in her argument.

Also missing from Maria's argument is a reason why this warrant supports the main obligation. It is not immediately obvious how the assertion that the angle bisectors form four congruent triangles implies that they meet in the center of the square. This warrant is not incorrect, but it incomplete.

In addition to this warrant of the main implication, Maria also provided a rebuttal. She said, “the square, it would obviously be in the middle, *but with a* parallelogram or a trapezoid *it wouldn’t exactly, like always*— [the intersection of the angle bisectors] wouldn’t be in the same spot” (MC070708, 220). The conjunction “but” marks the clause that follows it as a rebuttal to the clause that preceded it. Maria did not provide a warrant for this implication. The rebuttal is in the form of an implication. “With a” marks the data of this implication and “it wouldn’t exactly, like, always” marks the conclusion. The rebuttal that Maria provided, that the angle bisectors of a parallelogram and a trapezoid do not meet at a point, is true.

Maria’s arguments in response to Alpha’s comments

Maria elaborated the arguments in Figure 11 after watching Alpha present his conjecture and Beta refine that conjecture. The animated scenario was paused right before Gamma made her intervention. The researcher asked Maria, “So what do you think about that? So what do you think about Alpha’s statement?” Maria responded by creating the following arguments (see Figure 11).

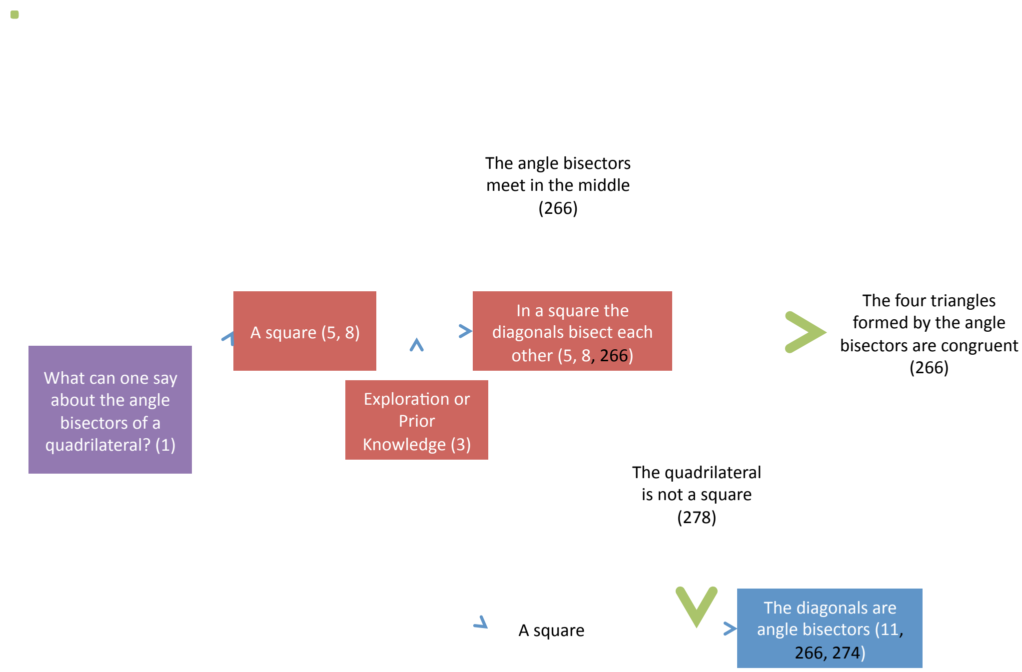

Figure 11: Maria's arguments in response to Alpha's comments

Maria's first move was to equate Alpha's claim, that the angle bisectors bisect each other, with her claim that in a square the angle bisectors meet in the middle. She said, "I guess *[Alpha] kinda has the idea* that the point meets in the middle…like, the diagonals bisect each other and that's something that *could be important in a proof*" (MC070708, 266). Here the phrase, "[Alpha] kinda has the idea" marks that the clause that follows as something that Maria attributes to Alpha. Maria went on to say, "that's basically what the angle bisectors do, *like* the diagonals bisect each other" (MC070708, 266). The word "like" equates Maria's statement that comes before, "the angle bisectors meet in the middle" to Alpha's statement that follows, "the diagonals bisect each other." Maria also said, "could be important in a proof" which marks "the diagonals bisect each other" as data in the argument. Maria then claimed that the four triangles formed by the diagonals are congruent. Maria said, "*from there you can go*, um, the triangles are all congruent" (MC070708, 266). "From there you can go" marks the clause that follows, "the triangles are all congruent" as a conclusion of the implication.

Here Maria reiterated two claims that she made in earlier arguments; that the angle bisector of the square "meet in the middle" and that the four triangles formed by angle bisectors are congruent. Again, both of these claims are true, although the arguments that Maria elaborated to support these claims are incomplete. Maria equated the first claim, that the angle bisectors "meet in the middle" with the claim that the diagonals bisect each other. Insofar as the diagonals of a square are the angle bisectors, and "in the middle" means "equidistant from the each vertex," this statement is true.

Maria built a separate argument by remembering what Beta said in the animated scenario, that the diagonals and the angle bisectors are the same. Maria said, "it's like the student in the back said, it's technically the same in my opinion" (MC070708, 266). Later Maria said, "the diagonals are the angle bisectors" (MC070708, 274). In both of these statements the data that she is discussing, a square, is implicit. The conclusion is that "it's technically the same" or "the diagonals are the angle bisectors." Maria completes the argument by adding the rebuttal, that the four triangles formed by the angle bisectors are only congruent in a square, not other quadrilaterals. Maria said, "Although, of course, they're not in the other quadrilaterals, but in a square they are" (MC070708, 278). "Although" marks the clause that follows as being in opposition to the previous claim, "that the diagonals are angle bisectors."

The implication of this argument is true, however, the rebuttal is incomplete. Maria asserted that the angle bisectors of quadrilaterals besides squares are not diagonals. In many cases this is true, for instance, in rectangles or trapezoids, but there are also quadrilaterals, such as rhombi, where the angle bisectors are diagonals. The power of this argument is in the observation that although the diagonals of a square are also angle bisectors, this is not always true.

Maria's argument in response to Lambda's comments (1)

Following viewing the first half of Lambda's description of his proof, Maria constructed the following argument parallel to Lambda's (see Figure 12). She began this argument with the conclusion and built up to the initial data. Figure 13 shows the corresponding geometric diagram.

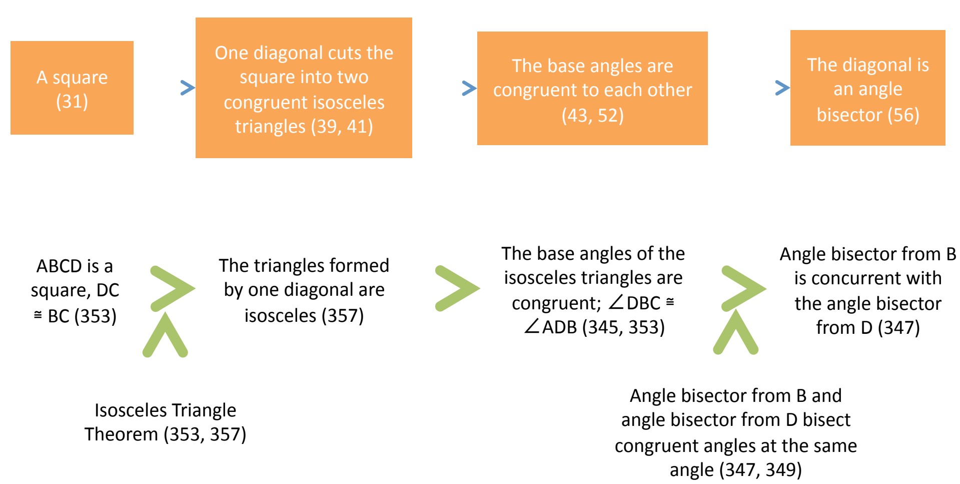

Figure 12: Maria's argument in response to Lambda's comments (1)

A B

D C

Figure 13: Diagram accompanying Lambda's comments

Maria built this argument in two stages. She began with the third implication, then she finished the argument by constructing the first implication. Maria elaborated the third implication to explain what Lambda argued in the animated scenario. The researcher asked Maria, "What do you think Lambda's talking about?" Maria replied, "I think *he's talking* about how angles DBC and ADB are congruent…*So* the angle bisector of D and B are both the same line *because* they both intersect the angles at the same angle" (MC070708, 343-347). "So" in the middle of this sentence marks the implication in this sentence with the data "angles DBC and ADB are congruent" and the conclusion is "the angle bisector of D and B are both the same line." Maria also gave a warrant, "they both intersect the angles at the same angle," marked by "because" before the clause.

Table 3: Maria's elaboration (2)

I think *he's talking* about how	Equate to Lambda's statement
angles DBC and ADB are congruent…	Data (3)
So	→ (3)
the angle bisector of D and B are both the same line	Conclusion (3)

because	
they both intersect the angles at the same angle	Warrant (3)

The data that and the conclusion that Maria elaborated for this implication are true, however the warrant that she provided is not sufficient to justify that the data imply the conclusion. She seemed to be taking for granted the claim that the angle bisector of B goes through D, and the angle bisector of D goes through B. This is understandable given that she was working with a diagram in which these objects (the angle bisectors and the opposite vertex) are concurrent. Maria also took for granted that she is working with a square. This assumption is made explicit when she constructed the first implication in this argument.

Next Maria built the first and second implication of this argument. The researcher asked her "Do you think that's [the argument just described] is what Lambda's saying, too?" Maria replied, "Yeah, *because* he said the base vertices of the isosceles [triangle] are congruent *because*, by, like, if you know the theorem for the isosceles triangle, like *you already have proven* that DC and BC are congruent *so* you know it's isosceles" (MC070708, 353). Here the two uses of "because" mark the two implications. The first "because" marks the previous argument as the conclusion of the current implication. The last implication begins with the data, "angles DCB and ADB are congruent," and these are the base vertices of the isosceles triangle that she referenced at the beginning of this utterance. The second "because" marks the second implication in the series. It marks the preceding clause, "the base vertices of the isosceles [triangle] are congruent" as the conclusion and "the theorem for isosceles triangle" as the warrant that justifies this conclusion. The clause following this "because" is a warrant, and not data, because it is a theorem and therefore in the form of a compound proposition. The phrase "you already have proven" marks the clause "DC and BC are congruent" as a statement that was known prior to the argument, and therefore the data of the implication. The final "so" in this sentence marks a reinforcement of the first implication, with the conclusion "it's isosceles" following from the data "DC and BC are congruent."

Table 4: Maria's elaboration (3)

Yeah,	
'*cause* he said	← (2)
the base of these isosceles the base uh vertices of the isosceles are congruent	Conclusion (2)
because by like if you know	← (2)
the theorem for the isosceles triangle	Warrant (1)
like you already have *proven* that	
DC and BC are congruent	Data (1)
So	→ (1)
you know	
it's isosceles.	Conclusion (1)

The first conclusion, that asserts that the triangles formed by one diagonal of the square are isosceles, is true based on the data, that the sides of the triangle are congruent. However, it is incorrectly warranted by the isosceles triangle theorem[5]. Rather, it would be more appropriately warranted by the definition of isosceles triangle[6]. The conclusion of the second implication, that the base angles of the isosceles triangles are congruent, could be seen to follow from the data, that the triangles formed by one diagonal are isosceles and that the figure is a square, but this implication is incomplete because Maria did not provide a warrant. A reasonable warrant would be to claim that the triangles are congruent, in addition to being isosceles, and therefore the base angles of the two triangles are congruent because of CPCTC[7].

Maria's argument in response to Lambda's comments (2)

The arguments that are mapped in Figure 14 were made in the beginning of Maria's second viewing of The Square. After watching the whole animated scenario of The Square the researcher asked Maria, "What about [Lambda's] argument? Did you follow his argument?" Maria responded by outlining the following argument (see Figure 14).

[5] The Isosceles Triangle Theorem states that if two sides of a triangle are congruent then the angles opposite those sides are also congruent.

[6] This definition of an Isosceles Triangle is a triangle with at least two congruent sides.

[7] CPCTC is an abbreviation for the theorem "corresponding parts of congruent triangles are congruent."

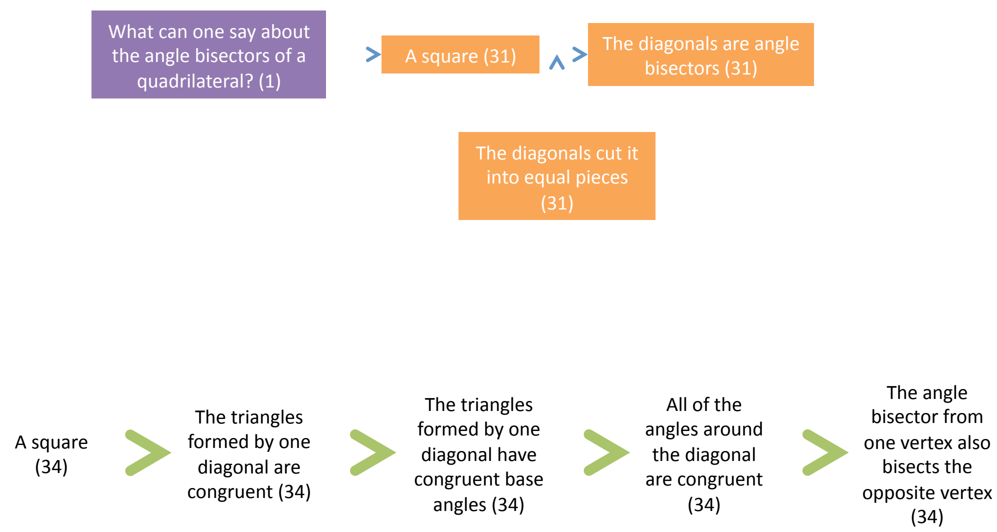

Figure 14: Maria's argument in response to the Lambda's comments (2)

Maria's argument began implicitly with the data that the quadrilateral is a square. She then claimed that the triangles formed by one diagonal are congruent. Instead of asserting that the triangles are isosceles, Maria directly claimed that the base angles of these congruent triangles are congruent. This assertion about congruent base angles was used to support the claim that the diagonal is surrounded by congruent angles. Maria then made her conclusion, that the angle bisector from one vertex also bisects the opposite angle. Maria said, "[Lambda] *proved* that the diagonals cut the square into two congruent triangles *and* they had congruent base angles *so* all of the angles around the diagonal were congruent *making it* the angle bisector of the opposite angle" (MC080108, 34). There are four implications in this sentence, marked by the processes "proved" and "making" and the conjunctions "and" and "so." These markers point to "the diagonals cut the square into two congruent triangles" as the conclusion for the first implication and the data for the second implication. "They had congruent base angles" is marked as the conclusion of the second implication and the data for the third implication by "and" and "so." "All of the angles around the diagonal were congruent" is marked as the conclusion of the third implication and the data for the fourth implication by "so" and "making it." Finally, " [the diagonal is] the angle bisector of the opposite angle" is marked as the conclusion of the fourth implication by "making it."

Table 5: Maria's elaboration (4)

[Lambda] *proved* that	→ (1)
the diagonals cut the square into two congruent triangles	Conclusion (1) Data (2)
and	→ (2)
they had congruent base angles	Conclusion (2) Data (3)

so	→ (3)
all of the angles around the diagonal were congruent	Conclusion (3) Data (4)
making it	→ (4)
the angle bisector of the opposite angle	Conclusion (4)

This argument, although it contains no warrants, is a nearly valid argument for the claim that the diagonals of a square are angle bisectors. The last implication would be better formed if it contained a justification for the switch from diagonals (which are the subject of the previous nodes) to angle bisectors (which are the subject of the final node). The first implication could be warranted by the side-side-side theorem of triangle congruence[8], the second implication could be warranted by CPCTC, the third implication is a restatement of the data to better position the fourth implication, and the fourth implication could be warranted by the definition of angle bisector.

From looking at these arguments one can see that Maria's arguments contain few rebuttals and backings, and that she often stung together several implications in a row to form a more complicated argument. As I explain in the discussion section, these characteristics of her arguments set them apart from Sonia's and they cause them to look similar to arguments that could be found in a high school geometry classroom. Below I

[8] The side-side-side theorem of triangle congruence states that if two triangles have three pairs of corresponding sides congruent then the triangles are congruent.

describe the arguments that were generated from conversations with the second participant in the study, Sonia, who had not taken a high school geometry class. Sonia also watched The Square twice, and her conversations with the researcher generated thirteen arguments. As can be seen below, Sonia's arguments do not share many characteristics with proofs that could be generated in a high school geometry classroom.

Sonia's arguments

The following section describes maps of arguments that were elaborated by Sonia. Sonia elaborated thirteen arguments, over two viewings of The Square. For each argument I describe the elaboration of the argument as stated by Sonia and discuss the validity of each argument.

Sonia's arguments in response to the angle bisectors problem (1)

Sonia watched the animated scenario until the moment when Alpha makes the conjecture that "about the square, they would have to bisect each other." Sonia heard Alpha say this and responded, "that's what I've been thinking, too" (SA070808, 445). The researcher asked Sonia, "How do you know, though?" and Sonia elaborated the argument that is mapped in Figure 15 in an attempt to show why the angle bisectors would bisect each other.

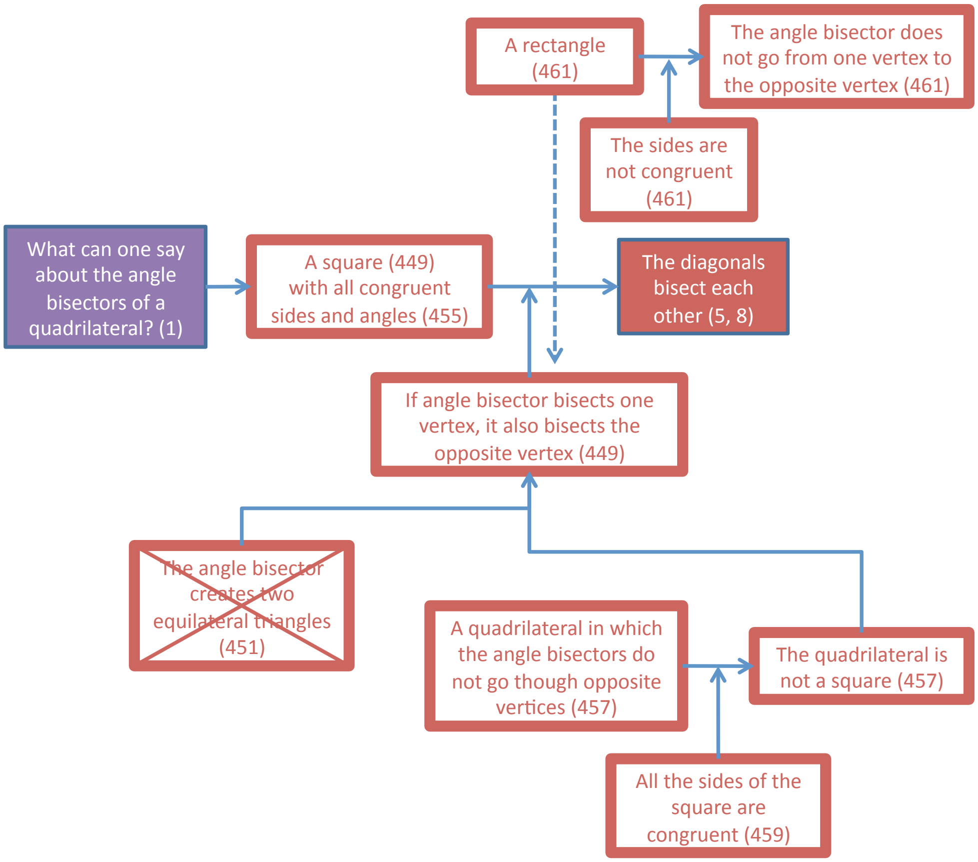

Figure 15: Sonia's arguments in response to the angle bisectors problem (1)

For data, Sonia used a square and she added that the square has all congruent sides and angles. Sonia said, "'*Because* it's a square" (SA070808, 449). The marker "because" tells us that "it's a square" is used as data in her argument. At the prompting of the researcher she went on to say the properties of a square, "all of [the square's] sides are the same and all of the angles are congruent" (SA070808, 455). Both of these properties follow from the definition of a square. Sonia asserted that these data imply Alpha's conjecture, that the diagonals bisect each other, based on the warrant that each angle bisector goes from each vertex of the square to the opposite vertex. Sonia said, "*because* you go from one corner to the other corner, and, like, across from it" (SA070808, 449).

Here Sonia used the marker "because" again, here one can infer that this is the warrant for her argument.

It is true that the data, a square, could support the conclusion that the diagonals bisect each other, as Sonia's argument asserts, however, the warrant for this implication does not involve angle bisectors. Sonia's use of this warrant points to the fact that she may have misinterpreted Alpha's conjecture "they bisect each other" to be a statement about angle bisectors instead of segment bisectors.

Sonia said, as backing for this warrant, "*since* it's a square if you cut it like that, don't you end up with an equilateral triangle?" (SA070808, 451). One can take this to be a backing because it is a proposition that she supplied in response to the researchers prompt to justify her warrant. Then Sonia changed her mind, saying "No, that can't be true" (SA070808, 453). The rejection of this node is marked with a [X] in the argument map. She went on to support her warrant, that the angle bisector of one vertex also bisects the opposite vertex, with a different argument. She claimed that if this were not the case, that is, in a quadrilateral in which the angle bisectors do not go through opposite corners, that the quadrilateral could not be a square because all the sides of a square are congruent. Sonia said, "if it didn't go through the corners *then* it wouldn't be a square *because* the sides are all the same" (SA070808, 457-459). In this concise statement one sees that Sonia provided a set of data marked by *"if"* ("*if* it didn't go through the corners"), a conclusion marked by *"then"* ("*then* it wouldn't be a square"), and a warrant for that implication marked by *"because"* ("*because* the sides are all the same"). This argument is similar to a contrapositive. Sonia began by assuming the negation of what she wants to conclude and then arrived at a conclusion that is the negation of what she was assuming in her larger argument.

In creating this backing, one sees Sonia correctly rejected the claim that the angle bisector of a square forms two equilateral triangles. Since they share a vertex with the square, these triangles contain an angle that measures 90° and therefore could not be equilateral (in which case all the angles would measure 60°). The implication that Sonia built to support her warrant contains true data and conclusion. She could have made a stronger claim, but her claim, that the quadrilateral is not a square, mirrors the data and warrant that she built in the main implication and therefore fits her argument.

Besides providing this warrant and backing for her argument, Sonia also provided a rebuttal to her warrant. This rebuttal is marked by the fact that it was preceded by the researcher's request for a counterexample to her claim. Sonia asserted that the claim, that the diagonals bisect each other, would not be true in a rectangle. She created the implication that since the sides in a rectangle are not congruent, the angle bisector would not go from one corner to the opposite corner. Sonia said, "*if* you had a rectangle and you bisected the angle *then* it wouldn't go through the other corner *because* the sides are different lengths" (SA070808, 461). In this statement the data is marked with *"if*", the conclusion is marked with *"then*," and the warrant is marked with *"because*."

Sonia's claim, that the angle bisectors of a rectangle do not bisect the opposite vertex is true. Also the warrant for this implication, that the sides are not congruent, can be seen to be appropriate, since if the other properties of the rectangle remained constant but the sides became congruent (the figure would be a square) the angle bisectors would bisect the opposite vertex. The non-congruent nature of the sides can be seen to be the essential characteristic of the rectangle that keeps the angle bisectors from bisecting opposite vertices.

Sonia's arguments in response to the angle bisectors problem (2)

In response to the angle bisectors problem, Sonia elaborated three arguments about the points of intersection of the angle bisectors of quadrilaterals (see Figure 16).

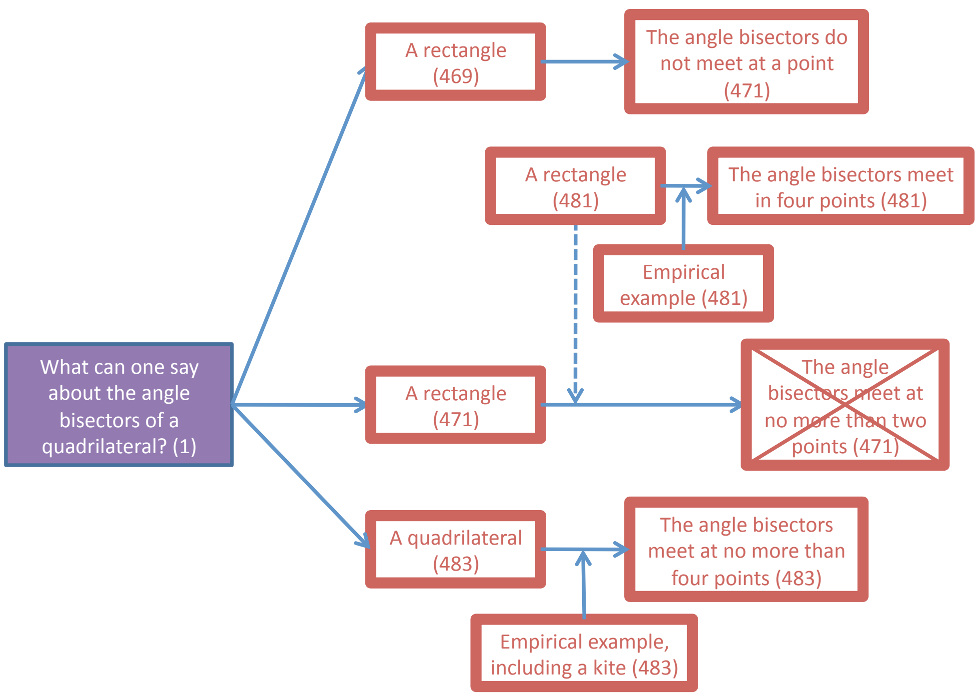

Figure 16: Sonia's arguments in response to the angle bisectors problem (2)

Sonia's first argument about points of intersection of the angle bisectors of quadrilaterals was that in a rectangle the angle bisectors do not meet at one point. She said, "*in* a rectangle, um, not all the angle bisectors would cross necessarily… Like, they wouldn't all go to one point" (SA070808, 469-471). Here the word "in" marks "a rectangle" as the data and there is no marker for the conclusion. As Sonia considered the points of intersection of a rectangle she also drew the diagram seen in Figure 17 of a rectangle and its angle bisectors. In this diagram the angle bisectors can only be seen to intersect in two points. Each angle bisector intersects once with one adjacent angle bisector and then ends at the boundary of the rectangle. From this diagram it is not apparent that the angle bisectors are parallel to the angle bisector opposite or that each angle bisector has another point of intersection with the other angle bisector adjacent.

The implication that Sonia elaborated here is true, however it is incomplete because it does not contain a warrant. It is also a very weak claim in comparison to the fact that the angle bisectors of a rectangle form a square, which Sonia is prevented from making by the diagram that does not show the extension of the angle bisectors beyond the edges of the rectangle.

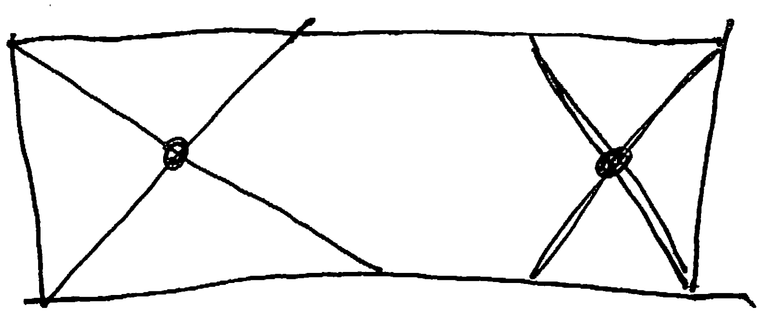

Figure 17: Sonia's first sketch of a rectangle and its angle bisectors

Using this diagram (see Figure 17), Sonia made another assertion about the angle bisectors of rectangles. She initially claimed that the angle bisectors meet at no more than two points. She said, “I think that *in* a rectangle it *has to*, [pause] there can’t be more than two points where angle bisectors cross" (SA070808, 471). *"In"* marks the conditions that Sonia uses as data, and then after a pause she made the assertion based on this data, that the angle bisectors of a rectangle can meet at no more than two points. This assertion is marked by the process *"has to [meet]."* Sonia then drew a sketch of another rectangle and revised her claim (see Figure 18). Her second diagram of a rectangle still had angle bisectors that ended at the boundary of the rectangle, but in this rectangle the angle bisectors intersected four times inside the rectangle.

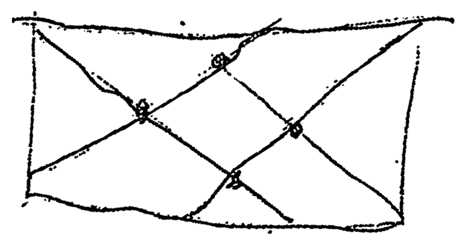

Figure 18: Sonia's second sketch of a rectangle and its angle bisectors

One can see here the power that the diagram has over the arguments that Sonia built. In the first diagram that Sonia drew the angle bisectors only intersected twice (as a result of the proportions of the rectangle and her choice to end the angle bisectors at the boundary of the rectangle), while in the second diagram that Sonia drew the angle bisectors intersected four times (as a result of a change in the proportions of the rectangle). This could have prompted the question of 'under what conditions do the angle bisectors of a rectangle intersect four times inside the rectangle?', based on a comparison of the two diagrams. Instead Sonia rejected the first diagram and made a conjecture based on the second.

Based on this new diagram, Sonia refuted her previous assertion by claiming that the angle bisectors of a rectangle can have angle bisectors that meet in four points. Sonia drew large dots on the four points of intersection and said, "here it crosses in lots of places" (SA070808, 481). The data in her argument is embedded in Sonia's gesture to her diagram, implying that she used a rectangle as her data. From this she asserted that there must be four point of intersection among the angle bisectors. Sonia did not attempt to justify these claims beyond empirical exploration of sketches of two rectangles.

Like the first argument elaborated here, Sonia's claims are true, but weak. Sonia was content asserting that the angle bisectors of a rectangle meet at four points but she did not look further at describing the properties of those points, which form a square.

Sonia then looked at the case of a general quadrilateral and wondered, "if you can get it to cross in more than four places" (SA070808, 483). She sketched a few quadrilaterals and said, "*I don't think* you can get it—an equilateral, I mean, a quadrilateral to cross in more than four places…That can't be possible…if you took a kite… There can't be more than four intersections" (SA070808, 483). *"I don't think*" marks the data in this argument, where "it" refers to a quadrilateral. At the end of her utterance Sonia stated the conclusion, "there can't be more than four intersections." Like with the previous argument, this argument is based on empirical exploration. When Sonia said, "if you took a kite" this could alternatively be modeled as an attempted rebuttal to the implication that in a quadrilateral the angle bisectors meet at more than four points. It is not modeled that way here because Sonia did not finish her thought. She may have draw a kite as she was talking and seen that the angle bisectors do not meet at more than four points. In fact, Figure 19 shows diagrams that Sonia sketched in the course of elaborating these arguments, and it appears that Sonia drew two diagrams of kites, each with diagonals instead of angle bisectors.

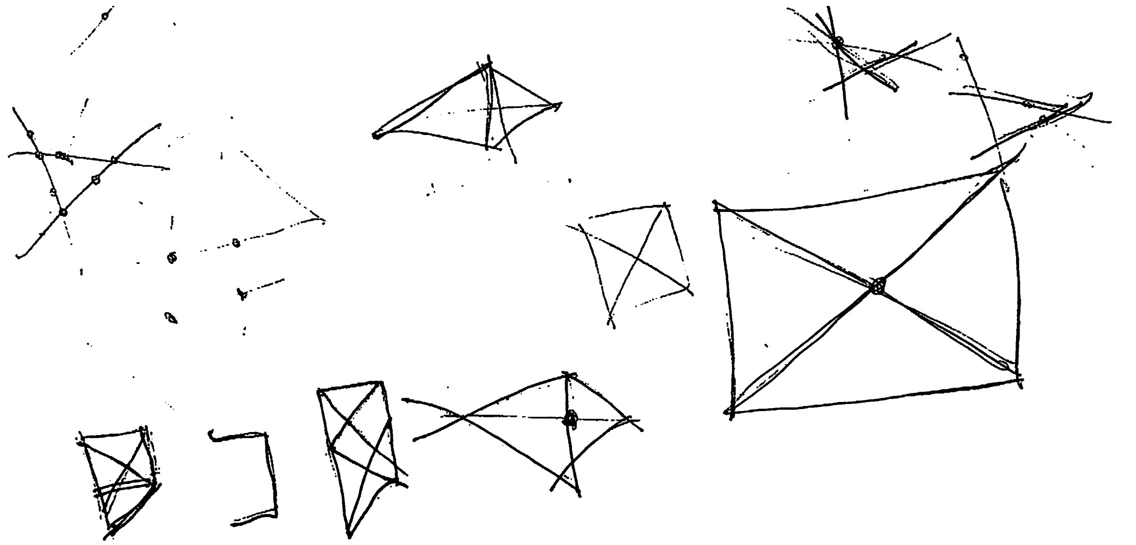

Figure 19: Sonia's sketches of quadrilaterals and their angle bisectors

This last argument that Sonia made is false. She asserted that the angle bisectors of a quadrilateral meet at no more than four points, however, in a general quadrilateral the angle bisectors meet at six points. Because Sonia did not continue to draw angle bisectors past the edge of the quadrilateral, and because she mistook the angle bisectors for diagonals in some cases, she did not see that the angle bisectors of a quadrilateral could meet at six points.

Sonia's argument in response to Alpha's comments

Sonia and the researcher watched the animated scenario until the moment when Alpha returns to his seat saying, "I just thought the diagonals cut the square in half." At this point the researcher paused the movie and asked Sonia, "So what do you think he might mean by that, 'diagonals cut the square in half'?" In response Sonia constructed the following arguments (see Figure 20).

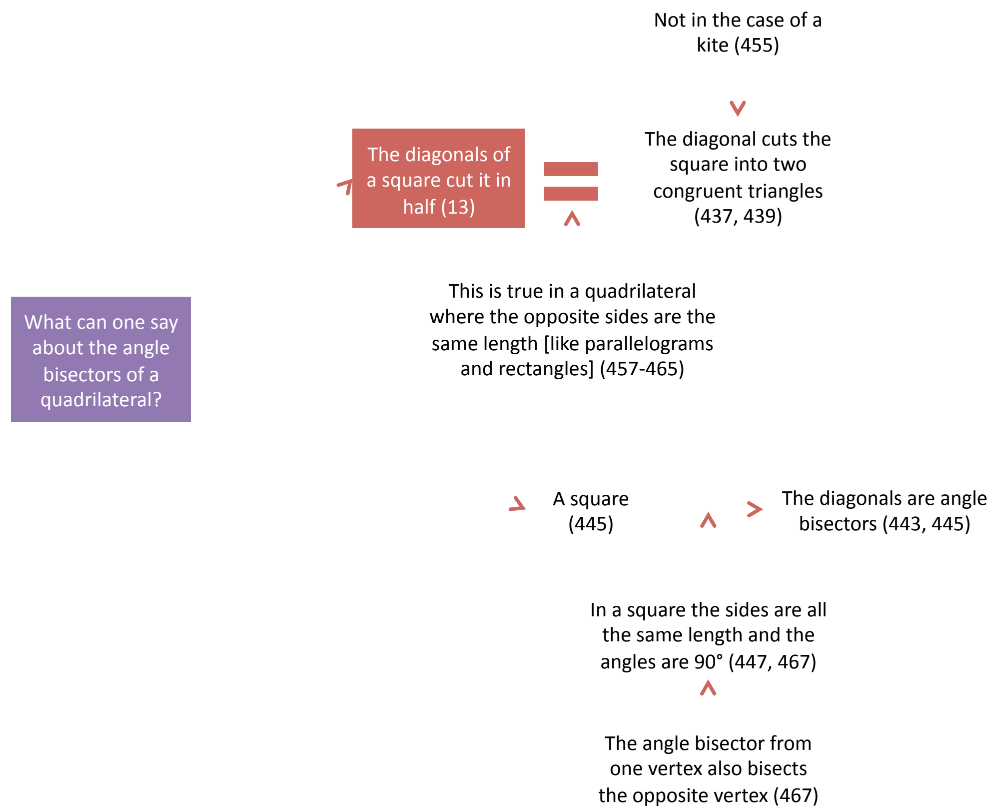

Figure 20: Sonia's argument in response to Alpha's comments

The first argument is abnormal because it does not contain an implication. Instead it contains an equality, where Sonia interpreted a statement from Alpha, an animated student. The researcher asked Sonia, "What do you think [Alpha] might mean by that, 'diagonals cut the square in half?'" Sonia replied, "Like it cuts it in half triangle wise." Sonia's response to the researcher's question marks her statement as being the same as Alpha's statement. Later Sonia provided a rebuttal to this interpretation, in response to the researcher's question, "do diagonals always cut the shape in half?" Sonia pointed to the case of a kite, "here you would have this small little kite thing that does divide…" (SA070908, 455). Sonia also provided a warrant explaining why her statement, "Like it cuts it in half triangle wise" is the same as Alpha's. After giving the counterexample of the kite Sonia said, "But *in* a shape where the opposite sides are the same length, and *then* they cut the shape in half" (SA070908, 457). Sonia drew examples of a rectangle and a parallelogram in which the diagonals cut the square in half. From her examples one can infer that what she meant by "cuts it in half triangle wise" is that the diagonals cuts the quadrilateral into two congruent triangles. To support this, she gave as examples the rectangle and the parallelogram as quadrilaterals that have diagonals that cut the shape in half and she gave the kite and its minor diagonal as a counter-example.

What Alpha says in the animated scenario, that the diagonals cut the square in half, and what Sonia said to interpret Alpha's statement, that the diagonals cut the square in half "triangle wise" are imprecise. However, if one interprets Sonia's restatement to mean that [both] the diagonals cut the quadrilateral into two congruent triangles, then the warrant that Sonia provided for her restatement, that it is true in a quadrilateral that has opposite sides that are the same length, is valid. Sonia's rebuttal, that the diagonals of a kite do not cut it in half "triangle wise" is also valid because the diagonal that connects congruent angles does not divide the kite into congruent angles.

To establish the implication at the heart of the second argument (see Figure 20) Sonia said, "the diagonals are the same thing" (SA070908, 433) meaning that in a square the diagonals are the same as the angle bisectors. Sonia said this in response to the researcher's question of how Alpha's statement, "the diagonals cut the square in half" was related to the animated teacher's questions about angle bisectors of a quadrilateral. It is clear that Sonia was considering the case of a square because she defended Alpha from the teacher by saying that "he's not wrong if it's a square" (SA070908, 455). The warrant for this implication is marked by the *"because*" the beginning of Sonia's statement, “*Because* they’re at 90˚ angles at – on the side – on the shape that has the same length on all sides” (SA070908, 447). Sonia continued to provide a backing for this, marked by the “*so* ” at the beginning of the statement, “*so* the diagonals and the angle bisectors are the same thing – or they go through the same path on the – square” (SA070908, 447).

The implication and warrant that Sonia provided in this argument are appropriate, in that both the data and conclusion are true, and the warrant points to the properties of the square that support the conclusion. Sonia also provided a justification of why these properties of a square support the conclusion, because the angle bisector from one vertex bisects the opposite vertex.

Sonia’s arguments in response to Gamma’s comments (1)

Sonia and the researcher watched the animated scenario until the moment when Gamma bids to come to the board to share her conjecture. At this point the researcher asked Sonia, “So what’s her point? What is she saying?” In response Sonia built an argument that she perceived to mirror Gamma’s argument (see Figure 21).

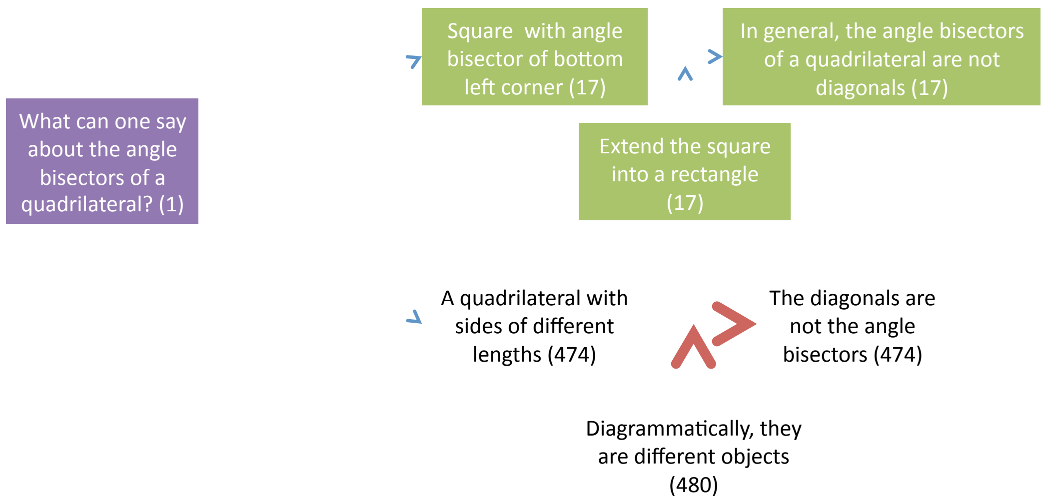

Figure 21: Sonia's arguments in response to Gamma's comments (1)

Sonia summarized Gamma's argument by saying, "it's the sides of different lengths and the diagonals aren't the same thing" (SA070908, 474). Sonia's language was missing the causal markers that would connect the data to the conclusion. Instead she used "and" to connect the two clauses. The fact that she stated the clause about sides of different lengths before the clause about diagonals hints that the first causally precedes the second. This implication was clear to Sonia based on her visualization of these objects. She said, "*Because* you could draw the rectangle and be like, 'diagonal, diagonal' but then you could, like, bisect it and it's not the same thing" (SA070908, 480). As she said this she traced out a rectangle, and its diagonals and angle bisectors in the air. Sonia's warrant is marked by "because" at the beginning of her demonstration.

In this argument Sonia's visualization skills served her well and she was able to correctly conclude that in a rectangle the diagonals are not the angle bisectors. She pinpointed the properties of a rectangle that make it different from a square (where the diagonals are the angle bisectors), that is, the sides are of different lengths. She appropriately warranted her implication on the visualization of the diagonals and angle bisectors of a rectangle.

Sonia's argument in response to Lambda's comments (1)

Sonia and the researcher paused the animated scenario when Lambda said, "you have to prove that the base angles on each triangle are equal to each other." In an attempt to describe what Lambda had said, Sonia made the following argument (see Figure 22).

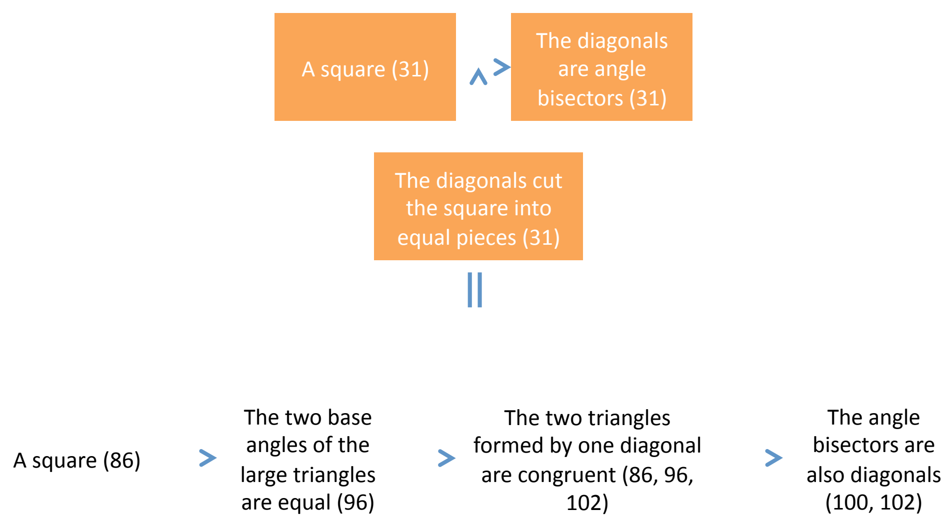

Figure 22: Sonia's argument in response to Lambda's comments (1)

Sonia began with the data, like Lambda did, that the quadrilateral under question is a square. Sonia said, "so *they have* a square" (SA071008, 86). The phrase "they have" is a marker for data because it implies that it is known before the argument is made. Sonia came to the conclusion that is the converse of Lambda, that is that the angle bisectors are diagonals, when Lambda concluded that the diagonals are angle bisectors. Sonia concluded, “then *it means* that [the angle bisectors] are also the diagonals, right?” (SA071008, 100). The phrase “then it means” marks the outcome of the argument and therefore the conclusion. Sonia provided two additional links in her argument that support her claim. These two nodes represent the claims that the two base angles of the large triangles are equal and therefore the two triangles formed by one diagonal are congruent. Sonia said, “[Lambda’s] trying to *prove* that the two base angles are equal so that he can *prove* that the two triangles are congruent” (SA071008, 96). The statement, “the two base angles are equal” is contained between the markers, “prove” that imply that it is both data and conclusion for different implications. When the researcher asked Sonia why she made those claims she responded that it’s part of Lambda’s proof that the bisectors are diagonals. This is a marker that the two previous statements are data for the final statement.

The first three nodes of Sonia’s argument are appropriate for supporting the claim that the diagonals of a square are also angle bisectors. However, the last node switches this order and makes the inverse claim, that the angle bisectors are diagonals. Since the first three nodes assume that the segment connecting opposite vertices are diagonals, the conclusion of the argument should be an assertion about diagonals, not angle bisectors.

Sonia’s arguments in response to Lambda’s comments (2)

Since the claim that Sonia made in the previous argument sounded as if it were the converse of Lambda’s claim, the researcher suggested that they watch Lambda’s argument again, in the hopes that Sonia would realize that her claim was different from Lambda’s. The pair watched the end of the animated scenario again, pausing it occasionally for the researcher to ask clarifying questions. When the animated scenario ended, the researcher asked Sonia, “What do you think the argument was? It’s a little different than what you said a minute ago.” Sonia replied by elaborating the following argument (see Figure 23).

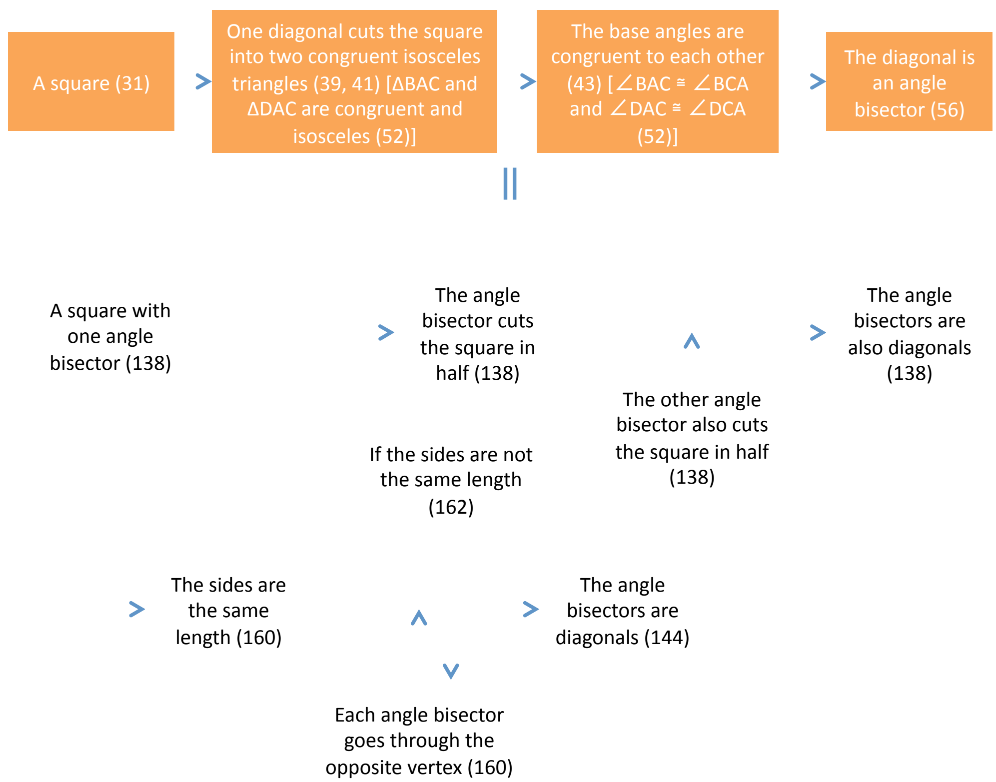

Figure 23: Sonia’s arguments in response to Lambda’s comments (2)

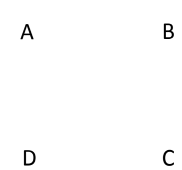

Figure 24: Diagram accompanying Lambda's comments

Sonia began her response to the researcher’s questions by saying, “Well, [Lambda] was saying that …” (SA071008, 138). So one can take the argument that she made as an interpretation of Lambda's argument in the animated scenario. Sonia went on to say, "*if you took* one diagonal, I mean, one angle bisector and *prove* that it cut the square in half, *then* the other angle bisector will also do that, *so* that means they're the diagonals" (SA071008, 138). "If you took" marks "one diagonal" the data for the argument because it implies that the statement that follows it was available before the argument was formed. Implicit in Sonia's utterance is that the diagonal belongs to a square. Next, the process "prove" marks "it cut the square in half" as the first intermediate conclusion in the argument. "Then" marks "the other angle bisectors will also do that" as either data or warrant. Here it is coded as warrant. Finally, "so" marks "they're the diagonals" as the final conclusion of the argument. This complete utterance, consisting of four connected statements, yields the main line of Sonia's argument. She began with a square and one angle bisector as her data. From this she made the implication that the angle bisector cuts the square in half. Then, she made the conclusion that the angle bisectors are diagonals. She warranted this last implication with the claim that the other angle bisector (the angle bisector from the opposite vertex, not that from an adjacent vertex) also cuts the square in half.

Table 6: Sonia's elaboration (1)

if you took	
one diagonal, I mean, one angle bisector	Data (1)
and *prove* that	→
it cut the square in half,	Conclusion (1)

	Data (2)
then	→ (2)
the other angle bisector will also do that,	Warrant (2)
so	→ (2)
that means they're the diagonals	Conclusion (2)

The argument is consistent in that it begins with the angle bisectors of a square and ends by making a conclusion about the angle bisectors of a square. It is also valid in the sense that each node could be seen in an argument that supports the conclusion. However, this is the inverse of Lambda's argument that Sonia was trying to explicate.

The researcher continued to press this distinction between angle bisectors and diagonals and said to Sonia, "So, start over. If in this corner [the angle bisector] divides those angles how do you know that it goes through this corner over here [making it also a diagonal]?" Sonia replied by creating the argument in the lower half of the figure above (see Figure 23). Sonia began again with the data that the figure is a square and went on to say that therefore the sides of the figure must all be the same length. She said, "because it's a square, because the sides are the same length" (SA071008, 160). Then she warranted the conclusion that the angle bisector is a diagonal by the statement that each angle bisector goes through the opposite corner. In her words, "because the sides are the same length then it has to go through that corner" (SA071008, 160). Sonia also gave an example where her warrant would not hold. She provided the rebuttal that her warrant would not hold if the sides of the figure were not the same length. She said, "if

[Lambda] made the sides longer then the angle bisectors no longer cross through opposite corners" (SA071008, 162).

Sonia's argument could be reduced to the claim, since all the sides of a square are the same length the angle bisector of one vertex also passes through the opposite vertex. This assertion is true. Sonia's argument, however, does not provide any justification for this assertion. Her warrant is simply a restatement of her implication, and her rebuttal to the warrant is a negation of her data. Sonia has done useful work in making this claim, but her work in justifying this claim is not useful.

Sonia's argument in response to Lambda's comments (3)

Sonia and the researcher re-watched Lambda's argument. When they reached the end of the animated scenario the researcher asked Sonia, "I realize that [Lambda] says 'you want to show that [the triangles formed by one diagonal of a square] are congruent' but he never actually says how you know that they're congruent…how could you show that they were congruent?" In response to this question Sonia created the following argument (see Figure 25).

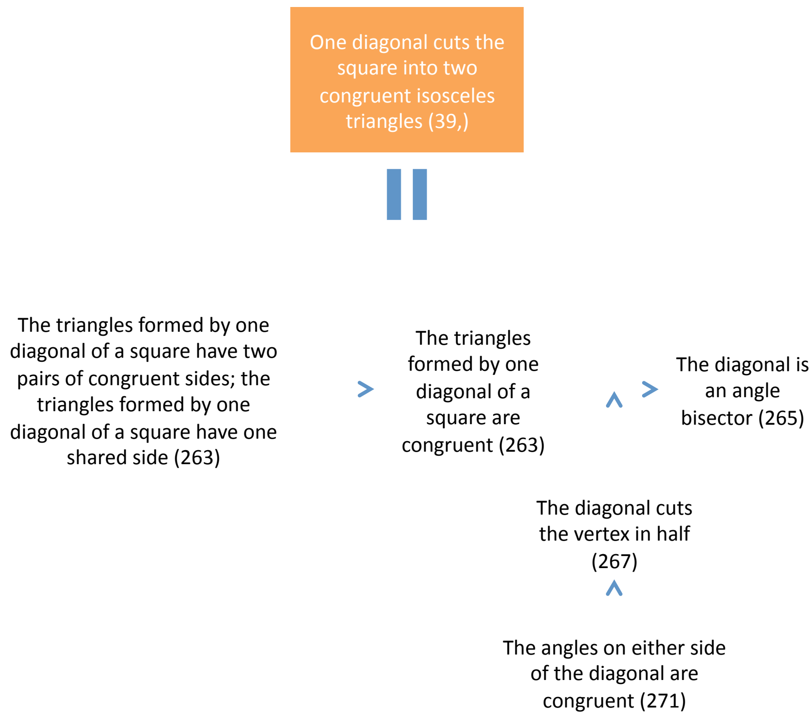

Figure 25: Sonia's argument in response to Lambda's comments (3)

In this argument Sonia used the claim that the two triangles formed by one diagonal of a square are congruent to conclude that the diagonal is an angle bisector. Sonia began her argument by claiming that the two triangles formed by one diagonal of a square are congruent because they have three pairs of congruent sides. Sonia used as data the fact that two pairs of sides are congruent because they are sides of a square, and the fact that the third pair of sides is congruent because it the diagonal of the square is a shared side between the two triangles. She did not give the warrant of the "side-side-side" theorem of triangle congruence. Sonia said, "*We know* that these two sides are congruent ... Then *we know* that this line here [the diagonal] is—they have it in common *so* they have to be equal [congruent]" (SA071008, 263). The "*we know*" at the beginning of both these sentences marks the first clause in both sentences as data. So "the sides are congruent" and “they have it in common” are both data. The “so” in the middle of the second sentence marks the clause that follows it as a conclusion. Therefore, the intermediate conclusion is that “they have to be equal.” From here Sonia made the implication that the diagonal is an angle bisector. In response to the researcher’s probe, “how did you know that the diagonal’s an angle bisector? Sonia said “it cuts it in half” (SA071008, 267). In this case the conclusion to the argument is marked by the researcher’s probe. Sonia further supported this with the backing that the angles on either side of the diagonal are congruent. Sonia said, “*because* the angles are both congruent” (SA071008, 271). The “*because*” and this statement’s position directly following the warrant marks this statement as a backing.

Table 7: Sonia's elaboration (2)

We know that	
these two sides are congruent	Data (1)
Then *we know* that	
this line here [the diagonal] is—they have it in common	Data (1)
so	→ (1)
They [the triangles] have to be equal [congruent]	Conclusion (1)

Here, by focusing on the claim that the triangles formed by the diagonal of a square are congruent, Sonia created an argument that matched Lambda’s argument and concluded with the claim that the diagonals are the angle bisectors, instead of the claim that the angle bisector is a diagonal. In addition to supporting the correct claim, the argument is valid. Sonia used triangle congruence to show that the angles on either side of the diagonal are congruent; therefore the diagonal is an angle bisector.

Sonia's summary of The Square

Sonia and the researcher revisited The Square in a session near the end of their meetings, eleven days after the last time that they viewed The Square. The researcher began the viewing by asking Sonia what she remembered from her pervious viewings of The Square, without reminding her of any of the details of the story. Sonia replied by giving a short chain of implications that supported the claim that the angle bisectors of a square meet at a point (see Figure 26).

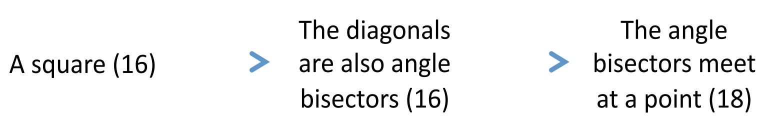

Figure 26: **Sonia's summary of The Square**

Sonia began with the data that the quadrilateral is a square and used this to imply that the diagonals of a square are angle bisectors. She said, "We were *trying to prove* that the diagonals of a square are also its angle bisectors" (SA082108, 16). The process "trying to prove" marks "the diagonals of a square are also its angle bisectors" as the conclusion of the implication. When the researcher pressed her further by asking, "why would we care about that?" Sonia extended this to saying that the angle bisectors of a square meet at a point. She said, "They were talking about that it meets at a point" (SA082108, 18). The researcher's question marks Sonia's response as a further conclusion.

Although this argument is incomplete because it does not contain warrants for the implications, it is valid because each node is consistent with the others and could be used to support the claim and the angle bisectors meet at a point. The first implication could be supported by an argument that shows that the diagonal cuts the square into two congruent triangles, which means that each diagonal is also an angle bisector. The second implication could be supported by the warrant that the diagonals of any convex quadrilateral intersect.

It is interesting that the argument that Sonia lays out captures the main points of the argument in the story. She did not remember Alpha's conjecture, Gamma's rebuttal, or the details of Lambda's argument, but she did remember the key mathematical points, that the angle bisectors of a square meet at a point because they are the diagonals.

Sonia's argument in response to Gamma's comments (2)

Sonia and the researcher watched the segment of The Square in which Gamma comes to the board and gives a counter argument to the claim that the angle bisectors are the diagonals. The animated scenario was paused at the moment when the teacher calls for a proof of Alpha's conjecture. The researcher then asked Sonia, "What about Gamma? … Can you say anything about her?" In response to this question Sonia summarized Gamma's argument (see Figure 27).

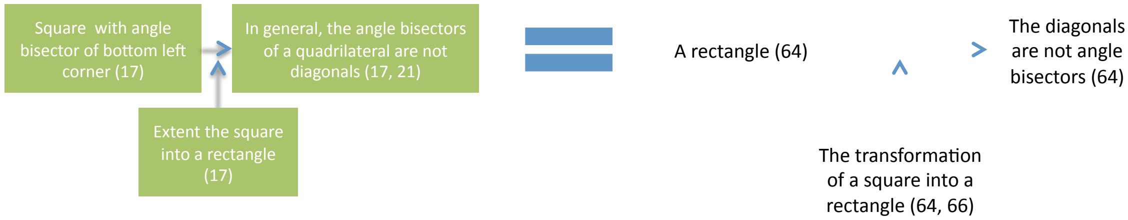

Figure 27: **Sonia's argument in response to Gamma's comments (2)**

Sonia saw that Gamma was making the claim that the angle bisectors are not the diagonals. Sonia said, "Well, she was *trying to show* that the angle bisectors aren't always diagonals" (SA082108, 64). The process "*trying to show*" marks the clause "the angle bisectors aren't always diagonals" as the conclusion of the argument. Sonia recognized that Gamma's argument was based on the observation of the transformation of a square into a rectangle. She said, "To *prove* that she turned the figure, the square on the board, into a rectangle…It just further proves that the diagonals and the angle bisectors aren't the same thing" (SA082108, 64-66). The phrase "*to prove that*" marks the warrant for the implication, which is her [mental] observation of the transformation of a square into a rectangle. The data of Sonia's argument was implicit in her utterances. Because her warrant was particular to rectangles, I took "a rectangle" as her data, however, it's possible that she was thinking of "all quadrilaterals" as data, and simply providing the case of a rectangle as an example.

Sonia's argument is a valid argument, based on perception, which shows that the diagonals of a rectangle are not its angle bisectors. Sonia used the same warrant as Gamma in the animated scenario, saying that as a square transforms into a rectangle the diagonals move so that they are no longer angle bisectors.

Sonia's argument in response to what the animated teacher wrote on the board

After Sonia elaborated the argument shown in Figure 27, the researcher asked Sonia, "So the teacher wrote that thing on the board right there ["In a □ the ang bis ☒"] and she said, 'how do we prove something like that?' Um, how do you think we would prove something like that?" Sonia gave a brief argument for how to show that the diagonals are the angle bisectors (see Figure 28). Like other arguments that she built during this viewing, she did not provide warrants, only a string of implications. Apparently, Sonia interpreted the animated teacher's statement as meaning that "in a square the diagonals are angle bisectors."

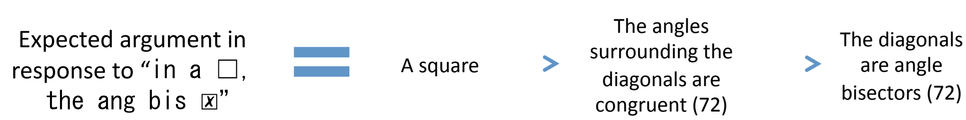

Figure 28: Sonia's argument in response to what the animated teacher wrote on the board

Sonia began by constructing the implication that in a square, the angles surrounding the diagonals are congruent. Sonia said, "if you *can prove* that the angles are all congruent…" referring to the angles that are formed at each vertex of the square by the diagonals. The process "*can prove*" marks the clause "the angles are all congruent" as a conclusion of the implication. She continued, "*then* you *can prove* that those are the angle bisectors" (SA082108, 72). Again, "*can prove*" marks the conclusion, that "those are all angle bisectors." The conjunction "*then*" between Sonia's sentences marks a continuation of the argument so the implications are in a series. Like in the previous argument, the data was implicit in Sonia's utterances. This argument is the outline of the argument that Lambda provides in the second half of the square. Here Sonia omitted the reasoning that supports the implication that in a square the angles formed by the diagonals are all congruent.

This is the sketch of a valid argument and a reasonable response to the statement that the animated teacher writes on the board. Sonia provided the data and conclusion as well as an intermediate node to support the conclusion. The first implication could be warranted by an argument about the triangles formed by one diagonal of a square. The second implication could be warranted by the definition of angle bisector.

Sonia's argument in response to Lambda's comments (4)

Sonia and the researcher watched Lambda begin his argument that the diagonals are the angle bisectors in a square. They paused the animated scenario when the teacher erases a diagonal at Lambda's request. The researcher asked Sonia, "So what about the argument that Lambda's trying to make? Um, you think he's doing a good job making an argument?" In response to this question Sonia elaborated the following argument that is parallel to Lambda's argument (see Figure 29).

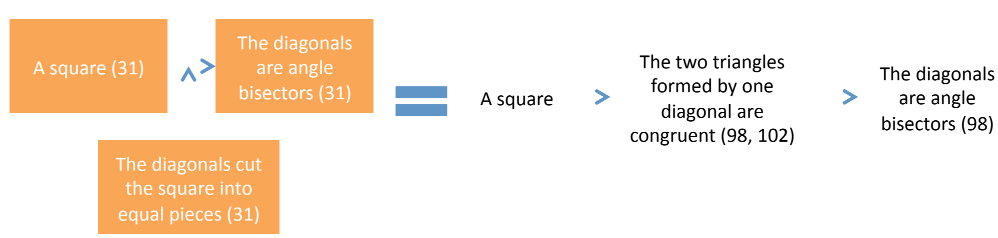

Figure 29: Sonia's argument in response to Lambda's comments (4)

In this argument, Sonia began with the data that the quadrilateral is a square and she endorsed the implication that since the two triangles formed by one diagonal are congruent then the diagonals must be angle bisectors. Sonia said, "it makes sense if you *can prove* that those triangles are congruent *and that* there's an angle bisector"

(SA082108, 98). The process "*can prove*" marks that "those triangles are congruent" as a conclusion of an implication, and Sonia's continuation "*and [you can prove] that*" marks that "there's an angle bisector" is another conclusion to an implication. The data for this argument is implicit, but it is clear that Sonia was talking about a square because she references "those triangles" which are the triangles formed by one diagonal in a square. Like in previous arguments, Sonia did not give warrants for these implications.

Sonia's elaboration of this argument matched Lambda's argument, and is valid, despite the lack of warrants. The first implication could be warranted by properties of the square and the side-side-side theorem for triangle congruence. The second implication could be warranted by CPCTC and the definition of angle bisectors. In general, the arguments that Sonia elaborated on her last viewing of the animated scenario are better formed than the arguments that she elaborated during her first and second viewing.

Looking across these arguments elaborated by Sonia, who had not taken a high school geometry class, one sees that she made arguments that had a very complex structure, often employing compound arguments, or arguments that had implications embedded within rebuttals or warrants. However, the complexity of her arguments is different than the complexity displayed by Maria's arguments. Sonia's arguments are less predictable in that the structure of each argument cannot be predicted from the arguments that come before. Below I describe the results of the analysis of the mathematical territory covered by the arguments elaborated by the participants.

Map of content

From Lakatos (1976) one sees the value of mathematical exploration, and the importance of making "conscious guesses" (p. 30). *Proofs and Refutations* highlights the interplay of exploration and justification in the form of making conjectures, or conscious guesses, and then attempting to support or refute them. The following figures show how the participants made conjectures and explored the mathematical territory surrounding the angle bisectors problem. Of course, these conjectures are not ends to themselves, but starting places for testing the validity of mathematical claims. In Figures 30 and 31 one sees the mathematical territory coved by Maria and Sonia through the conjectures that they made in response to the angle bisectors problem. One sees that Sonia covered more territory than Maria, through examining more types of quadrilaterals, and making more conjectures for each of these quadrilaterals.

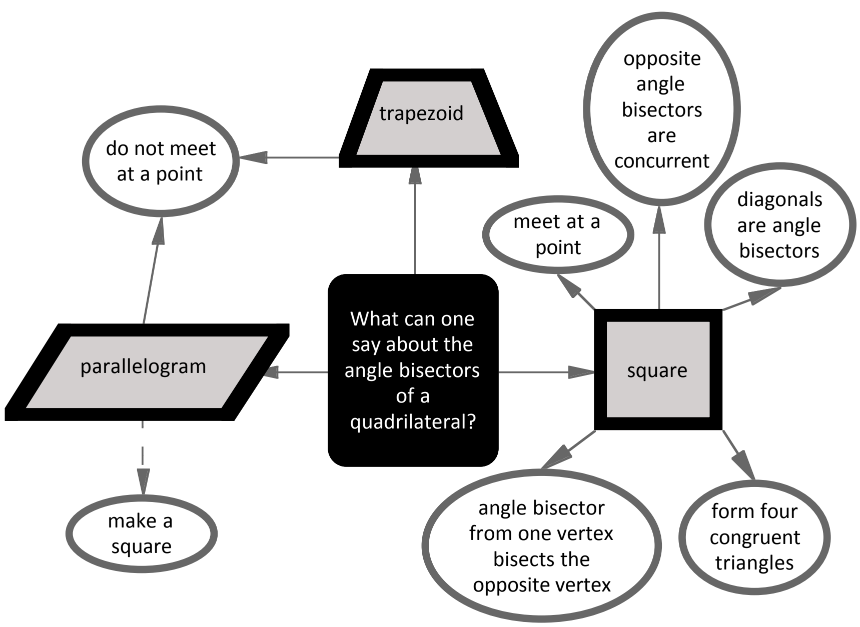

Figure 30: Mathematical territory of Maria's arguments

Figure 30 shows the mathematical territory covered by the arguments elaborated by Maria. From this map one sees that Maria explored the angle bisectors of three types of quadrilaterals, parallelograms, trapezoids, and squares. She drew one incorrect conclusion about parallelograms and one correct conclusion about both parallelograms and trapezoids. Maria's explorations were focused on the case of the square, and in particular she focused on the conclusions that are discussed in the animated scenario, The Square.

This map supports the claim that Maria interpreted the angle bisectors problem as a confirmatory problem, which called for her to prove that given statements are true. Instead of taking the problem as an opportunity to explore the mathematical territory surrounding the angle bisectors problem, she attempted to recall statements that she has heard in the past about the angle bisectors of a quadrilateral and then, in her arguments, attempted to support these claims.

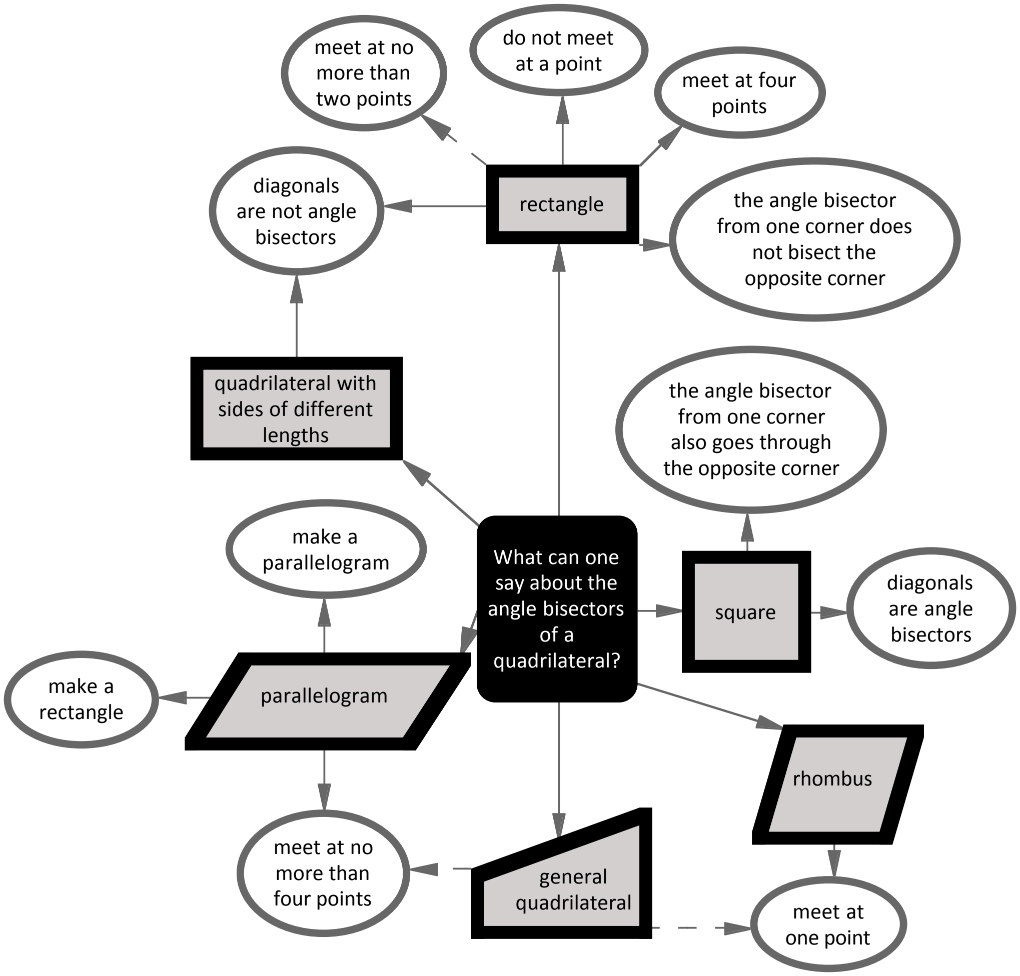

Figure 31: **Mathematical territory of Sonia's arguments**

There is a clear contrast between the map of the mathematical territory covered by Maria and the map of the mathematical territory covered by Sonia. In Figure 31 one sees that Sonia looked at more types of quadrilaterals than Maria, and that Sonia made more conjectures about each type of quadrilateral. Sonia explored the case of a rectangle, a square, a rhombus, a "general" quadrilateral, a parallelogram, and a quadrilateral with sides of different lengths. From the conjectures that Sonia made about each of these types of quadrilaterals one can see that, among other things, Sonia was interested in the question of how many times the angle bisectors of a quadrilateral would intersect. She was also able to make good progress in answering the angle bisectors problem by exploring many types of shapes and by creating conjectures for each of these shapes.

This map supports the claim that Sonia interprets the angle bisectors problem as an exploratory problem, which calls for the solver to discover some information. More so than Maria, Sonia is able to create conjectures in response to the angle bisectors problem. In the work of doing mathematics, this activity of generating conjectures is a substantial aspect of the work, and once the conjecture is made, doing the proof to justify the conjecture can be relatively straightforward (K. Smith, personal communication, October 5, 2010).

In the following section I discuss conclusions that can be drawn from the arguments elaborated by Maria and Sonia and the mathematical territory covered by each of them.

Discussion

In this discussion I support the claim that the arguments elaborated by Maria and Sonia are substantially different from each other. I show how the key difference between the arguments made by the two participants is that the arguments that are elaborated by Maria reflect the structure of the two-column proof that is pervasive in the high school geometry classroom, while the arguments elaborated by Sonia reflect features of the work of developing new knowledge in discipline of mathematics. By comparing Maria's arguments to those of Sonia one sees that the structure of the two-column proof allowed Maria some affordances that are not visible in Sonia arguments, as well as some constraints that did not appear to affect Sonia's arguments.

To focus the discussion I use the idea of disciplinary agency to examine the arguments elaborated by the two participants. I claim that the different forms of argumentation seen in arguments elaborated by the two participants can be traced to the disciplinary agency that the participants worked within to form the argument. From the literature on the creation of mathematical knowledge in high school geometry classrooms, I construct a description of what disciplinary agency entails in the high school geometry classroom. I then use this description of disciplinary agency to highlight differences between the arguments elaborated by the participants.

Disciplinary agency of the two-column proof

From the literature reviewed in this study, proof in high school geometry classrooms is dominated by two-column proofs, created by students inside the instructional situation of 'doing proofs.' Two column proofs are written in these situations using a very specific set of rules that can be related to disciplinary agency. Once a student picks (or is provided with) a set of assumptions, this disciplinary agency guides students through the process of applying known theorems, postulates and definitions to each statement in the proof to produce the next statement. Disciplinary agency also guides students through the process of proving triangles congruent, if there are any triangles in the diagram that accompanies the proof task.

This disciplinary agency for the high school geometry classroom describes the actions that students would take while they work on a proof, but it also has implications for the argument that could result from these actions. This disciplinary agency of the two-column proof consists of producing statements and reasons for the two-column proof, which results in an argument that consists of a string of implications, connected by warrants. There is not room in this process for the construction of rebuttals or backings.

Sonia's arguments are not as consistent or focused as Maria's because they do not appear to be guided by the disciplinary agency of the two-column proof. For Sonia, apparently she can draw geometric objects and infer relationships among the corresponding concepts from these diagrams. So her process for building arguments appears to consist of taking a new set of data, say a square and its angle bisectors, drawing it and inferring information from her diagram about these objects. She then supports these claims by translating her observations of relationships into rationale for her claims. This process is less disciplined than Maria's and therefore Sonia arrives at arguments that are not as organized as Maria's, or as valid.

Below I argue that the disciplinary agency of the two-column proof can be seen in Maria's arguments and that this agency is not as visible in the arguments elaborated by Sonia.

Arguments that contain chains of implications

Chains of implications are one of the main characteristics of the two-column proof. Of a total of six arguments elaborated by Maria, two arguments, or one-third of her arguments consisted of chains of implications longer than two implications. Sonia elaborated a total of fourteen arguments and only one of her arguments contained a chain of implications that contained three implications.

Chains of implications containing data, conclusions, and (possibly) warrants focus on the most important pieces of an argument. They convey information about the chain of implications and provide a justification for the conclusion of each implication. The creation of long chains of reasoning is essential for creating advanced arguments. To construct complex arguments students not only have to create different pieces of an argument (like the data, conclusion, and warrant) but they also have to tie these arguments together into larger arguments. From looking at the participants' arguments, it seems that arguments that are structured like a two-column proof support the construction of longer chains of reasoning. Considering that there is a limited amount of time for teaching proof it is understandable to focus on these aspects of the argument.

Arguments that do not contain rebuttals or backings

The two-column proof provides a very structured method for students to create arguments; one benefit of this is the affordance of creating chains of implications as discussed above. However, the two-column form encourages students to construct arguments by producing a series of statements and reasons, and does not have room for either rebuttals or backings.

Of Maria's six arguments, only 2 arguments, or one-thirds of her arguments, contain either rebuttals or backings. Of Sonia's fourteen arguments, six arguments, just under one-half, of her argument contain either rebuttals or backings. These rebuttals and backings are important pieces of arguments because they both point to the validity of an argument. Rebuttals point to cases when the implication would not hold and backings provide reasons why the warrant is an appropriate justification for the implication.

Arguments that are compound

Another structural outcome of the disciplinary agency of two-column proofs is the focus on one implication. Most of Maria's arguments focus on one implication, or one series of implications, whereas Sonia's arguments contain compound arguments, or arguments that involve multiple implications that are not in a series. In compound arguments, implications are found within a warrant, or rebuttal. Comparing arguments from the two participants one can see that Maria's arguments are built as a series of implications, while Sonia builds more on the periphery of the implication. Compound arguments result in arguments that are less analytic, but cover more mathematical territory.

Arguments that contain congruent triangles

A technique that is very common in high school mathematics classes, and related to the disciplinary agency of the two-column proof, is the strategy to prove that triangles are congruent. This allows students to make claims about pairs of congruent segments and angles that make up these triangles. Maria made claims about congruent triangles, which had not been prompted by the animated scenario in two of her arguments and Sonia never used congruent triangles except when they were suggested by the actions of the animated students in The Square.

Taken together these observations about the arguments elaborated by the participants show that the arguments elaborated by Maria would better fit into a high school geometry classroom. The arguments elaborated by Sonia would not fit as well into the high school geometry class as her arguments reflect a style of argumentation is not technically proficient with the tools of the two-column proof.

Arguments that cover mathematical territory

From looking at the maps of the mathematical territory that the participants covered with their arguments one sees that Sonia's arguments covered more territory than Maria's. Through her engagement with The Square, Sonia approached the angle bisectors problem as an exploratory problem that allowed her to test out conjectures and therefore cover new mathematical territory. Maria approached the angle bisectors problem as a confirmatory problem and this allowed her to confirm conjectures that she remembered from past experiences, but not generate new conjectures and cover new mathematical territory.

These two ways of approaching the problem are echoed in the validity of the arguments that were used to cover the mathematical territory. Maria's arguments were more likely to be valid than Sonia's, so even though Sonia covered more territory, the arguments that she used to cover that territory are often invalid. In the arguments made by these two participants, there seems to be a balance between the amount of territory covered and the validity of the arguments. As the amount of mathematical territory covered increases the validity of the arguments decreases, and vice versa.

Different modes of argumentation

I can now return to the metaphor for thinking about work done in mathematics classrooms as thinking of the class as an orchestra, conducted by the teacher and where each student plays a different instrument (Herbst & Balacheff, 2009). Each student adds ideas at different moments in time and each student's contribution is seen to consist of a mathematical idea along with the style of argumentation that is used to justify that idea. Together these disparate ideas and argumentation styles combine to form a performance that may embody a mathematical concept. In that view, Maria and Sonia represent two different instruments because they represent two different ways of arguing for ideas. In particular, Maria's arguments (instrument) resemble the arguments that coincide with the mathematical arguments that are supported by a high school geometry classroom.

Extending the metaphor, Maria's arguments would be harmonious with the arguments elaborated in class, while Sonia's arguments would be inharmonious within the context of the high school geometry classroom. That is not to say that Sonia's arguments are not valuable or productive, only that they do not fit well with the other arguments that exist in this context.

Conclusion

This study examines the mathematical arguments created by mathematically successful adolescent learners in response to an animated scenario of geometry instruction. These arguments are taken as examples of the creation of new mathematical knowledge. By looking at these arguments I have been able to show to what degree they resemble the arguments that would be expected in a high school geometry classroom. The form of argumentation displayed by Sonia, and expected in the high school geometry classroom, both affords and constrains the arguments that are created.

In terms of affordances for argumentation provided by the two-column proof, one sees that it supports arguments that are built from strings of implications. These strings of implications are essential for constructing advanced arguments that connect data to conclusions that do not follow directly. The two-column proof also provides the opportunity to practice applying the theorems of geometry and tools of argumentation, like triangle congruence. These lead to arguments that are both efficient and more likely to be valid.

In term of constrains for argumentation that stem from the two-column proof, one sees that it constrains arguers from including rebuttals or backings in their arguments. These are important pieces of arguments that point to the validity of the implication. Use of the two-column format also encourages arguers to approach problems as confirmatory problems instead of exploratory problems. This results in arguments that cover less ground because they confirm known statements instead of generate new statements.

Returning to the research question of “How can the mathematics being studied in high school geometry classrooms shape the position of the student in geometry instruction?” one sees that the key aspect of mathematics being studied that can shape the position of the geometry student is the two-column proof format and the disciplinary agency it can engender. Looking at the arguments elaborated by the two participants one sees that arguments that reflect the structure of the two-column proof show the affordances of strings of implications, and the productive use of triangle congruence. However, these arguments are also constrained by the lack of rebuttals and backings, and they do not contain compound arguments. From the mathematical territory covered by the arguments elaborated by the two participants one sees that the disciplinary agency of the two-column proof can result in approaching problems as confirmatory problems, not as exploratory problems. Interpreting problems in the latter way can allow for covering more mathematical territory and mirrors an important aspect of how mathematical work is done in the discipline.

The concepts of disciplinary and human agency can help highlight the difference between arguments that are elaborated in response to confirmatory problems and arguments that are elaborated in response to exploratory problems. If we think about the angle bisectors problem, there are ways of working on the problem that reflect human agency, that is ways that involve the human decision to generalize or specialize, and there are ways of working on the problem that reflect the disciplinary agency of the two-column proof, like producing a string of statements and reasons that lead from a set of data to a known conclusion. Looking at the arguments created by Maria and Sonia we see that Maria's arguments reflect the disciplinary agency of the two-column proof and Sonia's arguments reflect the human agency.

In addition to these observations, this study shows that Toulmin's model of argumentation can be a powerful tool for studying the arguments elaborated by students in the course of discussions. By developing codes for the linguistic markers that label parts of arguments in speech, I have been able to distill mathematical arguments from conversations about classroom interaction. These arguments avail themselves to comparison and analysis of their characteristics. The characteristics of arguments used in this case study came from the literature discussing the creation of mathematical knowledge in classrooms and in the discipline of mathematics.

This study raises questions about what is being taught in high school geometry classes. Returning to the quote from Bruner, "any subject can be taught effectively in some intellectually honest form to any child at any stage of development," (p. 33, 1960) one is pressed to ask the question, "are students in high school geometry classrooms learning an intellectually honest form of proof and argumentation?" This study illustrates the possibility that some learners may come into this class with some ways of arguing that are similar to those that are found in the discipline of mathematics but that are not supported by the two-column proof. However, learners may leave the high school geometry class making arguments that are more likely to be valid, employing some technically advanced tools for argumentation (like triangle congruence and common geometric theorems), and constructing arguments that connect data to conclusions that require a series of several implications to establish. These are key steps in learning how to build advanced arguments that the two-column proof can support.

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Chapter 3

Teachers' Perceptions of Geometry Students

A major (if not *the* major) goal of educational research is to improve student learning. This is often attempted through efforts to improve pedagogical practices, increase educational resources, or improve curriculum. This study seeks to improve student learning through improving instruction (Cohen, Raudenbush & Ball, 2003; Lampert, 2001), by first attempting to understand instruction. Instruction is conceived of as relationships between teachers, students, and content in environments. To fully understand instruction, one would need to understand each of these relationships along with their interaction with the environment. This study focuses on the relationship between teachers and students; in particular, I focus on understanding teachers' perception of their students.

Teachers' perception of their students is an important aspect of the relationship between teachers and students because teachers' perception of students is a key resource that teachers use to make instructional decisions. Teachers observe student behavior and then they interpret that behavior using their perception of students. Teachers use these interpretations to make decisions; about lesson planning, about how to conduct a lesson, about how to grade a test.

Much research on teaching has followed a unidirectional model in which teaching influences learning. But this relationship is at least circular, with students' actions also influencing teacher actions (Clark and Peterson, 1986). Instruction in classrooms is driven partially by students' actions, but it is the responsibility of the teacher to sustain this interaction (Brousseau, 1997). Hence, for the teacher, students' actions in classrooms are not just an outcome to produce (in the form of work on tasks and classroom discussions) but more generally, student actions are resources that are essential to the teacher's work. A better understanding of teachers' perceptions of students will lead to a better understanding of the work of teaching by understanding how teachers use students in that work.

A metaphor for thinking about the relationship between students and the work of teaching is musicians and the work of conducting an orchestra (Herbst & Balacheff, 2009). In this work the conductor is responsible for managing the work of the individual musicians to form a coherent performance of a piece of music. Each musician plays only one instrument, and only plays at specific times during the performance. However, from the point of view of an observer the contributions of each musician combine to form a unified performance of a piece of music. This is similar to the way that a teacher unites the work of her class of students to form a unified performance of a piece of mathematical work despite the fact that each student only has access to his or her own experience, and each student only publically participates at specific moments during the lesson.

Recognizing the importance of students in the teachers' work leads to the question of how teachers perceive their students. The way that the teacher eventually conducts instruction relies on the tools, including her students, which she perceives as available to her. In this study I ask the following question:

- What perceptions of students are instrumental in the work of teaching geometry?

I'm interested in uncovering the ways that teachers perceive their students that are instrumental in the work of teaching. That is, when they are engaged in the work of teaching, how do teachers characterize their students in ways that aid in their work? I will revisit this research question later in the paper with the assistance of the theoretical framework.

To address this question, I begin by looking at past research on teachers' perceptions of students. Although a significant amount of research has been done on teachers' perceptions of students these studies do not provide information on teachers' perceptions of students inside instruction. Instead, these studies provide valuable information about how teachers think about students outside of specific instructional scenarios, during activities such planning lessons and predicting students achievement. After discussing this literature I introduce the theoretical frameworks that I use to parse the work of teaching and teacher knowledge. The framework for teacher decision making, practical rationality (Herbst & Chazan, 2003), builds on the framework for the work of teaching to model teacher knowledge as dependent on the work of teaching. I then describe my methods for data collection. The data analyzed in this study consists of conversations among high school geometry teachers in response to an animated scenario of geometry instruction. Using the frameworks for the work of teaching and teacher knowledge, and systemic functional linguistics (Martin & Rose, 2003), I analyze the conversations among teachers to examine how the participants perceived students who are instrumental in the work of teaching. I present the results from this analysis and discuss implications for understanding the work of teaching.

Past research on teachers' perceptions of students

In the following section I review relevant literature on teachers' perceptions of students. I focus on studies that explore the categories of students that are salient to teachers. This literature provides a foundation for studying teachers' perceptions of students as well as providing examples of types of students that have been interesting to researchers in the past. I document a gap in this existing literature: teachers' perceptions of students within instructional contexts. A conclusion from my review is that the majority of work that has been done in this area focuses on characteristics of students that are not necessarily instrumental in the moment-to-moment enactment of instruction.

The discussion of literature is divided into two main sections, studies that explore teachers' perceptions of students that are generated by researchers, and studies that explore teachers' perceptions of students that are generated by teachers. In the first section there are studies that are survey based, studies that ask participants questions about hypothetical students described though a handful of characteristics, and a study in which the researcher attributes perceptions of students to a teacher through classroom observation. In the second section there are studies that are based on interviews with individual teachers about students, and observations of conversations among teachers about teaching.

Researcher generated categories

In the studies of teachers' perception of students discussed below the researchers begin their study with a set of categories of students already in hand. In these studies the researchers ask teachers to provide their opinion about these categories of students but the teachers have no part in developing the list of students.

Ideal pupil checklist

The *Ideal Pupil Checklist* consists of a list of several general traits of students, like, "adventurous," "does work on time," "non-conforming," "quiet," and "talkative," that teachers are asked to rate as desirable, neutral, or undesirable in a student. The traits listed on the survey "have been found through empirical studies to differentiate fully actualized or creative persons from less creative people" (Schaefer, 1973, p. 444). The aims of research utilizing this survey seem to be primarily descriptive in nature, looking for similarities in views among individual teachers and groups of teachers. Two studies that use the *Ideal Pupil Checklist* to study teachers' perception of students were conducted by Schaefer (1973) and Yamamoto (1969), both described below.

Schaefer's (1973) study found that the student traits most valued by teachers in a special education setting are "being considerate of others," "independent in judgment," "determination," "independent in thinking," "receptive to ideas of others," and "sense of humor;" while these teachers negatively valued the traits: "negativistic," "disturbs class organization," and "fault-finding." Yamamoto's (1969) study found that pre-service teachers positively valued a different set of traits: "courteous," "desires to excel," "does work on time," "obedient," "considerate of others," "industrious," and "sincere" and negatively valued the trait of being “unwilling to accept others.” Only one trait, “considerate of others” was valued highly by teachers in both studies.

Differences between the lists of highly valued traits can be attributed to the differences between the work settings of the two sets of teachers. The teachers in the Schaefer study were working with boys in a school that was designed to support students with difficulties in school, while the teachers in the Yamamoto study were pre-service teachers who had very little experience with students. It is interesting that the more experienced group of teachers included traits related to thinking and judgment on their list of positive traits while these types of traits where missing from the list of traits that were positivity valued by the pre-service teachers. The pre-service teachers seemed to value hard work and motivation over the content of the students’ ideas.

Hypothetical students

A method for studying teachers’ perceptions of students that was used in the studies below, is to present a teacher with a written description of a student that is then used to make predictions about that hypothetical student’s future achievement. In these studies the categories were provided by the researchers and embedded in the descriptions of hypothetical students. The research was aimed at determining which of these categories had an effect on teachers’ instructional predictions.

Shavelson et al (1977) gave teachers fictitious scenarios describing students in terms of their father’s occupation, student’s use of time, intelligence, academic ability, curiosity, and their attitude toward school. The researchers gathered information on the instructional decisions that teachers made using this information, and which of these types of information was the most persuasive to the teachers.

The researchers gave the participants a short scenario that read:

> "Michael is ten years old and beginning the fifth-grade. He lives with his parents, an older brother, and two younger sisters" (Shavelson et al, 1977, p. 89).

The researchers then gave the participants two more sets of information about Michael, asking them to make instructional decisions after each set of information. The information given at each staged varied in terms of valence (positive or negative) and reliability (from a reliable source or not). An example of (positive and reliable) information to be used in the first instructional decision is:

> "In an interview with his parents, his father gave his occupation as an engineer in an aerodynamics firm. In the interview his parents also noted that Michael spent about two hours each evening on his homework and reading books. On an individual intelligence test, Michael scored quite high" (Shavelson et al, 1977, p. 89).

An example of (negative and unreliable) information to be used in the second instructional decision is:

> "When interviewed, some of Michael's classmates said that they didn't particularly like him and that they thought he wasn't a very good student. Cathy Robbins, an education student at a nearby college, had been hired as a substitute aid in Michael's school. She had assisted in Michael's class for a few days and had decided to administer an inkblot test to the class. She interpreted the results to mean that Michael's curiosity led him to be easily distracted from academic

activities and that he had a negative attitude toward school" (Shavelson et al, 1977, p. 91).

The researchers found that the reliability of the information given affected the participants' decisions and that, in situations where participants were given conflicting information about Michael, they made decisions that were predicted by the more reliable information. The categories of students that were created for the use of this study are clearly related to the work of teaching, but the researchers assumed that the categories that matter to teachers' decision-making are static and do not vary between moments, contexts, or even academic subjects.

A study by Borko et al (1979) supports the claim that teachers' categories of students are instrumental in teachers' decision making. In this study, participants were given the same prompts as described in Shavelson et al (1977). The researchers then determined which of the types of information given in the hypothetical prompt were the most persuasive for participants. The researchers found that the most significant factor in participants' decision making was their perception of Michael's ability, which was established prior to the first instructional decision and then participants modified their perception based on the information they receive prior to the second instructional decision.

By looking closer at instances where information given about Michael seemed to have no effect on participants' decisions, the researchers found that teachers' decisions are also based on other factors: beliefs and attitude about education, the nature of instructional tasks, and the availability of alternative strategies and materials to the teacher.

Classroom observation

Morine-Dershimer's (1983) observations of whole class discussions showed how the teachers' evaluations of student contributions created groups of students who were seen as either "students to learn from" or "students who participate." "Students to learn from" were defined as students who contributed to the class discussion and received positive feedback from the teacher. "Students who participate" were defined as students who contributed to the classroom discussion. In the teachers' evaluation of student contributions the teacher was implicitly telling the students how well a particular student was doing his job of being a student. And in particular, the teacher was telling the students how well the student was engaging with the task as envisioned by the teacher. By giving feedback on students' contributions the teacher encouraged some types of task involvement and discouraged others. A finding of this study is that the teachers who engaged in this evaluation of student responses had classes that did better on measures of student achievement than classes of teachers who did not engage in this evaluation.

The studies described above are all examples of research on teachers' perceptions of students in which the researchers began the study with a system for categorizing students that they used as a prompt for opinions for teachers. Below I describe studies that are examples of research on teachers' perceptions of students in which the participants in the study determine the system for categorizing students.

Teacher generated categories

Below are studies of teachers' perceptions of students where the participants in the study generated the categories of students. Unlike the studies discussed above, the researchers did not begin the study with a list of categories of students. Rather, the categories of students emerged from the research.

Individual interviews about students

Morine-Dershimer (1978a) explored teachers' views of students by asking teachers to perform a "pupil sort task" in which participants sorted index cards with their students' names into piles according to similarities in behavior during a particular lesson or at a particular time in the school year. The researchers asked participants to sort their students five different times over the course of the school year. Each time the participants were given a similar prompt, to sort their students in relation to what they'd been observing about their students, but the observation focus was different each time. The first sort task was done in relation to the first day of school, the second was done shortly after teachers received the students' diagnostic reading scores, the third and fourth were in relation to a reading lesson, and the fifth was in relation to the end of the school year. This study shows teachers' perception of students to be dynamic over the course of the year and responsive to the context (in terms of Time of Year, Observational Setting, and Teacher's Curriculum-management System). The results showed that participants sorted their students across several characteristics, Ability/Achievement, Involvement in Instruction, Personality, Peer Relationships, Activity Orientation, and Growth/Progress. Importantly, the ways that teachers were attuned to these characteristics of students varied on the observational setting, e.g., if teachers were discussing their students in general or with respect to a particular lesson.

From the point of view of the current study, the most interesting finding is that Activity Orientation was seen as a category for sorting students with regards to specific lessons but never with regards to general (non-lesson specific) observations. This gives weight to the claim that teachers do see their students in a different light while they are engaged in specific instructional activities.

Morine-Dershimer (1978b) reported on the accuracy of teachers' predictions for their students' success. Twice, after the participant completed the pupil sort task described in the previous study, the interviewer asked the participant to predict student performance on an upcoming achievement test for each student. These predictions were made in September and again in November. In general, participants did better in predicting students' performance on the test in November. The researcher found that teachers' predictions were more accurate for students who they anticipated would succeed than for students that they anticipated would fail and that teachers were likely to predict that a student would be successful if they also rated the student highly on characteristics that they viewed as important for effective functioning within their classroom. Also associated with participants' accurate predictions were participants' categorization of students during the pupil sort task in terms of level of involvement in instruction, personality, ability/achievement, and peer relationships.

Individual interviews about teaching

A study by Mayer and Marland (1997) attempts to record the perception that teachers have of their students. Using data gathered from five participants through individual interviews about teaching, the researchers were able to sort the participants' perceptions of students into the following categories: work habits, abilities, previous schooling, personalities, attributes, interests, family/home background, in-class behavior, playground behavior and peer relationships. The categories that the researchers found in the data reflected the goals of the participants that were interviewed. Participants classified students in such a way as to leverage the categories to help them meet their teaching goals. The perception of students presented in the Mayer and Marland study are focused on relationship building and less focused on instruction.

In their study, Mayer and Marland mapped the perceptions that participants have of their students as well as how these perceptions related to their view of the role of the teacher. They also looked at how these perceptions were used in making instructional decisions. What one learns from Meyer and Marland's study is, in addition to the specific categories that came out of the analysis, that there is evidence that teachers sort their students along stable dimensions.

Observation of teacher conversations

Horn (2005, 2007) collected the categories of students that were apparent in conversations of teaching practice that centered on problems of teaching in particular lessons. In the department meetings observed by Horn, participants discussed problems that had arisen in their classes and together they worked on interpreting and understanding the problems of practice that the lesson displayed.

The ways that participants classified their students in the conversations presented by Horn (2005, 2007) were in terms of students' abilities and motivation. These categories were then used to decide upon the organization of courses and student placement in these courses. In one school, during a conversation around restructuring the algebra and geometry sequence participants talked about their students as "'regular,' 'not-quick,' 'lazy,' 'college-bound'" (Horn, 2005, p. 222). In another school, participants talked about "fast kids" and "slow kids" in relation to group activities that are suitable for both.

The other participants in this school challenged these categories noting that “fast kids” are not really fast at everything, just as “slow kids” are not really slow at everything.

Participants in both schools used their perception of students to think about how to best teach students, but participants at the second school had a much more dynamic view of students and their ability that led to more productive discussion about instructional decisions. “In both departments, teachers communicate their assumptions abut students, subjects matter, and teaching through the kinds of categories they invoked in conversation and the ways that they deployed these categories to model and solve their problems” (Horn, 2005, p. 225).

Horn’s contribution is to describe how participants’ categories of students are instrumental in shaping instruction and school wide curriculum because they were collected from discussions in departmental meetings and therefore impact departmental decision-making. However, I argue that since students are not discussed in the context of instructional scenarios, these descriptions of students do not necessarily impact instructional decision-making. Participants’ assumptions about students, subject matter, and teaching are visible in these categories and do shape the actions that participants see as possible in their classrooms. In Horn’s research these assumptions are framed with respect to the “Mismatch Problem,” or the perceived difference between students’ abilities and the intended school curriculum.

The studies described above show systems of categorizing students that come from the teachers in the study, instead of the researchers conducting the study. These studies tell us something about the importance of students in the work of teaching that is not visible in the studies in which the categorization of students is generated by the researchers. These studies allow us to see which characteristics of students are salient to teachers and how these characteristics of students matter in the work of teaching.

In looking across the literature on teaching geometry, one finds a large body of literature on the use of dynamic geometry software (Gawlick, 2002; Laborde et al 2006; Schwartz & Yerushalmy, 1987). However, much of this literature is focused on the affordances of the software and the interactions between the teacher and the software or the student and the software. This literature (with the exception of Lampert,1988, discussed below) does not pay close attention to how the software affects the interactions between the teacher and students or, in particular the teacher's perception of students.

Overall, the types of students that are explored in the studies reviewed here are not necessarily instrumental to the moment-to-moment enactment of instruction. Some examples of the categories of students discussed in these studies are gender, socioeconomic status, overall motivation and ability, and consideration of others. While all of these perceived characteristics may have consequences for students' behavior inside instruction these studies do not provide a direct connection between these labels and how these students might participate in instruction. For instance, if a student is determined to have a low ability level how does this affect their instructional behavior? Do they have difficulty understanding the resources of a task? Do they stop after writing the 'given' and 'prove' when they work on proofs? Do they disrupt the class when they become confused? The answers to these questions could be useful in designing effective instruction for this student. Simply labeling a student as "low ability" does not provide any information for how that student might act or what corresponding teacher action is called for.

The current study aims at answering these questions. I look at teachers' perceptions of students as they relate to the work that the teacher does to enact instruction.

In the following section I review two studies of teachers' perceptions of students that do look at teacher's perceptions of students inside instruction. The study, "Teachers' thinking about students' thinking about geometry" (Lampert, 1988) provides guidance for the current study in the way that it focuses on teachers' perception of students' cognition and behavior in instructional contexts. The idea of a 'modal student,' or the teacher's hypothetical partner in instruction (Herbst, 1998), points to the importance of teachers' perception of students while enacting instruction.

Motivation for study

There is little research on teachers' perceptions of students that highlights the importance of instruction in these perceptions. I would like to describe two exceptions that act as motivation for the current study. The first is Lampert's study, "Teachers' thinking about students' thinking about geometry" (1988). The second is Herbst's construct of the "modal student" (Herbst, 1998). Both of these are remarkable in the way that they foreground the work that teachers and students do in their description of how teachers think about their students.

Lampert (1988) looked at how teachers describe students during work in novel classroom situations and how they described this work as different than in traditional lessons. She describes teachers' reflections on geometry lessons that incorporated a dynamic geometry software package, The Geometric Supposer (Schwartz, & Yerushalmy, 1987). These reflections were gathered in individual interviews with teachers about their experience teaching with the Supposer. The teachers were teaching with curriculum materials designed around the use of the Supposer that made use of open-ended questions and had students make inductive claims about geometric relationships.

The teachers in the study reported several ways that this method of interacting with the content is different than in traditional lessons. Although Lampert does not explicitly discuss the students that the teachers in the study perceived, she does discuss the work that teachers see students doing, and from this one can infer descriptions of students in terms of their action. Here I give two examples of ways that teachers described the work that students do that highlight the instructional significance of these descriptions.

In the open-ended explorations of Supposer lessons the normal work of being a geometry student was disrupted in two ways. First, the teachers saw that students would make discoveries that came out of order with respect to the order of topics in a traditional geometry course. This lack of a stable order resulted in teachers perceiving students who made claims and discovered relationships that they did not have the tools to prove deductively. Second, because of the visualization of geometry concepts afforded by the Supposer, students would make geometric discoveries but they would not necessarily see the need to deductively prove their claims. From these descriptions of instruction employing the Supposer, one can formulate descriptions of students; students who discover relationships that could not be proven deductively, and students who do not see the need to produce deductive proofs. Both of these descriptions point to students who clash with the traditional work of a geometry classroom. In a traditional geometry classroom students would need to justify claims by making deductive links that connect their assumptions to their conclusion based on theorems that they have previously proven. The participants worried that the affordance of the software to support inductive reasoning would work against efforts to support these students in learning to make deductive arguments, which are seen by the participants as a cornerstone of the work of geometry students.

Herbst (1998) conceives of the "modal student." The modal student is an imagined rhetorical partner for the teacher while she is conducting instruction. Herbst defines the modal student as "a hypothetical person playing the role generalized across all students in the class" (Herbst, 1998, p. 150). That is, returning to the metaphor of the class as an orchestra, the modal student would be the equivalent of an instrument imagined by the conductor that played in the voice of any instrument. Classroom interaction is different than other interactions that individuals have outside of school because, even though it is constructed by, maybe, 30 individuals, the interactional pattern mirrors interaction between only two individuals; the teacher and the students. In this interactive pattern all of the students function as one voice in conversation with the teacher. This imagined conversation partner of the teacher is what Herbst conceives of as the modal student.

The modal student is useful for thinking about the work of teaching; because it makes clear that there are actions that some students must perform to allow instruction to proceed. Without students interacting with the teacher (even in the form of silent listening), the conversation between the teacher and students would collapse to a monologue constructed by the teacher, without an audience, and there could be no learning on behalf of the student. Part of my work in the current study could be conceptualized as describing this modal student, as it exists for the high school geometry teacher. That is, as geometry teachers conduct instruction, who are their rhetorical partners?

In the work of Lampert and Herbst one sees that it is important to the teachers' work to characterize the student in terms of the work that they *do*, not who they *are*. This means that there is no need for an inference about how the description of the student will correlate with the teachers' expectation for how these students will act in instruction. For example, if a teacher reports that a student is "bad at reasoning," which is a report on who the student is, it is not clear how the teacher expects this to be reflected in the students' actions and therefore not clear how this description leads to an instructional decision. However, if a teacher reports that a student "will make claims without seeing the need to prove them," which is a report on what the student does, then it is easier to see how the teacher expects this student to act during instruction, and therefore easier to see how this description could lead to an instructional decision.

The current study aims at collecting descriptions of students in terms that are relevant to instruction. My goal is to learn about teachers' perceptions of students that teachers see as instrumental to instruction. Because teachers are the agents who conduct instruction, using students as resources, understanding how teachers perceive students gives a better understanding of how instruction is enacted and how exactly particular teaching acts are dependent on students.

To achieve this goal of describing teachers' perceptions of students that are instrumental to the work of teaching, it is necessary to have a theoretical framework for describing this work. Also, because teachers' perception of students are a subset of teacher knowledge, it is necessary to have a theoretical framework for describing teacher knowledge. In the next section I describe the theoretical frameworks that I deploy in this study.

Theoretical framework

In this section I will describe particular frameworks for classroom interaction, the work of teaching, and teacher knowledge. Each of these frameworks builds on the previous, so the framework for the work of teaching is based on the framework for classroom interaction and the framework for teacher knowledge is based on the framework for the work of teaching. The frameworks for the work of teaching and for teacher knowledge will be used in the analysis of the data to describe the ways that teachers' perceive their students that are instrumental to instruction.

This section builds to a description of practical rationality (Herbst & Chazan, 2003), which is a framework for teacher decision-making based on the work of teaching. It hypothesizes that the "knowledge" used in teacher decision making is not necessarily true or verifiable, but that it is a rationale for teachers' action based on the dynamics of the work of teaching. Before I describe practical rationality I describe a framework for the work of teaching that highlights the work that teachers do to manage students' work on mathematical task and value this work with respect to the didactical contract. I begin this section with a description of the didactical contract, instructional situations, and mathematical tasks, which are tools for interpreting classroom interaction.

I use these frameworks for this study because they provide a coherent method for connecting teachers' descriptions of students, as a subset of teacher decision making, to the work of teaching, and to classroom interaction. Also, practical rationality conceptualizes of teacher knowledge not as something that is rational, as in correct, but rational in the sense that it is sensible, or justifiable based on the work that teachers do. This view allows for the contradictions and inconsistencies that are inherent in teacher decision-making.

Model of classroom interaction

To understand classroom interactions I use a model of classroom interaction developed by Herbst that is based on Bourdieu's notion of symbolic economy (1980, 1998) and Brousseau's notion of didactical contract (1997). According to this model, teachers and students act as if they are trading classroom work for claims that they have taught and learned a bit of the geometry curriculum. The foundational hypothesis is that inside educational institutions the teacher and her students enter into this economy because of their obligation to a didactical contract that brings students and teachers together to teach and learn geometry. A didactical contract specifies in rather general terms what it means to teach and learn geometry and what the geometry is that needs to be taught and learned.

The didactical contract can be thought of as a set of norms, or tacit rules for how an observer would conclude that instruction should proceed. Norms, or dispositions, are cultural resources that actors use to construct their performances in particular settings. According to Bourdieu, norms are "structured structures predisposed to function as structuring structures (p. 53)". That is, these structures for action are preexisting in the culture but when a particular actor enacts them in a particular moment they feel (to the actor and his companions) as if they are spontaneous improvisations in response to the current circumstances.

Teachers use these norms to construct their instructional moves and to value their work and the work of their students. Of, course not all moves that a teacher makes are according to the norms, and each decision made by the teacher feels unique, as if it were constructed solely by her own choices and circumstance, not as if it were scripted by a set of rules. Instruction that appears to be constructed according to these norms is considered "normal" instruction.

An important problem of the framework of the didactical contract, which an observer can infer from the actions of a teacher and her students, is to find out how the norms of the contract apply to specific chunks of work, or conversely, how their work on a specific task contributes to meeting the demands of the contract. Research on the use of specially designed tasks (e.g., Brousseau, 1997; Herbst, 2003) has shown that one way this problem is handled is by negotiating how the contract applies to the task when a task is implemented—this negotiation can be viewed as changes to the task itself or to the way in which the task is taken as contributing to the didactical contract. In the extreme the task can be dramatically changed or its place as part of the course of studies can be severely alienated.

Another way in which this problem can be seen to be handled is through the existence of instructional situations (Herbst, 2006). Instructional situations are recurrent patterns of activity that organize the actions of the students and teacher so that they can engage in work that exchanges for claims on the contract. In particular, tasks that are traditional in a mathematics course, such as "solve $2x - 1 = 3x + 4$" in algebra, do not often call for a negotiation of the task, since the word "solve" and the existence of one variable both act as cues to conjure up what the student is supposed to do (Chazan & Lueke, 2009). In general I hypothesize that these customary, recurrent patterns of activity make room for some canonical tasks saving people the need to negotiate how the contract applies for the task. In geometry the existence of the situations of "doing proofs" (Herbst & Brach, 2006) and "installing theorems" (Herbst, Nachlieli & Chazan, in press) has been documented. In this study I focus on the instructional situations of 'making conjectures' and 'doing proofs.' For a description of these situations see "The work of 'studenting' in high school geometry classrooms" (Aaron, this volume).

By using this model of classroom interaction, classroom activity is viewed as made up of tasks, often embedded in situations that facilitate the exchange of work on tasks (Herbst, 2006). Therefore, the opportunities for action in the classroom can be viewed through the lens of the tasks that are enacted in class. Below I describe how *task* is used in the current study.

Following Doyle, a task can be modeled by identifying a product or goal that students are expected to arrive at, a set of resources for students to utilize, and a set of operations students can enact to reach that goal (Doyle, 1983; Doyle, 1988). The goal and resources of the task make up the task milieu, or the environment that the student works within. The milieu provides feedback to the student when the student performs operations on resources (See Figure 32). A task could be a proof exercise where the product would be a proof that connects the premise that a triangle ABC is isosceles to the conclusion that its base angles are congruent. In this case the operations could include the introduction of an auxiliary line and the discovery of congruent triangles. The resources could include a diagram, a 'given' and 'prove', and the two-column proof format. If a student were provided a diagram as a resource and then performed the operation of drawing an auxiliary line, the milieu could provide feedback to the student in the form of showing a new diagram containing, for example, new triangles to use as resources.

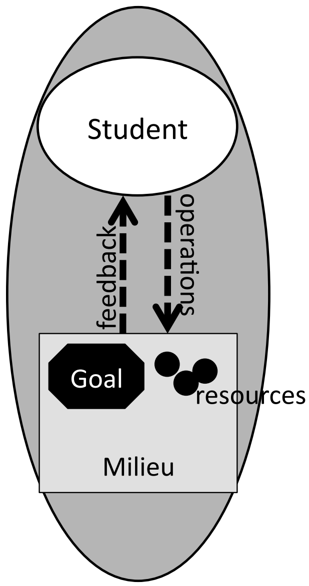

Figure 32: Students' work on tasks (adapted from Herbst, 2010)

As an example, consider the angle bisectors problem: What can one say about the angle bisectors of a quadrilateral? That problem could be part of several tasks. When students are working on the angle bisectors problem, if a dynamic geometry sketch of a quadrilateral were part of the task design then it could be expected that students might use this resource to explore many cases (by "dragging" the vertices of a given quadrilateral). In contrast, without the software it could be expected that students might only draw a handful of different quadrilaterals. Similarly, if students were working in small groups it could be expected that students might have more opportunities to produce counterexamples and counterarguments in response to their classmates' ideas than if they were working independently. Thus a task is not just the problem statement but the chain of anticipated, possible, or enacted operations using resources to achieve the end product stipulated in the statement.

In thinking about teachers' perception of students, I hypothesize that when teachers talk about their students within the context of instruction they will describe their students in terms that can be understood to relate to the task, situation, and contract. For instance, teachers discussing their students in terms related to the task could describe the resources that the student is using. A teacher could say, "she was thinking about a square," describing the resources that a student used on a task. Descriptions related to both the situation and contract involve the valuing of student work with relation to the contract. For instance, teachers discussing their students in terms related to the situation or contract could describe which student had a conjecture that was worth sharing with the class. A teacher could say, "I'd bring the kid who worked on a special case to the board" in relation to the norm for the situation 'making conjectures' that states that *the teacher should call a student to the board who has a conjecture that is apparently correct but that other students can build on*. Or a teacher could say, "I know lots of students made mistakes like this, so we should talk about it together" in relation to the contractual norm that states that *the teacher is responsible for recognizing and publicly identifying errors* (Herbst et al, 2010).

Now, building on this framework for classroom interaction, I describe a framework for the work of teaching. This framework illustrates the various pieces of work that a teacher would need to do to conduct classroom interaction as described above. This framework for the work of teaching will then be used to construct a framework for practical rationality.

Model of the work of teaching

Central to this framework for the work of teaching (Herbst, 2010) is students' work on tasks (See Figure 32). In the figure the solid arrows indicate the teachers' work. She is responsible for observing the tasks enacted by students, the milieu of these tasks, transactions between the students' work on and the mathematics to be learned, and the mathematics to be learned. The grey arrows show that the aspects of the teachers' work that are dependent only on the relationship between the teacher and the mathematics and do not directly involve students so they are less important to the current study. The dotted arrow represents the exchange of work that students do on tasks for claims on the didactical contract. The dashed arrows represent the student's interaction with the task milieu.

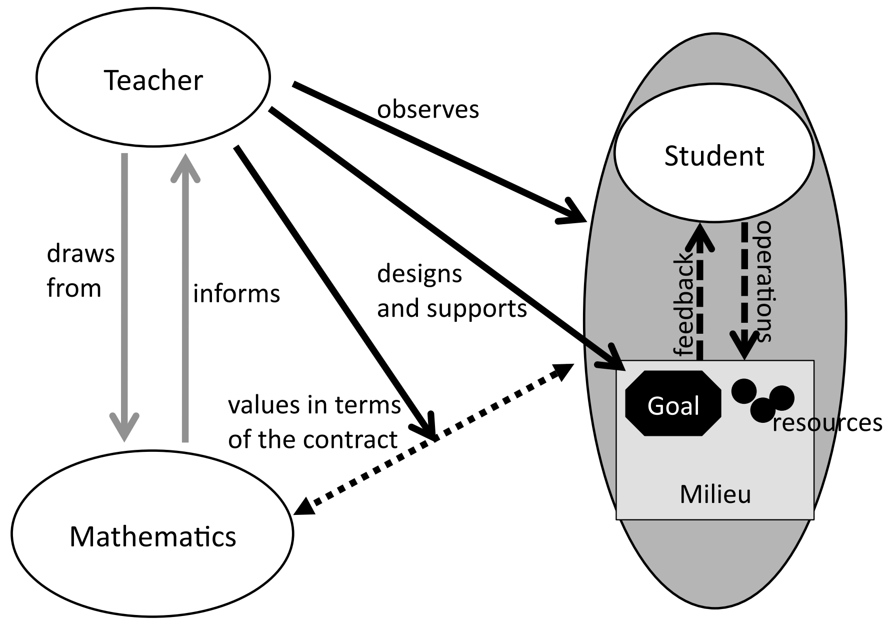

Figure 33: The work of teaching (adapted from Herbst, 2010)

Figure 32 shows the hypothesis that to support students' work on tasks, the teacher is responsible for doing work to design and support the milieu, observe students' performance of operations and the feedback they receive from the milieu, and value students' completed work in terms of the didactical contract. The work of valuing students' work on task in terms of the didactical contract is related to the instructional situation, which, if the class is working in an established situation, will guide this work. The teacher also is responsible for drawing from and being informed by the mathematical content that is at stake in the course; since this work does not interact directly with students it is not used in the current analysis.

Returning to the research questions about teachers' perception of students, from Figure 32 one can see that students who are described in terms of the resources or goals of a task could be instrumental to the work of designing and supporting the milieu. Students who are described in terms of the operations of a task could be instrumental to the work of observing the student's work on the task. Students who are described in terms of the situation or contract could be instrumental to the work of valuing students' work in terms of the contract.

An important goal of the current research is to explain how descriptions of students, which are related to classroom interaction, and which can be seen to be instrumental in the work of teaching, can be systematically organized within a framework for teacher knowledge. That is, how are these descriptions of students integrated into teachers' decision-making? The following section describes a framework for teacher decision making, practical rationality, which provides a guide for thinking about how teachers' perceptions of students are used in teacher decision-making.

Practical Rationality

The teachers' perceptions of students that are the focus of this paper are one facet of teachers' *practical rationality* (Fenstermacher & Richardson, 1993; Green, 1976; Herbst & Chazan, 2003). As conceived of in this paper, practical rationality includes dispositions to abide by norms that originate from the role a teacher needs to play in the instructional situations of a given contract, and dispositions to respond to professional obligations that originate from the position of "mathematics teacher," and it complements the personal resources that an individual brings to the work (See Figure 33, Herbst, 2010). Each of these types of knowledge contains some subset of knowledge about students, or encourages particular ways of perceiving students. That is, the norms of the situation lay out expectations for students' actions, the professional obligations of teaching oblige teachers to attend to their students in particular ways, and individual teachers bring beliefs about students and abilities to understand students' mathematical arguments.

Figure 33 shows teachers' how professional obligations and the work of teaching contribute to practical rationality. This practical rationality is mediated by teachers' personal resources before it results in action. Each of these constructs is explained below, but first I explain the structural relationships between these constructs. Following Herbst, I assume that the norms that guide the work of teaching, shown on the right-hand side of Figure 33, model normal classroom interaction, and stem from the frameworks for classroom interaction and the work of teaching outlined above. However, because the profession of teaching is enacted by people with obligations and resources, teaching a particular set of students, in a particular environment, instruction often does not follow these norms. To account for this deviation in the model of practical rationality, teachers' actions are also conceived of as being influenced by teachers' professional obligations, shown on the left-hand side of Figure 33. These professional obligations provide justification for perceived breaches of norms. That is, when instruction does not go normally, I hypothesize that teachers can attribute the perceived breaches to an obligation of the profession. I also hypothesize that all individuals who take on the role of "geometry teacher" share a familiarity with norms of instruction and professional obligations. However, within this group there is still variance in action, which is accounted for by individuals' personal resources, seen in the lower middle of Figure 33.

These resources are used to explain why two teachers, who share the same familiarity with the work of teaching geometry and professional obligations might act in different ways while conducting instruction. Below I explain these norms, professional obligations, and personal resources in more detail.

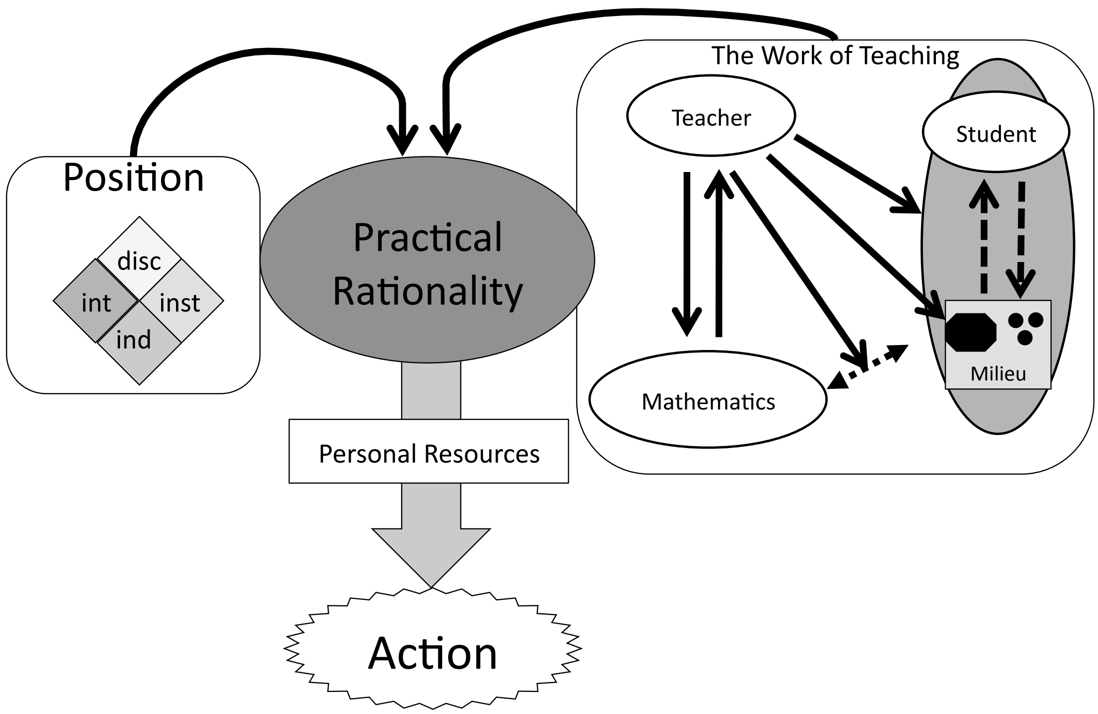

Figure 34: The practical rationality of teaching (adapted from Herbst 2010)

Situational norms are 'activated' by situational cues that prompt teachers to act as if a course of action is appropriate at a particular moment. These norms are embedded in instructional situations (Herbst, 2006). Depending on the instructional situation, teachers act as if they are compelled to make different instructional decisions. These norms are of three different types, accountability norms that dictate who (teacher or student) should do what work in the classroom, temporal norms that dictate when specific actions should be done, and exchange norms that dictate what work trades for what claims on the didactical contract. These norms are resources that teachers use to construct normal classroom interaction.

Professional obligations are hypothesized, tacit commitments that teachers have to the profession of teaching (Herbst & Balacheff, 2009). I assume that these commitments are normative for all members of the profession and are applicable at all times that an individual is in the role of teacher. These obligations are hypothesized to come from four sources, or stakeholders in educational endeavors; individual, interpersonal, disciplinary, and institutional. Individual obligations are obligations that teachers have to individual students. Interpersonal obligations are obligations that teachers have to their students as a group and to the smooth functioning of this group of individuals. Disciplinary obligations are obligations that the teacher has to the discipline of mathematics. Institutional obligations are obligations that the teacher has to the institution of the school that she works within (Herbst, 2010). As explained above, these obligations interact with situational norms to produce perceived breaches of norms in favor of commitments to one of the stakeholders listed here.

Each teacher brings her own personal set of resources to the work of teaching. This is a diverse set of resources that include, teacher beliefs (for example see Cooney, Shealy, Arvold, 1998; Leatham, 2006), mathematical knowledge for teaching (For example see Ball, Hill & Bass, 2005), educational characteristics (like the number of courses that they've taken), and professional characteristics (like the number of years that they've been teaching). These personal resources mediate the professional obligations and situational norms that guide teachers' instructional decision making. Although these personal resources are an important part of the model of practical rationality, the current study focuses on the collective aspects of practical rationality, and takes participants to be informants for the profession of teachers.

Now, in light of this framework for practical rationality, I reformulate the research question. Originally, I posed the question;

- What perceptions of students are instrumental in the work of teaching?

I can now ask the questions,

- What perceptions of students are instrumental in the work of designing and supporting the task milieu?
- What perceptions of students are instrumental in the work of observing students' work on a task?
- What perceptions of students are instrumental in the work of valuing students' work in terms of the didactical contract both in general and in the particular instructional situations of 'making conjectures' and 'doing proofs'?
- What perceptions of students are instrumental in encouraging the teacher to breach the norms of instruction in favor of a professional obligation?

The first three research questions inquire into instruction that unfolds according to the hypothesized instructional norms. They seek to understand how teachers' perceptions of students are used to create instruction. The last question inquires into instruction that is not normal, or not guided by the norms of instruction. It seeks to understand the teachers' perception of their students in terms of their professional obligations that encourage them to act in ways that could be perceived as a breach of an instructional norm.

In this section I have detailed the theoretical frameworks that guide this study. I began with a framework for viewing classroom interaction as comprised of mathematical tasks, instructional situations, and the didactical contract. Using this framework for classroom interaction I detailed a framework for viewing the work of teaching. For the current study I focus on the work of designing and supporting the task milieu, observing students' task operations, and valuing students' work on tasks in terms of the didactical contract. This framework for the work of teaching was then used to detail a framework for teacher rationality based on this work of teaching. Together, these frameworks allow me to reframe the research questions to look at teachers' perceptions of students that are instrumental in specific aspects of the work of teaching. I am also able to utilize the idea of breaches of perceived norms based on professional obligations, to see teachers' perceptions of students that are related to teachers' professional obligations. In the following section I describe the data that was used in the current study to inform these research questions.

Data

The data presented here were collected over the course of two school years, during which the ThEMaT (Thought Experiments in Mathematics Teaching) research project held study groups with experienced geometry teachers. Each year, two groups of five to twenty teachers met for three hours once per month. Participants came from a diverse group of schools including urban, suburban, and rural schools. Participants served a diverse group of students. In these study group sessions, participants watched and responded to animated classroom scenarios in conversations with fellow participants and members of the ThEMaT research team. Participants also engaged in other activities related to the animated scenarios, like working on mathematical tasks, looking at student work, and reading and writing scripts for classroom scenarios.

These sessions were video and audio recorded and then transcribed and indexed for analysis. To index the data corpus sessions were divided into intervals based on changes in the activity structure of the session (Herbst, Nachlieli, & Chazan, in press). An interval is a continuous length of time during a study group meeting in which participants are engaged in a particular activity or conversation. Herbst, Nachlieli & Chazan define, "An interval consists of segments of group interaction that participants construct as units of conversation by way of employing a combination of the organizational features." These features include who the active participants are, the specialized division of labor in the conversation, the labels that participants use to describe the theme being discussed, and length of interval (intervals are normally of the order of 2 to 8 minutes). These features result in intervals that cover the timeline of the session, but overlap at their boundaries.

These intervals are used as the units of analysis because I theorize that they represent self-contained mini-conversations within the larger conversations that constitute the study group sessions. This parsing collects turns of conversation that are, from the point of view of the participants, related, and separates turns of conversation that are, from the point of view of the participants, unrelated. That is, I use the flow of the conversation to determine which topics are related and which are not, instead of imposing a researcher's view of which topics are related or unrelated.

The data in this study consists of all the intervals in which the participants discussed one particular animated scenario, The Square[9]. The Square was watched in eight sessions, which are made up of 367 intervals. Of those intervals, The Square was discussed in 136 intervals. In the other intervals participants were discussing other animated scenarios, responding to prompts not related to the animated scenario, discussing logistics, or eating dinner. Table 8 provides a summary of the sessions included in this study, the participants who attended each session, the number of intervals in each session, the number of intervals in each session spent discussing The Square, and the total duration of each session.

Table 8: Summary of the Intervals Included in the Analysis

Session Name	Participants in Attendance	Number of Intervals in Session	Number of Intervals Discussing The Square	Length of Session
ThEMaT 081905	Carl, Lucille, Melissa, Esther, Karen, Lynne, James, Cynthia, Edwin, Glen, Mara, Penelope, Greg, Tina, Megan, Denise	21	13	1h 40m
ESP081905	Carl, Lucille, Melissa, Esther, Karen, Lynne, James	37	4	2h 22m
ITH081905	Cynthia, Edwin, Glen, Mara, Penelope, Greg, Tina,	51	11	2h 22m

[9] The Square and the rest of ThEMaT's animated classroom scenarios can be viewed at http://grip.umich.edu/themat.

	Megan, Denise			
ESP091305	Carl, Lucille, Melissa, Esther, Karen, Lynne, James	57	9	2h 40m
TMT110706	Edwin, Cynthia, Raina, Melanie, Stan, Jake, Megan	72	48	3h 1m
TMW111507	Madison, Mark, Tina, Denise, Karen, Tabitha	55	43	3h 10m
TMT062007	Edwin, Lucille, Raina, Melanie, Stan, Jake	41	6	1h 35m
TMW062007	Glen, Madison, Tina, Denise, Karen, Tabitha	33	2	1h 17m
Total		367	136	18h 7 m

When quotes from the transcripts of these sessions are displayed in the results section they are labeled with a parenthetical citation that contains the session name, the interval number within the session, and the turn number within the interval, that points to where the quote is located.

The sessions were conceived of as modified breaching experiments (Garfinkel, 1964; Herbst & Chazan 2003, 2009; Herbst, Nachlieli, & Chazan, in press). A breaching experiment is aimed at uncovering participants' tacit knowledge by confronting them with a scenario that breaks with their expectations for social interaction. I assume that much of teachers' knowledge about how to teach is tacit and therefore inaccessible by simple probing. By asking teachers to react to scenarios that display an action that we hypothesize will be perceived as a breach in normal action, teachers have the chance to respond to that hypothesized breach by narrating the scenario as it would normally occur, pointing to the norms for instruction, imagining the consequences of the actions in the scenario, or pointing to the rationale behind an action. A difference between the breaching experiments used in this study, and breaching experiments as they were originally conceived is that our teachers are not actually taking part in the scenario, but are experiencing it vicariously by engaging with an animated scenario of geometry instruction.

The animated scenarios used in the breaching experiments were developed explicitly for this purpose. I argue that these animated scenarios have unique features that allow teachers to become immersed in the scenarios that they depict (Aaron & Herbst, 2007). The animated scenarios invite this immersion through the graphics that compose the animated scenarios and through the story that is presented in the animated scenario. In terms of the graphics, the characters in the story are represented by blue, animated 2D-characters (see Figure 35). The use of non-realistic looking characters in the representations is hypothesized to prevent the viewer from rejecting the idea that they could be the actor in the scenario. Since the characters clearly do not represent any real person, anyone could fill their position. In terms of the story depicted in the animated scenario, important aspects of the narrative have been omitted, like who the characters are beyond their performance in this scenario, what comes before or after this story in time, and in what school, in what city, in what environment this story took place. All of these missing aspects of the narrative invite the viewer to project their own experiences onto the context of the story (Chazan & Herbst, in press; Herbst & Chazan, 2006; Herbst et al., in press).

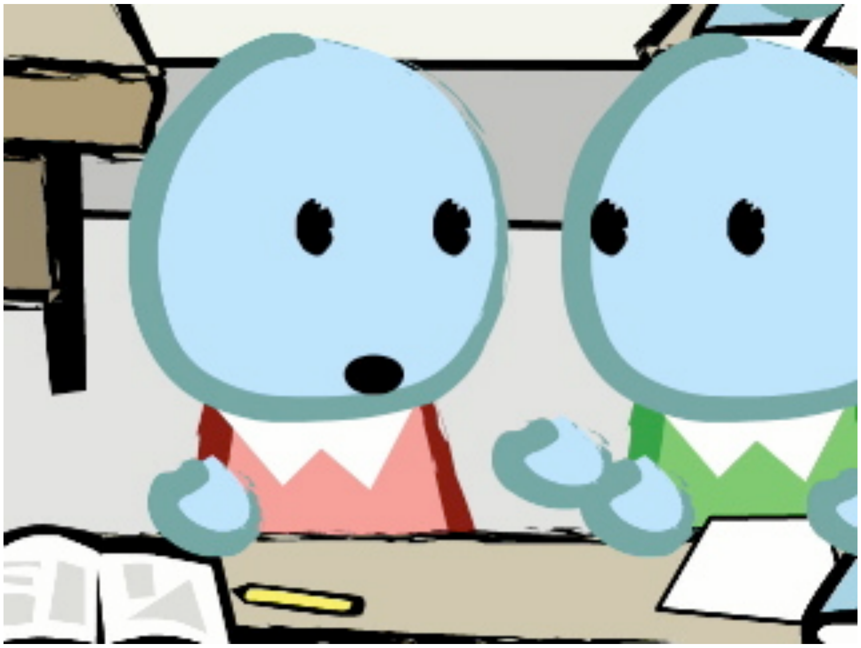

Figure 35: Characters in an animated representation of instruction

Description of animated scenario

The following is an analysis of the work that the animated class does in the animated scenario, The Square. This analysis divides the timeline of the animated scenario into "segments," according to the mathematical task that is underway. In particular, each segment of The Square is defined by the task product that the class is working towards and the task resources the class has available to work with. Within each segment students deploy operations, using resources, with the aim of reaching the task product, along with other things that make up the scenario but are not necessarily tied to the task. These tasks structure the flow of the work of the class and characterize the mathematical work that is being done. A segment is all of the activity that is depicted in the animated scenario during the time that a particular task is being deployed.

In The Square there are nine segments during which the class works on nine different, but related, tasks. The first six segments show tasks within the instructional situation of ‘making conjectures.’ Segments seven, eight, and nine show tasks within the instructional situation of ‘doing proofs.’ Within the situation of ‘making conjectures,’ (1) the class is given a conjecturing task about angle bisectors of a quadrilateral; (2) Alpha shares his conjecture about diagonals of a square; (3) the teacher rephrases the task to be about angle bisectors of a square; (4) the teacher asks the class if diagonals and bisectors are the same thing; (5) Gamma illustrates that diagonals and angle bisectors are different using the case of a rectangle; and (6) in light of Gamma’s counter-example the class reformulates Alpha’s conjecture. Within the situation of ‘doing proofs,’ (7) the teacher calls on Lambda to provide a proof for Alpha’s conjecture; (8) the teacher removes one diagonal from the diagram on the board to assist Lambda’s proof and; (9) the animated scenario ends with the teacher calling for a two-column proof of Alpha’s conjecture. Below I describe each of these segments in terms of the task that the class is working on.

Segment 1: The class is given a conjecturing task about angle bisectors of a quadrilateral

The mathematical story in The Square begins with the reminder that the angle bisectors of a triangle meet at a point and the statement of the problem, “What can one say about the angle bisectors of a quadrilateral?” The teacher asks the students to make conjectures in response to this problem, telling them that they will then try to prove some of those conjectures. The animated scenario shows students sitting at desks with a partner as they begin to discuss the problem.

This set up of the problem and the time that students spend working independently constitutes the first segment. There are a few resources that are available to the students to use on the task. From the statement of the problem one sees the mathematical resources, quadrilaterals and angle bisectors; in addition, from the reminder about triangles, the mathematical resource of point of intersection is also available to students. Students also have the resources of their partner to work with on this problem and the paper and pencil that students have on their desks.

Segment 2: Alpha shares his conjecture about diagonals of a square

At the end of the time that the teacher has allocated for working on the problem the teacher calls the class to order and asks Alpha to share his conjecture from the board. Alpha comes to the board, draws his diagram and says, “It’s a square and the diagonals. They bisect each other.”

When the teacher ends students’ work on the angle bisectors problem, she is changing the product of the task, from one of producing conjectures, to one of sharing and discussing conjectures. In terms of the mathematical task, there are two important aspects of Alpha's statement. First, he changes the focus of the problem, from one about general quadrilaterals to one about squares. This changes the resources of the task by introducing squares as a possible resource. Second, instead of talking about angle bisectors as the original task called for, Alpha is introducing a new mathematical resource to the task, diagonals, and not saying anything explicitly about angle bisectors.

Segment 3: The teacher rephrases the task to be about angle bisectors of a square

In response to Alpha’s conjecture, the teacher corrects Alpha by saying that the problem is about angle bisectors, not about diagonals. The teacher seems to be dismissing the possibility that either Alpha misspoke and did mean to talk about the angle bisectors, or that Alpha was implicitly making a claim about the relationship between the diagonals and angle bisectors of a square. The teacher changes the task by removing diagonals from the set of resources that are acceptable to make claims about. She also implicitly accepts the addition of squares to the set of resources that students have at their disposal.

In this segment I expect participants to see confusion around the resources of the task. In segment 2, Alpha introduced the diagonals as a resource and in this segment the teacher rejects diagonals as usable resources. This confusion is exacerbated by the fact that in the case of a square, which the class is looking at, the diagonals and angle bisectors are represented by the same objects.

Segment 4: The teacher asks the class if diagonals and bisectors are the same thing

The teacher asks the class, "What can one say about angle bisectors of a square?" and Alpha returns to his seat saying, "I just thought that the diagonals cut the square in half." The teacher writes "Alpha: the diagonal cuts the square in half" on the board and asks for elaboration. Beta elaborates on Alpha's comment by saying that the diagonals are also the angle bisectors. The teacher then asks the class if they agree with Beta's assertion, that the diagonals are also the angle bisectors.

In this segment the teacher quickly changes the product of the task twice. First she asks the students to elaborate on Alpha's statement, that the diagonals cut the square in half, and then she asks the students if they agree with Beta's statement, that the diagonals are the angle bisectors. The two statements made by Alpha and Beta, that the diagonals cut the square in half, and that the diagonals are the angle bisectors, are also added to the resources that the students have to use on the task.

Segment 5: Gamma illustrates that diagonals and angle bisectors are different using the case of a rectangle

Gamma disagrees with Beta's statement that the diagonals are the angle bisectors and brings up the example of a rectangle. She comes to the board to present a counter-example. She asks the class to imagine a square being elongated to form a rectangle. As this happens, the angle bisectors of the figure remain constant while the diagonals are deformed so the diagonals cannot be the same as the angle bisectors. Alpha defends his claim by saying that he was making a claim about squares, not rectangles.

In this segment Gamma can be seen to enlarge the task resources by talking about the case of a rectangle in addition (and contrast) to the case of a square. She also highlights the difference between the two mathematical resources of diagonals and angle bisectors. By showcasing the example of a rectangle, Gamma shows that the diagonals and the angle bisectors are represented by different objects in the case of a rectangle, therefore rejecting the idea that they are, in general, the same.

Segment 6: In light of Gamma's counter-example the class reformulates Alpha's conjecture

Gamma returns to her seat and Beta rejects her counter-example. Beta reformulates Alpha's conjecture by specifying the condition that the quadrilateral is a square; Beta says that Alpha meant, "in a square the diagonals meet at a point." This claim is quite different from Alpha's original conjecture that the diagonals of a square bisect each other. Gamma further clarifies Beta's statement by saying that "in a square the *angle bisectors* meet at a point." This reformation of Alpha's conjecture is about angle bisectors instead of diagonals and it specifies the case of a square so it is not vulnerable to Gamma's earlier counter-example. In response to Beta and Gamma's reformulation Alpha makes the statement, “in a square the angle bisectors meet at a point because they are the diagonals.” The teacher writes “in a □ the ang bis ⊠” on the board.

In this segment several more statements become available for students to use as resources on the task. Of the statements made by Beta, Gamma, and Alpha, the teacher writes only “in a □ the ang bis ⊠” on the board for the whole class to see. Because this statement is about angle bisectors, it is most closely related to Gamma’s statement, that in a square, the angle bisectors meet at a point. The confusion between angle bisectors and diagonals that was apparent in earlier segments is gone in this segment and the class acts as if they are clear about the importance of the square in Alpha’s conjecture. This is the last segment during which the class is working on making conjectures. In the next segment the teacher reframes the task and asks the class to begin working on doing proofs.

Segment 7: The teacher calls on Lambda to provide a proof for Alpha’s conjecture

This segment begins with the teacher asking Lambda to provide a proof of Alpha’s conjecture. Lambda’s proof begins with the claim that the diagonals cut the square into two equal pieces so the diagonals are the same as the angle bisectors. The teacher interprets Lambda as meaning that the four small triangles formed by both diagonals are congruent instead of the two larger triangles formed by one diagonal are congruent. Seeing that the teacher is not looking at the same triangles that he is, Lambda asks the teacher to remove one diagonal, saying that only one is required for the proof. The teacher responds that she will leave both diagonals on the diagram because squares have two diagonals.

Lambda continues his proof by saying that the two triangles formed by one diagonal are congruent and isosceles. At this point the teacher asks Lambda to state the claim that he is trying to prove. Lambda responds by saying that "the base angles of both triangles have to be equal to each other." Here Lambda is referring to the angles at the vertex of the square, formed by one diagonal. The teacher misinterprets this to mean the two base angles of one of the small triangles formed by two diagonals. Lambda again asks the teacher to remove one diagonal from the diagram. Beta expresses confusion at the idea of only having one diagonal. The teacher reminds Lambda that the proof the class is working on is about the intersection of the diagonals, therefore there must be two diagonals in the diagram, and otherwise there would be no intersection.

In this segment one sees Lambda trying to convince the teacher to remove one diagonal from the square. This would change the resources of the task by removing an element from the diagram. In general, the teacher is in control of these diagrammatic resources and so the animated teacher resists Lambda's attempt to change the task in this way. Reciprocally, the teacher makes a request that Lambda does not comply with. The teacher asks Lambda to clarify what it is that he is proving and Lambda ignores her request. The teacher had already written a proof statement on the board but she apparently thought that Lambda might be working on proving something besides this statement. If Lambda had admitted had he was proving a statement other than the one given by the teacher this would also change the resources of the task.

Segment 8: The teacher removes one diagonal from the diagram on the board to assist Lambda's proof

Lambda insists that his proof only needs one diagonal. The teacher gives into Lambda's requests and erases one diagonal from the diagram at the board. Once the teacher has removed one diagonal Lambda concludes that the base angles of both triangles are congruent because they are isosceles. He goes on to say that this can be used to show that the diagonal is also an angle bisector. Lambda asserts that a similar argument could be repeated to show that the other diagonal is an angle bisector.

In this segment that teacher removes one diagonal from the diagram of the square, changing the resources of the task. In removing the diagonal the teacher removes some embedded triangles from the diagram, but she also makes other embedded triangles easier to see. Also in this segment Lambda finishes his elaboration of the argument that in a square, the diagonals are angle bisectors. The teacher, however, does not treat this as the completion of the task, since, in the following segment, she asks the class for a proof of the claim.

Segment 9: The teacher calls for a two-column proof for Alpha's conjecture

The animated scenario ends with the teacher starting to summarize Lambda's argument. She asserts that Lambda's 'given' is that the figure is a square and Alpha offers the proof statement that the angle bisectors of a square meet at one point. Using these resources the teacher begins work on a two-column proof of Lambda's conjecture.

The actions of Alpha and the animated teacher in this segment recast the task, calling for a two-column proof. To set the stage for this proof the class establishes the resources of the 'givens' and the proof statement that will be used in the proof.

These segments of The Square are used to organize the descriptions of students that are collected from the conversations among participants (as described below in the methods section). Each description of a student is found in an interval of a study group conversation that participants had around The Square (see description of 'interval' above). These descriptions of students are then organized according the segments of the animated scenario. A description of a student is associated with a segment when the participant describes the student in relation to the work on the task that can be seen in a particular segment. In the following section I describe how the data that was collected in these sessions, in response to the animated scenario The Square, was analyzed.

Method

This section describes the methodology used in uncovering teachers' perceptions of students. First I give a brief overview of the overall method and then I describe each step in detail. The transcripts of intervals were first coded using participant analysis from systemic functional linguistics. Participant analysis focuses on the people and things that take part in the actions described in the conversation. This analysis resulted in a list of descriptions of students taken from the conversations among teachers. This list of descriptions of students was further coded for the components of classroom interaction (task, situation, and contract) or professional obligations (individual, interpersonal, institutional, or disciplinary) that were used in the description. From these descriptions in terms of classroom interaction, descriptions were mapped onto aspects of the work of teaching in which they are instrumental. Finally, descriptions were categorized in terms of the work of teaching. The results of this analysis are presented in the next section.

The data in this study were first analyzed using systemic functional linguistics. In particular, I used participant analysis and cohesion chains to find categories of perception (Herbst, Nachlieli & Chazan, in press). A category of perception refers to an object, material or mental, that participants are aware of with respect to the instructional context and is usually represented in language as a cohesion chain (Martin & Rose, 2003). That is, a category of perception is anything that the participants perceive as being relevant to the work of teaching. In coding the corpus of data our research group was interested in several specific groups of categories of perception. These groups are: time, space, student, class, teacher, task, solution, diagrams, mathematical objects, propositions, proofs, mathematical practices, teaching acts, material resources, other stakeholders, and curriculum. In this study I am only interested in categories of perception used to describe students.

Systemic functional linguistics views language primarily as a tool through which speakers and listeners construct meaning. It is concerned with the linguistic resources that speakers have available to them to construct various meanings about the world, and position themselves with respect to the world. The current analysis is an attempt to learn about how teachers construct descriptions of students through linguistic resources and infer from that what meaning these descriptions have for the work of teaching.

I hypothesize that the descriptions of students in the data will include two types of descriptions about students. The first type of descriptions is in terms of classroom interaction; in particular they are related to the task, situation, and didactical contract. Descriptions of students in terms of the task provide information about the products, operation and resources that student use in the task. With respect to the instructional situation, one learns about actions that the participants hold students accountable for, work that students could do that would trade for claims on the didactical contract, or the order of the work that the participants expect the students to engage with. Descriptions of students in terms of the didactical contract are similar to descriptions in terms of the instructional situation; except that descriptions in terms of the didactical contract describe students in terms of actions that could happen across instructional situations, not specifically during a particular instructional situation.

The second type of descriptions of students are in terms of the professional obligations that teachers respond to; in particular they are related to teachers' individual, interpersonal, institutional, or disciplinary obligations. Since these descriptions are in terms of obligations, and not in terms of classroom interaction, they are more general. Descriptions in terms of individual obligations provide information about the various cognitive abilities, emotional profiles, unique personalities, and behavior that participants attribute to students. Descriptions in terms of interpersonal obligations provide information about how participants perceive individual students interacting with the collective discursive space, physical space, and social space of the classroom. Descriptions of students in terms of institutional obligations provide information about how participants perceive students' relationship with the institution of school. These perceptions of students could be related to the time, curricular, assessment, etc., constraints of school. Descriptions of students in terms of disciplinary obligations provide information about how participants view their students' relationship with the discipline of mathematics. These views could be related to knowledge of mathematical statements, mathematical practices, or mathematical applications.

Categories of perception of students are coded in the transcript whenever participants talked about either an animated student from The Square, a real student from their classroom, a hypothetical student, or a general student. When participants describe any of these, the coding scheme records both the description that the participant gave and any additional information that participants gave about this student, like actions that they perform, challenges they present for the teacher, etc.

The unit of analysis for this coding is the interval, not the turn, the individual speaker, or the session. In particular, I am not looking at categories of students held by individuals but at the categories of students that have currency among the group of teachers. Each category of student was coded at most once per interval and if a category of student was discussed in two different intervals then it would be coded twice.

Once categories of perception had been coded for all the intervals in the data corpus the categories of students were sorted according to the theoretical framework for classroom interaction. These codes are described in Table 9 and are named students described by task, students described by situation, students described by contract, and students described by obligation.

Table 9: Descriptions of codes applied to categories of perception

Code	*Explanation of Code*
Students described by task	This code was applied to participants' comments in which students were described in terms of the task that they were engaged with. In particular, these are descriptions of students in relation to the task product, resource or operation.
Students described by situation	This code was applied to participants' comments in which students were described terms of the situation that they were working within. In particular, these are descriptions of students that are related to the

	norms of either 'making conjectures' or 'doing proofs.'
Students described by contract	This code was applied to participants' comments in which students were described in terms of their role with respect to the didactical contract. In particular, these are descriptions of students that are situated inside interactions between teachers and students, around mathematical content, in a classroom
Students described by obligation	This code was applied to participants' comments in which students were used to describe the teacher's professional obligations. In particular, these are descriptions that stem from one of four sources; individual, interpersonal, institutional, and/or disciplinary.

Participants' comments about students were coded as "students described by task" when the participants discussed students in terms of the task that the students were engaged with. These comments include information about the goals of the task that students are working towards, the resources that students have available to them as they work on the task, or the operations that students do to complete the task. For example, the comment, "students who are thinking about diagonals" would be coded after "students described by task" because it describes a student in terms of the resources of the task that he is engaged with.

Participants' comments about students were coded as "students described by situation" when the participants discussed students in terms of the situation that they were working within. For a description of the norms of 'making conjectures' and 'doing proofs' please see "The work of 'studenting' in high school geometry classrooms" (Aaron, this volume).

These comments include information about the division of labor between the teacher and students in a particular situation, the timing of events within the situation, and the exchange value of classroom work (what claims on the didactical contract could such work allow) in a particular situation. For example, a comment about, “students who know they are not supposed to ‘go by looks’ when they do proofs” would be coded after “students described by situation” because it describes a student in terms of the value that participants see in their work in the instructional situation of ‘doing proofs.’

Participants’ comments were coded as “students described by contract” when the participants discussed students in terms of their role in the general didactical contract. These comments were connected to the work of teaching but were not tied to a particular instructional situation. For example, a comment about “students who like to come to the board and dominate the conversation” would be coded after “students described by contract” because it describes a student in terms that are not situation specific. However, this description of the student is in terms of the work that students might do which separates it from “students described by obligation” listed below which are not situated inside interactions between teachers and students, around mathematical content, in a classroom.

Participants also talked about their students in terms of their professional obligations. These are comments that are not directly tied to instruction. Rather they point to the figurative stakeholders that classroom mathematics instruction is accountable to. These obligations stem from four sources: individual, interpersonal, instructional, and disciplinary. Below I describe the coding after each of these sources of obligations.

Participants' comments are coded after "students described by obligation: individual" if the comment reflects a teacher's obligation to honor the differences among and uniqueness of individual students. These comments may include descriptions of students as particular types of learners, like visual learners, or as individual students who behave in different ways, like getting frustrated. For example, a teacher could say, "most of my students are very shy."

Participants' comments were coded after "students defined by obligation: interpersonal" if the comment reflects an obligation to the class of students as a group. These descriptions of students are related to the teachers' obligation to manage the shared space, time, and other resources of the classroom. For example, a teacher could say, "Some students will talk all period and not leave room for anyone else, if you let them."

Participants' comments are coded after "students defined by obligation: institutional" if the comment reflects an obligation to adhere to the constraints that are imposed by the institution of the school. These descriptions of students are related to requirements like the fact that teachers assign grades based on students' work, and that teachers and students meet according to a preset schedule. For example, a teacher could say, "Students who failed Algebra get a new start in Geometry."

Participants' comments are coded after "students defined by obligation: disciplinary" if the comment reflects an obligation to faithfully represent the discipline of mathematics. These descriptions of students relate to the nature of mathematics and the understanding of truth and proof as it relates to mathematics. For example, a teacher could say, "my goal is to make sure that when students leave my class they appreciate the beauty of mathematics."

This coding of the categories of perception allowed me to map the participants' descriptions of students onto classroom interaction. As mentioned in the theoretical framework, students described in terms of each of these aspects of classroom interaction, tasks (by way of their goals, operations and resources), situations, and the contract can all be related to the work of teaching. Students described in terms of task goals and resources could be instrumental to the work of designing and supporting the task milieu. Students described in terms of the operations of a task could be instrumental in the work of observing a students' work on a task. Students described in terms of the instructional situation or didactical contract could be instrumental in the work of valuing students' work in terms of the didactical contract. Also students defined in terms of teachers' professional obligations could be instrumental in compelling a teacher to act in a way that could be perceived as a breach of a norm in response to a commitment to an educational stakeholder.

These methods allow for analysis of participants' perceptions of students and of the work of teaching more generally. They support a coherent connection between the description of teachers' knowledge (in the form of perceptions of students), the work that teachers do, and the dynamics of classroom interaction. In the next section I show the results of this analysis.

Results

Below I report on the results of the analysis. I use segments (described above in the data section) of the animated scenario, The Square, to organize descriptions of students that were uncovered in the analysis. The notion of "segment" is helpful because it connects the animated students' actions with the participants' perceptions. As described in the data section, a segment is a subdivision of time in the animated scenario in which the class is working on one particular instantiation of a mathematical task. In particular, a segment is a characterized by the task product that the class is working towards and the resources that the class has available to use in their progress toward that task product. Since not all the students described by the participants can been seen in the animated scenario (because many are hypothetical students, or students from the participants' classrooms), descriptions of students are assigned to the segment that the participants were discussing when they described that student.

For each segment I describe the significant features of the story that the participants viewed, explain the descriptions of students that participants perceived, provide evidence from the transcript that supports these descriptions, and connect these descriptions of students to the work of teaching. Within each segment, the participants discuss several different types of students and within each type of student the participants describe several particular students. Each section addressing a type of student contains a table with a heading that names the type of student, a left hand column that describes the work of teaching that these students are instrumental in, and a right hand column that lists the particular descriptions of students. The section ends with a list of all the descriptions uncovered in the analysis.

Segment 1: The class is given a conjecturing task about angle bisectors of a quadrilateral

In the first segment, before students begin working on or discussing the angle bisectors problem, the teacher poses the task for students to make conjectures. The teacher first reminds the class that the angle bisectors of a triangle meet at a point and then she poses the problem, "what can one say about the angle bisectors of a quadrilateral?"

After the animated teacher poses the angle bisectors problem the students begin to work on the task. Below are descriptions of students that participants noticed working on the original task. The animated teacher provided the resources "quadrilateral," and "angle bisector" in the statement of the task, and in the example about triangles before she posed the task she introduced "triangle" and "point." The animated teacher told the students that their goal was to make conjectures (that they would later prove), but she did not suggest any operations for students to use in the task.

By stating that the students' goal was to make conjectures, the animated teacher situates their work in the instructional situation of 'making conjectures.' One can anticipate that participants will talk about students in terms of the actions that they are accountable for performing while they are producing conjectures, like creating diagrams and proposing relationships between mathematical concepts. One can anticipate that participants talk about the value of students' work, and the timing of the actions that students perform while they are producing conjectures.

Students who use angle bisectors and diagonals as resources of the task

When participants talk about their students in relation to their work on the angle bisectors problem while they are making conjectures they focus on students' use of, and distinction between, angle bisectors and diagonals. Participants generally notice that students interchange these two mathematical concepts. Tina expects that if she talks about angle bisectors in class, “half the class is gonna be thinking ‘diagonal’” (TMW111506, 43, 1201). The participants saw confusion between angle bisectors and diagonals as being exacerbated by the animated teacher talking about both concepts in one conversation. James said that the animated teacher “had already thrown them off a little bit about saying ‘diagonal, angle bisector, what are we talking about?’” (ESP091305, 11, 210). This particular confusion is worsened by the fact that in a square the angle bisectors and the diagonals are the same object (the segment connecting opposite vertices of a square). Denise said that if she insisted to her students that diagonals and angle bisectors are different her students would look at a square and respond, “No, you don’t know what you’re talking about because they are the same” (TMW111506, 48, 1326).

Table 10: Students who use angle bisectors and diagonals as resources of the task

Students who are instrumental in the work of designing and supporting the milieu	• Student who thinks “diagonal” when the teacher says “angle bisector” • Student who is confused by the teacher talking about both concepts, angle bisectors and diagonals, in one conversation • Student who insists that the diagonals are the angle bisectors

Here the participants are describing students in terms of the resources that they use in the task, and in particular the participants’ are describing their concern with the fact that they do not see the students making a distinction between angle bisectors and diagonals. This shows that part of the teachers’ work is to monitor the resources that the students use in a task and it is important to the participants that the students understand the concepts that they are deploying in the task.

Students who use other mathematical objects as resources of the task

The participants talked about students in terms of the mathematical concepts, besides angle bisectors and diagonals, which students deploy as resources in the angle bisectors problem. Some of these mathematical concepts are square (Mark, TMW111506, 18, 504), rectangle (Greg, ThEMaT081905, 13, 168; Raina, TMT110706, 59, 847), parallelogram (Jillian, TMW111506, 29, 802), kite (Greg, ThEMaT081905, 13, 168; Megan, TMT110706, 13, 123), rhombus (Megan, ThEMaT081905, 17, 196), parallel lines (Megan, TMT110706, 60, 853), basic quadrilateral (Lynne, ThEMaT081905, 8, 125), and types of quadrilaterals (James, ThEMaT081905, 9, 142; Jillian, TMW111506, 30, 830). The participants also mentioned the diagrams that students utilize, such as the diagram of a rectangle with its diagonals and angle bisectors (Stan, TMT110706, 59, 831).

Table 11: Students who use other mathematical objects as resources of the task

Students who are instrumental in the work of designing and supporting the milieu	• Student who uses a square as a task resource • Student who uses a rectangle as a task resource • Student who uses a parallelogram as a task resource • Student who uses a kite as a task resource • Student who uses a rhombus as a task resource • Student who uses parallel lines as a task resource • Student who uses a basic quadrilateral as a task resource • Student who uses types of quadrilaterals as a task resource

- Student who uses a diagram of a quadrilateral and its angle bisectors as a task resource

Here one again sees the participants describing students in terms of the resources that they use in the task. The participants describe the particular quadrilaterals that students might use on this task. The task activates the concept “quadrilateral” as a resource, but the participants see that students would not use a general quadrilateral to make conjectures, but they would instead look at classes of quadrilaterals that they are familiar with. This gives more evidence to the claim that part of the teachers’ work is to monitor the resources that the students use in a task.

Students who use additional tools as resources of the task

There are several other tools that participants saw that the animated teacher could provide to students that could aid their work on the angle bisectors problem. The participants saw that students could work on the task with a partner and this partner could communicate the pair’s ideas to the class. Raina suggested, “maybe [Alpha’s] got the kid that he's sitting next to that he's working with him and they come up with this idea together” (TMT110706, 17, 206). They also saw that students could use dynamic geometry software to create the diagrams used in the task. Lucille recommended, “I think some kids like the computers or the calculators” (ESP091305, 4, 45). To help students navigate the hierarchy of quadrilaterals the participants saw that the animated teacher could provide the class with a worksheet that contained several examples of one type of quadrilateral, several different quadrilaterals, or a hierarchical list of the quadrilaterals (Jillian, TMW111506, 30, 830; Jillian, TMW111506, 30, 832; James, ThEMaT081905, 9, 142). The participants saw that the animated teacher could also provide additional resources by writing students' ideas on the board. Denise interpreted the animated teacher's actions as, "put[ting] all the points [students' ideas] up there [on the board] so you can come up with one big point [idea]" (TMW111506, 37, 985).

Table 12: Students who use additional tools as resources of the task

Students who are instrumental in the work of designing and supporting the milieu	• Student who uses a partner as a task resource • Student who uses dynamic geometry software as a task resource • Student who uses a worksheet listing several examples of one type of quadrilateral as a task resource • Student who uses a worksheet listing different types of quadrilaterals as a task resource • Student who uses a worksheet with a hierarchical list of quadrilaterals as a task resource • Student who uses ideas written on the board as a task resource

The earlier resources that the participants saw the students using were resources that students brought to the task, or chose to take up without intervention of the teacher. These resources are examples that the participants saw that the animated teacher could provide to her students to aid them in their work on the task. These resources show that part of the work of teaching is to monitor students' work on tasks and to provide them with additional resources that could aid their work.

From this analysis of the resources that the participants saw that students could use on this task, one sees that the participants described students who used the resources that the animated teacher provided her students with, but the participants also described several students who used resources other than the ones provided by the animated teacher. In particular, the participants brought up students who confused the concepts angle bisector and diagonal and they described students who looked at particular quadrilaterals instead of a general quadrilateral, as mentioned in the statement of the task. Also, the participants described students who used additional resources that the teacher could provide to aid the students in their work on the task.

Students who draw quadrilaterals as an operation of the task

While students are making conjectures in response to the angle bisectors problem, the participants saw the need for them to draw quadrilaterals. The participants report that some students might not know to draw a quadrilateral to begin work on the problem. These students might instead be involved in activities that are not related to the task. The participants would prompt these students to draw any quadrilateral. Tabitha narrated which students would need help, “I'd probably start with the kids who are sitting there, either talking to their neighbor or staring at the wall and say, ‘all right, well. Draw something with four sides. Draw in angle bisectors. Draw somethin' else with four sides’” (TMW111506, 12, 327). Participants report that other students might have the idea to draw a quadrilateral but they might not know how to strategically pick a quadrilateral so that they would be able to make a conjecture. If the participants saw a student drawing several of the same type of quadrilateral they would prompt the student to draw a different type of quadrilateral. Similarly, if the participants saw a student who had drawn several different types of quadrilaterals the participants said they would compliment the students and encourage her to draw conclusions from her diagrams. Tina said, “I'll have kids who'll draw a square three times in a row. ‘Well, draw something different than a square’… Other kids who have done, maybe done three different ones, you might just say, ‘Good job,’ you know, ‘Keep going’ you know, ‘Draw some conclusions’” (TMW111506, 13, 359). Without the teacher’s prompting the student might draw a conclusion that was less general than the student realized.

Table 13: Students who draw quadrilaterals as an operation of the task

Students who are instrumental in the work of observing student’ work on tasks	• Student who draws quadrilaterals • Student who does not have the idea to draw quadrilaterals • Student who sits in her seat, talks to her neighbor, or stares at the wall • Student who strategically picks quadrilaterals to use to make conjectures • Student who draws several of the same type of quadrilateral • Student who uses diagrams of quadrilaterals to make conjectures

Like with the descriptions of students in terms of the resources that participants saw that they could use on the task, here too one can infer something about the work of teaching. Here one sees the participants using the operations that students are performing, like sitting and talking to a neighbor, as cues that the teacher needs to prompt this student to perform some other operation so that the student can be successful on the task. This shows that part of the teachers’ work is to monitor the operations that the students use on tasks and it is important to the participants that the operations that students deploy will lead students to complete the task.

Students who draw angle bisectors and make conclusions as operations of the task

According the participants, during work on the angle bisectors problem, once students have drawn a diagram of a quadrilateral they must draw in the angle bisectors (not the diagonals) and they must interpret the diagram and make conjectures based on the diagram. Participants saw that many students would draw the diagonals of the quadrilateral instead of the angle bisectors. Mark narrated the story that he imagined unfolding in his classroom, "'Are we talking about angle bisectors today?' 'Yes.' 'Are we talking about diagonals?' 'No.' 'No,' it's like, 'ok, so are we going to look at any diagonals today?' 'No.' And then you still give the exercise and you still walk around, and they'll still be five Alphas out there who draw you a square and want to put diagonals through it" (TMW111506, 28, 769). The participants also saw that most students would not be able to infer meaning from complicated diagrams. Melissa said that students will never be able to effectively use diagrams because "some kids are just never gonna be able to use their imagination well in order to picture any of it" (ESP091305, 33, 775). This confusion is exacerbated by the fact that students who have trouble deciphering diagrams will not let the teacher know that they do not understand. Megan said, "The ones that don't get it won't complain" (TMW111506, 55, 769).

The participants saw Alpha using his diagram to perform two operations; noticing that the diagonals bisect each other and "guesstimating." These operations are not shown in the animated scenario but the participants inferred that Alpha performed these from the fact that he makes a conjecture. Tina hypothesized that Alpha thought, "'I had two diagonals and it worked [they bisect each other]'" (TMW111506, 27, 721). Denise saw Alpha guesstimating; she said, "he just, he's just estim-- guesstimating here, you know?"

(TMW111506, 28, 744). The participants made the observation that Alpha was able to make this conjecture but he was unable to explain it to the class.

Table 14: Students who draw angle bisectors and make conclusions as operations of the task

Students who are instrumental in the work of observing student' work on tasks	• Student who draws angle bisectors • Student who interprets the diagram • Student who makes conjectures based on the diagram • Student who draws diagonals (thinking they are angle bisectors) • Student who infers meaning from a complicated diagram • Student who notices that the diagonals of a square bisect each other • Student who "guesstimates"

Here the participants are describing students in terms of task operations that they see will lead to successful completion of the task. From this list of students one sees that participants anticipate that students will have trouble with the operations of drawing angle bisectors and drawing conclusions from their diagrams. In addition to the students that will have difficulty with drawing quadrilaterals, the participants described more students who they anticipated would have difficulty with this second phase of the task. Again this list of students in terms of the operations of the task supports the claim that part of the work of teaching is monitoring the operations that students use on tasks.

From this analysis of the operations that the participants saw students could use on this original task, one sees that the participants described students working on the task in two phases. The first phase involves drawing quadrilaterals and the second phase drawing angle bisectors of those quadrilaterals and using the resulting diagram to produce conjectures. The participants described students who would have trouble with the each phase of the task. One sees that the participants describe more students who have trouble with the operations of the task than students who are successful in deploying the operations of the task. This does not reflect the number of students who have trouble with the task, but the number of *ways* in which students could have trouble with the task.

In discussing this segment of the story the participants described students who deployed specific resources and operations in the completion of the task. From these descriptions one learns that part of the work of teaching is to monitor the resources and operations that student use during tasks. The participants' point to the fact that the teachers' work also includes providing additional resources that are not included in the statement of the task if they aid the students' work on the task. One also learns that the participants anticipate that conjecture tasks are completed in two phases, the first determines the overall shape of the diagram to be used in producing a conjecture (in the case of the angle bisectors problem is entails picking an appropriate quadrilateral), and the second phase completes the diagram and asks the student to infer claims from the diagram (in the case of the angle bisectors problem this entails drawing angle bisectors and producing a conjecture). The participants described students from each phase of the task who would have difficulty completing the task.

Segment 2: Alpha shares his conjecture about diagonals of a square

Once students have completed their individual work on the angle bisectors problem the animated teacher calls the class back together. The animated teacher begins the discussion of students' conjectures by calling Alpha to the board to share his conjecture.

At the animated teacher's request, Alpha shares his conjecture at the board. Initially he claims, "they bisect each other" and draws a diagram that he titles "a square and its diagonals." This comment could pose a challenge for the teacher because it could simultaneously be seen as helpful and detrimental to the classroom conversation.

There are ways to describe Alpha's presentation that recommend it as a good conjecture to share during 'making conjectures.' Alpha's diagram truthfully depicts the objects, squares, diagonals, and angle bisectors. His diagram is useful for illustrating that the angle bisectors of a square meet at a point and for illustrating that the angle bisectors of a square are also its diagonals. Both of these are claims that would be reasonable conjectures to make in response to the angle bisectors problem. Alpha's claim that the diagonals bisect each other could be used to show that each of the four triangles formed by the diagonals are pair-wise congruent and therefore the diagonals are angle bisectors. Also, since Alpha's diagram invites the claim that angle bisectors are diagonals, it allows for other students to react by providing counterexamples to this claim based on other quadrilaterals that they have explored.

There are also ways to describe Alpha's presentation that do not recommend it as a good conjecture to share during 'making conjectures.' Most notably, Alpha's conjecture is not a reasonable response to the angle bisectors problem because it is not about angle bisectors. The task asked students to make conjectures about angle bisectors and Alpha responded with a conjecture about diagonals. Also, Alpha chose a square as a quadrilateral that he would make conjectures about which is a very special case. So claims about squares would likely not allow other students to talk about similar claims with other quadrilaterals.

One can anticipate ways that the participants might describe Alpha to justify the decision to call him to the board. According to the hypothesized norms of the situation 'making conjectures,' the animated teacher should call someone to the board who has a conjecture that is apparently correct, but still allows for other students to comment.

From the comments below one can also learn about the characteristics of students that are important to teachers while they are managing students' work at the board. In terms of the task one can see that Alpha has used resources in producing his conjecture that were different than the resources provided by the animated teacher. From the norms of the situation one can expect that the participants will hold Alpha accountable for presenting a conjecture that is apparently correct and that will prompt other students' responses. Since Alpha's conjecture is apparently incorrect (because it involves diagonals instead of angle bisectors), one can expect that the participants will perceive that Alpha's conjecture does not satisfy this norm. Also, the contractual norm, that students are responsible for communicating their ideas, could elicit comments from participants about Alpha's actions regarding communicating his conjecture.

Students who have conjectures that should be shared at the board

One way that participants value the work of students within the situation of 'making conjectures' is to call students to the board. The participants reported that the teacher can call on some students to shape the conversation. Denise posed a question to the group, "What [students] have on their paper, would that matter who you would call on to give you an answer?" It is not that participants will only call students to the board who have correct conjectures, but that students' work must have characteristics that make it useful in the class discussion. Participants said they would call students to the board to share if they will be able to make a contribution to the discussion while still leaving room for other students to participate. One example of this is that participants will call on students who don't have a complete answer, or a conjecture that could be expanded on by other students. Denise said, "I would intentionally call on someone who didn't have a complete answer" (TMW111506, 14, 378) and Madison said that she would call on "Someone we can build on" (TMW111506, 14, 381). For example, while making conjectures in response to the angle bisectors problem, Tina said that she would call on a student who had a conjecture that he thought applied to all quadrilaterals but, in fact, only applied to a subset (TMW111506, 14, 383). Participants saw that this type of conjecture allows other students to point out counter-examples to and refine this student's conjecture. In the case of the angle bisectors problem, participants might not call on the student who had worked on the case of the square because it is a special case does not provide enough room for input from others.

Table 15: Students who have conjectures that should be shared at the board

Students who are instrumental in the work of valuing students' work in terms of the contract	• Student who does not have a complete answer • Student who had done work that can be built upon • Student who had an conjecture other students can refute • Student who worked on a special case

Here that participants report that they would comply with the perceived norm of bringing a student to the board that had produced a conjecture that is apparently correct, but still allows for other students to comment. Implicit in their comments is the assumption that the conjecture that one of these students would present would also be apparently correct. By bringing these students to the board the teacher is showcasing them as important examples of the work of making conjectures, and therefore valuing students' work on the task in terms of the didactical contract, which states that student are supposed to learn how to make conjectures (NCTM, 2000).

Besides the students listed above that conform with the norm of who the teacher should call to the board, the participants saw that there are other students that she could have called on. These students are described below in terms of the professional obligation that the teacher sees that she fulfills by calling on these students. The participants also saw that there are students who the teacher should not call on. These students are also described in terms of teachers' professional obligations.

When participants are faced with students bidding to speak who rarely talk in class they are inclined to call on them. Megan said that if a quiet student raises her hand in class the teacher might say, "Oh my god, they have an idea, I need to grab that" (TMT110706, 8, 74). This choice could be justified on the individual obligation that the teacher has to respond to her students' individual needs. However, the participants saw this as not an ideal choice because there is a chance that these students are not skilled at communicating their ideas and will confuse other students. Megan went on to say, "If people are already confused and you have this person who's not a great student and all of a sudden they have their hand up, you might not pick them" (TMT110706, 8, 74). This revision that Megan made to her choice of student to bring to the board could be justified by the interpersonal obligation that teachers have to respect the shared discursive space of the classroom or the individual obligation to not embarrass students. Participants recognize that some students are disruptive by asking random questions and the participants may choose not to call on these students. Megan said, "[students] in class start just asking these random questions, and it sucks up your whole time" (TMT110706, 8, 68). Not calling a disruptive student to the board could be justified by the teachers' interpersonal obligation to the shared discursive space. Finally, regarding picking students to share their conjectures in class, participants see that they need to give time to the student who has the conjecture that the teacher would like shared, since participants see that it is the teacher's responsibility to make sure that all the students hear the correct answer. Lucille said, "eventually you have to do that because that's our job" (ThEMaT081905, 6, 104). The choice to bring the student to the board to covey a particular piece of mathematical knowledge could be justified by the teachers' disciplinary obligation.

Table 16: Students who have conjectures that should be shared at the board

Students who are instrumental in the work of valuing students' work in terms of the contract	• Student who is quiet in class • Student who is not skilled at communicating their ideas • Student who is disruptive • Student who produced a conjecture that conveys a particular piece of mathematical knowledge

This list of students displays the students that the participants reported could influence their decision of who to call to the board, described in light of teachers' professional obligations. In particular, these obligations could encourage the teacher to deviate from the norm for whom to call to the board. These professional obligations could prevent the teacher from valuing students' work according to the instructional norm.

Students who share their conjecture at the board in terms of the resources of the task

Based on the conjecture that Alpha presented at the board, the participants noticed that Alpha was confused about the resources that he was using in the task. The problem is posed in terms of angle bisectors of a quadrilateral so this is the resource that participants expected him to use. Mark made the observation about Alpha, "He never used the term bisector at any time" (TMW111506, 27, 723). Instead Alpha makes claims about the diagonals of a square. Participants made a point of this distinction and pointed out that Alpha does not seem to be aware of the distinction. Tabitha said about Alpha, "The kid probably doesn't even understan-- realize the difference" (TMW111506 21, 563).

Table 17: Students who use resources at the board

Students who are instrumental in the work of designing and supporting the task milieu	• Student who uses the expected resources in a task (such as angle bisectors in the angle bisectors problem) • Student who does not use the expected resources in a task (such as diagonals in the angle bisectors problem)

Here the participants are holding the student at the board accountable for presenting a conjecture that used the appropriate resources. They see that Alpha does not do this, but they do not sanction Alpha, only note that he might not understand the difference between the resources that were provided for the task and the resources that he used on the task. These descriptions of students support the claim that part of the teachers work is to monitor students' presentations of conjectures to verify that the students used the appropriate resources during the task.

Students who share their conjecture at the board in terms of the instructional situation

The participants noticed that while the class was engaged in 'making conjectures' the animated teacher calls Alpha to the board and that for a stretch of time he is standing at the board. Tina saw that the animated teacher, "brought Alpha to the board" (TMW111506, 28, 740) and Melanie noted that she might "let Alpha sit back down" (TMT110706, 28, 358) since he was standing at the board while the class was discussing his conjecture. The participants also noticed that although Alpha stated his conjecture in front of the class, it was the animated teacher, and not Alpha, who wrote Alpha's conjecture on the board. Tina noticed that the animated teacher was the one to "write what Alpha said." (TMW111506, 38, 1042) The participants also noticed the animated teacher's goal in working with Alpha. Tina made the observation that, "what we were trying to do is to get Alpha to understand that the diagonals and the angle bisector of a square were the same thing" (TMW111506, 51, 1426). According to the participants, the animated teacher's goal in the discussion is to have Alpha both state and understand the conjecture that the angle bisectors of a square meet at a point.

Table 18: Students who share their conjecture at the board in terms of the instructional situation

Students who are instrumental in the work of valuing students' work on terms of	• Student who comes to the board to state his conjecture • Student who stays at the board for a stretch of time • Student who states his conjecture in front of the class • The teacher wrote this student's conjecture on the board

the contact	• Student who doesn't understand the resources of the task

These descriptions of students point to the actions that the participants hold a student at the board responsible for when they present a conjecture at the board. In terms of student work, the responsibilities of the students are low. The student is expected to come to the board, stay there for some time, and during this time they are expected to state their conjecture. According to the participants' comments there is no expectation for the student to explain their conjecture. The teacher is seen to have responsibilities in the form of making sure that the student has used the appropriate resources and writing the student's conjecture on the board.

The participants' descriptions of students in this segment showcase the students who are instrumental in the teachers' work to bring a student to the board to share his conjecture. Participants see these descriptions of students in terms of the situation as supporting their work in ways that they perceive to be complying with that norm that states who the teacher should bring to the board. They also see descriptions of students in terms of professional obligations that would prevent them from complying with that perceived norm.

Segment 3: The teacher rephrases the task to be about angle bisectors of a square

The focus of the mathematical work in the third segment of The Square is to evaluate Alpha's conjecture. In response to Alpha's conjecture the animated teacher said, "Alpha, the question is about the angle bisectors not about the diagonals." In the animated scenario, Alpha hangs his head, returns to his seat and says, "Well, I just thought that the diagonals cut the square in half." One can anticipate that the participants will respond to the teacher's evaluation of Alpha's conjecture by providing descriptions of Alpha that argue why the animated teacher should have found something positive in his conjecture, and descriptions of Alpha that argue why it was appropriate for the animated teacher to treat Alpha the way she did.

Students who would be hurt by the animated teachers' dismissal

The participants in the study groups were concerned about Alpha's emotional well-being. They saw that the animated teacher acted inappropriately towards Alpha by responding harshly to his conjecture. By responding to Alpha the way she did, the participants saw that the animated teacher encouraged Alpha to disengage from the discussion and she did not attending to his emotional needs. They saw that Alpha would feel pain in response to the animated teacher's reaction to his conjecture, that he didn't like being wrong, that he was devastated by the animated teacher's response, and that he would not be likely to come to the board in the future to share his ideas. When Cynthia heard the animated teacher's response to Alpha conjecture she said, "Ouch!" (TMT110706, 20, 227). Penelope observed that, "Alpha sure didn't like being wrong" (ITH081905, 11, 247). Madison went further and observed "the devastation" that the animated teacher's response had on Alpha (TMW111506, 26, 708).

However, the participants also ascribed some resilience to Alpha, saying that he began the lesson happy and that he recovered quickly from his hurt feelings. As Alpha walks to the board to share his conjecture, Stan saw that, "He's all happy" (TMT110706, 20, 234). And Penelope saw that Alpha was able to "get past his hurt feelings" (ITH081905, 11, 247). Melanie said that if she were in the place of the animated teacher, "I definitely would try to make sure he didn't feel out of place" (TMT110706, 2, 258).

Table 19: Students who would be hurt by the animated teachers' dismissal

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who is hurt by the teacher's reaction to his conjecture • Student who is devastated by the teacher's response • Student who is happy • Student who is resilient • Student who feels out of place

These descriptions of students show the participants' views of the students who would be the most affected by the animated teacher's perceived breach of the norm that states that the teacher should find something positive in students' conjectures. These students are all described in relation to the teachers' professional obligation to her individual students' emotional needs. The participants see that while the teacher is responding to students' conjectures she has an obligation to tend to her students' emotional needs.

Students who would not be hurt by the animated teacher's dismissal

The participants explained the animated teacher's perceived breach of the norm by justifying it using descriptions of Alpha. These descriptions showed that he could benefit from the animated teachers' rejection of his conjecture. If Alpha were a disruptive student who talked too much, or was "being stupid," then the participants said that the animated teacher acted reasonably when she responded harshly to his conjecture. Tina said that it is possible that the animated teacher doesn't "want him talking anymore" (TMT110706, 22, 267). The participants saw that it is her duty to control Alpha; to keep him from disrupting the class and to teach him that his behavior is inappropriate. On the other hand, the participants hypothesized that Alpha might be, an "A student." Greg said that “[Alpha] knows his stuff, he's probably an A student” (ITH081905, 5, 73). Greg went on to say that Alpha “was the brightest kid in the class, he kind of knew his stuff” (ITH081905, 5, 75). If this were the case then the participant saw that the animated teacher might also have acted reasonably when she responded harshly. In this case the animated teacher might be relying on Alpha’s strength to allow her to push his thinking further.

Table 20: Students who would not be hurt by the animated teachers’ dismissal

Students who are instrumental in the work of valuing students’ work on terms of the contact	• Student who is disruptive and talks too much • Student who is disruptive and is “being stupid” • Student who is an “A student” and is the brightest student in the class

The participants justified the animated teacher’s rejection of Alpha’s conjecture by describing Alpha in terms of the teachers’ professional obligations. The participants described Alpha in terms of both interpersonal obligations (“disruptive”) and in terms of obligations to students’ individual cognition (“brightest student in the class”). These descriptions of Alpha provide justification for the animated teacher’s perceived breach of the norm that states that the teacher should find something positive in students’ conjectures.

This segment showcases the animated teacher responding negatively to Alpha’s conjecture, which I hypothesized the participants will perceive as a breach of an instructional norm. The participants’ responses showcase the use of professional obligations to justify perceived breaches in norms. The participants provide justifications for both why the animated teacher could have and why the animated teacher should not breach this hypothesized norm by describing Alpha in terms of individual and interpersonal obligations. From these comments I gather evidence that when teachers respond to students' conjectures they are concerned with both the student's emotional well-being and the shared discursive space of the class.

Segment 4: The teacher asks the class if diagonals and bisectors are the same thing

The participants see Beta as the focal student in segment four. The participants notice Beta whispering while Alpha is standing at the board, saying that she thought that the diagonals and the angle bisectors were the same thing. Later, after Alpha has returned to his seat the participants notice Beta clarifying Alpha's conjecture. One can anticipate that the participants will talk about the resources that Beta deployed in her work on the task, especially because Beta's comments focus on the relationship between diagonals and angle bisectors, the two concepts that the participants saw Alpha having difficulty with. Because Beta whispered her comment, which is not a normal mode of communication in the classroom, one can expect participants to provide possible justification for her whispering instead of speaking at a normal volume.

Students who use propositions as resources

The participants described Beta in terms of the resources that she used on the task, however, unlike Alpha, the resources they saw Beta using were propositions about the relationship between angle bisectors and diagonals (therefore implying that Beta recognizes that they could be different). For instance, Tina heard Beta saying, "that the diagonals are also the angle bisectors of a square" (TMW111506, 37, 1016). And Megan said of Beta, "she was saying 'the angle bisector is the diagonal'" (TMT110706, 22, 271). The resources that the participants saw Alpha making use of are mathematical objects and concepts, while the resources that the participants saw Beta making use of include also mathematical propositions.[10]

Table 21: Student who use propositions as resources

Students who are instrumental in the work of designing and supporting the task milieu	• Students who use propositions about the relationship between diagonals and angle bisectors of a quadrilateral • Students who use the proposition, the diagonals are also the angle bisectors of a square • Students who use the proposition, the angle bisector is the diagonal

These comments about the resources that Beta used to work on the task provide further evidence for a claim about the work of teaching. This claim, which was also supported by comments that the participants made in response to the resources that Alpha used on the task, is that part of the work of teaching includes monitoring the resources that students use to complete a task. According to the participants, the teacher should be doing this work while students present conjectures at the board (like Alpha did) and while students are engaged in a discussion about conjectures (like Beta is engaged in.).

[10] Unlike mathematical concepts, like diagonals and angle bisectors, mathematical propositions, like "diagonals and angle bisectors are the same thing," describe some property of the mathematical concept involved or describe relations between mathematical concepts.

Students who whisper

The participants explained Beta's whispering in terms of unique qualities that she possessed. Noticing Beta's whisper in this way is related to teachers' professional obligations to individual students. The participants noticed that Beta whispered in class, and referred to her as "the whisper[er]" (ITH081905, 6, 92). They also interpreted Beta's actions in the animated scenario as meaning that she is shy and according to Denise, Beta is a "little smart-mouth" (TMW111506, 26, 700). In response to the content of her whisper, the participants judged that Beta is smart. Tabitha said, "Beta was smart here" (TMW111506, 37, 1026). Megan and Edwin said that even if Beta were shy it would have been ok for the teacher to call on her, because she had the correct answer (TMT110706, 22, 273-274).

Table 22: Students who whisper

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who whispers • Student who is a smart-mouth • Student who is smart • Student who is shy • Student who would give the correct answer if the teacher called on her

Here one sees participants' descriptions of Beta that explain why she would whisper in class in terms of teachers' professional obligations to individuals. These comments show that whispering in class is abnormal; otherwise Beta's mode of communication would not invoke comments from the participants (e.g., no participants commented on the volume of Alpha's voice). Because the participants saw that Beta is acting abnormally they attribute personality traits to her, like smart-mouth, smart, and shy. These first two descriptions of Beta explain why she might have felt the need to speak at this segment in the story and the third description of Beta explains why she would do so in a whisper, instead of at full volume. These comments provide evidence to support the norm that students should communicate their ideas in class, implying that they should also communicate these ideas in an appropriate manner. The perception of Beta as a whisperer shows that participants saw that they had a professional obligation to understand why it is that Beta would not feel comfortable complying with this perceived norm.

Students who clarify other student's conjectures

Beta's actions in the animated scenario evoked from participants concerns about students having the opportunity to talk to each other. They saw that students should have space in the classroom to talk to each other instead of always talking to the teacher. Karen said that she would rather have students talk to each other and she "worrie[s] about how much I'm interjecting between any two people who are talking" (ThEMaT081905, 19, 216). Also, participants saw that the language that students use is important in having other students understand them. Cynthia said that as a teacher, "you're trying to get [the student's] language to be right so the rest of the class can understand" (ThEMaT081905, 6, 102). This can be seen to be relevant to Beta's comments in the animated scenario because her rephrasing of Alpha's conjectures is an attempt to clarifying the language he is using and therefore help the animated teacher and the rest of the class understand his conjecture.

Table 23: Students who clarify other student's conjectures

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who makes comments in class • Student who talks to other students

Here the participants describe Beta in terms of their interpersonal obligations to the class. These comments point to another part of the work of teaching, that is, supporting students' communication to each other, instead of to the teacher. The participants' comments support the claim that within a discussion of a student's conjecture, the teacher is responsible for encouraging the students to address each other, not only the teacher, despite the fact that they are acting as if it is a norm for students to only address the teacher. This obligation provides justification for this action that participants mark as a breach of this norm.

In this segment, when a student whispers clarification of another student's conjecture, one sees that the participants were aware of at least two levels of classroom interaction. The first level is the level of the task, where students use resources to make conjectures, and the second level is the level of the instructional situation, where students are asked to use their experience making conjectures as a basis for discussing a particular conjecture. At the level of the task, the participants' comments support the claim that the teacher is responsible for monitoring the resources that students use in a task. At the level of the situation, the participants' comments support the claim that the teacher is responsible for encouraging students to discuss conjectures with each other, not only with the teacher.

Segment 5: Gamma illustrates that diagonals and angle bisectors are different using the case of a rectangle

In segment 5 the mathematical work is focused on Gamma's counter-example to Beta's claim that the diagonals are the angle bisectors. Upon hearing Beta say that the diagonals are the angle bisectors, Gamma replies from her seat, "obviously they are not," continuing the discussion about Alpha's conjecture. The animated teacher calls Gamma to the board to share her point and Gamma comes to the board saying, "I guess I don't have a point, but what I was saying is that, in general, if you have a quadrilateral the angle bisectors are not the diagonals."

In this segment one sees that the animated teacher failing to keep the classroom discussion on topic, which we hypothesize participants will perceive as a breach of a contractual norm. By inviting Gamma to the board to share her observation about a rectangle the animated teacher could be seen as changing the topic from a discussion about a square to a discussion about a rectangle.

One can anticipate ways that the participants might comment on the teacher's decision to bring Gamma to the board, especially because Gamma is disagreeing with another student, and bringing up the case of a new quadrilateral. The work that Gamma does to visualize the transformation of a square into a rectangle is not a common operation for the geometry classroom because it involves modifying a diagram and it required complex visualization skills. Gamma had to visualize the top and bottom sides of the rectangle elongating, displacing the vertical sides of the rectangle. One could also anticipate that participants might comment on this operation. Below are the participants' comments on Gamma's contribution to the discussion.

Students who refute the conjectures of other students in terms of the product of the task

When Gamma spoke up in class, the participants saw Gamma provide a counterexample to Alpha's conjectures by saying that Alpha's conjecture only holds for squares and that in general angle bisectors are not diagonals. Megan, talking about Gamma's response to Alpha's conjecture, said, "[Gamma] said it's not true for a rectangle" (TMT110706, 33, 248). The participants also noted that Gamma refuted Alpha's conjecture. James, talking for a student in the animated class who just heard Gamma's comments about Alpha's conjecture said, "'Ok they're not [the same], [Alpha] is way off base, forget about the square, those aren't angle bisectors, they're just diagonals that happen to be intersecting'" (ThEMaT081905, 18, 203).

Table 24: Students who have a goal to disagree

Students who are instrumental in the work of designing and supporting the task milieu	• Student who provides a counterexample for another student's conjecture • Student who refutes another student's conjecture

These descriptions of Gamma in terms of the product of the discussion task show that the participants noticed how Gamma's contribution shaped the conjecture that was being discussed. The participants reported that Gamma provided resistance to Alpha's poorly formulated conjecture in the form of a counterexample and refutation. This disagreement between Alpha and Gamma could be understood in many ways (e.g., Gamma doesn't like Alpha, Gamma is a know-it-all, Alpha and/or Gamma didn't understand the task) but the participants discuss this disagreement in terms of the products of the task that the animated students are engaged with.

Students who refute the conjectures of other students in terms of the situation

The participants gave two descriptions of Gamma in relation to the situation of 'making conjectures;' that Gamma was not working on making a conjecture, and that Gamma made a point that was so essential to the discussion that if Gamma had not been in class, , then the animated teacher would need to bring up the case of the rectangle. Megan seemed to devalue Gamma's contribution by saying, "[Gamma's] not trying to make a conjecture" (TMT110706, 47, 656). However, Tabitha said, "If you don't have a Gamma, you tell them [the class], 'Draw a rectangle, now draw in--' or if they just draw diagonals, you help them 'draw in angle bisectors'" (TMW111506, 40, 1140).

Table 25: Students who refute the conjectures of other students in terms of the situation

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who is not working on making a conjecture • Student who brings up an essential case

These two comments are contradictory in the sense that the first description places Gamma's contribution outside of the situation, but the second description positions

Gamma's comment as essential to the discussion. This contradiction can be resolved if one sees that, following Tabitha's comment, Gamma's contribution is more appropriate for the teacher to make than for the student. In the situation of 'making conjectures' the participants saw that the value of students' contributions is in the production of new conjectures but the evaluation of those conjectures is the responsibility of the teacher. The participants reported that Gamma's action is appropriate, however she was the inappropriate person to perform it.

Students who are hesitant to come to the board

The participants hypothesized that Gamma had a clear idea when she was sitting at her seat, but that she did not want to share this idea with the rest of the class. When Gamma said, "I guess I don't have a point," Madison interpreted her as thinking, "Oh you want me to share it? Well I don't know if it's really that big of a deal" (TMW111506, 39, 1098). Another hypothesis was that Gamma said this in case it turned out that she was incorrect. Madison said that Gamma could have said that "just to cover herself in case it didn't end up being right" (TMW111506, 39, 1098).

Table 26: Students who are hesitant to come to the board

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who did not want to share her idea with the rest of the class • Student is uncertain of her answer

In these comments one sees the participants describing Gamma in terms of the teacher's obligations to individual students. In the first description, Gamma is cast as a particular kind of individual, one who does not like to share her ideas with the group, reflecting an individual obligation. In the second description, Gamma is cast as a student who is uncertain of her answer and therefore hesitant to come to the board. In both of these descriptions Gamma is seen as complying with the norm, but she is also seen as resisting it. These professional obligations explain how the participants justify Gamma's resistance to comply with the perceived norm.

This segment showcases the students who are instrumental in the teachers' work of managing a discussion where a student refutes another students' conjecture, which involves a combination of supporting the task milieu and valuing students work in terms of the didactical contract. In particular, the teacher supports the task milieu by noticing the different task resources that are being used by the two students. The teacher values students' work in terms of the didactical contract by placing the work of refuting a conjecture outside of the domain of the student in the situation of 'making conjectures.'

Segment 6: In light of Gamma's counter-example the class reformulates Alpha's conjecture

In segment 6 the animated students discuss Alpha's conjecture in light of Gamma's counter-example, refine Alpha's conjecture to be about angle bisectors instead of diagonals, and they specify that the figure involved is a square. The push from Gamma, in the form of her counter-example, allows Alpha and the other students to reformulate Alpha's conjecture into a correct statement about the angle bisectors of a square. After the animated students discuss and clarify Alpha's conjecture the teacher represents Alpha's conjecture on the board by writing, "in a □ the ang bis ☒."

Even though there are several animated students at the heart of this segment the participants do not comment on the students in this segment. The lack of participant comments on students could be due to the fact that the key issues in this segment; the fact that the statement is not parsed into 'given' and 'prove', and the unusual register that the teacher uses to write the statement, do not involve students. These issues are more closely tied to normal teaching practice than to the students who participants see inhabiting the classroom.

Segment 7: The teacher calls on Lambda to provide a proof for Alpha's conjecture

Segments 7 and 8 contain Lambda's proof of Alpha's conjecture. The proof is separated into two segments because during segment 7 the diagram of a square on the board contains two diagonals and during segment 8 it only contains one diagonal. The change in the diagram reflects a change in the task that the class is working on.

After writing, "in a □ the ang bis ☒" on the board, the teacher calls on Lambda to produce a proof for the statement. From his seat Lambda verbally sketches a proof, using conceptual language, for why the diagonals of a square are its angle bisectors. During this proof sketch he requests that the teacher remove one of the diagonals from the square so that attention could be focused on the two triangles formed by only one diagonal of the square. When the teacher calls for a proof of Alpha's conjecture the instructional situation can be seen to change from 'making conjectures' to 'doing proofs.'

Prior to the teachers' call for a proof the animated students were working on the activity of conjecturing, which I see as guided by the norms of the situation 'making conjectures.' However, when the teacher asked Lambda to produce a proof for Alpha's conjecture the activity that the class was working on shifted to producing a proof, which could been seen as guided by the norms of 'doing proofs.' From an analysis of the story depicted in the animated scenario Lambda could be seen to be not complying with many of the norms for the situation of 'doing proofs.' In particular, his argument is not produced as a series of statements and reasons, and he attempts to modify the diagram associated with the proof by removing one diagonal. Also, one sees that in the animated scenario Lambda was not as precise in this description of the proof that the diagonals are angle bisectors as it would be expected when 'doing proofs.' In particular, when Lambda refers to objects in the diagram, like angles and triangles, he does not refer to them by their labels. Instead he talks in vague language about things like "top and bottom triangles." Lambda also leaves out some crucial steps of his proof, like stating which two triangles are congruent, that allow him to say that the angles on either side of the diagonal are congruent. The norms for the situation of 'doing proofs' provide a frame for viewing the actions that students perform while engaged with this activity.

One can anticipate ways that the participants might talk about how Lambda constructs his proof, in particular the language Lambda uses and his choice to talk about key ideas in the proof instead of providing sequential statements and reasons. Participants might also talk about the teacher's decision not to bring Lambda to the board, especially after the teacher had brought both Alpha and Gamma to the board to share their ideas. One can also anticipate that participants might respond to the teacher's work of choosing a proof statement.

Students who should not do a proof at the board

A piece of the work of teaching highlighted by the animated teacher's interaction with Lambda is managing students at the board. Generally, the participants saw this work as reduced to keeping disruptive students from coming to the board. The participants suspected that the animated teacher might have kept Lambda from the board because she knew that he could be a disruptive student. The participants saw that these students are disruptive because they enjoy the attention and do not focus on the mathematics, or they dominate the conversation without letting other students participate. Karen said, "several times [I] have had students that really want to get to the board and sort of dominate the conversation" (ThEMaT081905, 6, 90). Greg said that before he calls students to the board he asks himself, "is it the person who just always goes up there to get attention? Is it the designated person to let everybody off the hook?" (ThEMaT081905, 6, 106). Participants also reported that in some classes there are students who always come to the board and the other students do not like this. Greg said, "sometimes that person's met with animosity" (ThEMaT081905, 6, 106). Greg also said that the student at the board might begin to feel like he is doing all the work while his classmates observe. Greg went on to say that a student might be "tired of always having to bail everybody out and explain himself on the board" (ThEMaT081905, 6, 106).

Table 27: Students who should not do a proof at the board

Students who are instrumental in the work of valuing students' work in terms of the contact	• Disruptive student • Student who enjoys the attention of being at the board • Student who dominates the conversation at the exclusion of other students • Student who shares ideas that confuse the rest of the students • Student who always comes to the board

With these descriptions of students the participants provide a justification for why a teacher might breach the hypothesized contractual norm that states that students should be disposed to share ideas in public when so asked. Each of these students embodies either an interpersonal or individual obligation that would compel the teacher to not bring them to the board. Students who would disrupt the class, students who would dominate the conversation at the exclusion of other students, students who would confuse other students, and students who annoy other students by always coming to the board are all examples of students described in terms of the teacher's interpersonal obligation. A student who enjoys the attention of being at the board is example of a student described in terms of the teacher's individual obligation to her students.

Segment 8: The teacher removes one diagonal from the diagram on the board to assist Lambda's proof

Segment 8 begins with the teacher consenting to Lambda's request to remove one diagonal from the diagram of the square on the board and ends with Lambda finishing his sketch of the proof that the diagonals of a square are also angle bisectors. As mentioned above, in the description of segment 7, the participants perceive Lambda's actions as a breach of several norms of the situation 'doing proofs.'

Like in segment 7, one can anticipate the participants might talk about the language Lambda uses and his choice to talk about key ideas in the proof instead of providing sequential statements and reasons. Since Lambda's moves are so atypical, one can anticipate the participants to talk about how a student would typically produce a proof, pointing out the differences between Lambda's actions and what is expected.

Students who understand proofs

Participants described how their students related to understanding proofs. Participants said that linear and detail-oriented students need to see every detail of a proof. James said about linear, detail-oriented students, “if they don’t see each step, you know, I mean it’s the person who yells out, ‘isn’t that a segment?’ ‘Oh yeah, draw a line over it’” (ESP081905, 30, 584). James was pointing out that these students need every detail of a proof to be in place, including notation of segments. James insisted on using the correct order of letters to name objects when writing proofs because, “I just think that it stops students from interrupting when they get lost” (ESP081905, 25, 452). The participants also report that students have difficulty with changes being made to diagrams. Tina said about the animated teacher’s removal of one diagonal in the square, “A lot of kids are visual learners and that's [changing a diagram] a hard thing for them” (ThEMaT081905, 5, 54). The participants saw that students need to perform various operations that aid them in understanding the proof of the claim that the angle bisectors of a square meet at a point presented by Lambda. The participants expect that most students would be able to follow Lambda’s proof, but the fact that it was an oral argument and not written down might cause difficulty for some students. Denise said that she expected the other students to follow along with Lambda’s proof, “other students can kind of catch on along the way, you know going through each step” (ThEMaT081905, 6, 92). However, Megan said, “you need to write it down because some of the kids didn't get that” (TMT110706, 81, 1278).

Table 28: Students who understand proofs

Students who are	• Linear, detail-oriented student

instrumental in the work of valuing students' work in terms of the contact	• Student who will interrupt the teacher if she makes a notational error • Student who has difficulty with changes being made to diagrams • Student who has trouble following an oral argument

These descriptions of students are all in terms of the work that teachers do to value students' work in terms of the contract because they describe the ways that students can understand proofs, and doing proofs is a skill that students need to learn according to the didactical contract (NCTM, 2000). In particular, these descriptions describe the ways that participants perceive students having difficulty fulfilling this requirement of the contract.

Segment 9: The teacher calls for a two-column proof of Alpha's conjecture

Segment 9 comes after Lambda has articulated the argument supporting the claim that in a square the diagonals are angle bisectors. Instead of praising his work the teacher dismisses it and calls for a proof of the claim. The key action in this segment is the teacher's valuing of Lambda proof. Accordingly, one can anticipate that the participants will discuss students in terms of how they work on proofs and the value that these proofs have in terms of the didactical contract. Below are participants' descriptions of students related to valuing proofs and the work that students put into producing proofs.

Students who do pseudo-proofs

The participants describe Lambda in terms of the work that he did while doing a proof in response to the angle bisectors problem. They said that he "had a plan" (Edwin, TMT110706, 8, 79), "had an idea" (Carl, ESP091305, 12, 228), had a "separate idea"

(Lynne, ESP091305, 14, 311), “could have” written a proof (Jillian, TMW111506, 53, 1468), did a “fairly coherent proof” (Karen, ESP091305, 12, 218), and had done a proof “in his mind” (Melissa, ESP091305, 45, 254). All of these speak well of his ability and the participants’ perception of his ability. However they also pointed to the fact that his work was somewhat incomplete. The participants either credit him with a plan for a proof or some qualified proof, but not a full-fledged proof. In addition to these comments about the work that Lambda did, the participants said that in response to Lambda’s work the teacher should have called him to the board and asked him to prove Alpha’s conjecture.

Table 29: Students who do pseudo-proofs

Students who are instrumental in the work of valuing students’ work in terms of the contact	• Student who had a plan/idea/separate idea • Student who should take charge/could have written a proof • Student who did a fairly coherent proof/thinks that he has done a proof

These descriptions of students highlight the students who are instrumental in the work at a teacher does to value students’ work on mathematical tasks in terms of the didactical contract. The participants saw that all of these students are making progress on learning to do proofs, however they do not yet know how to produce what the participants would label as an unqualified proof. I label these as pseudo-proofs because as the participants talk about these, they are not simply proofs but some specific, qualified proof that teachers are compelled to distinguish from normal proofs.

Students who do produce steps in proofs

Participants assign value to different types of proofs and then characterize the students who wrote these proofs. For instance, participants saw students who are not able to focus on the main arguments in proofs by adding extra and irrelevant steps as clueless. Karen described, "the kids are just like totally clueless and they're just wandering around" (ESP081905, 22, 405). Students who turn in a proof that contains many more steps than it requires are seen as having trouble with proofs. Carl said that, "occasionally I'll have a kid who has trouble with proofs and they finally get proofs and then they turn in their test and theirs are twenty-five lines in a ten line proof" (ESP081905, 22, 360). On the other hand, students who write more abstract steps in their proofs are more advanced in their thinking process. Melissa said, "someone that's more advanced in their thinking process could be abstract" (ESP081905, 22, 370). There are students who participants saw as refusing to do proofs. These students would not write any more than they needed to, and Karen said that with some students all they would do is "write the given and the conclusion" (ESP081905, 22, 405).

Table 30: Students who do produce steps in proofs

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who loose the focus of an argument • Student who turns in a proof that contains more steps than are required • Student who writes more abstract steps in their proofs • Student who refuses to do proofs

These descriptions of students are related to the value that a teacher assigns to proofs that are produced by students. Here one sees some of the ways that students could produce proofs that contains steps that are abnormal in some regard, either too long, contains abstract statements, or incomplete. Since these are descriptions of ways of creating proofs that do not match with a proof as described in the situation of doing proofs, these students are considered to be instrumental in the work of valuing students' work in terms of the contract.

Students who are producing a proof

When the participants discussed students' work on 'doing proofs' they talked about actions that students are accountable for. One such action is for students to remember the names of theorems and postulates. Lucille said, "they are still struggling with remembering what the theorems and postulates are" (ESP091305, 32, 747). Jillian said, "I still want somebody to know the definition, and you know, knows their theorems and postulates" (TMW111506, 39, 1108). Participants expected to be able to direct students' attention to the parts of the diagram that are important for the proof. Megan said, about using color to highlight objects in a diagram, "It just shows [the students] 'look there's more'" (TMT110706, 63, 906). Participants didn't always expect students to be precise in their write up of proofs but participants took responsibility for showing students proofs that contain all the necessary details. Esther said, "I still show my students all those things" (ESP081905, 21, 325). Participants did, however, expect that students could follow the completed proofs of other students. Melissa said that for some students, "they get it when they see somebody writing it but they don't think of it themselves" (ESP081905, 26, 505). When students were unable to complete a complicated proof the participants expected students to raise their hand and see if the teacher or another student would complete it for them. Carl talked about getting stuck on a complicated proof, "You might as well just raise your hand and see if someone can bail you out" (ESP081905, 8, 114).

Table 31: Students who are producing a proof

Students who are instrumental in the work of valuing students' work in terms of the contact	• Student who has trouble remembering the names of theorems and postulates • Student who knows the definitions • Student who has attention directed to the parts of the diagram that are important for the proof • Student who is precise in the write up of a proof • Student who is shown proofs that contain all the necessary details • Student who is able to follow the completed proofs of other students

These descriptions of students are all in terms of the instructional situation 'doing proofs' because the actions that participants expect students to perform match with the accountability norms for the situation. Here one sees that participants expected students to know how to perform most of the necessary actions to produce a proof. However, participants recognize that students might have trouble remembering the name of theorems and postulates, and in some cases students may need to have their attention directed to important parts of the diagram. These students are all instrumental in the work that teachers do while observing student work on proof tasks.

Students who value proof

Participants saw their students not understanding the connections between mathematical knowledge and mathematical proof. According to the participants, students will look at a statement they are asked to prove and assert that since they are asked to prove this statement it must be true. Mara said, “I have kids that in a geometry classes think it's pointless to do all these proofs because these proofs have been proved, already proved” (ITH081905, 10, 199). The fact that students are only asked to prove true statements weakens the link between proof and truth in their minds. Karen said that students “have not yet connected truth and proof like this” (TMW111506, 50, 1395). The individual steps in a proof do not build to a convincing argument for students. Tabitha said that her students will look at a completed proof and say, “Did we finish?” instead of seeing the larger argument that they have built (TMW111506, 52, 1444).

Table 32: Students who value proof

Students who are instrumental in the work of valuing students’ work in terms of the contact	• Student who does not understand the connections between mathematical knowledge and mathematical proof • Student who looks at a prove statement and asserts that it must be true • Student who only attempts to prove true statements • Student who does not see how individual steps in a proof build to a convincing argument • Student who looks at a completed formal proof and says, “Did we finish?”

Here participants describe their students in terms of the understanding that students have of the connection between mathematical knowledge and mathematical proof. These descriptions describe the ways that students can misunderstand this relationship. Insofar as this understanding is something that teachers are expected to teach students according to the didactical contract, these descriptions of students are instrumental to the work of valuing students' work in terms of the didactical contract.

In this section I have detailed the participants' descriptions of students that are instrumental in instruction. These descriptions of students are organized by the segments that appear in the animated story, The Square, however, the students that are described here are relevant to the particular work of teaching that is exemplified in the animated scenario, regardless of the particular instructional context. The animated scenario is a useful tool for eliciting the participants' comments, but it is only that, a tool. The comments that it elicits stand on their own as observations about instruction. In the following section I discuss these results in terms of what these results say about the work of teaching.

Discussion

In the following section I discuss the findings of this study. I return to the research questions to guide this discussion. The research questions are:

- What perceptions of students are instrumental in the work of designing and supporting the task milieu?
- What perceptions of students are instrumental in the work of observing students' work on a task?

- What perceptions of students are instrumental in the work of valuing students' work in terms of the didactical contract?
- What perceptions of students are instrumental in encouraging the teacher to breach the norms of instruction in favor of a professional obligation?

I will address each of these questions below.

The first research question asks, "What perceptions of students are instrumental in the work of designing and supporting the task milieu?" Besides listing the students that were uncovered in the analysis (as shown in Table 33) I offer some general discussion of the students that this list contains. These comments capitalize on the difference between the situations of 'making conjectures' and 'doing proofs' to highlight features of the list of students.

Table 33: Students that are instrumental in the work of designing and supporting the task milieu

Students who do not understand the resources of the task	• Student who thinks "diagonal" when the teacher says "angle bisector" • Student who is confused by the teacher talking about both concepts, angle bisectors and diagonals, in one conversation • Student who insists that the diagonals are the angle bisectors
Students who use other mathematical objects as resources of the	• Student who uses a square as a task resource • Student who uses a rectangle as a task resource • Student who uses a parallelogram as a task resource • Student who uses a kite as a task resource • Student who uses a rhombus as a task resource

task	• Student who uses parallel lines as a task resource • Student who uses a basic quadrilateral as a task resource • Student who uses types of quadrilaterals as a task resource • Student who uses a diagram of a quadrilateral and its angle bisectors as a task resource
Students who use propositions as resources	• Students who use propositions about the relationship between diagonals and angle bisectors of a quadrilateral • Students who use the proposition, the diagonals are also the angle bisectors of a square • Students who use the proposition, the angle bisector is the diagonal
Students who use additional tools as resources of the task	• Student who uses a partner as a task resource • Student who uses dynamic geometry software as a task resource • Student who uses a worksheet listing several examples of one type of quadrilateral as a task resource • Student who uses a worksheet listing different types of quadrilateral as a task resource • Student who uses a worksheet with a hierarchical list of quadrilaterals as a task resource • Student who uses ideas written on the board as a task resource

Students who use resources at the board	• Student who uses the expected resources in a task (such as angle bisectors in the angle bisectors problem) • Student who does not use the expected resources in a task (such as diagonals in the angle bisectors problem)
Students who have a goal to disagree	• Student who provides a counterexample for another student's conjecture • Student who refutes another student's conjecture

These descriptions of students can be grouped into categories of students who use mathematical resources, students who use physical resources, students who use resources at the board, and students who have a goal to disagree. These categories of students suggest that within the situation of 'making conjectures' there is a wide range of resources that students could use, and differences in students' resource use are important to the teacher's work.

This is an interesting contrast to the situations of 'doing proofs' where teachers do not describe any students in terms of the work of designing and supporting the task milieu. I propose that a reason for this difference is that within the situation of 'making conjectures' students typically have many resources available to them and part of their responsibility is to choose the resources that they will use on the task, while in the situation of 'doing proofs' students are expected to use exactly the resources that are made available by the teacher. The fact that students are accountable for choosing resources causes teachers to be more attentive to the resources that they use.

The second research question asks, “What perceptions of students are instrumental in the work of observing students’ work on a task?” These students are listed below in Table 34.

Table 34: Students who are instrumental in the work of observing students’ work on a task

Students who draw quadrilaterals as an operation of the task	• Student who draws quadrilaterals • Student who does not have the idea to draw quadrilaterals • Student who sits in her seat, talks to her neighbor, or stares at the wall • Student who strategically picks quadrilaterals to use to make conjectures • Student who draws several of the same type of quadrilateral • Student who uses diagrams of quadrilaterals to make conjectures
Students who draw angle bisectors and infer conclusions as operations of the task	• Student who draws angle bisectors • Student who interprets the diagram • Student who makes conjectures based on the diagram • Student who draws diagonals (thinking they are angle bisectors) • Student who infers meaning from a complicated diagram • Student who notices that the diagonals of a square bisect each other • Student who “guesstimates”

There are only two categories of students that were found to be instrumental to this work; students who draw quadrilaterals, and students who draw angle bisectors and infer conclusions, and both of these categories of students were generated from participants' discussions about students involved in 'making conjectures,' none of these categories of students were generated from participants' discussions about students involved in 'doing proofs.' Similar to students described in terms of the work of designing and supporting the task milieu above, students' task operations seem to be more relevant in the situation of 'making conjectures' than in the situation of 'doing proofs.' Also from these categories of students one can hypothesize that student actions besides these are invisible to the participants, i.e. if a student performs a different type of move then the teacher may not notice, and therefore not be able to support the student in their work.

The third research question asks, "What perceptions of students are instrumental in the work of valuing students' work in terms of the didactical contract?" These students are listed below in Table 35 and 36. Table 35 contains students described within the situation of 'making conjectures' and Table 36 contains students described within the situation of 'doing proofs.'

Table 35: Students who are instrumental in the work of valuing students' work in terms of the didactical contract within the situation of 'making conjectures' (described with respect to the contract)

Students who have conjectures that should be shared at the board	• Student who does not have a complete answer • Student who had done work that can be built upon • Student who had a conjecture other students can refute • Student who worked on a special case

Students who share their conjecture at the board	• Student who comes to the board to state his conjecture • Student who stays at the board for a stretch of time • Student who states his conjecture in front of the class • The teacher wrote this student's conjecture on the board • Student who doesn't understand the resources of the task
Students who whisper	• Student who whispers • Student who is a smart-mouth • Student who is smart • Student who is shy • Student who would give the correct answer if the teacher called on her
Students who clarify other student's conjectures	• Student who makes comments in class • Student who talks to other students
Students who refute the conjectures of other students	• Student who was not working on making a conjecture • Student who brought up an essential case

This list of students who are instrumental in the work of valuing students' work in terms of the didactical contract within the situation of 'making conjectures' suggests that the value of students' work on making conjectures is more dependent on engaging students with the lesson then on learning any new mathematical knowledge or practice. Only four students in this list of eighteen descriptions of students are described in a way that is related to mathematics; "student who does not have a complete answer," "student who worked on a special case," "student who doesn't understand the resources of the task" and "student who brought up an essential case." Besides these four descriptions of students all the other descriptions are related to the participation structure of the class. This is very different from the list of students who are instrumental in the work of valuing students' work in terms of the didactical contract within the situation of 'doing proofs,' which is discussed below.

The following table contains the list of students who are instrumental in the work of valuing students' work in terms of the didactical contract within the situation of 'doing proofs.'

Table 36: Students who are instrumental in the work of valuing students' work in terms of the didactical contract within the situation of 'doing proofs' (described with respect to the contract)

Students who do pseudo-proofs	• Student who had a plan/idea/separate idea • Student who should take charge/could have written a proof • Student who did a fairly coherent proof/thinks that he has done a proof
Students who do produce steps in proofs	• Student who loose the focus of an argument • Student who turns in a proof that contains more steps than is required • Student who writes more abstract step in their proofs

	• Student who refuses to do proofs
Students who are producing a proof	• Student who has trouble remembering the names of theorems and postulates • Student who knows the definitions • Student who has attention directed to the parts of the diagram that are important for the proof • Student who is precise in the write up of a proof • Student who is shown proofs that contain all the necessary details • Student who is able to follow the completed proofs of other students
Students who understand proofs	• Linear, detail-oriented students • Student who will interrupt the teacher if she makes a notational error • Difficulty with changes being made to diagrams • Student who has trouble following an oral argument
Students who value proof	• Student who does not understand the connections between mathematical knowledge and mathematical proof • Student who looks at a "prove" statement and asserts that it must be true • Student who only attempts to prove true statements • Student who does not see how individual steps in a proof build to a convincing argument

	• Student who looks at a completed formal proof and says, "Did we finish?"

This list of students, who are instrumental in the work of valuing students' work on tasks in the situation of 'doing proofs,' is much more focused on the mathematics than the corresponding list for the situation of 'making conjectures.' Here one sees that the value of students' work is based on being able to produce and understand proofs. This list shows that teachers don't necessarily expect student to always be able to successfully do the work of 'doing proofs,' but, in general, teachers value students' work on proof based on a mathematical standard.

These lists of descriptions of students; students who are instrumental in the work of designing and supporting the task milieu, students who are instrumental in the work of observing students work on tasks, and students who are instrumental valuing students' work on tasks, describe the students that teachers perceive as being instrumental in the work of conducting instruction according to the norms. In the following section I describe the students that teachers perceive to be instrumental for conducting instruction in ways that they perceive to be a breach the instructional norms.

Tables 37 and 38 contains the list of students who are instrumental in compelling teachers to act in ways that they see as a breach of instructional norms within the situations of 'making conjectures' and 'doing proofs.' I discuss these descriptions of students below.

Table 37: Students who are instrumental in compelling teachers to act in ways that they perceive as a breach of instructional norms within the situation of 'making conjectures' (described with respect to professional obligations)

Students who	• Student who is quiet in class

have conjectures that should be shared at the board	• Student who is not skilled at communicating their ideas • Student who is disruptive • Student who produced a conjecture that conveys a particular piece of mathematical knowledge
Students who would be hurt by the animated teachers' dismissal	• Student who is hurt by the teacher's reaction to his conjecture • Student who is devastated by the teacher's response • Student who is happy • Student who is resilient • Student who feels out of place
Students who would not be hurt by the animated teachers' dismissal	• Student who is disruptive and talks too much • Student who is disruptive and is "being stupid" • Student who is an "A student" and is the brightest student in the class
Students who are hesitant to come to the board	• Student who did not want to share this idea with the rest of the class • Student is uncertain of her answer

Table 38: Students who are instrumental in compelling teachers to act in ways that they perceive as a breach of instructional norms within the situation of 'doing proofs' (described with respect to professional obligations)

Students who should not do a proof at the board	• Disruptive student • Student who enjoys the attention of being at the board • Student who dominates the conversation at the exclusion of other students • Student who shares ideas that confuse the rest of the students • Student who always comes to the board

Again, there is an interesting comparison between the students who teachers perceive as instrumental in compelling them to act in ways that they perceive as a breach of instructional norms within the situations of 'making conjectures' and 'doing proofs.' There are many more descriptions of students that participants see as calling for perceived breaches of instructional norms in the situation of 'making conjectures' than in the situation of 'doing proofs.' This makes sense in light of the claim that the value of students' work in 'making conjectures' is more dependent on engagement in the lesson and the value of students' work in 'doing proofs' is more dependent on learning mathematical practices. I propose the explanation that since the value of making conjectures is to engage students in the lesson, the teacher is more attuned to the individual and interpersonal obligations that she has to her students in this situation. When the teacher is conducting the situation of 'doing proofs' she is more attuned to the mathematical obligations that she might have (that do not surface in this study because they do not include perceptions of students).

This discussion highlights the different ways that teachers perceive their students between the situations of 'making conjectures' and 'doing proofs.' I propose that the value of students' work in the situation of 'making conjectures' is based on the engagement that students have with the lesson. Similarly, I propose that the value of students' work in the situation of 'doing proofs' is based on the possibility that students learn about the mathematical practice of proving. This difference in the value of students' work affects the ways that teachers perceive their students and the work that teachers do when they conduct instruction within these two situations.

Conclusion

In this paper I report on a study that looked at geometry teachers perception of students in instruction. The findings show that teachers' perceive students differently depending on the work that the teachers perceive the class to be engaging in. This is true in two senses. First, to the extent that teachers see their responsibility to conduct instruction, teachers perceive their students differently depending on which aspect of the work of teaching they are immersed in when they perceive the student. If teachers are immersed in the work of designing or supporting the task milieu, then they perceive students in terms of the task goals and resources that students are using. If teachers are immersed in the work of observing students' work on tasks, then teachers perceive students in terms of the task operations that students are performing. If teachers are immersed in the work of valuing students' work in terms of the didactical contract, then they perceive their students in terms of the instructional situation they are working within.

Second, to the extent that teachers see students engaged with various tasks, within different instructional situations, they perceive students differently. When students are seen as engaged in 'making conjectures' the teacher perceives students in relational terms that emphasize students engagement with the lesson. When students are seen as engaged in ‘doing proofs’ the teacher perceives students in mathematical terms that emphasize their progress in learning the process of proving.

Valuing work in ‘making conjectures’

The way that teachers perceive their students within the situation of ‘making conjectures’ is tied to tensions that are inherent in the teacher’s work in this situation (Herbst et al, 2010). Chazan (1995) points out that when students are working on making conjectures both teachers and students are unsure what exactly it means to “make conjectures.” Both students and teachers are comfortable with the more defined activity of gathering data to use in making conjectures, but they lose momentum when it comes time to generalize those data into conjectures.

Teachers’ and students’ uncertainty about this work leads to the teacher’s lack of tools for valuing students’ conjectures. On one hand, teachers want to value any conjecture that students come up with, placing the emphasis on the work that was done to arrive at the conjecture, instead of the conjecture itself. On the other hand, valuing all conjectures equally goes against the mathematical disposition that some conjectures are more relevant or interesting than others. Also, according to Chazan’s (1995) argument, when teachers only value some conjectures students feel as if the teacher is “fishing” and students’ best chance for success is guessing what the teacher has in mind.

Teachers describe students in terms of the resources that they use on conjecturing tasks and the operations that they use to gather information before they make generalizations, but they do not describe students in terms of the operations that are needed to generalize and come up with conjectures. This matches the results that Chazan (1995) found when he studied teachers work managing students working on conjecturing tasks.

These observations about managing students' work on making conjectures raises the question of what could be done to help teachers better value students' work on conjecturing. I see two difficult aspects to this work. First, is the need to structure students' work on 'generalizing.' Second is the need for teachers to respond to students' conjectures that they had not anticipated. To support teachers in structuring students' work on 'generalizing' teachers need to understand some discrete operations that students can use to achieve this goal beyond reducing this work to 'intuition.' To support teachers in responding to unanticipated student conjectures, curriculum materials could provide teachers with a map of the mathematical territory that students are exploring. This map could aid teachers in locating students conjectures that might be otherwise unexpected by teachers and help teachers value these conjectures.

Convergence of perception and experience

The categories that teachers employ to sort their students affect, and are affected by, the traits of their students that they find meaningful (Bowker & Star, 1999). That is, teachers form categories of students from their experience teaching, and once a teacher has established categories of students, these categories "create" students to be put into them. Bowker and Star label this phenomenon as *convergence*. This convergence allows systems of classification to disappear as categories emerge from experiences and then these categories shape subsequent experiences. In the words of Bowker and Star, "This blindness [to the classification system] occurs by changing the world such that the system's description of reality becomes true" (Bowker &Star, 1999, p. 49).

If one takes Bowker and Star's claims seriously, then these perceptions of students that I describe here are not only perceptions that exist in the conversations among teachers.

These perceptions of students shape the way that teachers see the world of their classroom, therefore shaping individual students into the types of students that they perceive. By being aware of teachers' perception of students we can take a first step toward building classrooms where students are capable and successful.

Connection to student learning

I began this paper with the claim that understanding teachers' perception of their students could lead to improving student learning. In this conclusion I revisit this claim in light of the findings of this study. Since teachers are responsible for conducting instruction, their perception of students open up or close down opportunities for student action.

Looking at the findings related to the students that teachers perceive when they are immersed in the work of designing and supporting the task milieu, one sees that teachers see many more types of students involved in the situation of 'making conjectures' than in 'doing proofs.' This points to the well-defined nature of proof tasks and the ill-defined nature of conjecturing tasks. I would like to see teachers be more aware of the different types of students that could be involved in this work while immersed in 'doing proofs.' The milieu of proof tasks could be designed so that students had an opportunity to make sense of the resources that are made available to them—to choose which given and prove statements work best for the task and which style of argumentation is the best suited to the proof. This could provide students with better opportunities to learn about the relationships between the assumptions and conclusions of an argument and to become more flexible in their mathematical argumentation style.

Similarly to the students that teachers perceive when they are immersed in the work of designing and supporting the task milieu, teachers also do not see any students with respect to the work of observing students' work on a task within the situation of 'doing proofs.' By studying proof tasks, teachers could become more aware of the possible work that students might do on a proof task and therefore be more aware of the types of students that they could see working on these tasks.

Looking at the findings related to the categories of students that teachers perceive when they are immersed in the work of valuing students' work in terms of the didactical contract, one sees that teachers do not perceive students doing valuable mathematical work within the situation of 'making conjectures.' In the situation of 'making conjectures' teachers value students' work in terms of how they engage with the lesson, while in the situation of 'doing proofs' teachers value students work in terms of students progress toward understanding the mathematical practice of producing proofs. This points to the claim that during the situation of 'making conjectures,' students are not supported in work that has a mathematical basis (Herbst et al, 2010). I argue, then, that it would be worthwhile to look at the situation of 'making conjectures' and think about ways to make students' work in this situation more valuable to teachers—either by changing the work that students are asked to do, or by helping teachers see the mathematical value of the work that students already do in this situation.

The findings about teachers' perceptions of students in terms of their professional obligations also supports this claim, that teachers do not value the mathematical work that students do in the situation of 'making conjectures.' I hypothesize that since teachers do not perceive the mathematical value of students' work in this situation, they focus instead on the relational work that can be done, often acting is ways that they see as breaching the norms of the situation to do so.

This study gives us a better sense of the work of teaching and the ways that teachers perceive their students as resources in that work. It also gives us a better sense of the work of being a student and that possibilities that are available or not, valued or not, depending on the perception of the teacher. The ways that teachers perceive their students shapes the ways that teachers design and support the task milieu that students engage with, it shapes they ways that teachers observe the work that students do on those tasks, and it shapes the ways that teachers value students work once it has been completed. Students are essential resources of the teacher's work, and a teacher's perception of her students is the mode through which she accesses those resources.

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Chapter 4

The Work of Studenting in High School Geometry Classrooms

This study seeks to better understand the actions that students perform during geometry instruction. Often, improvements in student learning are sought through improvements in teaching. However, the conception of instruction as interactions between teacher, students, and content, in environments (Cohen, Raudenbush & Ball, 2003) points to the fact that teaching is dependent on, and interactive with, the instructional actions of students. Therefore, to improve student learning, improvements in instruction must take student actions into account. It follows, then, that the path to improved student learning requires understanding students' actions in instruction. The paper attempts to develop that understanding.

This study is part of a larger project to understand the position of the student in geometry instruction. I hypothesize that this position of the student is shaped, in part, by the mathematical content that geometry students encounter in instruction, the perception that geometry teachers have of their students, and the way that students perceive the work of 'studenting' in geometry instruction. The current study looks at this last shaper of the position of the student in geometry instruction, students' perception of the work of 'studenting.'

As the actions that teachers take within instruction are often referred to as 'teaching' I look at the corresponding student actions, referred to as 'studenting' (Fenstermacher, 1986). I claim that studenting, while it may not be rational in the sense of correct, does reflect a rationality of sorts that comes from enacting the position of student. By position of student I mean that individual adolescents come into high schools and automatically are subject to expectations about how they should act in relation to the teacher, their peers, and the mathematical content they that they are expected to learn. Even students who do not act according to these expectations are still exposed to them and are aware of them. It is these expectations of how a student *should* act that I am concerned with in this study.

This paper contains both empirical and theoretical discussions of the question, 'what is the work of 'studenting' in high school geometry instruction.' The empirical discussion consists of an empirical study of the norms of studenting. I follow Bourdieu (1990, 1998) and Herbst (2003, 2006) in defining norms as cultural resources that individuals use to construct performances in social contexts. These resources are available to members of particular social groups, so, for instance, geometry students are inclined to construct their action in a different ways (using different cultural resources) from geometry teachers, or from elementary students. In this empirical study I look at students perception of the norms of geometry instruction.

The theoretical discussion of 'what is the work of 'studenting' in high school geometry instruction' builds on the results of the empirical study. In particular, the results of the empirical study show that there are some student actions that are not explained by the existence of instructional norms. Instead, participants' comments point to actions that represent departure from the norms. I use these cases, where students justify their actions using alternative justifications, as motivation to formulate a tentative model for the rationality of studenting that incorporates both the norms of instruction and other explanations of student actions that have been studied in prior research.

In the past, students' actions have been explained in many disparate ways; students' personal resources, like motivation, personality, and cognition, students' relationship to the school, students relationship with the work that this done in classrooms, and students' response to tacit norms for action. However, there is not much research on how these different ways of explaining student action relate to each other. This study attempts to do two things; to further explore the norms that guide students' actions in instruction and to integrate other research on students' actions into a more coherent theory to describe student action.

The following are the research questions I explore in this paper:

- How can hypothesized norms of instruction be used to justify student actions?
- What other justifications for instructional actions do students provide for their actions, when the action supported by the norm is deemed inappropriate?
- What is a model of 'studenting' that takes into account instructional norms for student actions as well as other research on students' actions?

The first question is the subject of the empirical study. This study begins with a list of hypothesized norms and examines the data to see how students' comments in response to an animated classroom scenario reflect adherence to or departure from these hypothesized norms. The second question looks at the perceptions of students that reflect justifications for actions that are expectations to the hypothesized norms. To answer this question I look for similarities across these alternative justifications and return to the literature to find support for these justifications from other research. Then, using what I learned from the current empirical study and a survey of the literature, I explore the final question by proposing a model for the rationality of the work of studenting, based on a model for the rationality of the work of teaching developed by Herbst (2010a, 2010b).

I begin by discussing past research on student actions in classrooms. I look at research that assumes that student actions are determined by students' personal resources, students relationship with the school, students relationship with the work done in classrooms, and previous work on 'studenting.' I then discuss the current study, which engaged classes of high school geometry students in conversations around an animated scenario from a high school geometry classroom. In these conversations students reacted to the actions of the animated students and provided evidence that either supported hypothesized instructional norms or provided alternative justifications to these actions. These alternative justifications were examined using previous research to understand the departures from the norms. The results from the empirical study and the examination of the literature to explain the evidence that supports alternative justifications yielded a tentative theoretical model for describing student actions within the instructional situations of 'making conjectures' and 'doing proofs.' These instructional situations and the theory of instructional exchanges that is key to this work is described in detail in "Teachers' perceptions of geometry students" (Aaron, this volume). In the following section I begin by describing past research on students' actions in classrooms.

Research on student actions

In the following section I survey research that has tried to explain why students act in the ways that they do. This is clearly an important topic to study because students' actions in instruction are, in many ways, determine students' success. Students outnumber any other group of people in a school, and their involvement and engagement are the primary reason that the school exists. The research below is divided into three areas; research that assumes that student actions are guided by personal resources held by individual students, research that assumes that student actions are guided by commitments that students hold due to their role as "student," and research that assumes that student actions are guided by norms of instructional interactions.

Looking across the literature on students' work in geometry class, one finds a large group of studies on students' work with dynamic geometry software (Forsythe, 2007; Galindo, 1998; Battista, 2002). I did not include this literature in the current review for two reasons. First, research on students' use of dynamic geometry software looks primarily at students' cognitive structures and epistemology. While these both could be seen to influence the actions that students take in instruction, they are much larger issues that are outside the scope of the current study. Second, the instructional situations being examined in the current study, 'making conjectures' and 'doing proofs' are greatly changed by the introduction of dynamic geometry software. I have decided to limit my study to instruction not including this powerful resource. Further studies are warranted to examine how these situations are affected by dynamic geometry software.

Literature on personal resources

The field of educational psychology has contributed greatly to the question of what personal resources are related to different educational actions, emotions, and cognitions. By personal resources I mean characteristics or traits of individuals that unique to specific individuals in a particular context. For this review of related research I am interested in concepts of multiple goals and goal orientation theories (Dweck & Leggett, 1988; Harackiewicz et al., 2002; Pintrich, 2000; Pintrich & Schunk, 1996) and attribution theory (Weiner, 1985). Multiple goals and goal orientation theories examine the question of which academic goals students hold and how those goals manifest in student action. Attribution theory examines the connections that students see between outcomes and perceived causes, in the form of events or behavior. These approaches view goals and attributions to be key factors that influence the actions that students take in schools. In the following section I discuss the relevant literature on students' goals and attributions.

Students' goals and view of their ability

Depending on students' outlook on classroom work, they can be described as having either mastery, performance-approach, or performance-avoidance goals (Pintrich & Schunk, 1996). Students with mastery goals are interested in developing their academic abilities. To these students, the expenditure of effort is a necessary part of their intellectual development and a sign of high ability. These students will therefore be likely to set goals for themselves based on a comparison of their current ability with their ability in the past. On the other hand, students who hold performance goals are focused on how their ability level is evaluated by their teacher and classmates. These students are interested in how their ability is perceived by others (in contrast to developing their ability). Expending effort is not seen as productive, but rather, a sign of low ability because high ability would imply that no effort was required to succeed. It is conceptually useful to divide performance goals into performance-approach and performance-avoidance goals (Schunk, Pintrich & Meece, 2008). Performance-approach goals reflect a student's interest in displaying his or her positive ability while performance-avoid goals reflect a student's interest in concealing his or her negative ability. Research has show the highest performance outcomes are correlated with holding both mastery and performance-approach goals (Harackiewicz et al., 1998).

Related to mastery and performance goals is the question of how students view the possibility of change in their own abilities (Dweck & Leggett, 1988). That is, do they see their ability as something that can be grown and developed, or something that is constant? Students who see their abilities as something that is constant and immune to development usually do not take up mastery goals. From their point of view effort has no effect on their abilities; either they are able or they are not. Since developing their ability is not an option, their goals for classroom work are likely to be to either display or conceal their ability level.

Students' attributions: interpretations of cause and effect

In students' attempts to make sense of their classroom environment, students attribute causes to events. Attributions are driven by the questions, "why did such-and-such happen?" These attributions are an important part of how students make meaning from the activities of their geometry class. In answering the question of what a geometry student might do, attribution theory helps explain how students make sense of the causes and effects of their actions on their environment and vice versa.

Attributions are parsed across three dimensions; stability, locus, and control (Pintrich & Schunk, 1996). The dimension of stability measures how much, in the eyes of the attributor, the effect was caused by something that is stable across time and people. The dimension of locus measures how much the effect was caused by something that is either internal or external to the actor. The dimension of control measures how much the effect was caused by something that is controllable.

The literature discussed above attempts to explain students' actions by way of their goals and attributions. One sees that one reason that students would pick a particular instructional action over another is that they perceive that one of these actions would better advance their goals. The way that students determine which actions would best advance their goals is explained by attribution theory.

Literature on students' commitments

The following studies are aimed at understanding how students' relationship inside the school can influence the actions that they make. In schools there are two key elements that students have a relationship with; the school as a whole (including the institution, the individual teachers, and other students), and the academic work that they do. Below I look at studies that fall into these two categories.

Relationship with the school

In the following section I describe literature that discusses students actions in schools, focusing on research that assumes that students' actions in schools are related to the relationship that students have with the institution of school. This studies look at the school as a whole, focusing on the curriculum, peer groups and relations, relationships with teachers and administrators, and the value students place on their experiences in school.

The high school ethnography, *Jocks and Burnouts* (Eckert, 1989) describes and attempts to explain the existence of two groups pervasive to American high schools, the Jocks and Burnouts. The Jocks are students who see themselves as part of the high school community and engage in school activities like sport teams and school dances. These students see the preparation—both social and academic, that they receive in high school as a valuable resource in their future. The Burnouts do not see themselves as part of the high school community, and see high school as a waste of time because it keeps them from activities, like working in their uncle's auto repair shop or taking care of their siblings, that they see as more relevant to their future and current life. *Jocks and Burnouts* highlights the interaction of the social groups as a key part of their continual regeneration. Jocks define themselves in terms of being "not burnouts" and Burnouts define themselves as "not jocks."

Life in classrooms consists of activities such as passing and failing tests, being frustrated, accomplishing challenging task, but these highlights exist in a vast sea of sitting, listening, walking to recess, and completing mundane tasks. Jackson (1968) examines the moment-to-moment activities of students in classrooms, focusing on the unremarked "daily grind" instead of the remarkable highlights of students' days. Jackson focuses on three aspects that contribute to the daily grind; crowds, praise and power. In response to these aspects of classrooms students develop strategies to pass the time in harmony with their teacher and fellow students, gain praise from their teacher (or at the least avoid censure), act in deference to the imbalance of power strongly skewed in favor of the teacher.

A result of the large number of students in schools is a “one size fits all” classroom environment (with a few exceptions). There may be a large amount of variety among courses that are available to students, but once a student is in a class, each member of the class receives the same instruction (Powell et al., 1985). The sorting process may be painstaking, but once it is completed most instructional individuality is lost. This does not mean that students do not experience class differently from the student next to them, but the institution of school treats them the same as long as they don’t stand out enough to be moved to a different class, as is the case with students who are labeled as needing ‘special education.’

In a study on the causes of students dropping out of high school, Farrell, Peguero, Lindsey, and White (1988) found that social pressures were all encompassing for students and school was adding stress without providing any payoff. Like other studies of students’ experience with school, these students reported the overwhelming feeling that school is boring and exerts pressure that is out of proportion to the pay offs that students get from being “good students.” Efforts of the teacher to place demands on students, and then to evaluate students’ response to those demands, make students less likely to engage with the material and more likely to experience boredom in the classroom.

Anagnostopoulos (2006) studied how students who were held back a grade were categorized by other students and the adults in the school. There are two distinct groups of students who were demoted a grade. The “true demotes” are students who were seen not to attempt to succeed in school. These true demotes were also seen as lazy and morally inferior to other students in the school. The name “true demote” reflects the view that these students deserved to be demoted a grade. The other category of demotes are “kids who got tripped up.” These students were seen to have extenuating circumstances that caused them to not do well in school, despite their best efforts. Teachers and their peers agreed that they did not really deserve to be held back, but their demotion was a necessary consequence of their situation.

A key difference between ‘true demotes” and “students who got tripped up” was their attitude towards school. True demotes were seen as disruptive and angry. Whether these students were disruptive and angry before they were demoted or demotion led them to be perceived as disruptive and angry is not clear. Also, it is not clear how accurate this description of true demotes is. Teachers described their regular classes as ‘teasing’ each other, while demote classes were described as ‘bickering.’ Teachers of demoted classes may have been inclined to see demoted students in a negative light.

The book *“Doing School”* (Pope, 2001) looks at how students interacted with schools in terms of their choice of classes and relationships with teachers. Pope interviewed five students, each of whom had developed their own ways to cope with the pressures of high school. Each of the students included in the book were recommended by teachers at the high school for being outstanding, either in terms of academics, sports, clubs, or work ethic.

Pope’s ethnography uncovered the stress that these students experienced that was invisible to their teachers and parents, as well as the strategies that these students had adopted for coping with the pressure and workload. The students' main goal was to get high paying jobs when they reached adulthood. To get these jobs they believed that they needed to attend a top tier university. And to be admitted to a top tier university they needed to have great grades and lots of extra curricular activities on their high school transcripts. To garner the praise of their teachers and coaches, these students were willing to cheat, do homework during class, "suck up" to teachers, and contest low grades. All of these behaviors are generally associated with "bad" students, not the stars of the student body.

Each of these studies looks at how students relate to the institution of school. They place importance on how students view the curriculum, the other members of the school, and the value that school has in their lives. In the studies below, the researchers focus instead on the relationships that students have to the work that they do in classrooms and how this can be seen to influence their actions.

Relationship to school work

In the studies described below, the researchers look at how the academic work that students are asked to do in school influences their actions. Many of these studies look at how students interpret the tasks that they are asked to do in classrooms, both in terms of what work the task requires of them and in terms of what that task is worth in their eyes.

Fried argues in his book, *The Game of School* that both students and teachers often do not engage in authentic teaching and learning, but rather play "the game of school." In this game, teachers give tasks that do not promote deep thinking on behalf of their students and students complete assignments in ways that will garner them passing grades without regard to what could be learned from the assignment. According to Fried, "The Game begins whenever we focus on getting through the school day rather than actually *learning*." (Fried, 2005, p. x) The game of school is the result of teachers and students focusing on the outcome of their school activities, like grades and performance evaluations, and ignoring the educational experience.

Lave's chapter, "The Culture of Acquisition and the Practice of Understanding" in *Situated Cognition* (1997) looks in detail at what it is that students do in classrooms in response to tasks. In Lave's observation, students left to complete mathematical work without direct supervision of the teacher would use strategies that they knew to work instead of the methods sanctioned by the teacher. Those methods were used because they had been shown to give the correct answers in the past and so it was safer to secretly use these methods than to risk using the new methods and arriving at incorrect answers. Lave labels this as an "appearance of understanding" as opposed to a demonstration of understanding as the teacher had in mind. By using methods to complete the task that are different than those expected by the teacher, students changed the task. However, because students arrived at the expected answer, the teacher was unaware of the change in the task.

Herbst (2003) gives an example of a task that differed for the students and the teacher within the context of using novel tasks to elicit new knowledge from students. The teacher's goal of having students generate new knowledge while working on the task resulted in a key difference between the task as envisioned by the teacher and the task as described to the students. The students in this example were given cardboard cutouts of eight triangles and asked to order them with respect to area. The students were reminded of the area formula for triangles, but they were told that they should use the formula as little as possible. In terms of the operations to be enacted within the task, the teacher envisioned the students using the area formula to compare the area of triangles by uncovering the multiplicative nature of the area formula (thus not "using" the area formula in the sense of calculating areas). This operation could not be explicitly told to the students because that would remove the possibility of students generating this as new knowledge. This example shows the importance of how students interpret a task on the subsequent work that students do. The students in this study acted as if they interpreted the teachers' instructions as meaning that they must not use the area formula for triangles at all and this substantially impacted the progress they were able to make on the task.

According to Boaler (1998), many schools do not usually facilitate the ways that girls learn mathematics. Boaler argues that mathematics instruction often rewards students who enact tasks with goals of speed and accuracy instead of understanding of concepts. During interviews, boys felt accomplishment when they completed a task quickly and accurately, whereas the girls' sense of accomplishment relied on the feeling that they understood the concepts being learned. The girls in Boaler's study were not lacking some key ability that kept them from doing mathematics as well as the boys, but these girls interpreted goals for the task based on conceptual understanding that did not match with the goals of the tasks as interpreted by the teacher.

Students from different backgrounds bring different approaches to being a student into the classroom. Sfard and Prusak (2005) found that recent immigrants from the former Soviet Union ("new comers") approached assignments very differently from their native Israeli peers ("old timers"). On an assignment where the teacher did not explicitly ask for students to turn their work in, the "new comers" were less likely than the "old timers" to have written any notes. In this context, note taking was a method for students to show the teacher that they had exerted effort on a task. The “new comers” were not concerned with showing the teacher their work. They engaged with tasks in a way that Sfard and Prusak argue was directed at understanding the mathematics; the “new comers” were able to show a deeper level of understanding than the “old timers.” The authors interpret the difference between the work of these two groups of students as a quest of a deeper understanding that is tied to conceptions of learning as inherently good, and not just for the grade or certificate at the end of the course.

Aaron & Herbst (in review) found that groups of students in the geometry classroom view the work of being a student differently. Some students come in attuned to the evaluation of the teacher, with a goal of getting high grades. Other students enter the classroom attuned to the content and focused on understanding the mathematics put before them. Still other students are not attuned to either of these. This last group of students simply do whatever is asked of them in the classroom so that they can avoid punishment. These are three different ways that individuals can conceive of the work of being a student, and each could lead to students acting differently during instruction.

Students’ trajectories are a useful way to describe students’ metaphorical paths through school and their relationship with particular classes (Chazan, 2000). The idea of trajectories takes into account ‘where students have been’ in terms of past experience at school, at home, and what they have seen done by their parents and others in their communities; and it also takes into account ‘where students are going’ in terms of their academic and career goals. A particular class that students are enrolled in can either fit with their perceived trajectory, by building on their past experiences and giving them new experiences that they anticipate to be valuable in the future, or a class can not fit in a student's perceived trajectory by seeming disjoint from past and future experiences. Chazan used the metaphor of student trajectories to explain why students in his Algebra I class would or would not engage with the material in class and complete homework. For students who perceived the class as coincident with their trajectory, the class work was meaningful and therefore they would engage with it. For students who perceive the class as non-coincident with their trajectory, the class work was irrelevant and therefore they would disengage.

Lampert (2001) discusses students' disposition to be 'people who learn in school' and the teacher's responsibility to help promote this disposition in his or her students. Without this disposition toward productive study of academic subjects, Lampert claims that students are less likely to engage with the material, less likely to learn anything from that engagement, and even if they did learn, they would be unlikely to use that knowledge in public. Lampert tells the story of how she assisted two students in developing this disposition. The first student, Richard, is supported through the process of learning how to make mistakes in front of his peers. The second student, Saundra, also learns how to make mistakes in front of her peers; in addition she learns how to interact with educational materials in a constructive way. These examples both showcase how the teacher can support students in developing positive relationships with the work of school.

These studies discussed above look at students' relationship to the work that they are asked to do in classrooms. The researchers look at the actions that students take in response to tasks and how students value these tasks. In the literature described below the authors look that the work of 'studenting' or the work of being an expert student.

Literature on studenting

In the following section I discuss studies that have looked at explanations of student actions that are based on the work of being a student, or studenting. I follow Fenstermacher in defining "studenting" as the set of actions performed by an individual in the position of student. According to Fenstermacher, studenting is defined as the activities that may allow learning to take place, such as "recite, practice, seek assistance, review, check, locate sources, and access material" (Fenstermacher, 1986, p. 39). In this light, learning is something that is a result of studenting, by way of studying, and only indirectly a result of teaching. Within the context of modern schools, studenting takes on a larger meaning than that given above. It must be expanded to include activities such as getting along with one's peers, teacher and parents, navigating textbooks, deciphering handouts from the teacher, as well as nonacademic aspects of life in school. In general, learning how to be a person who studies in school (Lampert, 2001).

Children are not born knowing how to be a student. Through their time in school they learn how to behave and what is expected of them. Students learn what they would like to get out of school and they learn what school would like to get out of them (Doyle, 1983). These expectations vary from subject-to-subject. By the time students reach high school they are adept at reading their teachers and scanning the content offered to see what matches with their goals for the course. Students are only able to focus in this way because they have learned how through years of experience in classrooms with teachers and content.

Brousseau & Warfield (1999) describe the interactions that Gaël, an eight year old boy, had with his tutor around subtraction problems. The interaction was designed to help

Gaël, as he was struggling in mathematics. During the course of the tutoring sessions it became apparent that Gaël had developed a strategy to avoid being in a position of not knowing; he would defer to the adult, or his conception of what was expected of him, instead of consulting his own reasoning and computations. For example, when the tutor asked Gaël how many chips are in the large bag, if there are 10 in the small bag, and 56 all together, Gaël counted to ten and stated the number 5, seemingly at random. Gaël acted as if the appropriate response to the tutor's question was to count out loud to some number (his favorite seemed to be ten) and then to offer some number as an answer. Gaël showed little evidence that he knew how to subtract, but he did show evidence that he knew how to 'play the game' of subtraction.

The case of Gaël shows an example of a student who could interpret the task that is posed to him, and was motivated to provide the correct answer, he may even have been capable of successfully completing the task. His actions seemed to be guided by some other source. Gaël could be seen to be "studenting" as an attempt to avoid thinking. He was performing some of the actions that are appropriate for his situation, but he was unsuccessful because he was not deploying any mathematical reasoning.

Some students become "expert students" similar to the expert teachers studied by Leinhardt and Greeno (1986). The expert teachers were found to execute routines while teaching. A key feature of these routines is that they place a low cognitive demand on the teacher, so that she can allocate cognitive resources to other demands of teaching. I hypothesize expert students also enact routines that allow them to free up cognitive resources for other activities. For example, a geometry student may look at the theorems introduced in the section before starting her homework. This routine reflects an assumption that tasks ordinarily require theorems recently learned as resources. This reduces the number of possible arguments there might be to solve a problem to only those arguments that hinge on recent theorems.

Schoenfeld discusses how students have a long history of doing mathematical tasks in schools,

> The student comes to the problem having solved a huge number (in the tens of thousands) of mathematics problems. Whether or not the student is conscious of it, this prior experience shapes the amount of time and effort that will be invested in this problem. Prior experience will determine what information the student thinks is relevant and what concepts the student thinks are appropriate to the problem. (Schoenfeld, 1989)

Each new task is not really a new task, but one task in a long string of tasks that the student faces. The student's experience with previous tasks shapes how the student will approach the current task. This gives weight to the claim that students learn how to "student" over their time in classrooms. Even though each year in a student's school career could mean a new teacher and new classmates, and surely new tasks, the student brings with her many experiences of prior teachers and classmates and tasks that influence how she acts in this new instructional context.

An example of a student using classroom time and the cover of a task assigned by the teacher to fulfill her own goals comes from Mehan's article, "The Competent Student." Mehan (1980) narrates a classroom episode in which a teacher assigns a student, Carolyn, the task of leaving the rug circle to check the cupboard for recess balls. To the teacher, this task will provide her and the rest of the class with information that will help plan the day's activities. For the student, this task is an opportunity to use the teacher's request as a cover to hang up her sweater in the closet next to the ball cupboard. Leaving the rug without permission from the teacher would have been against the classroom rules. Carolyn volunteered for this task knowing that she had a different goal than the teacher. She knew the classroom rules well enough to know that her goal was in opposition with the rules, but she also knew the rules well enough to know which activities she could best use to meet her needs. Although this is not an academic task, it highlights the ways that expert students can navigate the complex terrain of classroom tasks and use them to meet their own goals.

The norms that students are expected to work within are determined by the class that they are in, and the activity structure that students are engaged with. Herbst and Brach (2006) layout the accountability structure for the situation of 'doing proofs' inside high school geometry classrooms (for more information on this see "Teachers' Perceptions of Geometry Students," Aaron, this volume). This accountability structure is an example of how teachers and students shape the activities of teaching and learning 'proof' by acting as if they are following a set of norms for action. The teacher is accountable for posing problems with clear statements of what shall be taken as 'given' and what is the statement that is to be proved, as well as providing an accompanying diagram with all of the relevant geometric objects available for inspection. The student is responsible for marking known statements on the diagram through various markings and for laying out a sequence of "statements" and "reasons" in the form of a two-column proof.

All of these studies are focused on studenting, insofar as they provide explanations of students' actions that are dependent on the nature of the work that students do in classrooms. The current study adds to this literature by providing a theoretical framework describing "studenting" and by showing how this framework can explain students' actions in high school geometry classes. Below I describe the empirical study that was conducted to test hypothesized norms of geometry instruction, and the instructional situations of 'doing proofs' and 'making conjectures.'

Data

The data to be used in this study was collected during one-time focus group sessions with classes of high school geometry students. These data come from eight classes in two schools. School 1 is a high-achieving public school serving 2,800 students in grades 9-12. School 2 is a low-achieving public school serving 1,200 students in grades 9-12. At School 1 I collected data in two honors level classes taught by Megan and two regular level class taught by Madison. At School 2 I collected data in two regular level classes taught by Jack and two remedial level classes taught by Sharleen. These schools, teachers, and classes were chosen so that the data corpus would represent a diverse group of students and experiences.

Near the end of the 2007-2008 school year I met with each class of students for one class period (about 50 minutes). Each class of approximately thirty students was shown the animated scenario, The Square[11] in five short clips. Each of the clips was between one

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[11] The Square, along with all of ThEMaT's animated scenarios, can be viewed at http://grip.umich.edu/themat

and two minutes in length and highlighted the actions of a small group of animated students. The first clip was an exception because it featured only the animated teacher posing the angle bisectors problem to the class. The second clip focused on Alpha and Beta, the third clip focused on Gamma, and the fourth and fifth clip focused on Lambda. After viewing the first clip participants were given some time to work in small groups or individually on the angle bisectors problem. After viewing each of the clips the moderator lead a discussion, in the form of a semistructured interview (De Groot, 2002), among participants with the aim of collecting general comments about what the participants just viewed as well as collecting comments regarding the participants' views of the animated students they saw in the animated scenario.

While watching The Square, participants were potentially comparing and contrasting their experiences with the events displayed in the animated scenario. Participants worked to make sense of the actions of the animated teacher and students; reporting on the actions that drew their attention, evaluating those actions, providing justifications for those actions, and suggesting alternatives.

The classroom teachers were present in all eight sessions and participated to varying degrees in the conversation. The most common interventions made by the classroom teachers were to help participants work on the angle bisectors problem and to prompt participants for comments when participants were slow in answering the moderator's questions.

These sessions were video and audio recorded and then transcribed and indexed for analysis. This indexing consisted of dividing the transcript based on the clip of the animated scenario being discussed.

In the following section I describe the analytic methods that were used to examine the data and the methodology of collecting the data. I include a description of the animated classroom scenario, a list of hypothesized norms that were the object of the analytic coding, and describe how these codes were applied to the data.

Methods

The focus groups with students were cases of modified breaching experiments (Garfinkel, 1964; see Aaron, this volume for a full description of the use of animated scenarios as breaching experiments). These breaching experiments were used to allow participants to point to things that are abnormal in familiar situations. I hypothesized that by showing participants the animated scenarios, which were designed to display actions that the participants could perceive as breaches in the normal running of a high school geometry class, I would be able to elicit from them the tacit perceptions they had of the norms of the work of being a geometry student. Because these perceptions are tacit, participants might not have been able to share them with an interviewer in a traditional interview. The immersive quality of the animated scenarios coupled with the provocation of the embedded hypothesized breaches of normal instruction prompted participants to share their perception of the geometry classroom.

For the most part the stories shown in the animated scenarios conformed to the norms for the instructional situations that they depict. However, they also displayed actions that could be perceived as breaching these norms in a few meaningful ways. The participants' responses to these hypothesized breaches provided clues as to what the norm is that they perceive as being breached and reasons why a student might feel obliged to breach that norm.

The animated scenarios were created to be used as prompts in breaching experiments with teachers (Herbst & Chazan, 2003; see Aaron, "Teachers' perceptions of geometry students," this volume for a description of the creation and use of the animated scenarios). This study employs the animated scenario in the collection of data from students, instead of teachers, which required a slightly different use of the animated scenario. For instance, instead of asking the participants to stop the animated scenario at moments when they have comments (as was done with teacher participants), the moderator stopped the animated scenario at predetermined points and asked the participants open-ended questions about what they had watched. The differences in data collection protocol between the teacher participants, for which the animated scenario was designed, and the student participants reflects the different positions of the teacher and student in instruction; teachers are in a much more direct control of the flow of the class while students do not have much control of the overall activities of the class (despite having control over their individual actions). The differences also reflect the confidence and authority with which teachers are able to discuss instruction, and the hesitance that some students display when talking in front of their peers.

Hypothesized norms

The right-hand column of Tables 39, 40, and 41 lists the hypothesized norms for student action that were tested in this study. This is a partial list of the norms[12] of the didactical contract and the situations of 'making conjectures' and 'doing proofs.' To generate this list of norms I began with a partial list of norms for teacher action from prior research (Herbst et al, 2010; Herbst et al., 2009; Herbst & Brach, 2006) that are listed in the left-hand column of Tables 39, 40, and 41. Norms for teacher action are included in this list if they are visible in The Square and if they have implications for student actions. For instance, the norm *the teacher is responsible for asking questions that keep the classroom discussion on topic* is included in the list because it is related to the animated teacher's choice to ask Gamma to come to the board and because it suggests the corresponding norm for student action; *students' interventions should address the topic that the teacher was proposing.* I generated this list of corresponding norms for student action based on the chosen norms for teacher action. The corresponding norms for student action are designed so that the student action would normally occur in concert with the teacher action.

This list of norms that are examined in this study is much shorter than a comprehensive list of norms for geometry instruction, however, as can be seen from the results, it is long enough to provide valuable insight into how norms appear to guide student actions and shape the position of student in the geometry classroom. Primarily, this list allows for

[12] The hypothesized norms of the didactical contract and the instructional situation 'making conjectures' that are used in this paper are currently under development in the GRIP research group, under the direction of Patricio Herbst.

confirmation of some norms of geometry instruction, and for the preliminary exploration of other justifications that the participants provided for student action. The combination of the confirmation of norms and alternative justifications for action is used as a motivation for a more complex model of student action.

Below the norms that are explored in this study are listed and a short description of the situations 'making conjectures' and 'doing proofs' is provided. These norms are further described, and evidence for them is shown in the results section. Table 39 shows hypothesized norms of the didactical contract in the geometry classroom.

Table 39: Contractual norms for geometry instruction

Norm for teacher action	*Corresponding norm for student actions*
The teacher is responsible for eliciting students' ideas	Students should share their ideas that are different than other students' ideas
	Students should complete incomplete arguments given in class
The teacher is responsible for keeping the classroom discussions on topic	Students' interventions should address the topic that the teacher is proposing
Some student ides should be displayed for all students	Students should be disposed to share ideas in public when so asked
[There is no corresponding norm for teacher action]	Students in the audience should support the presenter in public

The teacher should assess student ideas	Students should be amenable to assessment

Table 40 shows hypothesized norms of the instructional situation of ‘making conjectures.’

Table 40: Situational norms for 'making conjectures'

Norm for teacher action	*Corresponding norm for student actions*
The teacher should enable students’ conjectures to become public	Students should communicate their conjectures to the class
The teacher should ask the class to agree or disagree	Students can respond to other students' conjectures
The teacher should end the activity once the desired conjecture has been stated	Students should stop talking about a conjecture once it has been agreed upon or refuted

Based on the norms that are hypothesized here, one can sketch how a normal instance of ‘making conjectures’’ would proceed. In the situation of ‘making conjectures’ the teacher is responsible for engaging with a task that asks students to make a conjecture. The teacher is also responsible for providing students with resources to use on the conjecture task. After students have time to work independently, with the teachers’ supervision, on the conjecture task, the teacher chooses a student to come to the board to share their conjecture. After the student has stated his conjecture the teacher is responsible for pointing out a positive aspect of this conjecture. Other students may then agree or disagree with this conjecture. Once the conjecture has been either agreed upon or rejected the class moves on to discussing another conjecture or to engaging in another activity. Table 41 shows hypothesized norms of the instructional situation of 'doing proofs.'

Table 41: Situational norms for 'doing proofs'

Norm for teacher action	*Corresponding norm for student actions*
The teacher provides proof tasks that are answerable as posed	Students should not make changes to the proof problem

In the situation of 'doing proofs' the teacher is accountable for posing problems that call for a proof as part of the response, with clear statements of what shall be taken as given and what is the statement that is to be proved, as well as providing an accompanying diagram with all of the relevant geometric objects available for inspection. The student is responsible for marking known statements on the diagram though various markings and for laying out a sequence of "statements" and "reasons" in the form of a two-column proof. Since these norms are not visible in The Square, and the participants did not bring them up in their discussions, they are not included in the discussion.

These hypothesized norms were used to code the transcripts from students. The purpose of the coding was to find evidence that would support the existence of these norms. Below I describe the animated scenario that was used in the current study. The description includes commentary on normal and abnormal behavior according to the norms of the high school geometry classroom and the instructional situations of 'making conjectures' and 'doing proofs.'

Analysis of The Square

Below is a sketch of the animated scenario, The Square. The outline of the plot is provided along with a description of some of the norms that could support the actions seen in the story.

The animated scenario begins with the teacher reminding the class that when they studied triangles they learned that the angle bisectors of a triangle meet at a point. She then poses an open-ended question, "what can one say about the angle bisectors of a quadrilateral?" She asks the class to make conjectures with the idea that they will later try to prove those conjectures. The problem that the teacher assigns to the class is abnormal because it combines the activities of making conjectures and proving. Normally students would be asked to either make conjectures or do proofs but not to prove conjectures that they make. Also, the teacher does not specify any particular resources or operations that the students are expected to use while working on this task. The only resources that are specified are the mathematical objects quadrilaterals and angle bisectors, as these are included in the problem statement. By bringing up the angle bisectors of triangles she implicitly invokes points of intersection as a resource that could be used in the task.

After the students work on the task for an unknown period of time the teacher asks Alpha to share his conjecture with the class. Here one can notice that the teacher did not ask for volunteers, and in particular, Alpha did not volunteer to share his conjecture. Despite not volunteering, Alpha comes to the board to share his conjecture. Alpha draws a diagram on the board and describes it as "a square and the diagonals, they bisect each other." Here Alpha could be seen to be complying with the norm that *students should be disposed to share ideas in public when so asked*, and *students should communicate their* *conjectures to the class*. On the surface Alpha appears to be acting in a way that could be perceived as breaching the norm *students interventions should address the topic that the teacher is proposing*, since his conjecture is stated about diagonals of a square instead of angle bisectors of a quadrilateral. Alpha's action could also be seen to be complying with the norm *students should share ambiguous ideas*, although it's unclear if Alpha realized that his idea was ambiguous.

The teacher comments on Alpha's conjecture by saying, "Alpha, the question is about angle bisectors, not about the diagonals." Speaking from her seat in the front row Beta whispers, "but it's the same thing, isn't it?" Beta, depending on how her comment is interpreted, could be seen as either complying or breaching the norm *students in the audience should support the presenter in public.* On one hand she could be seen as defending Alpha's answer, or, on the other hand, she could be seen as talking out of turn. Beta's action could also be related to several other norms, *students should complete incomplete arguments given in class, students can respond to other students' conjectures,* and *students should stop talking about a conjecture once it has been agreed upon or refuted.* The first of these two norms support Beta's action, while the last norm suggests that it was inappropriate for Beta to whisper since the teacher's reaction to Alpha conjecture could be interpreted as a rejection.

After Beta's whisper Alpha clarifies his conjecture by saying, "I just thought that the diagonals cut the square in half," which Beta further clarified by saying "the diagonals are also the angle bisectors." This statement prompts Gamma to come to the board, at the teacher's request, to show that this claim is not true for a rectangle. Gamma shows a diagram of a rectangle, which has angle bisectors that are different than its angle bisectors. By coming to the board Gamma could be seen to be complying with the norm, *students should be disposed to share ideas in public when so asked*. Gamma could also be seen to be complying with the norms *students should share ideas that are different than other students' ideas*, *students' interventions should address the topic that the teacher is proposing, students should communicate their conjectures to the class,* and *students can respond to other students' ideas.*

Gamma's response to Alpha's conjecture prompts Alpha to restate his conjecture as, "in a square the angle bisectors meet at a point because they are the diagonals." The teacher then writes on the board, "in a □ the ang bis ⊠" and calls for volunteers to do a proof. When no one volunteers the teacher calls on Lambda. The teacher writing this statement on the board and asking students to produce a proof signals a change in instructional situation from 'making conjectures' to 'doing proofs.'

Lambda stays in his seat and describes a proof of the claim that the diagonals of a square are also the angle bisectors. During the course of this proof he asks the teacher to remove one of the diagonals so that he can prove the claim about each diagonal separately. By staying in his seat Lambda could be seen to be breaching the norm *students should be disposed to share ideas in public when so asked*. His actions can also be interpreted as breaching the norm, *students should not make changes to the proof problem* (Herbst et al., 2009), when he asks the teacher to remove one diagonal from the diagram. Lambda's description of the proof, which highlights the key statements, but does not provide reasons for the statements, can also be seen to be inappropriate based on the norm *students produce profs by alternating between creating statements and reasons for those statements* (Herbst et al., 2009).

This animated scenario was shown to participants in five clips. The first clip ended with the teacher giving students time to work on the problem. The second clip ended with the teacher asking the class if they agree with Beta's statement that the diagonals are the angle bisectors. The third clip ended with the teacher calling for a proof of the statement "in a □ the ang bis ☒." The fourth clip ends with the teacher erasing one diagonal from the diagram of the square on the board. The fifth clip ends with the animated scenario and the end of Lambda's proof.

After viewing the first clip the participants were asked to work on the mathematical problem so that they would become familiar with the concepts involved in the problem. After each of the following clips participants were asked what they noticed about the students in the clip. Below is a description of the how the discussions of this animated scenario were coded.

Coding for norms

To analyze the data I look at participants' responses to the actions of the animated students. By examining these reactions one can see how well the animated students' actions are suited to the instructional situation in which their actions are embedded from the students' perspective. The participants' comments are examined to see if they reflect a norm of instruction.

First, each comment from the participants was tagged as being in reference to a particular moment dealing with an animated student or the animated teacher. Second, within these moments, comments were summarized and similar comments were compiled. Third, the summarized and compiled comments were then coded as reflecting any situational norms. This coding was based on the norms listed in Tables 39, 40, and 41.

Perceptions of animated students' actions, with relation to a norm, could be coded in one of four ways. The participants could see that the animated students complied with a norm and that this compliance was appropriate. The participants could also see that the animated students complied with a norm, however this compliance could be seen as inappropriate. That is, the participants thought that the animated student should have breached a norm. The participants could also report that a norm was breached by the animated students and that this breach was appropriate. Finally, the participants could report that a norm was breached by an animated student and that this breach was inappropriate, that is, the animated student should not have breached that norm.

If an animated student's action was coded as either an inappropriate compliance or an appropriate breach with respect to a norm then that perception of the action was revisited at the end of the coding to look for commonalities. These perceptions of either an inappropriate compliance or an appropriate breach of a norm point to an action that is supported by something other than the norm, but not a rejection of the norm. In the discussion section I discuss other possible rationale for these actions. These other rationales for student action are then used to develop a more complete framework for describing the factors that influence student action.

In the following section I share the results of the analysis. The results are organized by norm. For each norm I list the actions in the animated scenario that the participants perceived as being related to the norm. Each action could be listed more than once in the results, associated with a different norm. For each action associated with a norm, I describe the evidence from the data that supports the claims that participants saw the action of the animated students as an appropriate compliance, an inappropriate compliance, an appropriate breach, or an inappropriate breach of the norm.

Results

Below is a list of hypothesized norms for the geometry classroom. For each hypothesized norm I list animated student actions that activate this norm and evidence from the student focus groups that point to the existence of this norm. This list of hypothesized norms for student action was generated from a list of norms for teacher actions from prior research (as described in the methodology section). I used a coding scheme based on these norms to examine the transcripts for comments that provided evidence that these hypothesized norms played a role in guiding student actions. Some norms on the list do not have any evidence to support them. This does not mean that these are not norms for student action, only that the particular prompt that was used in data collection did not produce conversation about these norms. More studies would need to be done to look for confirming or refuting evidence for these norms.

Evidence from the transcript could support either a perceived breach or perceived compliance with a norm, and participants could see this breach or compliance as either appropriate or inappropriate. When participants mark a norm as either appropriately complied with or inappropriately breached, participants are reporting that, with regards to a particular action, student actions should be in line with the norm. When participants mark a norm as either inappropriately complied with or appropriately breached, participants are reporting that, with regards to a particular action, student actions should not be in line with the norm. I will return to these cases in the discussion section and use them as a motivation to build a framework to describe the influences on student action in geometry instruction.

Contractual norms

Below is evidence for the hypothesized contractual norms. These norms guide student action in the classroom but are not specific to any one instructional situation.

Students should share their ideas that are different than other students' ideas

The norm *students should share their ideas that are different than other students' ideas* says that when students are sharing ideas in class, it is appropriate for students to share their ideas that are mathematically different than other ideas that have been previously shared. Implicit in this norm is that students should not share ideas that are mathematically similar to ideas that have previously been shared. This norm was activated by one action that is listed in Table 42, along with evidence for the existence of the norm.

Table 42: Actions related to the norm, *students should share their ideas that are different than other students' ideas*

Action	**Perceived relation to norm**
A student comes to the board to share her idea	Appropriate Compliance

A student comes to the board to share her idea (Appropriate Compliance)

The participants saw the norm *students should share their ideas that are different than other students' ideas* to be related to the action of a student coming to the board to share her idea. They saw that, in The Square, Gamma appropriately complied with this norm when she came to the board to share her idea because her idea was different than the idea that had just been presented by Alpha. Roma from Sharleen's third period class said, "[Gamma] wanted herself, like, to show what she thought about the problem too. So, that's what she was trying to do." This response from Roma showed the opinion that it is appropriate for students to share ideas. He emphasized the fact that Gamma was sharing her idea "too," implying that it is appropriate for Gamma to share her idea despite the fact that another student had already shared their idea, presumably because Gamma's idea was different. Cal, also from Sharleen's third period class, said that Gamma came to the board "'cause there's like more than one way to answer a question and she's showing them her way and [Alpha] was showing his way." Cal was explicit in saying that the reason that it was appropriate for Gamma to share her idea is because it is different than the idea previously shared by Alpha.

Also related to this norm, participants said that although it is appropriate for students to share ideas that are different than the ideas of other students, students are not expected to say how their idea fits into the bigger picture of the lesson. Paul from Megan's third period class compared Gamma's contribution in The Square to a situation where students discovered something interesting but that was not necessarily connected to the lesson, "Like, when we were doing circles, like, we found that triangles inside of circles are similar but we weren't actually trying to find that out, we were just trying to find something else out. We just pointed it out." Here Paul was giving an example from his experience in geometry class, as a reason why it was appropriate for Gamma to share an idea but to not say how that idea is related to the other ideas that have been shared in class.

Each of these comments from participants shows evidence for the norm *students should share their ideas that are different than other students' ideas*. These participants saw that it was appropriate for Gamma to share her idea because it was different than Alpha's.

Students should complete incomplete arguments given in class

The norm *students should complete incomplete arguments given in class* says that when students are sharing ideas in class, it is appropriate for students to share ideas that fill in gaps in the ideas shared by other students. This norm provides students with an opportunity to speak in class, despite the fact that other student may appear to have control of the floor. This norm was activated by two actions that are listed in Table 43. Below is evidence from the data.

Table 43: Actions related to the norm, *students should complete incomplete arguments given in class*

Action	Perceived relation to norm
A student whispers a contribution to a solution	Appropriate Compliance
A student whispers a contribution to a solution	Inappropriate Breach

Participants saw the norm *students should complete incomplete arguments given in class* as relevant to the action of a student whispering a contribution to a solution. Some participants' comments reflected the opinion that this action is appropriate in light of this norm, while the comments of other participants reflected the opinion that this action represents an inappropriate breach of the norm. This difference of opinion can be explained by the fact that some participants consider a whispered contribution as an appropriate form of contributing in class while others do not.

A student whispers a contribution to a solution (Appropriate Compliance)

Some participants saw that the action, a student whispers a contribution to a solution, was appropriate in light of the norm *students should complete incomplete arguments given in class*. In The Square, this action was performed by Beta when she whispers from her seat while Alpha is at the board sharing his conjecture. The participants suggested that while the animated students worked on forming conjectures, Beta formed a similar conjecture to the one that Alpha presented at the board. In this case, they see that Beta had some responsibility to fill in the missing pieces of his argument. Seth from Sharleen's fifth period said, "[Beta] thought she was doing it the same way [Alpha] did it." According to Seth, when Beta whispered, "but it's the same thing, isn't it?" she was filling in a step of the work [stating that the diagonals and angle bisectors are the same] that she did to reach the same conclusion as Alpha [that the angle bisectors of a square meet at a point]. In this interpretation of Beta's action, the participants focused on the fact that Beta is making a contribution to Alpha's incomplete solution, and ignored the fact that she whispers instead of speaking at normal volume. Their comments reflect the opinion that she was complying with the norm because she shared an idea that completed the incomplete argument that Alpha was sharing at the board.

A student whispers a contribution to a solution (Inappropriate Breach)

Other participants' comments reflected the opinion that Beta's action of whispering a contribution was an inappropriate breach of the norm *students should complete incomplete arguments given in class*. Despite the fact that Beta did attempt to contribute to Alpha's incomplete solution, these participants interpreted her contribution as being negated by the fact that she whispered. Participants saw that it was inappropriate for Beta to breach this norm because if she had spoken louder, therefore complying with the norm, then her fellow classmates could have used her comment as a resource. Marlo from Sharleen's third period class said, "[Beta] said they were the same but she didn't say it loud enough." Marlo's comment supports the claim that Beta's action would be in compliance with the norm if she had spoken louder, but she spoke too quietly to be in compliance with the norm.

Participants also gave reasons why Beta should not have breached the norm by speaking so quietly. The participants saw that through sharing her idea, Beta had the opportunity to aid the whole class in building a solution to the problem that the teacher posed. Spencer from Madison's first period class said with respect to Beta's whispered contribution, "Well, I think sometimes you should take a chance and say it because you might be right." Spencer's comment reflects the idea that sharing a potentially correct answer is appropriate because it could be true and therefore helpful to the class. Pamela from Megan's fourth period class said, "I think that maybe if Beta spoke up then Alpha might, like, I don't know, get, like, a spark in his mind, like, maybe he had a little idea but maybe if she said that then he can get another idea and he can keep going on and the teacher would see that, 'Yeah, that actually does lead to the question she asked before.'" Pamela's comment can be seen as an elaboration of Spencer's comment. Pamela explicated the process through which Beta's contribution could be helpful to Alpha and the class. She saw that Beta's contribution could spark an idea in Alpha's mind, allowing him to form a solution that would be acceptable to the teacher.

Depending on how the participants interpreted Beta's action, either as primarily contributing to the classroom discussion or primarily preventing herself from contributing by whispering, the participants saw that she either complied with or breached the norm that says *students should complete incomplete arguments given in class*. The difference between the two interpretations is that in the first interpretation, the participants saw that Beta was acting appropriately by filling in gaps in Alpha's argument, and in the second activation the participants saw that Beta was acting inappropriately by not speaking loudly enough for her classmates to hear her.

Students' interventions should address the topic that the teacher was proposing

The norm *students' interventions should address the topic that the teacher was proposing* says that when students are responding to problems posed by the teacher, their responses should match that problem. This means that their responses should use the appropriate mathematical concepts, should connect their answer to the question, and students should not change the problem from how it is posed by the teacher. This norm was activated by two actions that are listed in Table 44. Below is evidence from the data.

'**Table 44: Actions related to the norm, *students' interventions should address the topic that the teacher was proposing***

Action	**Perceived relation to norm**
A student presents an ambiguous solution	Appropriate Breach
A student provides a counterexample to another student's idea	Appropriate Compliance

The actions related to this norm are a student presenting an ambiguous solution, and a student providing a counterexample to another student's idea. These actions are preformed at different moments in The Square. In the beginning of The Square Alpha can be seen presenting an ambiguous solution, and later Gamma can be seen providing a counterexample to Alpha's idea.

A student presents an ambiguous solution (Appropriate Breach)

The participants' comments reflected the opinion that the first of these actions, a student presents an ambiguous solution on the board, was an appropriate breach of the norm *students' interventions should address the topic that the teacher was proposing*. The participants justified this perceived breach based on the perception that Alpha had some cognitive difficulty while he was solving the problem, so unwittingly, his response changed the topic.

The participants saw that Alpha's action in The Square was a breach of the norm because it was not clear how his conjecture connected to the problem that the teacher posed. However, the participants suggested reasons why it was appropriate for Alpha to provide the response that he did. Each of these reasons points to a cognitive difficultly that participants saw that Alpha could have had while working on the problem. One cognitive difficulty suggested by participants was that Alpha could have been thinking about perpendicular bisectors instead of angle bisectors. Art from Madison's third period class said that Alpha was "thinking of, like, perpendicular bisectors [instead of] angle bisectors." If this were the case, then the participants claimed that Alpha could have been making the conjecture that the diagonals of a square are perpendicular bisectors of each other.

Another cognitive difficulty suggested by participants was that Alpha did not have a fully formed conjecture, but simply had the idea to look at the case of the square. Denise from Madison's third period class said Alpha "doesn't know the answer, he just went up there and did a square." Gordon from Megan's fourth period class said, "Sometimes it's hard to put your thoughts into words especially if you are, um, shaky on the topic, so I think that's kinda what happened to Alpha." Gordon explains Alpha's breach of the norm by hypothesizing that the material was taxing for Alpha and therefore it was difficult for him to communicate his idea.

These comments from participants support the norm that *students responses should address the problem* and provide an example of a cognitive difficulty that participants attribute to Alpha to explain why he was unable to comply with the norm. The participants see Alpha attempting to answer the animated teacher's questions, but he is lacking the mathematical understanding needed to provide an acceptable conjecture.

A student provides a counterexample to another student's idea (Appropriate Compliance)

The participants' comments reflected the opinion that the second of these actions, a student provides a counterexample to another student's idea, is an appropriate compliance with the norm *students' interventions should address the topic that the teacher was proposing*. Participants saw that Gamma's contribution of thinking about the diagonals and angle bisectors of a rectangle was a counterexample to Alpha's conjecture and had an important mathematical component in terms of answering the problem posed by the teacher. That is, Gamma noticed that Alpha's conjecture was only true for squares and that Alpha's conflation of angle bisectors and diagonals is only non-problematic in certain quadrilaterals. The participants recognized Gamma's point, that in rectangles the angle bisectors are not the diagonals, as a mathematically valid point to make because the problem posed by the teacher is about quadrilaterals in general, not only squares. Bob from Madison's third period class said, "[Gamma] was trying to make it more general because [Alpha] was just talking about a square." Bob's comment reflects the opinion that it is appropriate for Gamma to provide a more general response than Alpha, which was too specific to answer the question posed by the teacher.

Also in light of Gamma's counterexample and this norm, some participants said that special cases, like Alpha's square, are not useful or are irrelevant when responding to general questions, like the angle bisectors problem. Muna from Megan's third period class said, "[Gamma] was just trying to point out that, um, what Alpha was saying was sort of irrelevant because they were talking about all quadrilaterals not just squares. So, I guess, answer the actual question." This comment from Muna highlighted the view that Alpha's conjecture did not address the teacher's problem, so it was appropriate for Gamma to provide a counterexample that was more clearly connected to the teacher's problem.

Other participants agreed that Gamma was making a distinction between squares and other quadrilaterals, but unlike Muna, these participants do not see the hierarchical relationship between squares, rectangles, and quadrilaterals so they did not see that Gamma's counterexample was a move towards answering the teacher's problem. Mary from Madison's third period class said, "Like, I don't know if [Gamma] realized it, but she was kind of saying that the rules for the squares are different than the rules for the rest of the quadrilaterals." Mary's comments showed a recognition of Gamma's counterexample, but did not place a mathematical value on Gamma's move from squares to rectangles.

The norm *students' interventions should address the topic that the teacher was proposing* was activated twice. The participants saw that it was appropriately complied with once and appropriately breached once. In the first case the participants saw that this norm was appropriately complied with since a student provided a counterexample to another student's idea. The participants saw that Alpha provided an answer to the teachers' question that only addressed a special case, so Gamma appropriately expanded on Alpha's response to make it a better answer to the teacher's question. On the second case the participants saw that the norm was appropriately breached because a student did not have the cognitive capabilities to respond to the problem that the teacher posed.

Students should be disposed to share ideas in public when so asked

The norm *students should be disposed to share ideas in public when so asked* says that when students are sharing solutions to problems in class they should present these solutions from the board. Student ideas, or student reactions to other students' comment may be shared from the student's seat, but when a student is presenting a solution, which usually consists of several connected ideas and could last for an extended period of time, that student should be at the board. This norm was activated by two actions that are listed in Table 45. Below is evidence from the data.

Table 45: Actions related to the norm, *students should be disposed to share ideas in public when so asked*

Action	Perceived relation to norm

A student is hesitant to present his solution at the board	Inappropriate Compliance
A students shares his solution from his seat	Inappropriate Breach

The actions related to this norm are a student is hesitant to present his solution, and a student shares his solution from his seat. These actions are performed at different moments in The Square. In the beginning of The Square Alpha can be seen as hesitant to present his solution, and toward the end of The Square Lambda can be seen sharing his solution from his seat. The participants' comments reflect the opinion that Alpha's action was in compliance with the norm, however they saw this compliance as inappropriate in light of students' fear of presenting incorrect solutions at the board. The participants' comments also reflect the opinion that Lambda's action was a breach of the norm, and they saw this breach as inappropriate; he should have come to the board to share his solution.

A student is hesitant to present his solution at the board (Inappropriate Compliance)

After seeing the teacher call Alpha to the board to share his conjecture, the participants anticipated that Alpha would resist the animated teacher's call to the board. This resistance shows the tension between the norm that states that *students should be disposed to share ideas in public when so asked*, and the fear that student have of presenting incorrect solutions at the board. Since Alpha is unsure of his idea, he is resistant of the teacher's request that he share it will the class. Kenneth from Sharleen's third period class said, "I think he felt unsure because he had the idea but he didn't know if it was right or wrong" and Robin from Jack's fourth period class said that if she were in

Alpha's place she would feel "stupid" because, "I always get it wrong." Both of these students are sharing the concern that students should not share an incorrect solution.

In the scenario described by the participants, Alpha complies with the norm *students should present their ideas from the board* but his compliance with this norm is encumbered by his inclination to stay in his seat and not share a solution that may be incorrect. Alpha's lack of confidence in the correctness of his solution explains his hesitance to come to the board.

A student shares his solution from his seat (Inappropriate Breach)

The participants saw Lambda breaching the norm *students should present their ideas from the board*. Participants gave reasons why this breach was inappropriate, therefore providing evidence for the norm. Resha from Madison's first period class said, "I think [the teacher] should have let [Lambda] come to the board and show exactly what he meant." Here Resha was pointing to the perceived breach of the norm and saying that Lambda should have complied with the norm. According to the participants, this move would help both the teacher and the rest of the class better understand Lambda's idea. In particular, this would give Lambda control over the diagram on the board so that he would then be able to erase one of the diagonals. Resha goes on to say that if Lambda had gone to the board "he would have redrawn the square so that people could see it visually." Resha provides a reason that Lambda should comply with the norm and present his solution from the board; so that he could use the diagram to better communicate his solution with the class.

The norm *students should be disposed to share ideas in public when so asked* was activated by two moments in The Square. The participants saw that it was inappropriately complied with when a hesitant student came to the board. Since the student was unsure of his solution he should have stayed in his seat and not risked sharing an incorrect solution from the board. The participants also saw that this norm was inappropriately breached when a student shared his solution from his seat. The participants said that this student should have complied with the norm by presenting his solution from the board.

Students in the audience should support the presenter in public

The norm *students in the audience should support the presenter in public* says that when students watch other students present solutions at the board, students should act in ways that support the presenter in public. This could mean helping clarify their ideas, not disagreeing with them or staying quiet. This norm was activated by three actions that are listed in Table 46. Below is evidence from the data.

Table 46: Actions related to the norm, *students in the audience should support the presenter in public*

Action	Perceived relation to norm
A student whispers while another student is presenting a solution at the board	Appropriate Compliance
A student clarifies another student's idea	Appropriate Compliance
A student clarifies another student's idea	Inappropriate Breach

The actions related to this norm are a student whispers while another student is presenting a solution at the board and a student clarifies another student's idea. The participants see the first action as an appropriate compliance with the norm, and they see the second action as both an appropriate compliance and an inappropriate breach. These actions can both be seen in the same moment in The Square. While Alpha was presenting his conjecture at the board, the teacher told Alpha that the problem was about angle bisectors, not diagonals. From her seat Beta whispered, "but it's the same thing, isn't it." Some participants labeled Beta's action as 'whispering while another student is presenting a solution at the board' and other participants labeled Beta's action as 'clarifying another student's idea.'

A student whispers while another student is presenting a solution at the board (Appropriate Compliance)

As a member of the class, listening to Alpha present his conjecture at the board, the participants saw that Beta is entitled to some reaction to Alpha's conjecture; in particular, it is appropriate for Beta to whisper from her seat. Some participants interpreted Beta's whisper as an agreement with Alpha's conjecture and confusion at the teacher's rejection. Rebecca from Jack's second period class said that she would be likely to say something very similar to Beta's comment. She said, "When someone does something on the board that I think is correct but then someone says it's not, I tend to say, 'well, that doesn't sound wrong, it's the same thing, isn't it?'" This comment from Rebecca shows the position that while a student is presenting a solution at the board it is appropriate for other students to whisper reactions to this solution.

A student clarifies another student's idea (Appropriate Compliance)

Participants saw that Beta's whisper was a means of supporting Alpha by clarifying his idea for the teacher and acting as a bridge between Alpha and the teacher. Corinne from Madison's third period class said, "[Beta], like, tried to help [Alpha] out in, like he didn't, she didn't really try to make him feel bad or make him feel like he got the wrong answer but she tried to help him by telling what he probably meant to the teacher." Corinne's comment suggested that it was appropriate for Beta to whisper a comment while Alpha was at the board because her comment was meant to help the teacher understand what Alpha was saying. Paul from Megan's third period class made a similar comment. He said, "Pretty much all of us students pretty much can tell what each other is saying except in some instances when it's really, really unclear but I think most of the time it's hardest communicating that to the teacher." Paul pointed out the difficulty that students have communicating with the teacher, and used this as a rationale to support Beta's action. Paul seems to be saying that it was appropriate for Beta to whisper because she was helping Alpha overcome the difficulty of communicating his idea to the teacher.

Some participants saw that the teacher was wrong in thinking that Alpha had made an incorrect conjecture and, so, it was appropriate for Beta to help the teacher see the correctness of Alpha's conjecture. Julia, another student from Megan's third period class, said that Beta acted appropriately when she spoke on Alpha's behalf, "just to let the teacher know that that's maybe what he meant, maybe [the teacher] understood it wrong and just give [Alpha] a little support." Unlike the comments above, from Corinne and Paul, which focused on helping Alpha communicate his conjecture, this comment from Julia focused on helping the teacher understand Alpha's comment. The participants saw that Beta's whisper had the potential to show the teacher how Alpha's conjecture was related to the original problem by pointing out that the angle bisectors and diagonals are the same in a square.

Robert, also from Megan's third period class, made the interesting observation that students are not allowed to tell the teacher that she misunderstood Alpha, however students are allowed to ask questions that can surface the teacher's misunderstanding. Robert said, "You don't always get to say to the teacher like, 'You're wrong, you didn't notice this' you could just ask it, like, in a question, like, 'Well, aren't the ang- the diagonals the same as the angle bisectors?' like, implying and then she'd be, like, 'Oh, yeah." Robert's comments, suggest that Beta's whispered question was not genuine confusion, but an attempt to prompt the teacher to see her mistake in rejecting Alpha's conjecture.

These comments support the norm s*tudents in the audience should support the presenter in public* by showing how Beta's actions could be interpreted as helping Alpha explain his idea or helping the teacher understand Alpha's conjecture. Both of these interpretations of Beta's action are seen as compliance with the norm.

A student clarifies another student's idea (Inappropriate Breach)

Unlike the participants quoted above, who interpreted Beta's whisper as a means of supporting Alpha, other participants interpreted Beta's action as talking out of turn and not supporting Alpha while he was presenting his conjecture at the board. The teacher and Alpha were engaged in a conversation and participants saw that the role of the other students was to listen quietly. According to these participants, it was inappropriate for Beta to interrupt this conversation because she was not supporting Alpha in his position of 'presenter in public.' Neil from Megan's fourth period class said, "[Beta] shouldn't have said anything because it wasn't even her idea. And you don't' even know if [the teacher] heard her or not or was even paying attention so you can't really say anything. And it doesn't depend on the situation anyway because it's not [Beta's] responsibility at all." Neil's comment provided evidence that Beta's action was inappropriate. According to Neil, Beta should not have participated in the discussion between the teacher and Alpha for several reasons; the idea being discussed was not Beta's idea, the teacher may not have been paying attention to Beta, and Beta did not have any responsibility to support Alpha. Taynia from Sharleen's third period class made a comment that also gives evidence that Beta's comment was inappropriate. She said, "It's just a comment that you really keep to yourself." Taynia seemed to be saying that Beta's comment is not appropriate for the whole class to hear. These comments from Neil and Taynia point to the norm *the students in the audience should support the presenter in public* because they are saying that by virtue of the fact that Alpha is standing at the board he has the right to speak at this time and Beta does not. The participants saw that not only was Beta not required to participate in the discussion while Alpha is at the board, she should not have been allowed to speak while Alpha is at the board. By speaking while Alpha was at the board the participants saw that she breached the norm.

The norm *students in the audience should support the presenter in public* was activated during one moment that was interpreted three different ways. Depending on how the participants interpreted Beta's whisper while Alpha was sharing his conjecture at the board, they either saw that she was appropriately complying with this norm or inappropriately breaching it. Some participants saw that Beta was helping Alpha communicate his idea to the teacher, therefore supporting Alpha and complying with the norm; while other participants saw that Beta was interrupting a discussion between the teacher and Alpha, therefore not supporting Alpha and breaching the norm.

Students should be amenable to assessment

The norm *students should be amenable to assessment* says that when students have an idea that could be interpreted in multiple ways they should share it with the class. These ambiguous ideas could be seen initially as correct or incorrect, but their potential value to the discussion overrides their potential incorrectness. This norm was activated by one action that is listed in Table 47. Below is evidence from the data.

Table 47: Actions related to the norm, *students should be amenable to assessment*

Action	**Perceived relation to norm**
A student shares an ambiguous idea with the class	Appropriate Compliance

A student shares an ambiguous idea with the class (Appropriate Compliance)

In the beginning of The Square, Alpha made the conjecture that the diagonals of a square bisect each other. The participants saw that Alpha's conjecture was ambiguous because it could be interpreted as either right or wrong. That is, Alpha's conjecture was a true statement, but on the surface it did not look like an appropriate answer to the angle bisectors problem that Alpha was supposedly responding to. The participants saw that ambiguous ideas like Alpha's are appropriate to share in class. Noa from Megan's third period class said, "Like [Alpha] interpreted it in a different way so that what was, like, technically wrong was actually right in a different way." Noa's comment provided evidence for the norm since she seemed to be saying that it was appropriate for Alpha to share his conjecture since there is the possibility that it could be interpreted as being correct. Selena, also from Megan's third period class, agreed that an ambiguous conjecture could be a source of redemption for the student stating it. She said, "Well, then you're kinda thinking 'Oh, I did something wrong' and um, but at the end he was actually right so it didn't really matter." Selena's comment showed how a student who was initially thought to have presented an incorrect conjecture, could, in the end, be seen to have presented a correct conjecture. Marilyn from Madison's first period class had a similar interpretation of Alpha's conjecture when she said, "But then part of his idea was kind of right in the end, though, like they ended up being the same thing so he was kind of right in a way." These comments from Noa, Selena, and Marilyn, all provided evidence for the norm that *students should be amenable to assessment* based on the rationale that initially appearing incorrect is worth the benefit of later appearing to be correct.

Situational norms for 'making conjectures'

Below is evidence for the hypothesized situational norms for the instructional situation 'making conjectures.' The results are organized by the hypothesized norms and synthesized in tables within each section.

Students should communicate their conjectures to the class

The norm *students should communicate their conjectures to the class* says that after students work independently to make conjectures they should then share their conjectures with the class. The teacher is responsible for conducting this work, but the students are responsible for being willing and able to share their conjectures. This norm was activated by four actions that are listed in Table 48. Below is evidence from the data.

Table 48: Actions related to the norm, *students should communicate their conjectures to the class*

Action	Perceived relation to norm
A student is hesitant to share his idea at the board	Inappropriate Compliance
A student whispers while another student presents a solution at the board	Appropriate Breach
A student clarifies another student's idea	Appropriate Compliance

The actions related to this norm are; a student is hesitant to share his idea at the board, a student clarifies another student's idea, and a student whispers while another student presents a solution at the board. The participants saw that Alpha was hesitant to share his idea at the board, and this led them to interpret his action as an inappropriate compliance with the norm in light of his desire to keep his idea private. The participants' comments also reflected the opinion that the action, a student whispers while another student is at the board, which was enacted by Beta, is related to this norm. Beta's action was seen as an appropriate breach of the norm because although it would be appropriate to share her idea, since she is not sure that it is correct, it is better to keep it to herself. The participants saw the action, a student clarifies another student's idea, when Gamma made the point that in general, diagonals are not angle bisectors. This action was seen as an appropriate compliance with the norm.

A student is hesitant to share his idea at the board (Inappropriate Compliance)

Although Alpha came to the board to share his idea when the teacher asked him to, the participants reported that Alpha might prefer to keep his idea private and not share it with the class. According to the participants, it could be the case that Alpha came up with good ideas while working on forming conjectures but he would prefer to not share these ideas with the class. Brianna from Jack's second period class said, "I think that maybe he's that guy that, like, thinks things but like doesn't have—doesn't want to actually say them so someone else can hear them." Similarly, Tony from Megan's third period class said, if he were in Alpha's position he'd be thinking, "I just wanted to answer the question and now I have to go and, like, do a demonstration on the board." Both of these students were conveying the opinion that regardless of the fact that it would be appropriate for Alpha to share his idea at the board, Alpha would like to keep his idea out of the public arena.

Even though participants saw that Alpha complied with the norm *students should communicate their conjectures to the class* the participants saw reasons why he should breach it. The participants saw that it was reasonable for Alpha to respect his desire to not share his idea. The participants felt that it was their prerogative to keep their ideas to themselves even when the norms of the situation dictated that all conjectures should be made public.

A student whispers while another student presents a solution at the board (Appropriate Breach)

The participants' comments provided evidence for the norm, *students should communicate their conjectures to the class*, and an opposing reason for Beta to not communicate her idea, with respect to Beta's whisper while Alpha was at the board sharing his conjecture. If the participants had perceived Beta sharing her idea at normal volume, then this would have been seen as a compliance with the norm, but since she instead whispered her idea, it was perceived as a breach of the norm. On one hand, since Beta has an idea, it would be appropriate for her to share it with the class. On the other hand, the participants interpreted Beta's actions as showing that she was unsure about the truth of her idea, so she should not share her idea in case it is not true. Martha from Madison's first period class said, "If it was a new unit and I really wouldn't know for sure so I probably wouldn't say it unless I knew for sure." Martha's comment reflects the opinion that Beta should not have shared her idea because there was a chance that it was incorrect. The participants' comments reflect an appropriate breach because they said that a student would be uncomfortable expressing an idea that was possibly false and this resulted in Beta whispering instead of speaking at regular volume.

A student clarifies another student's idea (Appropriate Compliance)

Some participants interpreted Gamma's actions as an appropriate compliance with the norm since the participants perceived that he was trying to help the teacher and the other students understand Alpha's conjecture. By helping the teacher and the class understand Alpha's conjecture Beta was seen as helping Alpha communicate his conjecture to the class, in accordance with the norm that Alpha should communicate his conjecture. Samuel from Sharleen's third period class said that Gamma wanted to come to the board because "she thought that she could show everybody how to do it." That is, Samuel suggested that Gamma was trying to show the class how to solve the problem, expanding on Alpha initial attempt.

The norm *students should communicate their conjectures to the class* could be seen to be activated three times and twice the participants implied that it would be appropriate for the animated student to breach is norm. In the case of the action, a student is hesitant to share his idea at the board, the participants saw that it would be reasonable for Alpha to heed his preference t keep his idea private. In the case of the action, a student whispers while another student presents a solution at the board, the participants saw that it would be reasonable for Beta to not share her idea because she was uncertain that it was correct. The participants also saw in one case that if a student was unable to share his conjecture then it is appropriate for another student to explain the conjecture on his behalf.

Students can respond to other students' conjectures

The norm *students can respond to other students' conjectures* says that when a student shares his conjecture other students may publicly agree or disagree with that conjecture. This norm was activated by one action that is listed in Table 49. Below is evidence from the data.

Table 49: Actions related to the norm, *students can respond to other students' conjectures*

Action	Perceived relation to norm
A student provides a counterexample to another student's idea	Inappropriate Compliance

A student provides a counterexample to another student's idea

In the eyes of some participants, Gamma made a transgression when she shared her idea about the angle bisectors and diagonals of a rectangle. Gamma's action was seen as a transgression despite the fact that it was in line with the norm. Participants interpreted

Gamma's tone and the timing of her comment as meaning that she disagreed with Alpha and was refuting his conjecture. These participants interpreted Gamma's comments as meaning that she believed Alpha had said something that was not true and that Gamma had a desire to point out that he is wrong. Eric from Madison's first period class said, "[Gamma] thinks she's right and everybody else is wrong, like ego-ish." Eric seems to be interpreting Gamma as putting down the rest of the class and announcing that she has the correct answer. The participants report that it is inappropriate for Gamma to make declarative statements instead of asking questions, as would be appropriate for a student. Caroline, also from Madison's first period class said, "[Gamma] kinda bothered me because she seemed like she knew everything and, I don't know, I tend to think when people make a statement they ask it—they turn it into more of a question." According to Caroline's comment, it would have been more appropriate for Gamma to soften her blow by putting it in the form of a question or somehow downplaying her confidence to make the attack on Alpha less aggressive. The participants attributed Gamma's intervention more to Gamma's "ego" and her desire to put Alpha down, and less to Gamma's concern for sharing important mathematical ideas.

The participants' comments give evidence that Gamma is acting in compliance with this norm *students can respond to other students' conjectures* however, their comments point to the fact that Gamma's actions are inappropriate. I take this to mean that either the hypothesized norm is incorrect or there is another influence on student actions that compels students to act in ways that are perceived as a breach of this norm. Both of the quotes above refer to the social interactions of students in class and the importance for students to not act as if they know more, or are smarter, than their peers. Therefore this concern is related to the interpersonal relationships among students, and the fact that students expect other students to not put each other down, or act as if they are smarter than other students.

Students should stop talking about a conjecture once it has been agreed upon or refuted

The norm *students should stop talking about a conjecture once it has bee agreed upon or refuted* says that once the class has established that a conjecture is true or false the students should stop discussing that conjecture. The class could establish the truth of a conjecture by discussing it as a group, or the teacher could announce that a conjecture is true or false. This norm was activated by one action that is listed in Table 50. Below is evidence from the data.

Table 50: Actions related to the norm, *students should stop talking about a conjecture once it has been agreed upon or refuted*

Action	**Perceived relation to norm**
A student comments on a conjecture after it has been agreed up on or refuted	Inappropriate Breach

Participants saw the norm *students should stop talking about a conjecture once it has bee agreed upon or refuted* as relevant to the action of a student commenting on a conjecture after it has been agreed up on or refuted. In the beginning of The Square, some participants interpreted Beta's actions as making a comment about Alpha's conjecture after the teacher has announced that it was incorrect. The participants view this as an inappropriate breach of the norm.

A student comments on a conjecture after it has been agreed up on or refuted

Some participants said that that Beta should not have made any comments on Alpha's conjecture because the animated teacher had already said that Alpha's conjecture was incorrect. This opinion is in agreement with the norm. Neil from Megan's fourth period class said, "If somebody is up at there [at the board] and they're doing a problem and the teacher just says they're wrong, like the teacher did, even though you would, even though Alpha would have liked to have help I don't think anybody should've because the teacher doesn't usually change her opinion unless she sees something there and it doesn't seem like the teacher saw anything so if somebody jumped in they would have felt the same way as Alpha." Neil's comment was strongly in support of the norm, because it reflects the belief that, regardless of what Beta says, the teacher has decided that Alpha's conjecture is incorrect, so there no reason for Beta to give her opinion.

The norm *students should stop talking about a conjecture once it has been agreed upon or refuted* was activated once, with respect to Beta's action of commenting on Alpha's conjecture. This action was viewed as an inappropriate breach the participants perceived that the teacher had established the conjecture as false.

Situational norms for 'doing proofs'

Below is evidence for the hypothesized situational norms for the instructional situation 'doing proofs.' The results are organized by the hypothesized norms and synthesized in tables within each section.

Students should not make changes to the proof problem

The norm, *students should not make changes to the proof problem* says that student should work on the proof problem as it is posed by the teacher, and not modify the proof statement or the diagram accompanying the proof problem. This norm was activated by one action that is listed in Table 51. Below is evidence from the data

Table 51: Actions related to the norm, *students should not make changes to the proof problem*

Action	Perceived relation to norm
A student requests to modify a problem	Appropriate Breach

The norm, *students should not make changes to the proof problem,* was activated by the action, a student requesting to modify a problem. This action could be seen in the animated scenario when Lambda asked the teacher to remove one diagonal from the diagram of the square that this on the board.

A student requests to modify a problem (Appropriate Breach)

The participants' comments reflected the opinion that the action, a student requests to modify a problem, is an appropriate breach of the norm. The participants saw this breach of the norm as appropriate because, even though Lambda's intervention did not address the topic that the teacher proposed, it did address a true statement. The participants saw that by requesting that the teacher removed one diagonal Lambda was creating a new problem that was not the one posed by the teacher. Participants saw Lambda as reasonable in his desire to make these changes and that the teacher was unreasonable for not allowing these changes. The participants supported this breach of the norm by saying that Lambda had a valid mathematical method for solving the problem. Paul from Megan's third period class supported Lambda's actions and would go further to get his ideas across. He said, "I get pretty annoyed, I'd just say 'draw that there's one and let me prove that there's these two triangles are congruent.'" Paul's comment reflects the opinion that it is appropriate for Lambda to request a change to the problem since he would be able to prove a true statement. According to the participants, the teacher should either understand why this is an appropriate move or give Lambda the chance to show his proof anyway.

By asking the teacher to erase one diagonal from the diagram of the square Lambda was changing the problem that had been given by the teacher and therefore could be seen to be breaching the norm *students' interventions should address the topic that the teacher was proposing*. However, since Lambda believes his proof to be the best way to prove Alpha's conjecture the participants saw that it is appropriate for him to ask the teacher to modify the diagram to produce the proof.

Summary of results

To summarize the results, Table 52 lists all the norms for which the focus groups contained evidence along with the corresponding student actions that activated those norms. The first column of this table lists the scope of the instructional norm, either the didactical contract of the geometry classroom, the situation of 'making conjectures' or the situation of 'doing proofs.' The second column lists the norm. The third column lists the student actions related to each norm. The final column describes the relationship between the norm and the action in terms of appropriateness and compliance. In the case that an action suggests an departure from the norm (actions that are perceived as an inappropriate compliance or an appropriate breach), the justification for that departure from the norm is also listed in this column. This list provides a starting point for describing the work of studenting. The norms, actions, and relationships that participants perceived between these norms and actions, provide the basis for a framework describing students' actions in geometry instruction, both with respect to the general didactical contract and with respect to the particular instructional situations, 'making conjectures' and 'doing proofs.'

Table 52: Summary of norms and actions

	Norm	Action	Perceived relation to norm
Contract	*Students should share their ideas that are different than other students' ideas*	A student comes to the board to share her idea	Appropriate Compliance
	Students should complete incomplete arguments given in class	A student whispers a contribution to a solution	Appropriate Compliance
		A student whispers a contribution to a solution	Inappropriate Breach
	Students' interventions should address the topic that the teacher was proposing	A student presents an ambiguous solution	Appropriate Breach: Cognitive Difficulties
		A student provides a counterexample to	Appropriate

	another student's idea	Compliance
Students should be disposed to share ideas in public when so asked	A student is hesitant to present his solution at the board	Inappropriate Compliance: Worry of Incorrectness
	A students shares his solution from his seat	Inappropriate Breach
Students in the audience should support the presenter in public	A students whispers while another student is presenting a solution at the board	Appropriate Compliance
	A student clarifies another student's idea	Appropriate Compliance
	A student clarifies another student's idea	Inappropriate Breach
Students should be amenable to assessment	A student shares an ambiguous idea with the class	Appropriate Compliance

'Making Conjectures'	*Students should communicate their conjectures to the class*	A student is hesitant to share his idea at the board	Inappropriate Compliance: Desire to keep idea private
		A student whispers while another student presents a solution at the board	Appropriate Breach: Worry of Incorrectness
		A student clarifies another student's conjecture	Appropriate Compliance
	Students can respond to other students' conjectures	A student provides a counterexample to another student's idea	Inappropriate Compliance: Politeness
	Students should stop talking about a conjecture once it has been agreed upon or refuted	A student comments on a conjecture after it has been agreed up on or refuted	Inappropriate Breach

'Doing Proofs	*Students should not make changes to the proof problem*	A student requests to modify a problem	Appropriate Breach: Share true ideas

One sees that the relationship between each action and its associated norm is coded as an appropriate compliance, inappropriate compliance, appropriate breach, or inappropriate breach in relation to the norm. That is, participants' responses could either confirm the hypothesis that the norm guides student action, or the participants' responses could point to the fact that the norm should not be applied in a particular moment, and an action other than the one that is recommended by the norm is appropriate. The actions that participants perceived to be inappropriate compliance with the norm or appropriate breaches of the norm do not disconfirm the norm, rather this evidence provides support for the claim that, in addition to the norms, there are other influences on student action.

I use these norms as a basis for building a theoretical model of the influences on student action. I take the view that the primary explanations for student actions are social, that is behavioral norms and commitments that students have as members of the school. I also take into account the individual characteristics and traits of students that influence their actins in geometry instruction. In the discussion I describe this model, using these results as motivation, especially the need to explain the relationship between norms and actions that participants perceived as either inappropriate compliance with the norm or appropriate breach of the norm.

In the discussion section I discuss the implications of these particular norms, actions, and perceived relationship between the norms and actions. I begin by discussing what I learn from the confirmation of these norms. I then look at the cases in which students provide evidence that reflects justifications for action that are departures from the norm. I return to the literature to attempt to explain these justifications. I end with a tentative model for justifications for student action, motivated by this empirical study and the examination of the evidence that supports departures from the norms, and based on a model for the rationality of teacher action developed by Herbst (2010a, 2010b)

Discussion

In the following section I discuss the results of this empirical study and use these results as motivation for describing a tentative framework for the rationality of the work of studenting. To begin this discussion I look across the evidence for the hypothesized norms to make generalizations about the instructional norms that guide students' actions. I then look at the evidence that participants provided that point to moments that are departures from the norm and are justified by alternative means. I return to the literature to find support for these alternative justifications. Then, motivated by this empirical study and the findings from the literature I build a tentative model for the rationality of studenting, based on a parallel model for the rationality of teaching, developed by Herbst (2010a, 2010b).

Discussion of hypothesized norms

From these results one sees that the participants, while responding to the animated scenario, did provide evidence to support the hypothesized norms. This list of hypothesized norms is not a complete list of the norms for the model of interactions developed for geometry instruction (Herbst & Brach, 2006; Herbst et al., 2009), but it does reflect the primary norms that can be seen to be active in The Square. Unlike previous research on instructional norms in geometry classrooms, this study focuses both on perceptions of norms during instruction (unlike Herbst & Brach, 2006 which focused on students' perceptions of norms while working on proof tasks in isolation) and on instructional norms from the students' point of view (unlike Herbst et al. 2009, which primarily looks at instructional norms from the perspective of the teacher).

Conflating students and their ideas

Two concepts that get conflated in the participants' talk and in the resulting analysis are the concepts of an idea that is correct and the idea of a student that is correct. From these conversations one can see that students strongly associate persons with ideas. That is, if a student voices an idea that is wrong, then that student is wrong, conversely, if a student voices an idea that is correct, then that student is correct. This conflation highlights the need for students to become risk-takers (Lampert, 2001). Since voicing a wrong answer carries the possibility of making a student a 'wrong person' then it is understandable that students would be hesitant to take intellectual risks.

The only norm included in this study that gives a student the space to disagree with another student's idea, and by extension, with another student, *students can respond to other students' conjectures*, is supported by students only as long as it is seen as being breached. Because refuting a conjecture is essentially the same as telling another student that they are wrong, politeness seems to overrule the norm. Because students are unwilling to disagree with other students they are unable to disagree with ideas.

Another norm that is related to this conflation is the norm *students should communicate their idea to the class*. Participants report that this norm should also be breached. This norm creates an opportunity for students to publicly invite the teacher or other students to disagree with their conjecture and therefore disagree with them. To avoid this danger students report that it is appropriate to breach this norm if the student can not be sure that their idea is not incorrect.

Participants' comments regarding these norms point to the fact that correct ideas and students who voice those ideas have a privileged place in the geometry classroom. In their view, students should try to associate themselves with correct ideas so that it is not just that the idea is right, but the idea becomes their idea and so they are right.

Alternative justifications

In providing reasons why it would be appropriate to breach a norm the participants provided five alternative justifications. These alternative justifications are departures from the rules determined by the norms. They are; cognitive difficulties, worry of incorrectness (invoked twice), desire to keep an idea private, politeness, and desire to share true ideas. Below I return to the literature to see how previous research can help explain these alternative justifications.

The first of these alternative justifications, cognitive difficulties, could be understood at one level to the expression of inadequate personal resources. That is, each student comes into the classroom, and whether or not an individual has the ability to solve a problem is a personal matter. However, I would like to suggest that this could also be an expression of part of the student's relationship to school work. Since students are constantly being confronted with material that is new to them, it is expected that not all students will understand all the material all the time. So, part of being a student involves experiencing cognitive difficulties. Lampert (2001) expressed this same view when she stressed the importance of teaching students to be ‘people who study in school,’ and in particular to be people who take academic risks, which may lead to some cognitive difficulties.

The second of these alternative justifications, worry of incorrectness, could be understood as an expression of performance-avoid goals (Pintrich & Schunk, 1996), which is thought of as a personal resource of an individual student. Students who justify the breach of norms on account of this worry are concerned with how their abilities are perceived by others and they are interested in concealing any deficits in their ability. This same exception to the norm could be interpreted as a concern for the integrity of the mathematics being discussed instead of concern for the individual expressing the idea. A significant part of being a student is developing a commitment to truth, and to learning how to tell truth from falsehood. Using this interpretation, participants are not trying to hide their lack of ability, but they are concerned with promoting true ideas as opposed to false ones. This corresponds to a commitment that students have with respect to their relationship with the work done in classrooms.

The third of these alternative justifications, the desire to keep an idea private, could be understood as personal resource in the form of a personality trait. In general, students are expected to share their idea with their classmates, at the request of the teacher, but here, participants are saying that some students remove themselves from this sharing. This action of keeping an idea private does not seem to be perceived by the participants as part of the work of studenting, but a reason why a student would momentarily stop doing this work.

The fourth of these expectations, politeness, could be understood as an aspect of a student's relationship to the school. Since students are in school and in classrooms with so many other students, and since these students spend an extended amount of time together, it's reasonable for this closeness to compel students to attempt to be polite to each other. This is related to what Jackson refers to as "crowds" (Jackson, 1968). Part of being a student means getting along with your peers (and possibly the adults) in the school.

The fifth and final of these alternative justifications, sharing true ideas, is related to the commitment to truth that students are expected to develop. The participants see that it is reasonable to breach a norm in favor of sharing an idea that that is true. Much of students time and energy in classrooms is directed toward discovering true ideas and justifying those ideas, so when they encounter a true idea, even one that might not be directly connected to the current problem, the participants see this as a valid topic to bring up. Like, the worry of incorrectness, this can be seen to be related to the work that is done in classrooms.

One can see that the five alternative justifications for student actions to the instructional norms that can be seen in the participants' comments; cognitive difficulties, worry of incorrectness, desire to keep an idea private, politeness, and sharing true ideas, reflect personal resources of individuals, the relationship that students have with the school, and the work that students do inside classrooms. Below I will describe a framework that combines each of these and the norms to form a model for the rationality of studenting.

Framework for the rationality of studenting

The theory of practical rationality of teaching (Herbst, 2010a, 2010b) has been explored in detail, both conceptually, and empirically for instructional situations particular to high school algebra and geometry classrooms (Chazan & Lueke, 2009; Herbst & Brach, 2006; Herbst, Nachlieli, & Chazan, in press). In the following section I will briefly describe this conceptual framework of practical rationality of teachers, which is described fully in "Teachers' perceptions of geometry students" (Aaron, this volume). I will then discuss the reconceptualization of this framework to account for the rationality of students.

Practical rationality of teachers

The practical rationality of teaching is a framework for finding the justification for the actions that teachers take inside instruction. This framework begins with the assumption that classroom interaction is a type of symbolic economy (Bourdieu, 1990; 1998) in which actions on mathematical tasks are exchanged for claims on the didactical contract (Brousseau, 1997; Herbst, 2002). This symbolic economy is described in detail in "Teachers' perceptions of geometry students" (Aaron, this volume). From this view of teachers' work (see Figure 35), one sees that two major responsibilities of teachers in instruction are orchestrating students' work on mathematical tasks and managing the exchange of student work for claims on the didactical contract. Focusing on orchestrating students' work and managing the exchange is a simplified view of the work that teachers do, however it is sufficient to model the instructional situations examined in this study.

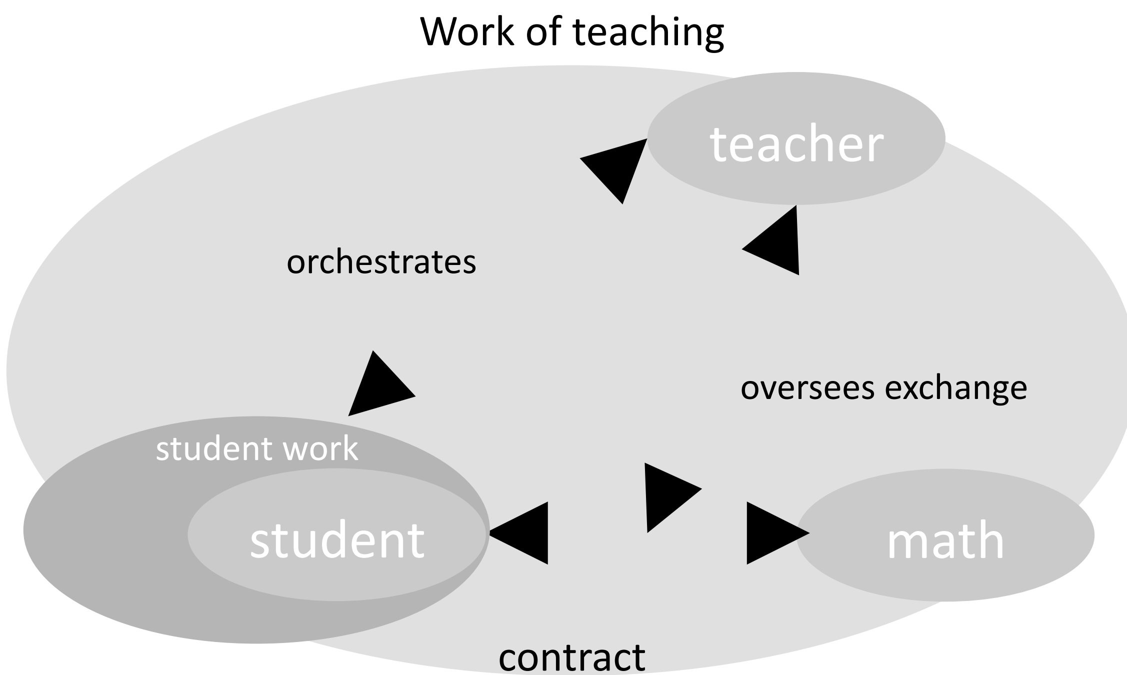

Figure 36: The work of teaching (adapted from Herbst 2010a)

This work that the teacher engages in gives rise to a rationale for decision-making, or a set of norms for action, that feeds teachers' decision-making. Another source of information for decision-making is teachers' professional obligations. These obligations are constraints on the work of teaching and come from four stakeholders (Herbst & Balacheff, 2009). These stakeholders are individual students, the class as a whole, the institution of school, and the discipline of mathematics. These obligations, historically, can be seen to contribute to the development of the norms of instruction, and during the course of instruction can override the teacher's inclination to act according to a particular norm.

For example, a hypothesized norm inside the situation of 'making conjectures' is *the teacher should make students' conjectures public*. This results in the action that, after students have had time to work independently making conjectures, the teacher will ask students to share their conjectures. However, the teacher's professional obligations could stop her from doing this. Responding to an individual obligation, a teacher could know that an individual student is shy, so she would not ask that student to share his conjecture. Or, in response to an interpersonal obligation, a teacher could worry about the effects on her class of validating conjectures that come from some students and rejecting conjectures that come from others, so she would not have any students share their conjectures. Or, responding to an institutional obligation, a teacher could decide that she does not have enough time to have students share their conjectures. Or, responding to a disciplinary obligation, a teacher could worry that since her students may come up with conjectures that are not correct she should not make space for them to be heard in class. In each of these examples the teacher is perceived to be breaching the norm in favor of a professional obligation.

Figure 37 shows the interaction of instructional norms and professional obligations that leads to teacher action (Herbst, 2010a). On the right hand side is the instructional norms for teacher action, based on the work of teaching. On the left hand side are the professional obligations that could compel a teacher to act in a way that could be seen as a breach of an instructional norm in favor of another action. These two, instructional norms and professional obligations contribute to teachers' practical rationality. Mediating this practical rationality is a teacher's personal resources. Personal resources are influences on teachers' action that are held by individual teachers (as opposed to norms which are cultural resources). An example of a personal resource is a teacher's level of mathematical knowledge for teaching (Ball et al., 2005). This framework provides a way of describing teachers' actions in instruction that is based on the work of teaching and on the professional obligations of teaching.

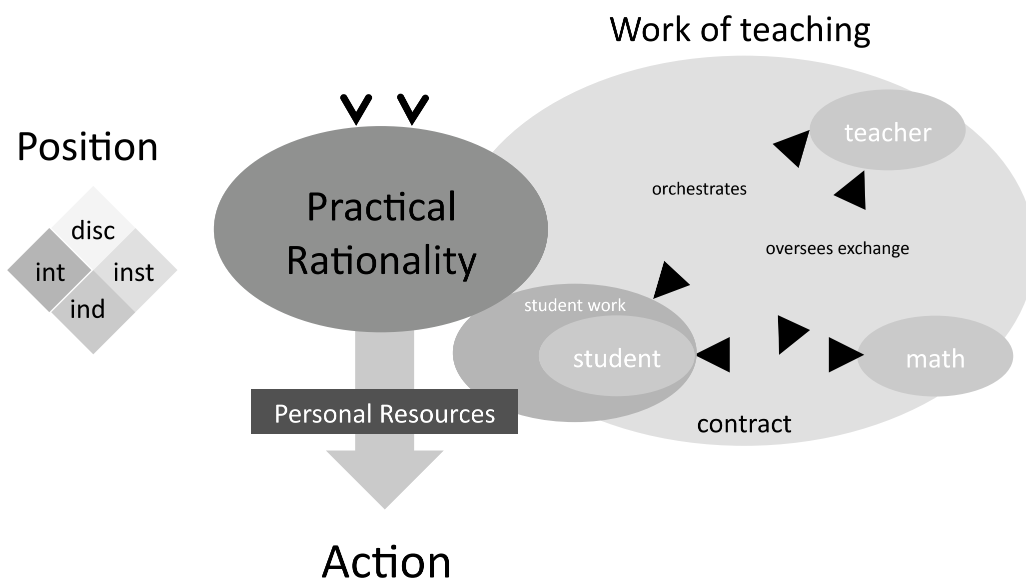

Figure 37: The practical rationality of teaching (adapted from Herbst 2010a)

Practical rationality of students

An aim of the current paper is to reconceptualize the framework for the practical rationality of teaching to account for the rationality of studenting. This involves mapping the work of the student, hypothesizing norms for student action, and hypothesizing obligations that students respond to, which could possibly override students' inclination to act according to these norms.

I argue that "studenting" is a practice (Cook & Brown, 1999). According to Cook and Brown, a practice consists of "the coordinated activities of individuals and groups in doing their 'real work' as it is informed by a particular organizational or group context" (p. 386-387). When students do work in classroom they are doing their 'real work' in the sense that as geometry students their major responsibility, as laid out by the didactical contract, is to do mathematical work in classrooms. They have learned how to do this work over the course of many years in school, and math classrooms in particular, and their work is informed by the judgment of their peers. One could imagine that when watching the animated scenario all of the participants could have said, "I don't know why Alpha acted that way. I'm not Alpha." But instead the participants felt that they had a right to speak on Alpha's behalf because they also engage in the practice of studenting.

Cook and Brown illustrate that medicine is a practice with the following example. "In the simplest case, if Vance's knee jerks, that is behavior. When Vance raps his knee with a physician's hammer to check his reflexes, it is behavior that has meaning, and thus is what we call action. If his physician raps his knee as part of an exam, it is practice. This is because the meaning of her action comes from the organized contexts of her training and ongoing work in medicine (where it can draw on, contribute to, and be evaluated in the work of others in her field)" (p. 387). Extending this illustration to the practice of studenting, if an individual student raises her hands to signal that she would like to share a counter-example to a conjecture that another student put forth, to show that another student has missed an important mathematical point, in the context of a classroom discussion with her peers, it is practice. This group of peers who give weight to the choices of individuals makes the work of studenting a practice.

I would like to argue that the practice of studenting rests on a rationality that is similar to the rationality that supports the practice of teaching. Over the course of a student's life in school, and in mathematics classrooms in particular, students become tacitly aware of the norms of the classroom and what they "ought to do" at a particular moment, within a particular instructional situation. Simultaneously students develop obligations of the position of student that will be such that they are occasionally compelled to act in a way that could be perceived as a breach of a norm in favor of acting in accordance with that obligation. For the sake of symmetry with Herbst's (2010a) model for the practical rationality of teaching, I refer to students' tacit knowledge for how to act during geometry instruction as the "practical rationality of geometry studenting," and I refer to students' commitments to the position of student "students' professional obligations."

Figure 38 reformulates instruction from the point of view of the student. The students' main responsibility is to do work on mathematical tasks. The student receives these tasks from the teacher and interacts with available resources to complete the task. Once the work is done, it is submitted to the teacher to be valued as either an instance of mathematical learning, or a chance to receive evaluation from the teacher (Aaron & Herbst, under review). Since the responsibility of exchanging students' work for claims on the contract is primarily in the domain of the teacher it is not included in this diagram of the work of studenting.

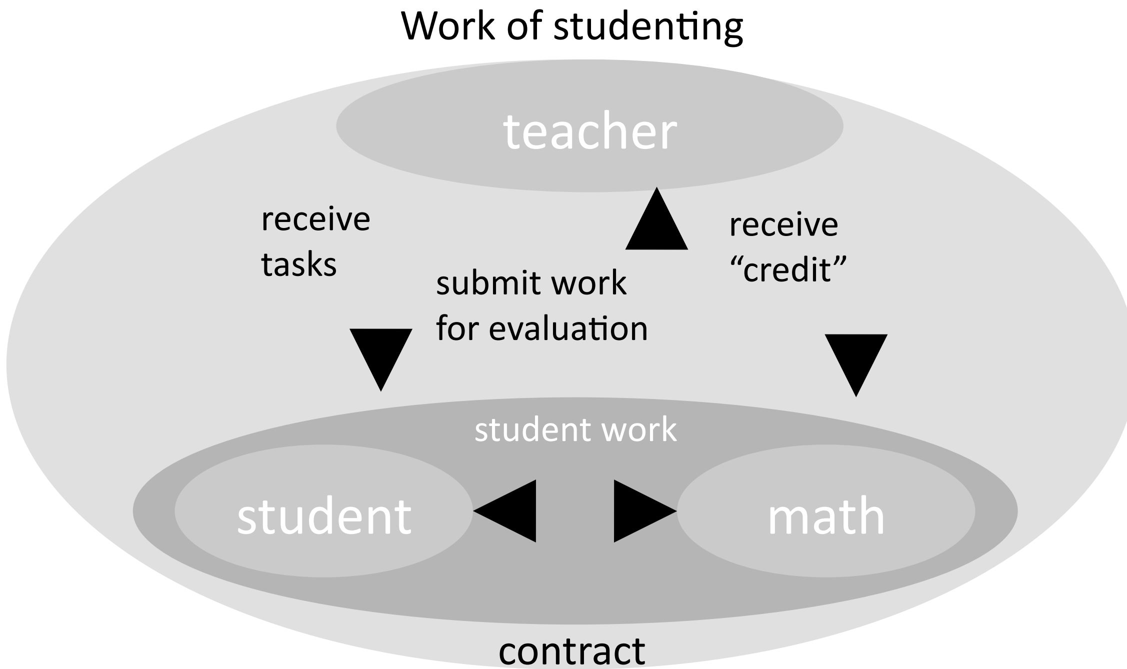

Figure 38: The work of studenting

Building on the description of the work of studenting in Figure 38, one can envision the corresponding description of the practical rationality of studenting. In Figure 39 the right hand side shows the work of studenting and contributes the instructional norms for student action to practical rationality. The left hand side shows the obligations that students respond to and could potentially override students' inclinations to act according to the norms for student action. The instructional norms and obligations combine to form the rationality of studenting, and individual student's personal resources mediate this rationality. An example of a personal resource would be student's goal orientation (Pintrich, 2000).

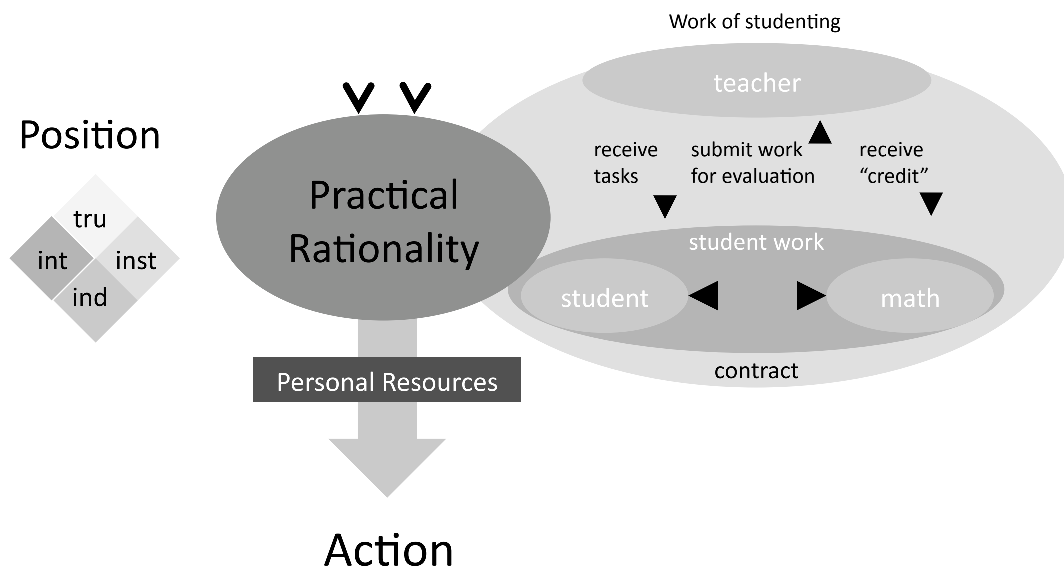

Figure 39: The practical rationality of studenting

In this model, the norms of instruction and the obligations constitute the practical rationality of studenting. Both the norms and obligations are shared social resources that students use to make meaning of instruction. Students' personal resources are factors that mediate these social resources and affect students' actions on an individual level. The norms are theorized to stem from the work of instruction, the obligations stem from the four sources of individual, interpersonal, institutional, and truth, and the personal resources stem from students' individual cognitive abilities, emotional dispositions, life experiences, etc.

In constructing this framework, I begin with the hypothesis that the obligations that students respond to correspond to the obligations teachers respond to. Since students are in the same classroom environment, with the same class, in the same institution, dealing with the same discipline as the teachers it is plausible that students would be obliged to the same stakeholders as the teachers. So I hypothesize that students respond to obligations from students as individuals, interpersonal relations in the classroom, and the institution of school. However, at least one adjustment is called for to fit the teachers' professional obligations to the students. Since students are learning about the discipline of mathematics, and do not yet know what obligations the discipline would impose on their actions, I replace the obligation to the discipline with a more general obligation to truth. I hypothesize that students do have tools to measure true and false and they support the endorsement of true ideas and reject the endorsement of false ideas.

From the empirical study of students' perception of instructional norms, one sees that the participants do act as if they respond to some set of obligations that act as departures from the norms. The findings showing alternative justifications lend support to the model for the rationality of studenting seen in Figure 39, in particular to the interaction between norms and obligations. The content of the obligations also is supported by the empirical study. In this model the obligations stem from four sources; individual students, interpersonal relations, the institution of school, and truth.

The five alternative justifications that were found in the empirical study; cognitive difficulties, worry of incorrectness, desire to keep an idea private, politeness, and desire to share true ideas, can be mapped onto these sources for obligations, and personal resources. Cognitive difficulties could be seen as an obligation that students have to individual students in the sense that each individual student is entitled, and expected, to have some cognitive difficulties as a result of being a student in school. The worry of being incorrect could be interpreted as either a personal resources in the form of a performance-avoid goal orientation, or it could be interpreted as an aspect of the students' obligation to truth in the sense that students should not endorse or support ideas that are incorrect. The desire to keep an idea private is an example of a personal resource because it reflects a desire for the student to step out of the student role temporarily. However, the participants saw that this is sometimes a reasonable move for a student to make. Politeness is as aspect of students' obligation to the interpersonal relations of the class; because the student is not just an individual student, but also a member of a class, it is required that students are polite to each other. Finally, a desire to share true ideas is another aspect of the obligation to truth. Like the worry of incorrectness, it reflects students' commitment to endorse and support true ideas.

This model for the rationality of studenting is useful because it combines several theories that have been used to explain student action. In particular, it describes how instructional norms, which in this model are the primary explanations for student behavior, can be overruled in particular moments by students' professional obligations. What I consider professional obligations have been considered in the literature in many ways, such as students' relationships with the schools and students' relationship to work in classrooms. These instructional norms and obligations combine to form the rationality of studenting, which, I claim, shapes the position of student in geometry instruction. This rationality is expected of all students (unlike the personal resources), and is directly related to the work of studenting. Practical rationality, before it results in action, is mediated by individual student's personal resources. The personal resources are not expected to be shared by all students, and they are not necessarily tied to the work of studenting. Much of the work in educational psychology, which looks at the cognitive, emotional, and behavioral actions of individual students, would fit into the category of personal resources.

Conclusion

The work of being a student is complex work, which is guided by a unique rationality based on this work. The current paper makes two contributions to understanding this work and this rationality. First, this paper contributes an empirical study of the instructional norms of geometry instruction. In this study classes of geometry students responded to an animated classroom scenario of geometry instruction, which allowed students to respond to particular instructional norms. These responses were coded according to a list of hypothesized norms, looking for confirmatory evidence for these norms, or evidence that students would act in a way that is a departure from the norm. This evidence of alternative justifications, which was unaccounted for by the theory of instructional norms, was than used as motivation to conceptualize a framework for the rationality of studenting using the framework for understanding the practical rationality of teaching developed by Herbst (2010a, 2010b). This framework for the rationality of studenting incorporates instructional norms, professional obligations, and personal resources. Each of these components is supported by past research.

As I mentioned in the beginning of this paper, the current study is part of a larger project aimed at better understanding the position of the student in geometry instruction. The framework described here adds to this understanding by describing the shared obligations and norms that students work within when they construct their actions in geometry instruction. The instructional norms explored in this empirical study, and the obligations hypothesized here and supported by the literature, are cultural resources that are available to individuals who take up the position of geometry student. Individuals become familiar with these resources as they spend time in schools and in geometry classrooms in particular.

Holland et al (1998) define positional identities as having to do with "the day-to-day and on-the-ground relations of power, deference and entitlement, social affiliation and distance—with the social-interactional, social-relational structures of the lived world" (p. 127). This definition of position highlights the importance of action in the "lived world" and how these actions construct our relationships with others. Within geometry instruction, students' actions construct their relationship with the content to be learned, to their peers, and to the teacher. The framework outlined here provides a theory for how these actions are constructed through instructional norms, professional obligations, and personal resources.

By understanding the student's position in geometry instruction we will be better able to understand geometry instruction as a whole. Understanding the complex system of geometry instruction is the first step to making informed improvements to the system. From this study it is clear that it is not enough to understand any one influence on the actions of geometry students, but these influences are all interconnected. A change in curriculum that is expected to produce a particular student action based on an understanding on instructional norms could result in an unexpected action that could be explained in terms of an obligation that is held by students. That is, changes to any part of the system should take into account their effect on other parts of the system, and these possible interactions can only by predicted by first understanding the system as a whole. The current paper represents an attempt in that direction.

Unlike teachers, students go through no professional training for how to be a student, so their actions are guided solely by the rationality that they develop from doing the work of studenting. There is, therefore, a circular relationship between studenting and the rationality of studenting. The actions that students engage in mold the rationality of studenting and the rationality of studenting shapes future actions. It is difficult, if not impossible, to change students' instructional actions directly, but if we can uncover the rationality of studenting we can use this as a lever for changing student action (see Lampert, 2001, chapter 10). The first step after understanding student rationality would be to design instructional activities where students' rationality could be use in favor of instruction. That is, in cases where students' actions are undesirable due to students' interpretation of a task or situation, that task or situation could be reengineered in such a way that the students' rationality would result in more beneficial instructional actions. As students spend time engaged in mathematical tasks, performing actions that embody more desirable mathematical work, then their rationality of studenting could possibility change to naturally support more productive mathematical work.

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Chapter 5

Conclusion

As a study of instruction, this dissertation provides insight into the position of the student inside instruction. This insight is gained through the combination of three studies, each looking at the position of student from a different instructional point of view. The first study looks at the position of the student from the viewpoint of mathematics, the second looks at the position of the student from the viewpoint of the teacher, and the third looks at the position of the student from the viewpoint of the student. Together these studies develop a view of the position of student that reflects its dependence on both the mathematics and the teacher, and reflects the interdependent nature of the elements of instruction. From this dissertation one gets a better understanding of the ways that the students' interaction with the teacher and the mathematics shape this position, and of the tacit norms that guide student action.

From a mathematical viewpoint, the first study provides examples of two different ways of making meaning from mathematical discussions. It also shows the virtue of the two-column proof in its affordance of supporting of chains of implications in arguments. However it also shows the drawback of the two-column proof in its lack of flexibility to support backings and rebuttals in students' arguments. One sees that learners bring resources for argumentation to the classroom that are not sufficiently supported, like creating backings and rebuttals, while the two-column proof does seem to provide useful support to students in shaping longer arguments.

This knowledge of the arguments that learners could produce when they are engaging with classroom discussions describes some of the mathematical space that individuals could occupy when they are in the position of student. One sees that this mathematical space affords the resources for building arguments in the form of long strings of implications, but that it restricts students from making arguments that are not of this form and arguments that include elements besides simple implications.

A finding of this study is that students, in instruction, elaborate arguments using different styles of argumentation. This study provides an illustration of two of these argumentation styles, but there could more styles worthy of study. For instance, it would be interesting to conduct a case study with students who are less successful in mathematics than the participants in the current study. Also, a longitudinal study that traced the development of a single student's arguments over the course of the high school geometry course could show how a familiarity with the argumentation style of the two-column proof develops over time. Another future project could be to look across the arguments of several students and build a conceptualization of different argumentation styles.

From the perspective of the teacher, the second study shows that teachers perceive students differently in the context of the instructional situations of 'making conjectures' and 'doing proofs.' In the situation of 'making conjectures' teachers perceive students in terms of engagement with the lesson. This focus on engagement ignores the mathematical value of students' work in this situation. In the situation of 'doing proofs' teachers perceive students in terms of the mathematical content at stake. These different ways that teachers perceive their students has implications for how students are supported in their mathematical work

Understanding these opportunities for action helps explain how the teacher shapes the position of the student. That is, because teachers are responsible for conducting instruction, teachers' perception of students shapes the work that is available for students to do. This description of how the teacher perceives her students while doing the work of teaching, inside instructional situations, describes the opportunities for action that the teacher makes available for students.

In future research it would be interesting to study teachers' perception of students in an algebra course, particularly the situations of 'solving equations' and 'doing word problems.' (Chazan & Lueke, 2009; Chazan, Sela & Herbst, in review) I would expect to find that teachers' perception of their students also varies across instructional situations in this course. A study of teachers' perception of students in algebra classrooms would also afford a comparison with teachers' perception of students in geometry classrooms. Since these courses appear to be so different in terms of the material for students to learn, it would be interesting to see if these differences were reflected in teachers' perception of students.

From the viewpoint of the student I conceptualize theoretical frameworks for studenting and the system of tacit knowledge that guides studenting, the practical rationality of studenting. This framework is based on the frameworks for the work of teaching, and the practical rationality of teaching (Herbst, 2010). I support the conceptualization of these frameworks for studenting and the practical rationality of studenting by investigating students' practical rationality in the high school geometry classroom. This practical rationality of studenting provides a description of the actions that students see as viable in instruction and therefore shapes the position of the student.

From the perspective of the student, the third study shows that the practical rationality of studenting consists of norms for student action and obligations that students hold to educational stakeholders. One sees that student actions can partially be accounted for by hypothesized instructional norms. When students' action are perceived as breaching instructional norms, these perceived breaches can be accounted for by obligations that students hold, based on their obligations to individuals, interpersonal relations, the institution of school, and truth. These norms and obligations of studenting can be seen to guide the work that students do in geometry instruction.

More research is called for to further develop the conceptualization of studenting. One aspect of research that is needed is to see if the results found in this study could be supported in additional conversations with students, possibly in smaller groups, so that more voices could be heard. Also, past research has shown that different students take on different instructional identities (Aaron & Herbst, in review) and research is needed to connect these instructional identities, which are related to students' overall stance towards the didactical contract, to the specific norms and obligations that are found in the current study.

Animated scenarios of classroom instruction proved to be valuable data collection tools in each of these studies. In the first study, on learners' mathematical arguments, the animated scenarios allowed participants access to classroom discussions that they could pause, rewind, or fast-forward. This ability to control the flow of time allowed them to examine the discussion in detail; dwelling on moments that caught their interest and moving quickly through moments that did not. The animated scenarios also provided the participants in this study with access to mathematical ideas that were presented through the voices of multiple characters, which highlighted the multiple views that go into developing a mathematical idea. The fact that these characters were simple animated characters instead of real students in a real classroom reduced the complexity of the representation so that the participants were more able to focus on the mathematical discussion.

The success of this method of engaging learners with mathematical ideas recommends it for use in future research aimed at understanding students' mathematical behavior and thinking, and for use in the teaching of students. For example, studies making use of animated scenarios could be aimed at understanding how learners take up the mathematical ideas of others as they are learning the subject. These animated scenarios could also be used with students in classrooms as supports for classroom discussions. Discussing mathematical ideas is a skill that students learn from experience. Students who have never had the opportunity to experience mathematical discussions could learn this skill from engaging with the animated scenarios. Eventually, the goal would be to have students discussing ideas with each other.

In the second and third studies, on teachers' perception of students and the practical rationality of studenting, the animated scenarios acted as the basis for breaching experiments that elicited teachers' and students' tacit knowledge about how to enact their respective positions in instruction. The animated scenarios allowed teachers and students to vicariously experience a familiar setting, being part of a high school geometry classroom, in which action did not proceed according to the normal rules for social interaction in that context. The abnormality of the interaction would prompt the teachers and students to respond in ways that could be seen to display the norms of practical rationality of teaching and studenting, respectively.

This method of vicarious breaching experiments could be used to uncover other tacit knowledge, besides the knowledge of teaching and studenting. For example, animated scenarios of medical diagnosis or courtroom proceedings could be developed to study the tacit knowledge of medical professionals or law professionals. These animated scenarios could be discussed by experienced practitioners and norms for action inferred from their conversations.

Overall, this dissertation offers a detailed description of the complex position of student that individuals take up in instruction. By looking at the position of student from the viewpoint of the teacher and mathematics, instead of only the student I develop a conceptualization of the student's position that is dependent not only on who the student is, but how the student interacts with the teacher and with the mathematics. This multifaceted view is essential in understanding the position of the student in instruction.

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