Generalized Lagrangian States and Their Propagation in Bargmann Space

by

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CHAPTER I

Introduction

Let's frame the discussion of the content of this work by first characterizing the broader field which the work relates to and sits within: semiclassical analysis. A one sentence description of the field would be that it is a conceptual and mathematical framework, with the goal of understanding the relationship between classical mechanical and quantum mechanical descriptions of physical systems in an asymptotic sense. In order to help unravel this brief description we can start by describing what we mean by classical mechanics and quantum mechanics.

Classical mechanics is a theoretical framework for modeling physical systems with a history dating back to the likes of Isaac Newton, Leonard Euler, Joseph Louis Lagrange, Pierre-Simon Laplace, and William Rowan Hamilton, among others. The theory was originally developed in order to understand the motions of bodies in the solar system, i.e. celestial mechanics. Since the discoveries of Einstein's theories of relativity as well as quantum theory we know that there are limits to the ability of classical mechanics to successfully predict the outcome of physical processes under certain conditions, specifically if the speed of an object is a significant fraction of the speed of light, an object is moving within a very strong gravitational field, or the system being studied is small, meaning on the level of molecules or smaller. Despite these limitations, classical mechanics is an extremely successful theory that is accurate at predicting the behavior of most of the phenomenon that humans encounter on a daily basis.

The basic object that is studied in classical mechanics is a point mass (and by extension, collections of point masses), and the goal is to understand the evolution (motion in time) of point mass(es). There exists several formulations of classical mechanics: Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics. For the purposes of this work, when discussing classical mechanics we always mean the Hamiltonian formulation. From a physical perspective, the backdrop of Hamiltonian mechanics is phase space, which is the collection of all possible positions and momenta of a physical system. The fundamental quantity that the framework of Hamiltonian mechanics needs in order to understand the dynamics of a point particle is the Hamiltonian function, H, of the particle (a.k.a. the energy function) which is a smooth function on phase space. The equations governing the evolution of a point particle in phase space are the so-called Hamilton equations of motion. For a single particle that is moving in m spatial dimensions (for say $m = 1, 2, 3$) there will correspond another m momentum degrees of freedom for a total phase space of $\mathbb{R}^{2m}$. If we have more than one particle then each particle will have m degrees of freedom for it's position and m degrees of freedom for its momentum. Thus for a multi-particle system we will have a phase space of $\mathbb{R}^{2n}$ for some positive integer n where the elements of this space are vectors of the form $(\mathbf{q}, \mathbf{p})$ where the n-vector $\mathbf{q}$ contains the n positions of the system, and $\mathbf{p}$ contains the corresponding n momenta. The Hamilton function will be $H = H(\mathbf{q}, \mathbf{p}) \in C^\infty(\mathbf{R}^{2n})$. Hamilton's equations are

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad\qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

where a dot denotes the time-derivative of the quantity. For discussions on Hamiltonian mechanics, and classical mechanics more generally, from a physical perspective consider the classsic (no pun intended) texts [16] and [27]. From a mathematical perspective we can understand Hamiltonian mechanics in the language of differential geometry, specifically the language of symplectic geometry. In the context of symplectic geometry, phase space is a smooth manifold M of even dimension that is endowed with a closed, non-degenerate 2-form ω. Such an M is called a symplectic manifold, and ω is called a symplectic form. Given a function $H \in C^\infty(M)$ we can define a vector field on M, the Hamiltonian vector field ξ_H, by the condition

$$\omega(\xi_H, \cdot) = -dH(\cdot),$$

where the two sides of this equation are equal as 1-forms on M. The integral curves of ξ_H define the Hamilton flow ϕ_t associated to H; and if one were to locally express the equations that define the flow (i.e. the integral curves of the Hamilton vector field) then one gets the Hamilton equations above. One final detail to note is that, within the theory of classical mechanics, an observable quantity (i.e. a physical quantity that can be measured) is represented as the values taken by smooth function on phase space. For discussions of Hamiltonian mechanics from a differential geometric perspective consider these classic texts [2] and [5].

Quantum mechanics is a theoretical framework that arose out of the discovery/realization that matter exhibits a dual nature with wave like and particle like properties, typically when it is examined on a sufficiently small scale. The foundations of quantum mechanics as a theory can be dated to the work of Neil Bohr, Werner Heisenberg, Max Planck, Louis de Broglie, Albert Einstein, Erwin Schrodinger, Max Born, John von Neumann, Paul Dirac, Wolfgang Pauli, David Hilbert, and Hermann

Weyl, among many others. Because of the small scale that must be probed in order observe evidence of the wave like property of matter we generally think of quantum theory as being a theory that describes matter on the molecular, atomic, and subatomic scales. Since quantum phenomenon is observable on a length scale that is far removed from the 'routine' experiences of humans, the conceptual understanding of quantum processes was shaped by experimental results, analogy to the well-known wave phenomenon in electricity-and-magnetism and optics, and though the mathematical framework itself. Whenever we are speaking of quantum mechanics in this work we are always speaking about the non-relativistic theory and phenomenon. The mathematical language that quantum theory is written in is functional analysis, and within this broad mathematical theory we have that the backdrop of quantum mechanics is a separable complex Hilbert space, $\mathcal{H}$, with inner-product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The fundamental objects that are studied are the elements of norm 1, often called state vectors since $\mathcal{H}$ is a vector space. Specifically, physical states will be represented by unit vectors in $\mathcal{H}$. The dimension of $\mathcal{H}$ will depend on the type of physical system being studied. For the purposes of this work, we will focus on the case where $\mathcal{H}$ is infinite-dimensional, and as such the state vector is often referred to as the wave function of the physical system. Working in non-relativistic quantum theory the evolution equation that governs the behavior of the wave function, ψ, is Schrodinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{H}\psi$$

where $\hbar$ is Planck's constant, and $\widehat{H}$ is the Hamiltonian, which is an unbounded self-adjoint operator on $\mathcal{H}$ for which the domain, $\mathrm{dom}(\widehat{H})$, is an everywhere dense subset of $\mathcal{H}$. The wave function, ψ, itself does not represent a physical quantity, rather

$|\psi|^2$ is interpreted as a probability density. Given an initial condition, the operator defined as the time t solution to the Schrodinger equation is known as the quantum propagator, often denoted as $\widehat{U}(t)$ and $e^{-\frac{i}{\hbar}\widehat{H}}$, which is a unitary linear operator on $\mathcal{H}$. The observable quantities in quantum theory are the expectation values of (possibly unbounded) self-adjoint operators on $\mathcal{H}$. The definition of the expectation value of the operator $\widehat{O}$ with respect to the state $\psi \in \mathrm{dom}(\widehat{O})$ is $\langle \psi, \widehat{O}\psi \rangle_{\mathcal{H}}$.

With the above 'working' definitions of classical mechanics and quantum mechanics we can now give a more detailed description of the field of semiclassical analysis. As stated above, semiclassical analysis was created toward first understanding the relationship between classical mechanical and quantum mechanical descriptions of physical systems in an asymptotic sense, and secondly to use this understanding to approximate the dynamics of certain quantum systems. A central piece of the theory of semiclassical analysis is the notion of quantization; which is a systematic means of defining a correspondence between a classical mechanical system and a corresponding quantum mechanical system, and vice-versa. The quantization correspondence will associate a classical phase space with a quantum Hilbert space (this correspondence is only 'one-way'), classical observables with quantum observables, and Hamilton flows with unitary evolution. There are many quantization schemes, each with its own unique set of benefits and drawbacks. Once a quantization procedure is prescribed then we can associate the classical behavior and the quantum behavior of our systems asymptotically in the limit $\hbar \longrightarrow 0$ in the sense that in this limit, classical dynamics becomes a more salient feature of quantum dynamics. A natural question to ask is 'What could one possibly mean for a constant $\hbar$ to go to zero?' The answer, adapted from [7], is essentially the following. For any particular physical system there are usually characteristic distances, masses, velocities, etc., from which a characteristic unit of action can be derived. We would expect that a classical mechanical description of a physical system will be successful if $\hbar$ divided by this unit is much less than 1, and the closer this ratio is to 1 the system will be less 'classical influenced'. In this sense $\hbar$ is considered a formal parameter. It should be mentioned at least once that since everything is made of molecules, atoms, and subatomic particles that means that 'classical' systems are really composed of quantum mechanical constituents, and so quantum theory is a more fundamental theory. In a real sense, there is no such thing as a classical system. Now, theoretical physics is about building and understanding successful (i.e. predictive) mathematical models of natural phenomenon, and this inevitably involves distilling down natural phenomenon to its most operable influences and disregarding negligible influences on a system. So, even though in principle all classical systems obey quantum principles we call systems classical when quantum phenomenon can be neglected when predicting the dynamics of the system. For a range of 'flavors' of semiclassical analysis see [14], [17], [28], and [42].

Now that we have characterized (at least a little) the general field that we are working in, let's discuss the specifics of this work.

Chapter II is titled 'Preliminaries', and just as the name suggests, this chapter is devoted to giving a fairly detailed exposition of the specific background material that is used throughout the rest of the work. This chapter is not meant as an introduction to semiclassical analysis, but rather a detailed reference of relevant background material. To someone new to the field it will likely seem like a disjoint collection of facts. Since semiclassical analysis seeks to understand the relationship between classical and quantum behavior, the most 'classical' of quantum states namely coherent states play a central role in the broader field in general and especially in the content of this work. For detailed discussions of coherent states see [4]. Since they are so important, we begin this chapter with a detailed treatment of coherent states in the standard representation, meaning that we are considering the quantum Hilbert space as $L^2(\mathbb{R}^n)$. After carefully detailing the results we'll need for coherent states in the standard representation we move on to detailing the quantum Hilbert space that we will be interested in using, namely Bargmann space. Originally proposed by V. Bargmann in [6], Bargmann space is a Hilbert space of states defined over the phase space $\mathbb{C}^n$. The section on Bargmann space begins with a discussion of Bargmann's a definition of the Hilbert spaces that now bear his name, then details some of the basic properties of the space that will be relevant for our purposes, and finally we see the connection between Bargmann space and the standard Hilbert space $L^2(\mathbb{R}^n)$ through the Bargmann transform. The work in this section is attributable to [6], [15], and [18]. Finally, we end the chapter with a discussion of the three most common types of quantization methods, Weyl quantization, Wick quantization, and Anti-Wick quantization. These are quantization procedures for associating observable on the phase space $\mathbb{R}^{2n}$ with the standard symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$ (or given a complex structure $\mathbb{C}^n$ with the corresponding symplectic form) with the observables on the standard Hilbert space $L^2(\mathbb{R}^n)$ in the case of Weyl quantization or Bargmann space in the case of the other two. We also detail the correspondence between these quantization procedures.

Chapter III is titled 'Generalized Lagrangian States in Bargmann Space', and in it we motivate, define, and study the main mathematical object studied in this work, a generalized Lagrangian state in Bargmann space. Such states are appropriately weighted continuous (i.e. integral) sums of coherent states, integrated over a Lagrangian submanifold of phase space. The idea of using such integral coherent state sums over a Lagrangian was first introduced independently by T.Paul and A. Uribe in [33], and by M.V. Karasev in [21] where the authors had the goal of constructing quasimodes of the time-independent Schrodinger equation using an ansatz that is quite similar to what we will refer to as generalized Lagrangian states in this work. D. Borthwick, T. Paul, and A. Uribe, then further studied such objects in [10], whereas M.V. Karasev continued his study of in [22]. The goal of Paul and Uribe in [33] was to provide a very explicit example of using such weighted integral sums of coherent states in constructing quasimodes that would also function as an introduction to the more general theory presented in [10]; quite similarly in goal, but more general in his approach, in [22], Karasev also constructs asymptotic approximations to stationary states of a Hamiltonian. Working in the general framework of the quantization of K$\ddot{a}$hler manifolds, the authors of [10] elucidate an abstract geometric theory of Legendrian distributions, with some applications to relative Poincare series given after the theory is introduced. We will see in chapter III that some of the semiclassical properties that are proven for Legendrian distributions will hold for our generalized Lagrangian states. Finally, in [22], Karasev extends the study of these integral coherent states sums to integrals over isotropic rather than Lagrangian submanifolds. In the context of this previous work, the generalized Lagrangian states presented here are another example of the utility of these integral coherent states sums, where in this case the coherent states are integrated over a Lagrangian submanifold of the standard phase space $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We will show that, in fact, such an object is impressively well suited to the two main tasks of semiclassical analysis, namely they provide a beautifully clever means of weaving together the geometric nature of classical mechanics and the analysis of quantum theory, and as we will see, they provide a powerful tool for incorporating classical mechanics to help approximate quantum dynamics. After generalized Lagrangian states are motivated and defined we then prove that under certain conditions we can analyze these states asymptotically, as $\hbar \longrightarrow 0$, using the method of stationary phase. With an asymptotic understanding provided by stationary phase we then study some semiclassical and microlocal properties of (certain) generalized Lagrangian states.

Chapter IV is titled 'Polynomial Hamiltonian Acting on Generalized Lagrangian States', and the goal of the chapter is right in the title. More descriptively, the goal of the chapter is to understand the action of the ring of pseudodifferential operators generated by $\widehat{Z}_j$ and $\widehat{Z}_j^*$ for $j = 1, \ldots, n$ (these operators are first introduced in II) on certain (well-behaved) generalized Lagrangian states. Even more specifically, we can characterize the goal of the chapter as finding an expression for the Weyl quantization of a polynomial symbol acting on one of our generalized states that is appropriately ordered in $\hbar$. Where 'appropriate' will be as an asymptotic expansion in generalized states, rather than a pointwise asymptotic expansion. We will see that under certain conditions on generalized state we can show that the collection of such states is closed under the action of the operators $\widehat{Z}_j$ and $\widehat{Z}_j^*$ for $j = 1, \ldots, n$, and from this we systematically build up to more complicated situations. The process will be to very carefully and systematically add in more (complicating) detail, until we arrive at the general results we'll need for the work to come later. The work in this chapter can best be characterized as a series of complicated and detailed calculations. There are two main reasons for such an investigation. One answer that is a bit of a 'cop-out', is that the question of how pseudodifferential operators act on generalized Lagrangian states is a natural one to answer considering the prominent role that such operators occupy in the semiclassical literature. A better answer, that is more focused on this work, is that such an understanding will be used in chapter

V in our main result concerning the quantum propagation of generalized Lagrangian states.

Chapter V is titled 'Propagation of Generalized Lagrangian States', and here we will prove some of the main results of this work, concerning (no surprise here, based on the title) how generalized Lagrangian states propagate quantum mechanically. The chapter begins with a discussion of classical dynamics, both on phase space and on the reduced Heisenberg group. Recall from chapter II that the reduced Heisenberg group has the structure of a principle fiber bundle with phase space as the base manifold, and the fibers being isomorphic to the circle S^1. The dynamics on phase space can be lifted to the reduced Heisenberg group in a natural way that will be key to proving the main result of this chapter. With these necessary facts concerning classical dynamics in place, we introduce the idea of a generalized Lagrangian state that is smoothly dependent on a real parameter; and then we specialize the discussion to certain type of parameter-dependent generalized states that incorporate ideas from classical mechanics, and where this smooth parameter is thought of as time. The results in this chapter will show that for these 'new' type of generalized Lagrangian states, with a compactly supported amplitude, their quantum evolution with respect to a Weyl quantized polynomial Hamiltonian is another such generalized Lagrangian state, modulo a semiclassically negligible term (i.e. a term that is $O(\hbar^\infty)$). We'll also see that if the amplitude of a state is not compactly supported that the process of proving the compact case will, at the very least, be adaptable to give a formal procedure for showing that an analogous result holds in this noncompact case (at least formally). The procedure of the proof can best be described as an asymptotic matching process that utilizes a key lemma from chapter III concerning the equality of generalized Lagrangian states translating into the equality of their amplitudes.

This chapter truly distinguishes this work from the previous work done on objects similar to generalized Lagrangian states (see the paragraph on chapter III above), in that the idea of propagating such weighted integral sums of coherent states has never previously been considered. From the perspective of the success in [33] and [21] of using similar weighted integral coherent states sums to asymptotically 'solve' the time-independent Schrodinger equation, it's natural to consider investigating such states with respect to the time-dependent Schrodinger equation.

Chapter VI is titled 'Applications of Generalized Lagrangian States', and here we'll use the main results from chapter V, either directly or as inspiration, to use generalized Lagrangian states to approximate the kernels of some operators on Bargmann space that are important in physics and physical chemistry. The chapter begins with a brief review of the definition of the quantum propagator, and it's kernel. With these basic definitions in place we propose an ansatz for the kernel of propagator that is a generalized Lagrangian state where the Lagrangian is the graph of the classical flow. After identifying some preliminary facts concerning such an ansatz, we give two examples of using it in the case of oscillator Hamiltonians. Though it is successful for these oscillator examples, it becomes clear that this ansatz will not work for more general Hamiltonians. To address this limitation we propose a modification that has the same general form as the original ansatz, but the Lagrangian is 'localized' to the part of phase space defined by an energy threshold. This new 'localized' ansatz is a generalized Lagrangian state where the Lagrangian has a boundary in the sense of manifolds. The Lagrangian having a boundary will require us to slightly modify our rules for the action of the ring of pseudodifferential operators generated by $\widehat{Z}_j$ and $\widehat{Z}_j^*$ for $j = 1, \ldots, n$. Once these modified rules are in place we will introduce a class of elements in Bargmann space, called semiclassically localized states, that are in a sense localized in a region of Bargmann space. Examples of such states include coherent states and the Hermite functions (i.e. monomials). With the 'localized' ansatz defining an approximation to the quantum propagator for the Weyl quantization of a polynomial Hamiltonian, and if this approximation acts on semiclassically localized states then we can prove a rigorous asymptotic error estimate for this approximation utilizing Duhamel's formula for the Schrodinger equation. Thus we can use a generalized Lagrangian state as a very successful approximation the kernel of the propagator when we are propagating an appropriate semiclassically localized state. Thus we've introduced a notion of localization in both the propagator and the states propagated that allows us to well approximate (certain) quantum dynamics semiclassically. While the original ansatz (i.e. the 'unlocalized' ansatz) cannot be used in most situations, it is interesting to note that it is intimately related to an approximation to the Schwartz kernel of the propagator on $L^2(\mathbb{R}^n)$ known as the Hermann-Kluk propagator (see [12], [23], [24], [40], and [35]). By related, we mean that the operators which are defined by these kernels are equal upon conjugation by the Bargmann transform. Thus, the use of generalized Lagrangian states can in a sense be thought of as a new derivation of the Hermann-Kluk propagator, as well as a way of tying the Hermann-Kluk propagator into the semiclassical analysis literature in a new and interesting way. Finally at the end of this chapter we will see that we can in fact use a generalized Lagrangian state to approximate the kernel of a 'forward-backward' pair of propagators. Such pairs of propagators arise in spectroscopic theory within correlation functions and optical response functions.

CHAPTER II

Preliminaries

2.1 Coherent States in the Standard Representation

The idea behind coherent states dates back to the days when the foundations of quantum theory were being codified. At the time, Erwin Schrödinger was aware that the new quantum theory must be by some means compatible with classical mechanics in some transitionary sense. Coherent states are a very natural language for discussing this transition, and as such they play a starring role in semiclassical analysis. In fact it has been remarked that "coherent states are the natural language of quantum mechanics". We'll need some facts and results about coherent states for our work, so we review some facts about them.

2.1.1 The Heisenberg Group

The discussion in this section is mainly adapted from ([15]).

Consider $\mathbb{R}^{2n+1}$ with a bracket operation between elements defined as follows,

$$\{(a_1,\dots,a_n,b_1,\dots,b_n,c),(a'_1,\dots,a'_n,b'_1,\dots,b'_n,c')\} = \left(\mathbf{0},\mathbf{0},\sum_{i=1}^{n} a_i b'_i - b_i a'_i\right)$$

(where the zeroes on the right-hand side are zero n-vectors). $\mathbb{R}^{2n+1}$ with this bracket operation forms a Lie algebra known as the Heisenberg Lie algebra. This Lie algebra is 'lurking' in both classical and quantum mechanics in the following way.

Consider the canonical coordinates on the standard classical phase space, $\mathbb{R}^{2n}$, q_1,..., q_n, p_1,...,p_n. The Poisson brackets of the collection of $2n+1$ smooth functions $\mathbb{R}^{2n}$ which include the $2n$ coordinate functions associated to these coordinates which we will (with only slight and standard abuse of notation) denote as q_1,..., q_n, p_1,...,p_n along with the constant function 1 are

$$\{q_j, q_k\} = 0, \qquad \{p_j, p_k\} = 0, \qquad \{p_j, q_k\} = \delta_{jk} \qquad \{q_j, 1\} = \{p_j, 1\} = 0.$$

Where in general, for $f, g \in C^1(\mathbb{R}^{2n})$

$$\{f, g\} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

The space span$\{q_1, \ldots, q_n, p_1, \ldots, p_n, 1\}$ along with the Poisson bracket defined above constitute a Lie algebra which is isomorphic to the Heisenberg Lie algebra.

Now, let's look at the (unbounded) canonical operators in quantum theory $\widehat{Q}_1$,..., $\widehat{Q}_n$, $\widehat{P}_1$,..., $\widehat{P}_n$ acting on the standard state space $L^2(\mathbb{R}^n)$ along with the identity operator I. Note that these "coordinate" operators are continuous if we restrict their domain to the Schwartz class $S(\mathbb{R}^n)$. The commutation relations between these operators are

$$\left[\widehat{Q}_j, \widehat{Q}_k\right] = 0, \qquad \left[\widehat{P}_j, \widehat{P}_k\right] = 0, \qquad \left[\widehat{Q}_j, \widehat{P}_k\right] = i\hbar\delta_{jk}I \qquad \left[\widehat{Q}_j, I\right] = \left[\widehat{P}_j, I\right] = 0.$$

where in general for two operators $\widehat{A}$, $\widehat{B}$, on $L^2(\mathbb{R}^n)$, $\left[\widehat{A}, \widehat{B}\right] \equiv \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$. These commutation relations are nearly identical to the Poisson bracket between the classical coordinate function above. Thus the space span$\{\widehat{Q}_1, ..., \widehat{Q}_n, \widehat{P}_1, ..., \widehat{P}_n, I\}$ along with the commutator bracket defined above is also a Lie algebra isomorphic to the Heisenberg Lie algebra.

For purposes that we'll see shortly, we would like to identify the Lie group associated to the Heisenberg Lie algebra. In order to do that we can consider a matrix Lie algebra which is isomorphic to the Heisenberg Lie algebra (they're coming out of the woodwork). A general element in h_n is a vector in $\mathbb{R}^{2n+1}$, denote such a vector by $(a_1, \ldots, a_n, b_1, \ldots, b_n, c)$. If we associate this element in h_n with the matrix

$$m(\mathbf{a}, \mathbf{b}, c) \equiv \begin{pmatrix} 0 & a_1 & \cdots & a_n & c \\ 0 & 0 & \cdots & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$, and to be clear all the entries of the matrix are zero except the first row and the last column. Then, the bracket operation on h_n can be expressed as the matrix commutator. Note that

$$m(\mathbf{a}, \mathbf{b}, c)m(\mathbf{a}', \mathbf{b}', c') = m(\mathbf{0}, \mathbf{0}, \mathbf{a} \cdot \mathbf{b}'),$$

from which we easily get

$$m(\mathbf{a}, \mathbf{b}, c)m(\mathbf{a}', \mathbf{b}', c') - m(\mathbf{a}', \mathbf{b}', c')m(\mathbf{a}, \mathbf{b}, c) = m(\mathbf{0}, \mathbf{0}, \mathbf{a} \cdot \mathbf{b}' - \mathbf{b} \cdot \mathbf{a}'),$$

as we expect.

This correspondence gives a Lie algebra isomorphism between h_n and $\{m(\mathbf{a}, \mathbf{b}, c) | (\mathbf{a}, \mathbf{b}, c) \in \mathbb{R}^{2n+1}\}$. To find the Lie group associated to h_n we can simply find the associated matrix Lie group by applying the matrix exponential map. Noting that

$$m(\mathbf{a}, \mathbf{b}, c)^2 = m(\mathbf{0}, \mathbf{0}, \mathbf{a} \cdot \mathbf{b})$$

and for $k \geq 3$

$$m(\mathbf{a}, \mathbf{b}, c)^k = m(\mathbf{0}, \mathbf{0}, 0) = 0,$$

we get

$$exp(m(\mathbf{a}, \mathbf{b}, c)) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(m(\mathbf{a}, \mathbf{b}, c))^n = I + m(\mathbf{a}, \mathbf{b}, c) + \frac{1}{2} m(\mathbf{0}, \mathbf{0}, \mathbf{a} \cdot \mathbf{b}),$$

thus explicitly

$$exp(m(\mathbf{a}, \mathbf{b}, c)) \equiv \begin{pmatrix} 1 & a_1 & \cdots & a_n & c + \frac{1}{2}(\mathbf{a} \cdot \mathbf{b}) \\ 0 & 1 & \cdots & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

It is a straight-forward calculation to see that

$$exp(m(\mathbf{a}, \mathbf{b}, c)) exp(m(\mathbf{a}', \mathbf{b}', c')) = exp\left(m\left(\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}', c + c' + \frac{1}{2}\left(\mathbf{a} \cdot \mathbf{b}' - \mathbf{a}' \cdot \mathbf{b}\right)\right)\right).$$

Thus, if we identify $(\mathbf{a}, \mathbf{b}, c) \in \mathbb{R}^{2n+1}$ with the matrix $exp(m(\mathbf{a}, \mathbf{b}, c))$ then we can make $\mathbb{R}^{2n+1}$ into a group with group law (the operation denoted by juxtaposition)

$$(\mathbf{a}, \mathbf{b}, c)(\mathbf{a}', \mathbf{b}', c') = \left(\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}', c + c' + \frac{1}{2}\left(\mathbf{a} \cdot \mathbf{b}' - \mathbf{a}' \cdot \mathbf{b}\right)\right).$$

We call this group the Heisenberg group and denote it by H_n. The exponential map between h_n and H_n is simply the identity (in the sense that $h_n \ni (\mathbf{a}, \mathbf{b}, c) \longrightarrow (\mathbf{a}, \mathbf{b}, c) \in H_n$). Also, the identity element is $(\mathbf{0}, \mathbf{0}, 0)$ and the inverse of $(\mathbf{a}, \mathbf{b}, c)$ is $(-\mathbf{a}, -\mathbf{b}, -c)$.

It is worth noting that

$$\{(\mathbf{0},\mathbf{0},c)|c\in\mathbb{R}\}$$

is the center and commutator subgroup of H_n. Also, Lebesgue measure on $\mathbb{R}^{2n+1}$ is a bi-invariant Haar measure on H_n. Note that we have the association regarding the classical phase space

$$\mathbb{R}^{2n}\cong H_n/\{(\mathbf{0},\mathbf{0},c)|c\in\mathbb{R}\}. \tag{2.1}$$

Thus we can think of H_n over the base space $H_n/\{(\mathbf{0},\mathbf{0},c)|c\in\mathbf{R}\}\cong\mathbb{R}^{2n}$ as a (principal) fiber bundle with structure group isomorphic to the real numbers.

2.1.2 Classical Translations

Suppose that the classical phase space of a system is $\mathbb{R}^{2n}$ and you have a classical Hamiltonian function of the form $H(\mathbf{p},\mathbf{q})=\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q}+c$ for $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ and $c\in\mathbb{R}$. Note that such a Hamiltonian is an element of (or at least can be associated with an element of) the Heisenberg Lie algebra. The corresponding classical equations of motion are

$$\dot{q}_j=\frac{\partial H}{\partial p_j}=a_j \qquad\qquad \dot{p}_j=-\frac{\partial H}{\partial q_j}=-b_j$$

the solution of which are the linear flow on $\mathbb{R}^{2n}$:

$$\phi_t(\mathbf{p},\mathbf{q})=(-t\mathbf{b}+\mathbf{p},t\mathbf{a}+\mathbf{q}).$$

The flow at $t=1$ is translation by $(-\mathbf{b},\mathbf{a})\in\mathbb{R}^{2n}$. Thus the flow of this classical system is translation.

This shows that we can think of translations in phase space as being (classically) generated by the flow associated to the elements of the Heisenberg Lie algebra.

2.1.3 The Definition of (Canonical) Coherent States

If we look at the corresponding quantum problem, with our Hilbert space of states being $L^2(\mathbb{R}^n_x)$, then we are thinking of solving the Schrödinger equation with the quantum Hamiltonian $\hat{H} = \mathbf{a}\cdot\hat{\mathbf{P}} + \mathbf{b}\cdot\hat{\mathbf{Q}} + cI$ which is the result of quantizing the above classical Hamiltonian with the standard rules from canonical quantization:

$$
\begin{aligned}
p_j &\longmapsto \widehat{P}_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j} \\
q_j &\longmapsto \widehat{Q}_j = x_j I \\
1 &\longmapsto I.
\end{aligned}
$$

Thus for the quantum case we look at solving the equation:

$$
i\hbar\frac{\partial u}{\partial t} = (\mathbf{a}\cdot\hat{\mathbf{P}} + \mathbf{b}\cdot\hat{\mathbf{Q}} + cI)u
$$

subject to the initial condition $u(\mathbf{x},t)|_{t=0} = u_0(\mathbf{x})$, where again $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Another, less physical more mathematical, but overall extremely useful way of thinking about what we are doing is that we are associating elements in the Heisenberg Lie algebra, $(\mathbf{a},\mathbf{b},c)$ with skew-Hermitian operators on $L^2(\mathbb{R}^n)$ (or more properly $S(\mathbb{R}^n)$) of the form $-\frac{i}{\hbar}(\mathbf{a}\cdot\hat{\mathbf{P}} + \mathbf{b}\cdot\hat{\mathbf{Q}} + cI)$. We will seek to exponentiate this representation of h_n in order to obtain a unitary representation of the Heisenberg group H_n.

Upon inserting the above forms for the operators involved we have

$$i\hbar\frac{\partial u}{\partial t} = \frac{\hbar}{i}\sum_{j=0}^{n} a_j\frac{\partial u}{\partial x_j} + \sum_{j=0}^{n} b_j x_j u + cu$$

again subject to the initial condition $u(\mathbf{x},t)|_{t=0} = u_0(\mathbf{x})$.

We can solve this equation using the method of characteristics. Let's reexpress the above equation as

$$\frac{\partial u}{\partial t} + \sum_{j=0}^{n} a_j\frac{\partial u}{\partial x_j} = \frac{-i}{\hbar}(\mathbf{b}\cdot\mathbf{x} + c)u.$$

Identifying the left side of the equation with the directional derivative of u along the constant vector field $\langle 1, \mathbf{a}\rangle$ we can think of the above equation as the total derivative of $G(t) \equiv u(\mathbf{x},t)$ along the characteristic curves (lines) $\mathbf{x}(t) \equiv \mathbf{x} + t\mathbf{a}$. Thus, via the chain rule in several variables, along the characteristic curves the Schrodinger equation becomes

$$G'(t) = \frac{-i}{\hbar}(\mathbf{b}\cdot\mathbf{x}(t) + c)u(\mathbf{x},t) = \frac{-i}{\hbar}(\mathbf{b}\cdot\mathbf{x}(t) + c)G(t).$$

This implies that

$$\begin{aligned}
G(t) &= G(0)exp\left(\frac{-i}{\hbar}\int_0^t (\mathbf{b}\cdot(\mathbf{x}+\tau\mathbf{a}) + c)d\tau\right)\\
&= u(\mathbf{x}(0),0)exp\left(\frac{-i}{\hbar}\left(\mathbf{b}\cdot\mathbf{x}t + \frac{1}{2}\mathbf{a}\cdot\mathbf{b}t^2 + ct\right)\right)\\
&= u(\mathbf{x},0)exp\left(\frac{-i}{\hbar}\left(\mathbf{b}\cdot\mathbf{x}t + \frac{1}{2}\mathbf{a}\cdot\mathbf{b}t^2 + ct\right)\right)\\
&= u_0(\mathbf{x})exp\left(\frac{-i}{\hbar}\left(\mathbf{b}\cdot\mathbf{x}t + \frac{1}{2}\mathbf{a}\cdot\mathbf{b}t^2 + ct\right)\right).
\end{aligned}$$

As in the classical case, we will let $t = 1$ and we get that

$$u(\mathbf{x}+\mathbf{a},1) = G(1) = u_0(\mathbf{x})e^{\frac{i}{\hbar}(-\mathbf{b}\cdot\mathbf{x}-\frac{1}{2}\mathbf{a}\cdot\mathbf{b}-c)}$$

Since we wish to have $u(\mathbf{x},1)$ (i.e. the solution to the Schrodinger equation at $t=1$) we make the change of variables $\mathbf{x}\to\mathbf{x}-\mathbf{a}$, and we get

$$u(\mathbf{x},1) = G(1) = u_0(\mathbf{x}-\mathbf{a})e^{\frac{i}{\hbar}(-\mathbf{b}\cdot(\mathbf{x}-\mathbf{a})-\frac{1}{2}\mathbf{a}\cdot\mathbf{b}-c)}$$

The above is telling us that

$$e^{-\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}+cI\right)}f = e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}f = e^{-\frac{i}{\hbar}c}e^{\frac{i}{\hbar}(-\mathbf{b}\cdot(\mathbf{x}-\mathbf{a})-\frac{1}{2}\mathbf{a}\cdot\mathbf{b})}f(\mathbf{x}-\mathbf{a}).$$

Let's make the associations $\mathbf{p}=-\mathbf{b}$, $\mathbf{q}=\mathbf{a}$, and define the family of (unitary) operators

$$\widehat{T}_{(\mathbf{p},\mathbf{q},c)} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \tag{2.2}$$

by

$$\widehat{T}_{(\mathbf{p},\mathbf{q},c)}f = e^{-\frac{i}{\hbar}\left(\mathbf{q}\cdot\widehat{\mathbf{P}}-\mathbf{p}\cdot\widehat{\mathbf{Q}}+cI\right)}f = e^{-\frac{i}{\hbar}c}e^{\frac{i}{\hbar}\frac{\mathbf{p}\cdot\mathbf{q}}{2}}e^{\frac{i}{\hbar}(\mathbf{p}\cdot(\mathbf{x}-\mathbf{q}))}f(\mathbf{x}-\mathbf{q}). \tag{2.3}$$

If we consider the association $\rho_\hbar : H_n \longrightarrow \mathcal{L}(L^2(\mathbb{R}^n))$ such that $\rho_\hbar(\mathbf{p},\mathbf{q},c) = \widehat{T}_{(\mathbf{p},\mathbf{q},c)}$, then this is the unitary representation of H_n we were looking to calculate.

Unfortunately this is not a faithful representation of H_n due to the nature of the contribution from 'c'. (Recall: A group representation in terms of linear operators on a vector space is called *faithful* if each group element is represented by a distinct linear operator.) We see that from our work above that every element of H_n of the form $(\mathbf{a}, \mathbf{b}, c + 2\pi k\hbar)$, with $k \in \mathbb{Z}$, will be represented by the same operator. We can alter our setup a bit in order to get a faithful representation.

If we note that the kernel of $\rho_\hbar$ (i.e. the set of group elements mapped to the identity operator) is

$$\{(\mathbf{0}, \mathbf{0}, 2\pi k\hbar) | k \in \mathbb{Z}\},$$

then it's clear that the elements, (which we will also denote by $(\mathbf{p}, \mathbf{q}, c)$), in the quotient space

$$H_n^{red} \equiv H_n / \{(\mathbf{0}, \mathbf{0}, 2\pi k\hbar) | k \in \mathbb{Z}\}, \tag{2.4}$$

which we will call the reduced Heisenberg group will provide a faithful representation of unitary operators if we regard $\rho_\hbar$ as a representation of H_n^{red}. It is worth noting that we can think of H_n^{red} over the base space of $H_n / \{(\mathbf{0}, \mathbf{0}, c) | c \in \mathbb{R}\}$ as a (principle) fiber bundle with structure group isomorphic to S^1.

Now, to use this construction to define the coherent states. We begin by noting that from the theory of Fourier Integral Operator's, see [13], [28], we know that the operator

$$e^{-\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}} + \mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}$$

moves the microsupport of a state in phase space along the classical flow of the Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{q} + c$ at $t = 1$. Thus if we begin with the state

$$\psi_{(0,0)}(\mathbf{x}) = \frac{1}{(\pi\hbar)^{n/4}} e^{-\frac{|\mathbf{x}|^2}{2\hbar}}$$

which is well known to have microsupport $(0,0)$ (and is as localized at that point as the uncertainty principle will allow) then we can define

$$\begin{aligned}\psi_{(-\mathbf{b},\mathbf{a},c)}(\mathbf{x}) &= e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}\psi_{(0,0)} \\ &= \frac{1}{(\pi\hbar)^{n/4}}e^{-\frac{i}{\hbar}c}e^{\frac{i}{\hbar}\left(-\mathbf{b}\cdot(\mathbf{x}-\mathbf{a})-\frac{1}{2}\mathbf{a}\cdot\mathbf{b}\right)}e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2\hbar}} \\ &= \frac{1}{(\pi\hbar)^{n/4}}e^{-\frac{i}{\hbar}c}e^{\frac{i}{\hbar}\frac{\mathbf{a}\cdot(-\mathbf{b})}{2}}e^{\frac{i}{\hbar}((-\mathbf{b})\cdot(\mathbf{x}-\mathbf{a}))}e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2\hbar}}\end{aligned}$$

(Note: the microsupport of $\psi_{(-\mathbf{b},\mathbf{a},c)}$ is at $\mathbf{p}=-\mathbf{b}$, $\mathbf{q}=\mathbf{a}$.)

Using this we define a family of states $\psi_{(\mathbf{p},\mathbf{q},c)}\in L^2(\mathbb{R}^n)$ by

$$\psi_{(\mathbf{p},\mathbf{q},c)}(\mathbf{x})=\widehat{T}_{(\mathbf{p},\mathbf{q},c)}\psi_{(0,0)}=\frac{1}{(\pi\hbar)^{n/4}}e^{-\frac{i}{\hbar}c}e^{\frac{i}{\hbar}\frac{\mathbf{p}\cdot\mathbf{q}}{2}}e^{\frac{i}{\hbar}(\mathbf{p}\cdot(\mathbf{x}-\mathbf{q}))}e^{-\frac{|\mathbf{x}-\mathbf{q}|^2}{2\hbar}}.$$

The elements in this family are parametrized by the elements of H_n^{red}. If we consider the subset of the above class associated with $c=0$ we arrive at the (canonical) coherent states. Considering these states individually we might disregard the phase associated with 'c', but we will see that if one considers sums of these states (in our case continuous sums in the form of integrals) that this phase can be extremely useful, in fact it will play a central role in our work.

2.2 Bargmann Space

The content of this section is adapted from [18], [6], and [15].

2.2.1 Motivation

The canonical operators which appear in the standard treatment of (nonrelativistic) quantum theory are the phase space "coordinate" operators which are linear, unbounded, and self-adjoint on $L^2(\mathbb{R}^n)$:

$$\widehat{P}_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j} \qquad\qquad \widehat{Q}_j = x_j I.$$

for which as we've seen, we have the communtation relations

$$\left[\widehat{Q}_j, \widehat{Q}_k\right] = 0, \qquad \left[\widehat{P}_j, \widehat{P}_k\right] = 0, \qquad \left[\widehat{Q}_j, \widehat{P}_k\right] = i\hbar\delta_{jk} I.$$

In the study of the harmonic oscillator it becomes convenient to define the so-called creation and annihilation operators (also known as the raising and lowering operators), respectively

$$\widehat{a}_j^* \equiv \frac{1}{\sqrt{2}}\left(\widehat{Q}_j - i\widehat{P}_j\right) \qquad\qquad \widehat{a}_j \equiv \frac{1}{\sqrt{2}}\left(\widehat{Q}_j + i\widehat{P}_j\right).$$

Note, some treatments may scale these definitions to incorporate $\hbar$ or masses appropriate to the system being studied. Using the commutation relations for $\widehat{P}_j$ and $\widehat{Q}_j$ and the linearity of the commutator one can easily compute the following commutation relations for $\widehat{a}_j$ and $\widehat{a}_j^*$:

$$[\widehat{a}_j, \widehat{a}_k] = 0, \qquad \left[\widehat{a}_j^*, \widehat{a}_k^*\right] = 0, \qquad [\widehat{a}_j, \widehat{a}_k^*] = \hbar\delta_{jk} I.$$

In 1928 V.A. Fock made the observation that if one considers the space $\mathcal{H}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$ and the operators

$$\widehat{Z}_j \equiv z_j I \quad \text{(multiplication by } z_j\text{)} \qquad \widehat{Z}_j^* = \hbar\frac{\partial}{\partial z_j},$$

then one arrives at commutation relations

$$\left[\widehat{Z}_j^*, \widehat{Z}_k^*\right] = 0, \qquad \left[\widehat{Z}_j, \widehat{Z}_k\right] = 0, \qquad \left[\widehat{Z}_j^*, \widehat{Z}_k\right] = \hbar\delta_{jk} I.$$

These are identical to the commutation relations for the creation and annihilation operators. (Note that at this point the two operators above cannot be considered adjoints since $\mathcal{H}(\mathbb{C}^n)$ is not a Hilbert space, thus we are just defining the symbols $\widehat{Z}_j$ and $\widehat{Z}_j^*$ and showing how their commutators behave.)

This compelling observation led V. Bargmann to seek out an inner product on $\mathcal{H}(\mathbb{C}^n)$ (or at least a subspace of it) that would make $\widehat{Z}_j$ and $\widehat{Z}_j^*$ adjoints of one another.

2.2.2 Defining Bargmann Space

One can find Bargmann's original work in [6]. In [18] and [15] one can find somewhat more readable treatments of Bargmann's original work. The space that Bargmann discovered is known as Bargmann space. It is also sometimes referred to as Segal-Bargmann space in an acknowledgment of the work of one of Bargmann's contemporaries I. Segal's whose work that was complementary to Bargmann's. Here we will give the definition(s) of Bargmann space and leave it to the next subsection to delve into its properties.

There are two (equivalent) definitions of Bargmann space, $\mathcal{B}$. Here we will call these two definitions $\mathcal{B}_1$, $\mathcal{B}_2$ for clarity.

Definition II.1. The space of functions known as Bargmann space, denoted $\mathcal{B}_1$, is defined as

$$\mathcal{B}_1 \equiv \left\{ g(\mathbf{z}) = f(\mathbf{z})e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \middle| \frac{\partial f}{\partial \overline{\mathbf{z}}} = 0 \text{ and } \int_{\mathbb{C}^n} |g(\mathbf{z})|^2 d\mathbf{z}_R d\mathbf{z}_I < \infty \right\},$$

where $\mathbf{z} \cdot \overline{\mathbf{z}} \equiv z_1^2 + \cdots + z_n^2$.

and/or equivalently

Definition II.2. (Alternative) The space of functions know as Bargmann space, denoted $\mathcal{B}_2$, is defined as

$$\mathcal{B}_2 \equiv \left\{ f(\mathbf{z}) \middle| \frac{\partial f}{\partial \bar{\mathbf{z}}} = 0 \text{ and } \int_{\mathbb{C}^n} |f(\mathbf{z})|^2 e^{-\frac{\mathbf{z}\cdot\bar{\mathbf{z}}}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I < \infty \right\}$$

where $\mathbf{z} \cdot \bar{\mathbf{z}} \equiv z_1^2 + \cdots + z_n^2$.

Where $\mathbf{z}_R \equiv Re(\mathbf{z})$ and similarly $\mathbf{z}_I \equiv Im(\mathbf{z})$. It is easy to see the equivalence of the two definitions. The first definition defines the elements of Bargmann space as functions on $\mathbb{C}^n$ that are a product of a holomorphic function with an exponential weight, the inner-product is the usual L^2 inner-product using Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The second definition is to define the elements of Bargmann space to be holomorphic functions on $\mathbb{C}^n$ that are square integrable with respect to a weighted Lebesgue measure where the weight is exactly the square of the exponential weight of the functions in the first definition, the inner-product is then an L^2 inner-product with respect to the weighted Lebesgue measure. Note that since we are defining the elements of Bargmann space to be holomorphic functions, or holomorphic functions multiplied by a smooth weight function, they are square integrable in the sense of the usual Riemann integral.

Each definition has advantages and disadvantages, the first definition has a more conceptual and theoretical value, mainly because it makes it essentially obvious that Bargmann space is a closed (complete) subspace of the Hilbert space $L^2(\mathbb{R}^{2n})$ and in fact putting the Bargmann weight with the function will be extremely important in our analysis. The second definition has a more practical use in terms of the mechanics of the definition and behavior of operators on the function space. In that when one wants to think of the behavior of $\widehat{Z}_j^*$ for instance, one does not want this operator acting on the weight part of elements of $\mathcal{B}_1$, but rather only the part of elements that is the holomorphic function. It is clear that there exists a unitary mapping between these two representations that consists of

$$\mathcal{B}_1 \ni g(\mathbf{z}) = f(\mathbf{z})e^{-\frac{\mathbf{z}\bar{\mathbf{z}}}{2\hbar}} \longmapsto e^{\frac{\mathbf{z}\bar{\mathbf{z}}}{2\hbar}} g(\mathbf{z}) = e^{\frac{\mathbf{z}\bar{\mathbf{z}}}{2\hbar}} f(\mathbf{z})e^{-\frac{\mathbf{z}\bar{\mathbf{z}}}{2\hbar}} = f(\mathbf{z}) \in \mathcal{B}_2$$

Remark II.3. During the discussion above we noted that when one includes the Bargmann weights in the definition of states in Bargmann space, definition $\mathcal{B}_1$, that this allows us to think of Bargmann space as a subspace of $L^2(\mathbb{C}^n)$. This is technically false, since the elements of Bargmann space are well-defined functions, and the elements of $L^2(\mathbb{C}^n)$ are equivalence classes of measure equivalent functions. Since we'll be making reference to this relationship later in this work I would like to clarify that technically what we mean when we say that $\mathcal{B}_1$ is a subspace of $L^2(\mathbb{C}^n)$ is that there is a subspace of $L^2(\mathbb{C}^n)$ such that each element (which are equivalence classes) can be represented by elements of $\mathcal{B}_1$. With this technicality addressed, we'll freely claim that $\mathcal{B}_1$ space is a subspace of $L^2(\mathbb{C}^n)$.

2.2.3 Properties

Bargmann space (in the context of definition $\mathcal{B}_2$) is an example of a holomorphic function space. We can flush out some very important properties of Bargmann space by framing part of the discussion in terms of abstract holomorphic function spaces. This general discussion of basic properties of holomorphic function spaces is adapted from [18].

To this end, let U be a non-empty open set in $\mathbb{C}^n$. Let $\mathcal{H}(U)$ denote the space of holomorphic (complex analytic) functions on U. Recall that a function of several complex variables, $f : U \longrightarrow \mathbb{C}$ is said to be holomorphic if f is continuous and holomorphic in each variable with the other variables fixed. Let α be a continuous strictly positive function on U.

Definition II.4. Let $\mathcal{H}L^2(U, \alpha)$ denote the space of L^2 holomorphic functions with respect to the weight α, that is,

$$\mathcal{H}L^2(U, \alpha) \equiv \left\{ F \in \mathcal{H}(U) \middle| \int_U |F(\mathbf{z})|^2 \alpha(\mathbf{z}) d\mathbf{z}_R d\mathbf{z}_I < \infty \right\}$$

where $d\mathbf{z}_R d\mathbf{z}_I$ denotes $2n$-dimensional Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Let's proceed with a series of results.

Proposition II.5. *$\forall\ \mathbf{z} \in U$, there exists a (real) constant $C_{\mathbf{z}}$ such that*

$$|F(\mathbf{z})| \leq C_{\mathbf{z}} \| \ F \ \|^2_{L^2(U,\alpha)}$$

$\forall\ F \in \mathcal{H}L^2(U, \alpha)$.

Proof. Define the "poly-disk"

$$P_r(\mathbf{z}) = \left\{ \mathbf{v} \in \mathbb{C}^n \middle| |v_k - z_k| < r, k = 1, \ldots, n \right\}.$$

If $\mathbf{z} \in U$, choose r small enough so that $\overline{P_R(\mathbf{z})} \subset U$. Then we will show that

$$F(\mathbf{z}) = \frac{1}{(\pi r^2)^n} \int_{P_r(\mathbf{z})} F(\mathbf{v}) d\mathbf{v}_R d\mathbf{v}_I.$$

To verify this claim let's first consider the case of $n = 1$. In this case we can note that F is a holomorphic function of a single complex variable, $P_r(z)$ is a disk of radius r, and standard complex function theory tells us we can expand F in a Taylor series centered at $v = z$, and so

$$F(v) = F(z) + \sum_{j=1}^{\infty} a_j (v - z)^j.$$

The series will converge uniformly to F on the compact set $\overline{P_r(z)} \subset U$. This justifies the following

$$\begin{aligned}
\frac{1}{(\pi r^2)} \int_{P_r(z)} F(v) dv_R dv_I &= \frac{1}{(\pi r^2)} \int_{P_r(z)} \left(F(z) + \sum_{j=1}^{\infty} a_j (v-z)^j \right) dv_R dv_I \\
&= \frac{1}{(\pi r^2)} \int_{P_r(z)} F(z) dv_R dv_I + \sum_{j=1}^{\infty} \frac{a_j}{(\pi r^2)} \int_{P_r(z)} (v-z)^j dv_R dv_I \\
&= \frac{1}{(\pi r^2)} F(z)(\pi r^2) + \sum_{j=1}^{\infty} \frac{a_j}{(\pi r^2)} \int_0^r \int_0^{2\pi} s^j e^{ij\theta} ds d\theta \\
&= F(z)
\end{aligned}$$

where in the second to last step we converted the integrals to polar coordinates, and we see that the angular integrals will give a result of zero for each term in the sum.

For the case $n > 1$ we simply perform the integral one complex variable at a time. By the $n = 1$ case, when we do, the v_k integral, this has the effect of setting $v_k = z_k$. After performing all n integrals we will arrive at the claim.

We can reexpress our result as follows

$$\begin{aligned}
F(\mathbf{z}) &= \frac{1}{(\pi r^2)^n} \int_{P_r(\mathbf{z})} F(\mathbf{v}) d\mathbf{v}_R d\mathbf{v}_I \\
&= \frac{1}{(\pi r^2)^n} \int_U \chi_{P_r(\mathbf{z})}(\mathbf{v}) \frac{1}{\alpha(\mathbf{v})} F(\mathbf{v}) \alpha(\mathbf{v}) d\mathbf{v}_R d\mathbf{v}_I \\
&= \frac{1}{(\pi r^2)^n} \left\langle \chi_{P_r(\mathbf{z})} \frac{1}{\alpha}, F \right\rangle_{L^2(U,\alpha)}
\end{aligned}$$

where χ_{P_r} is the characteristic function on P_r. By the Cauchy-Schwarz inequality we can then say

$$|F(\mathbf{z})|^2 \leq \frac{1}{(\pi r^2)^{2n}} \left\| \chi_{P_r(\mathbf{z})} \frac{1}{\alpha} \right\|^2_{L^2(U,\alpha)} \| F \|^2_{L^2(U,\alpha)}.$$

Noting that because $\overline{P_R(\mathbf{z})} \subset U$ and α is positive and continuous, $1/\alpha$ is bounded on $P_R(\mathbf{z})$, thus the first norm in the above equation is finite. Letting

$$C_{\mathbf{z}} \equiv \frac{1}{(\pi r^2)^{2n}} \left|\left| \chi_{P_r(\mathbf{z})} \frac{1}{\alpha} \right|\right|^2_{L^2(U,\alpha)}$$

which is independent of F we have our result. It's worth noticing that we may choose any value of r we like as long as long as $\overline{P_R(\mathbf{z})} \subset U$. □

What this (previous) proposition tells us is that pointwise evaluation is a linear functional on $\mathcal{H}L^2(U,\alpha)$.

Proposition II.6. *$\mathcal{H}L^2(U,\alpha)$ is a closed subspace of $L^2(U,\alpha)$, and is therefore a Hilbert space.*

Proof. From the proof of the previous proposition we see that for a given $\mathbf{z} \in U$ we can find a neighborhood V of $\mathbf{z}$ and a constant $\widetilde{C}_{\mathbf{z}}$ such that

$$|F(\mathbf{z})|^2 \leq \widetilde{C}_{\mathbf{z}} \| F \|^2_{L^2(U,\alpha)}$$

for all $\mathbf{v} \in V$ and for all $F \in \mathcal{H}L^2(U,\alpha)$. Suppose we have a sequence $\{F_k\}_{k=1}^{\infty}$ such that $F_k \in \mathcal{H}L^2(U,\alpha) \subset L^2(U,\alpha)$ and $\widetilde{F} \in L^2(U,\alpha)$ such that $F_k \longrightarrow \widetilde{F}$ in $L^2(U,\alpha)$. Now, $\{F_k\}_{k=1}^{\infty}$ being convergent implies that it is a Cauchy sequence in $L^2(U,\alpha)$. Then,

$$sup_{\mathbf{v}\in V} \left| F_n(\mathbf{v}) - F_m(\mathbf{v}) \right| \leq \sqrt{\widetilde{C}_{\mathbf{z}}} \left|\left| F_n - F_m \right|\right|_{L^2(U,\alpha)} \longrightarrow 0$$

as $n, m \longrightarrow \infty$. This tells us the for every $\mathbf{v} \in V$ we have that the sequence of complex numbers $\{F_k(\mathbf{v})\}_{k=1}^{\infty}$ is a Cauchy sequence. But we have more than this, since the above is really telling us that the sequence is uniformly Cauchy on V. A result from basic real analysis gives us that every uniformly Cauchy sequence has a limit function to which it converges to uniformly, call this function $\widetilde{G}$. We have that the sequence $\{F_k\}_{k=1}^{\infty}$ is both convergent in $L^2(U, \alpha)$ to $\widetilde{F}$ and it is uniformly convergent to $\widetilde{G}$. From standard measure theory we know that if a sequence of functions has an L^2 limit and a pointwise limit, then these limits must be equal almost everywhere, so up to a set of measure zero we have $\widetilde{G} = \widetilde{F}$. Furthermore we can use Morera's theorem from complex analysis (in each variable individually) to show that a locally uniform limit of holomorphic functions is holomorphic. This gives us that $\widetilde{F} \in \mathcal{H}L^2(U, \alpha)$, and so $\mathcal{H}L^2(U, \alpha)$ is a closed subspace of $L^2(U, \alpha)$ and so is a Hilbert space. □

Proposition II.7. *(Reproducing Kernel) Let* $\mathcal{H}L^2(U, \alpha)$ *be defined as above. Then there exists a function* $K(\mathbf{z}, \mathbf{w})$, $\mathbf{z}, \mathbf{w} \in U$, *with the following properties:*

1. $K(\mathbf{z}, \mathbf{w})$ *is holomorphic in* $\mathbf{z}$ *and anti-holomorphic in* $\mathbf{w}$ *and satisfies*

$$K(\mathbf{w}, \mathbf{z}) = \overline{K(\mathbf{z}, \mathbf{w})}$$

2. *For each fixed* $\mathbf{z} \in U$, $K(\mathbf{z}, \mathbf{w})$ *is square-integrable* $d\alpha(\mathbf{w})$. $\forall\ f \in \mathcal{H}L^2(U, \alpha)$

$$F(\mathbf{z}) = \int_U K(\mathbf{z}, \mathbf{w}) F(\mathbf{w}) \alpha(\mathbf{w}) d\mathbf{w}_R d\mathbf{w}_I.$$

3. *If $F \in L^2(U,\alpha)$, let PF denote the orthogonal projection of F onto the closed subspace $\mathcal{H}L^2(U,\alpha)$. Then*

$$PF(\mathbf{z}) = \int_U K(\mathbf{z},\mathbf{w})F(\mathbf{w})\alpha(\mathbf{w})d\mathbf{w}_R d\mathbf{w}_I.$$

4. $\forall\ \mathbf{z},\mathbf{v} \in U$

$$\int_U K(\mathbf{z},\mathbf{w})K(\mathbf{w},\mathbf{u})\alpha(\mathbf{w})d\mathbf{w}_R d\mathbf{w}_I = K(\mathbf{z},\mathbf{v}).$$

5. $\forall\ \mathbf{z} \in U$

$$|F(\mathbf{z})|^2 \leq K(\mathbf{z},\mathbf{z})\|\ F\ \|^2,$$

and the constant $K(\mathbf{z},\mathbf{z})$ is optimal in the sense that $\forall\ \mathbf{z} \in U$ there exists a non-zero $F_{\mathbf{z}} \in \mathcal{H}L^2(U,\alpha)$ for which equality holds.

6. *Given any $\mathbf{z} \in U$, if $\mu_{\mathbf{z}}(\cdot) \in \mathcal{H}L^2(U,\alpha)$ satisfies*

$$F(\mathbf{z}) = \int_U \overline{\mu_{\mathbf{z}}(\mathbf{w})}F(\mathbf{w})\alpha(\mathbf{w})d\mathbf{w}_R d\mathbf{w}_I$$

$\forall\ F \in \mathcal{H}L^2(U,\alpha)$, *then* $\overline{\mu_{\mathbf{z}}(\mathbf{w})} = K(\mathbf{z},\mathbf{w})$

Proof. In a previous proposition we showed that $\forall\ \mathbf{z} \in U$ that evaluation at $\mathbf{z}$ is a continuous linear functional on $\mathcal{H}L^2(U,\alpha)$. By the Riesz Representation Theorem, this linear functional can be represented uniquely as the inner product with some element in $\mathcal{H}L^2(U,\alpha)$. Let's denote this element by $\phi_{\mathbf{z}}(\mathbf{w}) \in \mathcal{H}L^2(U,\alpha)$. That is

$$F(\mathbf{z}) = \langle \phi_{\mathbf{z}}, F\rangle_{L^2(U,\alpha)} = \int_U \overline{\phi_{\mathbf{z}}(\mathbf{w})}F(\mathbf{w})\alpha(\mathbf{w})d\mathbf{w}_R d\mathbf{w}_I$$

So we set $K(\mathbf{z}, \mathbf{w}) \equiv \overline{\phi_{\mathbf{z}}(\mathbf{w})}$. (We adopt the physicists convention that inner-products are linear in the second entry, and conjugate linear in the first.) By construction we see that $K(\mathbf{z}, \mathbf{w})$ satisfies (2) of the proposition, and also since $\phi_{\mathbf{z}}(\mathbf{w})$ is a holomorphic function of $\mathbf{w}$, then $\overline{\phi_{\mathbf{z}}(\mathbf{w})}$ is anti-holomorphic in $\mathbf{w}$. To see that it satisfies (1) note that by it's definition we have the following

$$\begin{aligned}
\phi_{\mathbf{z}}(\mathbf{w}) &= \langle \phi_{\mathbf{w}}, \phi_{\mathbf{z}} \rangle_{L^2(U,\alpha)} \\
&= \overline{\langle \phi_{\mathbf{z}}, \phi_{\mathbf{w}} \rangle}_{L^2(U,\alpha)} \\
&= \overline{\phi_{\mathbf{w}}(\mathbf{z})},
\end{aligned}$$

so, $\overline{K(\mathbf{z}, \mathbf{w})} \equiv \overline{\overline{\phi_{\mathbf{z}}(\mathbf{w})}} = \overline{\phi_{\mathbf{w}}(\mathbf{z})} \equiv K(\mathbf{w}, \mathbf{z})$. Note that since also gives that $K(\mathbf{z}, \mathbf{w})$ is holomorphic in $\mathbf{z}$.

For (3) we consider two cases, noting that since $\mathcal{H}L^2(U, \alpha)$ is a closed subspace of $L^2(U, \alpha)$ we have $L^2(U, \alpha) = \mathcal{H}L^2(U, \alpha) \oplus \left(\mathcal{H}L^2(U, \alpha)\right)^{\perp}$. If $F \in \mathcal{H}L^2(U, \alpha)$ then (3) is saying the same thing as (2) since $PF = F$ and is therefore proven. If $F \in \left(\mathcal{H}L^2(U, \alpha)\right)^{\perp}$, then since $\phi_{\mathbf{w}}(\mathbf{z}) \in \mathcal{H}L^2(U, \alpha)$ we have that

$$\int_U K(\mathbf{z}, \mathbf{w}) F(\mathbf{w}) \alpha(\mathbf{w}) d\mathbf{w}_R d\mathbf{w}_I = \langle \phi_{\mathbf{w}}, F \rangle_{L^2(U,\alpha)} = 0.$$

So we see that the right side of the expression in (3) is equal to F if $F \in \mathcal{H}L^2(U, \alpha)$, and is equal to 0 if $F \in \left(\mathcal{H}L^2(U, \alpha)\right)^{\perp}$, and so it is exactly the orthogonal projector P onto $\mathcal{H}L^2(U, \alpha)$.

For (4) we note that this is simply (2) with $F(\mathbf{w}) = K(\mathbf{w}, \mathbf{v}) = \overline{\phi_{\mathbf{w}}(\mathbf{v})} = \phi_{\mathbf{v}}(\mathbf{w})$.

For (5) we should recall that for every $\mathbf{z} \in U$ the functional on $\mathcal{H}L^2(U, \alpha)$ that is evaluation at $\mathbf{z}$ is the inner-product with $\phi_{\mathbf{z}}$. Thus, we have from (2) via the Cauchy-Schwarz inequality

$$\begin{aligned}|F(\mathbf{z})|^2 &= |\langle \phi_{\mathbf{z}}, F\rangle_{L^2(U,\alpha)}|^2 \\ &\leq \| \phi_{\mathbf{z}} \|^2_{L^2(U,\alpha)} \cdot \| F \|^2_{L^2(U,\alpha)}.\end{aligned}$$

And finally, we have that

$$\| \phi_{\mathbf{z}} \|^2_{L^2(U,\alpha)} = \langle \phi_{\mathbf{z}}, \phi_{\mathbf{z}}\rangle_{L^2(U,\alpha)} = \phi_{\mathbf{z}}(\mathbf{z}) = K(\mathbf{z},\mathbf{z}).$$

The final thing to note is that for $F = \phi_{\mathbf{z}}$ one will get strict equality. Indeed,

$$\begin{aligned}|\phi_{\mathbf{z}}(\mathbf{z})|^2 &= |\langle \phi_{\mathbf{z}}, \phi_{\mathbf{z}}\rangle_{L^2(U,\alpha)^2}|^2 \\ &= |\langle \phi_{\mathbf{z}}, \phi_{\mathbf{z}}\rangle_{L^2(U,\alpha)^2}| \cdot |\langle \phi_{\mathbf{z}}, \phi_{\mathbf{z}}\rangle_{L^2(U,\alpha)^2}| \\ &= K(\mathbf{z},\mathbf{z})|\langle \phi_{\mathbf{z}}, \phi_{\mathbf{z}}\rangle_{L^2(U,\alpha)^2}| \\ &= K(\mathbf{z},\mathbf{z})\| \phi_{\mathbf{z}} \|^2_{L^2(U,\alpha)^2}\end{aligned}$$

Finally, for (6) note that if $\mu_{\mathbf{z}}(\cdot) \in \mathcal{H}L^2(U,\alpha)$ and satisfies $F(\mathbf{z}) = \langle \mu_{\mathbf{z}}, F\rangle_{L^2(U,\alpha)^2}$ then from (2) we know that

$$\langle \mu_{\mathbf{z}}, F\rangle_{L^2(U,\alpha)^2} = F(\mathbf{z}) = \langle \overline{K(\mathbf{z},\cdot)}, F\rangle_{L^2(U,\alpha)^2}.$$

Thus we have that for all $F \in \mathcal{H}L^2(U,\alpha)$,

$$\langle \mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)}, F\rangle_{L^2(U,\alpha)^2} = 0.$$

Since both $\overline{K(\mathbf{z},\cdot)}$ and $\mu_{\mathbf{z}}$ are element of $\mathcal{H}L^2(U,\alpha)$, we make take $F = \mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)}$. Thus

$$\langle \mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)}, \mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)} \rangle_{L^2(U,\alpha)^2} = \| \ \mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)} \ \|^2_{L^2(U,\alpha)^2} = 0,$$

so $\mu_{\mathbf{z}} - \overline{K(\mathbf{z},\cdot)} = 0$, or $\mu_{\mathbf{z}}(\mathbf{w}) = \overline{K(\mathbf{z},\mathbf{w})}$. □

Essentially, this proposition is flushing out the details that follow from the fact that pointwise evaluation is a continuous linear functional together with the Riesz Representation Theorem. We will see that the existence of a reproducing kernel in Bargmann space is fundamental to our work. The following theorem gives us a means of calculating the reproducing kernel.

Proposition II.8. *Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}L^2(U,\alpha)$. Then, for all $\mathbf{z},\mathbf{w} \in U$*

$$\sum_j |e_j(\mathbf{z})\overline{e_j(\mathbf{w})}| < \infty,$$

and

$$K(\mathbf{z},\mathbf{w}) = \sum_j e_j(\mathbf{z})\overline{e_j(\mathbf{w})}.$$

Proof. We begin the proof addressing issues of convergence. Consider for any $f \in \mathcal{H}L^2(U,\alpha)$ that Parseval's Theorem says that

$$\sum_j |\langle f, e_j \rangle_{L^2(U,\alpha)}|^2 = \| \ f \ \|^2_{L^2(U,\alpha)}$$

Now, for any $f, g \in \mathcal{H}L^2(U,\alpha)$ let's consider the Schwarz inequality in the Hilbert space l^2 of square-summable sequences, applied to the sequences $\{|\langle f, e_j \rangle_{L^2(U,\alpha)}|\}$ and $\{|\langle g, e_j \rangle_{L^2(U,\alpha)}|\}$:

$$\begin{aligned}
\langle |\langle f, e_j\rangle_{L^2(U,\alpha)}|, |\langle g, e_j\rangle_{L^2(U,\alpha)}|\rangle_{l^2} &= \sum_j |\langle f, e_j\rangle_{L^2(U,\alpha)}| \cdot \overline{|\langle g, e_j\rangle_{L^2(U,\alpha)}|} \\
&= \sum_j |\langle f, e_j\rangle_{L^2(U,\alpha)}| \cdot \overline{|\langle e_j, g\rangle_{L^2(U,\alpha)}|} \\
&= \sum_j |\langle f, e_j\rangle_{L^2(U,\alpha)} \cdot \langle e_j, g\rangle_{L^2(U,\alpha)}| \\
&\leq \| \; f \; \|_{L^2(U,\alpha)} \| \; g \; \|_{L^2(U,\alpha)}.
\end{aligned}$$

Specializing this result to the case $f = \phi_{\mathbf{z}}$ and $g = \phi_{\mathbf{w}}$ we get

$$\sum_j |e_j(\mathbf{z})\overline{e_j(\mathbf{w})}| \leq \| \; \phi_{\mathbf{z}} \; \|_{L^2(U,\alpha)} \| \; \phi_{\mathbf{w}} \; \|_{L^2(U,\alpha)} < \infty.$$

So the sum is absolutely convergent for each $\mathbf{z}$ and $\mathbf{w}$.

Now, let's think of the partial sums (inherently assuming j is a countable index, and therefore that $\mathcal{H}L^2(U,\alpha)$ is separable) as functions of $\mathbf{w}$ with $\mathbf{z}$ fixed. Thus, for each N define

$$\Upsilon^N_{\mathbf{z}}(\mathbf{w}) \equiv \sum_{j=1}^{N} e_j(\mathbf{z})\overline{e_j(\mathbf{w})} \in L^2(U_{\mathbf{w}}, \alpha)$$

Then, the orthogonality of the e_j's gives us for N, M that

$$
\begin{aligned}
\left\langle \Upsilon_{\mathbf{z}}^{N}, \Upsilon_{\mathbf{z}}^{M} \right\rangle_{L^2(U,\alpha)} &= \left\langle \sum_{j=1}^{N} e_j(\mathbf{z})\overline{e_j(\cdot)}, \sum_{k=1}^{M} e_k(\mathbf{z})\overline{e_k(\cdot)} \right\rangle_{L^2(U_{\mathbf{w}},\alpha)} \\
&= \sum_{j=1}^{N}\sum_{k=1}^{M} \left\langle e_j(\mathbf{z})\overline{e_j(\cdot)}, e_k(\mathbf{z})\overline{e_k(\cdot)} \right\rangle_{L^2(U_{\mathbf{w}},\alpha)} \\
&= \sum_{j=1}^{min\{N,M\}} \left\langle e_j(\mathbf{z})\overline{e_j(\cdot)}, e_j(\mathbf{z})\overline{e_j(\cdot)} \right\rangle_{L^2(U_{\mathbf{w}},\alpha)} \\
&= \sum_{j=1}^{min\{N,M\}} \parallel e_j(\mathbf{z})\overline{e_j(\cdot)} \parallel_{L^2(U_{\mathbf{w}},\alpha)}^2 \\
&= \sum_{j=1}^{min\{N,M\}} \parallel \langle \phi_{\mathbf{z}}, e_j \rangle_{L^2(U,\alpha)} \overline{e_j} \parallel_{L^2(U,\alpha)}^2 \\
&= \sum_{j=1}^{min\{N,M\}} \left| \langle \phi_{\mathbf{z}}, e_j \rangle_{L^2(U,\alpha)} \right|^2 \parallel e_j \parallel_{L^2(U,\alpha)}^2 \\
&= \sum_{j=1}^{min\{N,M\}} \left| \langle \phi_{\mathbf{z}}, e_j \rangle_{L^2(U,\alpha)} \right|^2 \\
&\leq \sum_{j=1} \left| \langle \phi_{\mathbf{z}}, e_j \rangle_{L^2(U,\alpha)} \right|^2 \\
&= \parallel \phi_{\mathbf{z}} \parallel_{L^2(U,\alpha)}^2 .
\end{aligned}
$$

Where we used the fact that $\parallel e_j \parallel_{L^2(U,\alpha)}^2 = 1$, and Parseval's Theorem in the last step. We are free to let $N, M \longrightarrow \infty$ and our bound is intact, thus we see that the series

$$
\sum_j e_j(\mathbf{z})\overline{e_j(\mathbf{w})},
$$

is norm convergent in $L^2(U, \alpha)$ as a function of $\mathbf{w}$ for a fixed value of $\mathbf{z}$. We find ourselves in the position (once again) of having a series that is both pointwise and norm convergent. We know, from a standard result in measure theory, that in this case we have that these two limits are equal almost everywhere. Call this limit (in the sense of equivalence classes of functions which agree almost everywhere) $\Upsilon_{\mathbf{z}}(\mathbf{w})$, then we have that this is the limit of the convergent (and therefore Cauchy) sequence of functions $\Upsilon_{\mathbf{z}}^N$. We can then use the estimate on pointwise evaluation proven in our first proposition (and a part of the proof) to argue that the convergence is not simply pointwise, but locally uniform. From their definition we have that the $\Upsilon_{\mathbf{z}}^N$'s are antiholomorphic in $\mathbf{w}$ and thus Morera's theorem will give us that their uniform limit, $\Upsilon_{\mathbf{z}}(\mathbf{w})$, will also be antiholomorphic in $\mathbf{w}$ (for each fixed $\mathbf{z}$). Going through this exact argument with the roles of $\mathbf{z}$ and $\mathbf{w}$ reversed we get that our our sum is holomorphic as a function of $\mathbf{z}$ with $\mathbf{w}$ fixed. To summarize we've shown that the sum

$$\sum_j e_j(\mathbf{z})\overline{e_j(\mathbf{w})},$$

is a holomorphic function of $\mathbf{z}$ and an antiholomorphic function of $\mathbf{w}$.

With convergence issues settled we can safely consider for any $F \in \mathcal{H}L^2(U, \alpha)$, employing several of the results in our Reproducing Kernel proposition we have

$$\begin{aligned}
F(\mathbf{z}) &= \langle \phi_{\mathbf{z}}, F \rangle_{L^2(U,\alpha)} \\
&= \sum_j \langle \phi_{\mathbf{z}}, e_j \rangle_{L^2(U,\alpha)} \langle e_j, F \rangle_{L^2(U,\alpha)} \\
&= \sum_j e_j(\mathbf{z}) \int_U \overline{e_j(\mathbf{w})} F(\mathbf{w}) \alpha(\mathbf{w}) d\mathbf{w}_R d\mathbf{w}_I \\
&= \int_U \left(\sum_j e_j(\mathbf{z}) \overline{e_j(\mathbf{w})} \right) F(\mathbf{w}) \alpha(\mathbf{w}) d\mathbf{w}_R d\mathbf{w}_I.
\end{aligned}$$

One can only conclude from this that the quantity in the parentheses in the last line must be $K(\mathbf{z}, \mathbf{w})$. $\square$

One can admire the theoretical value of this proposition, and simultaneously criticize the practical utility of the result. In general one must accept that it is not always possible to find an explicit orthonormal basis of a Hilbert space (especially a function space where we can expect that the dimension is often infinite). And even if we have such a basis we likely couldn't compute the sum. But, we will see that in the case of Bargmann space we have such a basis, and that we will be able to use it to compute the sum and arrive at the reproducing kernel.

In light of our general discussion of holomorphic function spaces we can refine our definition, $\mathcal{B}_2$, of Bargmann space

Definition II.9. The Bargmann space(s) are the holomorphic function spaces $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ where

$$\alpha_\hbar(\mathbf{z}) = \frac{1}{(\pi\hbar)^n} e^{-\frac{|\mathbf{z}|^2}{\hbar}}.$$

To the end of computing the reproducing kernel of Bargmann space we break down the job into a series of steps.

Let's begin with $n = 1$, and we'll begin by showing that $\{z^j\}_{j=0}^{\infty}$ is a basis for Bargmann space.

Proposition II.10. *The set of monomials $\{z^j\}_{j=0}^{\infty}$ is an orthogonal basis for $\mathcal{H}L^2(\mathbb{C}, \alpha_\hbar)$.*

Proof. Let's first check orthogonality. Here and in what follows we make extensive use of reexpressing integrals over $\mathbb{C}$ in polar coordinates (letting $z = re^{i\theta}$):

$$\begin{aligned}\langle z^k, z^j\rangle_{L^2(\mathbb{C},\alpha_\hbar)} &= \frac{1}{(\pi\hbar)}\int_{\mathbb{C}} \overline{z^k} z^j e^{-\frac{|z|^2}{\hbar}} dz_R dz_I \\ &= \frac{1}{(\pi\hbar)}\int_0^{2\pi}\int_0^{\infty} r^k e^{-ik\theta} r^j e^{ij\theta} e^{-\frac{r^2}{\hbar}} r dr d\theta \\ &= \frac{1}{(\pi\hbar)}\left(\int_0^{\infty} r^{k+j+1} e^{-\frac{r^2}{\hbar}} dr\right)\left(\int_0^{2\pi} e^{i(j-k)\theta} d\theta\right).\end{aligned}$$

Since $j,k \in \mathbb{N}\bigcup\{0\}$ the integral with respect to θ in the last line will be zero unless $j = k$. We have orthogonality.

Now we must check that in fact our states form a basis. From basic functional analysis we know that it suffices to show that if $F \in \mathcal{H}L^2(\mathbb{C},\alpha_\hbar)$ and $\langle z^n, F\rangle_{L^2(\mathbb{C},\alpha_\hbar)} = 0$ for all n, then $F = 0$. So, suppose $F \in \mathcal{H}L^2(\mathbb{C},\alpha_\hbar)$ and that $\langle z^n, F\rangle_{L^2(\mathbb{C},\alpha_\hbar)} = 0$ for all n. Since F is a holomorphic function of a single complex variable we can expand F in a power series

$$F(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Note, that we know that this series will converge uniformly on each compact subset of $\mathbb{C}$, specifically it will converge uniformly on any closed disk centered at the origin. Then,

$$
\begin{aligned}
\langle z^k, F\rangle_{L^2(\mathbb{C},\alpha_\hbar)} &= \int_{\mathbf{C}} \overline{z^k} F(z) e^{-\frac{|z|^2}{\hbar}} dz_R dz_I \\
&= \int_0^\infty \int_0^{2\pi} r^k e^{-ik\theta} F(re^{i\theta}) e^{-\frac{r^2}{\hbar}} r dr d\theta \\
&= \lim_{R\longrightarrow\infty} \int_0^R \int_0^{2\pi} r^{k+1} e^{-ik\theta} F(re^{i\theta}) e^{-\frac{r^2}{\hbar}} dr d\theta \\
&= \lim_{R\longrightarrow\infty} \int_0^R \int_0^{2\pi} r^{k+1} e^{-ik\theta} \left(\sum_{j=0}^\infty a_j (re^{i\theta})^j \right) e^{-\frac{r^2}{\hbar}} dr d\theta \\
&= \lim_{R\longrightarrow\infty} \sum_{j=0}^\infty \int_0^R \int_0^{2\pi} a_j r^{k+j+1} e^{i(j-k)\theta} e^{-\frac{r^2}{\hbar}} dr d\theta \\
&= \lim_{R\longrightarrow\infty} \sum_{j=0}^\infty a_j \left(\int_0^R r^{k+j+1} e^{-\frac{r^2}{\hbar}} dr \right) \left(\int_0^{2\pi} e^{i(j-k)\theta} d\theta \right) \\
&= 2\pi \lim_{R\longrightarrow\infty} \sum_{j=0}^\infty a_j \left(\int_0^R r^{2j+1} e^{-\frac{r^2}{\hbar}} dr \right)
\end{aligned}
$$

It is clear from this that the only way that $\langle z^k, F\rangle_{L^2(\mathbb{C},\alpha_\hbar)} = 0$ is if $a_k = 0$. Thus if $\langle z^k, F\rangle_{L^2(\mathbb{C},\alpha_\hbar)} = 0$ for all values of k, then $a_k = 0$ for all values of k, which would in turn imply that

$$
F(z) = \sum_{j=0}^\infty a_j z^j = 0. \qquad \square
$$

Now that we have established an orthogonal basis, we'll normalize this basis.

We proceed by induction on j. For $j = 0$ we note that

$$
\begin{aligned}
\frac{1}{(\pi\hbar)}\int_{\mathbb{C}}|1|^2 e^{-\frac{|\mathbf{z}|^2}{\hbar}}dz_R dz_I &= \frac{1}{(\pi\hbar)}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{\hbar}} r dr d\theta \\
&= \frac{2\pi}{(\pi\hbar)}\left(-\frac{\hbar}{2}\right)\lim_{R\longrightarrow\infty} e^{-\frac{r^2}{\hbar}}\Big|_0^R \\
&= -\lim_{R\longrightarrow\infty}\left[e^{-\frac{R^2}{\hbar}}-1\right] \\
&= 1.
\end{aligned}
$$

Now, for $j > 1$ we have

$$
\begin{aligned}
\| \, z^j \, \|^2_{L^2(\mathbb{C},\alpha_\hbar)} &= \frac{1}{(\pi\hbar)}\int_{\mathbb{C}}|z^j|^2 e^{-\frac{|z|^2}{\hbar}}dz_R dz_I \\
&= \frac{1}{(\pi\hbar)}\int_0^{2\pi}\int_0^{\infty} r^{2j+1}e^{-\frac{r^2}{\hbar}}dr d\theta \\
&= \frac{2}{\hbar}\int_0^{\infty} r^{2j}\left(e^{-\frac{r^2}{\hbar}}r\right)dr \\
&= \frac{2}{\hbar}\int_0^{\infty} r^{2j}\frac{d}{dr}\left(-\frac{\hbar}{2}e^{-\frac{r^2}{\hbar}}\right)dr \\
&= -\frac{2}{\hbar}\int_0^{\infty} 2jr^{2j-1}\left(-\frac{\hbar}{2}e^{-\frac{r^2}{\hbar}}\right)dr \\
&= (j\hbar)\frac{2}{\hbar}\int_0^{\infty} r^{2(j-1)+1}e^{-\frac{r^2}{\hbar}}dr \\
&= j\hbar\| \, z^{j-1} \, \|^2_{L^2(\mathbb{C},\alpha_\hbar)}.
\end{aligned}
$$

Combining this with our result for $j = 0$, we have that $\| \, z^j \, \|^2_{L^2(\mathbb{C},\alpha_\hbar)} = j!\hbar^j$, thus our orthonormal basis for $\mathcal{H}L^2(\mathbb{C},\alpha_\hbar)$ is

$$
\left\{\frac{z^j}{\sqrt{j!\hbar^j}}\right\}_{j=0}^{\infty}.
$$

Finally, for the $n = 1$ case we compute the Reproducing Kernel:

$$
K(z,w) = \sum_{j=0}^{\infty}\frac{z^j}{\sqrt{j!\hbar^j}}\overline{\frac{w^j}{\sqrt{j!\hbar^j}}} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{z\overline{w}}{\hbar}\right) = e^{\frac{z\overline{w}}{\hbar}}.
$$

With the $n = 1$ case completely under our belt we move on to $n > 1$.

Proposition II.11. *The set of monomials*

$$\left\{ \mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \middle| \alpha = (\alpha_1, \cdots, \alpha_n), \alpha_j \in \mathbb{N} \cup \{0\} \right\},$$

is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$.

Proof. As in the $n = 1$ case we'll check orthogonality first. Let α, β be multi-indices

$$\begin{aligned}
\langle \mathbf{z}^\alpha, \mathbf{z}^\beta \rangle_{L^2(\mathbb{C}^n, \alpha_\hbar)} &= \frac{1}{(\pi\hbar)^n} \int_{\mathbb{C}^n} \overline{\mathbf{z}^\alpha} \mathbf{z}^\beta e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I \\
&= \prod_{k=1}^n \frac{1}{(\pi\hbar)} \int_{\mathbb{C}} \overline{z^{\alpha_k}} z^{\beta_k} e^{-\frac{|z_k|^2}{\hbar}} dz_{kR} dz_{kI} \\
&= \prod_{k=1}^n \frac{1}{(\pi\hbar)} \int_0^{2\pi} \int_0^\infty r^{\alpha_k} e^{-i\alpha_k\theta} r^{\beta_k} e^{i\beta_k\theta} e^{-\frac{r^2}{\hbar}} r dr d\theta \\
&= \prod_{k=1}^n \frac{1}{(\pi\hbar)} \left(\int_0^\infty r^{\alpha_k + \beta_k + 1} e^{-\frac{r^2}{\hbar}} dr \right) \left(\int_0^{2\pi} e^{i(\beta_k - \alpha_k)\theta} d\theta \right),
\end{aligned}$$

and the θ integral will be zero unless $\beta_k = \alpha_k$ for each k. Since we have a product of these integrals we have that $\langle \mathbf{z}^\alpha, \mathbf{z}^\beta \rangle_{L^2(\mathbb{C}^n, \alpha_\hbar)} = 0$ unless $\alpha = \beta$.

To see that these states constitute a basis we will show that for $F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$, if $\langle \mathbf{z}^\beta, F \rangle_{L^2(\mathbb{C}^n, \alpha_\hbar)} = 0$ for all multi-indices β, then $F = 0$.

To this end let's note that for any $F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ we know that F is a holomorphic function of each of $\mathbf{z} = (z_1, \ldots, z_n)$ which means that it's a holomorphic function of each z_j with each of the others held constant. Similar to the $n = 1$ case, for holomorphic functions of a complex variable we have a multi-variable Taylor expansion

$$F(\mathbf{z}) = \sum_{|\alpha|=0}^{\infty} a_\alpha \mathbf{z}^\alpha$$

where the series converges absolutely for all $\mathbf{z} \in \mathbb{C}^n$, and converges uniformly in any compact subset of $\mathbb{C}^n$. Define the "poly-disk"

$$P_R(\mathbf{z}) = \left\{ \mathbf{v} \in \mathbb{C}^n \middle| |v_k - z_k| < R, k = 1, \ldots, n \right\},$$

then

$$\begin{aligned}
\langle \mathbf{z}^\beta, F \rangle_{L^2(\mathbb{C}^n, \alpha_\hbar)} &= \frac{1}{(\pi\hbar)^n} \int_{\mathbb{C}^n} \overline{\mathbf{z}^\beta} F(\mathbf{z}) e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \int_{P_R(\mathbf{0})} \overline{\mathbf{z}^\beta} F(\mathbf{z}) e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \int_{P_R(\mathbf{0})} \overline{\mathbf{z}^\beta} \left(\sum_{|\alpha|=0}^{\infty} a_\alpha \mathbf{z}^\alpha \right) e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{P_R(\mathbf{0})} \overline{\mathbf{z}^\beta} \mathbf{z}^\alpha e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}_R d\mathbf{z}_I \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \sum_{|\alpha|=0}^{\infty} a_\alpha \prod\nolimits_{k=1}^{n} \int_{B_R(0)} \overline{z_k}^{\beta_k} z_k^{\alpha_k} e^{-\frac{|z_k|^2}{\hbar}} dz_{kR} dz_{kI} \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \sum_{|\alpha|=0}^{\infty} a_\alpha \prod\nolimits_{k=1}^{n} \int_0^{2\pi} \int_0^R r^{\beta_k} e^{-i\beta_k\theta} r^{\alpha_k} e^{i\alpha_k\theta} e^{-\frac{|r|^2}{\hbar}} r dr d\theta \\
&= \lim_{R \longrightarrow \infty} \frac{1}{(\pi\hbar)^n} \sum_{|\alpha|=0}^{\infty} a_\alpha \prod\nolimits_{k=1}^{n} \left(\int_0^R r^{\alpha_k+\beta_k+1} e^{-\frac{|r|^2}{\hbar}} dr \right) \left(\int_0^{2\pi} e^{i(\alpha_k-\beta_k)\theta} d\theta \right).
\end{aligned}$$

Now, since the integral in r will be positive (for any $\alpha_k, \beta_k \geq 0$), and the integral in θ will be zero unless $\alpha_k = \beta_k$, we have that

$$a_\alpha \prod\nolimits_{k=1}^{n} \left(\int_0^R r^{\alpha_k+\beta_k+1} e^{-\frac{|r|^2}{\hbar}} dr \right) \left(\int_0^{2\pi} e^{i(\alpha_k-\beta_k)\theta} d\theta \right)$$

will be zero unless $\alpha_k = \beta_k$ for each value of k. So,

$$\langle \mathbf{z}^\beta, F\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} = \lim_{R\longrightarrow\infty} \frac{1}{(\pi\hbar)^n} \sum_{|\alpha|=0}^{\infty} 2\pi a_\alpha \prod\nolimits_{k=1}^{n} \left(\int_0^R r^{2\beta_k+1} e^{-\frac{|r|^2}{\hbar}} dr \right).$$

If $\langle \mathbf{z}^\beta, F\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} = 0$ then this will require that for each α, $a_\alpha = 0$, which in turn would imply that $F = 0$ since all the terms in its Taylor series expansion is zero. $\square$ $\square$

Now, to normalize this basis, we see that our calculation from the $n = 1$ is quite useful. We saw that (combining a few steps):

$$\begin{aligned}
\| z^j \|^2_{L^2(\mathbb{C},\alpha_\hbar)} &= \langle z^j, z^j\rangle_{L^2(\mathbb{C},\alpha_\hbar)} \\
&= \frac{1}{(\pi\hbar)} \left(\int_0^\infty r^{k+j+1} e^{-\frac{r^2}{\hbar}} dr \right) \left(\int_0^{2\pi} e^{i(j-k)\theta} d\theta \right) \\
&= \frac{2\pi}{(\pi\hbar)} \left(\int_0^\infty r^{2j+1} e^{-\frac{r^2}{\hbar}} dr \right) \\
&= j!\hbar^j.
\end{aligned}$$

Now using this in the $n > 1$ case we have

$$\begin{aligned}
\| \mathbf{z}^\alpha \|^2_{L^2(\mathbb{C},\alpha_\hbar)} &= \langle \mathbf{z}^\alpha, \mathbf{z}^\alpha\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= \prod_{k=1}^{n} \frac{2\pi}{(\pi\hbar)} \left(\int_0^\infty r^{2\alpha_k+1} e^{-\frac{r^2}{\hbar}} dr \right) \\
&= \prod_{k=1}^{n} \alpha_k!\hbar^{\alpha_k} \\
&= \alpha!\hbar^{|\alpha|},
\end{aligned}$$

where, recall, for a multi-index that $\alpha! = \alpha_1! \cdots \alpha_n!$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Thus, we have that

$$\left\{ \frac{\mathbf{z}^\alpha}{\sqrt{\alpha!\hbar^{|\alpha|}}} \middle| \alpha \in \{\mathbb{N} \cup \{0\}\right\}^n \}$$

is an orthonormal basis for $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$.

With no natural ordering to this basis it would be quite difficult to use our proposition for computing the reproducing kernel using an orthonormal basis. But, again, our $n = 1$ calculation is useful. It is by no means conclusive, but is suggestive, that we make the educated guess that the reproducing kernel for $n > 1$ might be

$$K(\mathbf{z}, \mathbf{w}) = e^{\frac{\mathbf{z}\cdot\overline{\mathbf{w}}}{\hbar}} = e^{\frac{z_1\overline{w_1}\cdots z_n\overline{w_n}}{\hbar}} = e^{\frac{z_1\overline{w_1}}{\hbar}} \cdots e^{\frac{z_n\overline{w_n}}{\hbar}}.$$

which we should note that are clearly holomorphic in each z_j (with all others held constant) and antiholomorphic in each w_j (with all others held constant) since they are the product of the reproducing kernels from $\mathcal{H}L^2(\mathbb{C}, \alpha_\hbar)$. This also gives

$$\left\langle e^{\frac{\overline{\mathbf{w}}\cdot(\cdot)}{\hbar}}, e^{\frac{\overline{\mathbf{w}}\cdot(\cdot)}{\hbar}} \right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} = \prod_{k=1}^n \left\langle e^{\frac{\overline{w_k}\cdot(\cdot)}{\hbar}}, e^{\frac{\overline{w_k}\cdot(\cdot)}{\hbar}} \right\rangle_{L^2(\mathbb{C},\alpha_\hbar)} = \prod_{k=1} e^{\frac{\overline{w_k}\cdot\overline{w_k}}{\hbar}} = e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{w}}{\hbar}} < \infty,$$

so our candidate is an element of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$.

To test whether our candidate is the reproducing kernel we can use the fact that if monomials form an orthonormal basis of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$, then we have that the set of polynomials

$$\sum_{|\alpha|=0}^{M} a_\alpha \mathbf{z}^\alpha$$

is a dense subset of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$.

Let's note that such states are square-integrable in each z_j (w.r.t weighted Lebesgue measure) with all the others held constant, so

$$\begin{aligned}
\left\langle e^{\frac{\overline{\mathbf{w}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha (\cdot)^\alpha \right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} &= \int_{\mathbb{C}^n} \overline{e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{v}}{\hbar}}} \left(\sum_{|\alpha|=0}^{M} a_\alpha (\mathbf{v})^\alpha \right) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v}_R d\mathbf{v}_I \\
&= \sum_{|\alpha|=0}^{M} a_\alpha \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{w}}{\hbar}} \mathbf{v}^\alpha e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v}_R d\mathbf{v}_I \\
&= \sum_{|\alpha|=0}^{M} a_\alpha \prod_{k=1}^{n} \int_{\mathbb{C}} e^{\frac{\overline{v_k}\cdot w_k}{\hbar}} {v_k}^{\alpha_k} e^{-\frac{|v_k|^2}{\hbar}} dv_{kR} dv_{kI} \\
&= \sum_{|\alpha|=0}^{M} a_\alpha \prod_{k=1}^{n} \langle \phi_{w_k}, {v_k}^{\alpha_k} \rangle_{L^2(\mathbb{C},\alpha_\hbar)} \\
&= \sum_{|\alpha|=0}^{M} a_\alpha \prod_{k=1}^{n} {w_k}^{\alpha_k} \\
&= \sum_{|\alpha|=0}^{M} a_\alpha \mathbf{w}^\alpha.
\end{aligned}$$

Thus our candidate for the reproducing kernel acts like the reproducing kernel on a dense subset of our space. Now we can appeal to a density argument. We know that for all $F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ and for all $\epsilon > 0$ there exists $a_\alpha = \langle \mathbf{z}^\alpha, F \rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)}$ and an $M = M(\epsilon)$, such that

$$\left\| F - \sum_{|\alpha|=0}^{M} a_\alpha (\cdot)^\alpha \right\|_{L^2(\mathbb{C}^n,\alpha_\hbar)} < \epsilon.$$

We can use this in two ways. First, let $\mathbf{z} \in \mathbb{C}^n$. Then by the continuity of pointwise evaluation we know that there exists $C_{\mathbf{z}}$ such that

$$\begin{aligned}
\left| F(\mathbf{z}) - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha (\cdot)^\alpha \right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} \right| &= \left| F(\mathbf{z}) - \sum_{|\alpha|=0}^{M} a_\alpha \mathbf{z}^\alpha \right| \\
&< \sqrt{C_{\mathbf{z}}} \left\| F - \sum_{|\alpha|=0}^{M} a_\alpha (\cdot)^\alpha \right\|_{L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&< \sqrt{C_{\mathbf{z}}}\,\epsilon.
\end{aligned}$$

We also have by virtue of the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\left|\left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F\right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)} - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)}\right| &= \left|\left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F - \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)}\right| \\
&\leq \left\|e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}\right\|^2 \left\|F - \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\|^2 \\
&\leq e^{\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{\hbar}}\epsilon.
\end{aligned}
$$

Thus the triangle inequality gives

$$
\begin{aligned}
\left|F(\mathbf{z}) - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F\right\rangle\right| &= \left|F(\mathbf{z}) - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle + \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F\right\rangle\right| \\
&\leq \left|F(\mathbf{z}) - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle\right| \\
&\quad + \left|\left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F\right\rangle - \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, \sum_{|\alpha|=0}^{M} a_\alpha(\cdot)^\alpha\right\rangle\right| \\
&\leq \left(e^{\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{\hbar}} + \sqrt{C_{\mathbf{z}}}\right)\epsilon.
\end{aligned}
$$

Now, since our choice ϵ is independent of $\mathbf{z}$ we are free here to make it as small as we like and thus we conclude that for each $\mathbf{z}$ that $F(\mathbf{z}) = \left\langle e^{\frac{\overline{\mathbf{z}}\cdot(\cdot)}{\hbar}}, F\right\rangle_{L^2(\mathbb{C}^n,\alpha_\hbar)}$. Finally, by condition (6) of our proposition on the reproducing kernel we see that this implies that for $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$,

$$
K(\mathbf{z}, \mathbf{w}) = e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}. \tag{2.5}
$$

2.2.4 The Bargmann Transform

Defining the Bargmann Transform

I would like to present the Bargmann transform in the form of a theorem from [18]. One can find formal calculations used justify the result, as well as a rigorous proof in [6], [15], and [18].

Theorem II.12. *Consider the map* $A_\hbar : L^2(\mathbb{R}^n, d\mathbf{x}) \longrightarrow \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ *given by*

$$(A_\hbar f)(\mathbf{z}) = \frac{1}{(\pi\hbar)^{\frac{n}{4}}} \int_{\mathbb{R}^n} e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar} f(\mathbf{x}) d\mathbf{x}.$$

Then,

1. *For all* $f \in L^2(\mathbb{R}^n, d\mathbf{x})$, *the integral is convergent and is holomorphic function of* $\mathbf{z} \in \mathbb{C}^n$.

2. *The map* $A_\hbar$ *is a unitary map of* $L^2(\mathbb{R}^n, d\mathbf{x})$ *onto* $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$.

3. *For* $k = 1, \ldots, n$

$$A_\hbar \left(\frac{\widehat{Q}_k + i\widehat{P}_k}{\sqrt{2}} \right) A_\hbar^{-1} = \hbar \frac{\partial}{\partial z_k} = \widehat{Z}_k^*,$$

and

$$A_\hbar \left(\frac{\widehat{Q}_k - i\widehat{P}_k}{\sqrt{2}} \right) A_\hbar^{-1} = z_k I = \widehat{Z}_k.$$

Proof. One step at a time. First let's note that for $f \in L^2(\mathbb{R}^n, d\mathbf{x})$ the integral

$$\int_{\mathbb{R}^n} e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar} f(\mathbf{x}) d\mathbf{x}$$

is well-defined for all values of $\mathbf{z} \in \mathbb{C}^n$ since the function $e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar}$ is square-integrable (w.r.t to $\mathbf{x}$) for each $\mathbf{z}$. Thus, $(A_\hbar f)(\mathbf{z})$ is a function of the complex variables $z_1, \ldots, z_n$. Considering each z_k separately (while thinking of the other's as constant) we see that since $e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x})/2\hbar}$ is holomorphic in each z_k we can use Morera's Theorem on $(A_\hbar f)(\mathbf{z})$ to see that it is holomorphic in z_k, and thus it is holomorphic in $\mathbf{z}$. So, we've shown that $A_\hbar$ maps $L^2(\mathbb{R}^n, d\mathbf{x})$ into $\mathcal{H}(\mathbb{C}^n)$. Putting the issue of further flushing out the image of $A_\hbar$ until we have a bit more with which to work.

Since we haven't proven that $A_\hbar$ is invertible we can partially prove point (3) above by showing that the following relations hold

$$A_\hbar\left(\frac{\widehat{Q}_k + i\widehat{P}_k}{\sqrt{2}}\right) = \widehat{Z}_k^* A_\hbar,$$

$$A_\hbar\left(\frac{\widehat{Q}_k - i\widehat{P}_k}{\sqrt{2}}\right) = \widehat{Z}_k A_\hbar$$

when $f \in \mathcal{S}(\mathbb{R}^n)$. Thinking of f as a Schwartz function (smooth and rapidly decay at infinity) the integral defining $(A_\hbar f)(\mathbf{z})$ is absolutely and uniformly convergent and thus we can freely differentiate inside the integral and neglect boundary terms (since they'll be zero) in the following integrations by parts:

$$\begin{aligned}
\frac{\partial}{\partial z_k}(A_\hbar f)(\mathbf{z}) &= \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} \frac{\partial}{\partial z_k} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} f(\mathbf{x})d\mathbf{x} \\
&= \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} \left(-\frac{z_k}{\hbar} + \frac{\sqrt{2}x_k}{\hbar}\right) e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} f(\mathbf{x})d\mathbf{x} \\
&= -\frac{z_k}{\hbar}(A_\hbar f)(\mathbf{z}) + \frac{\sqrt{2}}{\hbar} A_\hbar(x_k f(\mathbf{x}))(\mathbf{z}).
\end{aligned}$$

This shows that as operators on Schwartz functions

$$\frac{\partial}{\partial z_k} A_\hbar = -\frac{z_k}{\hbar} A_\hbar + \frac{\sqrt{2}}{\hbar} A_\hbar \widehat{Q}_k,$$

or equivalently

$$\left(\widehat{Z}_k^* + \widehat{Z}_k\right) A_\hbar = \sqrt{2} A_\hbar \widehat{Q}_k.$$

Now, using integration by parts we see

$$\begin{aligned}
A_\hbar \left(\frac{\partial f}{\partial x_k}\right)(\mathbf{z}) &= \frac{1}{(\pi\hbar)^{\frac{n}{4}}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_k} d\mathbf{x} \\
&= -\frac{1}{(\pi\hbar)^{\frac{n}{4}}} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar} f(\mathbf{x}) d\mathbf{x} \\
&= -\frac{1}{(\pi\hbar)^{\frac{n}{4}}} \int_{\mathbb{R}^n} \left(\frac{\sqrt{2} z_k}{\hbar} - \frac{x_k}{\hbar}\right) e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar} f(\mathbf{x}) d\mathbf{x}.
\end{aligned}$$

Which shows us that again as operators on Schwartz functions

$$A_\hbar \frac{\partial}{\partial x_k} = -\frac{\sqrt{2}}{\hbar} z_k A_\hbar + \frac{1}{\hbar} A_\hbar x_k I,$$

or equivalently

$$A_\hbar \frac{i}{\hbar} \widehat{P}_k = -\frac{\sqrt{2}}{\hbar} \widehat{Z}_k A_\hbar + \frac{1}{\hbar} A_\hbar \widehat{Q}_k,$$

or

$$A_\hbar i \widehat{P}_k = -\sqrt{2} \widehat{Z}_k A_\hbar + A_\hbar \widehat{Q}_k.$$

Doing a little algebra to rearrange the result here prove our less ambitious partial proof of (3). Consider the ground state of the harmonic oscillator (a friend we've met before),

$$\psi_{(0,0)}(\mathbf{x}) = \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{-\mathbf{x}^2/2\hbar}.$$

Up to a constant it is the unique function such that for each $k \in \{1, \ldots, n\}$ we have $\widehat{a}_k \psi_{(0,0)} = 0$. (Recall, $\widehat{a}_k = 2^{-1/2}(\widehat{Q}_k + i\widehat{P}_k)$.)

Notice that $\psi_{(0,0)} \in \mathcal{S}(\mathbb{R}^n)$ and that

$$\begin{aligned}
\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z}) &= \frac{1}{(\pi\hbar)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{(-\mathbf{z}^2 + 2\sqrt{2}\mathbf{x}\cdot\mathbf{z} - \mathbf{x}^2)/2\hbar} e^{-\mathbf{x}^2/2\hbar} d\mathbf{x} \\
&= \frac{1}{(\pi\hbar)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\left(\mathbf{z} - \sqrt{2}\mathbf{x}\right)^2/2\hbar} d\mathbf{x} \\
&= \frac{1}{(\pi\hbar)^{\frac{n}{2}}} \prod_{k=1}^{n} \int_{\mathbb{R}} e^{-\left(z_k - \sqrt{2}x_k\right)^2/2\hbar} dx_k.
\end{aligned}$$

Now, let's note that in each of the integrals in the product if it were the case that z_k were real-valued, then by a simple change of variables the integral would have no dependence on it. Thus when each z_k is restricted to the real axis in the $z_k - plane$ the above integrals are constant. Now, since $\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z})$ is holomorphic in each z_k by analytic continuation in each variable we see that it must be the case that $\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z})$ is independent (i.e. constant with respect to) each z_k simultaneously. Thus $\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z})$ is a constant function. To compute its value we can compute the Gaussian integral when each z_k is real-valued. We get $\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z}) = 1$.

This single function will give us a "beach-head" in Bargmann space from which we will be able to compute a surprising number of other states and transfer quite a lot of information from $L^2(\mathbb{R}^n, d\mathbf{x})$. To that end we note that combining the facts that $\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z}) = 1$, $\psi_{(0,0)} \in \mathcal{S}(\mathbb{R}^n)$, and our result (3) above to get

$$A_\hbar((\widehat{a}^{*\beta})\psi_{(0,0)}) = \widehat{Z}^\beta(1) = \mathbf{z}^\beta.$$

And, it is well known (see [15]) that the collection of functions $(\widehat{a}^{*\beta})\psi_{(0,0)}$ are the

Hermite functions, which are known to form an orthogonal basis for $L^2(\mathbb{R}^n, d\mathbf{x})$ with $\| (\widehat{a}^{*\beta})\psi_{(0,0)} \|^2_{L^2(\mathbb{R}^n,d\mathbf{x})} = \hbar^{|\beta|}\beta!$. We've seen that the collection of functions $\{\mathbf{z}^\beta | \beta \in \{\mathbb{N} \cup \{0\}\}^n\}$ forms an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ with $\| \mathbf{z}^\beta \|^2_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} = \hbar^{|\beta|}\beta!$. Thus we see that $A_\hbar$ maps an orthogonal basis of $L^2(\mathbb{R}^n, d\mathbf{x})$ onto an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ and preserves the norm. From this we immediately get that the image of $A_\hbar$ is $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$, is invertible, and unitary. The unitarity of $A_\hbar$ along with the partial proof of (3) gives us all of (3).

$\square$

<u>Note:</u> There are several different (equivalent) formulations of the Bargmann transform where the integral is expressed differently, or normalizations are changed. We choose the one above acknowledging that one may choose a different version and arrive at similar but not identical results.

The Action of the Transform on Certain Operators and States

Let's gather together some relevant facts that we showed (and used) in the previous section.

$$A_\hbar \left(\frac{\widehat{Q}_k + i\widehat{P}_k}{\sqrt{2}} \right) A_\hbar^{-1} = \hbar \frac{\partial}{\partial z_k} = \widehat{Z}_k^*,$$

and

$$A_\hbar \left(\frac{\widehat{Q}_k - i\widehat{P}_k}{\sqrt{2}} \right) A_\hbar^{-1} = z_k I = \widehat{Z}_k.$$

Also, we were able to show that for

$$\psi_{(0,0)}(\mathbf{x}) = \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{-\mathbf{x}^2/2\hbar},$$

we have

$$\left(A_\hbar \psi_{(0,0)}\right)(\mathbf{z}) = 1.$$

From this we showed that for the Hermite functions $(\widehat{a}^{*\beta})\psi_{(0,0)}$ that

$$A_\hbar((\widehat{a}^{*\beta})\psi_{(0,0)}) = \widehat{Z}^\beta(1) = \mathbf{z}^\beta.$$

At this point I would like to find a representation of the Heisenberg Group on $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ in the same way we found it on $L^2(\mathbb{C}^n, \alpha_\hbar)$ and in fact we will use our previous method for finding this 'new' representation directly. We can then use this representation to find the image of the coherent states under the Bargmann transform.

Using the linearity of the Bargmann transform, after a little bit of algebra we get

$$A_\hbar \hat{Q}_k A_\hbar^{-1} = \frac{1}{\sqrt{2}}(\hbar \frac{\partial}{\partial z_k} + z_k I)$$

$$A_\hbar \hat{P}_k A_\hbar^{-1} = \frac{1}{i\sqrt{2}}(\hbar \frac{\partial}{\partial z_k} - z_k I).$$

We will utilize this result by first starting with the Schrodinger equation with a linear Hamiltonian in $L^2(\mathbb{R}^n)$, which was the starting place of our derivation of coherent states previously:

$$i\hbar \frac{\partial u}{\partial t} = (\mathbf{a} \cdot \hat{\mathbf{P}} + \mathbf{b} \cdot \hat{\mathbf{Q}} + cI)u.$$

Acting on both sides of this equation by the Bargmann transform we get

$$
\begin{aligned}
i\hbar \frac{\partial u_B}{\partial t} &= A_\hbar(\mathbf{a}\cdot\hat{\mathbf{P}} + \mathbf{b}\cdot\hat{\mathbf{Q}} + cI)u \\
&= A_\hbar(\mathbf{a}\cdot\hat{\mathbf{P}} + \mathbf{b}\cdot\hat{\mathbf{Q}} + cI)A_\hbar^{-1}A_\hbar u \\
&= (\mathbf{a}\cdot A_\hbar\hat{\mathbf{P}}A_\hbar^{-1} + \mathbf{b}\cdot A_\hbar\hat{\mathbf{Q}}A_\hbar^{-1} + cI)u_B \\
&= \left(\mathbf{a}\cdot\frac{1}{i\sqrt{2}}\left(\hbar\frac{\partial}{\partial z_k} - z_k I\right) + \mathbf{b}\cdot\frac{1}{\sqrt{2}}\left(\hbar\frac{\partial}{\partial z_k} + z_k I\right) + cI\right)u_B \\
&= \left(\hbar\left(\frac{\mathbf{a}}{i\sqrt{2}} + \frac{\mathbf{b}}{\sqrt{2}}\right)\cdot\frac{\partial}{\partial \mathbf{z}} + \left(\frac{\mathbf{b}}{\sqrt{2}} - \frac{\mathbf{a}}{i\sqrt{2}}\right)\cdot\mathbf{z}I + cI\right)u_B \\
&= \left(\hbar\left(\frac{\mathbf{b}-i\mathbf{a}}{\sqrt{2}}\right)\cdot\frac{\partial}{\partial \mathbf{z}} + \left(\frac{\mathbf{b}+i\mathbf{a}}{\sqrt{2}}\right)\cdot\mathbf{z}I + cI\right)u_B \\
&= \left(\hbar\mathbf{w}\cdot\frac{\partial}{\partial \mathbf{z}} + \overline{\mathbf{w}}\cdot\mathbf{z}I + cI\right)u_B,
\end{aligned}
$$

where $u_B \equiv A_\hbar u$, $\mathbf{w} \equiv \frac{1}{\sqrt{2}}(\mathbf{b} - i\mathbf{a})$, and we have used in the first line above that the Bargmann transform will commute with differentiation in the time variable because the Bargmann transform is independent of time. We can rearrange the last form of the equation to read

$$
\left(\frac{\partial}{\partial t} + i\mathbf{w}\cdot\frac{\partial}{\partial \mathbf{z}}\right)u_B = -\frac{i}{\hbar}(\overline{\mathbf{w}}\cdot\mathbf{z} + c)u_B.
$$

Noting that the left-hand side of this equation can be interpreted as the directional derivative of the function u_B in the direction of the vector $\langle 1, i\mathbf{w}\rangle$. We can again utilize the method of characteristics and the characteristic curves will be lines. Let $\mathbf{z}(t) \equiv \mathbf{z} + i\mathbf{w}t$ (the characteristic curves in this case). Define

$$
\Gamma_B(t) \equiv u_B(\mathbf{z}(t), t)
$$

Along the characteristic curves our initial value problem reads

$$
\frac{d\Gamma_B}{dt} = -\frac{i}{\hbar}(\overline{\mathbf{w}}\cdot\mathbf{z}(t) + c)\Gamma_B.
$$

along the characteristic curves (lines in this case).

Thus

$$\begin{aligned}\Gamma_B(t) &= \Gamma_B(0)exp\left(-\frac{i}{\hbar}\int_0^t(\overline{\mathbf{w}}\cdot\mathbf{z}(\tau)+c)d\tau\right)\\ &= \Gamma_B(0)exp\left(-\frac{i}{\hbar}\int_0^t(\overline{\mathbf{w}}\cdot(\mathbf{z}+i\mathbf{w}\tau)+c)d\tau\right)\\ &= \Gamma_B(0)e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}t+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}t^2+ct\right)}\end{aligned}$$

So,

$$u_B(\mathbf{z}(t),t) = e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}t+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}t^2+ct\right)}.$$

The $t = 1$ solution is

$$\begin{aligned}u_B(\mathbf{z}+i\mathbf{w},1) &= \Gamma_B(0)e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}+c\right)}\\ &= u_B(\mathbf{z},0)e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}+c\right)}\end{aligned}$$

If we have an initial condition $u_B(\mathbf{z},0) = f(\mathbf{z})$, then

$$u_B(\mathbf{z}+i\mathbf{w},1) = e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}\right)}f(\mathbf{z})$$

which implies

$$
\begin{aligned}
u_B(\mathbf{z},1) &= e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot(\mathbf{z}-i\overline{\mathbf{w}})+\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}\right)}f(\mathbf{z}-i\mathbf{w}) \\
&= e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\overline{\mathbf{w}}\cdot\mathbf{z}-\frac{i}{2}\mathbf{w}\cdot\overline{\mathbf{w}}\right)}f(\mathbf{z}-i\mathbf{w}) \\
&= e^{-\frac{i}{\hbar}c}e^{\frac{\overline{(i\mathbf{w})}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}f(\mathbf{z}-i\mathbf{w})
\end{aligned}
$$

(Note: If $\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q}-i\mathbf{p})$ and $\mathbf{w} = \frac{1}{\sqrt{2}}(\mathbf{b}-i\mathbf{a})$, (thus $i\mathbf{w} = \frac{1}{\sqrt{2}}(\mathbf{a}-i\mathbf{b})$), so $\mathbf{z}-i\mathbf{w} = \frac{1}{\sqrt{2}}((\mathbf{q}-\mathbf{a})-i(\mathbf{p}+\mathbf{b}))$.)

By solving the "same" initial value problem on either side of the Bargmann transform we can conclude

$$
A_\hbar\left(e^{-\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}+cI\right)}\right)A_\hbar^{-1} = e^{-\frac{i}{\hbar}c}e^{-\frac{i}{\hbar}\left(\mathbf{w}\cdot\widehat{\mathbf{Z}}^*+\widehat{\mathbf{w}}\cdot\widehat{\mathbf{Z}}\right)}
$$

where, again $\mathbf{w} = \frac{1}{\sqrt{2}}(\mathbf{b}-i\mathbf{a})$.

Combine this with the fact that

$$
A_\hbar\psi_{(0,0)} = 1
$$

and we can see that the image of the (canonical) coherent states under the Bargmann transform.

Define, $\widehat{T}_{\mathbf{v},c} : \mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)$ by

$$
\begin{aligned}
\widehat{T}_{\mathbf{v},c} &= A_\hbar\widehat{T}_{(\mathbf{p},\mathbf{q},c)}A_\hbar^{-1} \\
&= A_\hbar\left(e^{-\frac{i}{\hbar}\left(\mathbf{q}\cdot\widehat{\mathbf{P}}-\mathbf{p}\cdot\widehat{\mathbf{Q}}+cI\right)}\right)A_\hbar^{-1}
\end{aligned}
$$

where for each $f \in \mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)$,

$$\widehat{T}_{\mathbf{v},c} f = e^{-\frac{i}{\hbar}c} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}} f(\mathbf{z}-\mathbf{v})$$

and to be clear $\mathbf{v} = i\frac{1}{\sqrt{2}}(\mathbf{p} - i\mathbf{q}) = \frac{1}{\sqrt{2}}(\mathbf{q} + i\mathbf{p})$.

Then we have

$$\begin{aligned}
\varphi_{\mathbf{v},c}(\mathbf{z}) &\equiv A_\hbar \psi_{(\mathbf{p},\mathbf{q},c)} \\
&= A_\hbar \left(\widehat{T}_{(\mathbf{p},\mathbf{q},c)} \psi_{(0,0)} \right) \\
&= \left(A_\hbar \widehat{T}_{(\mathbf{p},\mathbf{q},c)} A_\hbar^{-1} \right) \left(A_\hbar \psi_{(0,0)} \right) \\
&= \widehat{T}_{\mathbf{v},c}(1) \\
&= e^{-\frac{i}{\hbar}c} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}}
\end{aligned}$$

We saw that the (canonical) coherent states in $L^2(\mathbb{R}^n)$ were the states of the form $\widehat{T}_{(\mathbf{p},\mathbf{q},0)}\psi_{(0,0)}$. Thus the image of the (canonical) coherent states in Bargmann space will be those elements of the form

$$\varphi_{\mathbf{v}}(\mathbf{z}) = e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}},$$

where $\varphi_{\mathbf{v}}(\mathbf{z})$ is centered at $\mathbf{v} \in \mathbb{C}^n$. Note, this is big, the coherent states in $L^2(\mathbb{R}^n)$ get mapped to the reproducing kernel states in Bargmann space.

Finally, since the Schwartz functions

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \middle| f \in C^\infty(\mathbb{R}^n), \forall \alpha, \beta \in \{\mathbb{N} \cup \{0\}\}^n, \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta f(\mathbf{x})| < \infty \right\},$$

play a central role in $L^2(\mathbb{R}^n)$, being a dense subset upon which the operators $\widehat{Q}_k$ and $\widehat{P}_k$ are continuous, and clearly by definition for each k, we have $\widehat{Q}_k(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$ and $\widehat{P}_k(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$.

Note that if $f \in \mathcal{S}(\mathbb{R}^n)$ then from the fact that

$$\begin{aligned}
\frac{\partial}{\partial x_j}\left(e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar}\right) &= e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar}\frac{1}{2\hbar}\frac{\partial}{\partial x_j}(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2) \\
&= e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar}\frac{1}{2\hbar}(2\sqrt{2}z_j - 2x_j) \\
&= \frac{1}{\hbar}(\sqrt{2}z_j - x_j)e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar},
\end{aligned}$$

or equivalently

$$z_j e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} = \frac{\hbar}{\sqrt{2}}\left(x_j e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} + \frac{\partial}{\partial x_j}e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar}\right),$$

gives us

$$\begin{aligned}
\widehat{Z}_j(A_\hbar f) &= z_j \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} f(\mathbf{x})d\mathbf{x} \\
&= \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} z_j e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} f(\mathbf{x})d\mathbf{x} \\
&= \frac{\hbar}{\sqrt{2}}\Bigg(\frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} x_j f(\mathbf{x})d\mathbf{x} \\
&\qquad + \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} f(\mathbf{x})d\mathbf{x}\Bigg) \\
&= \frac{\hbar}{\sqrt{2}}\Bigg(\frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} x_j f(\mathbf{x})d\mathbf{x} \\
&\qquad - \frac{1}{(\pi\hbar)^{\frac{n}{4}}}\int_{\mathbb{R}^n} e^{(-\mathbf{z}^2+2\sqrt{2}\mathbf{x}\cdot\mathbf{z}-\mathbf{x}^2)/2\hbar} \frac{\partial f}{\partial x_j} d\mathbf{x}\Bigg) \\
&= A_\hbar\left(\frac{\hbar}{\sqrt{2}}\left(x_j f(\mathbf{x}) - \frac{\partial f}{\partial x_j}(\mathbf{x})\right)\right),
\end{aligned}$$

where we integrated by parts to go from the third to the fourth step. Now, since $f \in \mathcal{S}(\mathbb{R}^n)$ then $\frac{\hbar}{\sqrt{2}}\left(x_j f(\mathbf{x}) - \frac{\partial f}{\partial x_j}(\mathbf{x})\right) \in \mathcal{S}(\mathbb{R}^n)$. Thus,

$$\widehat{Z}_j\left(A_\hbar(\mathcal{S}(\mathbb{R}^n))\right) = A_\hbar(\mathcal{S}(\mathbb{R}^n)).$$

Also, we saw in the previous section (where we did an analogous computation to the one we just did, but for $\widehat{Z}_j^*$ that for a Schwartz function f,

$$\widehat{Z}_j^*(A_\hbar f) = -\widehat{Z}_j(A_\hbar f) + \frac{\sqrt{2}}{\hbar}(A_\hbar(x_j f)),$$

which combined with what we just showed gives us that

$$\widehat{Z}_j^*\left(A_\hbar(\mathcal{S}(\mathbb{R}^n))\right) = A_\hbar(\mathcal{S}(\mathbb{R}^n)).$$

So, this shows that the image of the Schwartz functions under the Bargmann transform is closed under the action of arbitrary (finite) compositions of $\widehat{Z}_j$'s and $\widehat{Z}_j^*$'s. More precisely, for all multi-indices $\alpha, \beta \in \{\mathbb{N} \cup \{0\}\}^n$ if $f \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{Z}^\alpha \widehat{Z}^{*\beta}(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar).$$

Now I'd like to flip the question around and ask 'What can we say about $f \in L^2(\mathbb{R}^n)$ if for all $\alpha, \beta \in \{\mathbb{N} \cup \{0\}\}^n$, $\widehat{Z}^\alpha \widehat{Z}^{*\beta}(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ and $\widehat{Z}^{*\beta} \widehat{Z}^\alpha(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$?'

From the relationship we know to hold by the theorem from the previous section

$$\widehat{Z}_j(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar) \Longrightarrow (A_\hbar^{-1} \widehat{Z}_j A_\hbar) f \in L^2(\mathbb{R}^n) \qquad \Longrightarrow \widehat{a}_j^* f \in L^2(\mathbb{R}^n);$$

and

$$\widehat{Z}_j^*(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar) \Longrightarrow (A_\hbar^{-1}\widehat{Z}_j^* A_\hbar)f \in L^2(\mathbb{R}^n) \qquad \Longrightarrow \widehat{a}_j f \in L^2(\mathbb{R}^n).$$

Thus we have that if for $f \in L^2(\mathbb{R}^n)$ that $\widehat{Z}^\alpha \widehat{Z}^{*\beta}(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ and $\widehat{Z}^{*\beta}\widehat{Z}^\alpha(A_\hbar f) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ for any and all multi-indices α and β, we have that

$$\widehat{a}^{*\alpha}\widehat{a}^\beta(f), \widehat{a}^\beta\widehat{a}^{*\alpha}(f) \in L^2(\mathbb{R}^n).$$

This tells us that for such an f we have that f remains in $L^2(\mathbb{R}^n)$ after the action of arbitrary (finite) compositions of $\widehat{a}_j$'s and $\widehat{a}_j^*$'s. This is equivalent to f remaining $L^2(\mathbb{R}^n)$ after the action of arbitrary (finite) compositions of $\widehat{Q}_j$'s and $\widehat{P}_j$'s. So, $f \in C^\infty(\mathbb{R}^n)$ and

$$\mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta f, \partial_{\mathbf{x}}^\beta(\mathbf{x}^\alpha f) \in L^2(\mathbb{R}^n)$$

A simple contradiction argument gives that the fact above can only be true if

$$\sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta f(\mathbf{x})| < \infty.$$

Combining all of this we arrive at the conclusion that

$$A_\hbar(\mathcal{S}(\mathbb{R}^n)) = \left\{ F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar) \middle| \forall \alpha, \beta \in \{\mathbb{N} \cup \{0\}\}^n, \widehat{Z}^\alpha \widehat{Z}^{*\beta}(F) \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar) \right\}.$$

For use later, let's define $\widetilde{S}(\mathbb{C}^n) \equiv A_\hbar(\mathcal{S}(\mathbb{R}^n))$. It is worth noting that since the coherent states and the Hermite functions in $L^2(\mathbb{R}^n)$ are Schwartz functions we see that their images under the Bargmann transform, the monomial basis and the reproducing kernel(s), will be elements in $\widetilde{S}(\mathbb{C}^n)$.

2.3 Quantization

A quantization (or a quantization method) is a correspondence between a classical system and a quantum system. Recall the following facts from the introduction:

The backdrop of classical mechanics is phase space. A classical system is completely specified by the positions and momenta of its constituents. This state space is known as phase space and in its most abstract form it is a smooth symplectic manifold. A classical mechanical system is given by a triple (M, ω, H), where M is a symplectic manifold, ω is the symplectic form on M, and $H \in C^\infty(M)$ is a Hamilton function which represents the energy of the system as a function of its configuration in M. From this all of the dynamics of a classical system can be specified as the system evolves via Hamilton's equations. Observable quantities in classical mechanics are smooth, real-valued functions on M, such as the Hamilton function, with the observable values being the values of the function. Thus classical mechanics is cast in the language of differential geometry, specifically symplectic geometry. For in depth perspective on the geometric nature of classical mechanics see the classic (no pun intended) texts [2] and [5].

The backdrop of quantum mechanics is a vector space, specifically a Hilbert space $\mathcal{H}$. Within the Hilbert space the vectors that represent the states of a physical system are the normalized vectors. Observable quantities are given by self-adjoint operators on $\mathcal{H}$, and the observable values of these quantities are the spectrum of eigenvalues of these operators. The dynamics of a system can be specified in quantum theory (as far as is possible in accordance with the probabalistic nature of quantum mechanics) with the knowledge of the Hamiltonian operator (or energy operator) $\widehat{H}$ on $\mathcal{H}$ via the Schrodinger equation. Quantum mechanics is cast in the language of functional analysis, specifically analysis on Hilbert spaces. See [28], [34], [37] among many others.

Thus a quantization is a means of systematically associating classical behavior of a classical system with the quantum behavior of a quantum system in two ways: (1) elements in the ring of smooth real-valued functions on a symplectic manifold are associated to (possibly unbounded) self-adjoint operators on a Hilbert space, and (2) Hamilton flows (classical evolution) are associated to unitary evolution (quantum evolution). Additional conditions to impose on a quantization procedure can come from considerations from physics (see [18]), but at a basic level this is what is involved.

A theorem that plays a central role in a discussion of an association of smooth functions on phase space with an operator on a Hilbert space (namely one of the function spaces we saw previously) is the Schwartz Kernel Theorem. This discussion is taken from [20]. As a motivation for this theorem let's note that if X_1, X_2 are open subsets of $\mathbb{R}^n$, then every $K \in C(X_1 \times X_2)$ defines an integral operator $\widehat{\mathcal{K}} : C_0(X_2) \longrightarrow C(X_1)$ by the following formula

$$(\widehat{\mathcal{K}}g)(\mathbf{x}) = \int_{\mathbb{R}^n} K(\mathbf{x}_1, \mathbf{x}_2) g(\mathbf{x_2}) d\mathbf{x}_2$$

The following theorem shows that this definition can be extended to arbitrary distributions K if g is restricted to C_0^∞, and $(\widehat{\mathcal{K}}g)$ is also a distribution. To this end we make note of the fact that when $K \in C(X_1 \times X_2)$ (and is thought of as a distribution)

$$\langle \widehat{\mathcal{K}}g_1, g_2 \rangle_{\mathcal{D}', \mathcal{D}} = K(g_1 \oplus g_2),$$

for $g_2 \in C_0^\infty(X_1)$ and $g_1 \in C_0^\infty(X_2)$.

Theorem II.13. *(Schwartz Kernel Theorem) Every $K \in D'(X_1 \times X_2)$ defines, according to*

$$\langle \widehat{\mathcal{K}} g_1, g_2 \rangle_{\mathcal{D}',\mathcal{D}} = K(g_1 \oplus g_2),$$

a linear map $\mathcal{K}$ from $C_0^\infty(X_2)$ to $\mathcal{D}'(X_1)$ which is continuous in the sense that $\mathcal{K}g_j \longrightarrow 0$ in $\mathcal{D}'(X_1)$ if $g_j \longrightarrow 0$ in $C_0^\infty(X_2)$. Conversely, to every such linear map $\mathcal{K}$ there is one and only one distribution K such that the above formula is valid. One calls K the kernel of $\mathcal{K}$.

Proof. See [**?**]. □

We won't have much need to explicitly use this theorem, but it will be the underpinning of the definitions to follow. When one cannot interpret the formulas to follow in a literal sense, then using the above theorem we may interpret them in the sense of distributions

2.3.1 Weyl Quantization

The first quantization scheme that we will present, known as Weyl quantization, is the central quantization procedure for our work because it is generally considered to be the most physically meaningful procedure in non-relativistic quantum theory. Though we will want to use the procedure on Bargmann space, it is most naturally defined on $L^2(\mathbb{R}^n)$. We'll then present two other quantization schemes that are naturally defined on Bargmann space, and the transfer the Weyl procedure over to Bargmann space by making use of these other procedures relation to Weyl quantization.

The Weyl quantization (named after Hermann Weyl) begins with the definition for $\rho(\mathbf{p}, \mathbf{q}) = e^{i(\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q})}$. We define

$$\widehat{\rho}_{weyl} \equiv e^{i\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)},$$

and recalling that this operator made an appearance when we were looking for a representation of the Heisenberg group in the standard representation and that it's action on $f \in L^2(\mathbb{R}^n)$:

$$\left(e^{i\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}f\right)(\mathbf{x}) = e^{\frac{i}{\hbar}(\mathbf{b}\cdot(\mathbf{x}+\mathbf{a})-\frac{1}{2}\mathbf{a}\cdot\mathbf{b})}f(\mathbf{x}+\mathbf{a}) = e^{\frac{i}{\hbar}(\mathbf{b}\cdot\mathbf{x}+\frac{1}{2}\mathbf{a}\cdot\mathbf{b})}f(\mathbf{x}+\mathbf{a}).$$

Putting that aside for a moment. Using the Fourier Inversion Theorem with the $\hbar$-Fourier Transform then, for $\rho \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$\rho(\mathbf{p},\mathbf{q}) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \widetilde{\rho}(\mathbf{a},\mathbf{b})e^{\frac{i}{\hbar}(\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q})}d\mathbf{a}d\mathbf{b},$$

where $\widetilde{\rho}$ denotes the $\hbar$-Fourier Transform of ρ. This leads us to the definition of the Weyl quantization of a general symbol to be

$$\widehat{\rho}(\widehat{\mathbf{P}},\widehat{\mathbf{Q}}) \equiv \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \widetilde{\rho}(\mathbf{a},\mathbf{b})e^{\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}d\mathbf{a}d\mathbf{b}.$$

Putting aside questions of convergence of the integral and the domain of definition for this operator (not unrelated issues), let's formally manipulate this expression into a less abstract form:

$$
\begin{aligned}
\left(\widehat{\rho}(\widehat{\mathbf{P}},\widehat{\mathbf{Q}})f\right)(\mathbf{x}) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \widetilde{\rho}(\mathbf{a},\mathbf{b})\left(e^{\frac{i}{\hbar}\left(\mathbf{a}\cdot\widehat{\mathbf{P}}+\mathbf{b}\cdot\widehat{\mathbf{Q}}\right)}f\right)(\mathbf{x})d\mathbf{a}d\mathbf{b}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \widetilde{\rho}(\mathbf{a},\mathbf{b})e^{\frac{i}{\hbar}(\mathbf{b}\cdot\mathbf{x}+\frac{1}{2}\mathbf{a}\cdot\mathbf{b})}f(\mathbf{x}+\mathbf{a})d\mathbf{a}d\mathbf{b}\\
&= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} e^{-\frac{i}{\hbar}(\mathbf{a}\cdot\xi+\mathbf{b}\cdot\mathbf{w})}\rho(\xi,\mathbf{w})e^{\frac{i}{\hbar}(\mathbf{b}\cdot\mathbf{x}+\frac{1}{2}\mathbf{a}\cdot\mathbf{b})}f(\mathbf{x}+\mathbf{a})d\xi d\mathbf{w}d\mathbf{a}d\mathbf{b}\\
&= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} \rho(\xi,\mathbf{w})e^{-\frac{i}{\hbar}\mathbf{a}\cdot\xi}e^{\frac{i}{\hbar}\mathbf{b}\cdot(\mathbf{x}+\frac{1}{2}\mathbf{a}-\mathbf{w})}f(\mathbf{x}+\mathbf{a})d\xi d\mathbf{w}d\mathbf{a}d\mathbf{b}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{3n}} \rho(\xi,\mathbf{w})e^{-\frac{i}{\hbar}\mathbf{a}\cdot\xi}\left(\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\mathbf{b}\cdot(\mathbf{x}+\frac{1}{2}\mathbf{a}-\mathbf{w})}d\mathbf{b}\right)f(\mathbf{x}+\mathbf{a})d\xi d\mathbf{w}d\mathbf{a}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{3n}} \rho(\xi,\mathbf{w})e^{-\frac{i}{\hbar}\mathbf{a}\cdot\xi}\delta\left(\mathbf{x}+\frac{1}{2}\mathbf{a}-\mathbf{w}\right)f(\mathbf{x}+\mathbf{a})d\xi d\mathbf{w}d\mathbf{a}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \rho\left(\xi,\mathbf{x}+\frac{1}{2}\mathbf{a}\right)e^{-\frac{i}{\hbar}\mathbf{a}\cdot\xi}f(\mathbf{x}+\mathbf{a})d\xi d\mathbf{a}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \rho\left(\xi,\mathbf{x}+\frac{1}{2}(\mathbf{y}-\mathbf{x})\right)e^{-\frac{i}{\hbar}(\mathbf{y}-\mathbf{x})\cdot\xi}f(\mathbf{y})d\xi d\mathbf{y}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(\mathbf{x}-\mathbf{y})\cdot\xi}\rho\left(\xi,\frac{\mathbf{x}+\mathbf{y}}{2}\right)f(\mathbf{y})d\xi d\mathbf{y}.
\end{aligned}
$$

These formulas beg for some qualifying detail. Note, that if one is to interpret the expression at face value then we must restrict ourselves to symbols, ρ, that are sufficient that the integrals will exist for every value of $\mathbf{x}$ for every f we wish to define the operators on. See [28],[14], and [15] for detailed discussions of these issues. Suffice it to say that if one interprets these formulas in the sense of distributions, as given by the Schwartz Kernel Theorem with a distribution kernel given by an oscillatory integral, then one can extend them to large classes of symbols and to act on very general state spaces.

In this work we'll be focusing on polynomial Hamiltonians, thus the following theorem is all we will need. See [15] for the proof.

Theorem II.14. *Let $\alpha,\beta \in \{\mathbb{N}\cup\{0\}\}^n$ be multi-indices, and $\rho(\mathbf{p},\mathbf{q}) = \mathbf{p}^\beta\mathbf{q}^\alpha$ then*

$$\widehat{\rho}_{weyl} = \frac{1}{(\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n)!} \sum_{\sigma \in S_{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n}} \sigma \cdot \left(\widehat{Q}_1^{\alpha_1} \cdots \widehat{Q}_n^{\alpha_n} \widehat{P}_1^{\beta_1} \cdots \widehat{P}_n^{\beta_n} \right),$$

where $S_{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n}$ is the permutation group on $\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$ objects, and $\sigma \cdot \left(\widehat{Q}_1^{\alpha_1} \cdots \widehat{Q}_n^{\alpha_n} \widehat{P}_1^{\beta_1} \cdots \widehat{P}_n^{\beta_n} \right)$ is schematic notation for what you get by permuting the $\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$ factors in $\widehat{Q}_1^{\alpha_1} \cdots \widehat{Q}_n^{\alpha_n} \widehat{P}_1^{\beta_1} \cdots \widehat{P}_n^{\beta_n}$.

2.3.2 Wick and Anti-Wick Quantization

The following discussion is adapted from [15]

Let's begin by considering our phase space $\mathbb{R}^{2n} = \{(\mathbf{q}, \mathbf{p}) | \mathbf{q}, \mathbf{p} \in \mathbb{R}^n\}$ which we identify with $\mathbb{C}^n$ by giving it the complex structure $\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$. We then have the conjugate complex coordinate $\overline{\mathbf{z}} = \frac{1}{\sqrt{2}}(\mathbf{q} + i\mathbf{p})$.

Let's first consider a general polynomial of $2n$ real variables expressed in the above complex variables

$$\rho(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta} \mathbf{z}^\alpha \overline{\mathbf{z}}^\beta$$

We define the operator $\widehat{\rho}_{wick}$ associated to ρ by

$$\widehat{\rho}_{wick} \equiv \sum_{\alpha,\beta} C_{\alpha,\beta} \widehat{Z}^\alpha \widehat{Z}^{*\beta}.$$

The so called Wick procedure (or Wick ordering) comes from taking the function $\rho(\mathbf{z}, \overline{\mathbf{z}})$ and substituting into the analytic expression $\widehat{Z}_j$ for each z_j and $\widehat{Z}_j^*$ for each $\overline{z}_j$. Since $\widehat{Z}_j$ and $\widehat{Z}_j^*$ do not commute we must impose an ordering on how the they act. In the Wick ordering we choose to have each $\widehat{Z}_j^*$ act before the $\widehat{Z}_j$'s act. The opposite ordering

We define the operator $\widehat{\rho}_{anti-wick}$ associated to ρ by

$$\widehat{\rho}_{anti-wick} \equiv \sum_{\alpha,\beta} C_{\alpha,\beta} \widehat{Z}^{*\beta} \widehat{Z}^{\alpha}.$$

The so called Anti-Wick procedure (or Anti-Wick ordering) comes from taking the function $\rho(\mathbf{z}, \overline{\mathbf{z}})$ and substituting into the analytic expression $\widehat{Z}_j$ for each z_j and $\widehat{Z}_j^*$ for each $\overline{z}_j$. Since $\widehat{Z}_j$ and $\widehat{Z}_j^*$ do not commute we must impose an ordering on how the they act. In the Anti-Wick ordering we choose to have each $\widehat{Z}_j$ act before the $\widehat{Z}_j^*$'s act.

It's straightforward to see that the Wick and Anti-Wick orderings each map the set of polynomials on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ onto the set of differential operators with polynomial coefficients.

In order to extend this correspondence to more general smooth functions. Let's first note that for $F \in \widetilde{S}(\mathbb{C}^n)$ (a good domain for considering such operators) we have from the reproducing kernel

$$F(\mathbf{z}) = \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} F(\mathbf{v}) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v} d\overline{\mathbf{v}}.$$

Then,

$$\begin{aligned}
(\widehat{\rho}_{wick} F)(\mathbf{z}) &= \sum_{\alpha,\beta} C_{\alpha,\beta} \int_{\mathbb{C}^n} \widehat{Z}^{\alpha} \widehat{Z}^{*\beta} \left(e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} \right) F(\mathbf{v}) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v} d\overline{\mathbf{v}} \\
&= \sum_{\alpha,\beta} C_{\alpha,\beta} \int_{\mathbb{C}^n} \mathbf{z}^{\alpha} \overline{\mathbf{v}}^{\beta} e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} F(\mathbf{v}) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v} d\overline{\mathbf{v}} \\
&= \int_{\mathbb{C}^n} \left(\sum_{\alpha,\beta} C_{\alpha,\beta} \mathbf{z}^{\alpha} \overline{\mathbf{v}}^{\beta} \right) e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} F(\mathbf{v}) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v} d\overline{\mathbf{v}} \\
&= \int_{\mathbb{C}^n} \rho(\mathbf{z}, \overline{\mathbf{v}}) e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} F(\mathbf{v}) e^{-\frac{|\mathbf{v}|^2}{\hbar}} d\mathbf{v} d\overline{\mathbf{v}}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
(\widehat{\rho}_{anti-wick}F)(\mathbf{z}) &= \sum_{\alpha,\beta} C_{\alpha,\beta}\widehat{Z}^{*\beta}\left(\widehat{Z}^{\alpha}F(\mathbf{z})\right)\\
&= \sum_{\alpha,\beta} C_{\alpha,\beta}\widehat{Z}^{*\beta}(\mathbf{z}^{\alpha}F(\mathbf{z}))\\
&= \sum_{\alpha,\beta} C_{\alpha,\beta}\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}\widehat{Z}^{*\beta}\left(e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}\right)\mathbf{v}^{\alpha}F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}}\\
&= \sum_{\alpha,\beta} C_{\alpha,\beta}\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}\overline{\mathbf{v}}^{\beta}e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}\mathbf{v}^{\alpha}F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}}\\
&= \frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}\left(\sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{v}^{\alpha}\overline{\mathbf{v}}^{\beta}\right)e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}}\\
&= \frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}\rho(\mathbf{v},\overline{\mathbf{v}})e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}}.
\end{aligned}
$$

To summarize, we have for $\rho(\mathbf{z},\overline{\mathbf{z}})$ the polynomial above:

$$
(\widehat{\rho}_{wick}F)(\mathbf{z}) = \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}\rho(\mathbf{z},\overline{\mathbf{v}})F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}},
$$

and

$$
(\widehat{\rho}_{anti-wick}F)(\mathbf{z}) = \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}}\rho(\mathbf{v},\overline{\mathbf{v}})F(\mathbf{v})e^{-\frac{|\mathbf{v}|^2}{\hbar}}d\mathbf{v}d\overline{\mathbf{v}}.
$$

With these expressions we can extend the definition of the Wick and Anti-Wick procedures to functions ρ that are not polynomials. Using the above expressions as the definitions of these quantization procedures we see that the restrictions on ρ will be what is necessary that the integrals defining these quantities are convergent. Thus, for the Wick procedure we would require that $\rho(\mathbf{z},\mathbf{v})$ be holomorphic in $\mathbf{z}$ and have suitable growth restrictions at infinity to ensure the integral is convergent for every value of $\mathbf{z}$. In contrast, for the Anti-Wick procedure we see that the above expression would only require ρ is integrable and again has suitable growth restrictions at infinity to allow for the convergence of the integral. Note that the relevant growth restrictions will be partially determined by what class of functions upon which the operator would be defined. In general one should expect these operators to be unbounded, as are $\widehat{Z}_j$ and $\widehat{Z}_j^*$.

We have displayed two ways of associating symbols to operators. We may ask about going in the 'opposite' direction in the case of the Wick and Anti-Wick procedures.

We begin with the Wick procedure. Suppose that $\widehat{T}$ is a (possibly unbounded) operator on $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ such that the domains of $\widehat{T}$ and $\widehat{T}^*$ contain the reproducing kernel(s) (which, for example, will be the case if we define the domains to be $\widetilde{S}(\mathbb{C}^n)$). Let's define $\Phi_{\mathbf{z}}(\mathbf{v}) \equiv \varphi_{\mathbf{z}}(\mathbf{v})e^{\frac{|\mathbf{z}|^2}{2\hbar}}$, i.e. $\Phi_{\mathbf{z}}$ is an unnormalized coherent state centered at $\mathbf{z}$. More directly $\Phi_{\mathbf{z}}(\mathbf{v}) = e^{\frac{\overline{\mathbf{z}}\cdot\mathbf{v}}{\hbar}}$, the reproducing kernel for Bargmann space. We define the Wick symbol of $\widehat{T}$ to be

$$\rho_T^W(\mathbf{z}, \overline{\mathbf{z}}) = e^{-\frac{|\mathbf{z}|^2}{\hbar}} \left\langle \Phi_{\mathbf{z}}, \widehat{T}\Phi_{\mathbf{z}} \right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} = e^{-\frac{|\mathbf{z}|^2}{\hbar}} (\widehat{T}\Phi_{\mathbf{z}})(\mathbf{z}),$$

or equivalently

$$\rho_T^W(\mathbf{z}, \overline{\mathbf{z}}) = \left\langle \varphi_{\mathbf{z}}, \widehat{T}\varphi_{\mathbf{z}} \right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} = e^{-\frac{|\mathbf{z}|^2}{2\hbar}} (\widehat{T}\varphi_{\mathbf{z}})(\mathbf{z}).$$

This is a smooth function on $\mathbb{C}^n$ which is the restriction to the set $\{\mathbf{w} = \mathbf{z}\}$ of the function

$$
\begin{aligned}
\rho_T^W(\mathbf{z},\overline{\mathbf{w}}) &= e^{-\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}\left\langle \Phi_{\mathbf{z}}, \widehat{T}\Phi_{\mathbf{w}}\right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= e^{-\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}\left\langle \widehat{T}^*\Phi_{\mathbf{z}}, \Phi_{\mathbf{w}}\right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= e^{-\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}\overline{\left\langle \Phi_{\mathbf{w}}, \widehat{T}^*\Phi_{\mathbf{z}}\right\rangle}_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= e^{-\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}\overline{\left(\widehat{T}^*\Phi_{\mathbf{z}}\right)(\mathbf{w})}.
\end{aligned}
$$

With this function defined

$$
\begin{aligned}
\left(\widehat{T}F\right)(\mathbf{z}) &= \left\langle \Phi_{\mathbf{z}}, \widehat{T}F\right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= \left\langle \widehat{T}^*\Phi_{\mathbf{z}}, F\right\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} \\
&= \int_{\mathbb{C}^n} \overline{\left(\widehat{T}^*\Phi_{\mathbf{z}}\right)(\mathbf{w})}F(\mathbf{w})e^{-\frac{|\mathbf{w}|^2}{\hbar}}d\mathbf{w}d\overline{\mathbf{w}} \\
&= \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}\rho_T^W(\mathbf{z},\overline{\mathbf{w}})F(\mathbf{w})e^{-\frac{|\mathbf{w}|^2}{\hbar}}d\mathbf{w}d\overline{\mathbf{w}}.
\end{aligned}
$$

This is exactly our expression defining the Wick procedure with $\rho = \rho_T^W$. Thus the map associating symbols with operators, $\rho \longrightarrow \widehat{\rho}_{wick}$, and $\widehat{T} \longrightarrow \rho_T^W$ are mutually inverse.

Turning our attention to addressing the Anti-Wick procedure we'll see that the nature of this procedure is quite different than the Wick procedure. We can characterize the procedure in words by beginning with a function ρ. The action of the corresponding Anti-Wick operator on a function F is to multiply F by ρ and then (recalling (3) from our theorem on the reproducing kernel) projecting this product into $\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)$. This type of operator is generally referred to as a *Toeplitz operator*. To be more precise, suppose ρ is a measurable function on $\mathbb{C}^n$ which satisfies the growth condition

$$|\rho(\mathbf{z},\overline{\mathbf{z}})| \leq Ce^{\varepsilon|\mathbf{z}|^2},$$

for some $\varepsilon < \frac{1}{2\hbar}$. If $F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$, and the product $pf \in L^2(\mathbb{C}^n, \alpha_\hbar)$ we can project this function into $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$ by integrating it against the reproducing kernel. This procedure is exactly our formula for the Anti-Wick procedure above. Thus we find

$$Dom(\widehat{\rho}_{anti-wick}) = \left\{ F \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar) \middle| \int_{\mathbb{C}^n} |\rho(\mathbf{z},\overline{\mathbf{z}})F(\mathbf{z})|e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}d\overline{\mathbf{z}} < \infty \right\},$$

which will include (among others) all polynomials and coherent states if ρ satisfies the above growth condition. The following lemma is necessary for our purposes

Lemma II.15. *If ρ_1 and ρ_2 satisfy the growth condition above and $\widehat{\rho_1}_{anti-wick} = \widehat{\rho_2}_{anti-wick}$, then $\rho_1 = \rho_2$.*

Proof. If we define $\rho \equiv \rho_1 - \rho_2$ then, $\widehat{\rho}_{anti-wick} = \widehat{\rho_1}_{anti-wick} - \widehat{\rho_2}_{anti-wick}$. From the proceeding description of $\widehat{\rho}_{anti-wick}$, then $\widehat{\rho}_{anti-wick}F = 0$ precisely when $\rho F \in L^2(\mathbb{C}^n, \alpha_\hbar)$ and $\langle G, \rho F\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} = 0$ for all $G \in \mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$. Taking $F(\mathbf{z}) = \mathbf{z}^\alpha$ and $G(\mathbf{z}) = \mathbf{z}^\beta$ (element of the monomial basis of $\mathcal{H}L^2(\mathbb{C}^n, \alpha_\hbar)$) then we have

$$\langle G, \rho F\rangle_{\mathcal{H}L^2(\mathbb{C}^n,\alpha_\hbar)} = 0 \Longrightarrow \int_{\mathbb{C}^n} \rho(\mathbf{z},\overline{\mathbf{z}})\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta e^{-\frac{|\mathbf{z}|^2}{\hbar}} d\mathbf{z}d\overline{\mathbf{z}} = 0$$

But, we know that $\operatorname{span}\{\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta|\alpha,\beta \in \{\mathbb{N}\cup\{0\}\}^n\}$ is dense in $L^2(\mathbb{C}^n, \alpha_\hbar)$ which follows directly from the theory of Hermite functions. Thus $\rho = 0$. $\square$

This lemma tells us that the operator $\widehat{\rho} = \widehat{\rho}_{anti-wick}$ is uniquely determined by the function ρ, we therefore can without reservation call ρ the Anti-Wick symbol of $\widehat{\rho}$. It is the case that not all operators, not even all bounded operators, will be able to be expressed as a Toeplitz operator. Consider the the operator that is the orthogonal projection onto the constant functions $\widehat{T}F = F(\mathbf{0})$. The only possible way to express $\widehat{T}$ in a Toeplitz form would be to take ρ as the delta-function at the origin. Thus, though this operator is bounded, we would need a distribution as our Anti-Wick symbol. Even broadening our allowable symbols to contain distributions wouldn't guarantee the existence of an Anti-Wick symbol in general.

Quantizations Unite!

We might wonder how the three standard quantization schemes, Weyl, Wick, and Anti-Wick, relate to each other. Specifically, let's think about how they relate on Bargmann space. Since we are only going to focus on polynomial Hamiltonians we'll focus our attention here on them.

Let's consider $\mathbb{R}^{2n}$ with global coordinates $(\mathbf{q}, \mathbf{p})$ and make the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (more details on this later) with a choice of complex structure $\mathbb{C}^n \ni \mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$. Suppose we have a real-valued polynomial function of $\mathbf{q}$ and $\mathbf{p}$, $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. We can consider H as a polynomial function of $\mathbf{z}$ and $\overline{\mathbf{z}}$ by noting that with $\mathbf{z}$ defined above we have $\mathbf{q} = \frac{1}{\sqrt{2}}(\mathbf{z} + \overline{\mathbf{z}})$ and $\mathbf{p} = \frac{i}{\sqrt{2}}(\mathbf{z} - \overline{\mathbf{z}})$.

By $\widehat{H}_{weyl}$ we mean the unbounded operator on Bargmann space that is given by first quantizing $H(\mathbf{q}, \mathbf{p})$ according to the Weyl quantization procedure (the symmetrizing process) given above, and then conjugating that operator, which acts on $L^2(\mathbb{R}^n)$, by the Bargmann transform to make it an operator on Bargmann space. By $\widehat{H}_{wick}$ and $\widehat{H}_{anti-wick}$ we mean the unbounded operators on Bargmann space given by the Wick and Anti-Wick quantization procedures respectively, note that these procedures yield operators on Bargmann space.

The relationship between $\widehat{H}_{weyl}$, $\widehat{H}_{wick}$, and $\widehat{H}_{anti-wick}$ is given by the following (rather amazing) result.

Proposition II.16. *Given a smooth polynomial Hamiltonian* $H = H(\mathbf{z}, \overline{\mathbf{z}})$ *on* $\mathbb{C}^n$*, then*

$$\widehat{H}_{anti-wick} = \left(e^{\frac{\hbar}{4}\Delta} H\right)_{weyl} \qquad \text{and} \qquad \widehat{H}_{weyl} = \left(e^{\frac{\hbar}{4}\Delta} H\right)_{wick}, \tag{2.6}$$

where Δ *is the Laplacian. Since* H *is a polynomial then we can understand the action of the operator* $e^{\frac{\hbar}{4}\Delta}$ *on* H *by the operator power series*

$$e^{\frac{\hbar}{4}\Delta} H = \sum_{j=0}^{\infty} \left(\frac{\hbar}{4}\right) \Delta^k H. \tag{2.7}$$

For a proof of this proposition see [15]. As we'll see in chapter IV the relationship between the Weyl and Wick quantization schemes will be fundamental to most of the calculations in this work.

CHAPTER III

Generalized Lagrangian States in Bargmann Space

In this chapter we begin by introducing the central object in this work, a generalized Lagrangian state on Bargmann space over $\mathbb{C}^n$. Then we prove some basic facts related to their microlocal/semiclassical properties. We will see that generalized Lagrangian states weave together the geometry of classical mechanics with the analysis of quantum mechanics in a very beautiful and useful way. Lagrangian submanifolds play a central role in the mathematical description of classical mechanics in terms of symplectic geometry. We will see that the creation of a generalized state in Bargmann space that is intrinsically associated to to a Lagrangian submanifold of $\mathbb{C}^n$ will give us a way to bridge the mathematical gap between the descriptions of classical mechanics, in terms of geometry, and quantum mechanics, in terms of analysis. These states were first introduced in [10], [33], [21].

3.1 Defining Generalized Lagrangian States

We begin the discussion of generalized states by defining them as a geometric object expressed in a coordinate free manner. Before we can give the definition we'll need some facts from differential geometry.

3.1.1 Results on Lagrangian Embeddings

Let's first recall the definition of a Lagrangian embedding.

Definition III.1. Suppose that Λ is an n-dimensional smooth manifold, and M is a $2n$-dimensional smooth symplectic manifold with symplectic form ω. A mapping $\mathbf{w} : \Lambda \longrightarrow M$ is a *Lagrangian embedding* if the following conditions are satisfied:

1. At each $\mathbf{x} \in \Lambda$ the differential map $\mathbf{w}_* : T_{\mathbf{x}}\Lambda \longrightarrow T_{\mathbf{w}(\mathbf{x})}M$ is injective.

2. $\mathbf{w}$ is a homoemorphism of Λ onto $\mathbf{w}(\Lambda)$, the image of $\mathbf{w}$.

3. The image of $\mathbf{w}$, $\mathbf{w}(\Lambda)$, is a Lagrangian submanifold of M with respect to the symplectic form ω. Thus, $\dim(\mathbf{w}(\Lambda)) = \frac{1}{2}\dim(M)$, and $\mathbf{w}^*\omega = 0$.

With the definition of a Lagrangian embedding in place we'll next introduce some facts about such embeddings that will be central in defining our generalized states. For the purposes in this work we won't need to consider general symplectic manifolds, specifically we'll take $M = \mathbb{C}^n$ by making the association $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Complex Structures on $\mathbb{R}^{2n}$

Let's consider some well known basic notions about linear complex structures on a vector space. See [29] and [43].

Suppose that V is a vector space over $\mathbb{R}$. A *complex structure* on V is an automorphism $J : V \longrightarrow V$ such that $J^2 = -I$, where I is the identity map on V. The set of all complex structures on V is denoted $\mathcal{J}(V)$. With such a map we can endow V with the structure of a complex vector space via the following relation defining multiplication by complex scalars

$$\mathbb{C} \times V \longrightarrow V$$

$$(s + it, v) \longmapsto sv + tJv.$$

With this structure we can consider V as a vector space over $\mathbb{C}$, which we will denote V_J.

Suppose the complex dimension of V_J is n. This means that there exist $e_1, \ldots, e_n \in V$ such that for all $v \in V_{\mathbb{C}}$ there are unique scalars $c_1, \ldots, c_n \in \mathbb{C}$ such that $v = c_1 e_1 + \cdots + c_n e_n$. If we denote $c_j = a_j + ib_j$ then, given the rule above for multiplication by complex scalars, we can conclude that for any $v \in V$ there exist unique scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ such that

$$\begin{aligned} v &= (a_1 e_1 + b_1 J e_1) + \cdots + (a_n e_n + b_n J e_n) \\ &= a_1 e_1 + \cdots + a_n e_n + b_1 J e_1 + \cdots + b_n J e_n. \end{aligned}$$

This implies that $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is a basis for V (over $\mathbb{R}$). Thus V is necessarily even dimensional. If we let $f_j \equiv Je_j$ then we have the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ which has the rules

$$\begin{aligned} Je_j &= f_j \\ Jf_j &= J^2 e_j = -e_j. \end{aligned}$$

Starting with these rules, extend the map J to the space $V \otimes \mathbb{C}$ by having it act linearly over $\mathbb{C}$, and call this new map $J_{\mathbb{C}}$. Note that $J_{\mathbb{C}}^2 = -I$, and that $V \otimes \mathbb{C}$ is a $2n$ dimensional complex vector space. If we consider the vectors $v_j \equiv \frac{1}{2}(e_j - if_j)$ and $\overline{v}_j \equiv \frac{1}{2}(e_j + if_j)$ in $V \otimes \mathbb{C}$ then we have

$$J_{\mathbb{C}}v_j = \frac{1}{2}\Big(J_{\mathbb{C}}e_j - iJ_{\mathbb{C}}f_j\Big) = \frac{1}{2}\Big(f_j - i(-e_j)\Big) = i\frac{1}{2}\Big(e_j - f_j\Big) = iv_j,$$

$$J_{\mathbb{C}}\overline{v}_j = \frac{1}{2}\Big(J_{\mathbb{C}}e_j + iJ_{\mathbb{C}}f_j\Big) = \frac{1}{2}\Big(f_j + i(-e_j)\Big) = -i\frac{1}{2}\Big(e_j + if_j\Big) = -i\overline{v}_j.$$

We can define the subspaces of $V \otimes \mathbb{C}$

$$V^{(1,0)} \equiv \operatorname{span}\{v_j | 1 \leq j \leq n\} \qquad\qquad V^{(0,1)} \equiv \operatorname{span}\{\overline{v}_j | 1 \leq j \leq n\}.$$

If $(v, z) \in V \otimes \mathbb{C}$ then the action (mapping) of complex conjugation is defined as $\overline{(v, z)} = (v, \overline{z})$ and extended linearly, so by definition $\overline{V^{(1,0)}} = V^{(0,1)}$. The rules above show that $J_{\mathbb{C}}$ acts as multiplication by i on $V^{(1,0)}$, and acts as multiplication by $-i$ on $V^{(0,1)}$. Also, $V \otimes \mathbb{C} = V^{(1,0)} \oplus V^{(0,1)}$ since the fact that $v_1, \ldots, v_n, \overline{v}_1, \overline{v}_n$ span $V \otimes \mathbb{C}$ follows from the fact that $e_1, \ldots, e_n, f_1, \ldots, f_n$ are a basis for V. This implies that $V^{(1,0)} = \{v \in V \otimes \mathbb{C} | J_{\mathbb{C}}v = iv\}$, and $V^{(0,1)} = \{v \in V \otimes \mathbb{C} | J_{\mathbb{C}}v = -iv\}$. In turn, this implies that these subspaces are independent of the basis $e_1, \ldots, e_n$, and so depend only on the map J.

The decomposition $V \otimes \mathbb{C} = V^{(1,0)} \oplus V^{(0,1)}$ gives a decomposition of the dual $(V \otimes \mathbb{C})^* = \big(V^{(1,0)}\big)^* \oplus \big(V^{(0,1)}\big)^*$, such that $\big(V^{(1,0)}\big)^* = \operatorname{span}\{v^j | 1 \leq j \leq n\}$ where

$$v^j(v_k) = \delta_{jk} \qquad \text{and} \qquad v^j(\overline{v}_k) = 0 \qquad \forall j, k,$$

and $\big(V^{(0,1)}\big)^* = \operatorname{span}\{\overline{v}^j | 1 \leq j \leq n\}$ where

$$\overline{v}^j(\overline{v}_k) = \delta_{jk} \qquad \text{and} \qquad \overline{v}^j(v_k) = 0 \qquad \forall j, k.$$

Notice that V_J is a complex vector space of (complex) dimension n such that the action of J is multiplication by i, and $V^{(1,0)}$ is a complex vector space of (complex) dimension n such that the action of $J_{\mathbb{C}}$ is multiplication by i. Therefore there is an obvious association to be made between these two spaces.

So far we've considered the basics of linear complex structures. We'll need the next few definitions in order to tie the idea of a complex structure to that of a symplectic structure.

Definition III.2. A real subspace $U \subset V$ is known as a *complex subspace* of V_J if $JU = U$.

Definition III.3. A real subspace $U \subset V$ is known as *totally real* if U contains no nontrivial subspaces which are complex.

Definition III.4. Let (V, ω) be a symplectic vector space. A complex structure $J \in \mathcal{J}(V)$ is said to be compatible with ω if

$$\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$$

for all $v_1, v_2 \in V$, and

$$\omega(v, Jv) > 0$$

for all nonzero $v \in V$. If J is compatible with ω then the pair (J, ω) is called *Kahler.*

With all of the necessary definitions in place we have the following proposition.

Proposition III.5. *Let $U \subset V$ be a Lagrangian subspace of the symplectic vector space (V, ω), and let $J \in \mathcal{J}(V)$ such that (J, ω) is Kahler. Then U is totally real.*

Proof. (By Contradiction) Suppose $W \subset U$ is a nonzero complex subspace of U. If $u \in W \backslash \{0\}$ then $Ju \in W \subset U$. Since U is Lagrangian we have that $\omega(u, Ju) = 0$, but this contradicts the fact that (J, ω) is Kahler. □

With all of this background on complex structures in place we are ready to prove the following lemma.

Lemma III.6. *Let (V, ω) be a symplectic vector space, and $J \in \mathcal{J}(V)$ that is compatible with ω. Suppose U is a Lagrangian subspace of V, and $E_1, \ldots, E_n$ are a basis for U. Since $V \subset V \otimes \mathbb{C}$, we can decompose each E_j so that $E_j = E_j^{(1,0)} + E_j^{(0,1)}$ where $E_j^{(1,0)} \in V^{(1,0)}$ and $E_j^{(0,1)} \in V^{(0,1)}$ then $E_1^{(1,0)}, \ldots, E_n^{(1,0)}$ are linearly independent.*

Proof. (By Contradiction) Suppose $E_j^{(1,0)} = \sum_{k \neq j} c_k E_k^{(1,0)}$ for some $j \in \{1, \ldots, n\}$ where $c_k = a_k + ib_k \in \mathbb{C}$. Then

$$\begin{aligned} E_j &= E_j^{(1,0)} + E_j^{(0,1)} \\ &= E_j^{(1,0)} + \overline{E_j^{(1,0)}} \\ &= \sum_{k \neq j} c_k E_k^{(1,0)} + \sum_{k \neq j} \overline{c_k} E_k^{(0,1)}. \end{aligned}$$

Which implies

$$\begin{aligned} JE_j &= \sum_{k \neq j} c_k J E_k^{(1,0)} + \overline{c_k} J E_k^{(0,1)} \\ &= \sum_{k \neq j} i c_k E_k^{(1,0)} - i \overline{c_k} E_k^{(0,1)} \\ &= \sum_{k \neq j} i(a_k + ib_k) E_k^{(1,0)} - i(a_k - ib_k) E_k^{(0,1)} \\ &= \sum_{k \neq j} i a_k \left(E_k^{(1,0)} - E_k^{(0,1)} \right) - b_k \left(E_k^{(1,0)} + E_k^{(0,1)} \right). \end{aligned}$$

Now,

$$i\left(E_k^{(1,0)} - E_k^{(0,1)}\right) = i\left(\frac{1}{2}\left(E_k - iJE_k\right) - \frac{1}{2}\left(E_k + iJE_k\right)\right)$$
$$= i\left(-\frac{i}{2}JE_k - -\frac{i}{2}JE_k\right)$$
$$= JE_k,$$

and similarly

$$E_k^{(1,0)} + E_k^{(0,1)} = \left(\frac{1}{2}\left(E_k - iJE_k\right) + \frac{1}{2}\left(E_k + iJE_k\right)\right) = E_k.$$

therefore

$$JE_j = \sum_{k\neq j} a_k JE_k + b_k E_k$$
$$= \sum_{k\neq j} a_k E_k + \left(\sum_{k\neq j} b_k E_k\right).$$

We know that $\sum_{k\neq j} a_k E_k$ and $\sum_{k\neq j} b_k E_k$ are both in U. Since (J,ω) is Kahler we know that $\omega\left(\sum_{k\neq j} a_k E_k, J\left(\sum_{k\neq j} a_k E_k\right)\right) > 0$. From this we can conclude that $J\left(\sum_{k\neq j} a_k E_k\right)$ cannot be in U since otherwise the fact that U is Lagrangian would imply that $\omega\left(\sum_{k\neq j} a_k E_k, J\left(\sum_{k\neq j} a_k E_k\right)\right)$ is zero.

Thus $JE_j = Jv_1 + v_2$ for some $v_1, v_2 \in U$. We know that $v_2 \neq 0$ since otherwise it would be the case that $JE_j = J\left(\sum_{k\neq j} a_k E_k\right)$, and since J is an automorphism this implies that $E_j = \sum_{k\neq j} a_k E_k$, which contradicts the fact that $E_1, \ldots, E_n$ are a basis of U. Also, we know that $v_1 \neq 0$ since the fact that (J,ω) is Kahler implies that

$$\begin{aligned} 0 &\neq \omega\left(E_k, JE_k\right) \\ &= \omega(E_k, Jv_1 + v_2) \\ &= \omega(E_k, Jv_1) + \omega(E_k, v_2) \\ &= \omega(E_k, Jv_1) \end{aligned}$$

and $\omega(E_k, Jv_1) \neq 0$ implies $Jv_1 \neq 0$. Note that we used the fact that since $E_k, v_2 \in U$ and U is Lagrangian we have $\omega(E_k, v_2) = 0$.

Finally, we note that

$$-E_j = J\left(JE_j\right) = J\left(Jv_1 + v_2\right) = -v_1 + Jv_2.$$

Thus $Jv_2 = v_1 - E_j \in U$. But since U is totally real and $v_2 \in U$ the only way this can hold is if $v_2 = 0$ which we've shown cannot be true. Thus, by contradiction

$$E_1^{(1,0)}, \ldots, E_n^{(1,0)} \qquad \text{are linearly independent.}$$

$\square$

Now let's specialize the discussion to $V = \mathbb{R}^{2n}$. By choosing a complex structure, J, on $\mathbb{R}^{2n}$, and following the above construction we can make the association $\left(\mathbb{R}^{2n} \otimes \mathbb{C}\right)^{(1,0)} \cong (\mathbb{R}^{2n})_J \cong \mathbb{C}^n$. This is our association of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$.

With this in mind we are ready to use the previous lemma to prove the following result.

Proposition III.7. *Let Λ be an n-dimensional smooth manifold, and $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ a Lagrangian embedding, where $\mathbb{C}^n$ is defined with respect to the association with*

$\mathbb{R}^{2n}$ along with the symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$, facilitated by a complex structure $J \in \mathcal{J}(\mathbb{R}^{2n})$ that is compatible with ω. Then

$$d\mathbf{w} \equiv \mathbf{w}^*(d\mathbf{z}) \tag{3.1}$$

is an n-form on Λ that is everywhere nondegenerate, where by definition $d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_n$.

Proof. On $\mathbb{C}^n$ (a.k.a. $(\mathbb{R}^{2n} \otimes \mathbb{C})^{(1,0)}$) we have the n-form

$$d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_n,$$

Define $d\mathbf{w} \equiv \mathbf{w}^*(d\mathbf{z})$, which is an n-form on Λ.

We can also think of Choose any $\mathbf{x}_0 \in \Lambda$. Let U be an open neighborhood of $\mathbf{x}_0$ with coordinates $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\mathbf{z}_0 \equiv \mathbf{w}(\mathbf{x}_0)$. So, $\mathbf{w}_* : T_{\mathbf{x}_0}\Lambda \longrightarrow T_{\mathbf{z}_0}\mathbf{C}^n$.

In order to understand $d\mathbf{w}$ we will need to understand the map $\mathbf{w}_*$, which is a map on tangent vectors to Λ at $\mathbf{x}_0$. To that end we'll do some basic differential geometry, see [2],[30],[38] for good discussions on this material. In order to characterize a general tangent vector to Λ at $\mathbf{x}_0$ we consider the velocity vector of a general curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \Lambda$ such that $\gamma(0) = \mathbf{x}_0$ and $\gamma((-\epsilon, \epsilon)) \subset U$. We can express γ in the local coordinates on U, so for $t \in (-\epsilon, \epsilon)$ we have $\gamma(t) = (x_1(t), \ldots, x_n(t))$. Suppose that in these local coordinates $\mathbf{x}_0 = (x_{01}, \ldots, x_{0n})$. For $f \in C^\infty(\Lambda)$, the action of the tangent vector $\gamma'(0)$ is

$$\begin{aligned}\gamma'(0)f &= \frac{d}{dt}(f\circ\gamma)\Big|_{t=0}\\ &= \frac{d}{dt}f(x_1(t),\ldots,x_n(t))\Big|_{t=0}\\ &= \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{dx_j}{dt}\Big|_{t=0}\\ &= \sum_{j=1}^{n}x_j'(0)\frac{\partial}{\partial x_j}\Big|_{\mathbf{x}_0}f.\end{aligned}$$

Thus we have the well-known result that, expressed in the local coordinates $\mathbf{x} = (x_1,\ldots,x_n)$, we have

$$\gamma'(0) = \sum_{j=1}^{n}x_j'(0)\frac{\partial}{\partial x_j}\Big|_{\mathbf{x}_0}.$$

This implies shows that

$$\frac{\partial}{\partial x_1}\Big|_{\mathbf{x}_0},\ldots,\frac{\partial}{\partial x_n}\Big|_{\mathbf{x}_0}\quad\text{are a basis for } T_{\mathbf{x}_0}\Lambda.$$

To understand $\mathbf{w}_*$, let's characterize $\mathbf{w}_*(\gamma'(0))g$ for an arbitrary smooth $g \in C^\infty(\mathbb{C}^n)$. By definition

$$\begin{aligned}\mathbf{w}_*(\gamma'(0))g &= \frac{d}{dt}\Big(g\circ(\mathbf{w}\circ\gamma,\overline{\mathbf{w}\circ\gamma})\Big)\Big|_{t=0}\\ &= \frac{d}{dt}\Big(g(\mathbf{z}(\mathbf{w}\circ\gamma),\overline{z}(\mathbf{w}\circ\gamma))\Big)\Big|_{t=0}\\ &= \sum_{k=1}^{n}\left[\frac{\partial g}{\partial z_k}\left(\sum_{j=1}^{n}\frac{\partial z_k}{\partial x_j}\frac{\partial x_j}{\partial t}\right)+\frac{\partial g}{\partial \overline{z}_k}\left(\sum_{j=1}^{n}\frac{\partial \overline{z}_k}{\partial x_j}\frac{\partial x_j}{\partial t}\right)\right]\Big|_{t=0}\\ &= \sum_{k=1}^{n}\left[\left(\sum_{j=1}^{n}\frac{\partial w_k}{\partial x_j}\frac{dx_j}{dt}\right)\frac{\partial}{\partial z_k}+\left(\sum_{j=1}^{n}\frac{\partial \overline{w}_k}{\partial x_j}\frac{dx_j}{dt}\right)\frac{\partial}{\partial \overline{z}_k}\right]g\Big|_{t=0}\end{aligned}$$

(where w_k and $\overline{w}_k$ denote $z_k\circ\mathbf{w}$ and $\overline{z}_k\circ\mathbf{w}$ respectively)

Thus we have the relation that

$$\mathbf{w}_*(\gamma'(0)) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} x_j'(0)\frac{\partial w_k}{\partial x_j}\right)\frac{\partial}{\partial z_k}\bigg|_{\mathbf{z}_0} + \left(\sum_{j=1}^{n} x_j'(0)\frac{\partial \overline{w}_k}{\partial x_j}\right)\frac{\partial}{\partial \overline{z}_k}\bigg|_{\mathbf{z}_0}.$$

which highlights the well-known result that

$$\frac{\partial}{\partial z_1}\bigg|_{\mathbf{z}_0}, \ldots, \frac{\partial}{\partial z_n}\bigg|_{\mathbf{z}_0}, \frac{\partial}{\partial \overline{z}_1}\bigg|_{\mathbf{z}_0}, \ldots, \frac{\partial}{\partial \overline{z}_n}\bigg|_{\mathbf{z}_0} \quad \text{are a basis for } T_{\mathbf{z}_0}\mathbb{C}^n.$$

Consider the special case where the curve γ is a curve of the form $\gamma(t) = (x_{01}, \ldots, x_j + t, \ldots, x_{0n})$. Then $\gamma'(0) = \frac{\partial}{\partial x_j}\Big|_{\mathbf{x}_0}$. The above result gives us that

$$\mathbf{w}_*\left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{x}_0}\right) = \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial x_j}\frac{\partial}{\partial z_k}\bigg|_{\mathbf{z}_0} + \frac{\partial \overline{w}_k}{\partial x_j}\frac{\partial}{\partial \overline{z}_k}\bigg|_{\mathbf{z}_0}\right).$$

Now, since $d\mathbf{w}$ is a multi-linear map on the tangent space at each point in Λ in order to characterize its behavior at our arbitrarily chosen $\mathbf{x}_0$ it suffices to understand how it acts on the basis

$$\frac{\partial}{\partial x_1}\bigg|_{\mathbf{x}_0}, \ldots, \frac{\partial}{\partial x_n}\bigg|_{\mathbf{x}_0}.$$

So,

$$
\begin{aligned}
d\mathbf{w}\left(\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{x}_0},\ldots,\left.\frac{\partial}{\partial x_n}\right|_{\mathbf{x}_0}\right)(\mathbf{x}_0) &= dz_1\wedge\cdots\wedge dz_n\left(\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{x}_0}\right),\ldots,\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_n}\right|_{\mathbf{x}_0}\right)\right)(\mathbf{z}_0)\\
&= dz_1\wedge\cdots\wedge dz_n\Bigg(\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial x_1}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}+\frac{\partial \overline{w}_k}{\partial x_1}\left.\frac{\partial}{\partial \overline{z}_k}\right|_{\mathbf{z}_0}\right),\ldots,\\
&\qquad \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial x_n}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}+\frac{\partial \overline{w}_k}{\partial x_n}\left.\frac{\partial}{\partial \overline{z}_k}\right|_{\mathbf{z}_0}\right)\Bigg)(\mathbf{z}_0)\\
&= dz_1\wedge\cdots\wedge dz_n\left(\sum_{k=1}^{n}\frac{\partial w_k}{\partial x_1}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0},\ldots,\sum_{k=1}^{n}\frac{\partial w_k}{\partial x_n}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}\right)(\mathbf{z}_0)\\
&\qquad \left(\because\ dz_j\left(\frac{\partial}{\partial \overline{z}_k}\right)=0\ \forall\, j,k\right)
\end{aligned}
$$

Now, since $\mathbf{w}$ is an embedding and $\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{x}_0},\ldots,\left.\frac{\partial}{\partial x_n}\right|_{\mathbf{x}_0}$ are a basis for $T_{\mathbf{x}_0}\Lambda$ we know that $\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{x}_0}\right),\ldots,\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_n}\right|_{\mathbf{x}_0}\right)$ are a basis for $T_{\mathbf{z}_0}(\mathbf{w}(\Lambda))$. Now, if we denote each pushed forward vector as $E_j\equiv\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_j}\right|_{\mathbf{x}_0}\right)$, then $E_1,\ldots,E_n$ are a basis of $T_{\mathbf{z}_0}(\mathbf{w}(\Lambda))$ which is a Lagrangian subspace of $T_{\mathbf{z}_0}\mathbb{C}^n$. We saw above that

$$
\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_j}\right|_{\mathbf{x}_0}\right)=\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial x_j}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}+\frac{\partial \overline{w}_k}{\partial x_j}\left.\frac{\partial}{\partial \overline{z}_k}\right|_{\mathbf{z}_0}\right).
$$

In light of the discussion above concerning the decomposition $\mathbb{C}^n\cong(\mathbb{R}^{2n})_{\mathbb{C}}=(\mathbb{R}^{2n})^{(1,0)}\oplus(\mathbb{R}^{2n})^{(0,1)}$ we have for $E_j=\mathbf{w}_*\left(\left.\frac{\partial}{\partial x_j}\right|_{\mathbf{x}_0}\right)$

$$
E_j^{(1,0)}\equiv\sum_{k=1}^{n}\frac{\partial w_k}{\partial x_j}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}\in\left(\mathbb{R}^{2n}\right)^{(1,0)}\qquad\text{and}\qquad E_j^{(0,1)}\sum_{k=1}^{n}\frac{\partial \overline{w}_k}{\partial x_j}\left.\frac{\partial}{\partial \overline{z}_k}\right|_{\mathbf{z}_0}\in\left(\mathbb{R}^{2n}\right)^{(0,1)}.
$$

The above lemma, III.6, tell us that $E_1^{(1,0)},\ldots,E_n^{(1,0)}$ are linearly independent. Thus, it follows from a well known fact about forms that $dz_1\wedge\cdots\wedge dz_n\left(E_1^{(1,0)},\ldots,E_n^{(1,0)}\right)(\mathbf{z}_0)\neq 0$, and therefore $d\mathbf{w}$ is a nondegenerate n-form on Λ.

$\square$

The following corollary follows easily from the proof above

Corollary III.8. *Given Λ, an n-dimensional smooth manifold, and $\mathbf{w} : \Lambda \longrightarrow \mathbb{R}^{2n}$ a Lagrangian embedding. Suppose that $\mathbb{R}^{2n}$ is given a complex structure that is compatible with $\omega = d\mathbf{p} \wedge d\mathbf{q}$, thus we can consider the embedding as $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. Let U be an open neighborhood in Λ with smooth coordinate system $\mathbf{x} = (x_1, \dots, x_n)$. Then to each $\mathbf{x}_0 \in \Lambda$ the complex-valued matrix $\frac{\partial \mathbf{w}}{\partial \mathbf{x}}(\mathbf{x}_0)$ is nondegenerate, where*

$$\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)_{jk}(\mathbf{x}_0) \equiv \frac{\partial w_j}{\partial x_k}(\mathbf{x}_0),$$

and w_j is the j^{th} complex coordinate of $\mathbf{w}$, restricted to U and expressed in the local coordinates $x_1, \dots, x_n$.

Proof. Following from the proof above we have

$$\begin{aligned} d\mathbf{w}\left(\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{x}_0}, \dots, \left.\frac{\partial}{\partial x_n}\right|_{\mathbf{x}_0}\right)(\mathbf{x}_0) &= dz_1 \wedge \cdots \wedge dz_n\left(\sum_{k=1}^{n} \frac{\partial w_k}{\partial x_1}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}, \dots, \sum_{k=1}^{n} \frac{\partial w_k}{\partial x_n}\left.\frac{\partial}{\partial z_k}\right|_{\mathbf{z}_0}\right)(\mathbf{z}_0) \\ &\qquad \left(\because\ dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0\ \forall\, j, k\right) \\ &= \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) dz_1 \wedge \cdots \wedge dz_n\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) \\ &= \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \end{aligned}$$

where $\frac{\partial \mathbf{w}}{\partial \mathbf{x}}$ denotes the complex valued matrix

$$\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \cdots & \frac{\partial w_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_n}{\partial x_1} & \cdots & \frac{\partial w_n}{\partial x_n} \end{pmatrix}.$$

The above proposition shows that $\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \neq 0$.

$\square$

The results above, (3.1) and (III.8), combine to show that with respect to a chosen compatible complex structure on $\mathbb{R}^{2n}$ we have that for an arbitrary $\mathbf{x}_0 \in \Lambda$ and coordinates $\mathbf{x}$ chosen in a coordinate chart containing $\mathbf{x}_0$ the local expression

$$d\mathbf{w}(\mathbf{x}_0) = \left(\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) d\mathbf{x}\right)(\mathbf{x}_0),$$

as n-forms on Λ.

3.1.2 Generalized Lagrangian States

We're now ready to introduce the idea of a generalized Lagrangian state on Bargmann space. We will first define these states in a coordinate free manner in terms of densities on Λ, and then we will introduce an expression using local coordinates which will be used extensively in the work that follows. For discussions on the theory of densities on manifolds see [9], [13], [17], and [42].

Before our main attraction we'll need two more preliminary definitions.

Definition III.9. Suppose that Λ is a smooth connected n-dimensional manifold. Consider $\mathbb{R}^{2n}$ with coordinates $(\mathbf{q}, \mathbf{p})$ as a symplectic manifold with the standard symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$. Suppose $\mathbf{w} : \Lambda \longrightarrow \mathbb{R}^{2n}$ is a Lagrangian embedding. A real-valued function $f : \Lambda \longrightarrow \mathbb{R}$ will be called a *lift function* if it satisfies the condition

$$df + \frac{1}{2}\mathbf{w}^* \left(\mathbf{q} \cdot d\mathbf{p} - \mathbf{p} \cdot d\mathbf{q}\right) = 0.$$

For each $\mathbf{x} \in \Lambda$ we can choose a local coordinates system $x_1, \ldots, x_n$ in a neighborhood, U, of $\mathbf{x}$, in which case we can consider $\mathbf{w}(U) = \left\{ (\mathbf{q}(x_1, \ldots, x_n), \mathbf{p}(x_1, \ldots, x_n)) \middle| (x_1, \ldots, x_n) \in U \right\}$, and so the above condition is locally equivalent to the system of n first-order partial differential equations

$$\frac{\partial f}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\left(q_k \frac{\partial p_k}{\partial x_j} - p_k \frac{\partial q_k}{\partial x_j}\right) = 0$$

for $j = 1, \ldots, n$.

Remark III.10. A (globally defined) lift f may not exist in general. The existence of a global lift is guaranteed by the cohomological criterion on Λ that the pullback of $\frac{1}{2}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q})$, a 1-form on $\mathbb{R}^{2n}$, to Λ is exact. In other words, there exists an $f \in C^{\infty}(\Lambda)$ such that

$$-\frac{1}{2}\mathbf{w}^{*}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q}) = df.$$

If we consider the 1-form on Λ used in the lift criterion, $-\frac{1}{2}\mathbf{w}^{*}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q})$, then it's exterior derivative is

$$\begin{aligned} d\left(-\frac{1}{2}\mathbf{w}^{*}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q})\right) &= \mathbf{w}^{*}\left(d\left(-\frac{1}{2}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q})\right)\right) \\ &= \mathbf{w}^{*}(\omega) \\ &= 0, \end{aligned}$$

where ω is the symplectic form, in the first line we used the fact that (appropriately understood) the exterior derivative commutes with pullbacks, and the last line follows from the definition of a Lagrangian embedding (III.1). Thus $\frac{1}{2}\mathbf{w}^{*}(\mathbf{q}\cdot d\mathbf{p} - \mathbf{p}\cdot d\mathbf{q})$ is a closed form on Λ; and so the Poincare Lemma (see [2]) tells us that in each coordinate chart on Λ there exists locally defined smooth functions that satisfy the lift condition. Thus the above system of partial differential equations that locally define a lift function can always be satisfied. We will concern ourselves with Λ's such that these local conditions can be satisfied globally on Λ.

If Λ is simply connected then such an f will always exist (globally). If Λ has more topological structure then additional conditions may be necessary in order to ensure it's existence. For the purposes of this work, see III.12 below, we are interested in the exponential of the lift function $e^{\frac{i}{\hbar}f}$. Thus in the case that Λ is topologically complicated enough that the above cohomological condition cannot be globally satisfied we may be able to impose additional conditions that will allow for $e^{\frac{i}{\hbar}f}$ to be well-defined globally on Λ.

As we'll see in the example at the end of this chapter concerning the basis states of the quantum harmonic oscillator, an example of such a condition is the restriction of the allowable values of $\hbar$ to specific sequences of values. Said another way, such a condition can require the values of $\hbar$ to be, in a sense, quantized. Such a condition is an example of a Bohr-Sommerfeld condition on Λ.

Definition III.11. Suppose that Λ is a connected smooth n-dimensional manifold and that $a : \Lambda \longrightarrow \mathbb{C}$ is a smooth $\hbar$-dependent function. Then a will be called an *amplitude function* if there exists smooth $\hbar$-independent functions $a_j : \Lambda \longrightarrow \mathbb{C}$ for $j = 1, 2, \ldots$ such that for every compact $U \subset \Lambda$, every $N \in \mathbb{N}$, and every $\beta \in \mathbb{N}^n$, there exist constants $C > 0$ and $\hbar_0 > 0$ (each dependent on the choice of both U, N, and β) such that for all $\mathbf{x} \in U$ and values $\hbar \in (0, \hbar_0]$

$$\left| \frac{\partial^\beta}{\partial x^\beta} \left(a(\mathbf{x}) - \sum_{j=0}^{N} \hbar^j a_j(\mathbf{x}) \right) \right| \leq C\hbar^{N+1}.$$

This asymptotic behavior will be denoted

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j.$$

We now have all the pieces we need to define the central object of this work. Cue the drum-roll.

Definition III.12. (Generalized Lagrangian State) Let Λ be an n-dimensional smooth manifold, and $\mathbf{w} : \Lambda \longrightarrow \mathbb{R}^{2n}$ a Lagrangian embedding with respect to the symplectic structure, $\omega = d\mathbf{p} \wedge d\mathbf{q}$, on $\mathbb{R}^{2n}$. We make the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with the choice of a compatible complex structure on $\mathbb{R}^{2n}$. With only a slight abuse of notation we then consider $\mathbf{w}(\Lambda)$ to be a Lagrangian submanifold of $\mathbb{C}^n$. Suppose that $a : \Lambda \longrightarrow \mathbb{C}$ is an amplitude function, and that Λ admits a lift condition defining a lift function $f : \Lambda \longrightarrow \mathbb{R}$.

A *generalized Lagrangian state* on Bargmann space, denoted $|\Lambda, a\rangle$, is defined as the $\mathbf{z} \in \mathbb{C}^n$ dependent integral

$$|\Lambda, a\rangle(\mathbf{z}) \equiv \int_\Lambda a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) |d\mathbf{w}|.$$

Where recall

- $K(\mathbf{z}, \mathbf{v}) = e^{\frac{\overline{\mathbf{v}} \cdot \mathbf{z}}{\hbar}} e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\mathbf{v} \cdot \overline{\mathbf{v}}}{2\hbar}}$ is the reproducing kernel in Bargmann space, (2.5),
- $|d\mathbf{w}|$ is the 1-density on Λ derived from the nondegenerate n-form $d\mathbf{w}(\mathbf{x}_0) = \left(\det \left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right) d\mathbf{x} \right) (\mathbf{x}_0)$ on Λ from (3.1)

Intimately related to the notion of a generalized Lagrangian state will be the idea of the symbol of a state.

Definition III.13. Given a generalized Lagrangian state $|\Lambda, a\rangle$ defined with respect to the Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. The 1/2-density $\delta_\mathbf{w} \equiv a|d\mathbf{w}|^{1/2}$, see 3.1, will be called the *symbol* of the generalized state. By assumption a has the asymptotic nature $a \sim \sum_{j=0}^\infty \hbar^j a_j$, (III.11). The terms $\delta^0_\mathbf{w} \equiv a_0|d\mathbf{w}|^{1/2}$ and $\delta^1_\mathbf{w} \equiv a_1|d\mathbf{w}|^{1/2}$ will be called the *principal symbol* and *sub-principal symbol* of the generalized state, respectively.

This is the central definition of this entire work, and as such it deserves some detailed discussion.

Remark III.14. First, let's address two matters of notation. Note that the notation for a generalized Lagrangian state, $|\Lambda, \alpha\rangle$, does not include the embedding $\mathbf{w}$ of Λ. Even though the embedding is not included in the notation, the state is intrinsically dependent on $\mathbf{w}$. The choice to not include $\mathbf{w}$ was made to prevent the notation from becoming too cumbersome. Second, we should also note that the condition defining the lift function f, see (III.9), will only define it up to an arbitrary additive integration constant. Thus a generalized Lagrangian state will be well-defined up to a multiplicative phase term of the form $e^{\frac{i}{\hbar}c}$, for a constant c. This is an oscillatory term in the limit $\hbar \longrightarrow 0$, so it doesn't have a completely trivial semiclassical nature, but it amounts to a global phase. In the spirit of a similar choice made commonly in quantum theory when faced with such global phases, we will assume the integration constant is zero. Also, the lift function f is also left out of the notation for the sake of compactness.

Remark III.15. Note that the type of mathematical object that any specific generalized state can be categorized as will be dependent on both Λ and the function a. Indeed for certain choices of Λ and a it will be the case that the formal integral will be well-defined as an integral in the sense of manifolds for each value of $\mathbf{z} \in \mathbb{C}^n$, and thus such a generalized Lagrangian state can be thought of as a holomorphic function on $\mathbb{C}^n$. On the other hand, for certain Λ and a, it will be the case that a generalized Lagrangian state cannot be thought of so literally, and must be thought of in the weaker sense as a linear functional on some subspace of the space of complex-valued functions. We will present a result shortly that will give some perspective concerning the nature of a generalized Lagrangian state for certain Λ and α.

We'll now introduce a more 'hands-on' form for Generalized Lagrangian states that will be used often in this work. A generalized Lagrangian states is an integral over Λ in the sense of manifolds. Since Λ has a manifold structure we know that one can find a locally-finite open cover of Λ, which we denote by $\{U_\gamma \subset \Lambda | \gamma \in \Gamma\}$, with the property that for each γ we have $\overline{U_\gamma}$ is contained in a coordinate patch of an atlas of Λ, where $\Gamma \subset \mathbb{N}$ is an indexing set. Furthermore, there exists a partition of unity subordinate to this open cover, $\{f_\gamma \in C_0^\infty(U_\gamma) | \gamma \in \Gamma\}$. On each coordinate patch we can consider a coordinate system $\mathbf{x}_\gamma = (x_{\gamma_1}, \ldots, x_{\gamma_n})$. We may then (formally if necessary) decompose the above integral such that

$$\begin{aligned}
|\Lambda, a\rangle(\mathbf{z}) &= \int_\Lambda a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) |d\mathbf{w}| \\
&= \sum_{\gamma \in \Gamma} \int_{U_\gamma} f_\gamma(\mathbf{x}_\gamma) a(\mathbf{x}_\gamma; \hbar) e^{\frac{i}{\hbar} f(\mathbf{x}_\gamma)} K(\mathbf{z}, \mathbf{w}(\mathbf{x}_\gamma)) \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}_\gamma}\right) \right| |d\mathbf{x}_\gamma| \\
&= \sum_{\gamma \in \Gamma} \int_{U_\gamma} f_\gamma(\mathbf{x}_\gamma) \sigma(\mathbf{x}_\gamma; \hbar) e^{\frac{i}{\hbar} f(\mathbf{x}_\gamma)} K(\mathbf{z}, \mathbf{w}(\mathbf{x}_\gamma)) |d\mathbf{x}_\gamma|,
\end{aligned}$$

where in each U_γ we define $\sigma(\mathbf{x}_\gamma; \hbar) \equiv a(\mathbf{x}_\gamma; \hbar) \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}_\gamma}\right) \right|$. Thus to each open U in Λ with (local) coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ there exists $a(\mathbf{x}; \hbar)$ such that $a(\mathbf{x}; \hbar) = \sigma(\mathbf{x}; \hbar) \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \right|^{-1}$, where this choice of σ depends on the choice of coordinates. Indeed, suppose that $\mathbf{y} = (y_1, \ldots, y_n)$ is another coordinate system on U and with respect to this coordinate system $a(\mathbf{y}; \hbar) = \widetilde{\sigma}(\mathbf{y}; \hbar) \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{y}}\right) \right|^{-1}$. If $\mathbf{s} \in U$, then we have

$$\begin{aligned}
\sigma(\mathbf{s};\hbar)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|^{-1}(\mathbf{s}) &= a(\mathbf{s};\hbar) \\
&= \widetilde{\sigma}(\mathbf{s};\hbar)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{y}}\right)\right|^{-1}(\mathbf{s}) \\
&= \widetilde{\sigma}(\mathbf{s};\hbar)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right)\right|^{-1}(\mathbf{s}) \\
&= \widetilde{\sigma}(\mathbf{s};\hbar)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|^{-1}(\mathbf{s})\left|\det\left(\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right)\right|^{-1}(\mathbf{s}) \\
&= \widetilde{\sigma}(\mathbf{s};\hbar)\left|\det\left(\frac{\partial\mathbf{y}}{\partial\mathbf{x}}\right)\right|(\mathbf{s})\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|^{-1}(\mathbf{s}).
\end{aligned}$$

So we conclude that $\sigma(\cdot;\hbar) = \widetilde{\sigma}(\cdot;\hbar)\left|\det\left(\frac{\partial\mathbf{y}}{\partial\mathbf{x}}\right)\right|(\cdot)$.

Remark III.16. For the purposes of performing calculations with generalized Lagrangian states we will use the coordinate infused expression

$$|\Lambda, a\rangle = \int_\Lambda \sigma(\mathbf{x};\hbar) e^{\frac{i}{\hbar}f(\mathbf{x})} e^{\frac{\overline{\mathbf{w}(\mathbf{x})\cdot\mathbf{z}}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} d\mathbf{x}$$

which will denote the expression above involving the decomposition of the integral in terms of a locally-finite open cover and a subordinate partition of unity, as well as a local expression for a and $d\mathbf{w}$. Thus the calculations that follow, involving multiplication and differentiation of the integrand of the above integral with respect to quantities defined in terms of a local coordinate, could be considered slightly formal. However, they can be made completely rigorous in a straight-forward manner by appealing to such a decomposition. For the sake of compactness of what are already very detailed calculations, as well as the author's overall sanity, we won't consider such decompositions again.

Remark III.17. Before we start calculating properties of generalized Lagrangian state we'll fix a complex structure in order to identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$. For the purposes of the rest of this work we choose the complex structure that results in the association

$$z_j = \frac{1}{\sqrt{2}}(q_j - ip_j) \text{ and } \overline{z}_j = \frac{1}{\sqrt{2}}(q_j + ip_j) \text{ for } j = 1, \ldots, n \tag{3.2}$$

on $\mathbb{R}^{2n}$. In chapter II we saw that this choice for a complex coordinates on $\mathbb{C}^n$ is a natural choice when considering Bargmann space. For instance, we saw that under conjugation by the Bargmann transform, $A_\hbar$, we have the operator relation

$$A_\hbar \left(\frac{\widehat{Q}_j - i\widehat{P}_j}{\sqrt{2}} \right) = z_j I.$$

This in turn led to the relationship between the representations of the Heisenberg group on $L^2(\mathbb{R}^n)$ and its representation on Bargmann space, that defined quantum translations in a sensible way. This is also the complex structure we chose when defining the nature of Wick and Anti-Wick quantization.

With respect to this complex structure the lift condition defining the function f in the definition of these generalized states becomes

$$df + \frac{1}{2i}(\mathbf{w} \cdot d\overline{\mathbf{w}} - \overline{\mathbf{w}} \cdot d\mathbf{w}) = 0, \tag{3.3}$$

where $\frac{1}{2i}(\mathbf{w} \cdot d\overline{\mathbf{w}} - \overline{\mathbf{w}} \cdot d\mathbf{w}) \equiv \mathbf{w}^* \left(\frac{1}{2i}(\mathbf{z} \cdot d\overline{\mathbf{z}} - \overline{\mathbf{z}} \cdot d\mathbf{z}) \right)$. Again, for each $\mathbf{x} \in \Lambda$ we can choose a local coordinates system $x_1, \ldots, x_n$ in a neighborhood, U, of $\mathbf{x}$, in which case we can consider

$$\mathbf{w}(U) = \left\{ \mathbf{z} = \mathbf{w}(x_1, \ldots, x_n) = \frac{1}{\sqrt{2}} \Big(\mathbf{q}(x_1, \ldots, x_n) - i\mathbf{p}(x_1, \ldots, x_n) \Big) \Big| (x_1, \ldots, x_n) \in U \right\},$$

and so the above condition is locally equivalent to the system of n first-order partial differential equations

$$\frac{\partial f}{\partial x_j} + \frac{1}{2i}\sum_{k=1}^{n}\left(w_k\frac{\partial \overline{w}_k}{\partial x_j} - \overline{w}_k\frac{\partial w_k}{\partial x_j}\right) = 0$$

for $j = 1, \ldots, n$.

3.2 Semiclassical Results

The primary mathematical tool that will be used to investigate the properties of generalized Lagrangian states is stationary phase for oscillatory integrals (as the formal integral defining these generalized states is as $\hbar \longrightarrow 0$). The specific stationary phase theorem that will be used in the investigation to follow, from [20], is stated below.

Theorem: (Stationary Phase) Let $U \subset \mathbb{R}^n$ be a compact set, V an open neighborhood of U and $k \in \mathbb{N}$. If $\sigma \in C_0^{2k}(U)$, $g \in C^{3k+1}(V)$, $Im(g) \geq 0$ in V, $Im(g(\mathbf{x}_o)) = 0$, $\nabla g(\mathbf{x}_o) = 0$, $\det(\text{Hess}(g)(\mathbf{x}_o)) \neq 0$, $\nabla g(\mathbf{x}_o) \neq 0$ in $U \setminus \{\mathbf{x}_o\}$ then

$$\left|\int_{\mathbb{R}^n}\sigma(\mathbf{x})e^{i\nu f(\mathbf{x})}d\mathbf{x} - e^{i\nu f(\mathbf{x}_o)}det\left(\nu\frac{Hess(g)(\mathbf{x}_o)}{2\pi i}\right)^{-1/2}\sum_{j<k}\nu^{-j}M_j\sigma\right| \leq C\nu^{-k}\sum_{|\alpha|\leq 2k} sup|D^{\alpha}\sigma|,$$

for $\nu > 0$. Here C is bounded when g stays in a bounded set in $C^{3k+1}(V)$ and $|\mathbf{x}-\mathbf{x}_o||\nabla g(\mathbf{x})|$ has a uniform bound. With

$$g_{\mathbf{x}_o}(\mathbf{x}) = g(\mathbf{x}) - g(\mathbf{x}_o) - \frac{1}{2}\langle \text{Hess}(g)(\mathbf{x}_o)(\mathbf{x}-\mathbf{x}_o), \mathbf{x}-\mathbf{x}_o\rangle,$$

which vanishes to third order at $\mathbf{x}_o$ we have

$$M_j\sigma = \sum_{\kappa-\mu=j}\sum_{2\kappa\geq 3\mu} i^{-j}2^{-\kappa}\frac{1}{\mu!\kappa!}\langle \text{Hess}(g)(\mathbf{x}_o)^{-1}D, D\rangle^{\kappa}(g_{\mathbf{x}_o}^{\kappa}\sigma)(\mathbf{x}_o). \tag{3.4}$$

This is a differential operator of order $2j$ acting on a at $\mathbf{x}_o$.

The first order of business in investigating these generalized states will be to show that under certain conditions on Λ and/or 'a' that the associated generalized Lagrangian states are in fact well-defined integrals and as such the above stationary phase theorem can be applied to their study.

Lemma III.18. *Suppose we have a generalized Lagrangian state, $|\Lambda, a\rangle$, defined with respect to an n-dimensional smooth manifold Λ, and a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. Further suppose that the amplitude function a is compactly supported. We can express this state in the form*

$$|\Lambda, a\rangle(\mathbf{z}) = \int_\Lambda \sigma(\mathbf{x}; \hbar) e^{\frac{i}{\hbar} g(\mathbf{x},\mathbf{z})} d\mathbf{x}$$

where

$$g(\mathbf{x}, \mathbf{z}) = f(\mathbf{x}) - i\overline{\mathbf{w}(\mathbf{x})} \cdot \mathbf{z} + \frac{i}{2}\left(\mathbf{w}(\mathbf{x}) \cdot \overline{\mathbf{w}(\mathbf{x})} + \mathbf{z} \cdot \overline{\mathbf{z}}\right)$$

.

Then the following hold:

1. *$Im(g) \geq 0$, and $Im(g) = 0$ iff there exists an $\mathbf{x} \in \Lambda$ such that $\mathbf{z} = \mathbf{w}(\mathbf{x})$,*
2. *$d_{\mathbf{x}} g = 0$ (i.e. $\nabla_{\mathbf{x}} g = 0$) iff there exists an $\mathbf{x} \in \Lambda$ such that $\mathbf{z} = \mathbf{w}(\mathbf{x})$,*
3. *$det(Hess_{\mathbf{x}}(g)) \neq 0$ if there exists an $\mathbf{x} \in \Lambda$ such that $\mathbf{z} = \mathbf{w}(\mathbf{x})$.*

Proof. First, note that if we separate out the imaginary part of g we get

$$\mathrm{Im}(g) = \mathrm{Im}(-i\overline{\mathbf{w}(\mathbf{x})} \cdot \mathbf{z}) + \frac{1}{2}(\mathbf{z} \cdot \overline{\mathbf{z}} + \mathbf{w} \cdot \overline{\mathbf{w}}) = \frac{1}{2}|\mathbf{z} - \mathbf{w}(\mathbf{x})|^2,$$

thus we have that $\mathrm{Im}(g) \geq 0$, and specifically the imaginary part of the phase is only equal to zero at an $\mathbf{x}_0$ such that $\mathbf{w}(\mathbf{x}_0) = \mathbf{z}$. Such an $\mathbf{x}_0$ exists, by assumption, and further since $\mathbf{w}$ is an embedding, we know that $\mathbf{x}_0$ is unique.

Next, we wish to show that the the point where $\mathrm{Im}(g) = 0$ corresponds to the point where $\nabla g = 0$ (i.e. the so-called stationary points of the phase). Choose a local coordinates system $x_1, \ldots, x_n$ in a neighborhood, U, of $\mathbf{x}_0$. In these local coordinates we can compute the gradient vector of g

$$\begin{aligned}
\frac{\partial g}{\partial x_k} &= \frac{\partial f}{\partial x_k} - i\mathbf{z} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} + \frac{i}{2}\left(\mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} + \overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial x_k}\right) \\
&= \frac{\partial f}{\partial x_k} + \frac{1}{2i}(2\mathbf{z}) \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \frac{1}{2i}\mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \frac{1}{2i}\overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial x_k} \\
&= \frac{\partial f}{\partial x_k} + \frac{1}{2i}\left((2\mathbf{z} - \mathbf{w}) \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial x_k}\right) \\
&= -\frac{1}{2i}\left(\mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial x_k}\right) + \frac{1}{2i}\left((2\mathbf{z} - \mathbf{w}) \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial x_k}\right) \\
&= \frac{1}{2i}\left((2\mathbf{z} - 2\mathbf{w}) \cdot \frac{\partial \overline{\mathbf{w}}}{\partial x_k}\right) \\
&= \frac{1}{i}\frac{\partial \overline{\mathbf{w}}}{\partial x_k} \cdot (\mathbf{z} - \mathbf{w})
\end{aligned}$$

The transition from the third line to the fourth line involved using the lift condition defining f, (3.3), expressed in the local coordinates $x_1, \ldots, x_n$. Note that the gradient of g vanishes under exactly the same condition that $\mathrm{Im}(g) = 0$, that is $\mathbf{z} \in \mathbf{w}(\Lambda)$.

Finally, we must verify that the Hessian matrix of the phase g is invertible at the stationary points. Let G denote the Hessian matrix of g, again, computing in the local coordinates used above we have

$$\begin{aligned}
G_{jk} &= \frac{\partial^2 g}{\partial x_j \partial x_k} \\
&= \frac{\partial^2 f}{\partial x_j \partial x_k} + \frac{1}{2i}\left(-\frac{\partial \mathbf{w}}{\partial x_j}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial x_k} + (2\mathbf{z}-\mathbf{w})\cdot\frac{\partial^2 \overline{\mathbf{w}}}{\partial x_j \partial x_k} - \frac{\partial \overline{\mathbf{w}}}{\partial x_j}\cdot\frac{\partial \mathbf{w}}{\partial x_k} - \overline{\mathbf{w}}\cdot\frac{\partial^2 \mathbf{w}}{\partial x_j \partial x_k}\right).
\end{aligned}$$

The condition defining f gives us

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_j \partial x_k} &= \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) \\
&= \frac{\partial}{\partial x_j}\left(-\frac{1}{2i}\left(\mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \overline{\mathbf{w}}\cdot\frac{\partial \mathbf{w}}{\partial x_k}\right)\right) \\
&= -\frac{1}{2i}\left(\mathbf{w}\cdot\frac{\partial^2 \overline{\mathbf{w}}}{\partial x_j \partial x_k} + \frac{\partial \mathbf{w}}{\partial x_j}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial x_k} - \overline{\mathbf{w}}\cdot\frac{\partial^2 \mathbf{w}}{\partial x_j \partial x_k} - \frac{\partial \overline{\mathbf{w}}}{\partial x_j}\cdot\frac{\partial \mathbf{w}}{\partial x_k}\right).
\end{aligned}$$

Inserting this expression into the above expression for G_{jk}, and evaluating this at the critical point $\mathbf{x}$ such that $\mathbf{w}(\mathbf{x}) = \mathbf{z}$ we get

$$G_{jk}|_{\mathbf{w}(\mathbf{x})=\mathbf{z}} = -i\frac{\partial \mathbf{w}}{\partial x_j}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial x_k}.$$

This shows that

$$G = -i\left(\frac{\partial \overline{\mathbf{w}}}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right). \tag{3.5}$$

Thus $\det(G) \neq 0$ follows from (III.8).

□

The fact that we can use stationary phase to study generalized Lagrangian states which have a compactly supported amplitude follows from this result and is stated as the following corollary.

Corollary III.19. *Suppose we have a generalized Lagrangian state, $|\Lambda, a\rangle$, defined with respect to an n-dimensional smooth manifold Λ, a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$, and an amplitude function a that is compactly supported. Suppose that Λ admits a lift function. If $\mathbf{z} \in \mathbf{w}(\Lambda)$ then $|\Lambda, a\rangle$ satisfies all of the necessary conditions to apply the stationary phase theorem above, and we can conclude that if we express our generalized state as*

$$|\Lambda, a\rangle(\mathbf{z}) = \int_\Lambda \sigma(\mathbf{x}; \hbar) e^{\frac{i}{\hbar} g(\mathbf{x}, \mathbf{z})} d\mathbf{x}$$

where

$$g(\mathbf{x}, \mathbf{z}) = f(\mathbf{x}) - i\overline{\mathbf{w}(\mathbf{x})} \cdot \mathbf{z} + \frac{i}{2}\left(\mathbf{w}(\mathbf{x}) \cdot \overline{\mathbf{w}(\mathbf{x})} + \mathbf{z} \cdot \overline{\mathbf{z}}\right),$$

then for M_l's given by (3.4),

$$|\Lambda, a\rangle(\mathbf{z}) \sim e^{\frac{i}{\hbar} g(\mathbf{w}^{-1}(\mathbf{z}), \mathbf{z})} \hbar^{n/2} det\left(\frac{Hess(g)(\mathbf{w}^{-1}(\mathbf{z}), \mathbf{z})}{2\pi i}\right)^{-1/2} \sum_{l=0}^{\infty} \hbar^l M_l(\sigma).$$

Specifically,

$$|\Lambda, a\rangle(\mathbf{z}) = \left(\frac{2\pi}{i}\right)^{n/2} \hbar^{n/2} e^{\frac{i}{\hbar} g(\mathbf{w}^{-1}(\mathbf{z}), \mathbf{z})} a(\mathbf{w}^{-1}(\mathbf{z})) + O(\hbar^{\frac{n}{2}+1}).$$

Proof. The only point that needs proving is the last assertion. Combining the fact that the operator M_0 simply evaluates at the stationary point, as well as the fact, from (3.5), that with respect to a local coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ about $\mathbf{w}^{-1}(\mathbf{z})$ we have $\det(\mathrm{Hess}_{\mathbf{x}}(g)) = (-i)^n \left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right|^2$, we have

$$
\begin{aligned}
|\Lambda, a\rangle(\mathbf{z}) &= e^{\frac{i}{\hbar}g(\mathbf{w}^{-1}(\mathbf{z}),\mathbf{z})}\det\left(\frac{\mathrm{Hess}(g)(\mathbf{w}^{-1}(\mathbf{z}),\mathbf{z})}{2\pi i}\right)^{-1/2}\hbar^{n/2}\sigma(\mathbf{w}^{-1}(\mathbf{z})) + O(\hbar^{\frac{n}{2}+1}) \\
&= \left(\frac{2\pi}{i}\right)^{n/2}\hbar^{n/2}e^{\frac{i}{\hbar}g(\mathbf{w}^{-1}(\mathbf{z}),\mathbf{z})}\left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right|^{-1}\sigma(\mathbf{w}^{-1}(\mathbf{z})) + O(\hbar^{\frac{n}{2}+1}) \\
&= \left(\frac{2\pi}{i}\right)^{n/2}\hbar^{n/2}e^{\frac{i}{\hbar}g(\mathbf{w}^{-1}(\mathbf{z}),\mathbf{z})}a(\mathbf{w}^{-1}(\mathbf{z})) + O(\hbar^{\frac{n}{2}+1}).
\end{aligned}
$$

□

Let's look at some of the properties of our generalized states.

Proposition III.20. *Suppose we have a generalized Lagrangian state, $|\Lambda, a\rangle$, defined with respect to an n-dimensional smooth manifold Λ, and a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. Further suppose that the amplitude function a is compactly supported. If $\mathbf{z} \notin \mathbf{w}(\Lambda)$, then $|\Lambda, a\rangle(\mathbf{z}) = O(\hbar^\infty)$.*

Proof.

$$
|\Lambda, a\rangle(\mathbf{z}) = \int_\Lambda \sigma(\mathbf{x};\hbar)e^{\frac{i}{\hbar}g(\mathbf{x},\mathbf{z})}d\mathbf{x}
$$

where

$$
g(\mathbf{x},\mathbf{z}) = f(\mathbf{x}) - i\overline{\mathbf{w}(\mathbf{x})}\cdot\mathbf{z} + \frac{i}{2}\left(\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})} + \mathbf{z}\cdot\overline{\mathbf{z}}\right)
$$

Separating out the imaginary part of g we calculate

$$
\mathrm{Im}(g) = \mathrm{Im}(-i\overline{\mathbf{w}(\mathbf{x})}\cdot\mathbf{z}) + \frac{1}{2}(\mathbf{z}\cdot\overline{\mathbf{z}} + \mathbf{w}\cdot\overline{\mathbf{w}}) = \frac{1}{2}|\mathbf{z} - \mathbf{w}(\mathbf{x})|^2.
$$

It is well known (see [20]) that in the absence of a point where $\mathrm{Im}(g) = 0$ that an integral of the form of $|\Lambda, a\rangle(\mathbf{z})$ will be rapidly decreasing in $\hbar$. Thus we conclude that if there does not exist an $\mathbf{x} \in \Lambda$ such that $\mathbf{w}(\mathbf{x}) = \mathbf{z}$ then $|\Lambda, \alpha\rangle = O(\hbar^\infty)$. (Note that this asymptotic estimate is point-wise in $\mathbf{z}$.) □

The next proposition will be vital to the main results of this work, but we first need a definition that's been lurking in the background.

Definition III.21. Suppose that a is an amplitude function on Λ. By definition this means that there exist smooth maps $a_j : \Lambda \longrightarrow \mathbb{C}$ that are independent of $\hbar$, such that $a \sim \sum_{j=0}^{\infty} \hbar^j a_j$ in the sense given in III.11. We extend the meaning of the symbol $\sim$ to generalized Lagrangian states in the following way. The state expansion of the generalized Lagrangian state $|\Lambda, a\rangle$ is defined as $|\Lambda, a_0\rangle + \hbar|\Lambda, a_1\rangle + \cdots + \hbar^k|\Lambda, a_k\rangle + \cdots = \sum_{j=0}^{\infty} \hbar^j |\Lambda, a_j\rangle$. Given such a state expansion we will write

$$|\Lambda, a\rangle \sim \sum_{j=0}^{\infty} \hbar^j |\Lambda, a_j\rangle.$$

We consider the following notation equivalent to the notion above

$$|\Lambda, a\angle = |\Lambda, a_0\rangle + \hbar|\Lambda, a_1\rangle + \cdots + \hbar^k|\Lambda, a_k\rangle + O(\hbar^{k+1}). \tag{3.6}$$

Remark III.22. While asymptotic in nature, the state expansion of a generalized Lagrangian state is not a point-wise asymptotic expansion. We can see this by noting that the stationary phase theory above shows that each term in the expansion can be expanded in a point-wise asymptotic expansion with infinitely many $\hbar$ contributions.

Note that the usage of the notation $O(\hbar^{\cdot})$ applied to generalized states in (3.6) is used to indicate a state with amplitude for which the standard usage of the 'big O' notation would apply. Indeed consider that in the asymptotic sense given in (III.11) we have that if $a \sim \sum_{j=0}^{\infty} \hbar^j a_j$, then it would be a standard usage of the 'big O' notation to say that $\left(a - \sum_{j=0}^{N-1} \hbar^j a_j\right) = O(\hbar^N)$. The meaning of the statement $|\Lambda, \nu\rangle = O(\hbar^N)$ is that $\nu = O(\hbar^N)$ in the standard usage of 'big O' notation.

Proposition III.23. *Given generalized Lagrangian states, $|\Lambda, a\rangle$ and $|\Lambda, b\rangle$, over a smooth n-dimensional manifold Λ, defined with respect to the same embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ with compactly supported a and b respectively. If $|\Lambda, a\rangle \equiv |\Lambda, b\rangle$ for all values of $\hbar$ then*

$$a = b + O(\hbar^\infty).$$

Proof. Let's first assume that for the generalized Lagrangian states $|\Lambda, a\rangle$, and $|\Lambda, b\rangle$ that a and b are independent of $\hbar$. For a general $\mathbf{x} \in \Lambda$ we choose $\mathbf{z} = \mathbf{w}(\mathbf{x})$, in this case we have, by III.19,

$$\begin{aligned}
0 &= |\Lambda, a\rangle(\mathbf{z}) - |\Lambda, b\rangle(\mathbf{z}) \\
&= |\Lambda, a - b\rangle(\mathbf{z}) \\
&= \left(\frac{2\pi}{i}\right)^{n/2} \hbar^{n/2} e^{\frac{i}{\hbar} g(\mathbf{w}^{-1}(\mathbf{z}), \mathbf{z})} \big(a(\mathbf{x}) - b(\mathbf{x})\big) + O(\hbar^{\frac{n}{2}+1})
\end{aligned}$$

Which implies

$$0 = (a(\mathbf{x}) - b(\mathbf{x})) + O(\hbar).$$

Letting $\hbar \longrightarrow 0$ we see that this can only hold true if $a(\mathbf{x}) = b(\mathbf{x})$. Since this holds for a general $\mathbf{x} \in \Lambda$, we conclude that $\alpha \equiv \beta$.

With the case were a and b are independent of $\hbar$, let's move on to the more general case where we assume that a and b have asymptotic expansions of the form

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j,$$

and

$$b \sim \sum_{j=0}^{\infty} \hbar^j b_j.$$

Thus we have the state expansions

$$|\Lambda, a\rangle = |\Lambda, a_0\rangle + \hbar|\Lambda, a_1\rangle + \cdots + \hbar^k|\Lambda, a_k\rangle + O(\hbar^{k+1})$$

and

$$|\Lambda, b\rangle = |\Lambda, b_0\rangle + \hbar|\Lambda, b_1\rangle + \cdots + \hbar^k|\Lambda, b_k\rangle + O(\hbar^{k+1})$$

for every positive integer k. Note that each term in this series is a generalized Lagrangian state with an amplitude that is independent of $\hbar$. Focusing, at first, on the leading level term, if $|\Lambda, a\rangle = |\Lambda, b\rangle$ for every value of $\hbar$, then we have

$$|\Lambda, a_0\rangle + O(\hbar^{\frac{n}{2}+1}) = |\Lambda, b_0\rangle + O(\hbar^{\frac{n}{2}+1}),$$

or more to the point

$$|\Lambda, a_0\rangle - |\Lambda, b_0\rangle = O(\hbar^{\frac{n}{2}+1}),$$

since the stationary phase theorem tells us that each Lagrangian state has contributions in $\hbar$ of $\hbar^{n/2+l}$ for $l = 0, 1, 2, 3, \ldots$. Applying the stationary phase theorem to the generalized Lagrangian states on both sides of this equation we get

$$\left(\frac{2\pi}{i}\right)^{n/2} \hbar^{n/2} e^{\frac{i}{\hbar} g(\mathbf{w}^{-1}(\mathbf{z}),\mathbf{z})} \big(a_0(\mathbf{x}) - b_0(\mathbf{x})\big) + O(\hbar^{\frac{n}{2}+1}) = O(\hbar^{\frac{n}{2}+1}),$$

which implies that

$$a_0(\mathbf{x}) - b_0(\mathbf{x}) = O(\hbar).$$

Letting $\hbar \longrightarrow 0$ we see that this can only hold true if $a_0(\mathbf{x}) = b_0(\mathbf{x})$. Again, since we choose $\mathbf{x} \in \Lambda$ with no conditions we have that the above holds true for each $\mathbf{x} \in \Lambda$, so $a_0 \equiv b_0$.

We proceed by induction, specifically, suppose that we have shown that $a_j \equiv b_j$ for $j = 0, 1, \ldots, k-1$. Then we combine this with the condition that $|\Lambda, a\rangle = |\Lambda, b\rangle$ for every value of $\hbar$ and we have

$$|\Lambda, a - a_0 - \hbar a_1 - \cdots - \hbar^{k-l} a_{k-l}\rangle = |\Lambda, b - b_0 - \hbar b_1 - \cdots - \hbar^{k-l} b_{k-l}\rangle$$

for every value of $\hbar$. Now, since

$$|\Lambda, a - a_0 - \hbar a_1 - \cdots - \hbar^{k-l} a_{k-l}\rangle = |\Lambda, a\rangle - |\Lambda, a_0\rangle - \hbar|\Lambda, a_1\rangle - \hbar^{k-l}|\Lambda, a_{k-l}\rangle,$$

and

$$|\Lambda, b - b_0 - \hbar b_1 - \cdots - \hbar^{k-l} b_{k-l}\rangle = |\Lambda, b\rangle - |\Lambda, b_0\rangle - \hbar|\Lambda, b_1\rangle - \hbar^{k-l}|\Lambda, b_{k-l}\rangle,$$

our state expansion gives us that

$$|\Lambda, a - a_0 - \hbar a_1 - \cdots - \hbar^{k-l} a_{k-l}\rangle = \hbar^k|\Lambda, a_{k-l}\rangle + O(\hbar^{k+1}),$$

and

$$|\Lambda, b - b_0 - \hbar b_1 - \cdots - \hbar^{k-l} b_{k-l}\rangle = \hbar^k|\Lambda, b_{k-l}\rangle + O(\hbar^{k+1}).$$

Thus we have that

$$|\Lambda, a - a_0 - \hbar a_1 - \cdots - \hbar^{k-l} a_{k-l}\rangle = |\Lambda, b - b_0 - \hbar b_1 - \cdots - \hbar^{k-l} b_{k-l}\rangle$$

implies

$$\hbar^k |\Lambda, a_k\rangle + O(\hbar^{k+1}) = \hbar^k |\Lambda, b_k\rangle + O(\hbar^{k+1}).$$

This, in turn, gives us

$$|\Lambda, a_k\rangle = |\Lambda, b_k\rangle + O(\hbar).$$

Applying the stationary phase theorem to this relation we have

$$\left(\frac{2\pi}{i}\right)^{n/2} \hbar^{n/2} e^{\frac{i}{\hbar} g(\mathbf{w}^{-1}(\mathbf{z}), \mathbf{z})} \big(a_k(\mathbf{x}) - b_k(\mathbf{x})\big) + O(\hbar^{\frac{n}{2}+1}) = O(\hbar^{\frac{n}{2}+1}),$$

which implies

$$a_k(\mathbf{x}) - b_k(\mathbf{x}) = O(\hbar).$$

Letting $\hbar \longrightarrow 0$ we see that this can only hold true if $a_k(\mathbf{x}) = b_k(\mathbf{x})$. Again, since we choose $\mathbf{x} \in \Lambda$ with no conditions we have that the above holds true for each $\mathbf{x} \in \Lambda$, so $a_k \equiv b_k$.

Thus we've shown that if $|\Lambda, a\rangle = |\Lambda, b\rangle$ for every value of $\hbar$, and a and b have asymptotic expansions

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j,$$

and

$$b \sim \sum_{j=0}^{\infty} \hbar^j b_j,$$

then $a_j \equiv b_j$ for all $j \in \mathbb{N}$. Following from the definition of an asymptotic expansion we have $a = b + O(\hbar^\infty)$.

□

Another result that gives us an idea of the nature of generalized Lagrangian states from a semiclassical perspective is the following:

Proposition III.24. *Suppose $|\Lambda, a\rangle$ is a generalized Lagrangian state defined with respect to an n-dimensional smooth manifold Λ, a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$, and compact amplitude function a that is independent of $\hbar$, then $|\Lambda, a\rangle \in \mathcal{B}_1(\mathbb{C}^n)$, and*

$$\langle \Lambda, a | \Lambda, a\rangle = (2\pi i)^{n/2}\, \hbar^{n/2} \int_\Lambda \delta_{\mathbf{w}} \overline{\delta_{\mathbf{w}}} + O(\hbar^{n/2+1}).$$

Where recall that

$$\delta_{\mathbf{w}} = a |d\mathbf{w}|^{1/2},$$

is the 1/2-density we are calling the symbol of the generalized state.

Proof. Recalling that

$$|\Lambda, a\rangle(\mathbf{z}) = \int_\Lambda a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) |d\mathbf{w}|,$$

we have that

$$|\Lambda, \sigma\rangle(\mathbf{z}) = F(\mathbf{z}) e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}},$$

for

$$F(\mathbf{z}) = \int_{\Lambda} \alpha e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) e^{\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}|,$$

where recall that $K(\mathbf{z}, \mathbf{w}(\cdot)) e^{\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} = \phi_{\mathbf{w}(\cdot)}(\mathbf{z}) = e^{\frac{\overline{\mathbf{w}(\cdot)}\cdot\mathbf{z}}{\hbar}}$ is holomorphic in $\mathbf{z}$. The fact that Λ is compact means that the above integral is well defined for every value of $\mathbf{z}$. That $F(\mathbf{z})$ is holomorphic is seen from

$$\begin{aligned}
\frac{\partial F}{\partial \overline{z}_j} &= \frac{\partial}{\partial \overline{z}_j} \int_{\Lambda} a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) e^{\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&= \int_{\Lambda} a e^{\frac{i}{\hbar} f} \left(\frac{\partial}{\partial \overline{z}_j} K(\mathbf{z}, \mathbf{w}) e^{\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \right) |d\mathbf{w}| \\
&= 0.
\end{aligned}$$

Where we are free to take the differential operator inside the integral since the integrand is smooth in everything and α is compactly supported.

Next, we consider

$$\begin{aligned}
\langle \Lambda, a | \Lambda, a \rangle &= \int_{\mathbb{C}^n} |\Lambda, a\rangle(\mathbf{z}) \overline{|\Lambda, a\rangle(\mathbf{z})} d\mathbf{z} \\
&= \int_{\mathbb{C}^n} \int_{\Lambda_{\mathbf{x}}} \int_{\Lambda_{\mathbf{y}}} \sigma(\mathbf{x}) \overline{\sigma(\mathbf{y})} e^{\frac{i}{\hbar}(f(\mathbf{x}) - f(\mathbf{y}))} e^{\frac{\overline{\mathbf{w}(\mathbf{x})}\cdot\mathbf{z}}{\hbar}} e^{\frac{\mathbf{w}(\mathbf{y})\cdot\overline{\mathbf{z}}}{\hbar}} e^{-\frac{(\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}) + \mathbf{w}(\mathbf{y})\cdot\overline{\mathbf{w}(\mathbf{y})})}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} d\mathbf{x} d\mathbf{y} d\mathbf{z} \\
&= \int_{\Lambda_{\mathbf{x}}} \int_{\Lambda_{\mathbf{y}}} \sigma(\mathbf{x}) \overline{\sigma(\mathbf{y})} e^{\frac{i}{\hbar}(f(\mathbf{x}) - f(\mathbf{y}))} \langle \phi_{\mathbf{w}(\mathbf{y})}, \phi_{\mathbf{w}(\mathbf{x})} \rangle_{L^2(\mathbb{C}^n)} e^{-\frac{(\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}) + \mathbf{w}(\mathbf{y})\cdot\overline{\mathbf{w}(\mathbf{y})})}{2\hbar}} d\mathbf{x} d\mathbf{y} \\
&= \int_{\Lambda_{\mathbf{x}}} \int_{\Lambda_{\mathbf{y}}} \sigma(\mathbf{x}) \overline{\sigma(\mathbf{y})} e^{\frac{i}{\hbar}(f(\mathbf{x}) - f(\mathbf{y}))} e^{\frac{\overline{\mathbf{w}(\mathbf{x})}\cdot\mathbf{w}(\mathbf{y})}{\hbar}} e^{-\frac{(\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}) + \mathbf{w}(\mathbf{y})\cdot\overline{\mathbf{w}(\mathbf{y})})}{2\hbar}} d\mathbf{x} d\mathbf{y} \\
&= \int_{\Lambda_{\mathbf{x}}} \sigma(\mathbf{x}) e^{\frac{i}{\hbar} f(\mathbf{x})} \left(\int_{\Lambda_{\mathbf{y}}} \overline{\sigma(\mathbf{y})} e^{-\frac{i}{\hbar} f(\mathbf{y})} e^{\frac{\overline{\mathbf{w}(\mathbf{x})}\cdot\mathbf{w}(\mathbf{y})}{\hbar}} e^{-\frac{\mathbf{w}(\mathbf{y})\cdot\overline{\mathbf{w}(\mathbf{y})}}{2\hbar}} e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} d\mathbf{y} \right) d\mathbf{x} \\
&= \int_{\Lambda_{\mathbf{x}}} \sigma(\mathbf{x}) e^{\frac{i}{\hbar} f(\mathbf{x})} \overline{I(\mathbf{x})} d\mathbf{x},
\end{aligned}$$

where

$$I(\mathbf{x}) = \int_{\Lambda_{\mathbf{y}}} \sigma(\mathbf{y}) e^{\frac{i}{\hbar} f(\mathbf{y})} e^{\frac{\overline{\mathbf{w}(\mathbf{y})} \cdot \mathbf{w}(\mathbf{x})}{\hbar}} e^{-\frac{\mathbf{w}(\mathbf{y}) \cdot \overline{\mathbf{w}(\mathbf{y})}}{2\hbar}} e^{-\frac{\mathbf{w}(\mathbf{x}) \cdot \overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} d\mathbf{y} = |\Lambda, a\rangle(\mathbf{w}(\mathbf{x})).$$

The fourth line in the above computation is clearly a convergent integral since $\Lambda_{\mathbf{x}} \times \Lambda_{\mathbf{y}}$ is compact, and the integrand is continuous. Thus we see that $|\Lambda, a\rangle \in \mathcal{B}_1$.

From the stationary phase result, III.19, we know that $I(\mathbf{x})$ has a stationary point when $\mathbf{w}(\mathbf{x}) \in \mathbf{w}(\Lambda)$, which is obviously satisfied, and because $\mathbf{w}$ is an embedding we know is happens at exactly one point, $\mathbf{x}$. The stationary phase result, III.19, tells us that

$$I(\mathbf{x}) = \left(\frac{2\pi}{i}\right)^{n/2} \hbar^{n/2} e^{\frac{i}{\hbar} g(\mathbf{x}, \mathbf{w}(\mathbf{x}))} a(\mathbf{x}) + O(\hbar^{\frac{n}{2}+1},)$$

where in general $g(\mathbf{x}, \mathbf{z}) = f(\mathbf{x}) - i\overline{\mathbf{w}(\mathbf{x})} \cdot \mathbf{z} + \frac{i}{2}\left(\mathbf{w}(\mathbf{x}) \cdot \overline{\mathbf{w}(\mathbf{x})} + \mathbf{z} \cdot \overline{\mathbf{z}}\right)$, so $g(\mathbf{x}, \mathbf{w}(\mathbf{x})) = f(\mathbf{x})$. Thus, we have

$$\begin{aligned}
\langle \Lambda, a | \Lambda, a \rangle &= \int_{\Lambda} \sigma(\mathbf{x}) e^{\frac{i}{\hbar} f(\mathbf{x})} \overline{I(\mathbf{x})} d\mathbf{x} \\
&= (2\pi i)^{n/2} \hbar^{n/2} \int_{\Lambda} \sigma(\mathbf{x}) \overline{a(\mathbf{x})} d\mathbf{x} + O(\hbar^{n/2+1}) \\
&= (2\pi i)^{n/2} \hbar^{n/2} \int_{\Lambda} \sigma(\mathbf{x}) \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \right|^{-1} \overline{a(\mathbf{x})} \left| \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \right| d\mathbf{x} + O(\hbar^{n/2+1}) \\
&= (2\pi i)^{n/2} \hbar^{n/2} \int_{\Lambda} a(\mathbf{x}) \overline{a(\mathbf{x})} |d\mathbf{w}| + O(\hbar^{n/2+1}) \\
&= (2\pi i)^{n/2} \hbar^{n/2} \int_{\Lambda} \delta_{\mathbf{w}} \overline{\delta_{\mathbf{w}}} + O(\hbar^{n/2+1})
\end{aligned}$$

□

This last proposition is the motivation for using the term 'state' in the name for the objects being studied.

3.3 The Geometry of The Lift Function

There is a very deep geometric foundation to our notion of a generalized Lagrangian states, specifically the lift function (III.9, 3.3, III.10) contains a lot of structure. The whole story of this geometry, indeed in it's fullest generality, is the work of Borthwick, Paul, and Uribe in [10]. The purpose of this section is not to repeat this work, but rather to give the reader some perspective on the geometry behind the lift function of a generalized state.

We begin by recalling from the discussion on the Heisenberg group from chapter II, that the coherent states are parametrized not by points in phase space $\mathbb{R}^{2n}$ ($\mathbb{C}^n$) with our choice of complex structure (3.2)), but rather by points in the reduced Heisenberg group

$$H_n^{red} \equiv H_n/\{(\mathbf{0}, \mathbf{0}, 2\pi k\hbar) | k \in \mathbb{Z}\}, \tag{3.7}$$

given in (2.4). And H_n^{red} has the structure of the principal circle bundle $P = \mathbb{R}^{2n} \times S^1$ ($P = \mathbb{C}^n \times S^1$) where we denote by $\pi : P \longrightarrow \mathbb{R}^{2n}(\mathbb{C}^n)$ the canonical projection. For general discussions on principal fiber bundles see [25] and [30], specifically [30] has a detailed (and very readable) discussion on circle bundles. We endow $P = H_n^{red}$ with the connection 1-form

$$\alpha = d\Theta + \frac{1}{2}(\mathbf{p} \cdot d\mathbf{q} - \mathbf{q} \cdot d\mathbf{p}), \tag{3.8}$$

in the real case, or in the complex case

$$\alpha = d\Theta + \frac{1}{2i}(\overline{\mathbf{z}} \cdot d\mathbf{z} - \mathbf{z} \cdot d\overline{\mathbf{z}}). \tag{3.9}$$

Note that with the standard symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$ ($\omega = id\mathbf{z} \wedge d\overline{\mathbf{z}}$) we have $d\alpha = \pi^*(\omega)$.

Definition III.25. Suppose that we have a smooth n-dimensional manifold Λ and $\mathbf{w} : \Lambda \longrightarrow \mathbb{R}^{2n}\,(\mathbb{C}^n)$ is a Lagrangian embedding (III.1). A map $F : \Lambda \longrightarrow P$ of the form $F(\mathbf{x}) = (\mathbf{w}(\mathbf{x}), -f(\mathbf{x}))$ will be called a *horizontal lift* of Λ into P if $F^*\alpha \equiv 0$.

If such a lift exists then the function f satisfies what we have called the lift condition (3.3) and is a lift function (III.9). We will also refer to the lift condition as the horizontality condition. Indeed, this abstract definition is the origin of the formulas we have in the definition of a lift function. As the result stationary phase result (III.19) shows, the condition we use to define a lift (3.3) is exactly the condition necessary for a generalized Lagrangian state to have an interesting semiclassical contribution, and now we can see that this is an entirely geometric condition on Λ.

3.4 Examples of Generalized Lagrangian States

Here we consider two examples of applications of where we find generalized Lagrangian states. For the first example, the basis states of the Harmonic oscillator, we work through the calculation of such states in detail

3.4.1 Harmonic Oscillator Basis States

H.O. Basis States in $n = 1$

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First let's consider $n = 1$. Our phase space will be $\mathbb{R}^2$ with the symplectic form

$$\omega = dp \wedge dq.$$

If we provide a complex structure to phase space that produces the identification $\mathbb{C} \cong \mathbb{R}^2$ where

$$(q,p) \longleftrightarrow z \equiv \frac{1}{\sqrt{2}}(q - ip),$$

then the corresponding symplectic form is

$$\omega = i dz \wedge d\overline{z}.$$

Focusing on the resonant harmonic oscillator (with m=1, w=1), the classical Hamiltonian is

$$H(p,q) = \frac{1}{2}(p^2 + q^2),$$

with a complex analog of

$$H(z,\overline{z}) = z\overline{z}.$$

Define Λ to be the energy surface (with value E) of H,

$$\Lambda \equiv \{z \in \mathbb{C} | H(z,\overline{z}) = |z|^2 = E\}$$

Note that Λ is a circle such that $|z|^2 = E$, or

$$\frac{1}{2}(p^2 + q^2) = E.$$

Since Λ is a 1-dimensional submanifold of a 2-dimensional symplectic manifold it is trivially Lagrangian. The embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}$ is just the inclusion map. If $z_0 = \frac{1}{\sqrt{2}}(q_0 - ip_0)$ such that $|z_0|^2 = E$ then we can cover Λ with the charts

$$U_1 = \{z = z_0 e^{i\Theta} | \Theta \in (0, 2\pi)\} \qquad U_2 = \{z = z_0 e^{i\Theta} | \Theta \in (\pi/2, 5\pi/2)\},$$

and it's with respect to these local parameters Θ that we will compute the lift function. The connection form with which we endow the Reduced Heisenberg group, P, (which in this case is simply $\mathbb{R}^2 \times S^1$, or $\mathbb{C} \times S^1$) is

$$\alpha = d\Theta + \frac{1}{2}(pdq - qdp)$$

in the real case, or in the complex case we get

$$\alpha = d\Theta + \frac{1}{2i}(\overline{z}dz - zd\overline{z}).$$

Note: In either formulation $d\alpha = \pi^*\omega$.

Using $\Lambda \subset \mathbb{C}$ as our manifold of interest, we lift it horizontally to P, by choosing the values of the S^1 variable to be a function of Θ such that the connection form will vanish on the resulting subspace of P. Thus we want to choose $f = f(\Theta)$ such that

$$0 = \frac{\partial f}{\partial \Theta} + \frac{1}{2i}\left(z\frac{\partial \overline{z}}{\partial \Theta} - \overline{z}\frac{\partial z}{\partial \Theta}\right)$$

where to be clear $z = z(\Theta) = z_0 e^{i\Theta}$ which is a parametrization of the inclusion map. Substituting the above form of $z(\Theta)$ into the lift condition from the connection form we get

$$\frac{\partial f}{\partial \Theta} = z_0\overline{z_0} = |z_0|^2 = E$$

Thus we get $f(\Theta) = f(0) + E\Theta$.

$$f(\Theta) = f(0) + E\Theta$$

This formula for f doesn't depend on the chart that you are computing within. But note that this formula for f is linear in the local parameter Θ, thus this function cannot be well defined on the circle. Recall in the discussion above, III.10, that if Λ is not simply connected, as the circle is not, then we may be able to impose additional conditions on the lift in order to guarantee that the exponential $e^{\frac{i}{\hbar}f}$ is well defined on Λ even if f itself is not.

First, let $N \equiv \frac{1}{\hbar}$. Then, using the expression we calculated for f above we have

$$e^{\frac{i}{\hbar}f} = e^{iN(f(0)+E\Theta)} = e^{iNf(0)}e^{iNE\Theta}.$$

If we choose to restrict the allowable values of N (i.e. $\hbar$) and possibly the values of E such that $NE \in \mathbb{N}$ then the exponential above will be well defined on the circle.

Now we are in a position to construct some interesting states, formally define the state ψ such that for $N \equiv \frac{1}{\hbar}$, and with the above as the lift function we get for ψ

$$\begin{aligned}
\psi(w) &= \int_\Lambda e^{\frac{i}{\hbar}f(\Theta)} e^{\overline{z(\Theta)}w/\hbar} e^{-\frac{(z(\Theta)\overline{z(\Theta)}+w\overline{w})}{2\hbar}} d\Theta \\
&= \int_0^{2\pi} e^{iNf} e^{N\overline{z}w} e^{-\frac{N}{2}(z\overline{z}+w\overline{w})} d\Theta \\
&= \int_0^{2\pi} e^{iN(f(0)+E\Theta)} e^{N\overline{z_0}e^{-i\Theta}w} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} d\Theta \\
&= e^{iNf(0)} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} \int_0^{2\pi} e^{iNE\Theta} e^{N\overline{z_0}we^{-i\Theta}} d\Theta \\
&= e^{iNf(0)} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} \int_0^{2\pi} e^{iNE\Theta} \left(\sum_{j=0}^{\infty} \frac{N^j \overline{z_0}^j w^j}{j!} e^{-ij\Theta} \right) d\Theta \\
&= e^{iNf(0)} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} \sum_{j=0}^{\infty} \frac{N^j \overline{z_0}^j w^j}{j!} \int_0^{2\pi} e^{iNE\Theta} e^{-ij\Theta} d\Theta \\
&= e^{iNf(0)} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} \sum_{j=0}^{\infty} \frac{N^j \overline{z_0}^j w^j}{j!} \int_0^{2\pi} e^{i(NE-j)\Theta} d\Theta
\end{aligned}$$

Note that the integral in the last line will be zero if $NE - j \neq 0$, and will be nonzero when $j = NE$. Thus we see that the condition we derived above that allows for the exponential of f to be well-defined is exactly the condition that's necessary in order for the state to be semiclassically non-negligible. When $j = NE$ the integral will be equal to 2π. Thus, under the assumption that NE (a quantization condition) is a positive integer we get

$$\psi(w) = 2\pi e^{iNf(0)} e^{-\frac{N}{2}(z_0\overline{z_0}+w\overline{w})} \frac{(N\overline{z_0})^{NE}}{(NE)!} w^{NE}$$

which is a monomial. This is an example of a Bohr-Sommerfeld condition. We require that $\hbar$ and E satisfy a quantization condition in order to allow for the a meaningful semiclassical contribution. Thus we get the un-normalized harmonic oscillator basis states as NE ranges over the nonnegative integers.

H.O. Basis States in $n > 1$

Now, let's consider a general n.

Our phase space will be $\mathbb{R}^{2n}$ with the symplectic form

$$\omega = d\mathbf{p} \wedge d\mathbf{q}.$$

If we provide a complex structure to phase space that produces the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ where

$$(\mathbf{q}, \mathbf{p}) \longleftrightarrow \mathbf{z} \equiv \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p}),$$

then the corresponding symplectic form is

$$\omega = i d\mathbf{z} \wedge d\overline{\mathbf{z}}.$$

Focusing on the resonant harmonic oscillator, the classical Hamiltonian is

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}(\mathbf{p}^2 + \mathbf{q}^2) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + q_j^2,$$

with a complex analog of

$$H(\mathbf{z},\overline{\mathbf{z}}) = \mathbf{z}\cdot\overline{\mathbf{z}} = \sum_{j=1}^{n} z_j\overline{z_j}.$$

Let $E_1,\ldots,E_n \in \mathbb{R}^+$, and define Λ to be

$$\Lambda \equiv \{\mathbf{z} \in \mathbb{C}^n | \mathbf{z} = (z_1,\ldots,z_n) \text{ where } .|z_j|^2 = E_j\}$$

The embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}$ is again just the inclusion map. If $\mathbf{z}_0 = \frac{1}{\sqrt{2}}(\mathbf{q}_0 - i\mathbf{p}_0)$ such that $\mathbf{z}_0 = (z_{01},\ldots,z_{0n})$ such that $|z_{0j}|^2 = E_j$ then we can cover Λ with the charts

$$U_1 = \{\mathbf{z}(\Theta_1,\ldots,\Theta_n) = (z_{01}e^{i\Theta_1},\ldots,z_{0n}e^{i\Theta_n})|\Theta_j \in (0,2\pi)\}$$

$$U_2 = \{\mathbf{z}(\Theta_1,\ldots,\Theta_n) = (z_{01}e^{i\Theta_1},\ldots,z_{0n}e^{i\Theta_n})|\Theta_j \in (\pi/2,5\pi/2)\},$$

and it's with respect to these local parameters Θ_j that we will compute the lift function. The connection form with which we endow the Reduced Heisenberg group, P, (which in this case is simply $\mathbb{R}^{2n}\times S^1$, or $\mathbb{C}^n \times S^1$) is

$$\alpha = d\Theta + \frac{1}{2}(\mathbf{p}\cdot d\mathbf{q} - \mathbf{q}\cdot d\mathbf{p})$$

in the real case, or in the complex case we get

$$\alpha = d\Theta + \frac{1}{2i}(\overline{\mathbf{z}}\cdot d\mathbf{z} - \mathbf{z}\cdot d\overline{\mathbf{z}}).$$

Note: Again, in either formulation $d\alpha = \pi^*\omega$.

Using $\Lambda \subset \mathbb{C}^n$ as our manifold of interest, we lift it horizontally to P, by choosing the values of the S^1 variable to be a function of Θ such that the connection form will vanish on the resulting subspace of P. Thus we want to choose $f = f(\Theta_1, \ldots, \Theta_n)$ such that

$$0 = \frac{\partial f}{\partial \Theta_j} + \frac{1}{2i}\left(\mathbf{z} \cdot \frac{\partial \overline{\mathbf{z}}}{\partial \Theta_j} - \overline{\mathbf{z}} \cdot \frac{\partial \mathbf{z}}{\partial \Theta_j}\right)$$

where to be clear $\mathbf{z}(\Theta_1, \ldots, \Theta_n) = (z_{01}e^{i\Theta_1}, \ldots, z_{0n}e^{i\Theta_n})$ which is a parametrization of the inclusion map.

Noting that $\frac{\partial \mathbf{z}}{\partial \Theta_j} = iz_{0j}e^{i\Theta_j}e_j$ and $\frac{\partial \overline{\mathbf{z}}}{\partial \Theta_j} = -i\overline{z}_{0j}e^{-i\Theta_j}e_j$ where the e_j's are the n-vectors such that $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, …, $e_n = (0, \ldots, 0, 1)$. Substituting the above form of $\mathbf{z}(\Theta_1, \ldots, \Theta_n)$ into the lift condition from the connection form we get

$$\begin{aligned}
0 &= \frac{\partial f}{\partial \Theta_j} + \frac{1}{2i}\left(\mathbf{z} \cdot (-iz_{0j}e^{-i\Theta_j}e_j) - \overline{\mathbf{z}} \cdot (i\overline{z}_{0j}e^{i\Theta_j}e_j)\right) \\
&= \frac{\partial f}{\partial \Theta_j} + \frac{1}{2i}\left(-iz_{0j}\overline{z}_{0j} - iz_{0j}\overline{z}_{0j}\right) \\
&= \frac{\partial f}{\partial \Theta_j} + \frac{1}{2i}\left(-2i|z_{0j}|^2\right) \\
&= \frac{\partial f}{\partial \Theta_j} - E_j
\end{aligned}$$

Thus we get for each j we have $f = E_j\Theta_j + K_j(\Theta_1, \ldots, \Theta_{j-1}, \Theta_{j+1}, \ldots, \Theta_n)$.

$$f(\Theta_1, \ldots, \Theta_n) = \sum_{j=1}^{n} E_j\Theta_j + K \qquad (K \quad \text{a constant}).$$

Again, as in the $n = 1$ case this formula for f doesn't depend on the chart that you are computing within. And also we note that this formula for f is linear in the local parameters Θ_j, thus this function cannot be well defined on Λ, which is the product of circles. Again, as in the discussion above, III.10, we may be able to impose additional conditions on the lift in order to guarantee that the exponential $e^{\frac{i}{\hbar}f}$ is well defined on Λ even if f itself is not.

First, let $N \equiv \frac{1}{\hbar}$. Then, using the expression we calculated for f above we have

$$e^{\frac{i}{\hbar}f} = e^{iN(K+E_1\Theta_1+\cdots+E_n\Theta_n)} = e^{iNK}e^{iNE_1\Theta_1}\cdots e^{iNE_n\Theta_n}.$$

If we choose to restrict the allowable values of N (i.e. $\hbar$) and possibly the allowable values of each E_j such that $NE_j \in \mathbb{N}$ then the exponential above will be well defined on the Λ.

We construct the state such that for each $\mathbf{w} \in \mathbb{C}^n$

$$\begin{aligned}
\psi(\mathbf{w}) &= \int_\Lambda e^{\frac{i}{\hbar}f(\Theta_1,\ldots,\Theta_n)} e^{\frac{\overline{\mathbf{z}(\Theta_1,\ldots,\Theta_n)}\cdot\mathbf{w}}{\hbar}} e^{-\frac{\mathbf{z}(\Theta_1,\ldots,\Theta_n)\cdot\overline{\mathbf{z}(\Theta_1,\ldots,\Theta_n)}+\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} d\Theta_1\cdots d\Theta_n \\
&= \int_\Lambda e^{iNf(\Theta_1,\ldots,\Theta_n)} e^{N\overline{\mathbf{z}(\Theta_1,\ldots,\Theta_n)}\cdot\mathbf{w}} e^{-\frac{N}{2}(\mathbf{z}(\Theta_1,\ldots,\Theta_n)\cdot\overline{\mathbf{z}(\Theta_1,\ldots,\Theta_n)}+\mathbf{w}\cdot\overline{\mathbf{w}})} d\Theta_1\cdots d\Theta_n \\
&= \int_0^{2\pi} e^{iN(K+E_1\Theta_1+\cdots+E_n\Theta_n)} e^{N(z_{01}w_1e^{i\Theta_1}+\cdots+z_{0n}w_ne^{i\Theta_n})} \\
&\qquad\qquad\qquad \times e^{-\frac{N}{2}(z_{01}\overline{z}_{01}+\cdots+z_{0n}\overline{z}_{0n}+w_1\overline{w}_1+\cdots+w_n\overline{w}_n)} d\Theta_1\cdots d\Theta_n \\
&= e^{iNK}\prod_{j=1}^n \int_0^{2\pi} e^{iNE_j\Theta_j} e^{Nz_{0j}w_je^{i\Theta_j}} e^{z_{0j}\overline{z}_{0j}+w_j\overline{w}_j} d\Theta_j.
\end{aligned}$$

Let

$$I_j \equiv \int_0^{2\pi} e^{iNE_j\Theta_j} e^{Nz_{0j}w_je^{i\Theta_j}} e^{z_{0j}\overline{z}_{0j}+w_j\overline{w}_j} d\Theta_j.$$

From the calculation in the $n = 1$ case above, we know that

$$I_j = \begin{cases} 0 & \text{if} \quad NE_j \notin \mathbb{N} \\ 2\pi e^{-\frac{N}{2}(z_{0j}\overline{z}_{0j}+w_j\overline{w}_j)} \frac{(N\overline{z}_{0j})^{NE_j}}{(NE_j)!} w^{NE_j} & \text{if} \quad NE_j \in \mathbb{N}. \end{cases}$$

Thus since

$$\psi(\mathbf{w}) = e^{iNK} \prod_{j=1}^{n} I_j,$$

we have that if N and each E_j satisfy $NE_j \in \mathbb{N}$ then

$$\psi(\mathbf{w}) = (2\pi)^n e^{iNK} e^{-\frac{N}{2}(\mathbf{z}_0 \cdot \overline{\mathbf{z}}_0 + \mathbf{w} \cdot \overline{\mathbf{w}})} \prod_{j=1}^{n} \frac{(N\overline{z}_{0j})^{NE_j}}{(NE_j)!} w^{NE_j}.$$

Which are well known to be the basis states of the harmonic oscillator in Bargmann space.

3.4.2 Quasimodes

In [33] the authors present an application of generalized Lagrangian states. With a phase space of $\mathbb{R}^2$, suppose we have a pseudo-differential operator $a(x, \hbar D_x)$ with Weyl symbol $a(x, \xi)$ that is polynomial in both x and ξ. Then $a(x, \hbar D_x)$ is an unbounded self-adjoint (differential) operator acting on $L^2(\mathbb{R})$ that is assumed to have a discrete specturm. Define the energy surface of a

$$\Omega_E \equiv \{(x, \xi) \in \mathbb{R}^2 | a(x, \xi) = E\}.$$

The authors show that for each connected component, Γ, of the energy surface and for every value of $N \in \mathbb{N}$ one can associate a state $\psi_N \in D(a(x, \hbar D_x)) \subset L^2(\mathbb{R})$, where ψ_N will have the general form

$$\psi = \int_0^T s(t; \hbar) e^{\frac{i}{\hbar} f(t)} \psi_{(x(t), \xi(t))} dt$$

and $(x(t), \xi(t))$ is a parametrization of the Hamilton flow of a. For some constant C_N and for infinitely many (specific) values of $\hbar$ having zero as a cluster point there exist E_N such that for ψ_N

$$||(a(x, \hbar D_x) - E_N)\psi_N||_{L^2(\mathbb{R})} \leq C_N \hbar^{N+1}.$$

3.4.3 Motivating Generalized Lagrangian States: Projecting Delta Functions

A motivation for the form of a generalized Lagrangian state comes from the notion of projecting a delta function. Recall our definition of Bargmann space that we labeled $\mathcal{B}_1$:

$$\mathcal{B}_1 \equiv \left\{ g(\mathbf{z}) = f(\mathbf{z}) e^{-\frac{\mathbf{z}\cdot\bar{\mathbf{z}}}{2\hbar}} \,\middle|\, \frac{\partial f}{\partial \bar{\mathbf{z}}} = 0, \text{and} \int_{\mathbb{C}^n} |g(\mathbf{z})|^2 d\mathbf{z}_R d\mathbf{z}_I < \infty \right\},$$

We can make the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with a choice of complex structure on $\mathbb{R}^{2n}$. With such a choice made we can make the association $L^2(\mathbb{R}^{2n}) \cong L^2(\mathbb{C}^n)$. As such we will now think of $\mathcal{B}_1$ as a Hilbert subspace of $L^2(\mathbb{C}^n)$. If we let $P_\hbar : L^2(\mathbb{C}^n) \longrightarrow \mathcal{B}_1$ denote the orthogonal projector onto Bargmann space, then we have shown in chapter II that this can be expressed as the inner-product with the reproducing kernel(s) of Bargmann space (i.e. the coherent states). Thus, for $f \in L^2(\mathbb{C}^n)$

$$(P_\hbar f)(\mathbf{z}) = \langle \varphi_{\mathbf{z}}, f \rangle_{L^2(\mathbb{C}^n)}.$$

Remark III.26. Recall that we are using the convention most often found in the physics literature where the inner product is conjugate linear in the first variable and linear in the second.

Now, for $\mathbf{z} \in \mathbb{C}^n$ we define the distribution $\delta(\mathbf{z}) : C(\mathbb{C}^n) \longrightarrow \mathbb{C}$ (Dirac's delta function) as

$$\delta(\mathbf{z})f \equiv f(\mathbf{z}),$$

for $f \in C(\mathbb{C}^n)$. Note that the use of the Greek letter δ here does not have any direct relation to the use of it in the context of the symbol of a generalized Lagrangian state.

Remark III.27. Now, to be clear, $C(\mathbb{C}^n)$ denotes the set of all continuous functions of $2n$ real-variables on $\mathbb{C}^n$. Here we are utilizing the fact that while distributions are technically defined on smooth compactly supported functions, since the delta function in the linear functional of point-wise evaluation it is a compactly supported distribution, and can be extended to a (much) larger class of functions on $\mathbb{C}^n$. In fact, the delta function is a perfectly well-defined linear functional on any (well-defined) function. I could not think of a universally accepted notation for a larger function class than the continuous functions.

The projection of the delta function gives a linear functional on $C(\mathbb{C}^n)$ (or $\mathcal{B}_1$ depending on how one would like to consider things) in that for $f \in C(\mathbb{C}^n)$

$$(P_\hbar \delta(\mathbf{z}))f \equiv \delta(\mathbf{z})(P_\hbar f) = (P_\hbar f)(\mathbf{z}) = \langle \varphi_{\mathbf{z}}, f \rangle_{L^2(\mathbb{C}^n)}.$$

In the above sense, first proposed by Berezin in [8], we say that the projection of a delta function on $C(\mathbb{C}^n)$ into Bargmann space is a coherent state in Bargmann space.

We can easily extend this notion to delta functions over measurable sets. To this end, let's recall that for Λ, a measurable subset of $\mathbb{C}^n$, and a measurable function $\alpha(\mathbf{z})$ we'll use the following formal calculation as the definition of the delta function on Λ with respect to α:

$$\left(\int_\Lambda \alpha(\mathbf{z})\delta(\mathbf{z})d\mathbf{z}d\overline{\mathbf{z}}\right)f = \int_\Lambda \alpha(\mathbf{z})\Big(\delta(\mathbf{z})f\Big)d\mathbf{z}d\overline{\mathbf{z}} = \int_\Lambda \alpha(\mathbf{z})f(\mathbf{z})d\mathbf{z}d\overline{\mathbf{z}}$$

for $f \in C(\mathbb{C}^n)$. Note that conditions on α sufficient for the above expression to be well defined depend on the nature of the set Λ. For instance if Λ is a bounded set then we would only need to require something like α is continuous. If Λ is an unbounded set then we must require α decay sufficiently at infinity for the above integral to make sense for all f we wish the functional to act on; we can relax the decay requirements of α if we require more decay from our set of f's (as in restrict our functional to acting on subspaces of $C(\mathbb{C}^n)$). A simple situation would be if $\alpha = 1$, in which case we would require $f \in C(\mathbb{C}^n) \cap L^1(\Lambda)$. It's not necessary to be exhaustive in a discussion about what requirements should be made on α; we can simply say that we need to be aware of the fact that the types of such function that will be allowed will depend on Λ as well as the types of f's that we'll have such functionals act upon, and that α needs to be chosen so that everything is well defined.

Projecting this type of functional into Bargmann space we have

$$\begin{aligned}\left(P_\hbar \int_\Lambda \alpha(\mathbf{z})\delta(\mathbf{z})d\mathbf{z}d\overline{\mathbf{z}}\right)f &= \left(\int_\Lambda \alpha(\mathbf{z})\delta(\mathbf{z})d\mathbf{z}d\overline{\mathbf{z}}\right)(P_\hbar f)\\ &= \int_\Lambda \alpha(\mathbf{z})\Big(\delta(\mathbf{z})P_\hbar f\Big)d\mathbf{z}d\overline{\mathbf{z}}\\ &= \int_\Lambda \alpha(\mathbf{z})\langle \varphi_{\mathbf{z}}, f\rangle_{L^2(\mathbf{C}^n)}d\mathbf{z}d\overline{\mathbf{z}}.\end{aligned}$$

So, in the context of the discussion above, we have as linear functionals

$$P_\hbar \int_\Lambda \alpha(\mathbf{z})\delta(\mathbf{z})d\mathbf{z}d\overline{\mathbf{z}} = \int_\Lambda \alpha(\mathbf{z})\varphi_{\mathbf{z}}d\mathbf{z}d\overline{\mathbf{z}}.$$

Thus, one can view a generalized Lagrangian state as the projection of a delta function on Λ where the function α above has a very specific form (and a very specific dependence on $\hbar$) related to an amplitude function and a lift function on Λ, specifically

$$\alpha(\mathbf{x}) = a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})}.$$

CHAPTER IV

Polynomial Hamiltonians Acting On Generalized Lagrangian States

The general goal of this chapter will be to understand the action of the ring of pseudodifferential operators generated by $\widehat{Z}_j$ and $\widehat{Z}_j^*$ for $j = 1, \ldots, n$ (these operators were first introduced in II) on certain (well-behaved) generalized Lagrangian states. Specifically, the goal will be to find an expression for $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ that is appropriately ordered in $\hbar$ for a general polynomial Hamiltonian $H(\mathbf{z}, \overline{\mathbf{z}})$. By 'appropriate' we mean an asymptotic expansion in terms of generalized Lagrangian states, rather than a pointwise asymptotic expansion. We begin by showing that classes of generalized Lagrangian states are closed under the action quantum Hamiltonians that are the Weyl, Wick, or Anti-Wick quantized polynomial symbols. Then, given a generalized state $|\Lambda, a\rangle$ with amplitude a we will derive explicit formulas for principal and sub-principal symbols, see (III.13), of $\widehat{H}_{weyl}|\Lambda, a\rangle$, and the process will allow for the computation of similar (but more complicated) formulas for higher order symbols as well. The formulas for the symbols of $\widehat{H}_{weyl}|\Lambda, a\rangle$ will involve differential operators acting on the symbols $|\Lambda, a\rangle$; these differential operators are intrinsically dependent on the complex structure of phase space.

For the sake of convenience let's institute some notation for organizing generalized Lagrangian states. Following the basic setup for our generalized states, suppose we fix a smooth n-dimensional manifold, Λ, and a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. Consider the set $J(\Lambda, \mathbf{w})$ to be

$$J(\Lambda, \mathbf{w}) \equiv \left\{ \left| \Lambda, a \right\rangle(\mathbf{z}) = \int_\Lambda a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) |d\mathbf{w}| \, \middle| \; a \in C_0^\infty(\Lambda) \right\}.$$

In this chapter we will assume that Λ is an n-dimensional smooth manifold with empty boundary (i.e. $\partial\Lambda = \emptyset$).

We will start the work in this chapter by showing that the class $J(\Lambda, \mathbf{w})$ is closed under the action of the operators $\widehat{Z}_j \equiv z_j I$ (i.e. multiplication by z_j), and $\widehat{Z}_j^* \equiv \frac{\partial}{\partial z_j}$. Before we can show this result we'll need the following preliminaries.

Jacobi's Formula

We'll need Jacobi's formula for the derivative of the determinant of a parameter dependent matrix for the work of evaluating the action of the above operators on our generalized states. The results in this section are well known in the sense that they have been known for a while, but not well-known in the sense that it is easy (or at least wasn't easy for me) to find them in a reference, so I include them here in detail.

We denote the trace of a square matrix A as $\mathrm{Tr}(A)$. Let's begin with the following lemma:

Lemma IV.1. *For any $n \times n$ matrices A and B, if the elements of A are denoted A_{ij} and the elements of B are denoted B_{ij}, then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij} = Tr(A^T B).$$

Proof. The matrix component $(AB)_{jk} = \sum_{i=1}^{n} A_{ji}B_{ik}$, which implies that $(A^T B)_{jk} = \sum_{i=1}^{n} A_{ij}B_{ik}$. Thus, by definition we have

$$\mathrm{Tr}(A^T B) = \sum_{j=1}^{n} (A^T B)_{jj} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A_{ij}B_{ij} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}B_{ij}.$$

□

Now, recall the basic definitions

Definition IV.2. For an $n \times n$ matrix B the (i, j) *minor matrix* of B, denoted M_{ij}, is the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column from B. The (i, j) *cofactor* of B, denoted C_{ij} is given by

$$C_{ij} = (-1)^{i+j} \det(M_{ij}).$$

The Laplace cofactor expansion for the determinant tells us that if the elements of B are denoted B_{ij}, then

$$\det(B) = B_{i1}C_{i1} + B_{i2}C_{i2} + \cdots + B_{in}C_{in} = B_{1i}C_{1i} + B_{2i}C_{2i} + \cdots + B_{ni}C_{ni}.$$

We are going to combine the work above with the definitions

Definition IV.3. The *cofactor matrix* of the matrix B is the $n \times n$ matrix C such that the (i, j) element of C is the (i, j) cofactor C_{ij}. The *adjugate* of B, denoted $\mathrm{adj}(B)$ is defined by

$$\mathrm{adj}(B) \equiv C^T \tag{4.1}$$

i.e. $(\mathrm{adj}(B))_{ij} = \left(C^T\right)_{ij} = C_{ji}$.

Note that by definition for an $n \times n$ matrix A:

$$(A\text{adj}(A))_{ij} = \sum_{k=1}^{n} A_{ik} \left(\text{adj}(A)\right)_{kj} = \sum_{k=1}^{n} A_{ik} C_{jk},$$

and

$$(\text{adj}(A)A)_{ij} = \sum_{k=1}^{n} \left(\text{adj}(A)\right)_{ik} A_{kj} = \sum_{k=1}^{n} C_{ki} A_{kj} = \sum_{k=1}^{n} A_{kj} C_{ki}.$$

The Laplace cofactor expansion gives us $\det(A) = A_{1i}C_{1i} + A_{2i}C_{2i} + \cdots + A_{ni}C_{ni}$, so if $i = j$ in the formulas above we have that

$$(\text{adj}(A)A)_{ii} = \det(A) = (A\text{adj}(A))_{ii};$$

and if $i \neq j$ we have

$$(\text{adj}(A)A)_{ij} = \sum_{k=1}^{n} A_{kj} C_{ki},$$

Note that when $i = j$ above then we can interpret $(\text{adj}(A)A)_{ii}$ as the determinant of A. When $i \neq j$, then we can interpret $(\text{adj}(A)A)_{ij}$ as the cofactor expansion (determinant) of the matrix with the i^{th} column of A is replaced by the j^{th} column's values. Thus this new matrix has two identical columns, the i^{th} and j^{th}, which implies the determinant of this matrix is zero. Thus

$$(\text{adj}(A)A)_{ij} = 0 \quad \text{if} \quad i \neq j,$$

and similarly

$$(A\text{adj}(A))_{ij} = 0 \quad \text{if} \quad i \neq j.$$

Combining this work we get

$$(\mathrm{adj}(A)A)_{ij} = \det(A)\delta_{ij} \tag{4.2}$$

$$(A\mathrm{adj}(A))_{ij} = \det(A)\delta_{ij}, \tag{4.3}$$

or expressed another way,

$$A\mathrm{adj}(A) = \det(A)I = \mathrm{adj}(A)A, \tag{4.4}$$

where I is the $n \times n$ identity matrix. Thus if A is invertible (i.e. $\det(A) \neq 0$) then we divide by $\det(A)$ in the above equation and conclude that

$$A\frac{\mathrm{adj}(A)}{\det(A)} = I = \frac{\mathrm{adj}(A)}{\det(A)}A,$$

which shows us that

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)} \quad \text{or} \quad \mathrm{adj}(A) = \det(A)A^{-1}. \tag{4.5}$$

Finally we are in a position to prove the Jacobi formula.

Proposition IV.4. *For a differentiable parameter dependent $n \times n$ matrix $A = A(t)$*

$$\frac{d}{dt}det(A) = Tr\left(adj(A)\frac{d}{dt}A\right). \tag{4.6}$$

Proof. Laplaces formula for $\det(A)$ gives

$$\det(A) = \sum_{j=1}^{n} A_{ij}\left(\mathrm{adj}(A)^T\right)_{ij}.$$

Considering the determinant as a function on the elements of a matrix we can write

$$\det(A) = F(A_{11}, A_{12}, \ldots, A_{21}, A_{22}, \ldots, A_{n1}, A_{n2}, \ldots, A_{nn}).$$

Then the chain-rule gives

$$\begin{aligned}
\frac{d}{dt}\det(A) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial t} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial A_{ij}} \left(\det(A)\right) \frac{\partial A_{ij}}{\partial t} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial A_{ij}} \left(\sum_{k=1}^{n} A_{ik} \left(\operatorname{adj}(A)^T\right)_{ik} \right) \frac{\partial A_{ij}}{\partial t} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial A_{ik}}{\partial A_{ij}} \left(\operatorname{adj}(A)^T\right)_{ik} + A_{ik} \frac{\partial}{\partial A_{ij}} \left(\operatorname{adj}(A)^T\right)_{ik} \right) \frac{\partial A_{ij}}{\partial t}.
\end{aligned}$$

Now $\frac{\partial A_{ik}}{\partial A_{ij}} = \delta_{jk}$, and $\frac{\partial}{\partial A_{ij}} \left(\operatorname{adj}(A)^T\right)_{ik}$ since $\left(\operatorname{adj}(A)^T\right)_{ik} = C_{ik}$, and by definition C_{ik} doesn't contain any elements of A in its i^{th} row and k^{th} column. Thus

$$\begin{aligned}
\frac{d}{dt}\det(A) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(\delta_{jk} \left(\operatorname{adj}(A)^T\right)_{ik} \right) \frac{\partial A_{ij}}{\partial t} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\operatorname{adj}(A)^T\right)_{ik} \frac{\partial A_{ik}}{\partial t} \\
&= \operatorname{Tr}\left(\operatorname{adj}(A) \frac{d}{dt} A \right).
\end{aligned}$$

Where the last step follows from the lemma.

□

Corollary IV.5. *If the parameter dependent matrix A is invertible then we have that*

$$\frac{d}{dt}det(A) = Tr\left(A^{-1}\frac{d}{dt}A\right)det(A). \tag{4.7}$$

Now we're in the position of having only one preliminary discussion to consider before we are ready to tackle the problem at hand.

The L_j Operators

Our basic setup is that we have the n-dimensional manifold Λ, along with have the Lagrangian embedding

$$\begin{aligned} \mathbf{w} :& \Lambda \longrightarrow \mathbb{C}^n \\ & \mathbf{x} \longmapsto \mathbf{w}(\mathbf{x}) = (w_1(\mathbf{x}), \ldots, w_n(\mathbf{x})). \end{aligned}$$

For each $j \in \{1, \ldots, n\}$ we have the map $w_j : \Lambda \longrightarrow \mathbb{C}$. For each $\mathbf{x} \in \Lambda$ we have that

$$dw_{j_{\mathbf{x}}} \in T^*_{\mathbf{x}}(\Lambda) \otimes \mathbb{C}.$$

If we consider local coordinates $x_1, \ldots, x_n$ in a neighborhood U of $\mathbf{x}$, then in these coordinates

$$dw_{j_{\mathbf{x}}} = \frac{\partial w_j}{\partial x_1}(\mathbf{x})dx_1 + \cdots + \frac{\partial w_j}{\partial x_n}(\mathbf{x})dx_n.$$

Note that the n-vector $\left(\frac{\partial w_j}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial w_j}{\partial x_n}(\mathbf{x})\right)$ is the j^{th} row of the complex Jacobi matrix, III.8, $\frac{\partial \mathbf{w}}{\partial \mathbf{x}}$. Since $\frac{\partial \mathbf{w}}{\partial \mathbf{x}}$ is invertible we know that it's rows are linearly independent. This implies that $dw_{1\mathbf{x}}, \ldots, dw_{n\mathbf{x}}$ are linearly independent. Since $\dim_{\mathbb{C}}(T^*_{\mathbf{x}}(\Lambda) \otimes \mathbb{C}) = n$ this means that $dw_{1\mathbf{x}}, \ldots, dw_{n\mathbf{x}}$ are a basis of $T^*_{\mathbf{x}}(\Lambda) \otimes \mathbb{C}$ for each $\mathbf{x} \in \Lambda$. Define the vector fields $R_1, \ldots, R_n$ such that for each point in Λ $R_{1\mathbf{x}}, \ldots, R_{n\mathbf{x}}$ is the dual basis of $dw_{1\mathbf{x}}, \ldots, dw_{n\mathbf{x}}$. A local expression of each $R_{j_{\mathbf{x}}}$ is

$$R_{j_{\mathbf{x}}} = \sum_{l=1}^{n} \left(\frac{\partial w_j}{\partial x_l}\right)^{-1} (\mathbf{x}) \frac{\partial}{\partial x_l}, \tag{4.8}$$

$\left(\frac{\partial w_j}{\partial x_l}\right)^{-1}$ is the $(jl)^{th}$ element of $\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^T\right)^{-1}$. To see this, note that at $\mathbf{x} \in \Lambda$ we have

$$\begin{aligned}
dw_{k\mathbf{x}}\left(R_{j_{\mathbf{x}}}\right) &= \sum_{m=1}^{n} \frac{\partial w_k}{\partial x_m}(\mathbf{x}) dx_m(R_{j_{\mathbf{x}}}) \\
&= \sum_{m=1}^{n} \frac{\partial w_k}{\partial x_m}(\mathbf{x}) dx_m (\sum_{l=1}^{n} \left(\frac{\partial w_j}{\partial x_l}\right)^{-1} (\mathbf{x}) \frac{\partial}{\partial x_l}) \\
&= \sum_{m=1}^{n} \sum_{l=1}^{n} \frac{\partial w_k}{\partial x_m}(\mathbf{x}) \left(\frac{\partial w_j}{\partial x_l}\right)^{-1} (\mathbf{x}) dx_m \left(\frac{\partial}{\partial x_l}\right) \\
&= \sum_{m=1}^{n} \sum_{l=1}^{n} \frac{\partial w_k}{\partial x_m}(\mathbf{x}) \left(\frac{\partial w_j}{\partial x_l}\right)^{-1} (\mathbf{x}) \delta_{ml} \\
&= \sum_{m=1}^{n} \left(\frac{\partial w_j}{\partial x_m}\right)^{-1} (\mathbf{x}) \frac{\partial w_k}{\partial x_m}(\mathbf{x}) \\
&= \left(\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^T\right)^{-1} \left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^T\right)_{jk}, \\
&= \delta_{jk}.
\end{aligned}$$

as we expect.

Indeed we have that with respect to the local coordinates $x_1, \ldots, x_n$ in U that

$$R_j(w_k) = \sum_{l=1}^{n} \left(\frac{\partial w_j}{\partial x_l}\right)^{-1} \left(\frac{\partial w_k}{\partial x_l}\right) \tag{4.9}$$

$$= \delta_{jk}. \tag{4.10}$$

If we define the vector field $\overline{R_j}$ as that having local expression

$$\overline{R_j} = \sum_{l=1}^{n} \left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} \frac{\partial}{\partial x_l}, \tag{4.11}$$

$\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}$ is the $(jl)^{th}$ element of the matrix $\left(\left(\frac{\partial \overline{\mathbf{w}}}{\partial \mathbf{x}}\right)^T\right)^{-1}$, then the above calculation implies that that $\overline{R_j}(\overline{w}_k) = \delta_{jk}$. This in turn gives

$$\hbar \overline{R_j}(e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}) = z_j.$$

For the work to follow we will need a formula for 'integration by parts' with respect to a vector field that we will apply to the field $\overline{R_j}$. To that end we recall some basic formulas from differential geometry.

The Cartan formula for the Lie derivative gives us that for a smooth vector field X, and a smooth form ω defined on a n-dimensional smooth manifold Λ

$$\mathcal{L}_X(\omega) = \iota_X(d\omega) + d(\iota_X\omega).$$

If ω is a top-degree form then $d\omega = 0$, and so then $\mathcal{L}_X(\omega) = d(\iota_X\omega)$. Now note that if ω is an n-form, then $\iota_X\omega$ is an $(n-1)$-form, and so $d(\iota_X\omega)$ is an n-form. Since the space of top-degree forms is 1-dimensional there exists a unique smooth function denoted $\mathrm{div}_\omega(X)$ such that $\mathcal{L}_X(\omega) = \mathrm{div}_\omega(X)\omega$. This function is called the divergence of X with respect to ω.

Since the Lie derivative is a derivation we have for a smooth function f that

$$\mathcal{L}_X(f\omega) = \mathcal{L}_X(f)\omega + f\mathcal{L}_X(\omega) = (Xf)\,\omega + f\mathcal{L}_X(\omega),$$

where we used the fact that the Lie derivative of a smooth function (a 0-form) is just the vector field acting on the function. Combining this result with the results above gives us for the top-degree form ω

$$\mathcal{L}_X(f\omega) = (Xf)\,\omega + fd(\iota_X\omega)$$

Now, if Λ is orientable and if $\partial\Lambda = \emptyset$ then Stokes theorem gives us for ω a top-degree form

$$\int_\Lambda \mathcal{L}_X(\omega) = \int_\Lambda d(\iota_X\omega) = \int_{\partial\Lambda} \iota_X\omega = 0.$$

If ω has the form $fg\Omega$ for smooth functions f, g on Λ and an n-form Ω, then

$$\begin{aligned}
\mathcal{L}_X(fg\Omega) &= fg\mathcal{L}_X(\Omega) + \mathcal{L}_X(fg)\Omega \\
&= fgd(\iota_X\Omega) + X(fg)\Omega \\
&= fgd(\iota_X\Omega) + (X(f)g + fX(g))\,\Omega \\
&= fg\mathrm{div}_\Omega(X)\Omega + X(f)g\Omega + fX(g)\Omega.
\end{aligned}$$

Combining this with the integral result above we have

$$\begin{aligned}
0 &= \int_\Lambda \mathcal{L}_X(fg\Omega) \\
&= \int_\Lambda fg\mathrm{div}_\Omega(X)\Omega + \int_\Lambda X(f)g\Omega + \int_\Lambda fX(g)\Omega.
\end{aligned}$$

This implies the (rather beautiful) formula for integrating a vector field by parts:

$$\int_\Lambda X(f)g\Omega = -\int_\Lambda fX(g)\Omega - \int_\Lambda fg\mathrm{div}_\Omega(X)\Omega.$$

This result readily extends to the case where Ω is a 1-density and it isn't necessary to consider the 'orientability' of Λ, see [32]. Thus for a smooth Λ with a Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ which yields the global 1-density $|d\mathbf{w}| \equiv |w^*(d\mathbf{z})|$, see (3.1), we have

$$\int_\Lambda X(f)g|d\mathbf{w}| = -\int_\Lambda fX(g)|d\mathbf{w}| - \int_\Lambda fg\mathrm{div}_{|d\mathbf{w}|}(X)|d\mathbf{w}|. \tag{4.12}$$

Letting $X = \overline{R}_j$ we have the rule

$$\int_\Lambda \overline{R}_j(f)g|d\mathbf{w}| = -\int_\Lambda f\overline{R}_j(g)|d\mathbf{w}| - \int_\Lambda fg\mathrm{div}_{|d\mathbf{w}|}(\overline{R}_j)|d\mathbf{w}|.$$

Thus, for the first order differential operator L_j defined by

$$L_j(g) \equiv -\left(\overline{R}_j(g) - \mathrm{div}_{|d\mathbf{w}|}(X)g\right) \tag{4.13}$$

we have

$$\int_\Lambda \overline{R}_j(f)g|d\mathbf{w}| = \int_\Lambda fL_j(g)|d\mathbf{w}|.$$

For practical purposes it will be useful to calculate a local expression for the L_j's. Consider restricting our attention to a coordinate chart U in Λ with local coordinates $\mathbf{x} = (x_1, \ldots, x_n)$. Then we can preform the above calculation more explicitly to get

$$\begin{aligned}
\int_U \overline{R_j}(f)g|d\mathbf{w}| &= \int_U \overline{R_j}(f)(\mathbf{x})g(\mathbf{x})\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}| \\
&= \int_U \left(\sum_{l=1}^n \left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\frac{\partial}{\partial x_l}f\right)(\mathbf{x})g(\mathbf{x})\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}| \\
&= \sum_{l=1}^n \int_U \frac{\partial f}{\partial x_l}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1} g(\mathbf{x})\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}| \\
&= \sum_{l=1}^n -\int_U f\frac{\partial}{\partial x_l}\left(\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1} g\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|\right)|d\mathbf{x}| \\
&= \int_U fL_j(g)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}|,
\end{aligned}$$

where we integrated by parts to go from the third line to the fourth line, and used the fact that by the definition of a chart neighborhood $\partial U = \emptyset$, and the operator L_j is defined by

(4.14)

$$L_j(g) \equiv -\sum_{l=1}^{n} \left(\frac{\partial}{\partial x_l} \left(\left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} g \right) + \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} g Tr \left(\mathrm{Re} \left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right)^{-1} \frac{\partial}{\partial x_l} \frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right) \right) \right)$$

(4.15)

$$= -\sum_{l=1}^{n} \left(\left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} \frac{\partial g}{\partial x_l} \right.$$

(4.16)

$$\left. + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} + \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} Tr \left(\mathrm{Re} \left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right)^{-1} \frac{\partial}{\partial x_l} \frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right) \right) \right) g \right)$$

(4.17)

$$= -\left(\overline{R_j}(g) + \mathrm{div}_{|d\mathbf{w}|}(\overline{R_j}) g \right),$$

where we have the local expression

$$\mathrm{div}_{|d\mathbf{w}|}(\overline{R_j}) = \frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} + \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} Tr \left(\mathrm{Re} \left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right)^{-1} \frac{\partial}{\partial x_l} \frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right) \right)$$

Remark IV.6. The piece of the formula for L_j involving the trace requires some explanation. After the integration by parts above we will have a term where a derivative will fall on the term involving the determinant of our complex Jacobian. Consider

$$\begin{aligned}
\frac{\partial}{\partial x_l}\left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right| &= \frac{\partial}{\partial x_l}\left(\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\overline{\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)}\right)^{1/2} \\
&= \frac{1}{2}\left(\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\overline{\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)}\right)^{-1/2} \\
&\quad \times\left(\left(\frac{\partial}{\partial x_l}\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)\overline{\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)} + \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\overline{\left(\frac{\partial}{\partial x_l}\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)}\right) \\
&= \frac{1}{2}\left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right|^{-1} \\
&\quad \times\left(\left(\frac{\partial}{\partial x_l}\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)\overline{\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)} + \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\overline{\left(\frac{\partial}{\partial x_l}\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)}\right) \\
&.
\end{aligned}$$

Jacobi's formula, (4.7), gives that

$$\frac{\partial}{\partial x_l}\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) = \mathrm{Tr}\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^{-1}\frac{\partial}{\partial x_l}\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right),$$

Combining these results with some basic facts about the trace of matrices and we have that

$$\frac{\partial}{\partial x_l}\left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right| = \mathrm{Tr}\left(\mathrm{Re}\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^{-1}\frac{\partial}{\partial x_l}\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)\left|\det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right|.$$

The work of this entire chapter rests on the following proposition.

Proposition IV.7. *The class of states $J(\Lambda, \mathbf{w})$ is closed under the action of the operators $\widehat{Z}_j \equiv z_j I$ and $\widehat{Z}_j^* \equiv \frac{\partial}{\partial z_j}$.*

Proof. Recall from our discussion of Bargmann space that in the representation where we include the Bargmann weights with the state it is the case that the operator $\widehat{Z}_j^*$ does not effect the weights.

Since $K(\mathbf{z},\mathbf{w}) = e^{\frac{i}{\hbar}f}e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}$, we have

$$\begin{aligned}
\widehat{Z}_j^*\left|\Lambda, a\right\rangle(\mathbf{z}) &= \hbar\frac{\partial}{\partial z_j}\int_\Lambda ae^{\frac{i}{\hbar}f}e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \int_\Lambda ae^{\frac{i}{\hbar}f}\hbar\frac{\partial}{\partial z_j}\left(e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}\right)e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \int_\Lambda a\overline{w}_je^{\frac{i}{\hbar}f}e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \left|\Lambda, \overline{w}_ja\right\rangle,
\end{aligned}$$

where w_j is the j^{th} component of $\mathbf{w}$. Thus $J(\Lambda,\mathbf{w})$ is closed under the action of $\widehat{Z}_j^*$. To see that $J(\Lambda,\mathbf{w})$ is closed under the action of $\widehat{Z}_j$, we use (4.12), consider

$$\begin{aligned}
\widehat{Z}_j|\Lambda, a\rangle &= z_j\int_\Lambda ae^{\frac{i}{\hbar}f}e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \int_\Lambda ae^{\frac{i}{\hbar}f}\left(z_je^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}\right)e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \int_\Lambda ae^{\frac{i}{\hbar}f}\hbar\overline{R}_j\left(e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}\right)e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\
&= \int_\Lambda e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}\hbar L_j\left(ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right)e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}|.
\end{aligned}$$

From the definition of L_j, (4.13), we have

$$\begin{aligned}
\hbar L_j\left(ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) &= -\hbar\overline{R}_j\left(ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) - \hbar\operatorname{div}_{|d\mathbf{w}|}(\overline{R_j})ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} \\
&= \hbar\left(-\overline{R}_j\left(a\right)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} - a\overline{R}_j\left(e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) - \operatorname{div}_{|d\mathbf{w}|}(\overline{R_j})ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) \\
&= \hbar L_j\left(a\right)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} - \hbar a\overline{R}_j\left(e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right).
\end{aligned}$$

Calculating locally we get

$$\begin{aligned}
\hbar\overline{R}_j\left(e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) &= \hbar\overline{R}_j\left(e^{\frac{i}{\hbar}(f+i/2\mathbf{w}\cdot\overline{\mathbf{w}})}\right)\\
&= \hbar\sum_{l=1}^{n}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\frac{\partial}{\partial x_l}\left(e^{\frac{i}{\hbar}(f+i/2\mathbf{w}\cdot\overline{\mathbf{w}})}\right)\\
&= \hbar\sum_{l=1}^{n}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\frac{i}{\hbar}\left(\frac{\partial f}{\partial x_l}+\frac{i}{2}\left(\mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial x_l}+\overline{\mathbf{w}}\cdot\frac{\partial\mathbf{w}}{\partial x_l}\right)\right)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= i\sum_{l=1}^{n}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\left(\frac{\partial f}{\partial x_l}+\frac{i}{2}\mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial x_l}+\frac{i}{2}\overline{\mathbf{w}}\cdot\frac{\partial\mathbf{w}}{\partial x_l}\right)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}
\end{aligned}$$

From the lift condition defining f, (3.3),

$$\frac{\partial f}{\partial x_l}=\frac{i}{2}\left(\mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial x_l}-\overline{\mathbf{w}}\cdot\frac{\partial\mathbf{w}}{\partial x_l}\right),$$

thus

$$\frac{\partial f}{\partial x_l}+\frac{i}{2}\mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial x_l}+\frac{i}{2}\overline{\mathbf{w}}\cdot\frac{\partial\mathbf{w}}{\partial x_l}=i\mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial x_l}=i\sum_{k=1}^{n}w_k\frac{\partial\overline{w}_k}{\partial x_l}$$

Thus,

$$\begin{aligned}
\hbar\overline{R}_j\left(e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) &= \left[-\sum_{k,l=1}^{n}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\frac{\partial\overline{w}_k}{\partial x_l}w_k\right]e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= -\sum_{k=1}^{n}\left(\sum_{k=1}^{n}\left(\frac{\partial\overline{w}_j}{\partial x_l}\right)^{-1}\frac{\partial\overline{w}_k}{\partial x_l}\right)w_k e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= -\sum_{k=1}^{n}\left(\left[\left(\frac{\partial\overline{\mathbf{w}}}{\partial\mathbf{x}}\right)^{-1}\right]^{T}_{jl}\left(\frac{\partial\overline{\mathbf{w}}}{\partial\mathbf{x}}\right)_{kl}\right)w_k e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= -\sum_{k=1}^{n}\left(\left[\left(\frac{\partial\overline{\mathbf{w}}}{\partial\mathbf{x}}\right)^{-1}\right]^{T}_{jl}\left(\frac{\partial\overline{\mathbf{w}}}{\partial\mathbf{x}}\right)^{T}_{lk}\right)w_k e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= -\sum_{k=1}^{n}\delta_{jk}w_k e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\\
&= -w_j e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}
\end{aligned}$$

Therefore

$$\hbar L_j\left(ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}\right) = w_j ae^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} + \hbar L_j(a)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}},$$

which gives us

$$\begin{aligned}\widehat{Z}_j|\Lambda, a\rangle &= \int_\Lambda e^{\frac{\overline{w}\cdot\mathbf{z}}{\hbar}}\left(w_j a + \hbar L_j(a)\right)e^{\frac{i}{\hbar}f}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}|d\mathbf{w}| \\ &= |\Lambda, w_j a + \hbar L_j(a)\rangle \\ &= |\Lambda, w_j a\rangle + \hbar|\Lambda, L_j(a)\rangle.\end{aligned}$$

□

Thus we have shown that $\widehat{Z}_j(J(\Lambda, \mathbf{w})) \subset J(\Lambda, \mathbf{w})$, and $\widehat{Z}_j^*(J(\Lambda, \mathbf{w})) \subset J(\Lambda, \mathbf{w})$, and furthermore we have the formulas

$$\widehat{Z}_j^*\Big|\Lambda, a\Big\rangle = \Big|\Lambda, \overline{w}_j a\Big\rangle \tag{4.18}$$

and

$$\widehat{Z}_j\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_j a\Big\rangle + \hbar\Big|\Lambda, L_j(a)\Big\rangle. \tag{4.19}$$

where L_j is given by (4.13) and (4.14).

This proposition immediately implies that the class of states $J(\Lambda, \mathbf{w})$ is closed under the action of all elements in the ring of pseudodifferential operators generated by $\widehat{Z}_j$ and $\widehat{Z}_j^*$. Specifically, we have the following corollary:

Corollary IV.8. *The class of states $J(\Lambda, \mathbf{w})$ is closed under the action of the Wick, Anti-Wick, and Weyl quantizations of a classical Hamiltonian function $H(\mathbf{z}, \overline{\mathbf{z}})$ that is polynomial in $\mathbf{z}$ and $\overline{\mathbf{z}}$.*

Moving forward, we will consider the Weyl quantization of symbols of the form $H(\mathbf{z}, \overline{\mathbf{z}})$ which are real-valued and polynomial in each z_j and $\overline{z}_j$.

Noting the above rule for the behavior of $\widehat{Z}_j$ and $\widehat{Z}_j^*$ it will be convenient for our purposes to have a 'Wick' ordering to our expression for $\widehat{H}_{weyl}$. This is easily accomplished by use of the following result that was proved in chapter II:

Lemma: Given a smooth function $H(\mathbf{z}, \overline{\mathbf{z}})$ that is polynomial in each z_j and $\overline{z}_j$ then we have for some (finite) value of R:

$$\widehat{H}_{weyl} = \left(\widehat{e^{\frac{\hbar}{4}\Delta}(H)}\right)_{wick} = \sum_{k=0}^{R} \left(\frac{\hbar}{4}\right)^k \widehat{(\Delta^k H)}_{wick}$$

Our goal is to derive an expression for the action of $\widehat{H}_{weyl}$ on our states for Hamiltonians of the form

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta} \mathbf{z}^{\alpha} \overline{\mathbf{z}}^{\beta}$$

α and β are multi-indices.

We will build up to this goal by working through the task methodically, adding in levels of complexity as we go.

Let's work through a simple example to give some perspective on the general work to follow.

Example IV.9. Recall our setup, that Λ is a smooth n-dimensional manifold, $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ is a Lagrangian embedding, and a compactly supported amplitude α. A general quadratic Hamiltonian on $\mathbb{C}^n$ has the form

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} z_j \overline{z}_k.$$

Then, (using some details about the Laplacian that we'll see shortly)

$$\begin{aligned}
(\Delta H)(\mathbf{z}, \overline{\mathbf{z}}) &= 2 \sum_{s=1}^{n} \Big(\sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} z_j \overline{z}_k \Big) \\
&= 2 \sum_{s,j,k=1}^{n} C_{jk} \frac{\partial^2}{\partial z_s \partial \overline{z}_s} (z_j \overline{z}_k) \\
&= 2 \sum_{s,j,k=1}^{n} C_{jk} \delta_{js} \delta_{ks} \\
&= 2 \sum_{s=1}^{n} C_{ss},
\end{aligned}$$

and $\Delta^l H = 0$ for $l > 1$. Then,

$$\begin{aligned}
\widehat{H}_{weyl} \Big| \Lambda, a \Big\rangle &= \Big(\widehat{e^{\frac{\hbar}{4}\Delta}(H)} \Big)_{wick} \\
&= \sum_{k=0}^{1} \left(\frac{\hbar}{4} \right)^k \widehat{(\Delta^k H)}_{wick} \Big| \Lambda, a \Big\rangle \\
&= \Big(\widehat{H}_{wick} + \frac{\hbar}{4} 2 \Big(\sum_{s=1}^{n} C_{ss} \Big) I \Big) \Big| \Lambda, a \Big\rangle \\
&= \Big(\sum_{j,k=1}^{n} C_{jk} \widehat{Z}_j \widehat{Z}_k^* + \frac{\hbar}{2} \Big(\sum_{s=1}^{n} C_{ss} \Big) I \Big) \Big| \Lambda, a \Big\rangle \\
&= \sum_{j,k=1}^{n} C_{jk} \widehat{Z}_j \Big| \Lambda, \overline{w}_k a \Big\rangle + \frac{\hbar}{2} \Big(\sum_{s=1}^{n} C_{ss} \Big) \Big| \Lambda, a \Big\rangle \\
&= \sum_{j,k=1}^{n} C_{jk} \Big(\Big| \Lambda, w_j \overline{w}_k a \Big\rangle + \hbar \Big| \Lambda, L_j(\overline{w}_k a) \Big\rangle \Big) + \Big| \Lambda, \frac{\hbar}{2} \Big(\sum_{s=1}^{n} C_{ss} \Big) a \Big\rangle \\
&= \Big| \Lambda, \Big(\sum_{j,k=1}^{n} C_{jk} w_j \overline{w}_k \Big) a \Big\rangle + \hbar \Big| \Lambda, \frac{1}{2} \Big(\sum_{s=1}^{n} C_{ss} \Big) a + \sum_{j,k=1}^{n} C_{jk} L_j(\overline{w}_k a) \Big\rangle \\
&= \Big| \Lambda, H(\mathbf{w}, \overline{\mathbf{w}}) a \Big\rangle + \hbar \Big| \Lambda, \frac{1}{4} (\Delta H)(\mathbf{w}, \overline{\mathbf{w}}) a + \sum_{j,k=1}^{n} C_{jk} L_j(\overline{w}_k a) \Big\rangle.
\end{aligned}$$

Now, if we add in the fact that as an amplitude (see III.11) we know that a asymptotically depends on $\hbar$ in the sense that there are smooth functions a_j such that

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j,$$

this yields the state expansion, (III.21),

$$\Big|\Lambda, a\Big\rangle \sim \sum_{j=0}^{\infty} \hbar^j \Big|\Lambda, a_j\Big\rangle.$$

We can similarly determine the generalized state expansion of $\widehat{H}_{weyl}\Big|\Lambda, \alpha\Big\rangle$. Indeed, if for each positive integer $l \geq 2$, $\Big|\Lambda, a\Big\rangle = \Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^l\Big|\Lambda, a_l\Big\rangle + O(\hbar^{l+1})$, then the above result shows

$$\begin{aligned}
\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \widehat{H}_{weyl}\Big|\Lambda, a_0\Big\rangle + \hbar\widehat{H}_{weyl}\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^l\widehat{H}_{weyl}\Big|\Lambda, a_l\Big\rangle + O(\hbar^{l+1} \\
&= \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_0\Big\rangle + \hbar\Big|\Lambda, \frac{1}{4}(\Delta H)(\mathbf{w}, \overline{\mathbf{w}})a_0 + \sum_{j,k=1}^{n} C_{jk}L_j(\overline{w}_k a_0)\Big\rangle \\
&\quad + \hbar\bigg(\Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_1\Big\rangle + \hbar\Big|\Lambda, \frac{1}{4}(\Delta H)(\mathbf{w}, \overline{\mathbf{w}})a_1 + \sum_{j,k=1}^{n} C_{jk}L_j(\overline{w}_k a_1)\Big\rangle\bigg) \\
&\quad + \cdots + \hbar^l\bigg(\Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_l\Big\rangle + \hbar\Big|\Lambda, \frac{1}{4}(\Delta H)(\mathbf{w}, \overline{\mathbf{w}})a_l + \sum_{j,k=1}^{n} C_{jk}L_j(\overline{w}_k a_l)\Big\rangle\bigg) \\
&\quad + O(\hbar^{l+1}).
\end{aligned}$$

We can organize this expression with respect to $\hbar$ and get, for the sake of this example up to the leading term and the next term in the expansion,

$$\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_0\Big\rangle$$
$$+\hbar\Big(\Big|\Lambda, \frac{1}{4}(\Delta H)(\mathbf{w},\overline{\mathbf{w}})a_0 + \sum_{j,k=1}^{n} C_{jk}L_j(\overline{w}_k a_0)\Big\rangle + \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_1\Big\rangle\Big)$$
$$+ O(\hbar^2).$$

The following sections will outline this sort of process in the more general setting of an arbitrary polynomial Hamiltonian.

4.0.4 Polynomial Hamiltonians Acting on Generalized Lagrangian States: $n = 1$

Let's begin this process by working in $n = 1$, so 'z' is a single complex variable, and α, and β are just indices and not multi-indices. Also note that in the case $n = 1$ our operators L_j above will simply be the single operator L. We will also assume initially that our amplitude $'a'$ is independent of $\hbar$.

We begin the process of extending the rules above with the following lemmas:

Lemma IV.10. *Suppose Λ is a smooth 1-dimensional manifold, and $w : \Lambda \longrightarrow \mathbb{C}$ is a Lagrangian embedding. For a compactly supported 'a' independent of $\hbar$, and $\alpha \geq 1$, then*

$$\widehat{Z^*}^{\alpha}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, \overline{w}^{\alpha}a\Big\rangle$$

Proof. A completely transparent induction.

$\square$

Lemma IV.11. *Suppose Λ is a smooth 1-dimensional manifold, and $w : \Lambda \longrightarrow \mathbb{C}$ is a Lagrangian embedding. For a compactly supported 'a' independent of $\hbar$, and $\alpha \geq 2$*

$$\widehat{Z}^{\alpha}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w^{\alpha}a\Big\rangle + \sum_{j=1}^{\alpha-1}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle,$$

where $\widetilde{L_j}^{\alpha}$ *is the operator given by a sum of all possible permutations of* M_w *(multiply by* $'w'$*) and* L *with 'j' copies of* L *and '*$\alpha - j$*' copies of* M_w*, so for instance*

$$\widetilde{L_1}^{3} = M_w \circ M_w \circ L + M_w \circ L \circ M_w + L \circ M_w \circ M_w;$$

or

$$\widetilde{L_1}^{3}(a) = w^2 L(a) + wL(wa) + L(w^2 a).$$

Proof. (By induction)

Consider the case $\alpha = 2$, then

$$\begin{aligned}
\widehat{Z}^{2}\Big|\Lambda, a\Big\rangle &= \widehat{Z}\left(\widehat{Z}\Big|\Lambda, a\Big\rangle\right) \\
&= \widehat{Z}\left(\Big|\Lambda, wa\Big\rangle + \hbar\Big|\Lambda, L(a)\Big\rangle\right) \\
&= \Big|\Lambda, w^2 a\Big\rangle + \hbar\Big|\Lambda, L(wa)\Big\rangle + \hbar\Big|\Lambda, wL(a)\Big\rangle + \hbar^2\Big|\Lambda, L^2(a)\Big\rangle \\
&= \Big|\Lambda, w^2 a\Big\rangle + \hbar\Big|\Lambda, \widetilde{L_1}^{2}(a)\Big\rangle + \hbar^2\Big|\Lambda, L^2(a)\Big\rangle
\end{aligned}$$

.

Now we assume that for $\alpha \geq 3$

$$\widehat{Z}^{\alpha-1}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w^{\alpha-1}\Big\rangle + \sum_{j=1}^{\alpha-2}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha-1}(a)\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, L_{\alpha-1}(a)\Big\rangle.$$

Then, using this assumption, and the linearity of the operator $\widehat{Z}$ we have

$$
\begin{aligned}
\widehat{Z}^{\alpha}\Big|\Lambda, a\Big\rangle &= \widehat{Z}\widehat{Z}^{\alpha-1}\Big|\Lambda, a\Big\rangle \\
&= \widehat{Z}\left(\Big|\Lambda, w^{\alpha-1}\Big\rangle + \sum_{j=1}^{\alpha-2}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha-1}(a)\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, L_{\alpha-1}(a)\Big\rangle\right) \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \sum_{j=1}^{\alpha-2}\hbar^{j}\Big|\Lambda, w\widetilde{L_j}^{\alpha-1}(a)\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, wL^{\alpha-1}(a)\Big\rangle \\
&\qquad + \hbar\Big|\Lambda, L(w^{\alpha-1}a)\Big\rangle + \hbar\sum_{j=1}^{\alpha-2}\hbar^{j}\Big|\Lambda, L(\widetilde{L_j}^{\alpha-1}(a))\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{k}(a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \hbar\Big|\Lambda, L(w^{\alpha-1})\Big\rangle + \sum_{j=1}^{\alpha-2}\hbar^{j}\Big|\Lambda, w\widetilde{L_j}^{\alpha-1}(a)\Big\rangle \\
&\qquad + \sum_{l=1}^{\alpha-2}\hbar^{l+1}\Big|\Lambda, L(\widetilde{L_l}^{\alpha-1}(a))\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, wL^{\alpha-1}(a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \hbar\Big|\Lambda, L(w^{\alpha-1}a)\Big\rangle + \hbar\Big|\Lambda, w\widetilde{L_1}^{\alpha-1}(a)\Big\rangle + \sum_{j=2}^{\alpha-2}\hbar^{j}\Big|\Lambda, w\widetilde{L_j}^{\alpha-1}(a)\Big\rangle \\
&\qquad + \sum_{l=1}^{\alpha-3}\hbar^{l+1}\Big|\Lambda, L(\widetilde{L_l}^{\alpha-1}(a))\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, L(\widetilde{L_{\alpha-2}}^{\alpha-1}(a))\Big\rangle + \hbar^{\alpha-1}\Big|\Lambda, wL^{\alpha-1}(a)\Big\rangle \\
&\qquad + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \hbar\left(\Big|\Lambda, L(w^{\alpha-1}a)\Big\rangle + \Big|\Lambda, w\widetilde{L_1}^{\alpha-1}(a)\Big\rangle\right) \\
&\qquad + \sum_{j=2}^{\alpha-2}\hbar^{j}\left(\Big|\Lambda, w\widetilde{L_j}^{\alpha-1}(a)\Big\rangle + \Big|\Lambda, L(\widetilde{L_{j-1}}^{\alpha-1}(a))\Big\rangle\right) \\
&\qquad + \hbar^{\alpha-1}\left(\Big|\Lambda, L(\widetilde{L_{\alpha-2}}^{\alpha-1}(a))\Big\rangle + \Big|\Lambda, wL^{\alpha-1}(a)\Big\rangle\right) + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \hbar\Big|\Lambda, \widetilde{L_1}^{\alpha}(a)\Big\rangle + \sum_{j=2}^{\alpha-2}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(a)\Big\rangle \\
&\qquad + \hbar^{\alpha-1}\Big|\Lambda, \widetilde{L_{\alpha-1}}^{\alpha}(a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}a\Big\rangle + \sum_{j=1}^{\alpha-1}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(a)\Big\rangle
\end{aligned}
$$

$\square$

Note: We can extend this formula from the lemma to the case $\alpha = 1$ if we interpret the summation term as zero (i.e. does not contribute, is an 'empty' sum) if $\alpha = 1$. We can see that in this case the formula agrees with our previous formula for the action of $\widehat{Z}$ on one of our states.

We can use this to find an expression for $\widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}\Big|\Lambda, a\Big\rangle$,

$$\begin{aligned}
\widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}\Big|\Lambda, a\Big\rangle &= \widehat{Z}^{\alpha}\Big|\Lambda, \overline{w}^{\beta} a\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}\overline{w}^{\beta} a\Big\rangle + \sum_{j=1}^{\alpha-1} \hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(\overline{w}^{\beta} a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, L^{\alpha}(\overline{w}^{\beta} a)\Big\rangle \\
&= \Big|\Lambda, w^{\alpha}\overline{w}^{\beta} a\Big\rangle + \sum_{j=1}^{\alpha} \hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(\overline{w}^{\beta} a)\Big\rangle.
\end{aligned}$$

So, for our (classical) Hamiltonians

$$H(z, \overline{z}) = \sum_{\alpha,\beta} C_{\alpha,\beta} z^{\alpha}\overline{z}^{\beta}$$

or more specifically since we are in $n = 1$ we have for non-negative integers N, and M,

$$H(z, \overline{z}) = \sum_{\beta=0}^{N}\sum_{\alpha=0}^{M} C_{\alpha,\beta} z^{\alpha}\overline{z}^{\beta},$$

we have

$$\widehat{H}_{wick} = \sum_{\beta=0}^{N}\sum_{\alpha=0}^{M} C_{\alpha,\beta}\widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}$$

and so utilizing the results of the lemmas above

$$
\begin{aligned}
\widehat{H}_{wick}\Big|\Lambda, a\Big\rangle &= \sum_{\beta=0}^{N}\sum_{\alpha=0}^{M} C_{\alpha,\beta}\Big|\Lambda, w^{\alpha}\overline{w}^{\beta}\Big\rangle + \sum_{\beta=0}^{N}\sum_{\alpha=1}^{M} C_{\alpha,\beta}\sum_{j=1}^{\alpha}\hbar^{j}\Big|\Lambda, \widetilde{L_j}^{\alpha}(\overline{w}^{\beta}a)\Big\rangle \\
&= \Big|\Lambda, H(w,\overline{w})a\Big\rangle \\
&\quad + \sum_{\beta=0}^{N}\sum_{\alpha=1}^{M} C_{\alpha,\beta}\Big(\hbar\Big|\Lambda, \widetilde{L_1}^{\alpha}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{2}\Big|\Lambda, \widetilde{L_2}^{\alpha}(\overline{w}^{\beta}a)\Big\rangle + \cdots \\
&\qquad + \hbar^{\alpha-1}\Big|\Lambda, \widetilde{L_{\alpha-1}}^{\alpha}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{\alpha}\Big|\Lambda, \widetilde{L_{\alpha}}^{\alpha}(\overline{w}^{\beta}a)\Big\rangle\Big) \\
&= \Big|\Lambda, H(w,\overline{w})a\Big\rangle + \sum_{\beta=0}^{N} C_{1\beta}\hbar\Big|\Lambda, \widetilde{L_1}^{1}(\overline{w}^{\beta}a)\Big\rangle \\
&\quad + \sum_{\beta=0}^{N} C_{2\beta}\left(\hbar\Big|\Lambda, \widetilde{L_1}^{2}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{2}\Big|\Lambda, \widetilde{L_2}^{2}(\overline{w}^{\beta}a)\Big\rangle\right) \\
&\quad + \sum_{\beta=0}^{N} C_{3\beta}\left(\hbar\Big|\Lambda, \widetilde{L_1}^{3}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{2}\Big|\Lambda, \widetilde{L_2}^{3}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{3}\Big|\Lambda, \widetilde{L_3}^{3}(\overline{w}^{\beta}a)\Big\rangle\right) \\
&\quad + \cdots + \sum_{\beta=0}^{N} C_{M\beta}\Big(\hbar\Big|\Lambda, \widetilde{L_1}^{M}(\overline{w}^{\beta}a)\Big\rangle + \hbar^{2}\Big|\Lambda, \widetilde{L_2}^{M}(\overline{w}^{\beta}a)\Big\rangle \\
&\qquad\qquad + \cdots + \hbar^{M}\Big|\Lambda, \widetilde{L_M}^{M}(\overline{w}^{\beta}a)\Big\rangle\Big) \\
&= \Big|\Lambda, H(w,\overline{w})a\Big\rangle + \hbar\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_1}^{j}(\overline{w}^{\beta}a)\Big\rangle \\
&\quad + \hbar^{2}\sum_{j=2}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_2}^{j}(\overline{w}^{\beta}a)\Big\rangle \\
&\quad + \cdots + \hbar^{M-1}\sum_{j=M-1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_{M-1}}^{j}(\overline{w}^{\beta}a)\Big\rangle \\
&\quad + \hbar^{M}\sum_{\beta=0}^{N} C_{M\beta}\Big|\Lambda, \widetilde{L_M}^{M}(\overline{w}^{\beta}a)\Big\rangle
\end{aligned}
$$

This give us a state expansion for $\widehat{H}_{wick}|\Lambda, a\rangle$.

Now we have the result that for $H(z,\overline{z}) = \sum_{\alpha,\beta} C_{\alpha,\beta} z^{\alpha}\overline{z}^{\beta}$,

$$\widehat{H}_{weyl} = \widehat{H}_{wick} + \frac{\hbar}{4}\widehat{(\Delta H)}_{wick} + \cdots + \left(\frac{\hbar}{4}\right)^R \widehat{(\Delta^R H)}_{wick}$$

where $R \equiv min\{max(\alpha), max(\beta)\}$. Using this we will extend our result above to the action of $\widehat{H}_{weyl}$ on our states.

Before continuing with the goal of the chapter let's explicitly detail some facts, about the Laplacian on $\mathbb{C}$, that will be used shortly. On $\mathbb{R}^2 = \{(q,p)|q,p \in \mathbb{R}\}$, for $f \in C^2(\mathbb{R}^2)$,

$$\Delta f \equiv \frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial p^2}$$

Now, if $z = \frac{1}{\sqrt{2}}(q - ip)$, then $\frac{\partial}{\partial z} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right)$, so $\frac{\partial}{\partial \overline{z}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p}\right)$, and thus $\frac{\partial}{\partial q} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right)$, and $\frac{\partial}{\partial p} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)$.
So,

$$\begin{aligned}\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} &= \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right) - \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \\ &= 2\frac{\partial^2}{\partial z \partial \overline{z}}.\end{aligned}$$

Thus if $H(z,\overline{z}) = \sum_{\alpha,\beta} C_{\alpha,\beta} z^\alpha \overline{z}^\beta$,

$$\Delta H = 2\frac{\partial^2}{\partial z \partial \overline{z}} H = \sum_{\alpha,\beta} 2\alpha\beta C_{\alpha,\beta} z^{\alpha-1}\overline{z}^{\beta-1}$$

and

$$\Delta^l H = \sum_{\alpha,\beta} 2^l \frac{\alpha!}{(\alpha-l)!}\frac{\beta!}{(\beta-l)!} C_{\alpha,\beta} z^{\alpha-l}\overline{z}^{\beta-l}.$$

So,

$$\left(\frac{\hbar}{4}\right)^l \Delta^l H = \frac{\hbar^l}{2^l}\sum_{\alpha,\beta}\frac{\alpha!}{(\alpha-l)!}\frac{\beta!}{(\beta-l)!}C_{\alpha,\beta}z^{\alpha-l}\overline{z}^{\beta-l}$$
$$= \hbar^l \sum_{\alpha,\beta}\widetilde{C}^l_{\alpha,\beta}z^{\alpha-l}\overline{z}^{\beta-l}$$

where $\widetilde{C}^l_{\alpha,\beta} \equiv \frac{1}{2^l}\frac{\alpha!}{(\alpha-l)!}\frac{\beta!}{(\beta-l)!}C_{\alpha,\beta}$.

It would be best to rearrange things a bit. If we let $v = \alpha - l$ and $r = \beta - l$, then

$$\left(\frac{\hbar}{4}\right)^l \Delta^l H = \hbar^l \sum_{v=1}^{\alpha-l}\sum_{r=1}^{\beta-l}\frac{1}{2^l}\frac{(v+l)!}{v!}\frac{(r+l)!}{r!}C_{(v+l)(r+l)}z^v\overline{z}^r$$
$$= \hbar^l \sum_{v=1}^{\alpha-l}\sum_{r=1}^{\beta-l}C^l_{vr}z^v\overline{z}^r,$$

where $C^l_{vr} \equiv \frac{1}{2^l}\frac{(v+l)!}{v!}\frac{(r+l)!}{r!}C_{(v+l)(r+l)}$.

With the facts about the Laplacian above, and using our general result for $\widehat{H}_{wick}|\Lambda, a\rangle$ (and relabeling some indices) we get

$$\left(\frac{\hbar}{4}\right)^l \widehat{(\Delta^l H)}_{wick} = \hbar^l \Bigg(\bigg|\Lambda, \left(\frac{1}{4}\right)^l \left(\Delta^l H\right)(w,\overline{w})a\bigg\rangle + \hbar \sum_{j=1}^{M-l}\sum_{\beta=1}^{N-l}C^l_{j\beta}\bigg|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\bigg\rangle$$
$$+ \cdots + \hbar^{M-l}\sum_{\beta=1}^{N-l}C^l_{(M-l)\beta}\bigg|\Lambda, \widetilde{L_{M-l}}^{M-l}(\overline{w}^\beta a)\bigg\rangle\Bigg)$$
$$= \hbar^l \bigg|\Lambda, \left(\frac{1}{4}\right)^l \left(\Delta^l H\right)(w,\overline{w})a\bigg\rangle + \hbar^{l+1}\sum_{j=1}^{M-l}\sum_{\beta=1}^{N-l}C^l_{j\beta}\bigg|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\bigg\rangle$$
$$+ \hbar^{l+2}\sum_{j=2}^{M-l}\sum_{\beta=1}^{N-l}C^l_{j\beta}\bigg|\Lambda, \widetilde{L_2}^j(\overline{w}^\beta a)\bigg\rangle$$
$$+ \cdots + \hbar^M \sum_{\beta=1}^{N-l}C^l_{(M-l)\beta}\bigg|\Lambda, \widetilde{L_{M-l}}^{M-l}(\overline{w}^\beta a)\bigg\rangle$$

Thus, for $H(z,\overline{z}) = \sum_{\alpha,\beta}C_{\alpha,\beta}z^\alpha\overline{z}^\beta$,

$$\begin{aligned}\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \sum_{l=0}^{N}\left(\frac{\hbar}{4}\right)\widehat{(\Delta^l H)}_{wick}\Big|\Lambda, a\Big\rangle \\ &= \widehat{H}_{wick}\Big|\Lambda, a\Big\rangle + \frac{\hbar}{4}\widehat{(\Delta H)}_{wick}\Big|\Lambda, a\Big\rangle \\ &\quad + \cdots + \left(\frac{\hbar}{4}\right)^l \widehat{(\Delta^l H)}_{wick}\Big|\Lambda, a\Big\rangle + \cdots + \left(\frac{\hbar}{4}\right)^R \widehat{(\Delta^R H)}_{wick}\Big|\Lambda, a\Big\rangle.\end{aligned}$$

Note that if we consider the contributions of the terms in this sum on the level of our generalized states (meaning that we won't decompose our generalized states asymptotically) then we see that each term in this sum has an $\hbar^M$ term in it. In fact the $l = 0$ term contains terms with contributions of $\hbar^0$, $\hbar^1$,..., and $\hbar^M$; the $l = 1$ term contains contributions of $\hbar^1$,..., and $\hbar^M$; and in general the $l = k$ term will have contributions of $\hbar^k$,..., and $\hbar^M$. Thus the state expansion of $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ has the following contributions:

<u>For $\hbar^0$:</u> 1 term

$$\Big|\Lambda, H(w, \overline{w})a\Big\rangle,$$

<u>For $\hbar^1$:</u> 2 terms

$$\frac{1}{4}\Big|\Lambda, \Delta H(w, \overline{w})a\Big\rangle + \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\Big\rangle,$$

<u>For $\hbar^2$:</u> 3 terms

$$\left(\frac{1}{4}\right)^2\Big|\Lambda, \Delta^2 H(w, \overline{w})a\Big\rangle + \sum_{j=2}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_2}^j(\overline{w}^\beta a)\Big\rangle + \sum_{j=1}^{M-1}\sum_{\beta=0}^{N} C_{j\beta}^1\Big|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\Big\rangle,$$

<u>For $\hbar^3$:</u> 4 terms

$$\left(\frac{1}{4}\right)^3\Big|\Lambda, \Delta^3 H(w,\overline{w})a\Big\rangle + \sum_{j=3}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_3}^j(\overline{w}^\beta a)\Big\rangle + \sum_{j=2}^{M-1}\sum_{\beta=0}^{N-1} C^1_{j\beta}\Big|\Lambda, \widetilde{L_2}^j(\overline{w}^\beta a)\Big\rangle$$
$$+ \sum_{j=1}^{M-2}\sum_{\beta=0}^{N-2} C^2_{j\beta}\Big|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\Big\rangle,$$

$$\vdots$$

<u>For $\hbar^k (k \leq M)$:</u> $k+1$ terms

$$\left(\frac{1}{4}\right)^k\Big|\Lambda, \Delta^k H(w,\overline{w})a\Big\rangle + \sum_{j=k}^{M}\sum_{\beta=0}^{N} C_{j\beta}\Big|\Lambda, \widetilde{L_k}^j(\overline{w}^\beta a)\Big\rangle + \sum_{j=k-1}^{M-1}\sum_{\beta=0}^{N-1} C^1_{j\beta}\Big|\Lambda, \widetilde{L_{k-1}}^j(\overline{w}^\beta a)\Big\rangle$$
$$+ \sum_{j=k-2}^{M-2}\sum_{\beta=0}^{N-2} C^2_{j\beta}\Big|\Lambda, \widetilde{L_{k-2}}^j(\overline{w}^\beta a)\Big\rangle$$
$$+ \cdots + \sum_{j=1}^{M-(k-1)}\sum_{\beta=0}^{N-(k-1)} C^{k-1}_{j\beta}\Big|\Lambda, \widetilde{L_1}^j(\overline{w}^\beta a)\Big\rangle.$$

Let's introduce the notation

$$\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle = D_0^H\Big|\Lambda, a\Big\rangle + \hbar D_1^H\Big|\Lambda, a\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a\Big\rangle,$$

where M is the 'order' of the Hamiltonian, H, which is the highest number of factors of 'z' in it, or the highest number of $\widehat{Z}$'s in the quantum Hamiltonian. Note that for the type of Hamiltonians that we are focusing on, such an M exists (more or less by definition). So, we have that

$$D_0^H\Big|\Lambda, a\Big\rangle \equiv \Big|\Lambda, H(w,\overline{w})a\Big\rangle,$$

and for $1 \leq k \leq M$,

$$D_k^H\Big|\Lambda, a\Big\rangle \equiv \left(\frac{1}{4}\right)^k\Big|\Lambda, \left(\Delta^k H\right) a\Big\rangle + \sum_{l=1}^{k}\sum_{j=l}^{M-k+l}\sum_{\beta=0}^{N-k+l} C_{j\beta}^{k-l}\Big|\Lambda, \widetilde{L}_l^{\,j}\left(\overline{w}^\beta a\right)\Big\rangle,$$

where $C_{j\beta}^0 \equiv C_{j\beta}$.

For all of this discussion we have been assuming that 'a' is independent of $\hbar$. This has been implicit in the work and hasn't had an explicit effect on anything that we've done. The reason for this assumption has been that our goal is to find an expression for $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ that is systematically ordered in $\hbar$ in the sense that rather than a point-wise asymptotic expansion we have an asymptotic expansion in terms of generalized Lagrangian states, a state expansion. Now, we'll add to our most recent result the final layer of complexity, namely the complications of letting 'a' be symbolically dependent on $\hbar$ in an asymptotic sense, so we assume that there exist functions a_j such that,

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j$$

(see III.11). Recall that the above relation for the amplitude of a generalized state will define the use of the asymptotic symbol, $\sim$, with respect to generalize states, in the following sense

$$\Big|\Lambda, a\Big\rangle \sim \Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots = \sum_{j=0}^{\infty} \hbar^j\Big|\Lambda, a_j\Big\rangle.$$

Thus for each value of r we have

$$\Big|\Lambda, a\Big\rangle = \Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right),$$

so, for each r

$$\begin{aligned}
\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \widehat{H}_{weyl}\Big|\Lambda, a_0\Big\rangle + \hbar\widehat{H}_{weyl}\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r\widehat{H}_{weyl}\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right) \\
&= D_0^H\Big|\Lambda, a_0\Big\rangle + \hbar D_1^H\Big|\Lambda, a_0\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_0\Big\rangle \\
&\quad + \hbar\left(D_0^H\Big|\Lambda, a_1\Big\rangle + \hbar D_1^H\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_1\Big\rangle\right) \\
&\quad + \cdots + \hbar^r\left(D_0^H\Big|\Lambda, a_r\Big\rangle + \hbar D_1^H\Big|\Lambda, a_r\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_r\Big\rangle\right) \\
&\quad + O\left(\hbar^{r+1}\right).
\end{aligned}$$

Again, gathering terms at each level of $\hbar$ we get:

<u>For $\hbar^0$:</u> 1 term

$$D_0^H\Big|\Lambda, a_0\Big\rangle$$

<u>For $\hbar^1$:</u> 2 terms

$$D_1^H\Big|\Lambda, a_0\Big\rangle + D_0^H\Big|\Lambda, a_1\Big\rangle$$

<u>For $\hbar^2$:</u> 3 terms

$$D_2^H\Big|\Lambda, a_0\Big\rangle + D_1^H\Big|\Lambda, a_1\Big\rangle + D_0^H\Big|\Lambda, a_2\Big\rangle$$

$$\vdots$$

<u>For $\hbar^s$ $(s \leq M)$:</u> $s+1$ terms

$$D_s^H\Big|\Lambda, a_0\Big\rangle + D_{s-1}^H\Big|\Lambda, a_1\Big\rangle + \cdots + D_0^H\Big|\Lambda, a_s\Big\rangle$$

Note, for each j, $\widehat{H}_{weyl}\left|\Lambda, a_j\right\rangle$ has terms (at the level of states) up to $\hbar^M$. Thus in the above, if $s > M$ there will be no term involving $\left|\Lambda, a_0\right\rangle$; if $s > M+1$, there will be no terms involving $\left|\Lambda, a_0\right\rangle$ and $\left|\Lambda, a_1\right\rangle$. In general if $s > M+k$ there will be no terms involving $\left|\Lambda, a_0\right\rangle, \ldots, \left|\Lambda, a_k\right\rangle$, thus

<u>For $\hbar^s$ $(s > M)$:</u> $M+1$ terms

$$D_0^H\left|\Lambda, a_s\right\rangle + D_1^H\left|\Lambda, a_{s-1}\right\rangle + \cdots + D_{M-1}^H\left|\Lambda, a_{s-(M-1)}\right\rangle + D_M^H\left|\Lambda, a_{s-M}\right\rangle$$

We now have a systematic way of expanding $\widehat{H}_{weyl}\left|\Lambda, a\right\rangle$ in terms of Lagrangian states in the case of $n = 1$.

Recalling the definition of the symbol of a generalized Lagrangian state, from (III.13), from the above calculations we have proven the following theorem:

Theorem IV.12. *Suppose one has a classical Hamiltonian, on $\mathbb{C}$, of the form*

$$H(z, \overline{z}) = \sum_{\beta=0}^{N}\sum_{\alpha=0}^{M} C_{\alpha,\beta} z^{\alpha}\overline{z}^{\beta},$$

for non-negative integers N, and M. Further, suppose Λ is a smooth 1-dimensional manifold, and $w : \Lambda \longrightarrow \mathbb{C}$ is a Lagrangian embedding. For an associated generalized Lagrangian state with a compactly supported amplitude 'a' then the following are true:

1. *The principal symbol of $\widehat{H}_{weyl}\left|\Lambda, a\right\rangle$ is*

$$\delta_w^0 = H\left(w, \overline{w}\right) a_0 |d\mathbf{w}|^{1/2};$$

2. The sub-principal symbol of $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ *is*

$$\delta^1_w = \left(\frac{1}{4}\left(\Delta H\right)\left(w,\overline{w}\right)a_0 + \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\widetilde{L}^j_1\left(\overline{w}^\beta a_0\right) + H\left(w,\overline{w}\right)a_1\right)|d\mathbf{w}|^{1/2}.$$

It's worth noting that the first statement in the above theorem tell us that if δ_w is the principal symbol of a generalized Lagrangian state $\Big|\Lambda, a\Big\rangle$, then for a polynomial Hamiltonian H the principal symbol of $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ is $H\Big|_{w(\Lambda)}\delta_w$. Similarly, the sub-principal symbol of $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ is the sum of three terms, one is (up to constants) $\Delta H\Big|_{w(\Lambda)}$ multiplying the principal symbol of $\Big|\Lambda, a\Big\rangle$, one is the the sub-principal symbol of $\Big|\Lambda, a\Big\rangle$ multiplied by $H\Big|_{w(\Lambda)}$, and finally a 1/2-density that is given by a first order differential operator acting on a_0, where the first order differential operator is intrinsically dependent on the complex structure.

4.0.5 Polynomial Hamiltonians Acting on Generalized Lagrangian States: $n > 1$

With this simpler case as a guide we can work to calculate the analogous result in the more complicated case of $n > 1$. We begin with our basic rules derived at the beginning of the chapter:

$$\widehat{Z}^*_j\Big|\Lambda, a\Big\rangle = \Big|\Lambda, \overline{w}_j a\Big\rangle$$

and

$$\widehat{Z}_j\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_j a\Big\rangle + \hbar\Big|\Lambda, L_j(a)\Big\rangle,$$

for $j = 1 \ldots n$.

Following the procedure in the case of $n = 1$ above, begin with the following lemmas:

Lemma IV.13. *Suppose Λ is a smooth n-dimensional manifold, and $w : \Lambda \longrightarrow \mathbb{C}^n$ is a Lagrangian embedding. For a compactly supported 'a' independent of $\hbar$, and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, such that $\alpha_i \in \mathbb{N} \cup \{0\}$,*

$$\widehat{Z^*}^{\alpha} \Big| \Lambda, a \Big\rangle (\mathbf{z}) = \Big| \Lambda, \overline{\mathbf{w}}^{\alpha} a \Big\rangle,$$

where $\widehat{Z^*}^{\alpha} \equiv \widehat{Z_1^*}^{\alpha_1} \cdots \widehat{Z_n^*}^{\alpha_n}$, *and* $\overline{\mathbf{w}}^{\alpha} \equiv \overline{w_1}^{\alpha_1} \cdots \overline{w_n}^{\alpha_n}$.

Proof. A completely transparent induction.

□

Lemma IV.14. *Suppose Λ is a smooth n-dimensional manifold, and $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}$ is a Lagrangian embedding. For a compactly supported 'a' independent of $\hbar$, and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned} \widehat{Z}^{\alpha} \Big| \Lambda, a \Big\rangle &\equiv \widehat{Z}_1^{\alpha_1} \cdots \widehat{Z}_n^{\alpha_n} \Big| \Lambda, a \Big\rangle \\ &= \Big| \Lambda, \mathbf{w}^{\alpha} a \Big\rangle + \sum_{j=1}^{|\alpha|-1} \hbar^j \Big| \Lambda, \widetilde{L}_j^{\alpha}(a) \Big\rangle + \hbar^{|\alpha|} \Big| \Lambda, L^{\alpha}(a) \Big\rangle, \end{aligned}$$

where $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $L^{\alpha}(a) \equiv L_1^{\alpha_1} \circ L_2^{\alpha_2} \circ \cdots \circ L_n^{\alpha_n}(a)$, and $\widetilde{L}_j^{\alpha}$ is the operator given by a sum of all possible permutations of M_{w_k} and L_i with 'j' factors of L_i in the sense that one takes 'j' factors from the list

$$\underbrace{L_1, \ldots, L_1}_{\alpha_1\text{-factors}}, \underbrace{L_2, \ldots, L_2}_{\alpha_2\text{-factors}}, \ldots, \underbrace{L_n, \ldots, L_n}_{\alpha_n\text{-factors}},$$

and '$|\alpha| - j$' factors of M_{w_k} in the sense that one takes '$|\alpha| - j$' factors from the list

$$\underbrace{M_{w_1}, \ldots, M_{w_1}}_{\alpha_1\text{-factors}}, \underbrace{M_{w_2}, \ldots, M_{w_2}}_{\alpha_2\text{-factors}}, \ldots, \underbrace{M_{w_n}, \ldots, M_{w_n}}_{\alpha_n\text{-factors}},$$

where the ordering is fixed.

For example suppose $n = 3$ *and* $\alpha = (1,2,1)$, *so* $|\alpha| = 4$, *then*

$$\widetilde{L}_1^{(1,2,1)}(a) = L_1(w_2w_2w_3a) + w_1L_2(w_2w_3a) + w_1w_2L_2(w_3a) + w_1w_2w_2L_3(a),$$

where we have chosen 1 *factor from the list*

$$L_1, L_2, L_2, L_3$$

and the complementary $4 - 1 = 3$ *factors from the list*

$$M_{w_1}, M_{w_2}, M_{w_2}, M_{w_3}$$

and then summing over all possible terms of this form where the ordering of the subindices is always $1, 2, 2, 3$.

Proof. (By induction)

It suffices to prove the result as follows, for $m \geq 2$, we want to show

$$\widehat{Z}_{j_1}\cdots\widehat{Z}_{j_m}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_{j_1}\cdots w_{j_m}a\Big\rangle + \sum_{k=1}^{m-1}\hbar^k\Big|\Lambda, \widetilde{L}_k^{(e_{j_1}+\cdots+e_{j_m})}(a)\Big\rangle + \hbar^m\Big|\Lambda, L_{j_1}\circ\cdots\circ L_{j_m}(a)\Big\rangle.$$

where the e_i's are the unit vectors in the standard basis.

Let's consider the case $m = 2$, $\alpha = e_j + e_k$,

$$
\begin{aligned}
\widehat{Z}_j \widehat{Z}_k \Big| \Lambda, a \Big\rangle &= \widehat{Z}_j \Big(\Big| \Lambda, w_k a \Big\rangle + \hbar \Big| \Lambda, L_k(a) \Big\rangle \Big) \\
&= \Big| \Lambda, w_j w_k a \Big\rangle + \hbar \Big| \Lambda, w_j L_k(a) \Big\rangle \\
&\quad + \hbar \Big| \Lambda, L_j(w_k a) \Big\rangle + \hbar^2 \Big| \Lambda, L_j(L_k(a)) \Big\rangle \\
&= \Big| \Lambda, w_j w_k a \Big\rangle + \Big| \Lambda, \widetilde{L}_1^{(e_j+e_k)}(a) \Big\rangle + \hbar^2 \Big| \Lambda, L_j(L_k(a)) \Big\rangle
\end{aligned}
$$

It will also be instructive to include the $m = 3$ case even though it is not necessary for the induction.

If $m = 3$ we'll consider $\alpha = e_l + e_j + e_k$,

$$
\begin{aligned}
\widehat{Z}_l\widehat{Z}_j\widehat{Z}_k\Big|\Lambda, a\Big\rangle &= \widehat{Z}_l\Big(\Big|\Lambda, w_j w_k a\Big\rangle + \hbar\Big|\Lambda, w_j L_k(a)\Big\rangle \\
&\quad + \hbar\Big|\Lambda, L_j(w_k a)\Big\rangle + \hbar^2\Big|\Lambda, L_j(L_k(a))\Big\rangle\Big) \\
&= \Big|\Lambda, w_l w_j w_k a\Big\rangle + \hbar\Big|\Lambda, w_l w_j L_k(a)\Big\rangle + \hbar\Big|\Lambda, w_l L_j(w_k a)\Big\rangle \\
&\quad + \hbar^2\Big|\Lambda, w_l L_j(L_k(a))\Big\rangle + \hbar\Big|\Lambda, L_l(w_j w_k a)\Big\rangle + \hbar^2\Big|\Lambda, L_l(w_j L_k(a))\Big\rangle \\
&\quad + \hbar^2\Big|\Lambda, L_l(L_j(w_k a))\Big\rangle + \hbar^3\Big|\Lambda, L_l(L_j(L_k(a)))\Big\rangle \\
&= \Big|\Lambda, w_l w_j w_k a\Big\rangle + \hbar\Big(\Big|\Lambda, w_l w_j L_k(a)\Big\rangle + \Big|\Lambda, w_l L_j(w_k a)\Big\rangle + \Big|\Lambda, L_l(w_j w_k a)\Big\rangle\Big) \\
&\quad + \hbar^2\Big(\Big|\Lambda, w_l L_j(L_k(a))\Big\rangle + \Big|\Lambda, L_l(w_j L_k(a))\Big\rangle + \Big|\Lambda, L_l(L_j(w_k a))\Big\rangle\Big) \\
&\quad + \hbar^3\Big|\Lambda, L_l(L_j(L_k(a)))\Big\rangle \\
&= \Big|\Lambda, w_l w_j w_k a\Big\rangle + \hbar\Big|\Lambda, \widetilde{L}_1^{(e_l+e_j+e_k)}(a)\Big\rangle \\
&\quad + \hbar^2\Big|\Lambda, \widetilde{L}_2^{(e_l+e_j+e_k)}(a)\Big\rangle + \hbar^3\Big|\Lambda, L_l(L_j(L_k(a)))\Big\rangle
\end{aligned}
$$

Now for the induction, let's assume that for $m \geq 3$

$$
\widehat{Z}_{j_2}\cdots\widehat{Z}_{j_m}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_{j_2}\cdots w_{j_m} a\Big\rangle + \sum_{k=1}^{m-2}\hbar^k\Big|\Lambda, \widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\Big\rangle + \hbar^{m-1}\Big|\Lambda, L_{j_2}\circ\cdots\circ L_{j_m}(a)\Big\rangle.
$$

then we have

$$
\begin{aligned}
\widehat{Z}_{j_1}\widehat{Z}_{j_2}\cdots\widehat{Z}_{j_m}\bigg|\Lambda, a\bigg\rangle &= \widehat{Z}_{j_1}\Bigg(\bigg|\Lambda, w_{j_2}\cdots w_{j_m}a\bigg\rangle + \sum_{k=1}^{m-2}\hbar^k\bigg|\Lambda, \widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\bigg\rangle \\
&\quad + \hbar^{m-1}\bigg|\Lambda, L_{j_2}\circ\cdots\circ L_{j_m}(a)\bigg\rangle\Bigg) \\
&= \widehat{Z}_{j_1}\bigg|\Lambda, w_{j_2}\cdots w_{j_m}a\bigg\rangle + \sum_{k=1}^{m-2}\hbar^k\widehat{Z}_{j_1}\bigg|\Lambda, \widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\bigg\rangle \\
&\quad + \hbar^{m-1}\widehat{Z}_{j_1}\bigg|\Lambda, L_{j_2}\circ\cdots\circ L_{j_m}(a)\bigg\rangle\Bigg) \\
&= \bigg|\Lambda, w_{j_1}w_{j_2}\cdots w_{j_m}a\bigg\rangle + \hbar\bigg|\Lambda, L_{j_1}(w_{j_2}\cdots w_{j_m}a)\bigg\rangle \\
&\quad + \sum_{k=1}^{m-2}\hbar^k\Bigg(\bigg|\Lambda, w_{j_1}\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\bigg\rangle + \hbar\bigg|\Lambda, L_{j_1}(\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a))\bigg\rangle\Bigg) \\
&\quad + \hbar^{m-1}\Bigg(\bigg|\Lambda, w_{j_1}L_{j_2}\circ\cdots\circ L_{j_m}(a)\bigg\rangle + \hbar\bigg|\Lambda, L_{j_1}(L_{j_2}\circ\cdots\circ L_{j_m}(a))\bigg\rangle\Bigg) \\
&= \bigg|\Lambda, w_{j_1}w_{j_2}\cdots w_{j_m}a\bigg\rangle \\
&\quad + \hbar\Bigg(\bigg|\Lambda, L_{j_1}(w_{j_2}\cdots w_{j_m}a)\bigg\rangle + \bigg|\Lambda, w_{j_1}\widetilde{L}_1^{(e_{j_2}+\cdots+e_{j_m})}(a)\bigg\rangle\Bigg) \\
&\quad + \sum_{k=2}^{m-2}\hbar^k\bigg|\Lambda, w_{j_1}\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\bigg\rangle \\
&\quad + \sum_{k=1}^{m-3}\hbar^{k+1}\bigg|\Lambda, L_{j_1}(\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a))\bigg\rangle \\
&\quad + \hbar^{m-1}\Bigg(\bigg|\Lambda, L_{j_1}(\widetilde{L}_{m-2}^{(e_{j_2}+\cdots+e_{j_m})}(a))\bigg\rangle + \bigg|\Lambda, w_{j_1}L_{j_2}\circ\cdots\circ L_{j_m}(a)\bigg\rangle\Bigg) \\
&\quad + \hbar^m\bigg|\Lambda, L_{j_1}(L_{j_2}\circ\cdots\circ L_{j_m}(a))\bigg\rangle
\end{aligned}
$$

Now, recombining things with an eye toward the result we want to show, we get

$$
\begin{aligned}
\widehat{Z}_{j_1}\widehat{Z}_{j_2}\cdots\widehat{Z}_{j_m}\Big|\Lambda, a\Big\rangle &= \Big|\Lambda, w_{j_1}w_{j_2}\cdots w_{j_m}a\Big\rangle \\
&\quad + \hbar\Big(\Big|\Lambda, L_{j_1}(w_{j_2}\cdots w_{j_m}a)\Big\rangle + \Big|\Lambda, w_{j_1}\widetilde{L}_1^{(e_{j_2}+\cdots+e_{j_m})}(a)\Big\rangle\Big) \\
&\quad \sum_{k=2}^{m-2}\hbar^k\Big(\Big|\Lambda, w_{j_1}\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a)\Big\rangle + \Big|\Lambda, L_{j_1}(\widetilde{L}_k^{(e_{j_2}+\cdots+e_{j_m})}(a))\Big\rangle\Big) \\
&\quad + \hbar^{m-1}\Big(\Big|\Lambda, L_{j_1}(\widetilde{L}_{m-2}^{(e_{j_2}+\cdots+e_{j_m})}(a))\Big\rangle + \Big|\Lambda, w_{j_1}L_{j_2}\circ\cdots\circ L_{j_m}(a)\Big\rangle\Big) \\
&\quad + \hbar^m\Big|\Lambda, L_{j_1}(L_{j_2}\circ\cdots\circ L_{j_m}(a))\Big\rangle \\
&= \Big|\Lambda, w_{j_1}w_{j_2}\cdots w_{j_m}a\Big\rangle \\
&\quad + \sum_{k=1}^{m-1}\hbar^k\Big|\Lambda, \widetilde{L}_k^{(e_{j_1}+e_{j_2}+\cdots+e_{j_m})}(a)\Big\rangle \\
&\quad + \hbar^m\Big|\Lambda, L_{j_1}\circ L_{j_2}\circ\cdots\circ L_{j_m}(a)\Big\rangle
\end{aligned}
$$

$\square$

Combining the results of these two lemmas we can say that

$$
\begin{aligned}
\widehat{Z}^\alpha\widehat{Z^*}^\beta\Big|\Lambda, a\Big\rangle &= \widehat{Z}^\alpha\Big|\Lambda, \overline{\mathbf{w}}^\beta a\Big\rangle \\
&= \Big|\Lambda, \mathbf{w}^\alpha\overline{\mathbf{w}}^\beta a\Big\rangle + \sum_{j=1}^{|\alpha|-1}\hbar^j\Big|\Lambda, \widetilde{L}_j^\alpha(\overline{\mathbf{w}}^\beta a)\Big\rangle + \hbar^{|\alpha|}\Big|\Lambda, L^\alpha(\overline{\mathbf{w}}^\beta a)\Big\rangle.
\end{aligned}
$$

Again, for a general polynomial Hamiltonian $H(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta$ we have

$$
\widehat{H}_{wick} = \sum_{\alpha,\beta} C_{\alpha,\beta}\widehat{Z}^\alpha\widehat{Z^*}^\beta,
$$

and thus from above (with the same conditions on the generalized Lagrangian state)

$$\begin{aligned}
\widehat{H}_{wick}\Big|\Lambda, a\Big\rangle &= \sum_{\alpha,\beta} C_{\alpha,\beta}\left(\Big|\Lambda, \mathbf{w}^{\alpha}\overline{\mathbf{w}}^{\beta}a\Big\rangle + \sum_{j=1}^{|\alpha|-1}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{|\alpha|}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right) \\
&= \Big|\Lambda, \sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{w}^{\alpha}\overline{\mathbf{w}}^{\beta}a\Big\rangle + \sum_{\alpha,\beta} C_{\alpha,\beta}\left(\sum_{j=1}^{|\alpha|-1}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{|\alpha|}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right) \\
&= \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle + \sum_{\alpha,\beta} C_{\alpha,\beta}\left(\sum_{j=1}^{|\alpha|-1}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{|\alpha|}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right)
\end{aligned}$$

It is a somewhat obvious, but nevertheless important observation that in the above expression it is the α terms that contributes powers of $\hbar$. This is rooted in the fact that from our original rules for the action of $\widehat{Z}_j$ and $\widehat{Z}_j^*$ on our states we see that it is the action of the $\widehat{Z}_j$'s that produces a higher order term in $\hbar$. Thus, if our goal is to come up with an expression for $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$ ordered in $\hbar$ then our next step should be to take the above expression and properly order the α's.

Thus,

$$\begin{aligned}
\widehat{H}_{wick}\Big|\Lambda, a\Big\rangle &= \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a\Big\rangle \\
&\quad + \sum_{\beta}\Bigg(\sum_{|\alpha|=1} C_{\alpha,\beta}\hbar\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle \\
&\qquad + \sum_{|\alpha|=2} C_{\alpha,\beta}\Big(\sum_{j=1}^{1}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{2}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Big) \\
&\qquad + \sum_{|\alpha|=3} C_{\alpha,\beta}\Big(\sum_{j=1}^{2}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{3}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Big) \\
&\qquad + \cdots \\
&\qquad + \sum_{|\alpha|=n} C_{\alpha,\beta}\Big(\sum_{j=1}^{n-1}\hbar^{j}\Big|\Lambda, \widetilde{L}_{j}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \hbar^{n}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Big)\Bigg) \\
&= \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a\Big\rangle \\
&\quad + \hbar\sum_{\beta}\Bigg(\sum_{|\alpha|=1} C_{\alpha,\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>1} C_{\alpha,\beta}\Big|\Lambda, \widetilde{L}_{1}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Bigg) \\
&\quad + \hbar^{2}\sum_{\beta}\Bigg(\sum_{|\alpha|=2} C_{\alpha,\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>2} C_{\alpha,\beta}\Big|\Lambda, \widetilde{L}_{2}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Bigg) \\
&\quad + \hbar^{3}\sum_{\beta}\Bigg(\sum_{|\alpha|=3} C_{\alpha,\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>3} C_{\alpha,\beta}\Big|\Lambda, \widetilde{L}_{3}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Bigg) \\
&\quad + \cdots \\
&\quad + \hbar^{M-1}\sum_{\beta}\Bigg(\sum_{|\alpha|=M-1} C_{\alpha,\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>M-1} C_{\alpha,\beta}\Big|\Lambda, \widetilde{L}_{M-1}^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\Bigg) \\
&\quad + \hbar^{M}\sum_{\beta}\sum_{|\alpha|=M} C_{\alpha,\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle,
\end{aligned}$$

where we assume that $|\alpha| = M$ is the largest such value in H. Let's summarize the result we've proven with the work above

Proposition IV.15. *Suppose Λ is a smooth n-dimensional manifold, and $\mathbf{w} : \Lambda \longrightarrow$*

$\mathbb{C}$ is a Lagrangian embedding. For a compactly supported 'a' independent of $\hbar$, and a general polynomial Hamiltonian

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta} \mathbf{z}^{\alpha} \overline{\mathbf{z}}^{\beta},$$

for which $M \equiv \max\{|\alpha|\}$, we have

$$\begin{aligned}
\widehat{H}_{wick}\Big|\Lambda, a\Big\rangle = \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle \\
&+ \hbar \sum_{\beta} \left(\sum_{|\alpha|=1} C_{\alpha,\beta} \Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle + \sum_{|\alpha|>1} C_{\alpha,\beta} \Big|\Lambda, \widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle \right) \\
&+ \hbar^2 \sum_{\beta} \left(\sum_{|\alpha|=2} C_{\alpha,\beta} \Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle + \sum_{|\alpha|>2} C_{\alpha,\beta} \Big|\Lambda, \widetilde{L}_2^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle \right) \\
&+ \hbar^3 \sum_{\beta} \left(\sum_{|\alpha|=3} C_{\alpha,\beta} \Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle + \sum_{|\alpha|>3} C_{\alpha,\beta} \Big|\Lambda, \widetilde{L}_3^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle \right) \\
&+ \cdots \\
&+ \hbar^{M-1} \sum_{\beta} \left(\sum_{|\alpha|=M-1} C_{\alpha,\beta} \Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle + \sum_{|\alpha|>M-1} C_{\alpha,\beta} \Big|\Lambda, \widetilde{L}_{M-1}^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle \right) \\
&+ \hbar^{M} \sum_{\beta} \sum_{|\alpha|=M} C_{\alpha,\beta} \Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta} a)\Big\rangle.
\end{aligned}$$

This gives us an appropriately ordered expression for $\widehat{H}_{wick}\Big|\Lambda, a\Big\rangle$. We need to leverage this and the known relationship between $\widehat{H}_{wick}$ and $\widehat{H}_{weyl}$ to arrive at an expression for $\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle$. Now, based on the discussion of the Laplacian in the $n = 1$ case it is immediately clear that for $f(\mathbf{z}, \overline{\mathbf{z}})$ sufficiently smooth,

$$\Delta f = 2\left(\frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}} + \cdots + \frac{\partial^2 f}{\partial z_n \partial \overline{z_n}} \right).$$

This gives

$$\Delta\left(\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right) = \sum_{j=1}^{n} 2\alpha_j\beta_j \mathbf{z}^{\alpha-\mathbf{e_j}}\overline{\mathbf{z}}^{\beta-\mathbf{e_j}},$$

where (again) the e_i's are the unit vectors in the standard basis. Working out a couple more such expressions in order to get some perspective in the 'run-up' to a general formula:

$$\begin{aligned}
\Delta^2\left(\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right) &= 2^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z_k}} \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} \left(\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right) \right) \\
&= 2^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z_k}} \left(\sum_{j=1}^{n} \alpha_j\beta_j \mathbf{z}^{\alpha-\mathbf{e_j}}\overline{\mathbf{z}}^{\beta-\mathbf{e_j}} \right) \\
&= 2^2 \sum_{k=1}^{n}\sum_{j=1}^{n} \alpha_j\beta_j \frac{\partial^2}{\partial z_k \partial \overline{z_k}} \left(\mathbf{z}^{\alpha-\mathbf{e_j}}\overline{\mathbf{z}}^{\beta-\mathbf{e_j}} \right) \\
&= 2^2 \sum_{k=1}^{n}\sum_{j=1}^{n} \alpha_j\beta_j(\alpha_k-\delta_{jk})(\beta_k-\delta_{jk}) \left(\mathbf{z}^{\alpha-\mathbf{e_j}-\mathbf{e_k}}\overline{\mathbf{z}}^{\beta-\mathbf{e_j}-\mathbf{e_k}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta^3\left(\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right) = 2^3 \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} &\alpha_j\beta_j(\alpha_k-\delta_{jk})(\beta_k-\delta_{jk})(\alpha_l-\delta_{jk}-\delta_{kl})(\beta_l-\delta_{jk}-\delta_{kl}) \\
&\times \left(\mathbf{z}^{\alpha-\mathbf{e_j}-\mathbf{e_k}-\mathbf{e_l}}\overline{\mathbf{z}}^{\beta-\mathbf{e_j}-\mathbf{e_k}-\mathbf{e_l}} \right).
\end{aligned}$$

For a general positive integer l

$$\Delta^l\left(\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta\right) = 2^l \sum_{j_l=1}^{n} \cdots \sum_{j_1=1}^{n} \alpha_{j_1}\beta_{j_1}(\alpha_{j_2}-\delta_{j_1j_2})(\beta_{j_2}-\delta_{j_1j_2})$$

$$\times \cdots \times (\alpha_{j_l}-\delta_{j_1j_l}-\delta_{j_2j_l}-\cdots-\delta_{j_{l-1}j_l})$$

$$\times (\beta_{j_l}-\delta_{j_1j_l}-\delta_{j_2j_l}-\cdots-\delta_{j_{l-1}j_l})$$

$$\times\, \mathbf{z}^{\alpha-\mathbf{e_{j_1}}-\mathbf{e_{j_2}}-\cdots-\mathbf{e_{j_l}}}\overline{\mathbf{z}}^{\beta-\mathbf{e_{j_1}}-\mathbf{e_{j_2}}-\cdots-\mathbf{e_{j_l}}}$$

$$= 2^l \sum_{i=1}^{l} \sum_{j_i=1}^{n} \alpha_{j_1}\beta_{j_1}(\alpha_{j_2}-\delta_{j_1j_2})(\beta_{j_2}-\delta_{j_1j_2})$$

$$\times \cdots \times (\alpha_{j_l}-\delta_{j_1j_l}-\delta_{j_2j_l}-\cdots-\delta_{j_{l-1}j})$$

$$\times (\beta_{j_l}-\delta_{j_1j_l}-\delta_{j_2j_l}-\cdots-\delta_{j_{l-1}j_l})$$

$$\times\, \mathbf{z}^{\alpha-\mathbf{e_{j_1}}-\mathbf{e_{j_2}}-\cdots-\mathbf{e_{j_l}}}\overline{\mathbf{z}}^{\beta-\mathbf{e_{j_1}}-\mathbf{e_{j_2}}-\cdots-\mathbf{e_{j_l}}}.$$

Now, let's introduce some notation to "clean up" this expression. Let

$$\alpha^{j_1\cdots j_k} \equiv \alpha - (e_{j_1} - e_{j_2} - \cdots - e_{j_k}),$$

and similarly,

$$\beta^{j_1\cdots j_k} \equiv \beta - (e_{j_1} - e_{j_2} - \cdots - e_{j_k}),$$

which will give (just to be clear)

$$(\alpha^{j_1\cdots j_k})_i = \alpha_i - (\delta_{ij_1} + \cdots + \delta_{ij_k})$$

and

$$(\beta^{j_1\cdots j_k})_i = \beta_i - (\delta_{ij_1} + \cdots + \delta_{ij_k}).$$

Using this new notation we get

$$\Delta^l\left(\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right) = 2^l\sum_{i=1}^{l}\sum_{j_i=1}^{n}\alpha_{j_1}\beta_{j_1}(\alpha^{j_1})_{j_2}(\beta^{j_1})_{j_2}(\alpha^{j_1j_2})_{j_3}(\beta^{j_1j_2})_{j_3}$$
$$\times\cdots\times(\alpha^{j_1\cdots j_{l-1}})_{j_l}(\beta^{j_1\cdots j_{l-1}})_{j_l}$$
$$\times\,\mathbf{z}^{\alpha^{j_1\cdots j_l}}\overline{\mathbf{z}}^{\alpha^{j_1\cdots j_l}}.$$

This, in turn for $H(\mathbf{z},\overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}$, gives

$$\begin{aligned}
\Delta^l H &= \Delta^l\left(\sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta}\right)\\
&= \sum_{\alpha\beta}\sum_{i=1}^{l}\sum_{j_i=1}^{n}2^l C_{\alpha,\beta}\alpha_{j_1}\beta_{j_1}(\alpha^{j_1})_{j_2}(\beta^{j_1})_{j_2}(\alpha^{j_1j_2})_{j_3}(\beta^{j_1j_2})_{j_3}\\
&\qquad\times\cdots\times(\alpha^{j_1\cdots j_{l-1}})_{j_l}(\beta^{j_1\cdots j_{l-1}})_{j_l}\\
&\qquad\times\,\mathbf{z}^{\alpha^{j_1\cdots j_l}}\overline{\mathbf{z}}^{\alpha^{j_1\cdots j_l}}\\
&= \sum_{\alpha\beta}\sum_{i=1}^{l}\sum_{j_i=1}^{n}\widetilde{C}_{\alpha\beta}^{j_1\cdots j_l}\mathbf{z}^{\alpha^{j_1\cdots j_l}}\overline{\mathbf{z}}^{\alpha^{j_1\cdots j_l}},
\end{aligned}$$

where

$$\widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \equiv 2^l C_{\alpha,\beta}\alpha_{j_1}\beta_{j_1}(\alpha^{j_1})_{j_2}(\beta^{j_1})_{j_2}(\alpha^{j_1j_2})_{j_3}(\beta^{j_1j_2})_{j_3}\times\cdots\times(\alpha^{j_1\cdots j_{l-1}})_{j_l}(\beta^{j_1\cdots j_{l-1}})_{j_l}$$

We can 'compress' things once more to

$$\Delta^l H = \sum_{\alpha^{j_1\cdots j_l},\beta^{j_1\cdots j_l}}\widetilde{C}_{\alpha\beta}^{j_1\cdots j_l}\mathbf{z}^{\alpha^{j_1\cdots j_l}}\overline{\mathbf{z}}^{\alpha^{j_1\cdots j_l}}.$$

Since, we have

$$\widehat{H}_{weyl} = \sum_{l=0}^{R} \left(\frac{\hbar}{4}\right)^{l} (\widehat{\Delta^l H})_{wick},$$

and if $M = \max|\alpha|$, we get

$$\begin{aligned}
\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \sum_{l=0}^{R} \left(\frac{\hbar}{4}\right)^{l} (\widehat{\Delta^l H})_{wick}\Big|\Lambda, a\Big\rangle \\
&= \sum_{l=0}^{R} \left(\frac{\hbar}{4}\right)^{l} \Bigg(\sum_{\alpha^{j_1\cdots j_l}, \beta^{j_1\cdots j_l}} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \widehat{Z}^{\alpha^{j_1\cdots j_l}} \widehat{Z^*}^{\alpha^{j_1\cdots j_l}} \Bigg)\Big|\Lambda, a\Big\rangle \\
&= \sum_{l=0}^{R} \left(\frac{\hbar}{4}\right)^{l} \Bigg(\Big|\Lambda, \left(\Delta^l H\right)(\mathbf{w}, \overline{\mathbf{w}}) a\Big\rangle \\
&\qquad + \hbar \sum_{\beta^{j_1\cdots j_l}} \Bigg(\sum_{|\alpha^{j_1\cdots j_l}|=1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, L^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle \\
&\qquad\qquad + \sum_{|\alpha^{j_1\cdots j_l}|>1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, \widetilde{L}_1^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle\Bigg) \\
&\qquad + \hbar^2 \sum_{\beta^{j_1\cdots j_l}} \Bigg(\sum_{|\alpha^{j_1\cdots j_l}|=2} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, L^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle \\
&\qquad\qquad + \sum_{|\alpha^{j_1\cdots j_l}|>2} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, \widetilde{L}_2^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle\Bigg) \\
&\qquad + \cdots \\
&\qquad + \hbar^{M-l+1} \sum_{\beta^{j_1\cdots j_l}} \Bigg(\sum_{|\alpha^{j_1\cdots j_l}|=M-l+1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, L^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle \\
&\qquad\qquad + \sum_{|\alpha^{j_1\cdots j_l}|>M-l+1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, \widetilde{L}_{M-l+1}^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle\Bigg) \\
&\qquad + \hbar^{M-l} \sum_{\beta^{j_1\cdots j_l}} \sum_{|\alpha^{j_1\cdots j_l}|=M-l} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_l} \Big|\Lambda, L^{\alpha^{j_1\cdots j_l}} \left(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_l}} a\right) \Big\rangle\Bigg),
\end{aligned}$$

where for the sake of clarity we should note that $\alpha^{j_1\cdots j_l} \equiv \alpha - (e_{j_1} + \cdots + e_{j_l})$, so if $|\alpha| = k$, then $|\alpha^{j_1\cdots j_l}| = k - l$. Also, we define $\alpha^{j_1\cdots j_l} = \alpha$ if $l = 0$.

Let's note that the $l = j$ term in this sum has terms in it with powers of $\hbar$ from j to M. As before, let's systematically gather together terms from the above expression in terms of their $\hbar$ contribution:

For $\hbar^0$: 1 term

$$\Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle$$

For $\hbar^1$: 2 terms

$$\left(\frac{1}{4}\right)\Big|\Lambda, (\Delta H)(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle + \sum_{\beta}\left(\sum_{|\alpha|=1} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>1} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}^{\alpha}_{1}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right)$$

For $\hbar^2$: 3 terms

$$\left(\frac{1}{4}\right)^2\Big|\Lambda, (\Delta^2 H)(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle + \sum_{\beta}\left(\sum_{|\alpha|=2} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle + \sum_{|\alpha|>2} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}^{\alpha}_{2}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right)$$
$$+ \sum_{\beta^{j_1}}\left(\sum_{|\alpha^{j_1}|=1} \widetilde{C}^{j_1}_{\alpha\beta}\Big|\Lambda, L^{\alpha^{j_1}}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\Big\rangle + \sum_{|\alpha_{j_1}|>1} \widetilde{C}^{j_1}_{\alpha\beta}\Big|\Lambda, \widetilde{L}^{\alpha^{j_1}}_{1}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\Big\rangle\right)$$

For $\hbar^3$: 4 terms

$$\left(\frac{1}{4}\right)^3 \left|\Lambda, (\Delta^3 H)(\mathbf{w},\overline{\mathbf{w}})a\right\rangle + \sum_{\beta}\left(\sum_{|\alpha|=3} C_{\alpha\beta}\left|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\right\rangle + \sum_{|\alpha|>3} C_{\alpha\beta}\left|\Lambda, \widetilde{L}_3^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\right\rangle\right)$$
$$+\sum_{\beta^{j_1}}\left(\sum_{|\alpha^{j_1}|=2} \widetilde{C}_{\alpha\beta}^{j_1}\left|\Lambda, L^{\alpha^{j_1}}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\right\rangle + \sum_{|\alpha_{j_1}|>2} \widetilde{C}_{\alpha\beta}^{j_1}\left|\Lambda, \widetilde{L}_2^{\alpha^{j_1}}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\right\rangle\right)$$
$$+\sum_{\beta^{j_1 j_2}}\left(\sum_{|\alpha^{j_1 j_2}|=1} \widetilde{C}_{\alpha\beta}^{j_1 j_2}\left|\Lambda, L^{\alpha^{j_1 j_2}}(\overline{\mathbf{w}}^{\beta^{j_1 j_2}}a)\right\rangle\right.$$
$$\left.+\sum_{|\alpha_{j_1 j_2}|>1} \widetilde{C}_{\alpha\beta}^{j_1 j_2}\left|\Lambda, \widetilde{L}_1^{\alpha^{j_1 j_2}}(\overline{\mathbf{w}}^{\beta^{j_1 j_2}}a)\right\rangle\right)$$

<u>For $\hbar^k$:</u> '$k+1$' terms

$$\left(\frac{1}{4}\right)^k \left|\Lambda, (\Delta^k H)(\mathbf{w},\overline{\mathbf{w}})a\right\rangle + \sum_{\beta}\left(\sum_{|\alpha|=k} C_{\alpha\beta}\left|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\right\rangle + \sum_{|\alpha|>k} C_{\alpha\beta}\left|\Lambda, \widetilde{L}_k^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\right\rangle\right)$$
$$+\sum_{\beta^{j_1}}\left(\sum_{|\alpha^{j_1}|=k-1} \widetilde{C}_{\alpha\beta}^{j_1}\left|\Lambda, L^{\alpha^{j_1}}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\right\rangle\right.$$
$$\left.+\sum_{|\alpha_{j_1}|>k-1} \widetilde{C}_{\alpha\beta}^{j_1}\left|\Lambda, \widetilde{L}_{k-1}^{\alpha^{j_1}}(\overline{\mathbf{w}}^{\beta^{j_1}}a)\right\rangle\right)$$
$$+\cdots$$
$$+\sum_{\beta^{j_1\cdots j_{k-1}}}\left(\sum_{|\alpha^{j_1\cdots j_{k-1}}|=1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_{k-1}}\left|\Lambda, L^{\alpha^{j_1\cdots j_{k-1}}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_{k-1}}}a)\right\rangle\right.$$
$$\left.+\sum_{|\alpha_{j_1\cdots j_{k-1}}|>1} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_{k-1}}\left|\Lambda, \widetilde{L}_1^{\alpha^{j_1\cdots j_{k-1}}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_{k-1}}}a)\right\rangle\right)$$

To 'clean things up' we can introduce the same notation used in the $n=1$ case and let

$$\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle = D_0^H\Big|\Lambda, a\Big\rangle + \hbar D_1^H\Big|\Lambda, a\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a\Big\rangle, \tag{4.20}$$

where

$$D_0^H\Big|\Lambda, a\Big\rangle \equiv \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle \tag{4.21}$$

and for $1 \leq k \leq M$

(4.22)

$$D_k^H\Big|\Lambda, a\Big\rangle \equiv \left(\frac{1}{4}\right)^k\Big|\Lambda, \left(\Delta^k H\right)(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle + \sum_{\beta}\left(\sum_{|\alpha|=k} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right.$$

$$\left. + \sum_{|\alpha|>k} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}_k^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\right) \tag{4.23}$$

$$+ \sum_{i=1}^{k-1}\sum_{\beta^{j_1\cdots j_i}}\left(\sum_{|\alpha^{j_1\cdots j_i}|=k-i} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_i}\Big|\Lambda, L^{\alpha^{j_1\cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_i}}a)\Big\rangle\right. \tag{4.24}$$

$$\left. + \sum_{|\alpha^{j_1\cdots j_i}|>k-i} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_i}\Big|\Lambda, \widetilde{L}_{k-i}^{\alpha^{j_1\cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_i}}a)\Big\rangle\right) \tag{4.25}$$

Now the last level of intricacy to add in is to let 'a' depend symbolically on $\hbar$, again in the sense that there exist functions a_j such that

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j;$$

which in turn gives the state expansion

$$\Big|\Lambda, a\Big\rangle \sim \Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots = \sum_{j=0}^{\infty} \hbar^j\Big|\Lambda, a_j\Big\rangle.$$

Thus for each value of r we have

$$\Big|\Lambda, a\Big\rangle = \Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right),$$

so, for each r

$$\begin{aligned}
\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \widehat{H}_{weyl}\Big|\Lambda, a_0\Big\rangle + \hbar\widehat{H}_{weyl}\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r\widehat{H}_{weyl}\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right) \\
&= D_0^H\Big|\Lambda, a_0\Big\rangle + \hbar D_1^H\Big|\Lambda, a_0\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_0\Big\rangle \\
&\quad + \hbar\Big(D_0^H\Big|\Lambda, a_1\Big\rangle + \hbar D_1^H\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_1\Big\rangle\Big) \\
&\quad + \cdots + \hbar^r\Big(D_0^H\Big|\Lambda, a_r\Big\rangle + \hbar D_1^H\Big|\Lambda, a_r\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_r\Big\rangle\Big) \\
&\quad + O\left(\hbar^{r+1}\right).
\end{aligned}$$

Again, gathering terms at each level of $\hbar$ we get:

<u>For $\hbar^0$:</u> 1 term

$$D_0^H\Big|\Lambda, a_0\Big\rangle$$

<u>For $\hbar^1$:</u> 2 terms

$$D_1^H\Big|\Lambda, a_0\Big\rangle + D_0^H\Big|\Lambda, a_1\Big\rangle$$

<u>For $\hbar^2$:</u> 3 terms

$$D_2^H\Big|\Lambda, a_0\Big\rangle + D_1^H\Big|\Lambda, a_1\Big\rangle + D_0^H\Big|\Lambda, a_2\Big\rangle$$

$$\vdots$$

<u>For $\hbar^s$ $(s \leq M)$:</u> $s + 1$ terms

$$D_s^H\Big|\Lambda,a_0\Big\rangle + D_{s-1}^H\Big|\Lambda,a_1\Big\rangle + \cdots + D_0^H\Big|\Lambda,a_s\Big\rangle$$

Note, for each j, $\widehat{H}_{weyl}\Big|\Lambda,a_j\Big\rangle$ has terms (at the level of states) up to $\hbar^M$. Thus in the above, if $s > M$ there will be no term involving $\Big|\Lambda,a_0\Big\rangle$; if $s > M+1$, there will be no terms involving $\Big|\Lambda,a_0\Big\rangle$ and $\Big|\Lambda,a_1\Big\rangle$. In general if $s > M+k$ there will be no terms involving $\Big|\Lambda,a_0\Big\rangle$, ..., $\Big|\Lambda,a_k\Big\rangle$, thus

<u>For $\hbar^s$ $(s > M)$:</u> $M+1$ terms

$$D_0^H\Big|\Lambda,a_s\Big\rangle + D_1^H\Big|\Lambda,a_{s-1}\Big\rangle + \cdots + D_{M-1}^H\Big|\Lambda,a_{s-(M-1)}\Big\rangle + D_M^H\Big|\Lambda,a_{s-M}\Big\rangle$$

We now have a systematic way of expanding $\widehat{H}_{weyl}\Big|\Lambda,a\Big\rangle$ in terms of Lagrangian states in the general case.

As stated at the beginning of the chapter, from these calculations we have proven the following theorem (and more):

Theorem IV.16. *Suppose one has a classical Hamiltonian, on $\mathbb{C}^n$, of the form*

$$H(\mathbf{z},\overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta,$$

for multi-indices α, and β. Further, suppose Λ is a smooth n-dimensional manifold, and $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ is a Lagrangian embedding. For an associated generalized Lagrangian state with a compactly supported amplitude 'a' then the following are true:

1. *The principal symbol of $\widehat{H}_{weyl}\Big|\Lambda,a\Big\rangle$ is*

$$\delta^0_{\mathbf{w}} = H\left(\mathbf{w}, \overline{\mathbf{w}}\right) a_0 |d\mathbf{w}|^{1/2};$$

2. The sub-principal symbol of $\widehat{H}_{weyl}\left|\Lambda, a\right\rangle$ *is*

$$\begin{aligned}\delta^1_{\mathbf{w}} = \Bigg(&\frac{1}{4}\left(\Delta H\right)\left(\mathbf{w}, \overline{\mathbf{w}}\right) a_0 + \sum_{\beta}\sum_{|\alpha|=1} C_{\alpha\beta} L^{\alpha}(\overline{\mathbf{w}}^{\beta} a_0) \\ &+ \sum_{\beta}\sum_{|\alpha|>1} C_{\alpha\beta}\widetilde{L}^{\alpha}_1(\overline{\mathbf{w}}^{\beta} a_0) + H\left(\mathbf{w}, \overline{\mathbf{w}}\right) a_1\Bigg)|d\mathbf{w}|^{1/2}.\end{aligned}$$

As we saw in the $n = 1$ case, we see that if if δ_w is the principal symbol of a generalized Lagrangian state $\left|\Lambda, a\right\rangle$, then for a polynomial Hamiltonian H the principal symbol of $\widehat{H}_{weyl}\left|\Lambda, a\right\rangle$ is $H\Big|_{w(\Lambda)}\delta_w$. Similarly, the sub-principal symbol of $\widehat{H}_{weyl}\left|\Lambda, a\right\rangle$ is the sum of three terms, one is (up to constants) $\Delta H\Big|_{w(\Lambda)}$ multiplying the principal symbol of $\left|\Lambda, a\right\rangle$, one is the the sub-principal symbol of $\left|\Lambda, a\right\rangle$ multiplied by $H\Big|_{w(\Lambda)}$, and finally a 1/2-density that is given by a first order differential operator acting on a_0, where the first order differential operator is intrinsically dependent on the complex structure.

CHAPTER V

Propagation of Generalized Lagrangian States

In the first part of this chapter we will introduce the idea of generalized Lagrangian states that depend on a parameter. We will use such parameter dependent states where the parameter is thought of as time with the goal of showing how such states behave under quantum evolution.

5.1 Parameter Dependent Generalized Lagrangian States

5.1.1 Defining a Parameter Dependent Generalized Lagrangian State

Suppose Λ is a smooth n-dimensional manifold, I is an open subset of $\mathbb{R}$, and that for every $t \in I$ we have a Lagrangian embedding $\mathbf{w}_t : \Lambda \longrightarrow \mathbb{R}^{2n}$ (see III.1) and an amplitude $a_t : \Lambda \longrightarrow \mathbb{C}$. Identifying $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by a choice of complex structure, for each value of t we can construct the generalized Lagrangian state

$$|\Lambda, a_t\rangle(\mathbf{z}) = \int_\Lambda a_t e^{\frac{i}{\hbar} f_t} K(\mathbf{z}, \mathbf{w}_t) |d\mathbf{w}_t| \tag{5.1}$$

where $f_t : \Lambda \longrightarrow \mathbb{R}$ is defined (up to a constant) by the lift condition (3.3)

$$df_t + \frac{1}{2i}(\mathbf{w}_t \cdot d\overline{\mathbf{w}}_t - \overline{\mathbf{w}_t} \cdot d\mathbf{w}_t) = 0.$$

We now wish to smoothly parameterize the family of states $\{|\Lambda, a_t\rangle\}_{t\in I}$, defined with respect to the family of Lagrangian embeddings $\{\mathbf{w}_t\}_{t\in I}$.

For every $\mathbf{x} \in \Lambda$ and every $t \in I$ define $\mathbf{w}(\mathbf{x}, t) \equiv \mathbf{w}_t(\mathbf{x})$. Thus $\mathbf{w}(\mathbf{x}, t) : \Lambda \times I \longrightarrow \mathbb{C}^n$, and we will call such a map a smooth parametrization of Lagrangian embeddings if $\mathbf{w} \in C^\infty(\Lambda \times I)$. Along this same line of reasoning we define the map $f : \Lambda \times I \longrightarrow \mathbb{C}^n$ by $f(\mathbf{x}, t) \equiv f_t(\mathbf{x})$. Now, we know that for each fixed value of t that the function f_t, being defined by the lift condition above is well-defined up to an arbitrary constant. This means that for each value of t, f_t is well-defined if we specify $f_t(\mathbf{x}_t)$ for some $\mathbf{x}_t \in \Lambda$. We can then say that f will be well-defined given a boundary condition whereby each f_t is specified at the same $\widetilde{\mathbf{x}} \in \Lambda$.

Concerning the symbol of parameter dependent states, note the symbol of each $|\Lambda, a_t\rangle$ is

$$\delta_{\mathbf{w}_t} \equiv a_t(\mathbf{x})|d\mathbf{w}_t|^{1/2}.$$

If, as above, we understand that the map $\mathbf{w}$ depends on t, and we define the map $a : \Lambda \times I \longrightarrow \mathbb{C}$ by $a(\mathbf{x}, t) \equiv a_t(\mathbf{x})$ then

$$\delta_{\mathbf{w}} \equiv a(\mathbf{x}, t)|d\mathbf{w}|^{1/2}$$

is a smooth 1/2-density on Λ for every value of t. Note that in this last expression $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$.

A crucial component of showing our main result will be an understanding of how parameter dependent generalized Lagrangian states vary with respect to the parameter. In order to motivate a general rule we consider the case where $a = a(\mathbf{x}, t)$ is compactly supported for each value of t. In this case (III.24) we've seen that a parameter dependent generalized state $|\Lambda, a\rangle$ is a holomorphic function of

$\mathbf{z} \in \mathbb{C}^n$ for each value of t, and is in fact an element in Bargmann space. Indeed, if $a(\cdot, t; \hbar), f(\cdot, t) \in C^1(I)$ then we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\Lambda, a\rangle &= \frac{\partial}{\partial t}\int_\Lambda a e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&= \int_\Lambda \frac{\partial}{\partial t}\left(a e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} |d\mathbf{w}|\right) e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \\
&= \int_\Lambda \frac{\partial a}{\partial t} e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&\quad + \int_\Lambda a \frac{i}{\hbar}\frac{\partial f}{\partial t} e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&\quad + \int_\Lambda a e^{\frac{i}{\hbar}f} \left(\frac{1}{\hbar}\mathbf{z}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial t}\right) e^{\frac{\overline{\mathbf{w}(\mathbf{x},t)}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&\quad + \int_\Lambda a e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} \left(-\frac{1}{2\hbar}\left(\frac{\partial\mathbf{w}}{\partial t}\cdot\overline{\mathbf{w}} + \mathbf{w}\cdot\frac{\partial\overline{\mathbf{w}}}{\partial t}\right)\right) e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} |d\mathbf{w}| \\
&\quad + \int_\Lambda a e^{\frac{i}{\hbar}f} e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \frac{\partial}{\partial t}\left(|d\mathbf{w}|\right).
\end{aligned}$$

Using Jacobi's formula, (4.7), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(|d\mathbf{w}|\right) &= \frac{\partial}{\partial t}\left(\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}|\right)\\
&= \left(\frac{\partial}{\partial t}\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|\right)|d\mathbf{x}|\\
&= \left(\frac{\partial}{\partial t}\left(\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\overline{\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)}\right)^{1/2}\right)|d\mathbf{x}|\\
&= \frac{1}{2}\left(\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\overline{\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)}\right)^{-1/2}\\
&\quad\times\left(\left(\frac{\partial}{\partial t}\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)\overline{\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)}+\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\overline{\left(\frac{\partial}{\partial t}\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)}\right)|d\mathbf{x}|\\
&= \frac{1}{2}\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right|^{-1}\\
&\quad\times\left(\left(\frac{\partial}{\partial x_l}\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)\overline{\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)}+\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\overline{\left(\frac{\partial}{\partial t}\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)}\right)|d\mathbf{x}|\\
&= \mathrm{Tr}\left(\mathrm{Re}\left(\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)^{-1}\frac{\partial}{\partial t}\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)\left|\det\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right||d\mathbf{x}|\\
&= \mathrm{Tr}\left(\mathrm{Re}\left(\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)^{-1}\frac{\partial}{\partial t}\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)|d\mathbf{w}|.
\end{aligned}
$$

It is a straightforward calculation, using the chain rule, to show that the quantity $\mathrm{Tr}\left(\mathrm{Re}\left(\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)^{-1}\frac{\partial}{\partial t}\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right)$ is in fact coordinate invariant. Denote this globally well-defined quantity by

$$
Y \equiv \mathrm{Tr}\left(\mathrm{Re}\left(\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)^{-1}\frac{\partial}{\partial t}\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\right) \tag{5.2}
$$

After organizing the terms, and using the rule (4.18) we get

$$
\begin{aligned}
\frac{\partial}{\partial t}|\Lambda, a\rangle &= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot \overline{\mathbf{w}} + \mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right) a\right\rangle \\
&\quad + \frac{1}{\hbar}\sum_{j=1}^{n} z_j \left|\Lambda, \frac{\partial \overline{w}_j}{\partial t} a\right\rangle + \left|\Lambda, Ya\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot \overline{\mathbf{w}} + \mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right) a\right\rangle \\
&\quad + \frac{1}{\hbar}\sum_{j=1}^{n} \widehat{Z}_j \left|\Lambda, \frac{\partial \overline{w}_j}{\partial t} a\right\rangle + \left|\Lambda, Ya\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot \overline{\mathbf{w}} + \mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right) a\right\rangle \\
&\quad + \frac{1}{\hbar}\sum_{j=1}^{n} \left(\left|\Lambda, w_j \frac{\partial \overline{w}_j}{\partial t} a\right\rangle + \hbar\left|\Lambda, L_j\left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle\right) + \left|\Lambda, Ya\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \frac{\partial f}{\partial t} a\right\rangle - \frac{1}{2\hbar}\left|\Lambda, \left(\overline{\mathbf{w}}\cdot \frac{\partial \mathbf{w}}{\partial t}\right) a\right\rangle - \frac{1}{2\hbar}\left|\Lambda, \left(\mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t}\right) a\right\rangle \\
&\quad + \frac{1}{\hbar}\left|\Lambda, \left(\mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t}\right) a\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle + \left|\Lambda, Ya\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}}\cdot \frac{\partial \mathbf{w}}{\partial t}\right)\right) a\right\rangle \\
&\quad + \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle + \left|\Lambda, Ya\right\rangle.
\end{aligned}
$$

This leads to the following result for the derivative of a parameter dependent generalized Lagrangian state, with a compactly supported amplitude, with respect to the parameter

$$
\frac{\partial}{\partial t}|\Lambda, a\rangle = \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w}\cdot \frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}}\cdot \frac{\partial \mathbf{w}}{\partial t}\right)\right) a\right\rangle \tag{5.3}
$$

$$
+ \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle + \left|\Lambda, Ya\right\rangle. \tag{5.4}
$$

The above calculation is for the case where the amplitude of the state is compactly supported. In the more general case, where one may need to consider the generalized state as a linear functional on Bargmann space, the above result will hold via a completely analogous calculation. Thus we have the above result will hold in general.

5.1.2 Generalized States of Interest

In this section we specialize to a specific type of (parameter dependent) generalized Lagrangian state where we interpret the parameter to be time, t. The basic setup begins with some discussion of classical time evolution that will be used in defining our particular generalized states of interest, as well as understanding how these states evolve quantum mechanically.

Some Facts About Time Evolution in Classical Mechanics

Consider $\mathbb{R}^{2n}$ with the standard symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$. Suppose that $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, such that $H \in C^\infty(\mathbb{R}^{2n})$. Associated to H is the Hamilton vector field which is defined by the condition

$$\omega(\xi_H, \cdot) = -dH. \tag{5.5}$$

Expressed in local coordinates we have

$$\xi_H = -\frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{q}} = -\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} + \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} \tag{5.6}$$

where in the last expression we invoke summation convention. Denote by $\phi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ the time t map given by the (Hamilton) flow of ξ_H. The flow, ϕ_t, is the time t solution to Hamilton's equations

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad\qquad \dot{q}_j = \frac{\partial H}{\partial p_j}. \tag{5.7}$$

Lifting The Dynamics to P

The quantum evolution of generalized Lagrangian states, that yields a non-trivial semiclassical result, requires us to consider classical dynamics on the reduced Heisenberg group, not just phase space. In the results in the 'Semiclassical Results' section in chapter (III), we saw that the choice of the lift function f plays a central role in determining the semiclassical 'size' of generalized states, this leads to the requirement that to propagate a generalized state in a semiclassically meaningful way will require a careful choice of lift function for each value of t. This leads us to evolve the horizontal lift of the our Lagrangian embedding in a very specific way. In this section for the sake of simplicity we will work in real coordinates on phase space.

Recall that the reduced Heisenberg group has the structure of a (trivial) principal fiber bundle with fibers isomorphic to S^1. We can endow this circle bundle with the connection form

$$\alpha = d\Theta + \frac{1}{2}(\mathbf{p} \cdot d\mathbf{q} - \mathbf{q} \cdot d\mathbf{p}). \tag{5.8}$$

Note that $d\alpha = \pi^*\omega$, where $\pi : P \longrightarrow \mathbb{R}^{2n}$ is the canonical projection map.

Definition V.1. The Kostant field associated with H is the field on the reduced Heisenberg group, $P = \mathbb{R}^{2n} \times S^1$, given by

$$K_H = \widetilde{\xi}_H + H\frac{\partial}{\partial\Theta} \tag{5.9}$$

where $\widetilde{\xi}_H$ is the horizontal lift of ξ_H (with respect to α) and $\frac{\partial}{\partial\Theta}$ is the infinitesimal generator of the S^1 action on P. (Note that the H above, and in what follows is $H \circ \pi$.)

It is straight-forward to check that the Kostant field expressed in local coordinates is

$$K_H = -\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} + \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} + \left(H - \frac{1}{2}\left(q_j\frac{\partial H}{\partial q_j} + p_j\frac{\partial H}{\partial p_j}\right)\right)\frac{\partial}{\partial \Theta} \tag{5.10}$$

Let $\widetilde{\phi}_t : P \longrightarrow P$ be the flow of K_H. Some basic properties of this setup are given in the following lemma.

Lemma V.2. *Suppose that $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ such that $H \in C^\infty(\mathbb{R}^{2n})$, with ξ_H the associated Hamilton vector field with flow ϕ_t and associated Kostant vector field K_H with flow $\widetilde{\phi}_t$. Then*

1. $\pi \circ \widetilde{\phi}_t = \phi_t \circ \pi$,
2. $\widetilde{\phi}_t^* \alpha = \alpha$,
3. $K_H = 0$ *iff* $H = 0$.
4. *For any appropriately smooth* $H, G : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, *one has*

$$[K_H, K_G] = K_{\{H,G\}}.$$

Proof. 1. $\pi \circ \widetilde{\phi}_t = \phi_t \circ \pi$ follows from the fact that K_H and ξ_H are π related.

2. Using the Cartan formula for the Lie derivative we have

$$
\begin{aligned}
\mathcal{L}_{K_H}\alpha &= \iota(K_H)d\alpha + d(\iota(K_H)\alpha) \\
&= d\alpha(K_H,\cdot) + d(\alpha(K_H)) \\
&= (\pi^*\omega)(K_H,\cdot) + d(H) \\
&= \omega(\pi_*(K_H),\pi_*(\cdot)) + d(H) \\
&= -d(H) + d(H) \\
&= 0.
\end{aligned}
$$

Where we used the fact that $\alpha(K_H) = \alpha(\widetilde{\xi}_H + H\partial\Theta) = H$. Thus α is constant along the flow of K_H, i.e. $\widetilde{\phi}_t^*\alpha = \alpha$.

3. From the local expression for K_H

$$K_H = -\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} + \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} + \left(H - \frac{1}{2}\left(q_j\frac{\partial H}{\partial q_j} + p_j\frac{\partial H}{\partial p_j}\right)\right)\partial\Theta$$

It is clear that $K_H = 0$ iff $H = 0$.

4. Is a straight-forward calculation using the definition of the Kostant field.

□

Finally we have the following proposition

Proposition V.3. *For* $G(\mathbf{x},t) \equiv \widetilde{\phi}_t(G(\mathbf{x})) = (\phi_t(\mathbf{w}(\mathbf{x})), -f(\mathbf{x},t))$ *we have*

$$\alpha(\dot{G}) = H, \tag{5.11}$$

or more precisely

$$\alpha_G(\dot{G}) = (H\circ\pi)(G). \tag{5.12}$$

Proof. By definition

$$\dot{G} = \frac{d}{dt}(\widetilde{\phi}_t(\mathbf{w}, -f)) = K_H\Big|_{\widetilde{\phi}_t(\mathbf{w},-f)}.$$

Thus,

$$\alpha(\dot{G}) = \alpha\left(K_H\Big|_{\widetilde{\phi}_t(\mathbf{w},-f)}\right) = H \circ \pi.$$

□

Classical Dynamics and Lagrangian States

When we consider our generalized Lagrangian states we are interested in the phase space with a complex structure, namely $\mathbb{C}^n$. For use in our main result (coming up soon) we'll reinterpret two specific results of the classical evolution above in the complex case. Recall our choice of complex structure $\mathbf{z} \equiv \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$. This choice gives, $\omega = id\mathbf{z} \wedge d\overline{\mathbf{z}}$, as a symplectic form on $\mathbb{C}^n$. Employing the abstract definition above for the Hamilton field associated with H, we find the following expressions for Hamilton's equations in the variables z_j and $\overline{z}_j$

$$\dot{z}_j = i\frac{\partial H}{\partial \overline{z}_j}, \tag{5.13}$$

$$\dot{\overline{z}}_j = -i\frac{\partial H}{\partial z_j}. \tag{5.14}$$

Now, on the complex phase space we have the Lagrangian embedding $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$. For the Hamiltonian, H, with (complex) Hamilton flow $\phi_t : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ if we define

$$\mathbf{w}(\mathbf{x},t) \equiv \phi_t(\mathbf{x}), \tag{5.15}$$

then we find from Hamilton's equations

$$\frac{\partial}{\partial t}\left(\overline{\mathbf{w}(\mathbf{x},t)}\right) = \frac{\partial}{\partial t}\left(\overline{\phi_t(\mathbf{w}(\mathbf{x}))}\right) = -i\frac{\partial H}{\partial \mathbf{z}}\bigg|_{\mathbf{z}=\mathbf{w}(\mathbf{x},t)}.$$

The connection form on the complex version of the reduced Heisenberg group, $P = \mathbb{C}^n \times S^1$, given our choice of complex structure, is

$$\alpha = d\Theta + \frac{1}{2i}(\overline{\mathbf{z}} \cdot d\mathbf{z} - \mathbf{z} \cdot d\overline{\mathbf{z}}). \tag{5.16}$$

Thus, for our Lagrangian $\mathbf{w}(\Lambda) = \left\{\mathbf{w}(\mathbf{x}) = \frac{1}{2}(\mathbf{q}(\mathbf{x}) - i\mathbf{p}(\mathbf{x})) \in \mathbb{C}^n \middle| \mathbf{x} \in \Lambda\right\}$, the lift into P is again given by $G(\mathbf{x},0) = (\mathbf{w}(\mathbf{x}), -f(\mathbf{x}))$ where $-f$ is defined by the horizontal lift condition

$$d(-f) + \frac{1}{2i}(\overline{\mathbf{w}} \cdot d\mathbf{w} - \mathbf{w} \cdot d\overline{\mathbf{w}}) = 0,$$

or more directly f is defined by the lift condition

$$df + \frac{1}{2i}(\mathbf{w} \cdot d\overline{\mathbf{w}} - \overline{\mathbf{w}} \cdot d\mathbf{w}) = 0.$$

The above result that $\alpha(\dot{G}) = (H \circ \pi)(G)$ expressed explicitly in this complex case is

$$\frac{d}{dt}(-f) + \frac{1}{2i}\left(\overline{\mathbf{w}} \cdot \frac{d\mathbf{w}}{dt} - \mathbf{w} \cdot \frac{d\overline{\mathbf{w}}}{dt}\right) = (H \circ \pi)(G),$$

which implies

$$\frac{df}{dt} + \frac{1}{2i}\left(\mathbf{w} \cdot \frac{d\overline{\mathbf{w}}}{dt} - \overline{\mathbf{w}} \cdot \frac{d\mathbf{w}}{dt}\right) = -(H \circ \pi)(G) = -H(\mathbf{w}, \overline{\mathbf{w}}). \tag{5.17}$$

It will help to pictorially display the above classical dynamics that will underwrite the quantum dynamics of our generalized Lagrangian states. The diagram below summarizes this classical dynamics:

$$\begin{tikzcd}
 & \mathbb{C}^n \times S^1 \arrow[r, "\widetilde{\phi}_t"] \arrow[d, "\pi"] & \mathbb{C}^n \times S^1 \arrow[d, "\pi"] \\
\Lambda \arrow[ur, "{(w,-f)}"] \arrow[r, "w"] & \mathbb{C}^n \arrow[r, "\phi_t"] & \mathbb{C}^n
\end{tikzcd}$$

we lift $\mathbf{w}(\Lambda)$ horizontally into $\mathbb{C}^n \times S^1$, and then we can propagate forward via the Hamilton flow and Kostant flow respectively.

The above facts will allow us to specialize our general result for $\frac{\partial}{\partial t}|\Lambda, a\rangle$. Indeed if we have generalized Lagrangian states where $\mathbf{w}(\mathbf{x}, 0) = \mathbf{w}(\mathbf{x})$, $f(\mathbf{x}, 0) = f(\mathbf{x})$ where f is defined by the condition $df + \frac{1}{2i}(\mathbf{w} \cdot d\overline{\mathbf{w}} - \overline{\mathbf{w}} \cdot d\mathbf{w}) = 0$, and finally that $(\mathbf{w}(\mathbf{x}, t), -f(\mathbf{x}, t)) = \widetilde{\phi}_t(\mathbf{w}(\mathbf{x}), -f(\mathbf{x}))$, then our previous result specializes to

$$\begin{aligned}
\frac{\partial}{\partial t}|\Lambda, a\rangle &= \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial t}\right)\right) a\right\rangle + \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j \left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle \\
&\quad + \left|\Lambda, Ya\right\rangle \\
&= \frac{i}{\hbar}\left|\Lambda, -H(\mathbf{w}, \overline{\mathbf{w}})a\right\rangle + \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j \left(-i\frac{\partial H}{\partial z_j}(\mathbf{w}, \overline{\mathbf{w}})a\right)\right\rangle + \left|\Lambda, Ya\right\rangle
\end{aligned}$$

Thus, for parameter dependent Lagrangian states where the parameter is interpreted as time, t, and the geometry is motivated by classical mechanics we have the rule

(5.18)

$$\frac{\partial}{\partial t}|\Lambda, a\rangle = -\frac{i}{\hbar}\left|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\right\rangle + \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle - i\left|\Lambda, \sum_{j=1}^{n} L_j \left(\frac{\partial H}{\partial z_j}(\mathbf{w}, \overline{\mathbf{w}})a\right)\right\rangle + \left|\Lambda, Ya\right\rangle.$$

5.2 Propagating a Generalized Lagrangian State

We now have all of the pieces in place to state and prove our main result.

We're now ready for the main result, in the case of a compact amplitude.

Theorem V.4. *Let $H : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a polynomial symbol, and let $\widehat{U}(t)$ be the fundamental solution of Schrodinger's equation*

$$i\hbar\frac{\partial}{\partial t}\psi = \widehat{H}\psi$$

where $\widehat{H}$, defined on some dense subset of Bargmann space, denotes the Weyl quantization of H. If Λ is a n-dimensional compact smooth manifold and $|\Lambda, b\rangle$ is a generalized Lagrangian state with embedding $\mathbf{w} : \Lambda \longrightarrow \mathbf{C}^n$ and amplitude $b = b(\mathbf{x}; \hbar)$, then

$$\widehat{U}(t)|\Lambda, b\rangle = |\Lambda, a\rangle + R$$

where $|\Lambda, a\rangle$ is a parameter dependent Lagrangian state with embedding $\phi_t(\mathbf{w})$, where $\phi_t : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the Hamilton flow of H, and R is semiclassically negligible (i.e. $\|R\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty)$).

With the goal of clarity of exposition, two proofs are provided. First the proof of the result in the case of $n = 1$, and then with that result as a guide, the proof for a general n.

Proof. $(n = 1)$

The proof proceeds by taking an ansatz for the solution to the Schrodinger equation of the form $|\Lambda, a\rangle$ where we assume $a = a(x, t; \hbar)$ is an amplitude in the sense (III.11), therefore it has an asymptotic expansion of the form

$$a(x,t;\hbar) \sim \sum_{j=0}^{\infty} \hbar^j a_j(x,t).$$

We will match terms in the generalized Lagrangian state expansions for

$$i\hbar \frac{\partial}{\partial t} |\Lambda, a\rangle,$$

and

$$\widehat{H}|\Lambda, a\rangle,$$

which will lead to O.D.E.'s defining each of the a_j terms in the asymptotic expansion of a.

We begin by recalling that in the case of $n = 1$ that the general form of a polynomial Hamiltonian is

$$H(z, \overline{z}) = \sum_{\beta=0}^{N} \sum_{\alpha=0}^{M} C_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta},$$

and we have previously derived, (4.20), that for every value of $r \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}
\widehat{H}\Big|\Lambda, a\Big\rangle &= \widehat{H}\Big|\Lambda, a_0\Big\rangle + \hbar \widehat{H}\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r \widehat{H}\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right) \\
&= D_0^H\Big|\Lambda, a_0\Big\rangle + \hbar D_1^H\Big|\Lambda, a_0\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_0\Big\rangle \\
&\quad + \hbar\Big(D_0^H\Big|\Lambda, a_1\Big\rangle + \hbar D_1^H\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_1\Big\rangle\Big) \\
&\quad + \cdots + \hbar^r\Big(D_0^H\Big|\Lambda, a_r\Big\rangle + \hbar D_1^H\Big|\Lambda, a_r\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_r\Big\rangle\Big) \\
&\quad + O\left(\hbar^{r+1}\right).
\end{aligned}$$

where for a state $|\Lambda, \nu\rangle$ where ν is independent of $\hbar$ that

$$D_0^H\Big|\Lambda, \nu\Big\rangle \equiv \Big|\Lambda, H(w, \overline{w})\nu\Big\rangle,$$

and for $1 \le k \le M$,

$$D_k^H\Big|\Lambda, \nu\Big\rangle \equiv \left(\frac{1}{4}\right)^k\Big|\Lambda, \left(\Delta^k H(w, \overline{w})\right)\nu\Big\rangle + \sum_{l=1}^{k}\sum_{j=l}^{M-k+l}\sum_{\beta=0}^{N-k+l} C_{j\beta}^{k-l}\Big|\Lambda, \widetilde{L}_l^{\,j}(\overline{w}^\beta\nu)\Big\rangle,$$

where $C_{j\beta}^0 \equiv C_{j\beta}$. Thus if we gather together terms from the state expansion of $\widehat{H}\Big|\Lambda, a\Big\rangle$ we will get

<u>For $\hbar^0$:</u> 1 term

$$D_0^H\Big|\Lambda, a_0\Big\rangle$$

<u>For $\hbar^1$:</u> 2 terms

$$D_1^H\Big|\Lambda, a_0\Big\rangle + D_0^H\Big|\Lambda, a_1\Big\rangle$$

<u>For $\hbar^2$:</u> 3 terms

$$D_2^H\Big|\Lambda, a_0\Big\rangle + D_1^H\Big|\Lambda, a_1\Big\rangle + D_0^H\Big|\Lambda, a_2\Big\rangle$$

$$\vdots$$

<u>For $\hbar^s$ $(s \le M)$:</u> $s+1$ terms

$$D_s^H\Big|\Lambda, a_0\Big\rangle + D_{s-1}^H\Big|\Lambda, a_1\Big\rangle + \cdots + D_0^H\Big|\Lambda, a_s\Big\rangle$$

<u>For $\hbar^s$ $(s > M)$:</u> $M+1$ terms

$$D_0^H\left|\Lambda, a_s\right\rangle + D_1^H\left|\Lambda, a_{s-1}\right\rangle + \cdots + D_{M-1}^H\left|\Lambda, a_{s-(M-1)}\right\rangle + D_M^H\left|\Lambda, a_{s-M}\right\rangle$$

A systematic expansion for $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$, using the result from the previous section, yields

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\Lambda, a\right\rangle &= i\hbar\frac{\partial}{\partial t}\left(\left|\Lambda, a_0\right\rangle + \hbar\left|\Lambda, a_1\right\rangle + \cdots + \hbar^r\left|\Lambda, a_r\right\rangle + O\left(\hbar^{r+1}\right)\right) \\
&= i\hbar\frac{\partial}{\partial t}\left|\Lambda, a_0\right\rangle + i\hbar^2\frac{\partial}{\partial t}\left|\Lambda, a_1\right\rangle + \cdots + i\hbar^{r+1}\frac{\partial}{\partial t}\left|\Lambda, a_r\right\rangle + O\left(\hbar^{r+2}\right) \\
&= \sum_{j=0}^{r}\left(\hbar^j\left|\Lambda, H(w,\overline{w})a_j\right\rangle + \hbar^{j+1}\left|\Lambda, i\frac{\partial a_j}{\partial t}\right\rangle + \hbar^{j+1}\left|\Lambda, L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_j\right)\right\rangle\right. \\
&\qquad\qquad \left. + \hbar^{j+1}\left|\Lambda, iYa_j\right\rangle\right) + O\left(\hbar^{r+2}\right)
\end{aligned}$$

Comparing these two expansions at each level in $\hbar$ one finds (for a Hamiltonian of degree M)

	$i\hbar\frac{\partial}{\partial t}\|\Lambda, a\rangle$
$\hbar^0$	$\left\|\Lambda, H(w,\overline{w})a_0\right\rangle$
$\hbar^1$	$\left\|\Lambda, H(w,\overline{w})a_1\right\rangle + \left\|\Lambda, i\frac{\partial a_0}{\partial t}\right\rangle + \left\|\Lambda, L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right)\right\rangle + \left\|\Lambda, iYa_0\right\rangle$
$\vdots$	$\vdots$
$\hbar^s$	$\left\|\Lambda, H(w,\overline{w})a_s\right\rangle + \left\|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\right\rangle + \left\|\Lambda, L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_{s-1}\right)\right\rangle + \left\|\Lambda, iYa_{s-1}\right\rangle$
$\vdots$	$\vdots$

and

	$\widehat{H}\|\Lambda, a\rangle$
$\hbar^0$	$\Big\|\Lambda, H(w,\overline{w})a_0\Big\rangle$
$\hbar^1$	$\Big\|\Lambda, H(w,\overline{w})a_1\Big\rangle + D_1^H\Big\|\Lambda, a_0\Big\rangle$
$\vdots$	$\vdots$
$\hbar^s\ (s \leq M)$	$D_s^H\Big\|\Lambda, a_0\Big\rangle + D_{s-1}^H\Big\|\Lambda, a_1\Big\rangle + \cdots + D_1^H\Big\|\Lambda, a_{s-1}\Big\rangle + \Big\|\Lambda, H(w,\overline{w})a_s\Big\rangle$
$\hbar^s\ (s \geq M)$	$D_M^H\Big\|\Lambda, a_{s-M}\Big\rangle + D_{M-1}^H\Big\|\Lambda, a_{s-(M-1)}\Big\rangle + \cdots + D_1^H\Big\|\Lambda, a_{s-1}\Big\rangle + \Big\|\Lambda, H(w,\overline{w})a_s\Big\rangle$
$\vdots$	$\vdots$

Note that these expansions of $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ and $\widehat{H}|\Lambda, a\rangle$ are decompositions of each, in terms of states with amplitudes that are independent of $\hbar$. If $|\Lambda, a\rangle$ is a solution to the Schrodinger equation then we would require that $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ be equal to $\widehat{H}|\Lambda, a\rangle$. The result is not to solve Schrodinger's equation exactly, but rather approximately in a semiclassical sense. To that end we will match the terms in the state expansions for $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ and $\widehat{H}|\Lambda, a\rangle$ at each level of $\hbar$.

Beginning on the level of $\hbar^0$ we see that the lowest order term in both expansions is $|\Lambda, H(w,\overline{w})a_0\rangle$. This is a rather 'miraculous' result that is directly attributable to our choice of propagating on P via the Kostant field. Indeed much of this work is quite literally dependent on this 'miracle. So, to lowest order both expansions agree. Moving on to the level of $\hbar^1$; matching terms gives us

$$\Big|\Lambda, H(w,\overline{w})a_1\Big\rangle + \Big|\Lambda, i\frac{\partial a_0}{\partial t}\Big\rangle + \Big|\Lambda, L\Big(\frac{\partial H}{\partial z}(w,\overline{w})a_0\Big)\Big\rangle + \Big|\Lambda, iYa_0\Big\rangle \tag{5.19}$$

$$= \Big|\Lambda, H(w,\overline{w})a_1\Big\rangle + D_1^H\Big|\Lambda, a_0\Big\rangle. \tag{5.20}$$

Canceling like terms on both sides and inserting the definition of $D_1^H|\Lambda, a_0\rangle$ yields

$$\left|\Lambda, i\frac{\partial a_0}{\partial t}\right\rangle + \left|\Lambda, L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right)\right\rangle + \left|\Lambda, iYa_0\right\rangle \tag{5.21}$$

$$= D_1^H|\Lambda, a_0\rangle = \left(\frac{1}{4}\right)\left|\Lambda, \left(\Delta H(w,\overline{w})\right)a_0\right\rangle + \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\left|\Lambda, \widetilde{L}_l^{\,j}(\overline{w}^\beta a_0)\right\rangle. \tag{5.22}$$

Let's calculate a more explicit formula for $D_1^H|\Lambda, \nu\rangle$. Using the the formula for L, (which is (4.14) in the $n = 1$ case), let

$$\widetilde{T} \equiv \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} Tr\left(Re\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\right)\right) \tag{5.23}$$

and calculating in local coordinates, consider

$$
\begin{aligned}
\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\left|\Lambda, \widetilde{L}_l^{\,j}(\overline{w}^{\beta}\nu)\right\rangle &= \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\sum_{l=0}^{j-1}\left|\Lambda, w^{j-l-1}L(w^{l}\overline{w}^{\beta}\nu)\right\rangle \\
&= \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\sum_{l=0}^{j-1}\left(\left|\Lambda, -w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\nu\right)\right\rangle\right. \\
&\qquad\qquad\qquad \left. - \left|\Lambda, w^{j-l-1}\widetilde{T} w^{l}\overline{w}^{\beta}\nu\right\rangle\right) \\
&= -\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\sum_{l=0}^{j-1}\left(\left|\Lambda, \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{j-1}\overline{w}^{\beta}\frac{\partial \nu}{\partial x}\right\rangle\right. \\
&\qquad \left. + \left|\Lambda, w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\right)\nu\right\rangle + \left|\Lambda, \widetilde{T} w^{j-1}\overline{w}^{\beta}\nu\right\rangle\right) \\
&= -\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\left(\left|\Lambda, \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\sum_{l=0}^{j-1} w^{j-1}\right)\overline{w}^{\beta}\frac{\partial \nu}{\partial x}\right\rangle\right. \\
&\qquad\qquad + \sum_{l=0}^{j-1}\left|\Lambda, w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\right)\nu\right\rangle \\
&\qquad\qquad \left. + \left|\Lambda, \widetilde{T}\left(\sum_{l=0}^{j-1} w^{j-1}\right)\overline{w}^{\beta}\nu\right\rangle\right) \\
&= -\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\left(\left|\Lambda, \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(j w^{j-1}\right)\overline{w}^{\beta}\frac{\partial \nu}{\partial x}\right\rangle\right. \\
&\qquad\qquad + \left|\Lambda, \sum_{l=0}^{j-1} w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\right)\nu\right\rangle\Bigg) \\
&\qquad\qquad \left. + \left|\Lambda, \widetilde{T}\left(j w^{j-1}\right)\overline{w}^{\beta}\nu\right\rangle\right) \\
&= -\left|\Lambda, \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j w^{j-1}\overline{w}^{\beta}\right)\frac{\partial \nu}{\partial x}\right\rangle \\
&\quad - \left|\Lambda, \left(\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\sum_{l=0}^{j-1} w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\right)\right)\nu\right\rangle \\
&\quad - \left|\Lambda, \widetilde{T}\left(\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j w^{j-1}\overline{w}^{\beta}\right)\nu\right\rangle
\end{aligned}
$$

Now, further calculation gives

$$
\begin{aligned}
\sum_{l=0}^{j-1} w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^l \overline{w}^\beta\right) &= \sum_{l=0}^{j-1} w^{j-l-1}\left(\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right) w^l \overline{w}^\beta + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} l w^{l-1}\left(\frac{\partial w}{\partial x}\right)\overline{w}^\beta \right.\\
&\qquad \left. + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^l \beta \overline{w}^{\beta-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\right)\\
&= \sum_{l=0}^{j-1}\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right) w^{j-1}\overline{w}^\beta + \sum_{l=0}^{j-1}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} l w^{j-2}\overline{w}^\beta\left(\frac{\partial w}{\partial x}\right)\\
&\qquad + \sum_{l=0}^{j-1}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\beta w^{j-1}\overline{w}^{\beta-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\left(\sum_{l=0}^{j-1} w^{j-1}\right)\overline{w}^\beta\\
&\qquad + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\sum_{l=0}^{j-1} l w^{j-2}\right)\overline{w}^\beta\left(\frac{\partial w}{\partial x}\right)\\
&\qquad + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\beta\left(\sum_{l=0}^{j-1} w^{j-1}\right)\overline{w}^{\beta-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\left(j w^{j-1}\right)\overline{w}^\beta\\
&\quad + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{1}{2}j(j-1)w^{j-2}\right)\overline{w}^\beta\left(\frac{\partial w}{\partial x}\right)\\
&\quad + \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\beta\left(j w^{j-1}\right)\overline{w}^{\beta-1}\left(\frac{\partial \overline{w}}{\partial x}\right)
\end{aligned}
$$

Thus we have

$$\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta}\sum_{l=0}^{j-1} w^{j-l-1}\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} w^{l}\overline{w}^{\beta}\right) = \left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j w^{j-1}\overline{w}^{\beta}$$
$$+\frac{1}{2}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial w}{\partial x}\right)\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j(j-1) w^{j-2}\overline{w}^{\beta}$$
$$+\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j\beta w^{j-1}\overline{w}^{\beta-1}$$
$$=\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\frac{\partial H}{\partial z}(w,\overline{w})$$
$$+\frac{1}{2}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 H}{\partial z^2}(w,\overline{w})$$
$$+\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\frac{\partial^2 H}{\partial z\partial \overline{z}}(w,\overline{w}).$$

Also,

$$\widetilde{T}\left(\sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} j w^{j-1}\overline{w}^{\beta}\right) = \widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w}).$$

The fact that in $n = 1$ we have Hamiltonians of the form

$$H(z,\overline{z}) = \sum_{j=1}^{M}\sum_{\beta=0}^{N} C_{j\beta} z^{j}\overline{z}^{\beta}.$$

was used explicitly.

This gives

$$D_1^H|\Lambda,\nu\rangle = \left(\frac{1}{4}\right)\left|\Lambda,\left(\Delta H(w,\overline{w})\right)\nu\right\rangle - \left|\Lambda,\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial H}{\partial z}(w,\overline{w})\frac{\partial \nu}{\partial x}\right\rangle$$
$$-\left|\Lambda,\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\frac{\partial H}{\partial z}(w,\overline{w})\nu\right\rangle - \frac{1}{2}\left|\Lambda,\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 H}{\partial z^2}(w,\overline{w})\nu\right\rangle$$
$$-\left|\Lambda,\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 H}{\partial z\partial \overline{z}}(w,\overline{w})\nu\right\rangle - \left|\Lambda,\widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w})\nu\right\rangle.$$

Invoking (III.23) we get that

$$\left|\Lambda, i\frac{\partial a_0}{\partial t}\right\rangle + \left|\Lambda, L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right)\right\rangle + \left|\Lambda, iYa_0\right\rangle = D_1^H|\Lambda, a_0\rangle$$

will be satisfied if

$$\begin{aligned}
&i\frac{\partial a_0}{\partial t} + L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right) + iYa_0 \\
&\quad = \left(\frac{1}{4}\right)(\Delta H(w,\overline{w}))\,a_0 - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial H}{\partial z}(w,\overline{w})\frac{\partial a_0}{\partial x} - \left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\frac{\partial H}{\partial z}(w,\overline{w})a_0 \\
&\qquad - \frac{1}{2}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 H}{\partial z^2}(w,\overline{w})a_0 - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\left(\frac{\partial \overline{w}}{\partial x}\right)\frac{\partial^2 H}{\partial z\partial \overline{z}}(w,\overline{w})a_0 \\
&\qquad - \widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w})a_0.
\end{aligned}$$

We can simplify this equation by finding a more explicit expression for $L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right)$. Notice

$$\begin{aligned}
L\left(\frac{\partial H}{\partial z}(w,\overline{w})a_0\right) &= -\frac{\partial}{\partial x}\left(\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial H}{\partial z}(w,\overline{w})a_0\right) - \widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w})a_0 \\
&= -\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\frac{\partial H}{\partial z}(w,\overline{w})a_0 - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}a_0\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial z}(w,\overline{w})\right) \\
&\quad - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial H}{\partial z}(w,\overline{w})\frac{\partial a_0}{\partial x} - \widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w})a_0 \\
&= -\left(\frac{\partial}{\partial x}\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)\frac{\partial H}{\partial z}(w,\overline{w})a_0 - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\frac{\partial H}{\partial z}(w,\overline{w})\frac{\partial a_0}{\partial x} \\
&\quad - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}a_0\frac{\partial^2 H}{\partial^2 z}(w,\overline{w})\left(\frac{\partial w}{\partial x}\right) - \left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}a_0\frac{\partial^2 H}{\partial z\partial \overline{z}}(w,\overline{w})\left(\frac{\partial \overline{w}}{\partial x}\right) \\
&\quad - \widetilde{T}\frac{\partial H}{\partial z}(w,\overline{w})a_0.
\end{aligned}$$

If we insert this expression into the equation above, and cancel common terms on both sides we get

$$i\frac{\partial a_0}{\partial t} = \left(\left(\frac{1}{4}\right)(\Delta H(w,\overline{w})) - iY + \frac{1}{2}\frac{\partial^2 H}{\partial^2 z}(w,\overline{w})\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1}\right)a_0. \tag{5.24}$$

The lemma III.23 tells us that if a_0 satisfies the above equation, then our state expansions agree to the order of $\hbar^1$. Note that this equation for a_0 is a first order ordinary differential equation.

Following the same sort of reasoning, if we match the terms in the state expansion at the level of $\hbar^s$ for $s \leq M$ we get

(5.25)
$$\Big|\Lambda, H(w,\overline{w})a_s\Big\rangle + \Big|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\Big\rangle + \Big|\Lambda, L\Big(\frac{\partial H}{\partial z}(w,\overline{w})a_{s-1}\Big)\Big\rangle + \Big|\Lambda, iYa_{s-1}\Big\rangle$$

(5.26)
$$= D_s^H\Big|\Lambda, a_0\Big\rangle + D_{s-1}^H\Big|\Lambda, a_1\Big\rangle + \cdots + D_1^H\Big|\Lambda, a_{s-1}\Big\rangle + \Big|\Lambda, H(w,\overline{w})a_s\Big\rangle$$

which implies

(5.27)
$$\Big|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\Big\rangle + \Big|\Lambda, L\Big(\frac{\partial H}{\partial z}(w,\overline{w})a_{s-1}\Big)\Big\rangle + \Big|\Lambda, iYa_{s-1}\Big\rangle$$

(5.28)
$$= D_1^H\Big|\Lambda, a_{s-1}\Big\rangle + D_2^H\Big|\Lambda, a_{s-2}\Big\rangle + \cdots + D_{s-1}^H\Big|\Lambda, a_1\Big\rangle + D_s^H\Big|\Lambda, a_0\Big\rangle.$$

Let's introduce some notation to help organize what we have here. Recall that by definition

$$D_k^H\Big|\Lambda, \nu\Big\rangle \equiv \left(\frac{1}{4}\right)^k\Big|\Lambda, \left(\Delta^k H(w,\overline{w})\right)\nu\Big\rangle + \sum_{l=1}^{k}\sum_{j=l}^{M-k+l}\sum_{\beta=0}^{N-k+l} C_{j\beta}^{k-l}\Big|\Lambda, \widetilde{L}_l^{\,j}(\overline{w}^\beta \nu)\Big\rangle.$$

where $\widetilde{L}_l^j(g)$ is a sum of all permutations of l copies of L and $j-l$ copies of M_w acting on g. Since each factor of L will result in a derivative falling on g, the amplitude of $D_k^H\Big|\Lambda,\nu\Big\rangle$ will have terms with derivatives up to order k acting on ν. Define $F_0^k(w,\overline{w}), \ldots, F_k^k(w,\overline{w})$ by

$$D_k^H \Big| \Lambda, \nu \Big\rangle = \Big| \Lambda, F_0^k(w,\overline{w})\nu + F_1^k(w,\overline{w})\frac{\partial \nu}{\partial x} + \cdots + F_k^k(w,\overline{w})\frac{\partial^k \nu}{\partial x^k} \Big\rangle. \tag{5.29}$$

Note that each F_j^k is a known function that can be computed given w and H. With this notation in hand we can extract an equation for amplitudes from our equation involving states via III.23. So, using this new notation, as well as noting that we can perform the same simplification for the terms involving a_{s-1} in this equation that we did for a_0 in the case of matching states at $\hbar$, we get the equation

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\left(\frac{1}{4}\right)(\Delta H(w,\overline{w})) - iY + \frac{1}{2}\frac{\partial^2 H}{\partial^2 z}(w,\overline{w})\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} \right) a_{s-1} \tag{5.30}$$

$$+ \sum_{k=2}^{s}\sum_{j=0}^{k} F_j^k(w,\overline{w})\frac{\partial^j a_{s-k}}{\partial x^j}. \tag{5.31}$$

Finally, for the case of matching our state expansions at the level of $\hbar^s$ for $s \geq M$ we have

$$\Big|\Lambda, H(w,\overline{w})a_s\Big\rangle + \Big|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\Big\rangle + \Big|\Lambda, L\Big(\frac{\partial H}{\partial z}(w,\overline{w})a_{s-1}\Big)\Big\rangle \tag{5.32}$$

$$= D_M^H\Big|\Lambda, a_{s-M}\Big\rangle + D_{M-1}^H\Big|\Lambda, a_{s-(M-1)}\Big\rangle \tag{5.33}$$

$$+ \cdots + D_1^H\Big|\Lambda, a_{s-1}\Big\rangle + \Big|\Lambda, H(w,\overline{w})a_s\Big\rangle \tag{5.34}$$

which will, upon following the procedure laid out, lead to the equation

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\left(\frac{1}{4}\right)(\Delta H(w,\overline{w})) + \frac{1}{2}\frac{\partial^2 H}{\partial^2 z}(w,\overline{w})\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial \overline{w}}{\partial x}\right)^{-1} \right) a_{s-1} \tag{5.35}$$

$$+ \sum_{k=2}^{M}\sum_{j=0}^{k} F_j^k(w,\overline{w})\frac{\partial^j a_{s-k}}{\partial x^j}. \tag{5.36}$$

Let's notice that by matching states at the level of $\hbar^{s+1}$ for $s \geq 1$ we arrive at a first order ordinary differential equation for a_s that involves explicitly known functions as well as $a_0, \ldots, a_{s-1}$. If we define a_j, for $j = 0, 1, \ldots$, as the solution to the j^{th} transport equation above with the initial condition that $a_j\Big|_{t=0} \equiv b_j$, then, by the well known Borel's Theorem (see [14]), we can sum the asymptotic series and choose a (unique up to a term of $O(\hbar^\infty)$) as one of the smooth functions such that

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j.$$

Finally, we need to address the issue of how well our procedure of asymptotically matching state expansions of $\widehat{H}|\Lambda, a\rangle$ and $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ does at solving the Schrodinger equation. If we could arrange so that $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle = \widehat{H}|\Lambda, a\rangle$ with $|\Lambda, a\rangle\Big|_{t=0} = |\Lambda, b\rangle$, then we would have $\widehat{U}(t)|\Lambda, b\rangle = |\Lambda, a\rangle$, but this is too much to ask of our procedure. The limitation of the process of matching terms at each level of $\hbar$ is that it is not sensitive to contributions that are of the order $O(\hbar^\infty)$. This introduces error into the process of propagating a generalized Lagrangian state in two ways, one on each 'end' of the process of propagation.

Our method of evaluating the size of the semiclassical size of these errors will be to invoke Duhamel's formula. With the goal of using Duhamel's formula, first define $\psi(z, t)$ as the solution to the Schrodinger equation with initial condition $|\Lambda, b\rangle$, thus

$$i\hbar\frac{\partial}{\partial t}\psi = \widehat{H}\psi \tag{5.37}$$

$$\psi(z, 0) = |\Lambda, b\rangle. \tag{5.38}$$

Denote by $\widehat{U}(t)$ the time evolution operator for this system. Now, the method by which we constructed $|\Lambda, a\rangle$ tells us that it satisfies an initial value problem of the form

$$i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle = \widehat{H}|\Lambda, a\rangle + E(z,t) \tag{5.39}$$

$$|\Lambda, a\rangle(z,0) = |\Lambda, b\rangle + E_0, \tag{5.40}$$

where by definition $E(z,t) = \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)|\Lambda, a\rangle$. Note that E is an element of Bargmann space. The time-independent term E_0 in the initial condition, will be discussed shortly, it too is in Bargmann space. Note that we can modify the above initial value problem to see that the quantity $\Gamma(z,t) \equiv |\Lambda, a\rangle - E_0$ satisfies

$$i\hbar\frac{\partial}{\partial t}\Gamma = \widehat{H}\Gamma + \widetilde{E}(z,t) \tag{5.41}$$

$$\Gamma(z,0) = |\Lambda, b\rangle, \tag{5.42}$$

where $\widetilde{E}(z,t) = \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\Gamma = \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)(|\Lambda, a\rangle - E_0) = E(z,t) + \widehat{H}E_0$, and $\widetilde{E}$ is in Bargmann space.

Duhamel's formula allows us to solve the initial value problem defining Γ given ψ. Specifically, it says that

$$\Gamma(z,t) = \psi(z,t) + \frac{\hbar}{i}\int_0^t \widehat{U}(t-s)\widetilde{E}(z,s)ds, \tag{5.43}$$

which implies that

(5.44)

$$\|\Gamma - \psi\|_{L^2(\mathbb{C}^n)} \leq \hbar\int_0^t \|\widetilde{E}\|_{L^2(\mathbb{C}^n)}(s)ds \leq \hbar\int_0^t \left(\|E\|_{L^2(\mathbb{C}^n)}(s) + \|\widehat{H}E_0\|_{L^2(\mathbb{C}^n)}\right)ds.$$

We are focusing on propagating a generalized Lagrangian state $|\Lambda, b\rangle$ where the amplitude, b has an asymptotic expansion of the form

$$b(x) \sim \sum_{j=0}^{\infty} \hbar^j b_j(x).$$

The above process of matching terms gave us transport equations we use to define each a_j, where the initial condition we choose for each equation is $a_j\Big|_{t=0} \equiv b_j$. Thus we can say

$$a(x,0) \sim \sum_{j=0}^{\infty} \hbar^j a_j(x,0) = \sum_{j=0}^{\infty} \hbar^j b_j(x) \sim b(x).$$

This shows that that $a(x,0) = b(x) + O(\hbar^\infty)$. Thus, we can at best conclude that

$$|\Lambda, a\rangle\Big|_{t=0} = |\Lambda, b + O(\hbar^\infty)\rangle = |\Lambda, b\rangle + |\Lambda, O(\hbar^\infty)\rangle.$$

Let's call the unknown $O(\hbar^\infty)$ term here $\widetilde{b}(x;\hbar)$.

It is worth noting that we could in fact choose a such that it agrees with b at $t = 0$, which would make the argument above seem unnecessary. The problem with such a maneuver lies in the fact that the time evolution of a is defined by the time evolution of the a_j's; meaning that to evolve a to time t one evolves each a_j via it's transport equation to time t, then a at that time is defined to be a summation of the asymptotic series of the a_j's at that time. Thus at any time step we have to 'resum' an asymptotic series, which if we were to take the result of this sum and consider taking the limit as time reverses we cannot guarantee the result would be equal to the value of a we with which we began. Thus the fact that at any time a is defined as one (of many) possible choices of the sum of an asymptotic series leads to an unavoidable ambiguity of order $O(\hbar^\infty)$. The above discussion acknowledges this ambiguity at $t = 0$.

Thus, with the potentially slight abuse of notation of considering $\widetilde{b}$ as an amplitude with trivial asymptotic expansion, we have

$$|\Lambda, a\rangle\Big|_{t=0} = |\Lambda, b + \widetilde{b}\rangle = |\Lambda, b\rangle + |\Lambda, \widetilde{b}\rangle$$

Thus, by definition

$$E_0 = |\Lambda, \widetilde{b}\rangle, \tag{5.45}$$

Furthermore, a is smooth by Borel's Theorem (see [14]), and b is smooth by assumption, thus $\widetilde{b} = a - b$ is smooth, thus since Λ is compact we have that as stated above E_0 is an element in Bargmann space. From the proof of ,(III.24), we have

$$\langle\Lambda, \widetilde{b}|\Lambda, \widetilde{b}\rangle = \int_{\Lambda_x}\int_{\Lambda_y} \widetilde{b}(x)\overline{\widetilde{b}(y)}e^{\frac{i}{\hbar}(f(x)-f(y))}e^{\frac{\overline{w(x)}\cdot w(y)}{\hbar}}e^{-\frac{(w(x)\cdot\overline{w(x)})+w(y)\cdot\overline{w(y)})}{2\hbar}}dxdy,$$

thus if $\widetilde{b} = O(\hbar^\infty)$ then since this integral is real-valued we have

$$\|E_0\|_{L^2(\mathbb{C}^n)} = \langle\Lambda, \widetilde{b}|\Lambda, \widetilde{b}\rangle \tag{5.46}$$

$$\leq \int_{\Lambda_x}\int_{\Lambda_y} |\widetilde{b}(x)| \cdot |\widetilde{b}(y)|e^{-\frac{1}{2\hbar}|w(x)-w(y)|}dxdy \tag{5.47}$$

$$\leq \int_{\Lambda_x}\int_{\Lambda_y} |\widetilde{b}(x)| \cdot |\widetilde{b}(y)|dxdy = O(\hbar^\infty). \tag{5.48}$$

Our rules for the action of a polynomial Hamiltonian on a generalized Lagrangian state with compactly supported amplitude, as E_0 is, tell us that when such a Hamiltonian acts on such a state it yields another generalized state with an amplitude of the same order in $\hbar$. Thus since $\widetilde{b} = O(\hbar^\infty)$ we can conclude that $\widehat{H}E_0$ will have an amplitude that is also $O(\hbar^\infty)$; the same argument that let us conclude that $\|E_0\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty)$ shows that $\|\widehat{H}E_0\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty)$.

Now let's consider the term $E(z,t) = \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)|\Lambda, a\rangle$. Again, since the process of asymptotic matching is 'blind' to terms that are beyond all orders of $\hbar$, we must acknowledge that $\left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)|\Lambda, a\rangle \neq 0$. Indeed, the best we can say is if for each N we define

$$\widetilde{a}_N \equiv a - \sum_{j=0}^{N} \hbar^j a_j,$$

then $a_N = O(\hbar^{N+1})$, and we have

$$\begin{aligned}
\left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\left|\Lambda, a\right\rangle &= \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\left|\Lambda, \sum_{j=0}^{N} \hbar^j a_j + \widetilde{a}_N\right\rangle \\
&= \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\left(\left|\Lambda, \sum_{j=0}^{N} \hbar^j a_j\right\rangle + \left|\Lambda, \widetilde{a}_N\right\rangle\right) \\
&= \left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\left|\Lambda, \widetilde{a}_N\right\rangle
\end{aligned}$$

where in the last line we used the fact that the definition of the $a_j^{\prime}s$ is such that the term with them vanishes. Now, our rules for the action of $i\hbar\frac{\partial}{\partial t}$ and $\widehat{H}$ on a generalized Lagrangian state show that they produce another generalized Lagrangian state whose amplitude is of the same order in $\hbar$ as the original, thus

$$\left(i\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\left|\Lambda, \widetilde{a}_N\right\rangle = \left|\Lambda, c_N\right\rangle,$$

such that $c_N = O(\hbar^{N+1})$. Then, we have

$$\langle\Lambda, c_N|\Lambda, c_N\rangle = \int_{\Lambda_x}\int_{\Lambda_y} c_N(x)\overline{c_N(y)}e^{\frac{i}{\hbar}(f(x)-f(y))}e^{\frac{\overline{w(x)}\cdot w(y)}{\hbar}}e^{-\frac{(w(x)\cdot\overline{w(x)})+w(y)\cdot\overline{w(y)})}{2\hbar}}dxdy,$$

so since $c_N = O(\hbar^{N+1})$

$$\|E\|_{L^2(\mathbb{C}^n)} = \langle \Lambda, c_N | \Lambda, c_N \rangle \leq \int_{\Lambda_x} \int_{\Lambda_y} |c_N(x)| \cdot |c_N(y)| e^{-\frac{1}{2\hbar}|w(x)-w(y)|} dx dy \tag{5.49}$$

$$\leq \int_{\Lambda_x} \int_{\Lambda_y} |c_N(x)| \cdot |c_N(y)| dx dy \tag{5.50}$$

$$= O(\hbar^{N+1}). \tag{5.51}$$

Since this is true for each N we can conclude that the error term $E(z,t) = \left(i\hbar \frac{\partial}{\partial t} - \widehat{H} \right) \Big| \Lambda, a \Big\rangle$ has norm $O(\hbar^\infty)$. Note that this norm is time-dependent, and so this asymptotic estimate will hold for times in a bounded interval.

Taking the results (5.46) and (5.49), and noting that in each that these norms are smoothly dependent on t (one trivially so), and combining them with our result from applying Duhamel's principle (5.44) we get for t in a bounded interval, and for each $N \in \mathbb{N}$ there exists a constant C_N such that

$$\|\Gamma - \psi\|_{L^2(\mathbb{C}^n)} \leq \hbar \int_0^t \left(\|E\|_{L^2(\mathbb{C}^n)}(s) + \|\widehat{H} E_0\|_{L^2(\mathbb{C}^n)}(s) \right) ds \tag{5.52}$$

$$\leq C_N \hbar^N. \tag{5.53}$$

Combining the fact that for t in a bounded interval we have $\|\Gamma - \psi\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty)$ with the fact that for $E_0 = \Gamma - |\Lambda, a\rangle$, we saw above in (5.46) that $\|E_0\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty)$ we can use the triangle inequality to see that

$$\|\psi - |\Lambda, a\rangle\|_{L^2(\mathbb{C}^n)} = O(\hbar^\infty) \tag{5.54}$$

for bounded t.

Thus

$$\widehat{U}(t)|\Lambda, b\rangle = |\Lambda, a\rangle + R$$

where $\| R \|_{L^2(\mathbb{C})} = O(\hbar^\infty)$.

$\square$

The proof for the case $n > 1$ will be the same procedure as the $n = 1$ case, but with more complicated calculations.

Proof. $(n > 1)$

As in the $n = 1$ case, the proof proceeds by taking an ansatz for the solution to the Schrodinger equation of the form $|\Lambda, a\rangle$ where we assume $a = a(\mathbf{x}, t; \hbar)$ is an amplitude in the sense of III.11, namely it has an asymptotic expansion of the form

$$a(\mathbf{x}, t; \hbar) \sim \sum_{j=0}^{\infty} \hbar^j a_j(\mathbf{x}, t).$$

We will match terms in the generalized Lagrangian state expansions for

$$i\hbar \frac{\partial}{\partial t}|\Lambda, a\rangle,$$

and

$$\widehat{H}|\Lambda, a\rangle,$$

which will lead to ordinary differential equations defining each of the a_j terms in the asymptotic expansion of a.

We begin by recalling that a general polynomial Hamiltonian will have the form

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\alpha,\beta} C_{\alpha,\beta}\mathbf{z}^\alpha \overline{\mathbf{z}}^\beta,$$

and for a general n we have previously derived, in chapter IV, that for every value of $r \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}
\widehat{H}_{weyl}\Big|\Lambda, a\Big\rangle &= \widehat{H}_{weyl}\Big|\Lambda, a_0\Big\rangle + \hbar \widehat{H}_{weyl}\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r \widehat{H}_{weyl}\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right) \\
&= D_0^H\Big|\Lambda, a_0\Big\rangle + \hbar D_1^H\Big|\Lambda, a_0\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_0\Big\rangle \\
&\quad + \hbar\bigg(D_0^H\Big|\Lambda, a_1\Big\rangle + \hbar D_1^H\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_1\Big\rangle\bigg) \\
&\quad + \cdots + \hbar^r\bigg(D_0^H\Big|\Lambda, a_r\Big\rangle + \hbar D_1^H\Big|\Lambda, a_r\Big\rangle + \cdots + \hbar^M D_M^H\Big|\Lambda, a_r\Big\rangle\bigg) \\
&\quad + O\left(\hbar^{r+1}\right).
\end{aligned}$$

where

$$D_0^H\Big|\Lambda, a\Big\rangle \equiv \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle$$

and for $1 \leq k \leq M$ (where M is the highest 'order' of $\mathbf{z}$'s in $H(\mathbf{z}, \overline{\mathbf{z}})$)

$$\begin{aligned}
D_k^H\Big|\Lambda, a\Big\rangle \equiv \left(\frac{1}{4}\right)^k\Big|\Lambda, \left(\Delta^k H\right)(\mathbf{w}, \overline{\mathbf{w}})a\Big\rangle &+ \sum_{\beta}\bigg(\sum_{|\alpha|=k} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle \\
&\qquad + \sum_{|\alpha|>k} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}_k^{\alpha}(\overline{\mathbf{w}}^{\beta}a)\Big\rangle\bigg) \\
&+ \sum_{i=1}^{k-1}\sum_{\beta^{j_1\cdots j_i}}\bigg(\sum_{|\alpha^{j_1\cdots j_i}|=k-i} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_i}\Big|\Lambda, L^{\alpha^{j_1\cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_i}}a)\Big\rangle \\
&\qquad + \sum_{|\alpha^{j_1\cdots j_i}|>k-i} \widetilde{C}_{\alpha\beta}^{j_1\cdots j_i}\Big|\Lambda, \widetilde{L}_{k-i}^{\alpha^{j_1\cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1\cdots j_i}}a)\Big\rangle\bigg).
\end{aligned}$$

If we gather like terms in the above state expansion of $\widehat{H}|\Lambda, a\rangle$ then we get

<u>For $\hbar^0$:</u> 1 term

$$D_0^H \Big|\Lambda, a_0\Big\rangle$$

<u>For $\hbar^1$:</u> 2 terms

$$D_1^H \Big|\Lambda, a_0\Big\rangle + D_0^H \Big|\Lambda, a_1\Big\rangle$$

<u>For $\hbar^2$:</u> 3 terms

$$D_2^H \Big|\Lambda, a_0\Big\rangle + D_1^H \Big|\Lambda, a_1\Big\rangle + D_0^H \Big|\Lambda, a_2\Big\rangle$$

$$\vdots$$

<u>For $\hbar^s$ $(s \leq M)$:</u> $s+1$ terms

$$D_s^H \Big|\Lambda, a_0\Big\rangle + D_{s-1}^H \Big|\Lambda, a_1\Big\rangle + \cdots + D_0^H \Big|\Lambda, a_s\Big\rangle$$

Note, for each j, $\widehat{H}_{weyl} \Big|\Lambda, a_j\Big\rangle$ has terms (at the level of states) up to $\hbar^M$. Thus in the above, if $s > M$ there will be no term involving $\Big|\Lambda, a_0\Big\rangle$; if $s > M+1$, there will be no terms involving $\Big|\Lambda, a_0\Big\rangle$ and $\Big|\Lambda, a_1\Big\rangle$. In general if $s > M+k$ there will be no terms involving $\Big|\Lambda, a_0\Big\rangle$, $\ldots$, $\Big|\Lambda, a_k\Big\rangle$, thus

<u>For $\hbar^s$ $(s > M)$:</u> $M+1$ terms

$$D_0^H \Big|\Lambda, a_s\Big\rangle + D_1^H \Big|\Lambda, a_{s-1}\Big\rangle + \cdots + D_{M-1}^H \Big|\Lambda, a_{s-(M-1)}\Big\rangle + D_M^H \Big|\Lambda, a_{s-M}\Big\rangle$$

A systematic expansion for $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ yields

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Big|\Lambda, a\Big\rangle &= i\hbar\frac{\partial}{\partial t}\left(\Big|\Lambda, a_0\Big\rangle + \hbar\Big|\Lambda, a_1\Big\rangle + \cdots + \hbar^r\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+1}\right)\right) \\
&= i\hbar\frac{\partial}{\partial t}\Big|\Lambda, a_0\Big\rangle + i\hbar^2\frac{\partial}{\partial t}\Big|\Lambda, a_1\Big\rangle + \cdots + i\hbar^{r+1}\frac{\partial}{\partial t}\Big|\Lambda, a_r\Big\rangle + O\left(\hbar^{r+2}\right) \\
&= \sum_{j=0}^{r}\left(\hbar^j\Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_j\Big\rangle + \hbar^{j+1}\Big|\Lambda, i\frac{\partial a_j}{\partial t}\Big\rangle + \hbar^{j+1}\Big|\Lambda, iYa_j\Big\rangle\right. \\
&\qquad \left. + \hbar^{j+1}\Big|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_j\right)\Big\rangle\right) + O\left(\hbar^{r+2}\right)
\end{aligned}
$$

where we used (5.18).

Comparing these two expansions at each level in $\hbar$ one finds (for a Hamiltonian of degree M)

$$
\begin{array}{c|c}
 & i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle \\
\hline
\hbar^0 & \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_0\Big\rangle \\
\hbar^1 & \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_1\Big\rangle + \Big|\Lambda, i\frac{\partial a_0}{\partial t}\Big\rangle + \Big|\Lambda, iYa_0\Big\rangle + \Big|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_0\right)\Big\rangle \\
\vdots & \vdots \\
\hbar^s & \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})a_s\Big\rangle + \Big|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\Big\rangle + \Big|\Lambda, iYa_{s-1}\Big\rangle + \Big|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_{s-1}\right)\Big\rangle \\
\vdots & \vdots
\end{array}
$$

and

	$\widehat{H}\vert\Lambda, a\rangle$
$\hbar^0$	$\Big\vert\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_0\Big\rangle$
$\hbar^1$	$\Big\vert\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_1\Big\rangle + D_1^H\Big\vert\Lambda, a_0\Big\rangle$
$\vdots$	$\vdots$
$\hbar^s\ (s \leq M)$	$D_s^H\Big\vert\Lambda, a_0\Big\rangle + D_{s-1}^H\Big\vert\Lambda, a_1\Big\rangle + \cdots + D_1^H\Big\vert\Lambda, a_{s-1}\Big\rangle + \Big\vert\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_s\Big\rangle$
$\hbar^s\ (s \geq M)$	$D_M^H\Big\vert\Lambda, a_{s-M}\Big\rangle + D_{M-1}^H\Big\vert\Lambda, a_{s-(M-1)}\Big\rangle + \cdots + D_1^H\Big\vert\Lambda, a_{s-1}\Big\rangle + \Big\vert\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_s\Big\rangle$
$\vdots$	$\vdots$

Note that these expansions of $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ and $\widehat{H}|\Lambda, a\rangle$ are decompositions of each, in terms of states with amplitudes that are independent of $\hbar$. If $|\Lambda, a\rangle$ is a solution to the Schrodinger equation then we would require that $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ be equal to $\widehat{H}|\Lambda, a\rangle$. The result is not to solve Schrodinger's equation exactly, but rather approximately in a semiclassical sense. To that end we will match the terms in the state expansions for $i\hbar\frac{\partial}{\partial t}|\Lambda, a\rangle$ and $\widehat{H}|\Lambda, a\rangle$ at each level of $\hbar$.

Beginning on the level of $\hbar^0$ we see that the lowest order term in both expansions is $|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_0\rangle$. Again, the same 'miracle' we saw in the $n = 1$ case. So, to lowest order both expansions agree. Moving on to the level of $\hbar^1$; matching terms gives us

$$\Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_1\Big\rangle + \Big|\Lambda, i\frac{\partial a_0}{\partial t}\Big\rangle + \Big|\Lambda, iY a_0\Big\rangle + \Big|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}})a_0\right)\Big\rangle \tag{5.55}$$

(5.56)

$$= \Big|\Lambda, H(\mathbf{w}, \overline{\mathbf{w}})a_1\Big\rangle + D_1^H\Big|\Lambda, a_0\Big\rangle.$$

Canceling like terms on both sides and inserting the definition of $D_1^H|\Lambda, a_0\rangle$ yields

(5.57)

$$\left|\Lambda, i\frac{\partial a_0}{\partial t}\right\rangle + \left|\Lambda, iYa_0\right\rangle + \left|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_0\right)\right\rangle$$

(5.58)

$$= \left(\frac{1}{4}\right)\left|\Lambda, (\Delta H)\,(\mathbf{w},\overline{\mathbf{w}})a_0\right\rangle + \sum_{\beta}\left(\sum_{|\alpha|=1} C_{\alpha\beta}\left|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0)\right\rangle + \sum_{|\alpha|>1} C_{\alpha\beta}\left|\Lambda, \widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0)\right\rangle\right)$$

Let's calculate a more explicit formula for $D_1^H|\Lambda,\nu\rangle$. Recalling that L_k are given globally by (4.13), and locally by (4.14). The local expression will be used below, it is

(5.59)

$$L_k(g) \equiv -\sum_{l=1}^{n}\left(\frac{\partial}{\partial x_l}\left(\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} g\right) + \left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} gTr\left(\mathrm{Re}\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^{-1}\frac{\partial}{\partial x_l}\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)\right)$$

(5.60)

$$= -\sum_{l=1}^{n}\left(\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\frac{\partial g}{\partial x_l}\right.$$

(5.61)

$$\left. + \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} + \left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} Tr\left(\mathrm{Re}\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^{-1}\frac{\partial}{\partial x_l}\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right)\right)g\right).$$

For the convenience of the calculations to come let's define

$$\widetilde{T}_{kl} \equiv \left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} Tr\left(\mathrm{Re}\left(\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)^{-1}\frac{\partial}{\partial x_l}\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right)\right). \tag{5.62}$$

Also, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $L^{\alpha}(a) \equiv L_1^{\alpha_1} \circ L_2^{\alpha_2} \circ \cdots \circ L_n^{\alpha_n}(a)$, and $\widetilde{L}_1^{\alpha}$ is the operator given by a sum of all possible permutations of M_{w_k} and L_j with '1' factor of L_j in the sense that one takes '1' factor from the list

$$\underbrace{L_1,\ldots,L_1}_{\alpha_1\text{-factors}},\underbrace{L_2,\ldots,L_2}_{\alpha_2\text{-factors}},\ldots,\underbrace{L_n,\ldots,L_n}_{\alpha_n\text{-factors}},$$

and '$|\alpha|-1$' (i.e. all but one) factors of M_{w_k} in the sense that one takes '$|\alpha|-1$' factors from the list

$$\underbrace{M_{w_1},\ldots,M_{w_1}}_{\alpha_1\text{-factors}},\underbrace{M_{w_2},\ldots,M_{w_2}}_{\alpha_2\text{-factors}},\ldots,\underbrace{M_{w_n},\ldots,M_{w_n}}_{\alpha_n\text{-factors}},$$

where the ordering is fixed.

Remark V.5. I would like to make a disclaimer here that the calculation to arrive at a more explicit formula for the term $D_1^H|\Lambda,\nu\rangle$ that follows is pretty involved. While I think it is important to include for the completeness of the result, the reader who is willing to take the final result as given may want to skip the calculation.

By definition we have

$$\begin{aligned}
\widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu) &= L_1(\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\nu)+w_1L_1(\mathbf{w}^{\alpha-2e_1}\overline{\mathbf{w}}^{\beta}\nu)+\cdots+w_1^{\alpha_1-2}L_1(\mathbf{w}^{\alpha-(\alpha_1-1)e_1}\overline{\mathbf{w}}^{\beta}\nu)\\
&\quad+w_1^{\alpha_1-1}L_1(\mathbf{w}^{\alpha-\alpha_1e_1}\overline{\mathbf{w}}^{\beta}\nu)+w_1^{\alpha_1}L_2(\mathbf{w}^{\alpha-\alpha_1e_1-e_2}\overline{\mathbf{w}}^{\beta}\nu)\\
&\quad+w_1^{\alpha_1}w_2L_2(\mathbf{w}^{\alpha-\alpha_1e_1-2e_2}\overline{\mathbf{w}}^{\beta}\nu)\\
&\quad+\cdots+w_1^{\alpha_1}w_2^{\alpha_2-2}L_2(\mathbf{w}^{\alpha-\alpha_1e_1-(\alpha_2-1)e_2}\overline{\mathbf{w}}^{\beta}\nu)\\
&\quad+w_1^{\alpha_1}w_2^{\alpha_2-1}L_2(\mathbf{w}^{\alpha-\alpha_1e_1-(\alpha_2)e_2}\overline{\mathbf{w}}^{\beta}\nu)\\
&\quad+\cdots+\mathbf{w}^{\alpha-\alpha_ne_n}L_n(w_n^{\alpha_n-1}\overline{\mathbf{w}}^{\beta}\nu)+\mathbf{w}^{\alpha-(\alpha_n-1)e_n}L_n(w_n^{\alpha_n-2}\overline{\mathbf{w}}^{\beta}\nu)+\cdots\\
&\quad+\mathbf{w}^{\alpha-(2)e_n}L_n(w_n\overline{\mathbf{w}}^{\beta}\nu)+\mathbf{w}^{\alpha-e_n}L_n(\overline{\mathbf{w}}^{\beta}\nu).
\end{aligned}$$

Now, substituting the form for the $L_k's$ we have

$$
\begin{aligned}
\widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu) = -\sum_{l=1}^{n}\Bigg[&\Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-e_1)_j\mathbf{w}^{\alpha-e_1-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\quad+\beta_j\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg]\\
&\quad+\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\nu\Bigg]+\widetilde{T}_{1l}\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\nu\\
&+w_1\Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-2e_1)_j\mathbf{w}^{\alpha-2e_1-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\quad+\beta_j\mathbf{w}^{\alpha-2e_1}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\mathbf{w}^{\alpha-2e_1}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg]\\
&\quad+\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-2e_1}\overline{\mathbf{w}}^{\beta}\nu\Bigg]+w_1\widetilde{T}_{1l}\mathbf{w}^{\alpha-2e_1}\overline{\mathbf{w}}^{\beta}\nu\\
&+\cdots\\
&+w_1^{\alpha_1-2}\Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-(\alpha_1-1)e_1)_j\mathbf{w}^{\alpha-(\alpha_1-1)e_1-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\quad+\beta_j\mathbf{w}^{\alpha-(\alpha_1-1)e_1}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\mathbf{w}^{\alpha-(\alpha_1-1)e_1}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg]\\
&\quad+\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-(\alpha_1-1)e_1}\overline{\mathbf{w}}^{\beta}\nu\Bigg]+w_1^{\alpha_1-2}\widetilde{T}_{1l}\mathbf{w}^{\alpha-(\alpha_1-1)e_1}\overline{\mathbf{w}}^{\beta}\nu\\
&+w_1^{\alpha_1-1}\Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-\alpha_1e_1)_j\mathbf{w}^{\alpha-\alpha_1e_1-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\quad+\beta_j\mathbf{w}^{\alpha-\alpha_1e_1}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\mathbf{w}^{\alpha-\alpha_1e_1}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg]\\
&\quad+\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-\alpha_1e_1}\overline{\mathbf{w}}^{\beta}\nu\Bigg]+w_1^{\alpha_1-1}\widetilde{T}_{1l}\mathbf{w}^{\alpha-(\alpha_1)e_1}\overline{\mathbf{w}}^{\beta}\nu\\
&+w_1^{\alpha_1}\Bigg[\left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-\alpha_1e_1-e_2)_j\mathbf{w}^{\alpha-\alpha_1e_1-e_2-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\quad+\beta_j\mathbf{w}^{\alpha-\alpha_1e_1-e_2}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\mathbf{w}^{\alpha-\alpha_1e_1-e_2}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg]\\
&\quad+\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-\alpha_1e_1-e_2}\overline{\mathbf{w}}^{\beta}\nu\Bigg]+w_1^{\alpha_1}\widetilde{T}_{2l}\mathbf{w}^{\alpha-(\alpha_1)e_1-e_2}\overline{\mathbf{w}}^{\beta}\nu\\
&+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbf{w}^{\alpha-\alpha_n e_n}\Bigg[\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}(\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n)_j \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{w}^{\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right) \\
&\qquad\qquad\qquad + \beta_j \mathbf{w}^{\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg) \\
&\qquad\qquad\qquad + \mathbf{w}^{\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg] \\
&\qquad\qquad\qquad + \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n}\overline{\mathbf{w}}^{\beta}\nu\Bigg]\Bigg] \\
&\qquad\qquad\qquad + \mathbf{w}^{\alpha-\alpha_n e_n}\widetilde{T}_{nl}\mathbf{w}^{\alpha_n-1}\overline{\mathbf{w}}^{\beta}\nu \\
&+\cdots \\
&+ \mathbf{w}^{\alpha-e_n}\Bigg[\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\Bigg[\Bigg(\sum_{j=1}^{n}\beta_j\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\Bigg)+\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg] \\
&\qquad\qquad + \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\right)\overline{\mathbf{w}}^{\beta}\nu\Bigg]\Bigg] + \mathbf{w}^{\alpha-e_n}\widetilde{T}_{nl}\overline{\mathbf{w}}^{\beta}\nu
\end{aligned}
$$

Simplifying we get

$$
\begin{aligned}
\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta \nu) = -\sum_{l=1}^n \Bigg[& \Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \Bigg[\Bigg(\sum_{j=1}^n (\alpha - e_1)_j \mathbf{w}^{\alpha - e_1 - e_j} \overline{\mathbf{w}}^\beta \nu \left(\frac{\partial w_j}{\partial x_l} \right) \\
& \quad + \beta_j \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^{\beta - e_j} \nu \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \Bigg) + \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \frac{\partial \nu}{\partial x_l} \Bigg] \\
& \quad + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \right) \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \Bigg] + \widetilde{T}_{1l} \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \\
& + \Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \Bigg[\Bigg(\sum_{j=1}^n (\alpha - 2e_1)_j \mathbf{w}^{\alpha - e_1 - e_j} \overline{\mathbf{w}}^\beta \nu \left(\frac{\partial w_j}{\partial x_l} \right) \\
& \quad + \beta_j \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^{\beta - e_j} \nu \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \Bigg) + \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \frac{\partial \nu}{\partial x_l} \Bigg] \\
& \quad + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \right) \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \Bigg] + \widetilde{T}_{1l} \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \\
& + \cdots \\
& + \Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \Bigg[\Bigg(\sum_{j=1}^n (\alpha - (\alpha_1 - 1)e_1)_j \mathbf{w}^{\alpha - e_1 - e_j} \overline{\mathbf{w}}^\beta \nu \left(\frac{\partial w_j}{\partial x_l} \right) \\
& \quad + \beta_j \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^{\beta - e_j} \nu \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \Bigg) + \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \frac{\partial \nu}{\partial x_l} \Bigg] \\
& \quad + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \right) \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \Bigg] + \widetilde{T}_{1l} \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \\
& + \Bigg[\left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \Bigg[\Bigg(\sum_{j=1}^n (\alpha - \alpha_1 e_1)_j \mathbf{w}^{\alpha - e_1 - e_j} \overline{\mathbf{w}}^\beta \nu \left(\frac{\partial w_j}{\partial x_l} \right) \\
& \quad + \beta_j \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^{\beta - e_j} \nu \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \Bigg) + \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \frac{\partial \nu}{\partial x_l} \Bigg] \\
& \quad + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_1}{\partial x_l} \right)^{-1} \right) \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \Bigg] + \widetilde{T}_{1l} \mathbf{w}^{\alpha - e_1} \overline{\mathbf{w}}^\beta \nu \\
& + \Bigg[\left(\frac{\partial \overline{w}_2}{\partial x_l} \right)^{-1} \Bigg[\Bigg(\sum_{j=1}^n (\alpha - \alpha_1 e_1 - e_2)_j \mathbf{w}^{\alpha - e_2 - e_j} \overline{\mathbf{w}}^\beta \nu \left(\frac{\partial w_j}{\partial x_l} \right) \\
& \quad + \beta_j \mathbf{w}^{\alpha - e_2} \overline{\mathbf{w}}^{\beta - e_j} \nu \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \Bigg) + \mathbf{w}^{\alpha - e_2} \overline{\mathbf{w}}^\beta \frac{\partial \nu}{\partial x_l} \Bigg] \\
& \quad + \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_2}{\partial x_l} \right)^{-1} \right) \mathbf{w}^{\alpha - e_2} \overline{\mathbf{w}}^\beta \nu \Bigg] + \widetilde{T}_{2l} \mathbf{w}^{\alpha - e_2} \overline{\mathbf{w}}^\beta \nu \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\left[\left(\sum_{j=1}^{n}(\alpha-\alpha_1 e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n)_j\right.\right.\right.\\
&\qquad\qquad\qquad\qquad \times \mathbf{w}^{\alpha-e_n-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right)\\
&\qquad\quad \left.+ \beta_j \mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\right)\\
&\qquad\quad \left.+ \mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\right]\\
&\qquad\quad \left.\left.+ \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu\right]\right]\\
&\qquad\quad + \widetilde{T}_{nl}\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \cdots\\
&+ \left[\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\left[\left(\sum_{j=1}^{n}\beta_j \mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\right) + \mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\right]\right.\\
&\qquad\qquad\qquad\qquad \left.+ \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\right)\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu\right]\Bigg]\\
&\qquad\qquad\qquad\qquad + \widetilde{T}_{nl}\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu
\end{aligned}
$$

Reorganizing the terms in a more sensible way we get

$$
\begin{aligned}
\widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu) = -\sum_{l=1}^{n}\Bigg[&\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\Bigg(\sum_{j=1}^{n}[(\alpha-e_1)_j+(\alpha-2e_1)_j+\cdots+(\alpha-\alpha_1 e_1)_j] \\
&\qquad\qquad\qquad\qquad \times \mathbf{w}^{\alpha-e_1-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right) \\
&\qquad + \sum_{j=1}^{n}\alpha_1\beta_j\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
&\qquad + \alpha_1\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg) \\
&\qquad + \alpha_1\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1}\right) + \alpha_1\widetilde{T}_{1l}\mathbf{w}^{\alpha-e_1}\overline{\mathbf{w}}^{\beta}\nu \\
&+\left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1}\Bigg(\sum_{j=1}^{n}[(\alpha-\alpha_1e_1-e_2)_j+(\alpha-\alpha_1e_1-2e_2)_j+\cdots+(\alpha-\alpha_1e_1-\alpha_2e_2)_j] \\
&\qquad\qquad\qquad\qquad \times \mathbf{w}^{\alpha-e_2-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right) \\
&\qquad + \sum_{j=1}^{n}\alpha_2\beta_j\mathbf{w}^{\alpha-e_2}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
&\qquad + \alpha_2\mathbf{w}^{\alpha-e_2}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg) \\
&\qquad + \alpha_2\mathbf{w}^{\alpha-e_2}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1}\right) + \alpha_2\widetilde{T}_{2l}\mathbf{w}^{\alpha-e_2}\overline{\mathbf{w}}^{\beta}\nu \\
&+\cdots \\
&+\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\Bigg(\sum_{j=1}^{n}[(\alpha-\alpha_1e_1-\cdots-\alpha_{n-1}e_{n-1}-e_n)_j+(\alpha-\alpha_1e_1-\alpha_{n-1}e_{n-1}-2e_n)_j \\
&\qquad\qquad +\cdots+(\alpha-\alpha_1e_1-\cdots-\alpha_ne_n)_j]\mathbf{w}^{\alpha-e_n-e_j}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial w_j}{\partial x_l}\right) \\
&\qquad + \sum_{j=1}^{n}\alpha_n\beta_j\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta-e_j}\nu\left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
&\qquad + \alpha_n\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\frac{\partial \nu}{\partial x_l}\Bigg) \\
&\qquad + \alpha_n\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1}\right) + \alpha_n\widetilde{T}_{nl}\mathbf{w}^{\alpha-e_n}\overline{\mathbf{w}}^{\beta}\nu.
\end{aligned}
$$

Now if we define $A_{1j} \equiv (\alpha-e_1)_j+(\alpha-2e_1)_j+\cdots+(\alpha-\alpha_1e_1)_j$, then

$$A_{1j} = (\alpha_j - \delta_{1j}) + (\alpha_j - 2\delta_{1j}) + \cdots + (\alpha_j - \alpha_1\delta_{1j}) = \begin{cases} (\alpha_1 - 1) + (\alpha_1 - 2) + \cdots + 1 + 0 & \text{if } j = 1 \\ \alpha_1\alpha_j & \text{if } j \neq 1 \end{cases},$$

or equivalently,

$$A_{1j} = \begin{cases} \frac{1}{2}\alpha_1(\alpha_1 - 1) & \text{if } j = 1 \\ \alpha_1\alpha_j & \text{if } j \neq 1 \end{cases}.$$

Similarly, if $A_{2j} \equiv (\alpha - \alpha_1 e_1 - e_2)_j + (\alpha - \alpha_1 e_1 - 2e_2)_j + \cdots + (\alpha - \alpha_1 e_1 - \alpha_2 e_2)_j$, then

$$A_{2j} = (\alpha_j - \alpha_1\delta_{1j} - \delta_{2j}) + (\alpha_j - \alpha_1\delta_{1j} - 2\delta_{1j}) + \cdots + (\alpha_j - \alpha_1\delta_{1j} - \alpha_2\delta_{2j}),$$

and so

$$A_{2j} = \begin{cases} 0 & \text{if } j = 1 \\ (\alpha_2 - 1) + (\alpha_2 - 2) + \cdots + 1 + 0 & \text{if } j = 2 \\ \alpha_2\alpha_j & \text{if } j \neq 1, 2 \end{cases},$$

thus

$$A_{2j} = \begin{cases} 0 & \text{if } j = 1 \\ \frac{1}{2}\alpha_2(\alpha_2 - 1) & \text{if } j = 2 \\ \alpha_2\alpha_j & \text{if } j \neq 1, 2 \end{cases}.$$

Finally, if $A_{nj} \equiv (\alpha - \alpha_1 e_1 - \cdots - \alpha_{n-1}e_{n-1} - e_n)_j + (\alpha - \alpha_1 e_1 - \alpha_{n-1}e_{n-1} - 2e_n)_j + \cdots + (\alpha - \alpha_1 e_1 - \cdots - \alpha_n e_n)_j$, then

$$A_{nj} = \begin{cases} 0 & \text{if } j \neq n \\ (\alpha_n - 1) + (\alpha_n - 2) + \cdots + 1 + 0 & \text{if } j = n \end{cases},$$

so

$$A_{nj} = \begin{cases} 0 & \text{if } j \neq n \\ \frac{1}{2}\alpha_n(\alpha_n - 1) & \text{if } j = n \end{cases}.$$

Let's consider our result in the the case where we are dealing with a classical Hamiltonian, H, that is a monomial, i.e.

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \mathbf{z}^{\alpha}\overline{\mathbf{z}}^{\beta},$$

then above expression for $\widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu)$ combined with the facts just shown is

$$
\begin{aligned}
\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta \nu) = -\sum_{l=1}^n \Bigg[& \left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1} \Bigg(\frac{1}{2}\frac{\partial^2 H}{\partial z_1^2}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial w_j}{\partial x_l}\right) + \sum_{j=1}^n \frac{\partial^2 H}{\partial z_1 \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
& \quad + \sum_{j=2}^n \frac{\partial^2 H}{\partial z_1 \partial z_j}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial w_j}{\partial x_l}\right) + \frac{\partial H}{\partial z_1}(\mathbf{w}, \overline{\mathbf{w}})\frac{\partial \nu}{\partial x_l}\Bigg) \\
& \quad + \frac{\partial H}{\partial z_1}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_1}{\partial x_l}\right)^{-1} + \widetilde{T}_{1l}\right) \\
& + \left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1} \Bigg(\frac{1}{2}\frac{\partial^2 H}{\partial z_2^2}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial w_j}{\partial x_l}\right) + \sum_{j=1}^n \frac{\partial^2 H}{\partial z_1 \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
& \quad + \sum_{j=3}^n \frac{\partial^2 H}{\partial z_2 \partial z_j}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial w_j}{\partial x_l}\right) + \frac{\partial H}{\partial z_2}(\mathbf{w}, \overline{\mathbf{w}})\frac{\partial \nu}{\partial x_l}\Bigg) \\
& \quad + \frac{\partial H}{\partial z_2}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_2}{\partial x_l}\right)^{-1} + \widetilde{T}_{2l}\right) \\
& + \cdots \\
& + \left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1} \Bigg(\frac{1}{2}\frac{\partial^2 H}{\partial z_n^2}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial w_j}{\partial x_l}\right) + \sum_{j=1}^n \frac{\partial^2 H}{\partial z_n \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial \overline{w}_j}{\partial x_l}\right) \\
& \quad + \frac{\partial H}{\partial z_n}(\mathbf{w}, \overline{\mathbf{w}})\frac{\partial \nu}{\partial x_l}\Bigg) \\
& \quad + \frac{\partial H}{\partial z_n}(\mathbf{w}, \overline{\mathbf{w}})\nu \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_n}{\partial x_l}\right)^{-1} + \widetilde{T}_{nl}\right)\Bigg].
\end{aligned}
$$

Combining the various terms in this summation in an orderly fashion we get

$$
\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta \nu) = -\sum_{l=1}^n \Bigg[\frac{1}{2}\nu \sum_{j=1}^n \frac{\partial^2 H}{\partial z_j^2}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} \tag{5.63}
$$

$$
+ \nu \sum_{k=2}^n \sum_{j=k}^n \frac{\partial^2 H}{\partial z_{k-1} \partial z_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.64}
$$

$$
+ \nu \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 H}{\partial z_k \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.65}
$$

$$
+ \frac{\partial \nu}{\partial x_l} \sum_{k=1}^n \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.66}
$$

$$
+ \nu \sum_{k=1}^n \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} + \widetilde{T}_{kl}\right). \tag{5.67}
$$

Recall that the reason that we are going to such lengths to understand $\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta\nu)$ is that in the defining expression for $D_1^H|\Lambda,\nu\rangle$ we have the term

$$\sum_\beta\sum_{|\alpha|>1} C_{\alpha\beta}\left|\Lambda, \widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta a_0)\right\rangle.$$

Since the above rearrangement process of unraveling $\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta\nu)$ with respect to the Hamiltonian, H, is linear with respect to terms in the Hamiltonian we see from this expression that if one is given a polynomial Hamiltonian of the type where each term is such that $|\alpha|>1$ then one would have

$$\sum_\beta\sum_{|\alpha|>1} C_{\alpha\beta}\widetilde{L}_1^\alpha(\overline{\mathbf{w}}^\beta\nu) = -\sum_{l=1}^{n}\left[\frac{1}{2}\nu\sum_{j=1}^{n}\frac{\partial^2 H}{\partial z_j^2}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\right. \tag{5.68}$$

$$+\,\nu\sum_{k=2}^{n}\sum_{j=k}^{n}\frac{\partial^2 H}{\partial z_{k-1}\partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.69}$$

$$+\,\nu\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 H}{\partial z_k\partial \overline{z}_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.70}$$

$$+\,\frac{\partial\nu}{\partial x_l}\sum_{k=1}^{n}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \tag{5.71}$$

$$\left.+\,\nu\sum_{k=1}^{n}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}+\widetilde{T}_{kl}\right)\right). \tag{5.72}$$

The other term we need to unravel, in order to understand $D_1^H|\Lambda,\nu\rangle$, is

$$\sum_{|\alpha|=1} C_{\alpha\beta}\left|\Lambda, L^\alpha(\overline{\mathbf{w}}^\beta\nu)\right\rangle.$$

Let's note that if $|\alpha|=1$, then $\alpha=e_j$ for some $j=1,\ldots,n$. Thus

$$\sum_{|\alpha|=1} C_{\alpha\beta}\left|\Lambda, L^\alpha(\overline{\mathbf{w}}^\beta\nu)\right\rangle = \sum_{j=1}^{n} C_{e_j\beta}\left|\Lambda, L_j(\overline{\mathbf{w}}^\beta\nu)\right\rangle.$$

Now, inserting explicit expressions gives us

$$\sum_{j=1}^{n} C_{e_j\beta}\left|\Lambda, L_j(\overline{\mathbf{w}}^\beta \nu)\right\rangle = \sum_{j=1}^{n} C_{e_j\beta}\left|\Lambda, -\sum_{l=1}^{n}\left[\frac{\partial}{\partial x_l}\left(\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\overline{\mathbf{w}}^\beta \nu\right) + \widetilde{T}_{kl}\overline{\mathbf{w}}^\beta \nu\right]\right\rangle \tag{5.73}$$

$$= -\sum_{j=1}^{n}\sum_{l=1}^{n}\left|\Lambda, C_{e_j\beta}\left[\overline{\mathbf{w}}^\beta \nu\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\right) + \overline{\mathbf{w}}^\beta\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\frac{\partial \nu}{\partial x_l}\right.\right. \tag{5.74}$$

$$\left.\left. + \sum_{k=1}^{n}\beta_k \overline{\mathbf{w}}^{\beta-e_k}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} + \widetilde{T}_{kl}\overline{\mathbf{w}}^\beta \nu\right]\right\rangle. \tag{5.75}$$

Now, if we have a classical Hamiltonian, H, of the form

$$H(\mathbf{z}, \overline{\mathbf{z}}) = \mathbf{z}^{e_j}\overline{\mathbf{z}}^\beta = z_j \overline{\mathbf{z}}^\beta,$$

then since

$$\frac{\partial^2 H}{\partial z_m \partial \overline{z}_k} = 0,$$

for all m and k, and

$$\frac{\partial H}{\partial \overline{z}_k} = 0$$

if $j \neq k$. Then we can say

$$L_j(\overline{\mathbf{w}}^\beta \nu) = -\sum_{l=1}^{n} \left[\frac{1}{2}\nu \sum_{j=1}^{n} \frac{\partial^2 H}{\partial z_j^2}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} \right. \tag{5.76}$$

$$+ \nu \sum_{k=2}^{n} \sum_{j=k}^{n} \frac{\partial^2 H}{\partial z_{k-1} \partial z_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.77}$$

$$+ \nu \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial z_k \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.78}$$

$$+ \frac{\partial \nu}{\partial x_l} \sum_{k=1}^{n} \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.79}$$

$$\left. + \nu \sum_{k=1}^{n} \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} + \widetilde{T}_{kl} \right) \right]. \tag{5.80}$$

Thus

$$\sum_{|\alpha|=1} C_{\alpha\beta} L^\alpha(\overline{\mathbf{w}}^\beta \nu) = \sum_{j=1}^{n} C_{e_j\beta} L_j(\overline{\mathbf{w}}^\beta \nu) \tag{5.81}$$

$$= -\sum_{l=1}^{n} \left[\frac{1}{2}\nu \sum_{j=1}^{n} \frac{\partial^2 H}{\partial z_j^2}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_j}{\partial x_l} \right)^{-1} \right. \tag{5.82}$$

$$+ \nu \sum_{k=2}^{n} \sum_{j=k}^{n} \frac{\partial^2 H}{\partial z_{k-1} \partial z_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial w_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.83}$$

$$+ \nu \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial z_k \partial \overline{z}_j}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_j}{\partial x_l} \right) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.84}$$

$$+ \frac{\partial \nu}{\partial x_l} \sum_{k=1}^{n} \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} \tag{5.85}$$

$$\left. + \nu \sum_{k=1}^{n} \frac{\partial H}{\partial z_k}(\mathbf{w}, \overline{\mathbf{w}}) \left(\frac{\partial}{\partial x_l} \left(\frac{\partial \overline{w}_k}{\partial x_l} \right)^{-1} + \widetilde{T}_{kl} \right) \right] \tag{5.86}$$

Now, noting linearity of this process with respect to H we can extend this result to general polynomial Hamiltonians (with one technicality that hasn't been considered that will be mentioned in a moment) we have

$$\begin{aligned} D_1^H|\Lambda,\nu\rangle &= \left(\frac{1}{4}\right)\Big|\Lambda, (\Delta H)\,(\mathbf{w},\overline{\mathbf{w}})\nu\Big\rangle \\ &\quad + \sum_{\beta}\left(\sum_{|\alpha|=1} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu)\Big\rangle + \sum_{|\alpha|>1} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu)\Big\rangle\right) \\ &= \left(\frac{1}{4}\right)\Big|\Lambda, (\Delta H)\,(\mathbf{w},\overline{\mathbf{w}})\nu\Big\rangle + \Big|\Lambda, \Upsilon(\mathbf{w},\overline{\mathbf{w}})\nu + \sum_{l=1}^{n}\Upsilon_l(\mathbf{w},\overline{\mathbf{w}})\frac{\partial\nu}{\partial x_l}\Big\rangle. \end{aligned}$$

Where

$$\begin{aligned} \Upsilon(\mathbf{w},\overline{\mathbf{w}})\nu + \sum_{l=1}^{n}\Upsilon_l(\mathbf{w},\overline{\mathbf{w}})\frac{\partial\nu}{\partial x_l} = -\sum_{l=1}^{n}\Bigg[&\frac{1}{2}\nu\sum_{j=1}^{n}\frac{\partial^2 H}{\partial z_j^2}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} \\ &+ \nu\sum_{k=2}^{n}\sum_{j=k}^{n}\frac{\partial^2 H}{\partial z_{k-1}\partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \\ &+ \nu\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 H}{\partial z_k\partial \overline{z}_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \\ &+ \frac{\partial\nu}{\partial x_l}\sum_{k=1}^{n}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} \\ &+ \nu\sum_{k=1}^{n}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1} + \widetilde{T}_{kl}\right)\Bigg]. \end{aligned}$$

Remark V.6. Note that we have considered polynomial Hamiltonians with $|\alpha| \geq 1$. In the case where $|\alpha| = 0$ the Hamiltonian would have the general form $H(\mathbf{z},\overline{\mathbf{z}}) = \overline{\mathbf{z}}^{\beta}$. For such an H it is obviously true that

$$\frac{\partial H}{\partial z_k} = 0,$$

for all $k = 1,\ldots,n$. Thus since in the above expression for $D_1^H|\Lambda,\nu\rangle$ every term involving H involves at least one derivative of H with respect to some z_k, the above expression will hold valid for Hamiltonians with no $z_k's$ as well as Hamiltonians with terms of the sort already considered. Thus it truly does hold for all polynomial Hamiltonians.

Note that we can combine some of the terms in the above expression by observing that since H is a polynomial we are guaranteed that

$$\frac{\partial^2 H}{\partial z_k \partial z_j} = \frac{\partial^2 H}{\partial z_j \partial z_k},$$

and so

$$\frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 H}{\partial z_j^2}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} + \sum_{k=2}^{n}\sum_{j=k}^{n} \frac{\partial^2 H}{\partial z_{k-1}\partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}$$

Recall that this whole calculation was done in order to better understand the terms in $D_1^H|\Lambda,\nu\rangle$ in the context of our matching equation at the level of $\hbar^1$. This matching led to the equation

$$\left|\Lambda, i\frac{\partial a_0}{\partial t}\right\rangle + \Big|\Lambda, iYa_0\Big\rangle + \left|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_0\right)\right\rangle \tag{5.87}$$
$$= \left(\frac{1}{4}\right)\Big|\Lambda, (\Delta H)(\mathbf{w},\overline{\mathbf{w}})a_0\Big\rangle + \sum_{\beta}\left(\sum_{|\alpha|=1} C_{\alpha\beta}\Big|\Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0)\Big\rangle\right. \tag{5.88}$$
$$\left. + \sum_{|\alpha|>1} C_{\alpha\beta}\Big|\Lambda, \widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0)\Big\rangle\right). \tag{5.89}$$

Our work thus far has made the right-hand side of the equation very explicit. For the left-hand side we should note that

$$\begin{aligned}\sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\nu\right) &= -\sum_{k=1}^{n}\left(\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\left(\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\nu\right)+\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\widetilde{T}_{kl}\nu\right)\\
&= -\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\left(\frac{\partial}{\partial x_l}\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}+\widetilde{T}_{kl}\right)\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\nu\right.\\
&\qquad + \left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})\frac{\partial \nu}{\partial x_l}\\
&\qquad + \sum_{j=1}^{n}\frac{\partial^2 H}{\partial z_j \partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\nu\\
&\qquad \left. + \sum_{j=1}^{n}\frac{\partial^2 H}{\partial \overline{z}_j \partial z_k}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\nu\right)\end{aligned}$$

The lemma, (III.23), tells us that the above matching equation will be satisfied precisely when the amplitudes agree, that is

$$i\frac{\partial a_0}{\partial t}+iYa_0+\sum_{k=1}^{n}L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_0\right) \tag{5.90}$$

$$= \frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}})a_0+\sum_{\beta}\sum_{|\alpha|=1}C_{\alpha\beta}L^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0)+\sum_{\beta}\sum_{|\alpha|>1}C_{\alpha\beta}\widetilde{L}_1^{\alpha}(\overline{\mathbf{w}}^{\beta}a_0), \tag{5.91}$$

which when we insert our explicit expressions we see that several terms will cancel, and we arrive at our transport equation for a_0,

$$i\frac{\partial a_0}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}})-iY+\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_0. \tag{5.92}$$

If we match the terms at the level of $\hbar^s$ for $s \leq M$, then we get (after canceling a common term on both sides)

$$\left|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\right\rangle+\left|\Lambda, iYa_{s-1}\right\rangle+\left|\Lambda,\sum_{k=1}^{n}L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_{s-1}\right)\right\rangle \tag{5.93}$$

$$= D_s^H\left|\Lambda, a_0\right\rangle+D_{s-1}^H\left|\Lambda, a_1\right\rangle+\cdots+D_1^H\left|\Lambda, a_{s-1}\right\rangle. \tag{5.94}$$

Recalling the definition of $D_k^H \Big| \Lambda, \nu \Big\rangle$ for $1 \le k \le M$ we have

$$
\begin{aligned}
D_k^H \Big| \Lambda, \nu \Big\rangle \equiv \left(\frac{1}{4}\right)^k \Big| \Lambda, \left(\Delta^k H\right)(\mathbf{w}, \overline{\mathbf{w}})\nu \Big\rangle + \sum_{\beta} \Bigg(\sum_{|\alpha|=k} C_{\alpha\beta} \Big| \Lambda, L^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu) \Big\rangle \\
+ \sum_{|\alpha|>k} C_{\alpha\beta} \Big| \Lambda, \widetilde{L}_k^{\alpha}(\overline{\mathbf{w}}^{\beta}\nu) \Big\rangle \Bigg) \\
+ \sum_{i=1}^{k-1} \sum_{\beta^{j_1 \cdots j_i}} \Bigg(\sum_{|\alpha^{j_1 \cdots j_i}|=k-i} \widetilde{C}_{\alpha\beta}^{j_1 \cdots j_i} \Big| \Lambda, L^{\alpha^{j_1 \cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1 \cdots j_i}}\nu) \Big\rangle \\
+ \sum_{|\alpha^{j_1 \cdots j_i}|>k-i} \widetilde{C}_{\alpha\beta}^{j_1 \cdots j_i} \Big| \Lambda, \widetilde{L}_{k-i}^{\alpha^{j_1 \cdots j_i}}(\overline{\mathbf{w}}^{\beta^{j_1 \cdots j_i}}\nu) \Big\rangle \Bigg).
\end{aligned}
$$

Referring to the definition of each piece of $D_k^H \Big| \Lambda, \nu \Big\rangle$ we can distill down the nature of this (complicated) operator as taking a generalized Lagrangian state $\Big| \Lambda, \nu \Big\rangle$ and transforming it into another generalized Lagrangian state with amplitude that involves explicitly calculable functions as well as partial derivatives of ν up to order k. Thus the term above

$$
D_s^H \Big| \Lambda, a_0 \Big\rangle + D_{s-1}^H \Big| \Lambda, a_1 \Big\rangle + \cdots + D_2^H \Big| \Lambda, a_{s-2} \Big\rangle.
$$

is a Lagrangian state with an amplitude involving explicitly calculable functions as well as partial derivatives of order s of a_0, order $s-1$ of a_1, $\ldots$, and 2 of a_{s-2}. If we denote the amplitude described here by $F_s(a_0, \ldots, a_{s-2})$ then we see that the above matching process yields the following equation for a_{s-1}:

(5.95)

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)\left(\mathbf{w},\overline{\mathbf{w}}\right) - iY + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1}$$

(5.96)

$$+ F_s(a_0, \ldots, a_{s-2}).$$

Finally, the matching process for the case $\hbar^s$ where $s > M$ will yield the relation (after canceling a common term):

(5.97)

$$\left|\Lambda, i\frac{\partial a_{s-1}}{\partial t}\right\rangle + \left|\Lambda, iYa_{s-1}\right\rangle + \left|\Lambda, \sum_{k=1}^{n} L_k\left(\frac{\partial H}{\partial z_k}(\mathbf{w},\overline{\mathbf{w}})a_{s-1}\right)\right\rangle$$

(5.98)

$$= D_M^H\left|\Lambda, a_{s-M}\right\rangle + D_{M-1}^H\left|\Lambda, a_{s-(M-1)}\right\rangle + \cdots + D_1^H\left|\Lambda, a_{s-1}\right\rangle.$$

After following the procedure above we arrive at the equation for a_{s-1} of

(5.99)

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)\left(\mathbf{w},\overline{\mathbf{w}}\right) - iY + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1}$$

(5.100)

$$+ F_M(a_{s-(M)}, \ldots, a_{s-2}).$$

Now that we have derived our transport equations that define the $a_j's$ the process of proving the the error estimates is exactly the same argument (and work) as in the case of $n = 1$.

$\square$

Remark V.7. It is well worth noting that in the case where Λ is non-compact the method of proof of the theorem above will yield a formal process for propagating a generalized Lagrangian state.

This propagation 'process' is computationally the same as in the compact case. Beginning with a generalized Lagrangian state ansatz for a solution to the Schrodinger equation, and formally following the process above will generate transport equations that define the evolution of the terms in the asymptotic expansion of the amplitude. These equations will have the same basic form as those in the compact case. Since Λ is not compact we cannot guarantee that a generalized Lagrangian state defined with respect to it will be in Bargmann space, and thus no meaningful norm exists to measure the size of the error terms.

CHAPTER VI

Applications of Generalized Lagrangian States

In this chapter we will investigate some applications of Lagrangian states to problems in physics.

6.1 Approximating The Kernel of the Propagator With A Generalized Lagrangian State

6.1.1 The Propagator and its Kernel

The quantum propagator, denoted $\widehat{U}(t)$, is the time evolution operator on the quantum Hilbert space, $\mathcal{H}$, that is defined by the operator equation

$$i\hbar\frac{\partial}{\partial t}\widehat{U}(t) = \widehat{H}\widehat{U}(t)$$

with the initial condition $\widehat{U}(0) = I$. Here I is the identity operator and $\widehat{H}$, the quantum Hamiltonian, is an (essentially) self-adjoint operator with a domain, $D(\widehat{H})$, that is dense in $\mathcal{H}$. It is well known (Stone's theorem) that $\widehat{U}(t)$ is unitary.

If we consider our Hilbert space to be $L^2(\mathbb{R}^n)$, then the kernel of $\widehat{U}(t)$, which is guaranteed to exist via the Schwartz Kernel Theorem, denoted as $\widetilde{K}_t$ is a distribution that has been studied since the early days of quantum theory. But as we will see next, the kernel of the propagator in Bargmann space, which we'll denote K_t, is a much less abstract object.

In Bargmann space we have the following result from [15] concerning the kernel of bounded operators:

Proposition VI.1. *If $\widehat{V}$ is a bounded operator on Bargmann space, $\mathcal{B}_1(\mathbb{C}^n)$, let $K_V(\mathbf{z},\overline{\xi}) \equiv \widehat{V}\varphi_{\mathbf{w}}(\mathbf{z})$. Then K_V is an entire function on $\mathbb{C}^{2n}$ that satisfies:*

1. $K_V(\cdot,\xi) \in \mathcal{B}_1(\mathbb{C}^n)$ *for all* $\xi \in \mathbb{C}^n$, *and* $K_V(\mathbf{z},\cdot) \in \mathcal{B}_1(\mathbb{C}^n)$ *for all* $\mathbf{z} \in \mathbb{C}^n$;
2. $\left| K_V\left(\mathbf{z},\overline{\xi}\right) \right| \leq \| \widehat{V} \|_{\mathcal{B}_1(\mathbb{C}^n)}$;
3. $$(\widehat{V}f)(\mathbf{z}) = \int_{\mathbb{C}^n} K_V(\mathbf{z},\overline{\xi})f(\xi)d\xi d\overline{\xi}$$

Since $\widehat{U}(t)$ is unitary it is a bounded operator, so the above proposition applies, and we can say that in Bargmann space the kernel of the propagator is a smooth function satisfying the properties above. For instance, the reproducing property of Bargmann space tells us that the kernel of the identity operator (a.k.a. $\widehat{U}(0)$) is the reproducing kernel. Thus if $\mathcal{H} = \mathcal{B}_1(\mathbb{C}^n)$ then

$$K_0(\mathbf{z},\overline{\xi}) = K_I = e^{\frac{\overline{\xi}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}. \tag{6.1}$$

Note that the above proposition tells us that for every $\psi_0 \in \mathcal{B}_1(\mathbb{C}^n)$:

$$\psi(\mathbf{z},t) \equiv \left(\widehat{U}(t)\psi_0\right)(\mathbf{z}) = \int_{\mathbb{C}^n} K_t(\mathbf{z},\overline{\xi})\psi_0(\xi)d\xi d\overline{\xi}. \tag{6.2}$$

Therefore, by the definition of $\widehat{U}(t)$

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\int_{\mathbb{C}^n} K_t(\mathbf{z},\overline{\xi})\psi_0(\xi)d\xi d\overline{\xi} &= i\hbar\frac{\partial}{\partial t}\psi \\
&= \widehat{H}\psi \\
&= \widehat{H}\int_{\mathbb{C}^n} K_t(\mathbf{z},\overline{\xi})\psi_0(\xi)d\xi d\overline{\xi},
\end{aligned}
$$

where $\widehat{H}$ is an operator with respect to the $\mathbf{z}$ variables. Since the operators $\frac{\partial}{\partial t}$ and $\widehat{H}$ do not effect the integration variables in the above equation we can (at least formally) take them inside the integral where they will act on $K_t(\mathbf{z},\overline{\xi})$. Thus we can make the following (potentially) weaker statement that will imply the above relation

$$
i\hbar\frac{\partial}{\partial t}K_t(\mathbf{z},\overline{\xi}) = \widehat{H}K_t(\mathbf{z},\overline{\xi}). \tag{6.3}
$$

This chapter is purposed toward approximating K_t, and it is this condition we will aspire to satisfy with our approximation of $K_t(\mathbf{z},\overline{\xi})$.

6.1.2 A Special Class of Generalized Lagrangian States

For the work in this chapter we will focus on generalized Lagrangian states with the particularly convenient feature that for some n our Λ's are n-dimensional submanifolds of $\mathbb{R}^n$ that will be embedded into $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The fact that our Λ's are submanifolds of a Euclidean space with the same dimension means that for the purposes of this chapter we will have global coordinates on our Λ's. Having global coordinates allows us to relax some of the geometric formality that went into defining generalized Lagrangian states (III.12).

Specifically, suppose that Λ is a smooth submanifold of $\mathbb{R}^n$ with dimension n, $\mathbf{w} : \Lambda \longrightarrow \mathbb{C}^n$ is a Lagrangian embedding (III.1), a is an amplitude (III.11) on Λ, and f is a lift function (III.9) on Λ. Choosing coordinates $\chi = (\chi_1, \ldots, \chi_n)$ on Λ we can express the parameter-dependent generalized Lagrangian state

$$|\Lambda, a\rangle = \int_\Lambda a e^{\frac{i}{\hbar} f} K(\mathbf{z}, \mathbf{w}) |d\mathbf{w}| \tag{6.4}$$

$$= \int_\Lambda a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} e^{\frac{\overline{\mathbf{w}(\chi)} \cdot \mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}(\chi) \cdot \overline{\mathbf{w}(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}} \left| \det \left(\frac{\partial \mathbf{w}}{\partial \chi} \right) \right| d\chi \tag{6.5}$$

$$= \int_\Lambda \alpha(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} e^{\frac{\overline{\mathbf{w}(\chi)} \cdot \mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}(\chi) \cdot \overline{\mathbf{w}(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}} d\chi, \tag{6.6}$$

where $\alpha(\chi, t; \hbar) = a(\chi, t; \hbar) \left| \det \left(\frac{\partial \mathbf{w}}{\partial \chi} \right) \right|$. Since the integrand is a smooth function this is an integral in the 'pedestrian' sense of Riemann (or Lebesgue). To go from the first line to the second we used the fact, (3.1), that the 1-density $|d\mathbf{w}|$ can be expressed as $\left| \det \left(\frac{\partial \mathbf{w}}{\partial \chi} \right) \right| |d\chi|$ where $d\chi = d\chi_1 \wedge \cdots \wedge d\chi_n$ is the volume form on Λ with respect to the coordinates $\chi_1, \ldots, \chi_n$. Since a is an amplitude, then α (which because of our global coordinates is a well defined quantity on Λ) is also an amplitude. For the purposes of the this chapter we will incorporate the term $\left| \det \left(\frac{\partial \mathbf{w}}{\partial \chi} \right) \right|$ into the notion of amplitude in the sense that given the setup above we will take the generalized Lagrangian state $|\Lambda, a\rangle$ to be

$$|\Lambda, a\rangle = \int_\Lambda a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} e^{\frac{\overline{\mathbf{w}(\chi)} \cdot \mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}(\chi) \cdot \overline{\mathbf{w}(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}} d\chi. \tag{6.7}$$

Suppose that Λ is compact and that $\partial\Lambda = \emptyset$. With this expression we can consider the structure of the general formulas and results from previous chapters, focusing on those that are relevant to the work in this chapter, in this specialized case.

From chapter III we will use the proposition (III.23) in order to eventually generate transport equations for the propagation of these states (6.7). In the case where the generalized states $|\Lambda, a\rangle$ and $|\Lambda, b\rangle$ are defined over a Λ with global coordinates we can use (III.23) to conclude that if $|\Lambda, a\rangle = |\Lambda, b\rangle$ for all values of $\hbar$ and we are using the same global coordinates on Λ in both states then $a = b + O(\hbar^\infty)$.

From chapter IV, if we follow the proofs of the formulas for the action of $\widehat{Z}_j \equiv z_j I$ (4.19) and $\widehat{Z}_j^* \equiv \frac{\partial}{\partial z_j}$ (4.18) on generalized states of the type (6.7) then it is straightforward to see that we will arrive at the following rather familiar formulas

$$\widehat{Z}_j^* \Big|\Lambda, a\Big\rangle = \Big|\Lambda, \overline{w}_j a\Big\rangle \tag{6.8}$$

and

$$\widehat{Z}_j \Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_j a\Big\rangle + \hbar\Big|\Lambda, L_j(a)\Big\rangle. \tag{6.9}$$

where we have a modified expression for L_j:

$$L_j(g) = -\sum_{l=1}^{n} \frac{\partial}{\partial x_l}\left(\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} g\right). \tag{6.10}$$

With this modified/different definition of the L_j operators (compare to (4.14)) we see that the basic structure of the formulas for the action of $\widehat{Z}_j \equiv z_j I$ and $\widehat{Z}_j^* \equiv \frac{\partial}{\partial z_j}$ are the same as those in chapter IV. Since the rest of the results in chapter IV rely only on this basic structure we can immediately conclude that with the modified definition of the L_j's that the rest of the work in chapter IV directly applies.

Finally, let's consider how the relevant work in chapter V changes. Using the modified formulas from chapter IV we will have the following modified formula for the 'parameter derivative' (see (5.18))

(6.11)

$$\frac{\partial}{\partial t}|\Lambda, a\rangle = \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}}\cdot\frac{\partial \mathbf{w}}{\partial t}\right)\right) a\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t} a\right)\right\rangle.$$

Combining this modified formula with the appropriately updated versions of the result for $\widehat{H}_{weyl}|\Lambda, a\rangle$, if we consider the problem of propagating a generalized state of the form (6.7), then we will arrive at the main result in chapter V but with modified transport equations. Putting the pieces together, and following the work we did to arrive at the general transport equations (5.92, 5.95, and 5.99), and employing the modified proposition from chapter III discussed above, we get the transport equations:

(6.12)

$$i\frac{\partial a_0}{\partial t} = \left(\frac{1}{4}(\Delta H)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_0,$$

for $s \leq M$

(6.13)

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}(\Delta H)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1}$$

(6.14)

$$+ F_s(a_0, \ldots, a_{s-2}),$$

and for $s > M$

(6.15)

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial z_k \partial z_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1}$$

(6.16)

$$+ F_M(a_{s-(M)},\ldots,a_{s-2}),$$

where recall that M is the order of the Hamiltonian, or the highest power of $\mathbf{z}$.

6.1.3 Approximating the Kernel: Formal Aspects

With the goal in mind of approximating the kernel of $\widehat{U}(t)$ on Bargmann space with a particular type of generalized Lagrangian state, we'll begin with some formal considerations where we won't worry about the convergence of integrals or proving estimates.

The Geometry of the Approximation

In order to use the work we did in chapter IV we consider a polynomial Hamiltonian $H(\mathbf{z},\overline{\mathbf{z}})$. The flow of this Hamiltonian on $\mathbb{C}^n$ is denoted $\phi_t : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, and $\widehat{H}$ will denote the Weyl quantization of H. For our purposes in this section we will take $\Lambda = \mathbb{C}^n$. We will embed this Λ into $\mathbb{C}^{2n} \cong \mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$ which denotes the $2n$-dimensional complex space with coordinates $(\mathbf{z},\overline{\xi})$ and symplectic form $\omega = i\left(d\mathbf{z}\wedge d\overline{\mathbf{z}} + d\overline{\xi}\wedge d\xi\right)$. We can embed $\Lambda = \mathbb{C}^n$ into $\mathbb{C}^{2n} \cong \mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$ via the Lagrangian embedding

$$\mathbf{w}_0 : \mathbb{C}^n \longrightarrow \mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$$

$$\chi \longmapsto (\chi,\overline{\chi})$$

We can consider $H = H(\mathbf{z},\overline{\mathbf{z}})$ as a Hamiltonian on $\mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$ that is simply independent of the $\overline{\xi}$ variables. The flow of H on $\mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$, denoted $\Phi_t : \mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}} \longrightarrow \mathbb{C}^n_{\mathbf{z}} \times \overline{\mathbb{C}^n_{\xi}}$ is defined by the $4n$ equations

$$\dot{z}_j = i\frac{\partial H}{\partial \overline{z}_j} \qquad\qquad \dot{\overline{z}}_j = -i\frac{\partial H}{\partial z_j}, \tag{6.17}$$

$$\dot{\overline{\overline{\xi}}}_j = \dot{\xi}_j = 0 \qquad\qquad \dot{\overline{\xi}}_j = 0. \tag{6.18}$$

Thus we have

$$\Phi_t(\mathbf{z}, \overline{\xi}) = (\phi_t(\mathbf{z}), \overline{\xi}). \tag{6.19}$$

This tells us that we can define a time-dependent Lagrangian embedding of $\Lambda = \mathbb{C}^n$

$$\mathbf{w}(\chi, t) \equiv \Phi_t(\mathbf{w}_0(\chi)) = (\phi_t(\chi), \overline{\chi}). \tag{6.20}$$

Finally, we'll want to horizontally lift the Lagrangian $\mathbf{w}(\mathbb{C}^n)$ with a choice of lift function f. Pictorially, these classical dynamics are displayed in the following diagram

$$\begin{tikzcd}
 & \mathbb{C}^n \times \overline{\mathbb{C}^n} \times S^1 \arrow[r, "\widetilde{\Phi}_t"] \arrow[d, "\pi"] & \mathbb{C}^n \times \overline{\mathbb{C}^n} \times S^1 \arrow[d, "\pi"] \\
\Lambda \cong \mathbb{C}^n \arrow[ur, "{(w,-f)}"] \arrow[r, "w"] & \mathbb{C}^n \times \overline{\mathbb{C}^n} \arrow[r, "\Phi_t"] & \mathbb{C}^n \times \overline{\mathbb{C}^n},
\end{tikzcd}$$

where $\widetilde{\Phi}_t$ is the Kostant field, (V.1), associated to H.

With our classical picture in place, consider the following ansatz (a generalized Lagrangian state (III.12), that is indeed 'generalized' since Λ is not compact) for an approximation to the kernel of the propagator

$$S_t(\mathbf{z},\overline{\xi}) = \int_{\mathbb{C}^n} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}e^{\frac{\overline{\phi_t(\chi)}\cdot\mathbf{z}}{\hbar}}e^{\frac{\overline{\xi}\cdot\chi}{\hbar}}e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}}e^{-\frac{\phi_t(\chi)\cdot\overline{\phi_t(\chi)}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}, \tag{6.21}$$

Note that $S_t(\mathbf{z},\overline{\xi})$ fits into the general form for our generalized Lagrangian states with $\mathbf{r} = (\mathbf{z},\overline{\xi}) \in \mathbb{C}^{2n}$, and $\mathbf{w}(\chi,t) = \mathbf{w}_t(\chi) = (\phi_t(\chi),\overline{\chi})$, i.e.

$$S_t(\mathbf{r}) = \int_{\mathbb{C}^n} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}e^{\frac{\overline{\mathbf{w}(\chi,t)}\cdot\mathbf{r}}{\hbar}}e^{-\frac{\mathbf{w}(\chi,\mathbf{t})\cdot\overline{\mathbf{w}(\chi,t)}}{2\hbar}}e^{-\frac{\mathbf{r}\cdot\overline{\mathbf{r}}}{2\hbar}}d\chi d\overline{\chi}.$$

Above we saw that K_t, the exact kernel of the propagator satisfies the Schrodinger equation. Thus, if S_t is to be an approximation to K_t we will seek to have it approximately satisfy the Schrodinger equation. Our goal will be to find some criteria such that a generalized Lagrangian state of the above form satisfies the Schrodinger equation (i.e. we wish to propagate a generalized Lagrangian state) subject to the initial condition that $K_0 = S_0$ (see 6.1). Employing the formal process behind the main result, in chapter V, will be our method of approximation.

The initial condition $K_0 = S_0$ will be satisfied if we require the the lift function f (III.9) satisfies

$$f(\chi,0) = 0 \tag{6.22}$$

and the expansion terms for the amplitude (III.11) satisfy

$$a_j(\chi,0) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \geq 0. \end{cases} \tag{6.23}$$

With these conditions note that we know that $a(\chi,0;\hbar) = 1 + O(\hbar^\infty)$. Combining this with the fact that $\phi_0(\chi) = \chi$ we have

$$
\begin{aligned}
S_0(\mathbf{z},\overline{\xi}) &= \int_{\mathbb{C}^n}(1+O(\hbar^\infty))e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}}e^{\frac{\overline{\xi}\cdot\chi}{\hbar}}e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}\\
&= \int_{\mathbb{C}^n} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}}e^{\frac{\overline{\xi}\cdot\chi}{\hbar}}e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}+O(\hbar^\infty)\\
&= \langle\varphi_{\mathbf{z}},\varphi_\xi\rangle_{L^2(\mathbb{C}^n)}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}+O(\hbar^\infty)\\
&= e^{\frac{\overline{\xi}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}+O(\hbar^\infty)\\
&= K_0(\mathbf{z},\overline{\xi})+O(\hbar^\infty).
\end{aligned}
$$

With the above condition imposed on f and if $a(\chi,0)=1$ then we have that the above ansatz agrees with K_0 exactly.

Because our Λ is not compact, the best we can do is employ the method used in the main result, chapter V, to arrive at the conclusion that if the lift function f satisfies the given initial condition, and the expansion coefficients a_j's satisfy the transport equations

(6.24)
$$
i\frac{\partial a_0}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}})+\frac{1}{2}\sum_{l=1}^n\sum_{j=1}^n\sum_{k=1}^n\frac{\partial^2 H}{\partial r_k\partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial\chi_l}\right)\left(\frac{\partial\overline{w}_k}{\partial\chi_l}\right)^{-1}\right)a_0,
$$

for $s\le M$

(6.25)
$$
i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}})+\frac{1}{2}\sum_{l=1}^n\sum_{j=1}^n\sum_{k=1}^n\frac{\partial^2 H}{\partial r_k\partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial\chi_l}\right)\left(\frac{\partial\overline{w}_k}{\partial\chi_l}\right)^{-1}\right)a_{s-1}
$$

(6.26)
$$
+F_s(a_0,\dots,a_{s-2}),
$$

and for $s>M$

(6.27)

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 H}{\partial r_k \partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_k}{\partial \chi_l}\right)^{-1}\right)a_{s-1}$$

(6.28)

$$+ F_M(a_{s-(M)},\ldots,a_{s-2}),$$

which follow from (6.24), (6.25), and (6.27) subject to the initial condition above, then this generalized Lagrangian state will semiclassically approximate the kernel of the propagator, formally. There are some abuses of notation in these equation that's being employed in order to keep already complicated equation from becoming absurdly complicated. By way of clarification, note $\mathbf{r} = (\mathbf{z}, \overline{\xi})$, and we double dip by representing χ as an element in $\mathbb{C}^n$ in the ansatz, but here the derivatives with respect to χ_l are really with respect to the $2n$ real variables from which χ is formed (i.e. the real and imaginary parts of each χ_l). Since H depends only on the $\mathbf{z}$ variables which in terms of $\mathbf{r}$ we have $r_j = z_j$ for $j = 1, \ldots, n$ the terms in the above equation involving derivatives of H with respect to r_j for $j = n+1, \ldots, 2n$ are zero. For the sake clarity note that since H depends on only the $\mathbf{z}$ variables, when H is evaluated at $\mathbf{w}$ and $\overline{\mathbf{w}}$ then we have that the components of $\mathbf{w}$ which H is evaluated at are the ϕ_t components.

An Explicit Formula for the Lift Function

In the case of the propagator we can find an explicit expression for the lift function. For the purposes of the immediate discussion we'll consider the real case where $\Lambda = \mathbb{R}^{2n}$ with global coordinates $(\mathbf{q}, \mathbf{p})$ since the complex case is a bit more difficult to deal with 'notationally' speaking. We will take the Lagrangian embedding of Λ into $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}_{-}$, with global coordinates $(\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P})$ and symplectic form

$\omega = d\mathbf{p} \wedge d\mathbf{q} + d\mathbf{Q} \wedge d\mathbf{P}$ as

$$\mathbf{w} : \Lambda \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}_{-} \tag{6.29}$$

$$(\mathbf{q}, \mathbf{p}) \longmapsto ((\mathbf{q}, \mathbf{p}), \phi_t(\mathbf{q}, \mathbf{p})). \tag{6.30}$$

Let's denote $\phi_t(\mathbf{q}, \mathbf{p}) = (\mathbf{q}_t, \mathbf{p}_t)$.

On $P \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}_{-} \times S^1$ with S^1 variable Θ we define the connection form

$$\alpha = d\Theta + \frac{1}{2}\left(\mathbf{p} \cdot d\mathbf{q} - \mathbf{q} \cdot d\mathbf{p} + \mathbf{Q} \cdot d\mathbf{P} - \mathbf{P} \cdot d\mathbf{Q}\right). \tag{6.31}$$

If we lift Λ to P then the lift function f will satisfy

$$\frac{\partial(-f)}{\partial q_k} = -\frac{1}{2}\left(p_k + \mathbf{q}_t \cdot \frac{\partial \mathbf{p}_t}{\partial q_k} - \mathbf{p}_t \cdot \frac{\partial \mathbf{q}_t}{\partial q_k}\right),$$

or more directly

$$\frac{\partial f}{\partial q_k} = \frac{1}{2} p_k - \sum_{j=1}^{n} \left(q_{tj} \frac{\partial p_{tj}}{\partial q_k} - p_{tj} \frac{\partial q_{tj}}{\partial q_k} \right), \tag{6.32}$$

and

$$\frac{\partial(-f)}{\partial p_k} = \frac{1}{2}\left(-q_k + \mathbf{q}_t \cdot \frac{\partial \mathbf{p}_t}{\partial p_k} - \mathbf{p}_t \cdot \frac{\partial \mathbf{q}_t}{\partial p_k}\right),$$

and again more directly

$$\frac{\partial f}{\partial p_k} = \frac{1}{2} q_k - \sum_{j=1}^{n} \left(q_{tj} \frac{\partial p_{tj}}{\partial p_k} - p_{tj} \frac{\partial q_{tj}}{\partial q_k} \right), \tag{6.33}$$

Then the following function satisfies these equations

(6.34)

$$f(\mathbf{q},\mathbf{p},t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1 \left(p_{ti}\frac{\partial q_{ti}}{\partial q_j}q_j + p_{ti}\frac{\partial q_{ti}}{\partial p_j}p_j - q_{ti}\frac{\partial p_{ti}}{\partial p_j}p_j + q_{ti}\frac{\partial p_{ti}}{\partial p_j}p_j \right)\Bigg|_{(\tau\mathbf{q},\tau\mathbf{p})} d\tau.$$

The following examples will help clarify our work thus far in this chapter.

Two Examples

Example VI.2. To illustrate this process with an example that will allow us to step back from these (rather) complicated formula, let's consider the example of the resonant harmonic oscillator. The classical Hamiltonian is $H(\mathbf{z},\overline{\mathbf{z}}) = \mathbf{z}\cdot\overline{\mathbf{z}}$. Hamilton's equations can be solved explicitly and the flow is $\phi_t(\chi_0) = \chi_0 e^{it}$. From a quantum perspective we have the Weyl quantization of H is $\widehat{H} = \widehat{\mathbf{Z}}\cdot\widehat{\mathbf{Z}}^* + \frac{\hbar n}{2}I$. The Schrodinger equation can be solved exactly to give

$$\psi(\mathbf{z},t) = \widehat{U}(t)\psi_0 = e^{-i\frac{n}{2}t}\psi_0(e^{-it}\mathbf{z}),$$

for initial condition ψ_0. Beginning with the ansatz above, we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Big|\Lambda, a\Big\rangle &= \Big|\Lambda, H(\mathbf{w},\overline{\mathbf{w}})\Big\rangle + \Big|\Lambda, i\hbar\frac{\partial a}{\partial t}\Big\rangle + \hbar\Big|\Lambda, \sum_{j=0}^{n} L_j\left(\frac{\partial H}{\partial z_j}(\mathbf{w},\overline{\mathbf{w}})a\right)\Big\rangle \\ &= \Big|\Lambda, \mathbf{w}\cdot\overline{\mathbf{w}}a\Big\rangle + \Big|\Lambda, i\hbar\frac{\partial a}{\partial t}\Big\rangle + \hbar\Big|\Lambda, \sum_{j=0}^{n} L_j\left(\overline{w}_j a\right)\Big\rangle \end{aligned}$$

and

$$\begin{aligned}
\widehat{H}\Big|\Lambda, a\Big\rangle &= \widehat{\mathbf{Z}}\cdot\widehat{\mathbf{Z}}^*\Big|\Lambda, a\Big\rangle + \frac{\hbar n}{2}\Big|\Lambda, a\Big\rangle \\
&= \sum_{j=0}^{n}\widehat{Z}_j\Big|\Lambda, \overline{w}_j a\Big\rangle + \frac{\hbar n}{2}\Big|\Lambda, a\Big\rangle \\
&= \sum_{j=0}^{n}\left(\Big|\Lambda, w_j\overline{w}_j a\Big\rangle + \hbar\Big|\Lambda, L_j\left(\overline{w}_j a\right)\Big\rangle\right) + \frac{\hbar n}{2}\Big|\Lambda, a\Big\rangle \\
&= \Big|\Lambda, \mathbf{w}\cdot\overline{\mathbf{w}}a\Big\rangle + \Big|\Lambda, \frac{\hbar n}{2}a\Big\rangle + \hbar\Big|\Lambda, \sum_{j=0}^{n}L_j\left(\overline{w}_j a\right)\Big\rangle.
\end{aligned}$$

Matching these two expansions we get that

$$i\hbar\frac{\partial}{\partial t}\Big|\Lambda, a\Big\rangle = \widehat{H}\Big|\Lambda, a\Big\rangle$$

$$\Longrightarrow\quad \Big|\Lambda, i\hbar\frac{\partial a}{\partial t}\Big\rangle = \Big|\Lambda, \frac{\hbar n}{2}a\Big\rangle,$$

which implies that

$$i\hbar\frac{\partial a_s}{\partial t} = \frac{\hbar n}{2}a_s \quad s \geq 0$$

and this in turn implies

$$a_s = C_s e^{-i\frac{n}{2}t} \quad s \geq 0,$$

for constants C_s. Applying the initial conditions

$$a_s(\chi, 0) = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \geq 0 \end{cases}$$

we see that $C_0 = 1$ and $C_s = 0$ for $s \geq 1$. Now that the $a_s's$ are computed the other piece we need for the ansatz is the lift function f. Recall that by definition f satisfies that condition $df + \frac{1}{2i}(\mathbf{w}\cdot d\overline{\mathbf{w}} - \overline{\mathbf{w}}d\mathbf{w}) = 0$. So, if $\chi = \chi_R + i\chi_I$, then the lift condition is equivalent to the system of $2n$ partial differential equations, for $j = 1, \dots, n$,

$$\frac{\partial f}{\partial \chi_{Rj}} = \frac{1}{2i}\left(\overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial \chi_{Rj}} - \mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial \chi_{Rj}}\right),$$

$$\frac{\partial f}{\partial \chi_{Ij}} = \frac{1}{2i}\left(\overline{\mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial \chi_{Ij}} - \mathbf{w} \cdot \frac{\partial \overline{\mathbf{w}}}{\partial \chi_{Ij}}\right).$$

where $\chi_j = \chi_{Rj} + i\chi_{Ij}$. Since $\mathbf{w}(\chi, t) = (\phi_t(\chi), \overline{\chi}) = (e^{it}\chi, \overline{\chi})$ we have

$$\frac{\partial \mathbf{w}}{\partial \chi_{Rj}} = (e^{it}e_j, e_j), \quad \frac{\partial \mathbf{w}}{\partial \chi_{Ij}} = (ie^{it}e_j, -ie_j), \quad \frac{\partial \overline{\mathbf{w}}}{\partial \chi_{Rj}} = (e^{-it}e_j, e_j), \quad \frac{\partial \overline{\mathbf{w}}}{\partial \chi_{Ij}} = (-ie^{-it}e_j, ie_j),$$

where $e_1, \dots, e_n$ are the n-vectors $e_1 = (1, 0, \dots, 0)$, $e_1 = (0, 1, \dots, 0)$, ..., $e_n = (0, \dots, 0, 1)$. The equations defining f are

$$\begin{aligned}\frac{\partial f}{\partial \chi_{Rj}} &= \frac{1}{2i}\Big((e^{-it}\overline{\chi}, \chi)\cdot(e^{it}e_j, e_j) - (e^{it}\chi, \overline{\chi})\cdot(e^{-it}e_j, e_j)\Big)\\ &= \frac{1}{2i}(\overline{\chi}\cdot e_j + \chi\cdot e_j - \chi\cdot e_j - \overline{\chi}\cdot e_j)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \chi_{Ij}} &= \frac{1}{2i}\Big((e^{-it}\overline{\chi}, \chi)\cdot(ie^{it}e_j, -ie_j) - (e^{it}\chi, \overline{\chi})\cdot(-ie^{-it}e_j, ie_j)\Big)\\ &= \frac{1}{2i}(i\overline{\chi}\cdot e_j - i\chi\cdot e_j - (-i)\chi\cdot e_j - i\overline{\chi}\cdot e_j)\\ &= 0.\end{aligned}$$

Thus $f(\chi, t) = C$ a constant. Since the initial condition for f is $f(\chi, 0) = 0$ we have that $f(\chi, t) = 0$. Thus our ansatz for the kernel of the propagator of the resonant harmonic oscillator is

$$S_t(\mathbf{z},\overline{\xi}) = \int_{\mathbb{C}^n} e^{-i\frac{n}{2}t} e^{\frac{\overline{e^{it}\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} e^{-\frac{e^{-it}\chi\cdot\overline{e^{-it}\chi}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi}$$
$$= \int_{\mathbb{C}^n} e^{-i\frac{n}{2}t} e^{\frac{\overline{e^{it}\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi}$$

Indeed, if $\psi_0 \in \mathcal{B}_1(\mathbb{C}^n)$ then

$$\int_{\mathbb{C}^n} S_t(\mathbf{z},\overline{\xi})\psi_0(\xi)d\xi d\overline{\xi} = \int_{\mathbb{C}^n_\xi}\int_{\mathbb{C}^n_\chi} e^{-i\frac{n}{2}t} e^{\frac{\overline{e^{it}\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} \psi_0(\xi) d\chi d\overline{\chi} d\xi d\overline{\xi}$$
$$= \int_{\mathbb{C}^n_\chi} e^{-i\frac{n}{2}t} e^{\frac{\overline{e^{it}\chi}\cdot\mathbf{z}}{\hbar}} \langle \varphi_\chi, \psi_0\rangle_{L^2(\mathbb{C}^n)} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} d\chi d\overline{\chi}$$
$$= \int_{\mathbb{C}^n} e^{-i\frac{n}{2}t} e^{\frac{\overline{e^{it}\chi}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} \psi_0(\chi) e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\chi d\overline{\chi}$$
$$= e^{-i\frac{n}{2}t}\langle \varphi_{e^{-it}\mathbf{z}}, \psi_0\rangle_{L^2(\mathbb{C}^n)}$$
$$= e^{-i\frac{n}{2}t}\psi_0(e^{-it}\mathbf{z}).$$

Thus, in the case of the resonant harmonic oscillator the ansatz gives the exact kernel of the propagator.

The next example is a slightly more complicated situation that should help to illustrate the more general process that one would need to go through in order to use the ansatz.

Example VI.3. Consider the case of the harmonic oscillator with a shifted potential and a frequency ω_e in $n = 1$. We want to use a generalized Lagrangian state $\left|\Lambda, a\right\rangle(\mathbf{r})$ for $\mathbf{r} \equiv (z,\overline{\xi}) \in \mathbb{C}\times\overline{\mathbb{C}}$ in an effort to approximate the kernel of the propagator for 1-D quantum evolution in Bargmann space. The classical Hamiltonian for the non-resonant, shifted oscillator is

$$H(p,q) = \frac{p^2}{2} + \frac{1}{2}\omega_e (q - q_0)^2.$$

With the complex structure $z = \frac{1}{\sqrt{2}}(q - ip)$ we have $\overline{z} = \frac{1}{\sqrt{2}}(q + ip)$ which implies that $q = \frac{1}{\sqrt{2}}(z + \overline{z})$ and $p = \frac{i}{\sqrt{2}}(z - \overline{z})$. Making these substitutions into $H(p,q)$ you get

$$\begin{aligned} H(z,\overline{z}) &= \frac{1}{2}\left(\frac{i}{\sqrt{2}}(z - \overline{z})\right)^2 + \frac{1}{2}\omega_e\left(\frac{1}{\sqrt{2}}(z + \overline{z}) - q_0\right)^2 \\ &= \frac{1}{2}(1 + \omega_e) z\overline{z} + \frac{1}{4}(\omega_e - 1)z^2 + \frac{1}{4}(\omega_e - 1)\overline{z}^2 - \frac{q_0}{2\sqrt{2}}z - \frac{q_0}{2\sqrt{2}}\overline{z} + \frac{1}{2}\omega_e q_0^2, \end{aligned}$$

Now, the relevant versions of the transport equations that were derived in the main result of this work, in V, and also listed in the last section will be

$$i\frac{\partial a_0}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2} \frac{\partial^2 H}{\partial z^2}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_1}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_1}{\partial \chi_l}\right)^{-1}\right)a_0,$$

for $s \leq M$:

$$\begin{aligned} i\frac{\partial a_{s-1}}{\partial t} = &\left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2} \frac{\partial^2 H}{\partial z^2}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_1}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_1}{\partial \chi_l}\right)^{-1}\right)a_{s-1} \\ &+ F_s(a_0, \ldots, a_{s-2}), \end{aligned}$$

and for $s > M$

$$\begin{aligned} i\frac{\partial a_{s-1}}{\partial t} = &\left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2} \frac{\partial^2 H}{\partial z^2}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_1}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_1}{\partial \chi_l}\right)^{-1}\right)a_{s-1} \\ &+ F_M(a_{s-(M)}, \ldots, a_{s-2}), \end{aligned}$$

where $\chi = \chi_1 + i\chi_2 \in \mathbb{C}$, and $a_s(\chi, 0) = 0$ for $s > 1$.

Let's calculate all of the pieces we'll need to first write out these equations more explicitly, and then solve them. From the form of H expressed in complex coordinates we have that

$$\Delta H = 2\left(\frac{\partial^2}{\partial z \partial \overline{z}} + \frac{\partial^2}{\partial \xi \partial \overline{\xi}}\right) H = 1 + \omega_e,$$

and

$$\frac{\partial^2 H}{\partial z^2} = \frac{1}{4}(\omega_e - 1).$$

For the matrix elements note that (borrowing a result that we calculate later in the chapter) the flow of H, with respect to just the z-variable and denoted ϕ_t^H is

$$\begin{aligned}\phi_t^H(z) = &\left(\cos(\sqrt{\omega_e}t) + \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t)\right)z \\ &+ \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t)\overline{z} \\ &+ \frac{1}{\sqrt{2}}\left(1 - \cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t)\right)q_0.\end{aligned}$$

Now, for $w(\chi, t) = (\phi_t^H(\chi), \overline{\chi})$ if we define $w_1(\chi, t) = \phi_t^H(\chi)$ and $w_2(\chi, t) = \overline{\chi}$ then

(6.35)

$$\left(\frac{\partial w}{\partial \chi}\right) = \begin{pmatrix} \frac{\partial w_1}{\partial \chi_1} & \frac{\partial w_1}{\partial \chi_2} \\ \frac{\partial w_2}{\partial \chi_1} & \frac{\partial w_2}{\partial \chi_2}\end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t) & i\left(\cos(\sqrt{\omega_e}t) + \frac{i}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t)\right) \\ 1 & -i\end{pmatrix}.$$

So,

$$det\left(\frac{\partial w}{\partial \chi}\right) = \left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t) - 2i\cos(\sqrt{\omega_e}t).$$

Which in turn gives us

$$\left(\overline{\left(\frac{\partial w}{\partial \chi}\right)^{-1}}\right)^T = \frac{1}{\overline{det\left(\frac{\partial w}{\partial \chi}\right)}} \begin{pmatrix} i & -1 \\ i\left(\cos(\sqrt{\omega_e}t) - \frac{i}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t)\right) & \cos(\sqrt{\omega_e}t) - i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t) \end{pmatrix}.$$

Putting these pieces together, and simplifying a little, we get the transport equation for a_0

$$i\frac{\partial a_0}{\partial t} = \frac{1}{4}\left(1 + \omega_e + \frac{1}{2}(\omega_e - 1)\frac{\left(\frac{1}{\sqrt{\omega_e}} - \sqrt{\omega_e}\right)\sin(\sqrt{\omega_e}t)}{\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t) + 2i\cos(\sqrt{\omega_e}t)}\right)a_0.$$

The solution, taking into account the initial condition $a_0(\chi, 0) = 1$, is

$$a_0(\chi, t) = exp\left(-\frac{i}{4}\int_0^t \left(1+\omega_e+\frac{1}{2}(\omega_e-1)\frac{\left(\frac{1}{\sqrt{\omega_e}} - \sqrt{\omega_e}\right)\sin(\sqrt{\omega_e}\tau)}{\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}\tau) + 2i\cos(\sqrt{\omega_e}\tau)}\right)d\tau\right).$$

Note that a_0 does not depend on χ.

The transport equation for a_1 is

$$i\frac{\partial a_1}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2}\frac{\partial^2 H}{\partial z^2}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_1}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_1}{\partial \chi_l}\right)^{-1}\right)a_1 + F_1(a_0).$$

Now, by definition, see (4.22) for $k = 2$, $F_1(a_0)$ is the amplitude of the generalized Lagrangian state $D_2^H\left|\Lambda, a_0\right\rangle$. Since a_0 is independent of χ, and $D_2^H\left|\Lambda, a_0\right\rangle$ will have an amplitude where derivatives with respect to χ_1 and χ_2 fall on a_0 in every term we have that $F_1(a_0) = 0$. Thus a_1 satisfies the same transport equation as a_0 except the initial condition for a_1 is $a_1(\chi, 0) = 0$; this gives that $a_1(\chi, t) \equiv 0$. Indeed, this pattern will hold for every a_s for $s \geq 1$. The transport equation are decoupled.

Thus we arrive at the conclusion that formally following the procedure for propagating a generalized Lagrangian state we get the following approximation to the kernel of the propagator of a shifted oscillator:

$$S_t(\mathbf{z},\overline{\xi}) = \int_{\mathbb{C}^n} a_0(\chi,t;\hbar) e^{\frac{i}{\hbar}f(\chi,t)} e^{\frac{\overline{\phi_t^H(\chi)}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} e^{-\frac{\phi_t^H(\chi)\cdot\overline{\phi_t^H(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi},$$

where a_0 is given above.

The unfortunate reality of this ansatz is that the equations defining the $a_j's$ for a more general polynomial Hamiltonian will often result in solutions with exponential growth properties. Combine this with an unbounded domain, say $\Lambda = \mathbb{C}^n$, and this will require that if the generalized Lagrangian state ansatz is the kernel of an operator then it will have to be thought of in some weak sense rather than as a smooth function, as we know the kernel of the propagator is in Bargmann space. Indeed, since Bargmann space has no compactly supported functions, defining what sort of weak sense one would need to think of the ansatz fulfilling would be complicated. Further, the domain of the operator associated to this kernel will not be all of Bargmann space. Modifications or more assumptions (or both) will be necessary in order to establish a rigorous and useful approximation. One such modification, motivated by the notion of localizing in phase space, is proposed next.

6.1.4 Approximating the Kernel: Rigorous Estimates

The generalized Lagrangian state ansatz proposed above is a globally defined object on all of phase space $\Lambda = \mathbb{C}^n$. We have seen a situation, the resonant harmonic oscillator, where this global object is precisely the correct kernel of the propagator. In more complex situations, classical Hamiltonians with higher order terms, the global nature of the ansatz will lead to problems due to the nature of the transport equations defining the expansion coefficients of the asymptotic expansion of the amplitude a. There are several possible avenues to address such a problem. For instance, one possible modification is to simply restrict the types of Hamiltonians considered in order to circumvent the growth properties in these equations. The case of the resonant harmonic oscillator shows this is possible. Indeed some authors have used this notion on similar problems (see [35]) and [40], but the form of the transport equations suggests that such a process will require a restriction to Hamiltonians that are subquadratic. While this may avoid mathematical difficulties, this is not a physically meaningful modification, nor is it particularly useful for application.

The modification pursued here will be motivated by the physical notion that the dynamics of semiclassical quantum mechanical system (recall from the introduction that, modulo a lot of details, this essentially means a quantum system exhibiting dynamics that are in some sense highly correlated to the dynamics of an analogous classical system) should be localized in phase space in the sense that if we begin with an initial condition that is well localized in a region of phase space Ω, then semiclassically its quantum evolution should yield data that is well localized in the region of phase space given by the Hamilton flow of Ω.

Consider the following notion of localization that includes both a modification of the ansatz as well as focusing on certain states in Bargmann space. Given a polynomial Hamiltonian H with E a regular value of H, define

$$\Omega_E \equiv \left\{ \mathbf{z} \in \mathbb{C}^n \middle| H(\mathbf{z}, \overline{\mathbf{z}}) \leq E \right\}. \tag{6.36}$$

Note that because H is preserved by its flow, we have that $\phi_t(\Omega_E) = \Omega_E$ where ϕ_t is the flow of H. Also, Ω_E is compact.

Restricting the Ansatz

Let's begin the process of arriving at a restricted ansatz by first considering $\Lambda = \Omega_E$ and the Lagrangian state

$$S_t^E(\mathbf{z},\overline{\xi}) = \int_{\Omega_E} a(\chi,t;\hbar) e^{\frac{i}{\hbar}f(\chi,t)} e^{\frac{\overline{\phi_t(\chi)}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} e^{-\frac{\phi_t(\chi)\cdot\overline{\phi_t(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} . e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi}. \tag{6.37}$$

Lemma, III.20, tells us that this Lagrangian state will semiclassically concentrate inside Ω_E. Note that if $\Lambda = \Omega_E$, then $\mathbf{w}(\Omega_E)$ will be a Lagrangian submanifold of the graph of the flow of H with a boundary. This differs from our previous work in that we will need to consider the implication of $\partial\Lambda \neq \emptyset$. If we consider defining an operator whose kernel is (or at least very nearly) S_t^E then we will see that the concentration of S_t^E will carry over to the behavior of the operator; and this concentration will be one ingredient in our recipe of localization. Indeed, there is something very physically intuitive about choosing to localize in phase space by choosing an energy threshold.

Now, for reasons that will be made clear later, when we work on operator estimates, we will still require that our ansatz satisfy the condition that at $t = 0$ it is equal to the reproducing kernel. Since

$$S_0^E(\mathbf{z},\overline{\xi}) = \int_{\Omega_E} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi}, \tag{6.38}$$

we cannot choose S_t^E alone as the ansatz and still satisfy this initial condition. Instead we need to add to S_t^E an initial term that is time independent, but which will give the desired result at $t = 0$. The term we need to add, denoted $T_0(\mathbf{z},\overline{\xi})$ is

$$T_0(\mathbf{z},\overline{\xi}) = \int_{\mathbb{C}^n\setminus\Omega_E} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi}. \tag{6.39}$$

The operator whose kernel is given by T_0 is the Toeplitz operator on Bargmann space with symbol $\eta_{\mathbb{C}^n\setminus\Omega_E}$ (the characteristic function on $\mathbb{C}^n\setminus\Omega_E$), denoted $\widehat{T}_{\mathbb{C}^n\setminus\Omega_E}$, since for $\psi(\xi) = \widetilde{\psi}(\xi)e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} \in \mathcal{B}_1(\mathbb{C}^n)$ we have

$$\begin{aligned}
\left(\widehat{T}_{\mathbb{C}^n\setminus\Omega_E}\psi\right)(\mathbf{z}) &= \int_{\mathbb{C}^n} T_0(\mathbf{z},\overline{\xi})\psi(\xi)d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n\setminus\Omega_E} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} \psi(\xi)d\chi d\overline{\chi} d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n\setminus\Omega_E} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \langle\varphi_\chi,\widetilde{\psi}\rangle_{\mathcal{B}_2} d\chi d\overline{\chi} \\
&= \int_{\mathbb{C}^n} e^{\frac{\overline{\chi}\cdot\mathbf{z}}{\hbar}} \eta_{\mathbb{C}^n\setminus\Omega_E}\psi(\chi) e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} d\chi d\overline{\chi} \\
&= \left(\Pi_{\mathcal{B}_1}(\eta_{\mathbb{C}^n\setminus\Omega_E}\psi)\right)(\mathbf{z}),
\end{aligned}$$

where $\Pi_{\mathcal{B}_1}$ denotes the orthogonal projector onto Bargmann space. If we denote by $\widehat{U}_E(t)$ the operator defined by the kernel S_t^E, then if $\widehat{V}_E(t)$ is the operator defined by the kernel of $S_t^E + T_0$ we have $\widehat{V}_E(t) = \widehat{U}_E(t) + \widehat{T}_{\mathbb{C}^n\setminus\Omega_E}$.

The operator $\widehat{V}_E(t)$ will be our approximation to the propagator when we restrict their action to specific types of states. The goal of the next subsection will be to derive an operator estimate comparing $\widehat{V}_E(t)$ to the exact propagator when acting on certain states of interest. Before delving into that work, we first need to understand how the operators $\frac{\partial}{\partial t}$, $\widehat{Z}_j$, and $\widehat{Z}_j^*$ act on the kernels S_t^E and T_0.

First consider how these operator will act on S_t^E. We can adapt the rules derived in (IV.7) to the case when $\partial\Lambda \neq \emptyset$ by following the proof and modifying it appropriately. Consider for a generalized Lagrangian state of the form (6.7):

$$\begin{aligned}
\widehat{Z}_j^*\Big|\Lambda, a\Big\rangle &= \hbar\frac{\partial}{\partial z_j}\int_\Lambda a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})} e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\mathbf{x} \\
&= \int_\Lambda a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})}\hbar\frac{\partial}{\partial z_j}\left(e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}}\right) e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\mathbf{x} \\
&= \int_\Lambda a(\mathbf{x};\hbar)\overline{w}_j(\mathbf{x})e^{\frac{i}{\hbar}f(\mathbf{x})} e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\mathbf{x} \\
&= \Big|\Lambda, \overline{w}_j a\Big\rangle.
\end{aligned}$$

Thus the behavior of $\widehat{Z}_j^*$ on a Lagrangian state is independent of the nature of the boundary:

$$\widehat{Z}_j^*\Big|\Lambda, a\Big\rangle \tag{6.40}$$

is not dependent on the existence of a boundary of the Lagrangian.

Recall that the vector field $\overline{R_j}$ (4.11) defined by the expression

$$\overline{R_j} = \sum_{l=1}^{n} \left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} \frac{\partial}{\partial x_l},$$

has the property that

$$\hbar \overline{R_j}(e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}) = z_j.$$

With this in mind we'll adapt the work in (IV.7) concerning the action of $\widehat{Z}_j$. For convenience define $V(\mathbf{x};\hbar) \equiv a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})}\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1} e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}}$. Then we have for a generalized state of the form (6.7)

$$\begin{aligned}
\widehat{Z}_j\Big|\Lambda, a\Big\rangle &= z_j \int_\Lambda a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})}e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}\,d\mathbf{x} \\
&= \int_\Lambda a(\mathbf{x};\hbar)e^{\frac{i}{\hbar}f(\mathbf{x})}\overline{R_j}(e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}})e^{-\frac{\mathbf{w}(\mathbf{x})\cdot\overline{\mathbf{w}(\mathbf{x})}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}\,d\mathbf{x} \\
&= \sum_{l=1}^{n} \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda V(\mathbf{x};\hbar)\frac{\partial}{\partial x_l}\left(e^{\frac{\overline{w(\mathbf{x},t)}\cdot\mathbf{z}}{\hbar}}\right)d\mathbf{x} \\
&= \sum_{l=1}^{n} \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda \frac{\partial}{\partial x_l}\left(V(\mathbf{x};\hbar)e^{\frac{\overline{w(\mathbf{x},t)}\cdot\mathbf{z}}{\hbar}}\right)d\mathbf{x} - \sum_{l=1}^{n} \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda \frac{\partial}{\partial x_l}\left(V(\mathbf{x};\hbar)\right)e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}}\,d\mathbf{x}.
\end{aligned}$$

Simplify we get

$$\widehat{Z}_j\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_j a\Big\rangle + \hbar\Big|\Lambda, L_j(a)\Big\rangle + \sum_{l=1}^{n} \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda \frac{\partial}{\partial x_l}\left(V(\mathbf{x};\hbar)e^{\frac{\overline{w(\mathbf{x},t)}\cdot\mathbf{z}}{\hbar}}\right) d\mathbf{x},$$

for L_j defined in (6.10). Taking the sum inside the integral and using Stokes Theorem for the last term we get

$$\begin{aligned}\sum_{l=1}^{n} \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda \frac{\partial}{\partial x_l}\left(V(\mathbf{x};\hbar)e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}}\right) d\mathbf{x} &= \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_\Lambda \sum_{l=1}^{n} \frac{\partial}{\partial x_l}\left(V(\mathbf{x};\hbar)e^{\frac{\overline{w(\mathbf{x})}\cdot\mathbf{z}}{\hbar}}\right) d\mathbf{x} \\ &= \hbar e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} \int_{\partial\Lambda} V(\iota(\mathbf{v});\hbar)e^{\frac{\overline{w(\iota(\mathbf{v}))}\cdot\mathbf{z}}{\hbar}} d\mathbf{v} \\ &= \hbar\Big|\partial\Lambda, a\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\circ\iota\Big\rangle,\end{aligned}$$

where $\iota : \partial\Lambda \longrightarrow \Lambda$ is the inclusion map. Thus we have the rule that if $\partial\Lambda \neq \emptyset$, then

$$\widehat{Z}_j\Big|\Lambda, a\Big\rangle = \Big|\Lambda, w_j a\Big\rangle + \hbar\Big|\Lambda, L_j(a)\Big\rangle + \hbar\Big|\partial\Lambda, a\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\circ\iota\Big\rangle. \tag{6.41}$$

With this rule in place we can combine the proof of (5.18) to understand the action of $\frac{\partial}{\partial t}$ on a time dependent generalized Lagrangian state where the Lagrangian has a boundary

$$\begin{aligned}\frac{\partial}{\partial t}|\Lambda, a\rangle &= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot\overline{\mathbf{w}} + \mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right)a\right\rangle + \frac{1}{\hbar}\sum_{j=1}^{n} z_j\left|\Lambda, \frac{\partial \overline{w}_j}{\partial t}a\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot\overline{\mathbf{w}} + \mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right)a\right\rangle + \frac{1}{\hbar}\sum_{j=1}^{n} \widehat{Z}_j\left|\Lambda, \frac{\partial \overline{w}_j}{\partial t}a\right\rangle \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{i}{2}\left(\frac{\partial \mathbf{w}}{\partial t}\cdot\overline{\mathbf{w}} + \mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t}\right)\right)a\right\rangle + \frac{1}{\hbar}\sum_{j=1}^{n}\left(\left|\Lambda, w_j\frac{\partial \overline{w}_j}{\partial t}a\right\rangle\right. \\
&\quad \left. + \hbar\left|\Lambda, L_j\left(\frac{\partial \overline{w}_j}{\partial t}a\right)\right\rangle + \hbar\left|\partial\Lambda, a\frac{\partial \overline{w}_j}{\partial t}\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\circ\iota\right\rangle\right) \\
&= \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}}\cdot\frac{\partial \mathbf{w}}{\partial t}\right)\right)a\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t}a\right)\right\rangle \\
&\quad + \sum_{j=1}^{n}\left|\partial\Lambda, a\frac{\partial \overline{w}_j}{\partial t}\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\circ\iota\right\rangle.\end{aligned}$$

More succinctly we have

(6.42)

$$\frac{\partial}{\partial t}|\Lambda, a\rangle = \left|\Lambda, \frac{\partial a}{\partial t}\right\rangle + \frac{i}{\hbar}\left|\Lambda, \left(\frac{\partial f}{\partial t} + \frac{1}{2i}\left(\mathbf{w}\cdot\frac{\partial \overline{\mathbf{w}}}{\partial t} - \overline{\mathbf{w}}\cdot\frac{\partial \mathbf{w}}{\partial t}\right)\right)a\right\rangle + \left|\Lambda, \sum_{j=1}^{n} L_j\left(\frac{\partial \overline{w}_j}{\partial t}a\right)\right\rangle$$

(6.43)

$$+ \sum_{j=1}^{n}\left|\partial\Lambda, a\frac{\partial \overline{w}_j}{\partial t}\left(\frac{\partial \overline{w}_j}{\partial x_l}\right)^{-1}\circ\iota\right\rangle.$$

Remark VI.4. Let's distill what will be needed later concerning these new rules. By the definition of a manifold with boundary $\partial(\partial\Lambda) = \emptyset$. Thus, if we want to iterate the rules above to generate algebraic rules for generalized Lagrangian states with $\partial\Lambda \neq \emptyset$ under the action of the ring of pseudodifferential operators generated by $\widehat{Z}_j$ and $\widehat{Z}_j^*$ (i.e. polynomial Hamiltonians), then we will get the same results that we got in chapter IV when $\partial\Lambda = \emptyset$ plus a finite number of terms of the general form $|\partial\Lambda, s\rangle$ where s is a smooth function that is polynomial in $\hbar$. Since there will be a finite number of such terms, we can combine them all into one term of the same general type, which we denote by $|\partial\Lambda, s\rangle$. This un-detailed characterization of the contribution from $\partial\Lambda$ is all we are going to need.

The Estimate

The goal here will be to characterize how well the operator $\widehat{V}_E(t)$ approximates $\widehat{U}(t)$ when we restrict their action to a specific class of quantum states that we will call semiclassically localized. The first step toward this characterization is to define the specific quantum states of interest.

Definition VI.5. A state $\psi(\mathbf{z};\hbar) \in L^2(\mathbb{C}^n)$ such that for each fixed $\hbar$, $\psi \in \mathcal{B}_1(\mathbb{C}^n)$, is *semiclassically localized* inside the closed ball of radius R if there exists positive constants C, δ, γ, $\hbar_0$ such that for all $\mathbf{z} \in \mathbb{C}^n$ with $|\mathbf{z}| > R$ we have

$$|\psi(\mathbf{z};\hbar)| \leq Ce^{-\frac{\delta}{\hbar}(|\mathbf{z}|-R)^\gamma}, \tag{6.44}$$

for all $\hbar \in (0, \hbar_0]$.

Keeping with the theme of developing a notion of localization of quantum phenomenon in phase space for semiclassical quantum systems we should be interested not only in localizing our evolution operator, but also localizing the type of states on which it acts. States that are semiclassically localized are localized inside of a compact subset of phase space in the sense that they decay rapidly outside such a set. The states above have the property that outside a ball (which is equivalent to outside a compact set) they pointwise decay at least exponentially, and that decay happens 'faster' as $\hbar \longrightarrow 0$. Recall that the nature of Bargmann space, as a holomorphic function space, is that it contains no elements of compact support; which makes sense from a quantum intuitive perspective; the uncertainty principle tells us that we cannot strictly localize a quantum state in phase space. Semiclassically localized states are perhaps as close to compactly supported as we can hope for when working in Bargmann space.

Consider some examples and some characterizations of such states:

- Coherent States: If $\varphi_{\mathbf{v}}(\mathbf{z}) = e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} e^{-\frac{\overline{\mathbf{v}}\cdot\mathbf{v}}{2\hbar}} e^{-\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{2\hbar}}$, then for any $R > 0$ if $\mathbf{v} \in B_R(0)$ and $|\mathbf{z}| > R$,

$$|\varphi_{\mathbf{v}}(\mathbf{z})| = e^{-\frac{|\mathbf{z}-\mathbf{v}|^2}{2\hbar}} \leq e^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}}.$$

  Thus, such a coherent states, $\varphi_{\mathbf{v}}$ are semiclassically localized within the ball of radius R.

- Hermite Functions: If $H_\alpha(\mathbf{z}) = \frac{1}{\sqrt{\alpha!\hbar^{|\alpha|}}}\mathbf{z}^\alpha e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}$, then for any $R > 0$ if $\mathbf{v} \in B_R(0)$ and $|\mathbf{z}| > R$,

$$|H_\alpha(\mathbf{z})| = \frac{1}{\sqrt{\alpha!\hbar^{|\alpha|}}}|\mathbf{z}|^\alpha e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{4\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{4\hbar}} \leq Ce^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{4\hbar}} \leq Ce^{-\frac{(|\mathbf{z}|-R)^2}{4\hbar}}$$

  for $\hbar \in (0, \hbar_0]$ for some $\hbar_0 > 0$ since $\frac{1}{\sqrt{\alpha!\hbar^{|\alpha|}}}|\mathbf{z}|^\alpha e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{4\hbar}}$ is a bounded quantity where $\mathbf{z} \in \mathbb{C}^n$ and $\hbar > 0$ are allowed to take on any values in these domains. Indeed this argument can be generalized to any state of the form $P(\mathbf{z})e^{-\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{2\hbar}}$ where P is a polynomial.

- Suppose that Λ is a compact n-dimensional manifold. Consider the generalized Lagrangian state

$$|\Lambda, a\rangle(\mathbf{z}) = \int_\Lambda ae^{\frac{i}{\hbar}f}K(\mathbf{z},\mathbf{w})|d\mathbf{w}| = \int_\Lambda ae^{\frac{i}{\hbar}f}e^{\frac{\overline{\mathbf{w}}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\mathbf{w}\cdot\overline{\mathbf{w}}}{2\hbar}}|d\mathbf{w}|,$$

$$\implies \Big||\Lambda, a\rangle(\mathbf{z})\Big| \leq \int_\Lambda |a|e^{-\frac{|\mathbf{z}-\mathbf{w}|^2}{2\hbar}}|d\mathbf{w}|.$$

Since Λ is compact we know that $\mathbf{w}(\Lambda)$ is compact. Thus there exists $\widetilde{R} > 0$ (say $\widetilde{R} = \sup_{\mathbf{x}\in\Lambda} |\mathbf{w}(\mathbf{x})|$) such that for all $R > \widetilde{R}$ if $|\mathbf{z}| > R$ then $|\mathbf{z} - \mathbf{w}(\mathbf{x})| \leq |\mathbf{z}| - R$ for all $\mathbf{x} \in \Lambda$.

Now, if $|\mathbf{z}| > R$ then

$$e^{-\frac{|\mathbf{z}-\mathbf{w}|^2}{2\hbar}} \leq e^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}}$$

$$\therefore \left| |\Lambda, a\rangle(\mathbf{z}) \right| \leq \int_\Lambda |a| e^{-\frac{|\mathbf{z}-\mathbf{w}|^2}{2\hbar}} |d\mathbf{w}| \leq \left(\int_\Lambda |a| \cdot |d\mathbf{w}| \right) e^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}},$$

again for $|\mathbf{z}| > R$.

The above equation is almost enough to show that these generalized states are semiclassically localized. The final piece will be to see that even though $\int_\Lambda |a| \cdot |d\mathbf{w}|$ is $\hbar$-dependent (since a is $\hbar$-dependent), that it can be bounded above by a constant that is independent of $\hbar$. To this end, since Λ is compact the definition of an amplitude (III.11) tells us that for every N there exists a constant C_N and $\hbar_0$ such that for all $\mathbf{x} \in \Lambda$

$$\left| a(\mathbf{x}; \hbar) - \sum_{j=0}^{N-1} \hbar^j a_j(\mathbf{x}) \right| \leq C_N \hbar^N,$$

for all $\hbar \in (0, \hbar_0]$. For $j \geq 1$

$$\begin{aligned}
|a| &= \left| a - \sum_{j=0}^{N-1} \hbar^j a_j(\mathbf{x}) + \sum_{j=0}^{N-1} \hbar^j a_j(\mathbf{x}) \right| \\
&\leq \left| a - \sum_{j=0}^{N-1} \hbar^j a_j(\mathbf{x}) \right| + \left| \sum_{j=0}^{N-1} \hbar^j a_j(\mathbf{x}) \right| \\
&\leq C_N \hbar^N + \sum_{j=0}^{N-1} \hbar^j |a_j(\mathbf{x})| \\
&\leq C_N \hbar^N + \sum_{j=0}^{N-1} \hbar^j \widetilde{C}_j,
\end{aligned}$$

where $\widetilde{C}_j = \sup_{\mathbf{x}\in\Lambda} |a_j(\mathbf{x})|$ (which exists because Λ is compact). Thus

$$|a| \leq j\widetilde{C}$$

where $\widetilde{C} = \max\{\widetilde{C}_1, \ldots, \widetilde{C}_{N-1}, C_N\}$. Let $C \equiv \int_\Lambda j\widetilde{C}|d\mathbf{w}|$, a constant that is independent of $\hbar$. Thus

$$|\Lambda, a\rangle(\mathbf{z}) \leq \left(\int_\Lambda |a| \cdot |d\mathbf{w}| \right) e^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}} \leq Ce^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}},$$

for all $\mathbf{z}$ such that $|\mathbf{z}| > R$.

- If $\psi = \Pi_{\mathcal{B}_1}(g \cdot \varrho)$ where $f \in \mathcal{B}_1(\mathbb{C}^n)$ such that $\| \varrho \|_{L^2(\mathbb{C}^n)} \leq C$ for a constant C and $g \in C_0^\infty(\mathbb{C}^n)$, then if $R \equiv \sup_{\mathbf{z}\in \mathrm{supp}(g)} |\mathbf{z}|$, and if $|\mathbf{z}| > R$, then

$$
\begin{aligned}
|\psi(\mathbf{z})| &= |\Pi_{\mathcal{B}_1}(g \cdot \varrho)(\mathbf{z})| \\
&= \left| e^{-\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{2\hbar}} \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} g(\mathbf{v})\varrho(\mathbf{v}) e^{-\frac{\overline{\mathbf{v}}\cdot\mathbf{v}}{2\hbar}} d\mathbf{v}d\overline{\mathbf{v}} \right| \\
&= \int_{\mathbb{C}^n} e^{-\frac{|\mathbf{v}-\mathbf{z}|^2}{2\hbar}} |g(\mathbf{v})||\varrho(\mathbf{v})| d\mathbf{v}d\overline{\mathbf{v}} \\
&= \sup_{\mathbf{v}\in \mathrm{supp}(g)} |g(\mathbf{v})| \int_{\mathrm{supp}(g)} e^{-\frac{|\mathbf{v}-\mathbf{z}|^2}{2\hbar}} |\varrho(\mathbf{v})| d\mathbf{v}d\overline{\mathbf{v}} \\
&= \sup_{\mathbf{v}\in \mathrm{supp}(g)} |g(\mathbf{v})| \int_{\mathrm{supp}(g)} e^{-\frac{|\mathbf{v}-\mathbf{z}|^2}{2\hbar}} \parallel \varrho \parallel_{L^2(\mathbb{C}^n)} d\mathbf{v}d\overline{\mathbf{v}} \\
&= C\left(\sup_{\mathbf{v}\in \mathrm{supp}(g)} |g(\mathbf{v})| \right) \int_{\mathrm{supp}(g)} e^{-\frac{|\mathbf{v}-\mathbf{z}|^2}{2\hbar}} d\mathbf{v}d\overline{\mathbf{v}}
\end{aligned}
$$

where the reproducing property tell us that $|\varrho(\mathbf{v})| \leq \parallel \varrho \parallel_{L^2(\mathbb{C}^n)} \leq C$. Now, if $\mathbf{v} \in \mathrm{supp}(g)$ and $\mathbf{z} \in \mathbb{C}^n \backslash B_R(0)$ then $|\mathbf{v} - \mathbf{z}| > |\mathbf{z}| - R$, thus we have

$$
|\psi(\mathbf{z})| = C\left(\sup_{\mathbf{v}\in \mathrm{supp}(g)} |g(\mathbf{v})| \right) \lambda(\mathrm{supp}(g)) e^{-\frac{(|\mathbf{z}|-R)^2}{2\hbar}}
$$

where λ denotes Lebesgue measure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

- Suppose $F(\mathbf{z}) \in \mathcal{B}_1(\mathbb{C}^n)$ such that $F(\mathbf{z}) = f(\mathbf{z}) e^{-\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{2\hbar}}$ and for some γ' it is true that $|f(\mathbf{z})| \leq C e^{\frac{|\mathbf{z}|^{\gamma'}}{\hbar}}$. Then

$$
\begin{aligned}
|F(\mathbf{z})| &= |f(\mathbf{z})| e^{-\frac{|\mathbf{z}|^2}{2\hbar}} \\
&\leq C e^{\frac{|\mathbf{z}|^{\gamma'}}{\hbar}} e^{-\frac{|\mathbf{z}|^2}{2\hbar}} \\
&= C e^{\frac{|\mathbf{z}|^{\gamma'}}{\hbar}} e^{-\frac{|\mathbf{z}|^2}{4\hbar}} e^{-\frac{|\mathbf{z}|^2}{4\hbar}}.
\end{aligned}
$$

As long as $\gamma' < 2$ the term $e^{\frac{|\mathbf{z}|^{\gamma'}}{\hbar}} e^{-\frac{|\mathbf{z}|^2}{4\hbar}}$ will be bounded. Then for some $\hbar \in (0, \hbar_0]$ it is the case that for any $R > 0$ we can rather crudely estimate F

$$|F(\mathbf{z})| \leq C' e^{-\frac{|\mathbf{z}|^2}{4\hbar}} \leq C' e^{-\frac{(|\mathbf{z}|-R)^2}{4\hbar}}.$$

- Suppose $F(\mathbf{z}) \in \mathcal{B}_1(\mathbb{C}^n)$ such that $F(\mathbf{z}) = f(\mathbf{z}) e^{-\frac{\overline{\mathbf{z}}\cdot\mathbf{z}}{2\hbar}}$ and for some δ it is true that $|f(\mathbf{z})| \leq C e^{\frac{\delta|\mathbf{z}|^2}{\hbar}}$. Then

$$|F(\mathbf{z})| = |f(\mathbf{z})| e^{-\frac{|\mathbf{z}|^2}{2\hbar}} \leq C e^{\frac{(\delta-1/2)|\mathbf{z}|^2}{\hbar}} \leq C e^{\frac{(\delta-1/2)(|\mathbf{z}|-R)^2}{\hbar}},$$

  for any $R > 0$. Therefore, if $\delta < \frac{1}{2}$ then such states are semiclassically localized inside any ball of radius R.

It also follows directly from the definition that any finite linear combination of semiclassically localized states will also be semiclassically localized. The linear combination will be localized within the largest ball of the terms in the sum, and the decay coefficient γ being the smallest of all of the states in the linear combination. Combining this fact with the fact that the Hermite functions (monomials) are of this type and we see that the collection of all semiclassically localized states are, in a sense, dense in Bargmann space (though not uniformly in the various parameters used to define them).

With the quantum states of interest presented, the next step will be to estimate how well $\widehat{V}_E(t)$ approximates $\widehat{U}(t)$ when we consider their action on appropriately chosen semiclassically localized states. The basic setup of localization will involve:

1. Choose a region of phase space Ω, such that $\overline{\Omega}$ is bounded. This is the region of interest for the classical dynamics.

2. Next, choose an energy threshold, E, such that $\Omega \subset\subset \Omega_E$.

3. Note this implies that there exists an $\epsilon > 0$ such that

$$\Omega + \epsilon \equiv \left\{ \mathbf{z} \in \mathbb{C}^n \middle| \exists\, \mathbf{v} \in \Omega \text{ such that } |\mathbf{z} - \mathbf{v}| < \epsilon \right\} \subset\subset \Omega. \tag{6.45}$$

4. Choose to approximate the kernel of the propagator with a generalized Lagrangian state over Ω_E and consider this approximations behavior on a semi-classically localized state which is localized inside Ω.

Before implementing this procedure we should note that the standard tool used to analyze the effectiveness of an approximation of operators of the type defined as the time t solution of a partial differential equation is Duhamel's formula for operators.

Theorem: (The Duhamel Formula for the Schrodinger Equation)

The (exact) propagator satisfies the operator equation

$$i\hbar \frac{\partial}{\partial t} \widehat{U}(t) = \widehat{H}\widehat{U}(t)$$

subject to the initial condition $\widehat{U}(0) = I$.

Suppose $\widehat{V}(t)$ is an approximation to $\widehat{U}(t)$ that satisfies

$$i\hbar \frac{\partial}{\partial t} \widehat{V}(t) = \widehat{H}\widehat{V}(t) + \widehat{R}(t)$$

subject to the initial condition $\widehat{V}(0) = I$. Then the difference between the approximation and the exact propagator is

$$\widehat{V}(t) - \widehat{U}(t) = -\frac{i}{\hbar} \int_0^t \widehat{U}(t-s)\widehat{R}(s)ds.$$

With this we can gauge the difference between our approximation $\widehat{V}_E(t)$ and the exact propagator $\widehat{U}(t)$. Indeed, it is because we want to apply Duhamel's formula to our approximation that we added the Toeplitz contribution to the initial condition of the approximation in order to guarantee that initially $\widehat{V}_E(0) = I$. We will need to understand the error term $\widehat{R}(t)$ corresponding to $\widehat{V}_E(t)$, which by definition is $\widehat{R}(t) = i\hbar\frac{\partial}{\partial t}\widehat{V}_E(t) - \widehat{H}\widehat{V}_E(t)$. Now, since $\widehat{V}_E(t) = \widehat{U}_E(t) + \widehat{T}_{\mathbb{C}^n\backslash\Omega_E}$ we have that

$$\widehat{R}(t) = i\hbar\frac{\partial}{\partial t}\widehat{V}_E(t) - \widehat{H}\widehat{V}_E(t) = i\hbar\frac{\partial}{\partial t}\widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) - \widehat{H}\widehat{T}_{\mathbb{C}^n\backslash\Omega_E}.$$

Note that the time derivative of the Toeplitz contribution is zero. So, in order to characterize the error in the approximation scheme we will need to understand the nature of the terms $i\hbar\frac{\partial}{\partial t}\widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t)$ and $\widehat{H}\widehat{T}_{\mathbb{C}^n\backslash\Omega_E}$. The focus of the effort will be to understand the nature of these terms acting on semiclassically localized states that are localized inside a set Ω such that $\Omega \subset\subset \Omega_E$.

Theorem VI.6. *Suppose that $H = H(\mathbf{z}, \overline{\mathbf{z}})$ is a polynomial Hamiltonian on $\mathbb{C}^n$ and let $\widehat{H}$ denote its Weyl quantization. Suppose that E is a regular value of H, and let*

$$\Omega_E \equiv \{\mathbf{z} \in \mathbb{C}^n | H(\mathbf{z}, \overline{\mathbf{z}}) \leq E\}.$$

For each amplitude function $a : \Omega_E \longrightarrow \mathbb{C}$ let $\widehat{U}^a_E(t)$ denote the operator whose kernel is given by $|\Omega_E, a\rangle$, defined with respect to the time dependent embedding $\mathbf{w} : \Omega_E \longrightarrow \mathbb{C}^n \times \overline{\mathbb{C}^n}$ given by $\mathbf{w}(\mathbf{x}, t) = (\phi_t(\mathbf{x}), \overline{\mathbf{x}})$, where ϕ_t is the Hamilton flow of H. Let $\widehat{T}_{\mathbb{C}^n\backslash\Omega_E}$ denote the Toeplitz operator on Bargmann space with symbol given by the characteristic function on $\mathbb{C}^n\backslash\Omega_E$. Let $\widehat{V}^\alpha_E(t) \equiv \widehat{U}^\alpha_E(t) + \widehat{T}_{\mathbb{C}^n\backslash\Omega_E}$. For a particular choice of a (that is well-defined modulo a term of $O(\hbar^\infty)$) for a fixed $T > 0$ for every ψ that is semiclassically localized in the ball of radius R such that for some $\Omega \subset \mathbb{C}^n$ we have $B_R(0) \subset \Omega \subset\subset \Omega_E$ there exists an $\hbar_0 > 0$ such that for each N, for every $\epsilon > 0$, there exists C_N such that for all $\hbar \in (0, \hbar_0]$

$$\| (\widehat{V}_E^a(t) - \widehat{U}(t))\psi \|_{L^2(\mathbb{C}^n)} \leq C_N \hbar^N \left(\| \psi \|_{L^2(\mathbb{C}^n)} + \| \psi \|^{1-\epsilon}_{L^2(\mathbb{C}^n)} \right) \tag{6.46}$$

for all $|t| < T$.

Proof. Let ψ be semiclassically localized inside the ball of radius R such that $B_R(0) \subset\subset \Omega_E$. The procedure will be to systematically examine each term that contributes to

$$\widehat{R}(t)\psi = \left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi - \widehat{H}\widehat{T}_{\mathbb{C}^n \setminus \Omega_E} \psi. \tag{6.47}$$

Specifically, to use Duhamel's formula we will seek to estimate $\left\| \widehat{R}(t)\psi \right\|_{L^2(\mathbb{C}^n)}$. The triangle inequality gives us that $\left\| \widehat{R}(t)\psi \right\|_{L^2(\mathbb{C}^n)} \leq \left\| \left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi \right\|_{L^2(\mathbb{C}^n)} + \left\| \widehat{H}\widehat{T}_{\mathbb{C}^n \setminus \Omega_E} \psi \right\|_{L^2(\mathbb{C}^n)}$, and it's these norms that we will estimate.

Let's begin by finding an estimate for the term $\left\| \left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi \right\|_{L^2(\mathbb{C}^n)}$. With this task in mind, recall that by definition

$$\left(\widehat{U}_E(t)\psi \right)(\mathbf{z}) = \int_{\mathbb{C}^n} S_t^E(\mathbf{z}, \overline{\xi}) \psi(\xi) d\xi d\overline{\xi},$$

for the generalized Lagrangian state with boundary

$$S_t^E(\mathbf{z}, \overline{\xi}) = \int_{\Omega_E} a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi,t)} e^{\frac{\overline{\phi_t(\chi)} \cdot \mathbf{z}}{\hbar}} e^{\frac{\overline{\xi} \cdot \chi}{\hbar}} e^{-\frac{\chi \cdot \overline{\chi}}{2\hbar}} e^{-\frac{\phi_t(\chi) \cdot \overline{\phi_t(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z} \cdot \overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi \cdot \overline{\xi}}{2\hbar}} d\chi d\overline{\chi}. \tag{6.48}$$

Understanding the term $\left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi$ amounts to characterizing

$$\int_{\mathbb{C}^n} \left(i\hbar \frac{\partial}{\partial t} S_t^E(\mathbf{z}, \overline{\xi}) \right) \psi(\xi) d\xi d\overline{\xi} - \int_{\mathbb{C}^n} \left(\widehat{H} S_t^E(\mathbf{z}, \overline{\xi}) \right) \psi(\xi) d\xi d\overline{\xi}.$$

or

$$\int_{\mathbb{C}^n} \left(i\hbar \frac{\partial}{\partial t} S_t^E(\mathbf{z},\bar{\xi}) - \widehat{H} S_t^E(\mathbf{z},\bar{\xi}) \right) \psi(\xi) d\xi d\bar{\xi}. \tag{6.49}$$

Understanding the part of the integrand $i\hbar \frac{\partial}{\partial t} S_t^E(\mathbf{z},\bar{\xi}) - \widehat{H} S_t^E(\mathbf{z},\bar{\xi})$ is a familiar situation. In calculating this term there will be contributions of two types. The first type of contribution from this part of the integrand will be the generalized Lagrangian state for which $\Lambda = \Omega_E$, and the second type will be a generalized Lagrangian state for which $\Lambda = \partial\Omega_E$.

Let's denote the part of 6.49 that is a generalized Lagrangian state with $\Lambda = \Omega_E$ as $\left|\Omega_E, s_1\right\rangle$, and the part of 6.49 that is a generalized Lagrangian state with $\Lambda = \partial\Omega_E$ as $\left|\partial\Omega_E, s_2\right\rangle$. Thus,

$$\begin{aligned} &\int_{\mathbb{C}^n} \left(i\hbar \frac{\partial}{\partial t} S_t^E(\mathbf{z},\bar{\xi}) - \widehat{H} S_t^E(\mathbf{z},\bar{\xi}) \right) \psi(\xi) d\xi d\bar{\xi} = \int_{\mathbb{C}^n} \left|\Omega_E, s_1\right\rangle(\mathbf{z},\bar{\xi}) \psi(\xi) d\xi d\bar{\xi} && (6.50) \\ &\qquad + \int_{\mathbb{C}^n} \left|\partial\Omega_E, s_2\right\rangle(\mathbf{z},\bar{\xi}) \psi(\xi) d\xi d\bar{\xi}. && (6.51) \end{aligned}$$

Let's first focus on the term involving $\left|\Omega_E, s_1\right\rangle$. Indeed this term is what the entire expression of (6.49) would be equal to in the case that $\partial\Omega_E = \emptyset$. Thus, as we have seen in our main result, if we choose the $a_j's$ in the asymptotic expansion of the amplitude a in S_t^E to satisfy the transport equations 6.24, 6.25, 6.27, then the proof of (V.4) gives us that $\left|\Omega_E, s_1\right\rangle$ will have norm $O(\hbar^\infty)$, and more specifically the amplitude s_1 will be $O(\hbar^\infty)$. It is this process of choosing the a_j's that determines an appropriate a from the statement of the theorem. For the purpose of approximating the propagator, the question is how these considerations influence the operator that is defined with respect to the kernel $\left|\Omega_E, s_1\right\rangle$, phrased another way, how large is the term

$$\Psi_1(\mathbf{z}) \equiv \int_{\mathbb{C}^n} \Big|\Omega_E, s_1\Big\rangle(\mathbf{z}, \overline{\xi})\psi(\xi)d\xi d\overline{\xi}$$

in a semiclassical sense. Consider

$$\begin{aligned}
\Psi_1(\mathbf{z}) &= \int_{\mathbb{C}^n} \Big|\Omega_E, s_1\Big\rangle(\mathbf{z}, \xi)\psi(\xi)d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n} \left(\int_{\Omega_E} s_1(\chi, t; \hbar) e^{\frac{i}{\hbar}f(\chi,t)} e^{\frac{\overline{\phi_t(\chi)}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} e^{-\frac{\phi_t(\chi)\cdot\overline{\phi_t(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\chi d\overline{\chi} \right) \psi(\xi) d\xi d\overline{\xi} \\
&= \int_{\Omega_E} s_1(\chi, t; \hbar) e^{\frac{i}{\hbar}f(\chi,t)} e^{\frac{\overline{\phi_t(\chi)}\cdot\mathbf{z}}{\hbar}} \left(e^{-\frac{\chi\cdot\overline{\chi}}{2\hbar}} \int_{\mathbb{C}^n} e^{\frac{\overline{\xi}\cdot\chi}{\hbar}} \psi(\xi) e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\xi d\overline{\xi} \right) e^{-\frac{\phi_t(\chi)\cdot\overline{\phi_t(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\xi d\overline{\xi} \\
&= \int_{\Omega_E} s_1(\chi, t; \hbar) e^{\frac{i}{\hbar}f(\chi,t)} e^{\frac{\overline{\phi_t(\chi)}\cdot\mathbf{z}}{\hbar}} \psi(\chi) e^{-\frac{\phi_t(\chi)\cdot\overline{\phi_t(\chi)}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\chi d\overline{\chi}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\int_{\mathbb{C}^n} \left|\Psi_1(\mathbf{z})\right|^2 d\mathbf{z}d\overline{\mathbf{z}} &= \int_{\mathbb{C}^n} \Psi_1(\mathbf{z})\overline{\Psi_1(\mathbf{z})}d\mathbf{z}d\overline{\mathbf{z}} \\
&= \int_{\mathbb{C}^n}\int_{\Omega_E}\int_{\Omega_E} s_1(\chi_1,t;\hbar)\overline{s_1(\chi_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\chi_1,t)-f(\chi_2,t))}e^{\frac{\overline{\phi_t(\chi_1)}\cdot\mathbf{z}}{\hbar}}e^{\frac{\phi_t(\chi_2)\cdot\overline{\mathbf{z}}}{\hbar}} \\
&\qquad \times \psi(\chi_1)\overline{\psi(\chi_2)}e^{-\frac{\phi_t(\chi_1)\cdot\overline{\phi_t(\chi_1)}}{2\hbar}}e^{-\frac{\phi_t(\chi_2)\cdot\overline{\phi_t(\chi_2)}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}}d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}d\mathbf{z}d\overline{\mathbf{z}} \\
&= \int_{\Omega_E}\int_{\Omega_E} s_1(\chi_1,t;\hbar)\overline{s_1(\chi_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\chi_1,t)-f(\chi_2,t))}\left\langle \varphi_{\phi_t(\chi_2)},\varphi_{\phi_t(\chi_1)}\right\rangle_{\mathcal{B}_2} \\
&\qquad \times e^{-\frac{\phi_t(\chi_1)\cdot\overline{\phi_t(\chi_1)}}{2\hbar}}e^{-\frac{\phi_t(\chi_2)\cdot\overline{\phi_t(\chi_2)}}{2\hbar}}\psi(\chi_1)\overline{\psi(\chi_2)}d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2} \\
&= \int_{\Omega_E}\int_{\Omega_E} s_1(\chi_1,t;\hbar)\overline{s_1(\chi_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\chi_1,t)-f(\chi_2,t))}e^{\frac{\overline{\phi_t(\chi_1)}\cdot\phi_t(\chi_2)}{\hbar}} \\
&\qquad \times e^{-\frac{\phi_t(\chi_1)\cdot\overline{\phi_t(\chi_1)}}{2\hbar}}e^{-\frac{\phi_t(\chi_2)\cdot\overline{\phi_t(\chi_2)}}{2\hbar}}\psi(\chi_1)\overline{\psi(\chi_2)}d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2} \\
&= \Bigg| \int_{\Omega_E}\int_{\Omega_E} s_1(\chi_1,t;\hbar)\overline{s_1(\chi_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\chi_1,t)-f(\chi_2,t))}e^{\frac{\overline{\phi_t(\chi_1)}\cdot\phi_t(\chi_2)}{\hbar}} \\
&\qquad \times e^{-\frac{\phi_t(\chi_1)\cdot\overline{\phi_t(\chi_1)}}{2\hbar}}e^{-\frac{\phi_t(\chi_2)\cdot\overline{\phi_t(\chi_2)}}{2\hbar}}\psi(\chi_1)\overline{\psi(\chi_2)}d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}\Bigg| \\
&\leq \int_{\Omega_E}\int_{\Omega_E} |s_1(\chi_1,t;\hbar)|\cdot|s_1(\chi_2,t;\hbar)|e^{-\frac{|\phi_t(\chi_1)-\phi_t(\chi_2)|}{2\hbar}} \\
&\qquad \times |\psi(\chi_1)|\cdot|\psi(\chi_2)|d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}
\end{aligned}
$$

Now, recall that elements in Bargmann space are point-wise bounded by their norm (II.5 and more specifically point number (5) in II.7), which implies that if $\psi \in \mathcal{B}_1(\mathbb{C}^n)$ then we have for any $\mathbf{v} \in \mathbb{C}^n$ the bound $|\psi(\mathbf{v})| \leq \|\psi\|_{L^2(\mathbb{C}^n)}$. Using this fact in the last line above we can conclude that

$$\int_{\mathbb{C}^n} \left|\Psi_1(\mathbf{z})\right|^2 d\mathbf{z}d\overline{\mathbf{z}} \leq \int_{\Omega_E}\int_{\Omega_E} |s_1(\chi_1,t;\hbar)|\cdot|s_1(\chi_2,t;\hbar)|e^{-\frac{|\phi_t(\chi_1)-\phi_t(\chi_2)|}{2\hbar}}$$
$$\times |\psi(\chi_1)|\cdot|\psi(\chi_2)|d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}$$
$$\leq \Big(\parallel \psi \parallel^2_{L^2(\mathbb{C}^n)} \Big) \int_{\Omega_E}\int_{\Omega_E} |s_1(\chi_1,t;\hbar)|\cdot|s_1(\chi_2,t;\hbar)|$$
$$\times\, e^{-\frac{|\phi_t(\chi_1)-\phi_t(\chi_2)|}{2\hbar}} d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}$$

Now, if we define

$$I_1(t,\hbar) \equiv \left(\int_{\Omega_E} |s_1(\chi_1,t;\hbar)|\cdot|s_1(\chi_2,t;\hbar)|e^{-\frac{|\phi_t(\chi_1)-\phi_t(\chi_2)|}{2\hbar}} d\chi_1 d\overline{\chi_1}d\chi_2 d\overline{\chi_2}\right)^{1/2}$$

Then we can say that since the t dependence arises through the flow ϕ_t, that I_1 smoothly depends on t. So, if the values of t are constrained to a bounded interval (say $[-T,T]$ for some $T>0$), then we can combine this with the fact that s_1 is $O(\hbar^\infty)$ and we have two 'copies' of s_1 in the integral to argue that $I_1 = O(\hbar^\infty)$ in the sense that for some $\hbar_0 > 0$ for every N there exists a constant C_N (independent of $t \in [-T,T]$ and $\hbar \in (0,\hbar_0]$) such that $I_1 \leq C_N \hbar^N$. Therefore the work above shows that

$$\|\Psi_1\|_{L^2(\mathbb{C}^n)} \leq I_1(t,\hbar) \parallel \psi \parallel_{L^2(\mathbb{C}^n)} . \tag{6.52}$$

Next, consider the term related to the generalized Lagrangian state over $\partial\Omega_E$. Define

$$\Psi_2(\mathbf{z}) \equiv \int_{\mathbb{C}^n} \Big|\partial\Omega_E, s_2\Big\rangle(\mathbf{z},\overline{\xi})\psi(\xi)d\xi d\overline{\xi}.$$

Recalling that $\iota : \partial\Lambda \longrightarrow \Lambda$ denotes the inclusion map, we have

$$\begin{aligned}
\Psi_2(\mathbf{z}) &= \int_{\mathbb{C}^n} \Big|\partial\Omega_E, s_2\Big\rangle(\mathbf{z},\xi)\psi(\xi)d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n} \left(\int_{\partial\Omega_E} s_2(\nu,t;\hbar) e^{\frac{i}{\hbar}f(\iota(\nu),t)} e^{\frac{\overline{\phi_t(\iota(\nu))}\cdot\mathbf{z}}{\hbar}} e^{\frac{\overline{\xi}\cdot\iota(\nu)}{\hbar}} e^{-\frac{\iota(\nu)\cdot\overline{\iota(\nu)}}{2\hbar}} e^{-\frac{\phi_t(\iota(\nu))\cdot\overline{\phi_t(\iota(\nu))}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\nu d\overline{\nu} \right) \\
&\qquad\qquad \times \psi(\xi)d\xi d\overline{\xi} \\
&= \int_{\partial\Omega_E} s_2(\nu,t;\hbar) e^{\frac{i}{\hbar}f(\iota(\nu),t)} e^{\frac{\overline{\phi_t(\iota(\nu))}\cdot\mathbf{z}}{\hbar}} \left(e^{-\frac{\iota(\nu)\cdot\overline{\iota(\nu)}}{2\hbar}} \int_{\mathbb{C}^n} e^{\frac{\overline{\xi}\cdot\iota(\nu)}{\hbar}} \psi(\xi) e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}} d\xi d\overline{\xi} \right) \\
&\qquad\qquad \times e^{-\frac{\phi_t(\iota(\nu))\cdot\overline{\phi_t(\iota(\nu))}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\nu d\overline{\nu} \\
&= \int_{\partial\Omega_E} s_2(\nu,t;\hbar) e^{\frac{i}{\hbar}f(\iota(\nu),t)} e^{\frac{\overline{\phi_t(\iota(\nu))}\cdot\mathbf{z}}{\hbar}} \psi(\iota(\nu)) e^{-\frac{\phi_t(\iota(\nu))\cdot\overline{\phi_t(\iota(\nu))}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}} d\nu d\overline{\nu}.
\end{aligned}$$

This gives

$$\begin{aligned}
\int_{\mathbb{C}^n}\left|\Psi_2(\mathbf{z})\right|^2 d\mathbf{z}d\overline{\mathbf{z}} &= \int_{\mathbb{C}^n}\Psi_2(\mathbf{z})\overline{\Psi_2(\mathbf{z})}d\mathbf{z}d\overline{\mathbf{z}} \\
&= \int_{\mathbb{C}^n}\int_{\partial\Omega_E}\int_{\partial\Omega_E} s_2(\iota(\nu_1),t;\hbar)\overline{s_2(\iota(\nu_2),t;\hbar)}e^{\frac{i}{\hbar}(f(\iota(\nu_1),t)-f(\iota(\nu_2),t))}e^{\frac{\overline{\phi_t(\iota(\nu_1))}\cdot\mathbf{z}}{\hbar}} \\
&\quad\times e^{\frac{\phi_t(\iota(\nu_2))\cdot\overline{\mathbf{z}}}{\hbar}}\psi(\iota(\nu_1))\overline{\psi(\iota(\nu_2))}e^{-\frac{\phi_t(\iota(\nu_1))\cdot\overline{\phi_t(\iota(\nu_1))}}{2\hbar}}e^{-\frac{\phi_t(\iota(\nu_2))\cdot\overline{\phi_t(\iota(\nu_2))}}{2\hbar}} \\
&\quad\times e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}}d\nu_1 d\overline{\nu_1}d\nu_2 d\overline{\nu_2}d\mathbf{z}d\overline{\mathbf{z}} \\
&= \int_{\partial\Omega_E}\int_{\partial\Omega_E} s_2(\iota(\nu_1),t;\hbar)\overline{s_2(\iota(\nu_2),t;\hbar)}e^{\frac{i}{\hbar}(f(\iota(\nu_1),t)-f(\iota(\nu_2),t))} \\
&\quad\times\left\langle\varphi_{\phi_t(\iota(\nu_2))},\varphi_{\phi_t(\iota(\nu_1))}\right\rangle_{\mathcal{B}_2}e^{-\frac{\phi_t(\iota(\nu_1))\cdot\overline{\phi_t(\iota(\nu_1))}}{2\hbar}}e^{-\frac{\phi_t(\iota(\nu_2))\cdot\overline{\phi_t(\iota(\nu_2))}}{2\hbar}} \\
&\quad\times\psi(\iota(\nu_1))\overline{\psi(\iota(\nu_2))}d\nu_1 d\overline{\nu_1}d\nu_2 d\overline{\nu_2} \\
&= \int_{\partial\Omega_E}\int_{\partial\Omega_E} s_2(\nu_1,t;\hbar)\overline{s_2(\nu_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\iota(\nu_1),t)-f(\iota(\nu_2),t))}e^{\frac{\overline{\phi_t(\iota(\nu_1))}\cdot\phi_t(\iota(\nu_2))}{\hbar}} \\
&\qquad\times e^{-\frac{\phi_t(\iota(\nu_1))\cdot\overline{\phi_t(\iota(\nu_1))}}{2\hbar}}e^{-\frac{\phi_t(\iota(\nu_2))\cdot\overline{\phi_t(\iota(\nu_1))}}{2\hbar}} \\
&\qquad\times\psi(\iota(\nu_1))\overline{\psi(\iota(\nu_2))}d\nu_1 d\overline{\nu_1}d\nu_2 d\overline{\nu_2} \\
&= \Bigg|\int_{\partial\Omega_E}\int_{\partial\Omega_E} s_2(\nu_1,t;\hbar)\overline{s_2(\nu_2,t;\hbar)}e^{\frac{i}{\hbar}(f(\iota(\nu_1),t)-f(\iota(\nu_2),t))}e^{\frac{\overline{\phi_t(\iota(\nu_1))}\cdot\phi_t(\iota(\nu_2))}{\hbar}} \\
&\qquad\times e^{-\frac{\phi_t(\iota(\nu_1))\cdot\overline{\phi_t(\iota(\nu_1))}}{2\hbar}}e^{-\frac{\phi_t(\iota(\nu_2))\cdot\overline{\phi_t(\iota(\nu_2))}}{2\hbar}} \\
&\qquad\times\psi(\iota(\nu_1))\overline{\psi(\iota(\nu_2))}d\nu_1 d\overline{\nu_1}d\nu_2 d\overline{\nu_2}\Bigg| \\
&\leq \int_{\partial\Omega_E}\int_{\partial\Omega_E}|s_2(\nu_1,t;\hbar)|\cdot|s_2(\nu_2,t;\hbar)|e^{-\frac{|\phi_t(\iota(\nu_1))-\phi_t(\iota(\nu_2))|}{2\hbar}} \\
&\qquad\qquad\times|\psi(\iota(\nu_1))|\cdot|\psi(\iota(\nu_2))|d\nu_1 d\overline{\nu_1}d\nu_2 d\overline{\nu_2}.
\end{aligned}$$

Up to this point the calculation involving $\left|\partial\Omega_E, s_2\right\rangle$ has been (essentially) identical to the calculation involving $\left|\Omega_E, s_2\right\rangle$, but this last step of observing the semiclassical size of the above term is very different. Recall that ψ is semiclassically localized in the ball, $B_R(0)$, contained in a subset of phase space Ω such that $\Omega\subset\subset\Omega_E$. Recall that this setup let us conclude that there exists $\epsilon>0$ such that $B_R(0)\subset\Omega\subset\Omega+\epsilon\subset\subset\Omega_E$.

This tells us that for any $\mathbf{y} \in \partial\Omega_E$ that $|\mathbf{y}| - R > \epsilon$. Given this setup, for constants $C, \delta, \gamma, \hbar_0 > 0$ appropriate to ψ, since $\iota(\nu) \in \partial\Omega_E$ we have

$$\left|\psi(\iota(\nu))\right| \leq Ce^{-\frac{\delta\epsilon^\gamma}{\hbar}} \tag{6.53}$$

for all $\nu \in \partial\Omega_E$ and all $\hbar \in (0, \hbar_0]$, see (VI.5). Combining this with the reproducing property gives us that for every ϵ'

$$\begin{aligned}
\int_{\mathbb{C}^n} \left|\Psi_2(\mathbf{z})\right|^2 d\mathbf{z}d\overline{\mathbf{z}} \leq & \int_{\partial\Omega_E}\int_{\partial\Omega_E} |s_2(\nu_1, t; \hbar)| \cdot |s_2(\nu_2, t; \hbar)| e^{-\frac{|\phi_t(\iota(\nu_1)) - \phi_t(\iota(\nu_2))|}{2\hbar}} \\
& \qquad \times |\psi(\iota(\nu_1))| \cdot |\psi(\iota(\nu_2))| d\nu_1 d\overline{\nu_1} d\nu_2 d\overline{\nu_2} \\
& \int_{\partial\Omega_E}\int_{\partial\Omega_E} |s_2(\nu_1, t; \hbar)| \cdot |s_2(\nu_2, t; \hbar)| e^{-\frac{|\phi_t(\iota(\nu_1)) - \phi_t(\iota(\nu_2))|}{2\hbar}} \\
& \qquad \times |\psi(\iota(\nu_1))|^{1-\epsilon'} |\psi(\iota(\nu_1))|^{\epsilon'} \cdot |\psi(\iota(\nu_2))| d\nu_1 d\overline{\nu_1} d\nu_2 d\overline{\nu_2} \\
& \leq Ce^{-\frac{\delta'\epsilon^\gamma}{\hbar}} \parallel \psi \parallel^{2-\epsilon'}_{L^2(\mathbb{C}^n)} \int_{\partial\Omega_E}\int_{\partial\Omega_E} |s_2(\nu_1, t; \hbar)| \cdot |s_2(\nu_2, t; \hbar)| \\
& \qquad \times e^{-\frac{|\phi_t(\iota(\nu_1)) - \phi_t(\iota(\nu_2))|}{2\hbar}} d\nu_1 d\overline{\nu_1} d\nu_2 d\overline{\nu_2} \\
& \leq C_2 e^{-\frac{\delta'\epsilon^\gamma}{\hbar}} \parallel \psi \parallel^{2-\epsilon'}_{L^2(\mathbb{C}^n)},
\end{aligned}$$

for all $\hbar \in (0, \hbar_0]$, where $\delta' \equiv \delta\epsilon'$, and where the last step follows because the amplitude s_2 is smooth and polynomial in $\hbar$, thus the integral is convergent (note $\partial\Omega_E$ is compact). Note that since s_2 is polynomial in $\hbar$ we can choose the constant C_2 to be independent of of $\hbar$, and if we restrict the values of t to a bounded interval then we can choose C_2 to be independent of t as well. In other 'words' we've shown

$$\|\Psi_2\|_{L^2(\mathbb{C}^n)} \leq C_2 e^{-\frac{\delta'\epsilon^\gamma}{2\hbar}} \parallel \psi \parallel^{1-\epsilon'/2}_{L^2(\mathbb{C}^n)} \tag{6.54}$$

An invocation of the triangle inequality gives us that there exists an $\hbar_0$ such that for every N there exists a constant $\widetilde{C}_N$ such that

$$\left\| \left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi \right\|_{L^2(\mathbb{C}^n)} \leq \|\Psi_1\|_{L^2(\mathbb{C}^n)} + \|\Psi_2\|_{L^2(\mathbb{C}^n)} \tag{6.55}$$

$$\leq \widetilde{C}_N \hbar^N \left(\| \psi \|_{L^2(\mathbb{C}^n)} + \| \psi \|_{L^2(\mathbb{C}^n)}^{1-\epsilon'/2} \right). \tag{6.56}$$

Let's consider the final part of the error term, the Toeplitz contribution $\widehat{H}\widehat{T}_{\mathbb{C}^n \backslash \Omega_E}\psi$, with the same ψ as above.

Let's first consider the Toeplitz contribution for the Hamiltonian $\widehat{H} = \widehat{\mathbf{Z}}^\alpha \widehat{\mathbf{Z}^*}^\beta$. In this case we have by definition

$$\begin{aligned}
(\widehat{H}\widehat{T}_{\mathbb{C}^n \backslash \Omega_E}\psi)(\mathbf{z}) &= \widehat{H} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} \eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v})\psi(\mathbf{v}) e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}} d\mathbf{v}d\overline{\mathbf{v}} \\
&= \widehat{\mathbf{Z}}^\alpha \widehat{\mathbf{Z}^*}^\beta e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} \int_{\mathbb{C}^n} e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} \eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v})\psi(\mathbf{v}) e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}} d\mathbf{v}d\overline{\mathbf{v}} \\
&= e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} \int_{\mathbb{C}^n} \mathbf{z}^\alpha \overline{\mathbf{v}}^\beta e^{\frac{\overline{\mathbf{v}}\cdot\mathbf{z}}{\hbar}} \eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v})\psi(\mathbf{v}) e^{-\frac{\mathbf{v}\cdot\overline{\mathbf{v}}}{2\hbar}} d\mathbf{v}d\overline{\mathbf{v}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\mathbb{C}^n} |(\widehat{H}\widehat{T}_{\mathbb{C}^n \backslash \Omega_E}\psi)(\mathbf{z})|^2 d\mathbf{z}d\overline{\mathbf{z}} &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathbf{z}^\alpha \overline{\mathbf{v_1}}^\beta \mathbf{z}^\alpha \overline{\mathbf{v_2}}^\beta e^{\frac{\overline{\mathbf{v_1}}\cdot\mathbf{z}}{\hbar}} e^{\frac{\mathbf{v_2}\cdot\overline{\mathbf{z}}}{\hbar}} \eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v_1})\eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v_2}) \\
&\qquad \times \psi(\mathbf{v_1})\overline{\psi(\mathbf{v_2})} e^{-\frac{\mathbf{v_1}\cdot\overline{\mathbf{v_1}}}{2\hbar}} e^{-\frac{\mathbf{v_2}\cdot\overline{\mathbf{v_2}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} d\mathbf{v_1}d\overline{\mathbf{v_1}}d\mathbf{v_2}d\overline{\mathbf{v_2}}d\mathbf{z}d\overline{\mathbf{z}} \\
&= \Bigg| \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathbf{z}^\alpha \overline{\mathbf{v_1}}^\beta \mathbf{z}^\alpha \overline{\mathbf{v_2}}^\beta e^{\frac{\overline{\mathbf{v_1}}\cdot\mathbf{z}}{\hbar}} e^{\frac{\mathbf{v_2}\cdot\overline{\mathbf{z}}}{\hbar}} \eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v_1})\eta_{\mathbb{C}^n \backslash \Omega_E}(\mathbf{v_2}) \\
&\qquad \times \psi(\mathbf{v_1})\overline{\psi(\mathbf{v_2})} e^{-\frac{\mathbf{v_1}\cdot\overline{\mathbf{v_1}}}{2\hbar}} e^{-\frac{\mathbf{v_2}\cdot\overline{\mathbf{v_2}}}{2\hbar}} e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{\hbar}} d\mathbf{v_1}d\overline{\mathbf{v_1}}d\mathbf{v_2}d\overline{\mathbf{v_2}}d\mathbf{z}d\overline{\mathbf{z}} \Bigg| \\
&\leq \int_{\mathbb{C}^n}\int_{\mathbb{C}^n \backslash \Omega_E}\int_{\mathbb{C}^n \backslash \Omega_E} |\mathbf{v}_1|^\beta \cdot |\mathbf{v}_2|^\beta |\mathbf{z}|^{2\alpha} e^{-\frac{|\mathbf{v_1}-\mathbf{z}|^2}{2\hbar}} e^{-\frac{|\mathbf{v_2}-\mathbf{z}|^2}{2\hbar}} \\
&\qquad \times |\psi(\mathbf{v_1})| \cdot |\psi(\mathbf{v_2})| d\mathbf{v_1}d\overline{\mathbf{v_1}}d\mathbf{v_2}d\overline{\mathbf{v_2}}d\mathbf{z}d\overline{\mathbf{z}}.
\end{aligned}$$

Because ψ is semiclassically localized (VI.5) inside $B_R(0)$ where $B_R(0) \subset \Omega \subset\subset \Omega_E$, for all $\mathbf{v} \in \mathbb{C}^n \backslash \Omega_E$ and for all $\hbar \in (0, \hbar_0]$

$$|\psi(\mathbf{v})| \leq Ce^{-\frac{\delta(|\mathbf{v}|-R)^\gamma}{\hbar}},$$

and in fact we know that $|\mathbf{v}| - R > \epsilon > 0$. Again, since ψ is in Bargmann space, we have $|\psi(\mathbf{v})| \leq \| \psi \|_{L^2(\mathbb{C}^n)}$. Combining these two facts we have

$$\begin{aligned}
|\psi(\mathbf{v_1})| \cdot |\psi(\mathbf{v_2})| &= |\psi(\mathbf{v_1})|^{1-\epsilon'/2} \cdot |\psi(\mathbf{v_1})|^{\epsilon'/2} |\psi(\mathbf{v_2})|^{1-\epsilon'/2} \cdot |\psi(\mathbf{v_2})|^{\epsilon'/2} \\
&\leq \| \psi \|_{L^2(\mathbb{C}^n)}^{1-\epsilon'/2} \cdot |\psi(\mathbf{v_1})|^{\epsilon'/2} \| \psi \|_{L^2(\mathbb{C}^n)}^{1-\epsilon'/2} \cdot |\psi(\mathbf{v_2})|^{\epsilon'/2} \\
&= \| \psi \|_{L^2(\mathbb{C}^n)}^{2-\epsilon'} |\psi(\mathbf{v_1})|^{\epsilon'/2} |\psi(\mathbf{v_2})|^{\epsilon'/2} \\
&= \| \psi \|_{L^2(\mathbb{C}^n)}^{2-\epsilon'} |\psi(\mathbf{v_1})|^{\epsilon'/4} |\psi(\mathbf{v_2})|^{\epsilon'/4} |\psi(\mathbf{v_1})|^{\epsilon'/4} |\psi(\mathbf{v_2})|^{\epsilon'/4} \\
&\leq \| \psi \|_{L^2(\mathbb{C}^n)} Ce^{-\frac{\delta'(|\mathbf{v_1}|-R)^\gamma}{4\hbar}} Ce^{-\frac{\delta'(|\mathbf{v_2}|-R)^\gamma}{4\hbar}} Ce^{-\frac{\delta'(|\mathbf{v_1}|-R)^\gamma}{4\hbar}} Ce^{-\frac{\delta'(|\mathbf{v_2}|-R)^\gamma}{4\hbar}} \\
&\leq C^4 \| \psi \|_{L^2(\mathbb{C}^n)}^{2-\epsilon'} e^{-\frac{\delta' \epsilon^\gamma}{2\hbar}} e^{-\frac{\delta'(|\mathbf{v_1}|-R)^\gamma}{4\hbar}} e^{-\frac{\delta'(|\mathbf{v_2}|-R)^\gamma}{4\hbar}}.
\end{aligned}$$

Thus for $\widehat{H} = \widehat{\mathbf{Z}}^\alpha \widehat{\mathbf{Z}^*}^\beta$,

$$\begin{aligned}
\int_{\mathbb{C}^n} |(\widehat{H} \widehat{T}_{\mathbb{C}^n \backslash \Omega_E} \psi)(\mathbf{z})|^2 d\mathbf{z} d\overline{\mathbf{z}} &= C^4 \| \psi \|_{L^2(\mathbb{C}^n)}^{2-\epsilon'} e^{-\frac{\delta(\epsilon)^\gamma}{2\hbar}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n \backslash \Omega_E} \int_{\mathbb{C}^n \backslash \Omega_E} |\mathbf{v}_1|^\beta \cdot |\mathbf{v}_2|^\beta |\mathbf{z}|^{2\alpha} \\
&\qquad \times e^{-\frac{|\mathbf{v_1}-\mathbf{z}|^2}{2\hbar}} e^{-\frac{|\mathbf{v_2}-\mathbf{z}|^2}{2\hbar}} e^{-\frac{\delta'(|\mathbf{v_1}|-R)^\gamma}{4\hbar}} \\
&\qquad \times e^{-\frac{\delta'(|\mathbf{v_2}|-R)^\gamma}{4\hbar}} d\mathbf{v_1} d\overline{\mathbf{v_1}} d\mathbf{v_2} d\overline{\mathbf{v_2}} d\mathbf{z} d\overline{\mathbf{z}} \\
&= C^4 \| \psi \|_{L^2(\mathbb{C}^n)}^{2-\epsilon'} e^{-\frac{\delta'(\epsilon)^\gamma}{2\hbar}}.
\end{aligned}$$

As we showed in (4.20), if our (quantum) Hamiltonian is the Weyl quantization of a polynomial symbol, then through the relationship between Weyl quantization and Wick quantization we can express it as a sum of terms of the form $\widehat{\mathbf{Z}}^\alpha \widehat{\mathbf{Z}^*}^\beta$. Thus the above error estimate for the Toeplitz contribution can easily be extended to all Hamiltonians which are the Weyl quantization of a polynomial symbol. This tell us that

$$\| \widehat{H}\widehat{T}_{\mathbb{C}^n \setminus \Omega_E} \|_{L^2(\mathbb{C}^n)} \leq C^4 e^{-\frac{\delta'(\epsilon)^{\gamma}}{2\hbar}} \| \psi \|^{2-\epsilon'}_{L^2(\mathbb{C}^n)} . \tag{6.57}$$

Combining this result with the work for our previous term we have that there exists an $\hbar_0 > 0$ and an $\epsilon > 0$ such that for every N there exists a constant C_N such that for all $\hbar \in (0, \hbar_0]$

$$\left\| \widehat{R}(t)\psi \right\|_{L^2(\mathbb{C}^n)} \leq \left\| \left(i\hbar \frac{\partial}{\partial t} \widehat{U}_E(t) - \widehat{H}\widehat{U}_E(t) \right) \psi \right\|_{L^2(\mathbb{C}^n)} + \left\| \widehat{H}\widehat{T}_{\mathbb{C}^n \setminus \Omega_E} \psi \right\|_{L^2(\mathbb{C}^n)} \tag{6.58}$$

$$\leq C_N \hbar^N \left(\|\psi\|_{L^2(\mathbb{C}^n)} + \|\psi\|^{1-\epsilon}_{L^2(\mathbb{C}^n)} \right). \tag{6.59}$$

Finally, invoking Duhamel's formula we have

$$\| (\widehat{V}^a_E(t) - \widehat{U}(t))\psi \|_{L^2(\mathbb{C}^n)} \leq \hbar \int_0^t \left\| \widehat{R}(s)\psi \right\|_{L^2(\mathbb{C}^n)} ds \tag{6.60}$$

$$\leq tC_N \hbar^N \left(\| \psi \|_{L^2(\mathbb{C}^n)} + \| \psi \|^{1-\epsilon}_{L^2(\mathbb{C}^n)} \right). \tag{6.61}$$

Restricting the time t to a bounded interval we can bound tC_N by TC_N, thus absorbing this into C_N, we have proven the result.

$\square$

Tempered Semiclassically Localized States, and Their Propagation

Here we will further restrict the class of initial conditions for which we consider approximating the quantum propagation. With this restriction we will have as a corollary of the theorem above a stronger approximation result. First consider the following definition.

Definition VI.7. The norm of an element $\psi \in \mathcal{B}_1(\mathbb{C}^n)$ will be called $\hbar$-*tempered* if there exist $M_1, M_2 \in \mathbb{Z}$ and $C_{M_1}, C_{M_2} > 0$ such that

$$0 < C_{M_1}\hbar^{M_1} \leq \| \psi \|_{L^2(\mathbb{C}^n)} \leq C_{M_2}\hbar^{M_2}. \tag{6.62}$$

Given the definition of semiclassically tempering the norm of a state we have the following proposition that relates this notion to our previous work

Proposition VI.8. *Suppose that* $\psi(\mathbf{z};\hbar) \in L^2(\mathbb{C}^n)$ *is semiclassically localized inside the ball of radius* R *and that the norm of* ψ *is* $\hbar$*-tempered. Then* $\frac{\psi}{\|\psi\|}$ *is semiclassically localized in the ball of radius* $R + \epsilon$ *for any* $\epsilon > 0$.

Proof. By definition (**??**), since ψ is semiclassically localized inside the ball of radius R we know that there exists constants C, δ, γ, $\hbar_0$ such that for all $\mathbf{z} \in \mathbb{C}^n$ with $|\mathbf{z}| > R$ we have

$$|\psi(\mathbf{z};\hbar)| \leq Ce^{-\frac{\delta}{\hbar}(|\mathbf{z}|-R)^\gamma},$$

for all $\hbar \in (0, \hbar_0]$. Thus, for $|\mathbf{z}| > R$ and $\hbar \in (0, \hbar_0]$ and some $M_1 \in \mathbb{Z}$ and $C_{M_1} > 0$ we have

$$\begin{aligned}\left|\frac{\psi}{\parallel \psi \parallel}(\mathbf{z})\right| &\leq \frac{C}{\parallel \psi \parallel} e^{-\frac{\delta}{\hbar}(|\mathbf{z}|-R)^{\gamma}} \\ &= \frac{C}{\parallel \psi \parallel} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} \\ &= \frac{C}{C_{M_1}\hbar^{M_1}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} \\ &= \widetilde{C}\hbar^{-M_1} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}},\end{aligned}$$

for $\widetilde{C} = \frac{C}{C_{M_1}}$. Now, if $|\mathbf{z}| > R + \epsilon$ then since $\gamma > 0$

$$\begin{aligned}&|\mathbf{z}| - (R+\epsilon) < |\mathbf{z}| - R \\ \implies &(|\mathbf{z}| - (R+\epsilon))^{\gamma} < (|\mathbf{z}| - R)^{\gamma} \\ \implies &e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-(R+\epsilon))^{\gamma}} > e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}}.\end{aligned}$$

Therefore for $|\mathbf{z}| > R + \epsilon$ and for $\hbar \in (0, \hbar_0]$

$$\begin{aligned}\left|\frac{\psi}{\parallel \psi \parallel}(\mathbf{z})\right| &\leq \widetilde{C}\hbar^{-M_1} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-R)^{\gamma}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-(R+\epsilon))^{\gamma}} \\ &\leq \widetilde{C}\hbar^{-M_1} e^{-\frac{\delta}{2\hbar}\epsilon^{\gamma}} e^{-\frac{\delta}{2\hbar}(|\mathbf{z}|-(R+\epsilon))^{\gamma}}.\end{aligned}$$

For sufficiently small values of $\hbar$ (say values in the interval $(0, \widetilde{\hbar}]$ for some $\widetilde{\hbar}$) we have that $\hbar^{-M_1} e^{-\frac{\delta}{2\hbar}\epsilon^{\gamma}}$ is bounded by a constant that is independent of $\hbar$. Thus for $|\mathbf{z}| > R + \epsilon$ and $\hbar \in (0, \widetilde{\widetilde{\hbar}}]$ where $\widetilde{\widetilde{\hbar}} = \min\{\hbar_0, \widetilde{\hbar}\}$ we have

$$\widetilde{\widetilde{C}} e^{-\frac{\delta'}{\hbar}(|\mathbf{z}|-(R+\epsilon))^{\gamma}},$$

where $\delta' = \delta/2$.

$\square$

Following directly from (VI.6) we have the following corollary:

Corollary VI.9. *Suppose that $H = H(\mathbf{z}, \overline{\mathbf{z}})$ is a polynomial Hamiltonian on $\mathbb{C}^n$ and let $\widehat{H}$ denote its Weyl quantization. Suppose that E is a regular value of H, and let*

$$\Omega_E \equiv \{\mathbf{z} \in \mathbb{C}^n | H(\mathbf{z}, \overline{\mathbf{z}}) \leq E\}.$$

For each amplitude function $a : \Omega_E \longrightarrow \mathbb{C}$ let $\widehat{U}^a_E(t)$ denote the operator whose kernel is given by $|\Omega_E, a\rangle$, defined with respect to the time dependent embedding $\mathbf{w} : \Omega_E \longrightarrow \mathbb{C}^n \times \overline{\mathbb{C}^n}$ given by $\mathbf{w}(\mathbf{x}, t) = (\phi_t(\mathbf{x}), \overline{\mathbf{x}})$, where ϕ_t is the Hamilton flow of H. Let $\widehat{T}_{\mathbb{C}^n \backslash \Omega_E}$ denote the Toeplitz operator on Bargmann space with symbol given by the characteristic function on $\mathbb{C}^n \backslash \Omega_E$. Let $\widehat{V}^\alpha_E(t) \equiv \widehat{U}^\alpha_E(t) + \widehat{T}_{\mathbb{C}^n \backslash \Omega_E}$. For a particular choice of a (that is well-defined modulo a term of $O(\hbar^\infty)$) for a fixed $T > 0$ for every ψ that is semiclassically localized in the ball of radius R such that for some $\Omega \subset \mathbb{C}^n$ and each ϵ such that $0 < \epsilon < \epsilon_0$ we have $B_R(0) \subset B_{R+\epsilon_0}(0) \subset \Omega \subset\subset \Omega_E$ there exists an $\hbar_0 > 0$ such that for each N there exists C_N such that for all $\hbar \in (0, \hbar_0]$

$$\| (\widehat{V}^a_E(t) - \widehat{U}(t))\psi \|_{L^2(\mathbb{C}^n)} \leq C_N \hbar^N \| \psi \|_{L^2(\mathbb{C}^n)}, \tag{6.63}$$

for all $|t| < T$.

6.1.5 The Ansatz And The $L^2(\mathbb{R}^n)$ Side of the Bargmann Transform

The Hermann-Kluk Propagator

First proposed by Michael F. Herman and Edward Kluk in [19] the Herman-Kluk propagator (HK-propagator for short) is a semiclassical approximation to the Schwartz kernel of the propagator when the Hilbert space of states is $L^2(\mathbb{R}^n)$. The HK-propagator is given by the following formula (modulo an overall multiplicative constant that depends on convention)

$$K_t^{HK}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^{2n}} R_t(\mathbf{p},\mathbf{q}) e^{\frac{i}{\hbar}S_t(\mathbf{p},\mathbf{q})} e^{-\frac{i}{\hbar}\frac{\mathbf{p}_t\cdot\mathbf{q}_t}{2}} \psi_{(\mathbf{p}_t,\mathbf{q}_t)}(\mathbf{x}) e^{\frac{i}{\hbar}\frac{\mathbf{p}\cdot\mathbf{q}}{2}} \overline{\psi_{(\mathbf{p},\mathbf{q})}(\mathbf{y})} d\mathbf{p}d\mathbf{q}, \tag{6.64}$$

where recall from the discussion in chapter II on coherent states in the standard representation that

$$\psi_{(\tilde{\mathbf{p}},\tilde{\mathbf{q}})}(\mathbf{x}) = \widehat{T}_{(\tilde{\mathbf{p}},\tilde{\mathbf{q}})}\psi_{(0,0)} = \frac{1}{\pi\hbar^{n/4}} e^{\frac{i}{\hbar}\frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}}{2}} e^{\frac{i}{\hbar}(\tilde{\mathbf{p}}\cdot(\mathbf{x}-\tilde{\mathbf{q}}))} e^{-\frac{|\mathbf{x}-\tilde{\mathbf{q}}|^2}{2\hbar}},$$

is a canonical coherent state, $(\mathbf{p}_t,\mathbf{q}_t) \equiv \phi_t(\mathbf{p},\mathbf{q})$ is the classical flow; the (so-called) HK-prefactor R_t is given by

$$R_t(\mathbf{p},\mathbf{q}) = \frac{1}{2^{n/2}}\left[det\left(\frac{\partial \mathbf{p_t}}{\partial \mathbf{p}} - i\frac{\partial \mathbf{q_t}}{\partial \mathbf{p}} + i\frac{\partial \mathbf{p_t}}{\partial \mathbf{q}} + \frac{\partial \mathbf{q_t}}{\partial \mathbf{q}}\right)\right]^{\frac{1}{2}}, \tag{6.65}$$

where for instance

$$\left(\frac{\partial \mathbf{p_t}}{\partial \mathbf{q}}\right)_{ij} = \frac{\partial p_{ti}}{\partial q_j};$$

and finally the phase function S_t (not at all related to the previous use of S_t in the recent discussion of the estimate above) is the classical action defined by

$$S_t(\mathbf{p},\mathbf{q}) = \int_0^t [\mathbf{p}_\tau \dot{\mathbf{q}}_\tau - H(\mathbf{p}_\tau,\mathbf{q}_\tau)]d\tau. \tag{6.66}$$

It will be relevant shortly that the classical action satisfies the equations

$$\frac{\partial S_t}{\partial q_j} = \mathbf{p}_t \cdot \frac{\partial \mathbf{q}_t}{\partial q_j} - p_j, \tag{6.67}$$

$$\frac{\partial S_t}{\partial p_j} = \mathbf{p}_t \cdot \frac{\partial \mathbf{q}_t}{\partial p_j}. \tag{6.68}$$

Remark VI.10. Recent treatments of the HK-propagator in the physical chemistry literature generalize the above expression by replacing the coherent states with the more general squeezed states. Also in the chemistry literature it is often the case that a width parameter γ is used in the definition of coherent states; the canonical coherent states above correspond to a width parameter of $\gamma = 1/2$.

Conjugating the Ansatz with $\Lambda = \mathbb{C}^n$

A natural question to investigate is, what type of approximation to the Schwartz kernel of the propagator, on $L^2(\mathbb{R}^n)$, does one get by conjugating the unlocalized ansatz by the Bargmann transform? To answer this question recall that the ansatz for the kernel of the propagator on Bargmann space with $\Lambda = \mathbb{C}^n$ is

$$V_t(\mathbf{z}, \overline{\xi}) = \int_{\mathbb{R}^{2n}} a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} \varphi_{\phi_t(\chi)}(\mathbf{z}) \overline{\varphi_\chi(\xi)} d\chi d\overline{\chi} \tag{6.69}$$

where recall from the discussion on coherent states in Bargmann space in II, in general

$$\varphi_{\mathbf{v}}(\mathbf{w}) \equiv e^{\overline{\mathbf{v}} \cdot \mathbf{w}/\hbar} e^{-\mathbf{v} \cdot \overline{\mathbf{v}}/2\hbar} e^{-\mathbf{w} \cdot \overline{\mathbf{w}}/2\hbar}$$

which are the normalized coherent states in Bargmann space, $\mathcal{B}_1(\mathbb{C}^n)$. If $\widehat{V}(t)$ is the operator whose kernel is V_t then for $g(\xi) \in \mathcal{B}_1(\mathbb{C}^n)$ we have

$$\begin{aligned}
(\widehat{V}(t)g)(\mathbf{z}) &= \int_{\mathbb{C}^n} V_t(\mathbf{z}, \overline{\xi}) g(\xi) d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} \varphi_{\phi_t(\chi)}(\mathbf{z}) \overline{\varphi_\chi(\xi)} g(\xi) d\chi d\overline{\chi} d\xi d\overline{\xi} \\
&= \int_{\mathbb{C}^n} a(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} \varphi_{\phi_t(\chi)}(\mathbf{z}) \left(\langle \varphi_\chi, g \rangle_{\mathcal{B}} \right) d\chi d\overline{\chi}.
\end{aligned}$$

We consider the analog of $\widehat{V}(t)$ on the $L^2(\mathbb{R}^n)$ side of the Bargmann transform, by conjugating by the Bargmann transform, $A_\hbar$. $A_\hbar^{-1}\widehat{V}(t)A_\hbar$. For $h \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
A_\hbar^{-1}\widehat{V}(t)A_\hbar h &= \int_{\mathbb{R}^{2n}} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}A_\hbar^{-1}\varphi_{\phi_t(\chi)}(\mathbf{z})\left(\langle\varphi_\chi, A_\hbar h\rangle_{\mathbb{B}}\right)d\chi d\overline{\chi} \\
&= \int_{\mathbb{R}^{2n}} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}A_\hbar^{-1}\varphi_{\phi_t(\chi)}(\mathbf{z})\left(\langle A_\hbar^{-1}\varphi_\chi, h\rangle_{L^2}\right)d\chi d\overline{\chi} \\
&= \int_{\mathbb{R}^{2n}} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}A_\hbar^{-1}\varphi_{\phi_t(\chi)}(\mathbf{z})\left(\langle A_\hbar^{-1}\varphi_\chi, h\rangle_{L^2}\right)d\chi d\overline{\chi} \\
&= \int_{\mathbb{R}^{2n}} a(\mathbf{p},\mathbf{q},t;\hbar)e^{\frac{i}{\hbar}f(\mathbf{p},\mathbf{q},t)}\psi_{(\mathbf{p}_t,\mathbf{q}_t)}(\mathbf{x})\left(\langle\psi_{(\mathbf{p},\mathbf{q})}, h\rangle_{L^2}\right)d\mathbf{p}d\mathbf{q} \\
&= \int_{\mathbb{R}^n} \widetilde{V}_t(\mathbf{x},\mathbf{y})h(\mathbf{y})d\mathbf{y}
\end{aligned}$$

where along the way we've relabeled the integration variables to $\mathbf{p}$ and $\mathbf{q}$ since our integrals were with respect to Lebesgue measure so we were integrating over these, the real and imaginary parts of χ in each line all along just in complex notation, and where

$$\widetilde{V}_t(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^{2n}} a(\mathbf{p},\mathbf{q},t;\hbar)e^{\frac{i}{\hbar}f(\mathbf{p},\mathbf{q},t)}\psi_{(\mathbf{p}_t,\mathbf{q}_t)}(\mathbf{x})\overline{\psi_{(\mathbf{p},\mathbf{q})}(\mathbf{y})}d\mathbf{p}d\mathbf{q}.$$

The question we have before us is how the above expression compares to the HK-propagator. The HK-propagator is

$$K_t^{HK}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^{2n}} R_t(\mathbf{p},\mathbf{q})e^{\frac{i}{\hbar}S_t(\mathbf{p},\mathbf{q})}e^{-\frac{i}{\hbar}\frac{\mathbf{p}_t\cdot\mathbf{q}_t}{2}}\psi_{(\mathbf{p}_t,\mathbf{q}_t)}(\mathbf{x})e^{\frac{i}{\hbar}\frac{\mathbf{p}\cdot\mathbf{q}}{2}}\overline{\psi_{(\mathbf{p},\mathbf{q})}(\mathbf{y})}d\mathbf{p}d\mathbf{q}$$

Putting aside the issue of the amplitudes, we see that the comparison of the phases comes down to the relationship between $f(\mathbf{p},\mathbf{q},t)$, our lift function, and $\widetilde{f}(\mathbf{p},\mathbf{q},t) \equiv S_t(\mathbf{p},\mathbf{q}) - \frac{1}{2}\mathbf{p}_t\cdot\mathbf{q}_t + \frac{1}{2}\mathbf{p}\cdot\mathbf{q}$. Let's first note that we have imposed the initial condition on f that $f(\mathbf{p},\mathbf{q},0)=1$. Using the above formula for the action S_t we see that it is also the case that $\widetilde{f}(\mathbf{p},\mathbf{q},0)=1$

From the lift condition (III.9) defining f (where the connection form is represented on $\mathbb{R}^{2n}$) we have

$$\frac{\partial f}{\partial q_j}=-\frac{1}{2}p_j+\frac{1}{2}\left(\mathbf{p}_t\cdot\frac{\partial \mathbf{q}_t}{\partial q_j}-\mathbf{q}_t\cdot\frac{\partial \mathbf{p}_t}{\partial q_j}\right), \tag{6.70}$$

$$\frac{\partial f}{\partial p_j}=\frac{1}{2}q_j+\frac{1}{2}\left(\mathbf{p}_t\cdot\frac{\partial \mathbf{q}_t}{\partial p_j}-\mathbf{q}_t\cdot\frac{\partial \mathbf{p}_t}{\partial p_j}\right). \tag{6.71}$$

Comparing to the equations for the classical action S_t above, we get that the space derivatives of f and $\widetilde{f}$ satisfy the same equations. Also, adapting (5.17) to the case where we think of f as a function on $\Lambda=\mathbb{R}^{2n}$ we have the relation

$$\frac{\partial f}{\partial t}+\frac{1}{2}\left(\mathbf{q}_t\cdot\frac{\partial \mathbf{p}_t}{\partial t}-\mathbf{p}_t\cdot\frac{\partial \mathbf{q}_t}{\partial t}\right)=-H(\mathbf{p}_t,\mathbf{q}_t). \tag{6.72}$$

Again utilizing the form for S_t we will get that $\frac{\partial \widetilde{f}}{\partial t}$ satisfies the same relationship. So the time derivatives of f and $\widetilde{f}$ satisfy the same relation.

The fact that these two phase functions satisfy the same initial condition and the their derivatives are equal we can say that they are identically equal. Given that the phases are the same, then we can invoke (III.23) to conclude that the HK-prefactor R_t and a_0 are equal (modulo a term of $O(\hbar^\infty)$). Thus, the HK-propagator is the analog of the lowest order contribution to the generalized Lagrangian state ansatz with $\Lambda=\mathbb{C}^n$ on the L^2 side of the Bargmann transform. Thus, when we take the notion of generalized Lagrangian states over to the L^2 side of the Bargmann transform we see that they provide a new, and more natural, derivation of the H.K.-propagator than currently exists in the literature.

For a modern perspective on the HK-propagator with an emphasis on formal derivations and discussions of applications one can find in the physical chemistry literature [12], [23], and [24]. The HK-propagator has recently begun to be studied rigorously in the semiclassical analysis commmunity, see [35], [36], and [40].

6.2 Forward-Backward Propagation

The discussion in this section focusing on extending the use of generalized Lagrangian states to the quantization of slightly more general symplectomorphisms of phase space than the Hamilton flow. Specifically we'll show that when one has a particular type of product of propagators that satisfy a Schrodinger type equation then we can use the formalism we developed in the case of a single propagator to approximate the kernel of the product with a generalized Lagrangian state. It is important to note that the method will not be to simply compose the kernel of each propagator in the product, but rather attack the problem of approximating the kernel of the product in 'one fell swoop'. The motivation for this extension will come from the theory behind the spectroscopic investigation of systems in quantum chemistry.

6.2.1 Correlation Functions: Motivating Forward-Backward Propagation

All spectroscopic techniques are based on the interaction of matter with electromagnetic radiation. Spectroscopic studies of physical systems are, in a "nut-shell", when you use electromagnetic fields (often laser pulses) to excite a system, and then you detect electromagnetic signals that are released from the system as a result of the evolution of the system induced by the field (often relaxation). These emitted signals can be used to infer the structural as well as dynamical information of the system being studied. From the theory of spectroscopy a large number of spectroscopic observables can be computed from (quantum) time correlation functions. Thus correlation functions (and their generalization optical response functions) are an object of central theoretical interest. Various forms that correlation function can take are

$$C(t) = Tr(\widehat{A}e^{\frac{i}{\hbar}t\widehat{H}}\widehat{B}e^{-\frac{i}{\hbar}t\widehat{H}}),$$

where $\widehat{A}$ and $\widehat{B}$ are observables, and $\widehat{H}$ is the Hamiltonian of the system of interest; another is

$$C(t) = Tr(e^{-\beta\widehat{H}}\widehat{A}e^{\frac{i}{\hbar}t\widehat{H}}\widehat{B}e^{-\frac{i}{\hbar}t\widehat{H}});$$

a third is

$$C(t) = Tr(e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}\widehat{\rho}_g),$$

where $\hat{H}_e$, and $\hat{H}_g$ are ground state and excited state Hamiltonians, and $\hat{\rho}_g$ is a system operator, and a final example of a more complicated optical response function is

$$Tr\left(e^{-\frac{i}{\hbar}(t_1-t_2)\widehat{H}_e}e^{-\frac{i}{\hbar}(t_2-t_3)\widehat{H}_g}e^{-\frac{i}{\hbar}(t_3-t_4)\widehat{H}_e}e^{-\frac{i}{\hbar}(t_4-t_1)\widehat{H}_e}\widehat{\rho}_g\right).$$

Recall that the trace of an operator $\widehat{O}$ on a separable Hilbert space $\mathcal{H}$, denoted $Tr(\widehat{O})$, is defined as $Tr(\widehat{O}) = \sum_{j=0}^{\infty}\langle\widehat{O}e_j, e_j\rangle_{\mathcal{H}}$ where $\{e_j\}_{j=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$, see [34].

These examples are not to be taken as an exhaustive list, but rather to give a "flavor" for the type of mathematical object with which we are dealing. For a detailed discussion on correlation functions, see [31]. Focusing on the last example, and specifically on the operator being traced, we can use a generalized Lagrangian state as an ansatz in order to semiclassically approximate the operator kernel of a product of propagators.

6.2.2 Generalized Lagrangian States and Forward-Backward Propagation

We'll use a generalized Lagrangian state as an ansatz for semiclassically approximating the operator kernel of a product of propagators, one which propagates forward in time and the other propagates backward in time. Let's focus attention on the following unitary operator:

$$\widehat{W}(t) \equiv e^{\frac{i}{\hbar}t\widehat{H}_e} e^{-\frac{i}{\hbar}t\widehat{H}_g} \tag{6.73}$$

where $\widehat{H}_g$ and $\widehat{H}_e$ are distinct Hamiltonians. We'll show that the kernel of this product of propagator will satisfy a Schrodinger equation of sorts. First we can calculate the evolution equation for $\widehat{W}(t)$,

$$\begin{aligned}
\frac{d}{dt}\widehat{W}(t) &= \left(\frac{d}{dt}e^{\frac{i}{\hbar}t\widehat{H}_e}\right)e^{-\frac{i}{\hbar}t\widehat{H}_g} + e^{\frac{i}{\hbar}t\widehat{H}_e}\left(\frac{d}{dt}e^{-\frac{i}{\hbar}t\widehat{H}_g}\right) \\
&= e^{\frac{i}{\hbar}t\widehat{H}_e}\left(\frac{i}{\hbar}\widehat{H}_e\right)e^{-\frac{i}{\hbar}t\widehat{H}_g} + e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}\left(-\frac{i}{\hbar}\widehat{H}_g\right) \\
&= \frac{i}{\hbar}\widehat{H}_e e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g} - \frac{i}{\hbar}e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}\widehat{H}_g \\
&= \frac{i}{\hbar}\widehat{H}_e\widehat{W}(t) - \frac{i}{\hbar}\widehat{W}(t)\widehat{H}_g,
\end{aligned}$$

where we used the fact that, from the Spectral Theorem, for a self-adjoint $\widehat{H}$ it is true that $[\widehat{H}, e^{-\frac{i}{\hbar}t\widehat{H}}] = 0$ for all values of t. Thus we have the evolution equation

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\widehat{W}(t) = \widehat{H}_e\widehat{W}(t) - \widehat{W}(t)\widehat{H}_g, \tag{6.74}$$

with the initial condition $\widehat{W}(0) = I$. It will be useful to express this equation with a slight change to bring it in line with the form of a Schrodinger equation, namely,

$$i\hbar\frac{\partial}{\partial t}\widehat{W}(t) = -\widehat{H}_e\widehat{W}(t) + \widehat{W}(t)\widehat{H}_g, \tag{6.75}$$

again, with the initial condition $\widehat{W}(0) = I$.

Just as in the previous parts of this work with propagating generalized Lagrangian states we will take the operator equation for $\widehat{W}(t)$ and translate it into a condition on its kernel. Let's denote the kernel of $\widehat{W}(t)$ (which is a bounded operator on Bargmann space) as $W_t(\mathbf{z}, \overline{\xi})$, where we know that W is holomorphic in $\mathbf{z}$ and anti-holomorphic in ξ. Because $\widehat{H}_e$ acts after (to the left of) $\widehat{W}(t)$ it acts with respect to the $\mathbf{z}$ variables, and since $\widehat{H}_g$ acts before (to the right of) $\widehat{W}(t)$ it will act with respect to the ξ variables. If we focus, again, on Hamiltonians that are polynomial, then for some constants $A_{\alpha\beta}$ and $B_{\alpha\beta}$ for multi-indices α and β

$$\widehat{H}_e = \sum_{\alpha,\beta} A_{\alpha\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta \qquad\qquad \widehat{H}_g = \sum_{\alpha,\beta} B_{\alpha\beta}\xi^\alpha\overline{\xi}^\beta.$$

Suppose that $\psi \in \mathcal{B}_1(\mathbf{C}^n)$. With such Hamiltonians the action of H_e on W_t is generated by the action of $\widehat{Z}_j = z_j I$ and $\widehat{Z}_j^* = \hbar\frac{\partial}{\partial z_j}$ via

$$\widehat{Z}_j^*\left(\widehat{W}(t)\psi\right)(\mathbf{z}) = \widehat{Z}_j^*\int_{\mathbf{C}^n} W_t(\mathbf{z}, \overline{\xi})\psi(\xi)d\xi d\overline{\xi} = \int_{\mathbf{C}^n}\left(\hbar\frac{\partial}{\partial z_j}W_t(\mathbf{z}, \overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi}, \tag{6.76}$$

$$\widehat{Z}_j\left(\widehat{W}(t)\psi\right)(\mathbf{z}) = \widehat{Z}_j\int_{\mathbf{C}^n} W_t(\mathbf{z}, \overline{\xi})\psi(\xi)d\xi d\overline{\xi} = \int_{\mathbf{C}^n}\left(z_j W_t(\mathbf{z}, \overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi}. \tag{6.77}$$

The analogous results that generate the action of H_g require a bit more work to derive. Since $\psi \in \mathcal{B}_1(\mathbf{C}^n)$ we have that $\psi(\xi) = \widetilde{\psi}(\xi)e^{-\frac{\xi\cdot\bar{\xi}}{2\hbar}}$ where for each $j = 1, \ldots, n$ we have $\frac{\partial}{\partial\bar{\xi}_j}\widetilde{\psi}(\xi) = 0$ (i.e. $\widetilde{\psi} \in \mathcal{B}_2(\mathbb{C}^n)$). Also, by (VI.1), there is a $\widetilde{W}_t(\mathbf{z}, \xi)$ such that $W_t(\mathbf{z}, \bar{\xi}) = \widetilde{W}_t(\mathbf{z}, \bar{\xi})e^{-\frac{\xi\cdot\bar{\xi}}{2\hbar}}$. Recall, from the definition of Bargmann space in chapter II, that operators on Bargmann space do not act on the Bargmann weights. The action of H_g on W_t is generated by the action of $\widehat{\Xi}_j = \xi_j I$ and $\widehat{\Xi}_j^* = \hbar\frac{\partial}{\partial\xi_j}$ via the calculations

$$
\begin{aligned}
\left(\widehat{W}(t)\widehat{\Xi}_j^*\psi\right)(\mathbf{z}) &= \int_{\mathbf{C}^n} W_t(\mathbf{z}, \bar{\xi})\left(\widehat{\Xi}_j^*\psi(\xi)\right)d\xi d\bar{\xi} \\
&= \int_{\mathbf{C}^n} W_t(\mathbf{z}, \bar{\xi})\left(\hbar\frac{\partial}{\partial\xi_j}\psi(\xi)\right)d\xi d\bar{\xi} \\
&= \int_{\mathbf{C}^n} \widetilde{W}_t(\mathbf{z}, \bar{\xi})\left(\hbar\frac{\partial}{\partial\xi_j}\widetilde{\psi}(\xi)\right)e^{-\frac{\xi\cdot\bar{\xi}}{\hbar}}d\xi d\bar{\xi} \\
&= -\hbar\int_{\mathbf{C}^n} \frac{\partial}{\partial\xi_j}\left(\widetilde{W}_t(\mathbf{z}, \bar{\xi})e^{-\frac{\xi\cdot\bar{\xi}}{\hbar}}\right)\widetilde{\psi}(\xi)d\xi d\bar{\xi} \\
&= -\hbar\int_{\mathbf{C}^n} \widetilde{W}_t(\mathbf{z}, \bar{\xi})\frac{\partial}{\partial\xi_j}\left(e^{-\frac{\xi\cdot\bar{\xi}}{\hbar}}\right)\widetilde{\psi}(\xi)d\xi d\bar{\xi} \\
&= -\hbar\int_{\mathbf{C}^n} \widetilde{W}_t(\mathbf{z}, \bar{\xi})\left(-\frac{1}{\hbar}\bar{\xi}_j e^{-\frac{\xi\cdot\bar{\xi}}{\hbar}}\right)\widetilde{\psi}(\xi)d\xi d\bar{\xi} \\
&= \int_{\mathbf{C}^n} \left(\bar{\xi}_j W_t(\mathbf{z}, \bar{\xi})\right)\psi(\xi)d\xi d\bar{\xi} \\
&= \int_{\mathbf{C}^n} \left(\bar{\widehat{\Xi}}_j W_t(\mathbf{z}, \bar{\xi})\right)\psi(\xi)d\xi d\bar{\xi},
\end{aligned}
$$

or more compactly

$$
\left(\widehat{W}(t)\widehat{\Xi}_j^*\psi\right)(\mathbf{z}) = \int_{\mathbf{C}^n} \left(\bar{\widehat{\Xi}}_j W_t(\mathbf{z}, \bar{\xi})\right)\psi(\xi)d\xi d\bar{\xi}; \tag{6.78}
$$

and

$$
\begin{aligned}
\left(\widehat{W}(t)\widehat{\Xi}_j\psi\right)(\mathbf{z}) &= \int_{\mathbf{C}^n} W_t(\mathbf{z},\overline{\xi})\left(\widehat{\Xi}_j\psi(\xi)\right)d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n} W_t(\mathbf{z},\overline{\xi})\left(\xi_j\psi(\xi)\right)d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n} \widetilde{W_t}(\mathbf{z},\overline{\xi})\xi_j\widetilde{\psi}(\xi)e^{-\frac{\xi\cdot\overline{\xi}}{\hbar}}d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n} \widetilde{W_t}(\mathbf{z},\overline{\xi})\widetilde{\psi}(\xi)\left(-\hbar\frac{\partial}{\partial\overline{\xi}_j}e^{-\frac{\xi\cdot\overline{\xi}}{\hbar}}\right)d\xi d\overline{\xi} \\
&= -(-\hbar)\int_{\mathbf{C}^n} \frac{\partial}{\partial\overline{\xi}_j}\left(\widetilde{W_t}(\mathbf{z},\overline{\xi})\widetilde{\psi}(\xi)\right)e^{-\frac{\xi\cdot\overline{\xi}}{\hbar}}d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n} \hbar\frac{\partial}{\partial\overline{\xi}_j}\left(\widetilde{W_t}(\mathbf{z},\overline{\xi})\right)\widetilde{\psi}(\xi)e^{-\frac{\xi\cdot\overline{\xi}}{\hbar}}d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n} \left(\widehat{\overline{\Xi}}^{*}_j W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi},
\end{aligned}
$$

thus

$$
\left(\widehat{W}(t)\widehat{\Xi}_j\psi\right)(\mathbf{z}) = \int_{\mathbf{C}^n} \left(\widehat{\overline{\Xi}}^{*}_j W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} \tag{6.79}
$$

where $\widehat{\overline{\Xi}}_j = \overline{\xi}_j I$ and $\widehat{\overline{\Xi}}^{*}_j = \hbar\frac{\partial}{\partial\overline{\xi}_j}$. Note that since the operators $\widehat{Z}_j$, $\widehat{Z}^*_j$, $\widehat{\Xi}_j$, $\widehat{\Xi}^*_j$, $\widehat{\overline{\Xi}}_j$, and $\widehat{\overline{\Xi}}^{*}_j$ are unbounded operators (on their respective Bargmann spaces), to make the above calculations rigorous as opposed to formal we need to impose the condition that $\psi(\xi) \in \mathcal{B}_1(\mathbb{C}^n_{\mathbf{z}}) \cap Dom\left(\widehat{\Xi}_j\right) \cap Dom\left(\widehat{\Xi}^*_j\right)$ and that (invoking VI.1) $W_t(\mathbf{z},\cdot) \in \mathcal{B}_1(\mathbb{C}^n_{\mathbf{z}}) \cap Dom\left(\widehat{Z}_j\right) \cap Dom\left(\widehat{Z}^*_j\right)$, as well as $W_t(\cdot,\overline{\xi}) \in \mathcal{B}_1(\mathbb{C}^n_{\mathbf{z}}) \cap Dom\left(\widehat{\overline{\Xi}}_j\right) \cap Dom\left(\widehat{\overline{\Xi}}^{*}_j\right)$.

We can extend these basic rules to conclude that under appropriate domain conditions on ψ and W_t, for Wick-quantized monomials $\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta$ and $\xi^\alpha\overline{\xi}^\beta$ (defined with respect to the correspondences $z_j \longrightarrow \widehat{Z}_j, \overline{z}_j \longrightarrow \widehat{Z}^*_j, \xi_j \longrightarrow \widehat{\Xi}_j, \overline{\xi}_j \longrightarrow \widehat{\Xi}^*_j$):

$$\left(\widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}\widehat{W}(t)\psi\right)(\mathbf{z}) = \widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}\int_{\mathbf{C}^n}\left(W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} \tag{6.80}$$

$$= \int_{\mathbf{C}^n}\left(\widehat{Z}^{\alpha}\widehat{Z^*}^{\beta}W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} \tag{6.81}$$

$$= \int_{\mathbf{C}^n}\left(\mathbf{z}^{\alpha}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\beta}W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} \tag{6.82}$$

(6.83)

and

$$\left(\widehat{W}(t)\widehat{\Xi}^{\alpha}\widehat{\Xi^*}^{\beta}\psi\right)(\mathbf{z}) = \int_{\mathbf{C}^n}W_t(\mathbf{z},\overline{\xi})\left(\widehat{\Xi}^{\alpha}\widehat{\Xi^*}^{\beta}\psi\right)(\xi)d\xi d\overline{\xi} \tag{6.84}$$

$$= \int_{\mathbf{C}^n}\left(\widehat{\overline{\Xi}}^{\beta}\widehat{\overline{\Xi}^*}^{\alpha}W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} \tag{6.85}$$

$$= \int_{\mathbf{C}^n}\left(\overline{\xi}^{\beta}\left(\frac{\partial}{\partial \overline{\xi}}\right)^{\alpha}W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi}. \tag{6.86}$$

The above results will now easily extend to the Wick quantization of any polynomial symbol, namely, if $H = H(\mathbf{z},\overline{\mathbf{z}})$ is a polynomial, then the above shows that

(6.87)

$$\left(\widehat{H}_{wick}\widehat{W}(t)\psi\right)(\mathbf{z}) = \widehat{H}_{wick}\int_{\mathbf{C}^n}\left(W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi} = \int_{\mathbf{C}^n}\left(\widehat{H}_{wick}W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi}$$

(6.88)

This is the Wick quantization with respect to the basic quantization rules $z_j \longrightarrow \widehat{Z}_j$ and $\overline{z}_j \longrightarrow \widehat{Z}_j^*$. Expressed in another way, this is Wick quantization of $H = H(\mathbf{z},\overline{\mathbf{z}})$ where the holomorphic coordinate (the one corresponding to the multiplication operator upon quantization) is taken to be $\mathbf{z}$.

The more complicated result will involve quantizing the polynomial symbol $\widetilde{H} = \widetilde{H}(\xi,\overline{\xi})$ in two different ways. First, suppose we Wick quantize this symbol H where the holomorphic coordinates are taken to be the ξ_j's. This corresponds to the basic rules $\xi_j \longrightarrow \widehat{\Xi}_j = \xi_j I$ and $\overline{\xi}_j \longrightarrow \widehat{\Xi}_j^* = \frac{\partial}{\partial \xi_j}$. Call the resulting operator $\widehat{H}_1$. Second, suppose we Wick quantize the symbol H where the holomorphic coordinates are taken to be the $\overline{\xi}$'s. This corresponds to the basic rules $\overline{\xi}_j \longrightarrow \widehat{\overline{\Xi}}_j = \overline{\xi}_j I$ and $\xi_j \longrightarrow \widehat{\overline{\Xi}}_j^* = \frac{\partial}{\partial \overline{\xi}_j}$. Call the resulting operator $\widehat{H}_2$. The above result shows us that

$$\begin{aligned}
\widehat{W}(t)\left(\widehat{H}_1\psi\right)(\mathbf{z}) &= \int_{\mathbf{C}^n} W_t(\mathbf{z},\overline{\xi})\left(\widehat{H}_1\psi(\xi)\right)d\xi d\overline{\xi} \\
&= \int_{\mathbf{C}^n}\left(\widehat{H}_2 W_t(\mathbf{z},\overline{\xi})\right)\psi(\xi)d\xi d\overline{\xi}.
\end{aligned}$$

We can immediately extend these relations to Weyl-quantized symbols by invoking $\widehat{H}_{weyl} = \left(\widehat{e^{\frac{\hbar}{4}\Delta}H}\right)_{wick}$, (4.20).

This allows us to interpret the operator expression $-\widehat{H}_e\widehat{W}(t) + \widehat{W}(t)\widehat{H}_g$, where $\widehat{H}_e$ and $\widehat{H}_g$ are the Weyl quantization of the polynomial symbols above, as a condition on the kernel of $\widehat{W}(t)$, $W_t(\mathbf{z},\overline{\xi})$, namely

$$\left(-\widehat{H}_e\widehat{W}(t) + \widehat{W}(t)\widehat{H}_g\right)\psi = \int_{\mathbb{C}^n}\left[\left(-\widehat{H}_e + \widehat{H}_g\right)W_t(\mathbf{z},\overline{\xi})\right]\psi(\xi)d\xi d\overline{\xi},$$

where $\widehat{H}_e$ is quantized with respect to the basic rules $z_j \longrightarrow \widehat{Z}_j = z_j I$ and $\overline{z}_j \longrightarrow \widehat{Z}_j^* = \frac{\partial}{\partial z_j}$ and $\widehat{H}_g$ is quantized with respect to the basic rules $\overline{\xi}_j \longrightarrow \widehat{\overline{\Xi}}_j = \overline{\xi}_j I$ and $\xi_j \longrightarrow \widehat{\overline{\Xi}}_j^* = \frac{\partial}{\partial \overline{\xi}_j}$.

This work has proven the following proposition:

Proposition VI.11. *Suppose that $\widehat{H}_e$ and $\widehat{H}_g$ are the Weyl-quantizations of the polynomial Hamiltonians*

$$\widehat{H}_e = \sum_{\alpha,\beta} A_{\alpha\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta \qquad\qquad \widehat{H}_g = \sum_{\alpha,\beta} B_{\alpha\beta}\xi^\alpha\overline{\xi}^\beta.$$

Therefore, if $W_t(\mathbf{z},\overline{\xi})$ *is the kernel of the operator* $\widehat{W}(t) \equiv e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}$*, then the evolution equation of* $\widehat{W}(t)$ *leads to the following evolution equation for* W_t*,*

$$i\hbar\frac{\partial}{\partial t}W_t(\mathbf{z},\overline{\xi}) = \left(-\widehat{H}_e + \widehat{H}_g\right)W_t(\mathbf{z},\overline{\xi}). \tag{6.89}$$

The initial condition $\widehat{W}(0) = I$ *is equivalent to the condition that at* $t = 0$ *the kernel* W_t *is equal to the reproducing kernel, i.e.*

$$W_0(\mathbf{z},\overline{\xi}) = e^{\frac{\overline{\xi}\cdot\mathbf{z}}{\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}. \tag{6.90}$$

The Classical Picture Of Forward-Backward Propagation

Since the goal of this section is to approximate the operator kernel of the product of propagators, $\widehat{W}(t) \equiv e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}$, with a generalized Lagrangian state, we will need a corresponding classical dynamics to pair with the quantum dynamics. The classical symplectomorphism that is analogous to the operator $\widehat{W}(t)$ is $\phi^e_{-t}\circ\phi^g_t : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, where ϕ^e_t is the Hamilton flow of H_e, and ϕ^g_t is the Hamilton flow of H_g. To begin to frame this situation in more familiar terms, suppose we choose $\Lambda = \mathbb{C}^n$ and define $\mathbf{w} : \mathbf{C}^n \longrightarrow \mathbf{C}^{2n}$ as $\mathbf{w}(\chi) = (\chi,\overline{\chi})$.

Define Φ_t as the Hamilton flow of the polynomial Hamiltonian $H = -H_e(\mathbf{z},\overline{\mathbf{z}}) + H_g(\xi,\overline{\xi})$, then define $\mathbf{w}_t : \mathbf{C}^n \longrightarrow \mathbb{C}^{2n}$ by $\mathbf{w}_t(\chi) = \Phi_t(\mathbf{w}(\chi))$ where again, as in the case of approximating the kernel of the propagator, we will make the association $\mathbb{C}^{2n} \cong \mathbb{C}^n \times \overline{\mathbb{C}^n}$ (meaning if $(\mathbf{z},\overline{\xi})$ are coordinates in $\mathbb{C}^{2n}$ then the symplectic form will be $\omega = id\mathbf{z}\wedge d\overline{\mathbf{z}} + id\overline{\xi}\wedge d\xi$). Now, note that since $H = -H_e(\mathbf{z},\overline{\mathbf{z}}) + H_g(\xi,\overline{\xi})$ then its flow $\Phi_t : \mathbb{C}^n \times \overline{\mathbb{C}^n} \longrightarrow \mathbb{C}^n \times \overline{\mathbb{C}^n}$ is defined by the $4n$ equations

$$\dot{z}_j = i\frac{\partial H}{\partial \overline{z}_j} = i\frac{\partial}{\partial \overline{z}_j}\left(-H_e\right) = -i\frac{\partial H_e}{\partial \overline{z}_j},$$

$$\dot{\overline{z}}_j = -i\frac{\partial H}{\partial z_j} = -i\frac{\partial}{\partial z_j}\left(-H_e\right) = i\frac{\partial H_e}{\partial z_j},$$

$$\dot{\overline{\xi}}_j = i\frac{\partial H}{\partial \xi_j} = i\frac{\partial}{\partial \xi_j}\left(H_g\right) = i\frac{\partial H_g}{\partial \xi_j},$$

$$\dot{\xi}_j = -i\frac{\partial H}{\partial \overline{\xi}_j} = -i\frac{\partial}{\partial \overline{\xi}_j}\left(H_g\right) = -i\frac{\partial H_g}{\partial \overline{\xi}_j}.$$

The $2n$ equations that define the dynamics of the z_j and $\overline{z}_j$ represent the backward flow of H_e, and the $2n$ equations that represent the dynamics of $\overline{\xi}_j$ and ξ_j variables represents the forward flow of H_g. Thus, $\Phi_t(\mathbf{z}_0, \overline{\xi}_0) = (\phi^e_{-t}(\mathbf{z}_0), \phi^g_t(\overline{\xi_0})) = (\phi^e_{-t}(\mathbf{z}_0), \overline{\phi^g_{-t}(\xi_0)})$. thus $\mathbf{w}(\chi, t) = \Phi_t(\mathbf{w}(\chi)) = (\phi^e_{-t}(\chi), \overline{\phi^g_{-t}(\chi)})$.

The fact that the composite Hamilton flow is a symplectomorphism guarantees that with respect to this choice of symplectic form, $\mathbf{w}(\Lambda)$ is indeed a Lagrangian submanifold of $\mathbb{C}^{2n}$. With this choice of Lagrangian, we can visualize the classical dynamics that will be underwriting the quantum dynamics pictorially,

$$\begin{tikzcd}
& \mathbb{C}^n \times \overline{\mathbb{C}^n} \times S^1 \arrow[r, "\widetilde{\phi}_t"] \arrow[d, "\pi"] & \mathbb{C}^n \times \overline{\mathbb{C}^n} \times S^1 \arrow[d, "\pi"] \\
\Lambda \cong \mathbb{C}^n \arrow[ur, "{(w,-f)}"] \arrow[r, "w"] & \mathbb{C}^n \times \overline{\mathbb{C}^n} \arrow[r, "\phi^H_t"] & \mathbb{C}^n \times \overline{\mathbb{C}^n}
\end{tikzcd}$$

With the classical dynamics in place, consider the following ansatz (a generalized Lagrangian state, that is indeed 'generalized' since Λ is not compact) for an approximation to the kernel of the propagator

(6.91)

$$\digamma_t(\mathbf{z},\overline{\xi}) = \int_{\mathbb{C}^n} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}e^{\frac{\overline{\phi^e_{-t}(\chi)}\cdot\mathbf{z}}{\hbar}}e^{\frac{\overline{\chi}\cdot(\phi^g_{-t}(\chi))}{\hbar}}e^{-\frac{\phi^g_{-t}(\chi)\cdot\overline{\phi^g_{-t}(\chi)}}{2\hbar}}e^{-\frac{\phi^e_{-t}(\chi)\cdot\overline{\phi^e_{-t}(\chi)}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}$$

(6.92)

$$= \int_{\mathbb{C}^n} b(\upsilon,t;\hbar)e^{\frac{i}{\hbar}f(\upsilon,t)}e^{\frac{\overline{\phi^e_{-t}(\phi^g_t(\upsilon))}\cdot\mathbf{z}}{\hbar}}e^{\frac{\overline{\xi}\cdot\upsilon}{\hbar}}e^{-\frac{\upsilon\cdot\overline{\upsilon}}{2\hbar}}e^{-\frac{\phi^e_{-t}(\phi^g_t(\upsilon))\cdot\overline{\phi^e_{-t}(\phi^g_t(\upsilon))}}{2\hbar}}e^{-\frac{\mathbf{z}\cdot\overline{\mathbf{z}}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\upsilon d\overline{\upsilon}.$$

where we can go from the first to the second line by making the change of variables $\chi \longrightarrow \phi^g_t(\upsilon)$, which, since ϕ^g_t is a symplectomorphism the Jacobian of this change of variables is 1. The lift function f satisfies the lift condition, III.9, associated to the parameter-dependent Lagrangian embedding $\mathbf{w}(\chi,t) = \phi^H_t(\mathbf{w}(\chi))$. And, technically, $b(\chi,t;\hbar) \equiv a(\phi^g_{-t}(\chi),t;\hbar)$, where we assume that b is an amplitude, III.11, thus it has an asymptotic expansion of the form

$$b \sim \sum_{j=0}^{\infty} \hbar^j b_j,$$

such that the $b_j's$ are as smooth as we require. $\digamma_t(\mathbf{z},\overline{\xi})$ fits into the general form for our generalized Lagrangian states with $\mathbf{r} = (\mathbf{z},\overline{\xi}) \in \mathbb{C}^{2n}$, and $\mathbf{w}(\chi,t) = \mathbf{w}_t(\chi) = (\phi^e_{-t}(\chi),\overline{\phi^g_{-t}(\chi)})$, i.e.

$$\digamma_t(\mathbf{r}) = \int_{\mathbb{C}^n} b(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}e^{\frac{\overline{\mathbf{w}(\chi,t)}\cdot\mathbf{r}}{\hbar}}e^{-\frac{\mathbf{w}(\chi,\mathbf{t})\cdot\overline{\mathbf{w}(\chi,t)}}{2\hbar}}e^{-\frac{\mathbf{r}\cdot\overline{\mathbf{r}}}{2\hbar}}d\chi d\overline{\chi}.$$

Above we concluded that can we require W_t, the exact kernel of $\widehat{W}(t)$ to satisfy the Schrodinger equation. Thus, if $\digamma_t$ is to be an approximation to W_t we will seek to have it approximately satisfy the Schrodinger equation. The calculations for $\digamma_t$ will follow the calculations done for the propagator (above) exactly.

Remark VI.12. In computing with the ansatz $\digamma_t$ one should compute with the first form in 6.91 and then make the change of variables to the second form once the transport equations for the a_j's have been calculated. The reason for this is that the calculational tools used to derive the transport equations are predicated on the form of $\mathcal{F}_t$ given in the first expression (i.e. before the change of variables).

Viewing the task of approximating the kernel of $\widehat{W}(t)$ in the light of propagating a generalized Lagrangian state we apply the process of deriving the transport equations that was outlined as part of the main result in V; we have the general form for the expansion coefficients of 'a', namely a_0 satisfies the first order ordinary differential equation

$$i\frac{\partial a_0}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{2n}\frac{\partial^2 H}{\partial r_k \partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_k}{\partial \chi_l}\right)^{-1}\right)a_0,$$

with initial condition $a_0(\chi, 0) = 1$. Where $\mathbf{r} = (\mathbf{z}, \overline{\xi})$, and with a slight abuse of notation we double dip by representing χ as an element in $\mathbb{C}^n$ in the ansatz, but here the derivatives with respect to χ_l are really with respect to the $2n$ real variables from which χ is formed. For the other terms in the expansion of a we have for $s \leq M$:

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{2n}\frac{\partial^2 H}{\partial r_k \partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1} + F_s(a_0, \ldots, a_{s-2}),$$

and for $s > M$

$$i\frac{\partial a_{s-1}}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w},\overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{2n}\frac{\partial^2 H}{\partial r_k \partial r_j}(\mathbf{w},\overline{\mathbf{w}})\left(\frac{\partial w_j}{\partial x_l}\right)\left(\frac{\partial \overline{w}_k}{\partial x_l}\right)^{-1}\right)a_{s-1} + F_M(a_{s-(M)}, \ldots, a_{s-2}),$$

.

and $a_s(\chi, 0) = 0$ for $s > 0$.

An Example With Oscillators

Let's consider the case where both $\widehat{H}_g$ and $\widehat{H}_e$ are oscillator Hamiltonians, and we wish to use a generalized Lagrangian state to approximate the kernel of $\widehat{W}(t) \equiv e^{\frac{i}{\hbar}t\widehat{H}_e}e^{-\frac{i}{\hbar}t\widehat{H}_g}$. Specifically, suppose these Hamiltonians are the Weyl quantization of the (classical) Hamiltonians (expressed as functions over $\mathbb{R}^{2n}$)

$$H_g(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\omega_g\mathbf{q}^2, \qquad H_e(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\omega_e(\mathbf{q}-\widetilde{\mathbf{q}})^2.$$

Since the salient feature of the frequencies is that $\omega_g \neq \omega_e$ we can, without loss of generality, assume that $\omega_g = 1$. We can analytically solve Hamilton's equations to compute the flow associated to these harmonic potentials. Let's first calculate the classical flows that will be necessary for the task at hand.

To solve for the flow of H_g we have

$$\dot{p}_j = -\frac{\partial H_g}{\partial q_j} = -q_j \quad \text{and} \quad \dot{q}_j = \frac{\partial H_g}{\partial p_j} = p_j.$$

The solutions to these equations are

$$p_j(t) = p_{0_j}\cos(t) - q_{0_j}\sin(t) \quad \text{and} \quad q_j(t) = q_{0_j}\cos(t) + p_{0_j}\sin(t).$$

With our choice of complex structure on $\mathbb{C}^n$, $\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$, we have

$$\begin{aligned}
\mathbf{z}(t) &= \frac{1}{\sqrt{2}}(\mathbf{q}(t) - i\mathbf{p}(t)) \\
&= \frac{1}{\sqrt{2}}(\mathbf{q}_0\cos(t) + \mathbf{p}_0\sin(t) - i\mathbf{p}_0\cos(t) + i\mathbf{q}_0\sin(t)) \\
&= \frac{1}{\sqrt{2}}\Big((\mathbf{q}_0 - i\mathbf{p}_0)\cos(t) + i(\mathbf{q}_0 - i\mathbf{p}_0)\sin(t)\Big) \\
&= \mathbf{z}_0 e^{it} \qquad \left(\text{where } \mathbf{z}_0 = \frac{1}{\sqrt{2}}(\mathbf{q}_0 - i\mathbf{p}_0)\right),
\end{aligned}$$

thus we have the well known fact (that has been used at several points previously) $\phi_t^g(\mathbf{z}) = e^{it}\mathbf{z}$.

To solve for the flow of H_e we have

$$\dot{p}_j = -\frac{\partial H_g}{\partial q_j} = -\omega_e(q_j - \widetilde{q}_j) \quad \text{and} \quad \dot{q}_j = \frac{\partial H_g}{\partial p_j} = p_j.$$

The solutions to these equations are

$$p_j(t) = p_{0j}\cos(\sqrt{\omega_e}t) - \sqrt{\omega_e}(q_{0j} - \widetilde{q}_j)\sin(\sqrt{\omega_e}t),$$

and

$$q_j(t) = (q_{0j} - \widetilde{q}_j)\cos(\sqrt{\omega_e}t) + \frac{p_{0j}}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t) + \widetilde{q}_j.$$

Again, with the choice of complex structure on $\mathbb{C}^n$, $\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$, we have

$$\begin{aligned}
\mathbf{z}(t) &= \frac{1}{\sqrt{2}}(\mathbf{q}(t) - i\mathbf{p}(t)) \\
&= \frac{1}{\sqrt{2}}\Bigg((\mathbf{q}_0 - \widetilde{\mathbf{q}})\cos(\sqrt{\omega_e}t) + \frac{\mathbf{p}_0}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t) + \widetilde{\mathbf{q}} - i\mathbf{p}_0\cos(\sqrt{\omega_e}t) \\
&\qquad\qquad + i\sqrt{\omega_e}(\mathbf{q}_0 - \widetilde{\mathbf{q}})\sin(\sqrt{\omega_e}t)\Bigg) \\
&= \frac{1}{\sqrt{2}}(\mathbf{q}_0 - i\mathbf{p}_0)\cos(\sqrt{\omega_e}t) - \frac{\widetilde{\mathbf{q}}}{\sqrt{2}}\cos(\sqrt{\omega_e}t) + i\sqrt{\frac{\omega_e}{2}}\mathbf{q}_0\sin(\sqrt{\omega_e}t) \\
&\quad - i\sqrt{\frac{\omega_e}{2}}\widetilde{\mathbf{q}}\sin(\sqrt{\omega_e}t) + \frac{1}{\sqrt{2\omega_e}}\mathbf{p}_0\sin(\sqrt{\omega_e}t) + \frac{\widetilde{\mathbf{q}}}{\sqrt{2}} \\
&= \mathbf{z}_0\cos(\sqrt{\omega_e}t) + i\sqrt{\frac{\omega_e}{2}}\frac{1}{\sqrt{2}}(\mathbf{z}_0 + \overline{\mathbf{z}}_0)\sin(\sqrt{\omega_e}t) \\
&\quad + \frac{1}{\sqrt{2\omega_e}}\frac{i}{\sqrt{2}}(\mathbf{z}_0 - \overline{\mathbf{z}}_0)\sin(\sqrt{\omega_e}t) + \frac{1}{\sqrt{2}}(\cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t))\widetilde{\mathbf{q}} \\
&\quad + \frac{\widetilde{\mathbf{q}}}{\sqrt{2}} \\
&= \left(\cos(\sqrt{\omega_e}t) + \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t))\right)\mathbf{z}_0 \\
&\quad + \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t))\overline{\mathbf{z}}_0 \\
&\quad + \frac{1}{\sqrt{2}}\Big(1 - \cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t)\Big)\widetilde{\mathbf{q}},
\end{aligned}$$

where $\mathbf{z}_0 = \frac{1}{\sqrt{2}}(\mathbf{q}_0 - i\mathbf{p}_0)$, so $\mathbf{q}_0 = \frac{1}{\sqrt{2}}(\mathbf{z}_0 + \overline{\mathbf{z}}_0)$ and $\mathbf{p}_0 = \frac{i}{\sqrt{2}}(\mathbf{z}_0 - \overline{\mathbf{z}}_0)$. Note that if $\omega_e = 1$ and $\widetilde{\mathbf{q}} = 0$ then this formula indeed reduces the to above case for H_g. Thus, the (complexified) flow of the oscillator Hamiltonian H_e is

$$\begin{aligned}
\phi_t^e(\mathbf{z}) = &\left(\cos(\sqrt{\omega_e}t) + \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t))\right)\mathbf{z} \\
&+ \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t))\overline{\mathbf{z}} \\
&+ \frac{1}{\sqrt{2}}\Big(1 - \cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t)\Big)\widetilde{\mathbf{q}}.
\end{aligned}$$

The classical analog of the quantum evolution defined by $\widehat{W}(t)$ is the composite flow $\phi_{-t}^e(\phi_t^g(\mathbf{z}))$. Explicitly, we have the fact that we will use later, that

$$
\begin{aligned}
\phi^e_{-t}(\phi^g_t(\mathbf{z})) &= \left(\cos(\sqrt{\omega_e}(-t)) + \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}(-t))\right)\phi^g_t(\mathbf{z}) \\
&\quad + \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}(-t))\overline{\phi^g_t(\mathbf{z})} \\
&\quad + \frac{1}{\sqrt{2}}\left(1 - \cos(\sqrt{\omega_e}(-t)) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}(-t))\right)\widetilde{\mathbf{q}} \\
&= \left(\cos(\sqrt{\omega_e}t) - \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t)\right)e^{it}\mathbf{z} \\
&\quad - \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t))e^{-it}\overline{\mathbf{z}} \\
&\quad + \frac{1}{\sqrt{2}}\left(1 - \cos(\sqrt{\omega_e}t) - i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t)\right)\widetilde{\mathbf{q}}.
\end{aligned}
$$

Remark VI.13. Note that the choice of complex structure $\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} - i\mathbf{p})$ used in associating $\mathbb{R}^{2n} \cong \mathbb{C}^n$ makes the flow of H_g, ϕ^g_t, holomorphic. However, the flow of H_e, ϕ^e_t is not holomorphic with respect to this complex structure. The fact that H_e is an oscillator means that we could define another complex structure (incorporating ω_e and $\widetilde{\mathbf{q}}$) which would make ϕ^e_t holomorphic, but there isn't a complex structure that would allow for both, and hence the composite flow, to be holomorphic.

Now, for the sake of making this an example that well illustrates the process of using a generalized Lagrangian state to approximate the kernel of $\widehat{W}(t)$, let's now specialize to the case of n=1. As stated in the remark VI.12 we should perform the calculations to derive the transport equations with the following ansatz

$$
F_t(z,\overline{\xi}) = \int_{\mathbb{C}} a(\chi,t;\hbar)e^{\frac{i}{\hbar}f(\chi,t)}e^{\frac{\overline{\phi^e_{-t}(\chi)}\cdot z}{\hbar}}e^{\frac{\overline{\chi}\cdot(\phi^g_{-t}(\chi))}{\hbar}}e^{-\frac{\phi^g_{-t}(\chi)\cdot\overline{\phi^g_{-t}(\chi)}}{2\hbar}}e^{-\frac{\phi^e_{-t}(\chi)\cdot\overline{\phi^e_{-t}(\chi)}}{2\hbar}}e^{-\frac{z\cdot\overline{z}}{2\hbar}}e^{-\frac{\xi\cdot\overline{\xi}}{2\hbar}}d\chi d\overline{\chi}.
$$

Now with an eye toward calculating the other quantities relevant in the transport equations we have the Hamiltonians H_g and H_e expressed in the appropriate complex coordinates

$$H_g(\xi, \overline{\xi}) = \xi\overline{\xi},$$

$$H_e(z, \overline{z}) = \frac{1}{2}(1+\omega_e)z\overline{z} + \frac{1}{4}(\omega_e - 1)z^2 + \frac{1}{4}(\omega_e - 1)\overline{z}^2 - \frac{\widetilde{q}}{2\sqrt{2}}z - \frac{\widetilde{q}}{2\sqrt{2}}\overline{z} + \frac{1}{2}\omega_e\widetilde{q}^2.$$

Then, for $H(z, \overline{z}, \xi, \overline{\xi}) = -H_e(z, \overline{z}) + H_g(\xi, \overline{\xi})$

$$\Delta H(z, \overline{z}, \xi, \overline{\xi}) = 2\left(\frac{\partial^2}{\partial z \partial \overline{z}} + \frac{\partial^2}{\partial \xi \partial \overline{\xi}}\right)(-H_e(z, \overline{z}) + H_g(\xi, \overline{\xi})) = 2\left(-\frac{1}{2}(1+\omega_e) + 1\right) = 1 - \omega_e.$$

Since $r = (z, \overline{\xi})$ then

$$\frac{\partial^2}{\partial r_j \partial r_k} H = \delta_{1j}\delta_{1k}\left(\frac{\partial^2}{z^2}\right) H = \frac{1}{4}(\omega_e - 1)\delta_{1j}\delta_{1k},$$

since all the other combinations other than $j = 1$ and $k = 1$ are zero. Finally, recall that the Lagrangian embedding we are using for this ansatz is $w(\chi, t) = (\phi^e_{-t}(\chi), \overline{\phi^g_{-t}(\chi)})$. So, if w_1 and w_2 are defined as

$$\begin{aligned}
w_1(\chi, t) &\equiv \phi^e_{-t}(\mathbf{z}) \\
&= \left(\cos(\sqrt{\omega_e}t) - \frac{i}{2}\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t)\right)\chi \\
&\quad - \frac{i}{2}\left(\sqrt{\omega_e} - \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t)\overline{\chi} \\
&\quad + \frac{1}{\sqrt{2}}\left(1 - \cos(\sqrt{\omega_e}t) - i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t)\right)\widetilde{\mathbf{q}}.
\end{aligned}$$

and

$$w_2(\chi, t) = \overline{\phi^g_{-t}(\chi)} = \overline{\chi}e^{it},$$

and we use the notation that $\chi = \chi_1 + i\chi_2$ then we have

$$\left(\frac{\partial w}{\partial \chi}\right) = \begin{pmatrix} \frac{\partial w_1}{\partial \chi_1} & \frac{\partial w_1}{\partial \chi_2} \\ \frac{\partial w_2}{\partial \chi_1} & \frac{\partial w_2}{\partial \chi_2} \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\omega_e}t) - i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t) & i\left(\cos(\sqrt{\omega_e}t) - \frac{i}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t)\right) \\ e^{it} & -ie^{it} \end{pmatrix}.$$

Then

$$det\left(\frac{\partial w}{\partial \chi}\right) = e^{it}\left(\left(\frac{1}{\sqrt{\omega_e}} - \sqrt{\omega_e}\right)\sin(\sqrt{\omega_e}t) + 2i\cos(\sqrt{\omega_e}t)\right).$$

Which in turn gives us

$$\left(\overline{\left(\frac{\partial w}{\partial \chi}\right)^{-1}}\right)^T = \frac{1}{\overline{det\left(\frac{\partial w}{\partial \chi}\right)}} \begin{pmatrix} ie^{-it} & e^{-it} \\ i\left(\cos(\sqrt{\omega_e}t) + \frac{1}{\sqrt{\omega_e}}\sin(\sqrt{\omega_e}t)\right) & \cos(\sqrt{\omega_e}t) + i\sqrt{\omega_e}\sin(\sqrt{\omega_e}t) \end{pmatrix}.$$

Putting these pieces together, and simplifying a little, we get the transport equation for a_0

$$i\frac{\partial a_0}{\partial t} = \frac{1}{4}\left(1 - \omega_e + \frac{1}{2}(\omega_e - 1)\frac{\left(\frac{1}{\sqrt{\omega_e}} + \sqrt{\omega_e}\right)\sin(\sqrt{\omega_e}t) + 2i\cos(\sqrt{\omega_e}t)}{e^{2it}\left(\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}t) + 2i\cos(\sqrt{\omega_e}t)\right)}\right)a_0.$$

The solution, taking into account the initial condition $a_0(\chi, 0) = 1$, we have

$$a_0(\chi, t) = exp\left(-\frac{i}{4}\int_0^t \left(1+\omega_e+\frac{1}{2}(\omega_e-1)\frac{\left(\frac{1}{\sqrt{\omega_e}} + \sqrt{\omega_e}\right)\sin(\sqrt{\omega_e}\tau) + 2i\cos(\sqrt{\omega_e}t\tau)}{e^{2i\tau}\left(\left(\sqrt{\omega_e} + \frac{1}{\sqrt{\omega_e}}\right)\sin(\sqrt{\omega_e}\tau) + 2i\cos(\sqrt{\omega_e}\tau)\right)}\right)d\tau\right).$$

Note that a_0 does not depend on χ.

The transport equation for a_1 is

$$i\frac{\partial a_1}{\partial t} = \left(\frac{1}{4}\left(\Delta H\right)(\mathbf{w}, \overline{\mathbf{w}}) + \frac{1}{2}\sum_{l=1}^{2}\frac{\partial^2 H}{\partial z^2}(\mathbf{w}, \overline{\mathbf{w}})\left(\frac{\partial w_1}{\partial \chi_l}\right)\left(\frac{\partial \overline{w}_1}{\partial \chi_l}\right)^{-1}\right)a_1 + F_1(a_0).$$

Now, by definition, see (4.22) for $k = 2$, $F_1(a_0)$ is the amplitude of the generalized Lagrangian state $D_2^H \left| \Lambda, a_0 \right\rangle$. Since a_0 is independent of χ, and $D_2^H \left| \Lambda, a_0 \right\rangle$ will have an amplitude where derivatives with respect to χ_1 and χ_2 fall on a_0 in every term we have that $F_1(a_0) = 0$. Thus a_1 satisfies the same transport equation as a_0 except the initial condition for a_1 is $a_1(\chi, 0) = 0$; this gives that $a_1(\chi, t) \equiv 0$. Indeed, this pattern will hold for every a_s for $s \geq 1$.

Thus we arrive at the conclusion that following the procedure for propagating a generalized Lagrangian state we get an approximation to the kernel of the product of propagators

$$\begin{aligned} \digamma_t(z, \overline{\xi}) &= \int_{\mathbb{C}^n} a_0(\chi, t; \hbar) e^{\frac{i}{\hbar} f(\chi, t)} e^{\frac{\overline{\phi^e_{-t}(\chi) \cdot z}}{\hbar}} e^{\frac{\overline{\chi} \cdot (\phi^g_{-t}(\chi))}{\hbar}} e^{-\frac{\phi^g_{-t}(\chi) \cdot \overline{\phi^g_{-t}(\chi)}}{2\hbar}} e^{-\frac{\phi^e_{-t}(\chi) \cdot \overline{\phi^e_{-t}(\chi)}}{2\hbar}} e^{-\frac{z \cdot \overline{z}}{2\hbar}} e^{-\frac{\xi \cdot \overline{\xi}}{2\hbar}} d\chi d\overline{\chi} \\ &= \int_{\mathbb{C}^n} a_0(\upsilon, t; \hbar) e^{\frac{i}{\hbar} f(\upsilon, t)} e^{\frac{\overline{\phi^e_{-t}(\phi^g_t(\upsilon)) \cdot \mathbf{z}}}{\hbar}} e^{\frac{\xi \cdot \upsilon}{\hbar}} e^{-\frac{\upsilon \cdot \overline{\upsilon}}{2\hbar}} e^{-\frac{\phi^e_{-t}(\phi^g_t(\upsilon)) \cdot \overline{\phi^e_{-t}(\phi^g_t(\upsilon))}}{2\hbar}} e^{-\frac{z \cdot \overline{z}}{2\hbar}} e^{-\frac{\xi \cdot \overline{\xi}}{2\hbar}} d\upsilon d\overline{\upsilon}. \end{aligned}$$

where to go from the first line to the second one performs the volume preserving change of variables $\upsilon = \phi^g_{-t}(\chi)$, and it should be noted that the amplitude, a_0, didn't change because it is independent of χ.

BIBLIOGRAPHY

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[1] Abers, Ernest, S., *Quantum Mechanics*, Pearson Education, (2004)

[2] Abraham, Ralph; Marsden, Jerrold E., *Foundations of Mechanics*, Second Edition, AMS Chelsea Publishing, (1978)

[3] Alexandrova, Ivana, *Semi-Classical Wavefront Set and Fourier Integral Operators* arXiv:math/0407460v1 [math.AP] 27 July 2004

[4] Ali, Syed Twareque; Antoine, Jean-Pierre; Gazeau, Jean-Pierre, *Coherent States, Wavelets and Their Generalizations*, Springer, (1999)

[5] Arnold, V.I., *Mathematical Methods of Classical Mechanics*, Second Edition, Springer, (1989)

[6] Bargmann, V., *On a Hilbert Space of Analytic Functions and an Associated Integral Transform, Part 1*, Communications in Pure and Applied Mathematics, volume 14, pages 187-214, (1961)

[7] Bates, Sean; Weinstein, Alan, *Lectures on the Geometry of Quantization*

[8] Berezin, F.A., *Covariant and Contravariant Symbols of Operators*, Math. USSR Izvestija, volume 6, No. 5, (1972)

[9] Bott, Raoul; Tu, Loring W., *Differential Forms in Algebraic Topology*, Springer, (1982)

[10] Borthwick, D.; Paul, T.; Uribe, A., *Legendrian distributions with applications to relative Poincaré series*, Inventiones mathematicae, volume 122, pages 359-402, (1995)

[11] Butler, J., *Global* $\hbar$ *Fourier Integral Operators With Complex-Valued Phase Functions*, Bulletin of the London Mathematical Society, volume 34, pages 479-489, (2002)

[12] Deshpande, Sarin A.; Ezra, Gregory S., *On the Derivation of the Herman-Kluk Propagator*, Journal of Physics A, volume 39, pages 5067-5078, (2006)

[13] Duistermaat, J.J., *Fourier Integral Operators*, Progress in Mathematics Volume 130, Birkh́'auser, 1996

[14] Evans, Lawrence C.; Zworski, Maciej, *Lectures on Semiclassical Analysis: Version 0.75*, University of California at Berkeley

[15] Folland, Gerald B., *Harmonic Analysis in Phase Space*, Princeton Univsersity Press, (1989)

[16] Goldstein, Herbert; Poole, Charles; Safko, John, *Classical Mechanics*, Third Edition, Pearson Education, (2001)

[17] Guillemin, Victor; Sternberg, Schlomo, *Semi-classical Analysis*, (2004)

[18] Hall, Brian C., *Holomorphic Methods in Analysis and Mathematical Physics*, Contemporary Mathematics, volume 260, pages 1-59, (2000)

[19] Herman, Michael F.; Kluk, Edward, *A Semiclassical Justification For The Use Of Non-Spreading Wavepackets In Dynamics Calculations*, Chemical Physics, volume 91, pages 27-34, (1984)

[20] H́ormonder, Lars, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer, Classics In Mathematics Series, (2003)

[21] Karasev, M.V., *Connections on Lagrangian Submanifolds and Some Problems in Quasiclassical Approximation I*, Journal Of Mathematical Sciences, volume 59, No. 5, pages 1053-1062, (1989)

[22] Karasev, M.V., *Integral Representations over Isotropic Submanifolds and Equations of Zero Curvature*, Advances in Mathematics, Volume 135, No. 2, pages 220-286, (1998)

[23] Kay, Kenneth G., *Semiclassical Initial Value Treatments of Atoms and Molecules*, Annual Review of Physical Chemistry, volume 56, pages 255-280, (2005)

[24] Kay, Kenneth G., *The Herman-Kluk approximation: Derivation and Semiclassical Corrections*, Chemical Physics, volume 322, pages 3-12, (2006)

[25] Kobayashi, Shoshichi; Nomizu, Katsumi, *Foundations of Differential Geometry Volume I*, Wiley Classics Library, John Wiley and Sons, Inc., (1996)

[26] Laptev, A.; Sigal, I.M., *Global Fourier Integral Operators and Semiclassical Asymptotics*, Review of Mathematical Physics, volume 12, No. 5, pages 749-766, (2000)

[27] Landau, L.D.; Lifshitz, E.M., *Mechanics*, Course of Theoretical Physics Volume I, Third Edition, Butterworth Heinemann, (1976)

[28] Martinez, Andre, *An Introduction to Semiclassical and Microlocal Analysis*, Springer, (2002)

[29] McDuff, Dusa; Salamon, Dietmar, *Introduction to Symplectic Topology*, Second Edition, Oxford University Press, (1998)

[30] Morita, Shigeyuki, *Geometry of Differential Forms*, Iwanami Series in Modern Mathematics: Translations of Mathematical Monographs Volume 201, AMS, (1998)

[31] Mukamel, Shaul, *Principles of Nonlinear Optical Spectroscopy*, Oxford University Press, USA, (1999)

[32] Nicolaescu, Liviu I., *Geometry of Manifolds*, World Scientific, (1996)

[33] Paul, T.; Uribe, A., *A Construction of Quasi-modes Using Coherent States*, Ann. Inst. Henri Poincare, volume 59, No. 4, pages 357-381, (1993)

[34] Reed, Michael; Simon, Barry, *Methods Of Modern Mathematical Physics I: Functional Analysis*, Academic Press, (1980)

[35] Robert, Didier, *On the Herman-Kluk Semiclassical Approximation* arXiv:0908.0847 [math-ph] 6 Aug 2009

[36] Rousse, V., *Semiclassical Initial Value Representations*, arXiv:0904.0387v2 [math-ph] 1 July 2009

[37] Sakurai, J.J., *Modern Quantum Mechanics*, The Benjamin/Cummings Publication Company, (1985)

[38] Spivak, Michael, *A Comprehensive Introduction to Differential Geometry: Volume I*, Third Edition, Publish or Perish, Inc., (2005)

[39] Rousse, V.; Swart, T., *Global L^2-Boundedness Theorems for Semiclassical Fourier Integral Operators with Complex Phase*, arXiv:0710.4200v1 [math-ph] 23 October 2007

[40] Rousse, V.; Swart, T., *A Mathematical Justification for the Hermann-Kluk Propagator*, Communications in Mathematical Physics, Volume 286, Number 2, 725-750, (2009)

[41] Tyurin, Andrei, *Quantization, Classical and Quantum Field Theory, and Theta Functions*, American Mathematical Society, (2003)

[42] Woodhouse, N.M.J., *Geometric Quantization*, Second Edition, Oxford University Press, (1991)

[43] Zheng, Fangyang, *Complex Differential Geometry*, Studies in Advanced Mathematics Volume 18, AMS/International Press, (2000)