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General Image Method in a Plane-Layered Elastostatic Medium

Introduction

The study of elastic deformation fields in plane-layered media due to localized elastic disturbance (e.g., cracks) is relevant to several areas of applied mechanics. Theses areas include the study of deformations near fault zones in layered earth models and the study of the structural strength and micromechanics of layered composites. When studying such problems it is often important to determine the elastic response due to disturbances localized at a point (e.g., point forces, dilatation sources, and nuclei of strain, hereafter referred to as point sources) or along a line (e.g., line forces and dislocations, hereafter referred to as line sources). These point source and line source solutions often lead to an appropriate theoretical formulation of a variety of problems in the layered media under consideration, which could then be efficiently solved either analytically or numerically.

This paper describes a new general algorithm for obtaining the elastic response to point and line source disturbances in either of two mediums. The first medium consists of two bonded elastic half spaces having different elastic moduli (Fig. 1(*a*)) and the second medium consists of an elastic layer perfectly bonded to two elastic half spaces of different moduli (Fig. 1(*b*)) hereafter referred to as mediums A and B, respectively. When the elastic response due to an antiplane line source disturbance (e.g., a screw dislocation) is required, the algorithm is seen to correspond to the scalar image method. Hence, the present algorithm for obtaining the elastic fields due to point and line sources will be referred to as the generalized image method.

The image method for antiplane and plane strain problems has been applied to obtain the screw dislocation solution (Chou, 1966) in mediums A and B, the edge dislocation solution (Head, 1953) in medium A and (Stagni and Lizzio, 1987) in medium B. Various other researchers have studied the edge dislocation problem in special cases of medium B, where either one or both of the interfaces are either traction-free or frictionless (e.g., Lee and Dundurs, 1973; Nabarro and Kostlan, 1978; Moss and Hoover, 1978; Stagni and Lizzio, 1986).

Other known point and line source elastic fields include the line force parallel to a free surface (special case of medium A (Mellan, 1932)), point force in an elastic half space with a free surface (special case of medium A (Mindlin, 1936)), and point force in medium A (Rongved, 1955).

A systematic formulation for the derivation of point sources in a multilayered medium (medium B being a special case) was presented by Ben-Menahem and Singh (1968a,b) and later refined by Singh (1970) and independently by Sato (1971). Sato and Matsu'ura (1973) and Jovanovich et al. (1974a,b) made use of the Sato and Singh formulations respectively to calculate surface deformations for a variety of nuclei of strain disturbances and a variety of layered media having a free surface. The Ben-Menahem and Singh and the Sato formulations determine the required elastic fields in terms of inverse integral transforms which have to be numerically inverted for specific numerical parameters. Rundle and Jackson (1977) presented an approximate solution (as the sum of an infinite series) corresponding to a shearing nucleus of strain where the shearing plane is parallel to the interfaces in a special case of medium B and where a free surface is substituted for one of the half spaces. It is to be noted that the nuclei of strain elastic fields in medium A (or any elastic medium) can be obtained from the corresponding point force solution by means of differentiations and simple algebraic operations (e.g., Maruyama, 1964).

The scalar image method can be described as follows. Assume that the solution field due to a localized line or point source disturbance is known in an infinite homogeneous medium, and call that solution the fundamental field. The scalar image method (e.g., when dealing with Laplace's equation) applied to plane-layered mediums determines the solution field to a line or point surface disturbance separately for each layer. The solution field in the layer containing the disturbance is equal to the fundamental field plus one (with a single interface) or an infinite series (with multiple interfaces) of some constant factors multiplying fundamental fields located at shifted positions. The shifted positions correspond to (direct or indirect) reflections of the actual position of disturbance source with respect to the interfaces (an analogy is the reflections of an object placed between parallel partially refracting mirrors). The factors multiplying the image fundamental fields are determined from satisfying boundary con-

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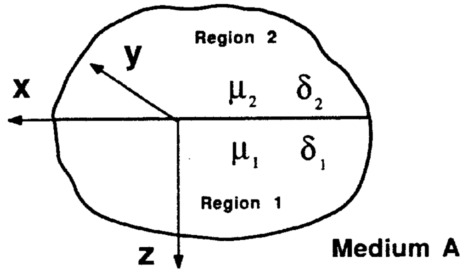

Fig. 1(a)

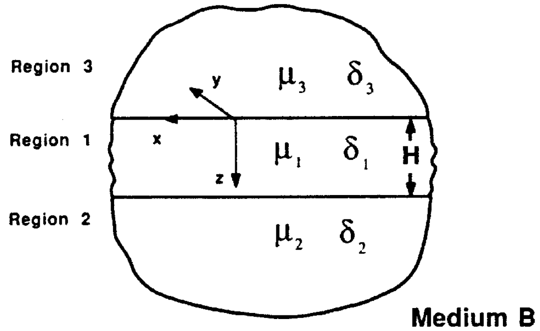

Fig. 1(b)

ditions at the plane interfaces. The solution field in the layers not containing the disturbance is equal to one (with a single interface) or an infinite series (with multiple interfaces) of some constant factors multiplying fundamental fields located at shifted positions. The shifted positions correspond to (direct or indirect) transmission of the actual position of the disturbance source with respect to the interfaces. Figure 2 is a graphical interpretation of the reflection-transmission of the fundamental field.

Stagni and Lizzio (1987) presented a two-dimensional image method in medium B. Since the field equations of two-dimensional elasticity are not scalar, boundary conditions at the interfaces cannot be satisfied by assuming the reflected and transmitted fields are constant factors multiplying shifted fundamental fields. Using Muskhelishvili potentials (Muskhelishvili, 1975), Stagni and Lizzio assumed that there does exist image fields whose potentials are then expanded in a general Laurent's series plus a logarithmic term. By matching boundary conditions at the two interfaces, a recursion relation is obtained which determines the image fields. Finally, Aderogba (1977) presents a three-dimensional method in medium A in terms of Papkovich-Neuber potentials (hereafter referred to as P–N potentials). Although the extension of Aderogba's algorithm to a medium B is in principle possible, the calculation of image P–N potentials involve evaluating multiple indefinite integrals whose multiplicity increases (linearly) with the number of reflections. In general, these indefinite integrals may not be obtained in closed form. The algorithm to be presented is in terms of Hansen potentials (used by Ben-Menahem and Singh, 1968a,b) whose relations to the P–N potentials are described in the next section. The general image algorithm presented in terms of the Hansen potentials does not require the evaluation of indefinite integrals, although multiple partial differentiations with respect to the coordinate perpendicular to the interface are required. The need for multiple partial differentiations (with respect to all coordinates) is required when the image method involves the P–N potentials. The Hansen potentials can be viewed as a linear operation (involving differentiation and integration) on the P–N potentials which obviates the need for indefinite integration when specifying an image algorithm.

In this paper the algorithm of the generalized image method is presented for mediums A and B. Application of this algorithm to a point dilatation source is then illustrated (other examples can be found in Fares, 1987). The present algorithm is suitable for generalization to viscoelastic mediums.

The Algorithm

Any elastic field can be represented in terms of Hansen potentials (Ben-Menahem and Singh, 1968a,b). The Hansen potentials are three harmonic (in the case of no body forces) scalar functions (ϕ_1, ϕ_2 ϕ_3) defined in the region of interest such that the elastic displacement field ($\underline{u}$) is given as:

$$\underline{u} = \nabla\phi_1 + \left(2\hat{e}_z\frac{\partial}{\partial z}\phi_2 - \nabla\phi_2 - 2\delta z\nabla\frac{\partial}{\partial z}\phi_2\right) + \nabla x(\hat{e}_z\phi_3) \qquad (1)$$

where ∇ is the gradient operator, ∇x is the curl operator and $\delta = 1/(3-4\nu)$ (ν is Poisson's ratio, μ is the shear modulus). The P–N potentials (Ω and Ψ) are defined as:

$$\underline{u} = (1/2\mu)\cdot(\nabla\Omega - \frac{1}{\delta}\cdot\underline{\Psi} + \underline{r}\cdot\nabla\Psi) \qquad (2)$$

The Hansen potentials are therefore related to the P–N potentials as:

$$\phi_1 = (1/2\mu)\cdot\Omega$$
$$\frac{\partial^2}{\partial z^2}\phi_2 = -\frac{1}{4\mu\delta}\nabla\cdot\underline{\Psi} \qquad (3)$$
$$\frac{\partial^2}{\partial z^2}\phi_3 = -\frac{1+\delta}{2\mu\delta}e_z\cdot\nabla x\underline{\Psi}$$

In order to derive a general image method for a multilayered medium, it is sufficient to describe an algorithm which derives the reflected and transmitted field of an arbitrary point or line source in a medium with a single interface. The procedure to obtain the elastic field in a multilayered medium would then be formally similar to Chou's method (1966) with image fields obtained by repeated use of the single interface general image algorithm (instead of factors multiplied by the original fundamental field).

Consider the single interface of Fig. 1(a) (medium A) with a point or line source located at $x=y=0$ and $z=h$. Let the Hansen potentials of the point or line source (located at $x=y=z=0$) in the absence of region 2 with an infinite extension of region 1 be given by ($\phi_1{}^0$, $\phi_2{}^0$, $\phi_3{}^0$). Assuming the Hansen potentials of the reflected and transmitted elastic fields are scalar multiplies of ($\phi_1{}^0$, $\phi_2{}^0$, $\phi_3{}^0$), $(\partial/\partial z)(\phi_1{}^0, \phi_2{}^0, \phi_3{}^0)$ and $(\partial^2/\partial z^2)(\phi_1{}^0, \phi_2{}^0, \phi_3{}^0)$, the algorithm is obtained (after a lot of algebra) by satisfying displacement and traction boundary conditions along the interface $z=0$ (see Fares, 1987, for an alternate derivation). Define the reflected and transmitted potentials by ($\phi_1{}^R$, $\phi_2{}^R$, $\phi_3{}^R$) and ($\phi_1{}^T$, $\phi_2{}^T$, $\phi_3{}^T$), respectively. The reflected and transmitted potentials are then given by:

$$\begin{bmatrix}\phi_1^R\\ \phi_2^R\end{bmatrix} = \underline{\underline{R}}\cdot\begin{bmatrix}\phi_1^{0*}\\ \phi_2^{0*}\end{bmatrix} \qquad \begin{bmatrix}\phi_1^T\\ \phi_2^T\end{bmatrix} = \underline{\underline{T}}\cdot\begin{bmatrix}\phi_1^{0}\\ \phi_2^{0}\end{bmatrix}$$

$$\phi_3^R = \left(\frac{1-\gamma}{1+\gamma}\right)\cdot\phi_3^{0*} \qquad \phi_3^T = \left(\frac{2}{1+\gamma}\right)\cdot\phi_3^0 \qquad (4)$$

$$\text{where:}\quad \phi^*(x,y,z) = \phi(z,y,-z) \qquad (5)$$

$$\underline{\underline{R}}=\begin{bmatrix} 2\cdot\delta_1\cdot\dfrac{\gamma-1}{\delta_1+\gamma}h\cdot\dfrac{\partial}{\partial z}\dfrac{\gamma\delta_2-\delta_1}{\gamma\delta_2+1}-4\cdot\delta_1^2\cdot\dfrac{\gamma-1}{\delta_1+\gamma}h^2\cdot\dfrac{\partial^2}{\partial z^2} \\ \dfrac{\gamma-1}{\delta_1+\gamma} & -2\cdot\delta_1\cdot\dfrac{\gamma-1}{\delta_1+\gamma}h\cdot\dfrac{\partial}{\partial z} \end{bmatrix} \tag{6}$$

$$\underline{\underline{T}}=\begin{bmatrix} \dfrac{\delta_1+1}{\delta_1+\gamma} & -2\cdot(\delta_1+1)\cdot\left(\dfrac{\delta_2}{\gamma\delta_2+1}-\dfrac{\delta_1}{\delta_1+\gamma}\right)\cdot h\cdot\dfrac{\partial}{\partial z} \\ 0 & \dfrac{\delta_1+1}{\gamma\delta_2+1} \end{bmatrix} \tag{7}$$

$$\gamma=\mu_2/\mu_1 \tag{8}$$

The Hansen potentials for regions 1 and 2 are then given by $(\phi_1{}^0(x,y,z-h)+\phi_1{}^R(x,y,z+h),\ \phi_2{}^0(x,y,z-h)+\phi_2{}^R(x,y,z+h),\ \phi_3{}^0(x,y,z-h)+\phi_3{}^R(x,y,z+h))$ and $(\phi_1{}^T(x,y,z-h),\ \phi_2{}^T(x,y,z-h),\ \phi_3{}^T(x,y,z-h))$, respectively. Note that **R** and **T** are matrix operators dependent on δ_1, δ_2, γ, and h.

The single interface image algorithm can now be used to obtain an image algorithm for multiple interfaces (Rice, 1985, private communications). Consider medium B (refer to Fig. 1(*b*)) with a point or line source located in region 1 at $x=y=0$ and $z=h$ (the case of the source lying in region 2 is given in Fares, 1987). Let the single interface reflection matrix operator related to the upper and lower interfaces be denoted as $\mathbf{R}^+(h)$ and $\mathbf{R}^-(-H+h)$, respectively (H is the thickness of region 1), and define $\gamma^+=\mu_3/\mu_1$ and $\gamma^-=\mu_2/\mu_1$. An image field reflected by the upper interface will have to be reflected again by the lower interface and vice versa as depicted in Fig. 2. These repeated reflections (in two directions) with the associated transmission of fields will lead to Hansen potentials defined as an infinite series of potentials in each of regions 1, 2, and 3 (in a manner analogous to Chou's results). The Hansen potentials in region 1 (potentials for region 2 are given in Fares, 1987) are then given by:

$$\phi=\begin{bmatrix}\phi_1^1\\ \phi_2^1\end{bmatrix}$$

$$\phi=\phi^0(x,y,z-h)+\sum_{m=1}^{\infty}[\phi_{m1}(x,y,z+2(m-1)H+h)$$
$$+\phi_{m2}(x,y,z-2mH-h)+\phi_{m3}(x,y,z-2mH+h)$$
$$+\phi_{m4}(x,y,z+2mH-h)] \tag{9}$$

where:

$$\phi_{02}=\phi_{04}=\phi^0$$

$$\phi_{m1}(x,y,z)=\underline{\underline{R}}^+((2m-2)\cdot H+h)\cdot\phi^*_{(m-1)2}(x,y,z)$$

$$\phi_{m2}(x,y,z)=\underline{\underline{R}}^-((-2m+1)\cdot H\text{-}h)\cdot\phi^*_{m1}(x,y,z)$$

$$\phi_{m3}(x,y,z)=\underline{\underline{R}}^-((-2m+1)\cdot H+h)\cdot\phi^*_{(m-1)4}(x,y,z)$$

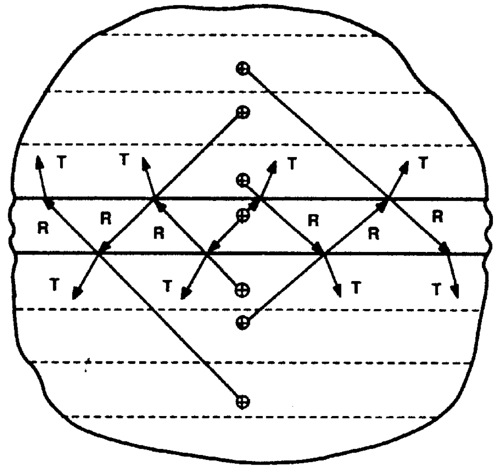

Fig. 2

$$\phi_{m4}(x,y,z)=\underline{\underline{R}}^+(2mH-h)\cdot\phi^*_{m3}(x,y,z) \tag{10}$$

The ϕ_3 potentials are given by (9) with $\phi=\phi_3$ and $\mathbf{R}^+$ and $\mathbf{R}^-$ replaced by $(1-\gamma^+)/(1+\gamma^+)$ and $(1-\gamma^-)/(1+\gamma^-)$, respectively in equation (10). When $\phi_1=\phi_2=0$ the scalar image method used by Chou is obtained, and hence, ϕ_3 can be associated with the antiplane component of an elastic deformation.

The convergence of the displacements or stresses corresponding to the series of potentials given by (9) depends on both the material contrasts between regions 1, 2, and 3 and on the singularity of the point or line source being considered. Given a fixed type of line or point source, the contribution of the farther images (images with higher "m" in equation (9)) decreases to zero as the medium approaches a homogeneous or bimaterial state. In addition, given fixed material properties in regions 1, 2, and 3, the contribution of the farther images decreases algebraically with the location of those images. Limited convergence studies on antiplane line sources (Fares, 1987) suggest that if the singularity of the line source is at least as strong as that corresponding to a dislocation, then the series giving the displacements and stresses converge for a wide range of material contrasts between regions 1, 2, and 3. Specifically, the series corresponding to the displacements due to a screw dislocation converges to the analytic closed form solution given by Tse and Rice (1986).

Example

In this section the elastic field of a dilatation source in a half space with a traction-free surface (special case of medium A), and the elastic field of a dilatation source in an elastic plate with two traction-free surfaces (special case of medium B) will be derived using the general image method. (Examples of several point and line source disturbances in a medium A, and nuclei of strain in a medium B are derived using the general image method in Fares, 1987.)

The P-N potentials for a dilatation source of expansion of net volume "V" are given by:

$$\Omega=\frac{\mu V}{2\pi}\cdot\frac{1}{r}$$
$$\underline{\Psi}=\underline{0} \tag{11}$$

where $r^2=x^2+y^2+z^2$.

Therefore, by using (3), the Hansen potentials can be obtained as:

$$\phi_1=\frac{V}{4\pi}\cdot\frac{1}{r}$$

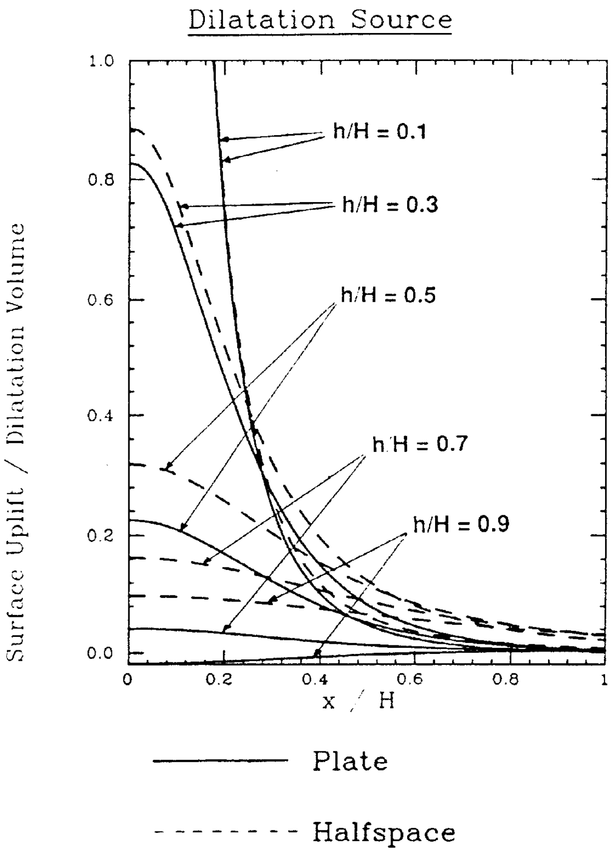

Fig. 3

$$\phi_2 = \phi_3 = 0 \tag{12}$$

Using the general image method relations (4) and (6) with $\gamma = 0$ for a traction-free half space, the reflected image potentials due to a dilatation source located at $x = y = 0$ and $z = h$ are then given by:

$$\phi_1^R = \frac{V}{4\pi} \bullet \left(-2 \bullet h \bullet \frac{z}{r^3}\right)$$

$$\phi_2^R = \frac{V}{4\pi} \bullet \left(\frac{1}{\delta} \bullet \frac{1}{r}\right)$$

$$\phi_3^R = 0 \tag{13}$$

Using both (12) and (13) and relation (1), the displacement field for a dilatation source in a half space is obtained:

$$\underline{u}/(V/4\pi) = e_x \bullet \left[\frac{x}{r^3} + 6 \bullet h \bullet \frac{x \bullet (z+h)}{r_2^5} + \frac{x}{\delta \bullet r_2^3} - 6 \bullet x \bullet \frac{(z+h)^2}{r_2^5}\right] + e_y \bullet \left[\frac{y}{r^3} + 6 \bullet h \bullet \frac{y \bullet (z+h)}{r_2^5} + \frac{y}{\delta \bullet r_2^3} - 6 \bullet y \bullet \frac{(z+h)^2}{r_2^5}\right] + e_z \bullet \left[\frac{(z-h)}{r^3} - 2 \bullet h \bullet \frac{1}{r_2^3} + 6 \bullet h \bullet \frac{(z+h)^2}{r_2^5} + \left(2 - \frac{1}{\delta}\right) \bullet \frac{(z+h)}{r_2^3} - 6 \bullet \frac{(z+h)^3}{r_2^5}\right] \tag{14}$$

where:

$$r^2 = x^2 + y^2 + (z-h)^2$$

$$r_2{}^2 = x^2 + y^2 + (z+h)^2 \tag{15}$$

The Hansen potentials representations for a dilatation source in a plate with two free surfaces has an infinite series representation. The displacement field obtained from these series for points located at finite distances from the dilatation source converges. The convergence is assured because the successive image displacement fields are located successively farther from the plate (with distance growing algebraically as "m" defined in (9) increases) while having functional variations which vary as $1/r^2$ (where "r" is the distance from the image source location to a point in the plate). In practice, however, the infinite series representations of the Hansen potentials have to be truncated. Since the calculation of image Hansen potentials with successively higher "m" in (9) and (10) is time consuming, an extrapolation (using Richardson's extrapolation, e.g., Bender and Orszag, 1978) is used to accelerate the convergence. By extrapolating and using Hansen potentials up to $m = 4$ in (9) and (10), very accurate elastic fields can be obtained at distances within three plate thicknesses from the source disturbance as evidenced by numerical studies based on antiplane deformation fields (Fares, 1987). Figure 3 shows the displacement in the z-direction at $z = 0$ when a dilatation source is located in a plate at $x = y = 0$ and $z/H = 0.1$, 0.3, 0.5, 0.7, and 0.9, (solid line). The corresponding displacement in the z-direction when a dilatation source is located in a half space is also shown (dashed line). Since the displacement field is radially symmetric, only displacements with $y = 0$ and a range of x values (taken to be $0 < x < H$ since the displacement field falls off rapidly with distance) need to be shown.

Referring to Fig. 3, the displacement magnitude above the dilatation source decreases with the depth of the source. Also, the dilatation source is more effective at causing surface uplift when the supporting foundation is more rigid, which explains the higher uplift values in the half space as compared to the plate. Finally, the surface deformations at $z = 0$ in a plate can be negative when the dilatation source is very close to the lower free surface ($z = H$). This can be explained as follows: The dilatation source in the lower portion of the plate causes local compressive stresses which have to be compensated by tensile stresses in the upper portion of the plate. This stress distribution is equivalent to a bending moment in the plate which produces negative surface uplift when the dilatation source is at a depth lower than $0.5H$. In addition to the bending moment effect of the dilatation source, expansion of the dilatation source produces displacements which are pointing away from the source producing positive surface uplift. The net surface deformation is a resultant of these two effects.

Conclusions

A general image method used to obtain elastostatic fields in plane-layered media has been presented. The image method relies on using potentials to represent elastic fields. For the case of a single interface, the image method gives the displacement fields in closed form, and can be applied to antiplane, plane, and three-dimensional problems. For the case of multiple interfaces, the image method gives the displacement fields in terms of infinite series. The convergence of the series can be accelerated which improves the efficiency of the method.

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