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Technical Report

DEGENERATE AUTOMATA: SOME RELATIONSHIPS INVOLVING SEMIGROUP ORDER AND REGULAR EVENTS

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Title: "Degenerate Automata: Some Relationships Involving Semigroup Order and Regular Events," B. P. Zeigler, University of Michigan Technical Report No. 03105-45-T.

Background: The Logic of Computers Group of the Communication Sciences Department of The University of Michigan is investigating the application of logic and mathematics to the design of computing automata. The application of algebraic techniques to the study of automata forms a part of this investigation.

Condensed Report Contents: This report investigates some relationships involving the order of the semigroup of an automaton and a class of automata for which this order takes on its smallest value relative to the number of states.

This class, called degenerate, is a limiting class in the sense that the semigroup order of any connected machine equals the number of states if it is degenerate, and is strictly greater than the state cardinality otherwise. Further, we show by counter-example that this result does not necessarily hold for disconnected machines even when they are reduced in appropriately defined manner. The lower bound on semigroup order is strengthened in the case of strongly connected automata. It is also shown that the class of degenerate automata, as herein defined, properly includes a variety of semigroup and group type automata studied in the literature.

The relevance of semigroup order to the number of subclasses and the minimum lengths of strings in an acceptor class are related to the semigroup order.

For Further Information: The complete report is available in the major Navy technical libraries and can be obtained from the Defense Documentation Center. A few copies are available for distribution by the author.

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Preliminary Notation, Definitions, and Theorems

1. We shall be concerned with transition systems (hereafter called machines) of the form

$$T = \langle S, Q, M \rangle$$

where S = input alphabet set

Q = internal state set

$M : Q \times S \longrightarrow Q$ = the transition function

2. S^* = set of all words on S, typically $x, y \in S^*$. $\ell(x)$ = length of x.

3. $\Phi = \langle \{\phi_x\}, \circ, \phi \rangle$ is the monoid (hereafter loosely called "semi-group") associated with T where

$$\phi_x(q) \stackrel{def}{=} M(q,x) \text{ for all } q \in Q,$$

(with M extended to S^* via $M(q,xs) = M(M(q,x),s)$)

and Λ is the null word, so that

$$\phi_\Lambda(q) = M(q,\Lambda) = q.$$

4. E is an equivalence relation on S^* where

$$xEy \text{ iff } \forall a(\phi_x(q) = \phi_y(q)).$$

E is a congruence relation, i.e.,

$$[x] \circ [y] = [xy]$$

where $[\ ]$ denotes a class of E, and $\circ$ denotes concatenation.

5. The monoid $S^*/E = \langle \{[x]\}, \circ, [\ ] \rangle$ is isomorphic to ϕ, symbolically

$$S^*/E \xrightarrow{iso} \Phi.$$

6. R_q is a right invariant (equivalence) relation on S where

$$xR_{q_i}y \text{ iff } \phi_x(q_i) = \phi_y(q_i).$$

Clearly

$$xEy \text{ iff } (\forall q \in Q)(xR_q y).$$

7. The following are relations on the set of all right invariant relations over S^*

a. R_a refines R_b iff $R_a \leq R_b$

$$\text{iff } xR_a y \Longrightarrow xR_b y \text{ for all } x,y \in S^*.$$

b. R_a comp. (comparable) R_b iff $R_a \leq R_b \vee R_b \leq R_a$.

c. R_a inc. (incomparable) R_b iff $\sim(R_a \text{ comp. } R_b)$

$$\text{iff } (\exists x,y)(xR_a y \wedge \sim xR_b y)$$

$$(\exists x',y')(x'R_a y' \wedge \sim x'R_b y').$$

8. A (accessibility) is a reflexive, transitive relation on Q where

$$q_1 \text{ A } q_2 \text{ iff } (\exists z)(M(q_1,z) = q_2).$$

9. Q_i is set of states accessible from q_1, i.e.,

$$Q_i = \{q/q_i \text{ A } q\}.$$

Clearly

$$q_1 \text{ A } q_2 \Longrightarrow Q_1 \supseteq Q_2.$$

10. C (communication) is an equivalence relation on Q where

$$q_1 \text{ C } q_2 \text{ iff } q_1 \text{ A } q_2 \wedge q_2 \text{ A } q_1$$

C_1 denotes the equivalence class of q_1.

11. $K(\alpha)$ is the cardinality of set α.

12. The order of a monoid $\beta = \langle B, \cdot, e \rangle$

$$O(\beta) = K(B).$$

13. T is assumed to have n states, i.e.,

$$K(Q) = n.$$

14. A transition system $T = \langle S, Q, M, q_1 \rangle$ with initial state q_1 is <u>connected</u> iff $Q_1 = Q$.

We quote the following well-known ideas from Nelson's [7] development of the Hartmanis decomposition theory:

Definition 3.5.7. Let $T_i = \langle S, Q_i, q_1^o, M_i \rangle$ and $T_j = \langle S, Q_j, q_j^o, M_j \rangle$ be two connected transition systems. The mapping $h : Q_i \rightarrow Q_j$ is a transition homomorphism if and only if

1) $h(q_1^o) = q_j^o$.

2) $h(M_i(q_i, s)) = M_j(h(q_i), s)$ for all $s \in S$, $q_i \in Q_i$.

Furthermore, if h is one-one, onto, it is a transition isomorphism.

Corollary 3.5.8. The followings are properties of h:

a) Clause 2) of Definition 3.5.7 may be replaced by

$h(M_i(q_i, x)) = M_j(h(q_i), x)$ for all $x \in S^*$.

b) There is at most one onto homomorphism between T_i and T_j.

c) There is a decision procedure for determining whether any map h is a transition homomorphism.

Introduction

This report investigates some relationships involving the order of the semigroup of an automaton and a class of automata for which this order takes on its smallest value, relative to the number of states.

This class, called degenerate automata in Definition 1.1, has the property that the order $O(\Phi) \leq n$ (where n is the number of states)[1]. In fact we shall show (Theorems 1.9 and 1.10)[2] that for any connected machine $O(\Phi) = n$ if it is degenerate, and $O(\Phi) > n$ if it is not degenerate. Hence the connected degenerate automata are precisely those automata whose semigroup order achieves its smallest value. Further, we show by counter-example that this result does not necessarily hold for disconnected machines even when they are reduced (in the sense of Definition 1.3). Theorem 1.12 strengthens the lower bound for strongly connected automata.

The class of degenerate automata properly includes a variety of automata by Trauth [2] is such that Φ is isomorphic to a group of automorphisms (i.e., isomorphisms from the machine onto itself) which characterize the automaton. Trauth defines the class of quasi-perfect automata as those group-type automata which are connected. Perfect automata turn out to be quasi-perfect automata with abelian groups. It is the perfect automata which are investigated by Fleck [3] and Weeg [4] and which appear as strongly connected commutative machines in Laing [5].

The group-type automata in turn belong to a class of state-independent

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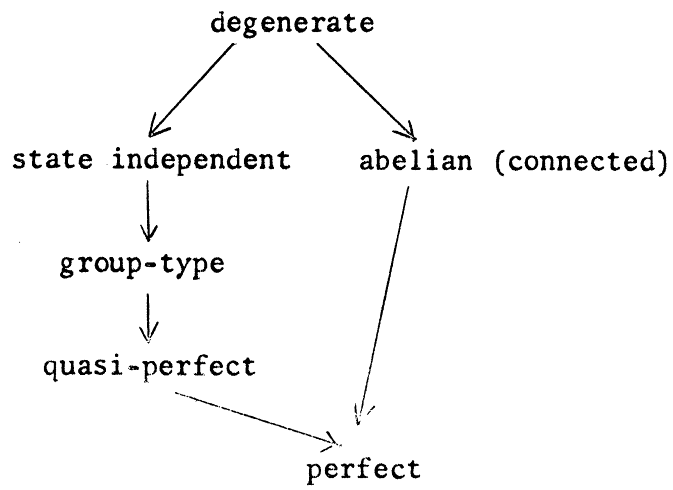
[1] All symbols are defined in the preliminary definitions.

[2] These results provide a converse for a theorem of Oehmke [1] which showed that the order of the semigroup of endomorphisms (see above) of a connected machine never exceeds n.

automata investigated by Beatty [6].[1] Finally, the connected degenerate automata turn out to be those automata for which Φ is isomorphic to a semigoup of endomorphisms (i.e., homomorphisms of the machine into itself) (Theorem 2) and includes the state independent machines as a proper subclass.

In sum, the following inclusion relations may be established.



[1] See also recent work by R. Bayer, "Automorphism Groups and Quotients of Strongly Connected Automata and Monadic Algebras," IEEE 1966 Symposium on Switching and Automata Theory.

0. Some Relationships Involving Semi-group Order and Regular Events

Before further investigating the degenerate automata, we pause to consider the relationship of semi-group order to the behavior of an automaton viewed as an acceptor. We consider a notion of "fineness of discrimination" and relate this to the lengths of the minimum representatives of the congruence classes S^*/E. We tentatively identify the longest of these as measuring the "effective memory span" of the machine. The arguments presented are not complete and are meant only to suggest the relevance of these notions to the understanding of automata behavior.

In regard to the "fineness of discrimination" and refering to the preliminary definitions we note that

$$xEy \Longleftrightarrow (\forall q_i)(xR_{q_i}y).$$

To interprete this statement we think of our machine $T = \langle S, Q, M\rangle$ as an acceptor automaton with initial state q_i, i.e., $T_{q_i} = \langle S, Q, M, q_i\rangle$ (with final states unspecified). The statement then says that if two words x,y are not distinguished by the congruence E, i.e., xEy, then it is impossible that they are distinguished by any acceptor T_{q_i} (since for any right congruence R_{q_1}, it must be that $xR_{q_1}y$). Conversely, noting that

$$\sim xEy \Longrightarrow (\exists q_i)(\sim xR_{q_i}y)$$

we have that if two words <u>are</u> distinguished by E, then there exists an acceptor which will also distinguish them. In other words, a bound on the fineness of discrimination of T viewed as an acceptor is imposed by the fineness of the partition S^*/E. Now, one measure of the fineness of a partition is the number of classes in it. (In fact, an ordering of partitions based on this number preserves the order induced by a partial ordering based on class inclusion.) Further the cardinality of S^*/E is just the order of the semi-group, $O(\Phi)$ and for a machine with n states, $1 \leq O(\Phi) \leq n^n$.

Notice, then, that an ordering based on $O(\Phi)$ over machines with the same number of states may provide additional information to that obtained from the partial ordering induced by homomorphism which relates machines with different numbers of states.[3]

We see the semi-group order determines the number of classes belonging to S*/E which, when united, form classes of S^*/R_{q_1} for some q_1. We interpret the "fineness of discrimination" as related to the number of such sub-classes comprising any acceptor class. We now argue that such discrimination involves the length of words.

Definition 0.1. $\hat{x}$ is a minimum representative of [x], a class induced by E, iff x is among the shortest of all words in [x], i.e.,

$$\ell(\hat{x}) = \min_{[x]} \ell(x) .$$

Definition 0.2. T has effective memory span, $\ell(\overset{*}{x})$, iff $\overset{*}{x}$ is among the longest of the minimum representatives of all classes of S*/E, i.e.,

$$\ell(\overset{*}{x}) = \max_{S^*/E} \ell(\hat{x}) .$$

The designation "effective memory span" is introduced here only as a suggestive name for $\ell(\overset{*}{x})$. Whether this concept fares well as a measure of memory must await further investigation.

The following theorem establishes bounds placed on the effective memory span by the order of the semigroup.

Theorem 0.1. For $T = \langle S, Q, M \rangle$ and Φ the semi-group of T, let $K(Q) = n$, $K(S) = m$.

[3] Consider for example, set of all compositions obtained by interconnecting a fixed number of machines. Each of these compositions has the same number of states but the orders of the transition semigroups may differ markedly. The relation of the semigroup order to the connection pattern might then be explored.

Then

$$\ell(\overset{*}{x}) < O(\Phi) \leq \min\{\frac{m^{\ell(\overset{*}{x})} - 1}{m - 1}, n^n\}$$

or equivalently

$$\log_m O'(\Phi) \leq \ell(\overset{*}{x}) < O(\Phi)$$

where $O'(\Phi) = \frac{O(\Phi)(m-1) + 1}{m} \approx O(\Phi)$ for m sufficiently large.

Proof. Let $\overset{*}{x} = s_1 s_2 \ldots s_{\ell(\overset{*}{x})}$. We shall show that the $\ell(\overset{*}{x}) + 1$ classes associated with heads

$$x_i = s_1 s_2 \ldots s_i \qquad (i = 0, 1, \ldots \ell(\overset{*}{x}), x_0 \equiv \Lambda)$$

are distinct. Suppose to the contrary $x_i E x_j$, $i < j$. Then since E is a congruence we may add the tail, $s_{j+1} \ldots s_{\ell(\overset{*}{x})}$ corresponding to the head x_j to both x_i, x_j, thus obtaining

$$x_i E x_j \Longrightarrow x_i s_{j+1} \ldots s_{\ell(\overset{*}{x})} \; E x_j s_{j+1} \ldots s_{\ell(\overset{*}{x})}$$

$$\Longrightarrow x_i s_{j+1} \ldots s_{\ell(\overset{*}{x})} \; E \overset{*}{x}$$

But also $\ell(x_i) < \ell(x_j) \Longrightarrow \ell(x_i s_{j+1} \ldots s_{\ell(\overset{*}{x})}) < \ell(\overset{*}{x})$ so that $\overset{*}{x}$ is not a minimum representative contrary to hypothesis. This establishes that $\ell(\overset{*}{x}) < K(S^*/E) = O(\Phi)$. The upper bound on $O(\Phi)$ is obtained by assuming that all words of length less than or equal to $\ell(\overset{*}{x})$ are non-equivalent minimum representatives.

Q.E.D.

We conclude that the semigroup order brackets the effective memory span of a machine. Thus, for example, flip-flops (degenerate) and delays (non-degenerate) are types of 2-state, 2-symbol connected machines with effective memory span of 1. That of a 3-state (2-symbol, connected) degenerate machine is always less than that of a 3-state machine with $O(\Phi) \geq 15$. Notice that the effective memory span may be less than the degree of definiteness [10]. Thus, while a flip-flop has indefinite storage (degree of definiteness = ∞) its effective memory span is only 1.

Continuing our attempt to relate the relations E, R_q we observe that E-induced subclasses of S^*/R_q are those satisfying left as well as right invariance, hence the well-known

<u>Theorem 0.2.</u> Let T_{q_1} be connected. Then

$$\begin{aligned} xEy &\Longleftrightarrow (\forall q)(M(q,x) = M(q,y)) \\ &\Longleftrightarrow (\forall z)(M(M(q_1,z))x) = M(M(q,z),y) \\ &\Longleftrightarrow (\forall z)(zxR_{q_1}zy) \end{aligned}$$

The E-induced subclasses of R_{q_1} then appear as those which are preserved under prefixing of any word. Alternately these classes are invariant under time translation (with $\ell(z)$ the shift interval).

Clearly one and only one of the classes of S^*/R_{q_1} contains the subclass associated with the longest minimum representative, $[\overset{*}{x}]$. If S^*/R_{q_1} has more than one class, T_{q_1} is therefore able to define a set of words containing a time invariant class of words at least $\ell(\overset{*}{x})$ long.

For degenerate automata, Definition 1.1 states that the classes of S^*/R_{q_1} are just those of S^*/E. It follows that all classes of S^*/R_{q_1} are time invariant, i.e.,

$$xR_{q_1}y \Longleftrightarrow (\forall z)(zxR_{q_1}zy).$$

Further, since $O(\Phi) \leq n$, it follows that the longest minimum representative of these classes can be no longer than n symbols in length.

1.0. Degenerate Machines

Definition 1.1. A machine $T = \langle S, Q, M \rangle$ is degenerate iff there is a state $q_1 \varepsilon Q$ such that

$$xR_{q_1}y \Longleftrightarrow xEy$$

for all $x, y \varepsilon S^*$ (universal quantification will be assumed whenever $x, y \varepsilon S^*$ appear unquantified).

Since in general $K(S^*/R_{q_i}) = K(Q_1) \leq n$ we have immediately from Definition 1.1 that

$$0(\Phi) = K(S^*/E) = K(S^*/R_{q_1}) \leq n.$$

We shall presently return to the converse of this statement, i.e., to the question of whether degenerate machines are the only ones for which $0(\Phi) \leq n$.

Let us refer to a state q_1 appearing in Definition 1.1 as a critical state, the motivation being that equivalence of words under the right invariance relation of a critical state is necessary and sufficient for equivalence under the congruence E.

Since from preliminary Definition 6 we have $xEy \Longleftrightarrow (\forall q)(xR_q y)$ we have the following equivalent but useful definition of degeneracy.

Definition 1.1'. A machine $T = \langle S, Q, M \rangle$ is degenerate iff there is a critical state in Q, i.e., a state q_1, such that

$$R_{q_1} \leq R_q \text{ for all } q \varepsilon Q.^1$$

Models of degenerate machines are generated by Caley representations of monoids.

Let $\langle S, \circ, 1 \rangle$ be a monoid; the corresponding Caley machine is defined as

[1] Note that there may be more than one critical state. If all states are critical, T is state independent, i.e. $R_{q_i} = R_{q_j}$ for all $q_i, q_j \varepsilon Q$

$$T(S) = \langle S, S, M \rangle$$

where $M(s,s') = s \cdot s'$ for all $s, s' \in S$.

It has been established (Myhill [8]) that T(S) is well defined and that its transition monoid is isomorphic to S. Moreover, we can show that T(S) is degenerate with critical state 1. This is so since

$$M(1,x) = M(1,y) \Longleftrightarrow 1x = 1y$$
$$\Longrightarrow ax = ay$$
$$\Longrightarrow M(a,x) = M(a,y)$$

for all state $a \in S$.

Now consider the set of submachines of T generated by members of Q considered as initial states.

<u>Definition 1.2.</u> For each $q_i \in Q$ we can define an initial state machine $T_i = \langle S, Q_i, q_i, M_i \rangle$ where

$Q_i = \{q/q_i \text{ A } q\}$

q_i = initial state of T_i

M_i is M restricted to Q_i.

It is easy to check that T_i is a well defined connected machine. Note too that R_{q_i} is the right invariance relation corresponding to T_i.

We now apply preliminary Theorem 3.5.1.8, to a pair q_1, q_i of a degenerate machines. $R_{q_1} \leq R_{q_i}$ iff there is a homomorphism from T_1 onto T_i, i.e., a map $h_{q_i} : Q_1 \longrightarrow Q_i$ such that

$$h_{q_i}(q_1) = q_i$$

and

$$M(h_{q_i}(q), s) = h_{q_i}(M(q_1, s))$$

for all $s \in S$ and $q \in Q_1$. Since the domain of h_{q_i} is Q_1 and not necessarily the total set Q, h_{q_i} is not a proper endomorphism, but what we shall refer to as a <u>sub-endomorphism</u>. Clearly if T is connected with q_1 the initial and critical state, then h_{q_i} is indeed an endomorphism. We thus have proved the

<u>Theorem 1.1.</u> T is degenerate with q_1 a critical state iff for each $q_i \in Q$ there is a sub-endomorphism $h_{q_i} : Q_1 \xrightarrow{onto} Q_i$ such that $h_{q_i}(q_1) = q_i$.

The following statements are immediate properties of sub-endomorphisms.

<u>Lemma 1.1.0.</u> 1. $h_{q_i}(M(q,y)) = M(h_{q_i}(q),y)$ for all $y \in S^*$ (from Cor. 3.5.8.a).

2. A sub-endomorphism defined on Q_1 is uniquely specified by the q_1 image, i.e.,

$$(h_i(q_1) = h_j(q_1)) \Longrightarrow h_{q_i} = h_{q_j}$$

3. For the sub-endomorphisms of Theorem 1.1 we have further

$$q_i = q_j \Longleftrightarrow h_{q_i} = h_{q_j}$$

<u>Proof of 2:</u> $h_i(q_1) = h_j(q_1) \Longrightarrow M(h_i(q_1),x) = M(h_j(q_1),x)$

$$\Longrightarrow h_i(M(q_1,x)) = h_j(M(q_1,x)), \text{ for all } x \in S^*$$

$$\Longrightarrow h_i = h_j$$

Q.E.D

We note that functional composition of sub-endomorphisms is not generally possible because of the restriction of the domain. Such composition is possible however for the subset $\{h_{q_i} / q_i \in Q_1\}$ since $q_i \in Q_1 \Longrightarrow Q_i \subseteq Q_1$ so that h_{q_i} may be taken to map Q_1 into Q_1, we shall refer to such a set of sub-endomorphisms as a <u>critical</u> set. A critical set with critical state q_1 can be written in the form

$$H_1 = \{h_{M(q_1,x)} | h_{M(q_1,x)}(M(q,y)) = M(h_{M(q_1,x)}(q_1),y)\}$$

<u>Theorem 1.2.</u> A critical set of endomorphisms forms a monoid under

functional composition, i.e., $H_1 = \langle H_1, \circ, h_{q_1} \rangle$ is a monoid.

Proof. Clearly h_{q_1} is an identity and functional composition is associative. For the rule of composition we have

$$h_{M(q_1,x)} h_{M(q_1,y)}(M(q_1,z)) = h_{M(q_1,x)} \circ M(h_{M(q_1,y)}(q_1),z)$$

$$= h_{M(q_1,x)} \circ M(q_1,yz)$$

$$= M(h_{M(q_1,x)}(q_1),yz))$$

$$= M(q_1,xyz)$$

$$= M(M(q_1,xy),z)$$

$$= h_{M(q_1,xy)}(M(q_1,z))$$

which demonstrates closure.

Q.E.D.

We are now ready to establish the equivalence between degenerate machines and those for which the transition monoid Φ is isomorphic to a monoid of sub-endomorphisms. As noted in the Introduction a variety of machines studied in the literature satisfy the latter description.

Theorem 1.3. T is degenerate iff there is a monoid of sub-endomorphisms defined on a connected subset of states which is isomorphic to the monoid of transitions, Φ.

Proof. In the forward direction assume T is degenerate with critical state q_1. According to Theorem 1.2 we have the existence of a monoid $H_1 = \langle \{h_{M(q_1,x)}\}, \circ, h_{q_1} \rangle$. We shall show that $g : \Phi \longrightarrow H_1$ such that

$$g(\phi_x) = h_{M(q_1,x)}$$

is an isomorphism.

Clearly g is onto; it is moreover, one-one since

$$h_{M(q_1,x)} = h_{M(q_1,y)} \Longrightarrow M(q_1,x) = M(q_1,y)$$
$$\Longrightarrow xR_{q_1}y$$
$$\Longrightarrow xEy \qquad \text{(degeneracy)}$$
$$\Longrightarrow \phi_x = \phi_y$$

Finally to show commutivity, we have $g(\phi_x \circ \phi_y) = g(\phi_{xy})$

$$= h_{M(q_1,xy)}$$
$$= h_{M(q_1,x)}h_{M(q_1,y)} \quad \text{(Thm. 1.2.)}$$
$$= g(\phi_x)\ g(\phi_y)$$

Hence g is an isomorphism.

For the converse let $H = \langle\{h_i\}, \cdot, h_1\rangle$ be a monoid of sub-endomorphisms whose domain and range are Q_1, the set of states accessible from a state, q_1. By assumption Φ is isomorphic to H. Let us first show that for every state $q_i \in Q_1$ there is an $h_i \in H$ such that $h_i(q_1) = q_i$, in other words that $\{h_i(q_1)\} = Q_1$.

First note that since there are at least as many transition maps in Φ as there are distinct q_1 images,

$$0(\Phi) \geq K\{\phi_x(q_1)\} = K(Q_1) \tag{1}$$

Because of isomorphism between H and Φ we have using (1) that

$$0(H) \geq K(Q_1) \tag{2}$$

From Lemma 1.1.0 we have that for

$$h_i \neq h_j \Longrightarrow h_i(q_1) \neq h_j(q_1)$$

hence that $K(\{h_i(q_1)\}) \geq 0(H)$ and from (2)

$$K(\{h_i(q_1)\}) \geq K(Q_1).$$

But since the range of every h_i is Q_1, any assumption other than $\{h_i(q_1)\} = Q_1$ involves a contradiction.

Now consider the submachine

$$T_1 = \langle S, Q_1, M_1 \rangle.$$

Since for each q_i there is a sub-endomorphism h_i such that $h_i(q_1) = q_i$ we have that from Theorem 1.1 that T_1 is degenerate with q_1 a critical state.

Let Φ_1 be the transition monoid for T_1. From the first part of the present theorem, Φ_1 is isomorphic to H and hence Φ_1 is isomorphic to Φ.

Since in general

$$xEy \Longrightarrow xE^1y$$

(where E^1 is the congruence relation for T_1) $\Phi_1 \xrightarrow{\text{iso}} \Phi$ implies that

$$xEy \Longleftrightarrow xE^1y \qquad (3)$$

But since T_1 is degenerate

$$xR_{q_1}y \Longleftrightarrow xE^1y$$

hence from (3)

$$xR_{q_1}y \Longleftrightarrow xEy,$$

i.e., the original machine T is degenerate.

Q.E.D.

Corollary 1.3.1. For a connected machine T, T is degenerate iff there is a monoid of endomorphisms which is isomorphic to the monoid of transitions, Φ.

Theorem 1.4. A connected machine having an abelian semigroup is degenerate.

Proof. Let q_1 be an initial state such that $Q_1 = Q$. We shall show that q_1 is a critical state.

Using the right invariant property of R_{q_1}, we have

$$xR_{q_1}y \Longrightarrow (\forall z)(xzR_{q_1}yz)$$

Further, it follows easily from the fact that Φ is abelian that

$$xzR_{q_1}zx$$

and

$$yzR_{q_1}zy \tag{3}$$

Since R_{q_1} is an equivalence relation (3) and (4) yield

$$\begin{aligned} xR_{q_1}y &\Longrightarrow (\forall z)(zxR_{q_1}zy) \\ &\Longrightarrow xEy \end{aligned}$$

(using Theorem 0.2). Since the converse implication always holds, Definition 1.0 is satisfied, hence the associated machine is degenerate.

<u>Theorem 1.5.</u> For connected machines

$$H = \Phi \text{ with } h_{M(q_1,x)} = \Phi_x \text{ iff } \Phi \text{ is abelian.}$$

<u>Proof.</u> Corollary 1.2.0 statement 4 states that

$$h_{M(q_1,x)}\Phi_y(q) = \Phi_y h_{M(q_1,x)}(q)$$

which holds for all $q \in Q$ since T is connected.

Now if $h_{M(q_1,x)} = \Phi_x$ then

$$\Phi_x\Phi_y = \Phi_y\Phi_x$$

so that Φ is abelian.

Conversely, if Φ is abelian, then by Theorem 1.4 a set of endomorphisms exists.

Let $q = M(q_1,y)$, then

$$\begin{aligned} h_{M(q_1,x)}(q) &= h_{M(q_1,x)}(M(q_1,y)) \\ &= M(q_1,xy) \\ &= M(q_1,yx) \\ &= M(M(q_1,y),x) \\ &= \Phi_{x(q)} \end{aligned}$$

Q.E.D.

1.1. A Definition of Machine Reduction

We have shown that for a degenerate machine $O(\Phi) \leq n$. We are now interested in the subclass of degenerate machines for which $O(\Phi) = n$. To do this we develop a method of reducing the number of states of a machine while keeping its transition semigroup invariant.

Definition 1.3. A machine is reduced iff for every $q_1, q_2 \in Q$

$$R_{q_1} \leq R_{q_2} \wedge C_1 \neq C_2 \Longrightarrow (\exists q \notin C_2)(q \; A \; q_2).$$

In words, for any two non-communicating states q_1, q_2 if R_{q_1} refines R_{q_2} then q_2 is accessible from some state not in its communicating class.

Our definition differs from the standard one in that we do not assume that T is connected and we allow the possibility that every state has a distinct output. As seen from Theorem 1.7, the present definition coincides with the standard one for connected minimal machines.

The sum of any machine with an isomorphic copy of itself is an example of an unreduced machine.[1]

Lemma 1. $R_{q_1} \leq R_{q_2} \Longrightarrow (\forall q_2' \in Q_2)(\exists q_1' \in Q_1)(R'_{q_1} \leq R'_{q_2})$.

Proof. $q_2' \in Q_2 \Longrightarrow$ there is a z such that

$$M(q_2, z) = q_2'.$$

Take $q_1' = M(q_1, z) \in Q_1'$.

Now

$$\begin{aligned} xR'_{q_1}y &\Longrightarrow M(q_1', x) = M(q_1', y) \\ &\Longrightarrow M(q_1, zx) = M(q_1, zy) \\ &\Longrightarrow zxR_{q_1}zy \\ &\Longrightarrow M(q_2, zx) = M(q_2, zy) \\ &\Longrightarrow M(q_2', x) = M(q_2', y) \\ &\Longrightarrow xR'_{q_2}y \end{aligned}$$

Q.E.D.

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[1] Figure 1, page 19 is an example of a machine which is reduced but not connected.

<u>Theorem 1.6.</u> If a machine T is not reduced there is a machine T' with fewer states such that

$$E = E'$$

hence

$$\Phi \xrightarrow{\text{iso}} \Phi' \text{ and } O(\Phi) = O(\Phi').$$

<u>Proof.</u> Let $T = \langle S, Q, M \rangle$ not be reduced. Then there are q_1, q_2 such that

1. $R_{q_1} \leq R_{q_2}$
2. $C_1 \neq C_2$
3. $\sim(\exists q \notin C_2)(q \text{ A } q_2)$ (Definition 3)

The idea behind the proof is that 1) R_{q_1} refines R_{q_2} and by Lemma 1, all states in q_2's communicating class $C_2 (\subseteq Q_2)$ and 2) C_2 is inaccessible to all states external to it, hence the whole class C_2 may be eliminated without affecting the word semigroup of the machine.

Formally we shall show that

$$xEy \text{ iff } xE'y$$

where $T' = \langle S, Q - C_2, M' \rangle$ where M' is M restricted to $Q - C_2$.

Note that $Q \neq C_2$ since if $Q = C_2$ then $q_1 \text{ C } q_2$, i.e., $C_1 = C_2$ violating condition 2 above.

Furthermore $M(Q-C_2, x) \subseteq Q - C_2$, since by condition 3 no state in $Q - C_2$ can access C_2 (for if $q \text{ A } q_2' \text{ A } q_2' \text{ C } q_2$ then $q \text{ A } q_2$). Thus T' is well defined.

Now

$$\begin{aligned} xEy &\Longrightarrow (\forall q)(\phi_x(q) = \phi_y(q)) \\ &\Longrightarrow (\forall q \in (Q-C_2))(\phi_x(q) = \phi_y(q)) \\ &\Longrightarrow xE'y \end{aligned}$$

Now note that $Q_1 \subseteq Q - C_2$ since if $q_1 \text{ A } q$ and also $q \in C_2$ then by

Condition 3 $q_1 \varepsilon C_2$ but then $C_1 = C_2$ violating Condition 2.

Since $C_2 \subseteq Q_2$, and $Q_1 \subseteq Q - C_2$, Lemma 1 tells us that if R_{q_1} refines R_{q_2} then every state in C_2 is refined by some state in $Q - C_2$. Thus if xR_qy for every $q \varepsilon Q - C_2$, then xR_qy for every $q \varepsilon C_2$.

Thus

$$\begin{aligned} xE'y &\Longrightarrow (\forall q \ \varepsilon \ (Q - C_2))(xR_qy) \\ &\Longrightarrow (\forall q \ \varepsilon \ (Q - C_2))(xR_qy) \wedge (\forall q \ \varepsilon \ C_2)(xR_qy) \\ &\Longrightarrow (\forall q \ \varepsilon \ Q)(xR_zy) \\ &\Longrightarrow xEy \end{aligned}$$

Q.E.D.

Noting that at least $q_2 \varepsilon C_2$, T' contains at least one fewer state than T.

Finally isomorphism clearly follows from equality of the congruence relation.

<u>Theorem 1.7.</u> If a machine is connected then it is reduced.

<u>Proof.</u> Assume T is not reduced. Then $C_2 \neq C_1$ and $R_{q_1} \leq R_{q_2}$ but $(\exists q \ \varepsilon \ C_2)(q \ A \ q_2)$. In particular, $q_1 \varepsilon C_2$ (since $C_1 \neq C_2$). So $q_1 \ A \ q_2$

Q.E.D.

<u>Theorem 1.8.</u> A degenerate machine is reduced iff it is connected.

<u>Proof.</u> Let T be degenerate with $R_{q_1} \leq R_q$ for all q. Assume T is reduced. Clearly $q_1 \ A \ q_1$. Let $q_2 \neq q_1$. If $q_2 \ C \ q_1$ then the theorem is proved. If not, then by Condition 3 of Definition 1.3 there is a q_3 which accesses q_2 but is not accessible from q_2. Now either $q_3 \ C \ q_1$ in which case the theorem is proved, otherwise by Condition 3 again, there is a q_4 such that $q_4 \ A \ q_3 \wedge \sim q_3 \ A \ q_4$. Now it cannot be that $q_2 \ A \ q_4$ (since then $q_2 \ A \ q_3$) thus $q_4 \notin (Q_2 \cup Q_3)$ or since $Q_3 \supset Q_2$ (the inclusion

is proper since $\sim q_2 \; A \; q_3) \; q_4 \notin Q_3$. Evidently this process must stop since at every stage j: $q_{j+1} \notin Q_j$ while $Q_j \supset Q_{j-1} \ldots \supset Q_2$. So that at most $Q_n = Q$. Thus for some, j, $q_j = q_1$ and furthermore

$$q_1 \; A \; q_{j-1} \wedge q_{j-1} \; A \; q_{j-2} \ldots q_3 \; A \; q_2 \Longrightarrow q_1 \; A \; q_2.$$

The converse follows from Theorem 1.7.

Q.E.D.

1.2. Semigroup Order and Degenerate Machines

Theorem 1.9. A degenerate machine T has $O(\Phi) = n$ if

1. it is connected.

or 2. it is reduced.

Otherwise $O(\Phi) < n$.

Proof. Connectedness $\Longleftrightarrow$ reduced by Theorem 1.6. If T is connected then $K(Q_1) = n$ and by Theorem 1.3 $O(H) = O(\Phi) = n$. If T is not reduced then by Theorem 1.6, there is a machine T', with fewer states such that $O(\Phi) = O(\Phi')$. But T' is also degenerate since for all states $q \in Q - C_2$ it is still true that $R_{q_1} \leq R_q$. Therefore $O(\Phi') \leq n-1$.

Q.E.D.

In regard to the converse of Theorem 1.7 it might be conjectured that for a reduced non-degenerate machine, $O(\Phi) \geq n+1$. Since reduction does not necessarily imply connectedness for non-degenerate machines it may happen in fact that $O(\Phi) < n$ as the following example shows.

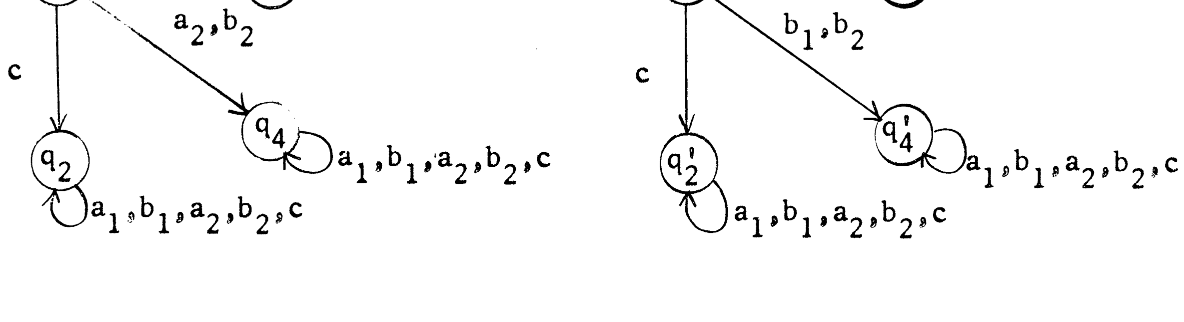

Figure 1.

The machine of Figure 1 is composed of two separate reduced degenerate components which do not enter into any homomorphic relation since $a_1 R_{q_1} b_1 \wedge \sim a_1 R_{q_2} b_1$ and $\sim a_1 R_{q_1} a_2$ while $a_1 R_{q_j} a_2$. It is easy to verify that while having 8 states and being non-degenerate and reduced, the semigroup of the machine has only 6 distinct functions. The dead states $q_2, q_3, q_4, q_2', q_3', q_4'$ are of course to blame for the small order. Regarding the machine as an acceptor and applying the appropriate reduction would remove many of these states with the concomitant effect of altering the word semigroup.

__Theorem 1.10.__ A __connected__ non-degenerate machine has $O(\Phi) \geq n+1$.

__Proof.__ Assume q_1 is the initial state. Then there are n distinct functions, ϕ_x differing at least in the q_1-images. Because R_{q_1} cannot refine all R_q, $q \in Q$, (otherwise T would be degenerate), there is a q_2, say such that $x_1 R_{q_1} x_2$ while $\sim x_1 R_{q_2} x_2$. In other words $\phi_{x_1}(q_1) = \phi_{x_2}(q_1)$ while $\phi_{x_1}(q_2) \neq \phi_{x_2}(q_2)$. Thus there are n-1 distinct functions differing at least in the q_1-images, and at least 2 functions distinct from the n-1 others having identical q_1-images but different q_2-images. Hence there are n+1 distinct functions.

Q.E.D.

Figure 2 displays a connected non-degenerate machine of 3 states having $O(\Phi) = 4$.

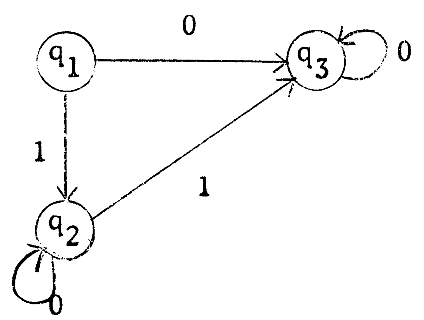

Figure 2.

Theorem 1.10 can be strengthened for strongly connected machines. First we have the following lemmas.

<u>Lemma 2.</u> $K(S^*/R_{q_1}) = Q_1$.

<u>Lemma 3.</u> If R_1 inc. R_2 and $K(S^*/R_1) \leq K(S^*/R_2)$ then there are at least 2 classes in S^*/R_1 which are split in S^*/R_2.

<u>Proof.</u> Since R_1 and R_2 are incomparable at least one class in S^*/R_1 is split in S^*/R_2. If only one class is split, then since $K(S^*/R_1) \leq K(S^*/R_2)$, it follows that $R_2 \leq R_1$. But this contradicts the hypothesis that R_1 inc. R_2.

We develop the following lemmas by using the fact that

1. $q_1 \text{ A } q_2 \Longrightarrow K(Q_1) \geq K(Q_2)$.
2. $q_1 \text{ A } q_2 \wedge \sim q_2 \text{ A } q_1 \Longrightarrow K(Q_1) > K(Q_2)$

<u>Lemma 4.</u> R_{q_1} comp. $R_{q_2} \wedge q_1 \text{ A } q_2 \Longrightarrow R_{q_1} \leq R_{q_2}$.

<u>Lemma 5a.</u> $(R_{q_1}$ comp. $R_{q_2}) \wedge q_1 \text{ C } q_2 \Longrightarrow R_{q_1} = R_{q_2}$.

<u>Lemma 5b.</u> $R_{q_1} = R_{q_2} \Longrightarrow (q_1 \text{ A } q_2 \Longleftrightarrow q_1 \text{ C } q_2)$.

Since $q_1 \text{ C } q_2$ iff $q_1 \text{ A } q_2 \quad q_2 \text{ A } q_1$, apply Lemma 4 twice, obtaining 5a.

<u>Theorem 1.11.</u> If T is strongly connected and for every pair q_1, q_2, R_{q_1} comp. R_{q_2}, then T is degenerate.

<u>Proof.</u> Lemma 5a applies to all pairs of states, hence $R_{q_1} = R_{q_2} = \ldots = R_{q_n}$. But then by Theorem 1.1, T is degenerate, in fact state independent.

Q.E.D.

<u>Theorem 1.12.</u> A strongly connected non-degenerate machine T, has $O(\Phi) \geq n+2$.

<u>Proof.</u> Theorem 1.11 is contradicted unless there are q_1, q_2 such that R_{q_1} inc. R_{q_2}. Noting that $K(Q_1) = K(Q_2)$ and applying Lemmas 2 and 3, there are at least 2 classes in S^*/R_{q_1} which are split in S^*/R_{q_2}.

The argument of Theorem 1.10 goes through except that now there are n-2 distinct functions differing at least in there q_1-images, at least 2 functions distinct from then n-2 others having identical q_1-images but different q_2-images, and at least 2 functions distinct from the others having the same q_1-images but different q_2 images.

Q.E.D.

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This report investigates some relationships involving the order of the semi-group of an automaton and a class of automata for which this order takes on its smallest value relative to the number of states. (U)

This class, called degenerate, is a limiting class in the sense that the semi-group order of any connected machine equals the number of states if it is degenerate, and is strictly greater than the state cardinality otherwise. Further, we show by counter-example that this result does not necessarily hold for disconnected machines even when they are reduced in appropriately defined manner. The lower bound on semi-group order is strengthened in the case of strongly connected automata. It is also shown that the class of degenerate automata, as herein defined, properly includes a variety of semi-group and group type automata studied in the literature. (U)

The relevance of semi-group order to the acceptance properties of automata is suggested. In particular, the number of subclasses and the minimum lengths of strings in an acceptor class are related to the semi-group order. (U)

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