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Notes on Plasma Physics

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FOREWORD

The information contained in this report was originally developed by Dr. Richard K. Osborn in connection with his work as Professor of Nuclear Engineering at the University of Michigan.

Because of the interest of the Jet Propulsion Laboratory in the subject, Dr. Osborn was invited to make this material available to interested research scientists and to conduct further studies relative to the subject. The Jet Propulsion Laboratory is pleased to publish his work for distribution to other interested agencies. It is the intention of the author to incorporate this material in a textbook at a later date.

These notes comprise a preliminary sketch of the plasma physics portion of a course in thermonuclear theory now being developed in the Department of Nuclear Engineering at the University of Michigan. Because they are preliminary, they must be regarded as incomplete; because they are drawn from an exposition of subject matter that is merely in its infancy, they must be interpreted as giving only a provisional viewpoint.

First Chapter

FORMULATION OF THE STATISTICAL PROBLEM

I. INTUITIVE DERIVATION OF THE NEUTRAL GAS EQUATION

Practically all the macroscopic observables of systems of the kind presently under consideration are statistical in character. Thus, descriptions of such systems, potentially useful for ultimate prediction, should perhaps be formulated in terms of the descriptions of a small but sufficient number of statistical quantities, each of which is invested with no more information than that necessary for quantitative analysis of envisaged experiments.

Consequently it would appear adequate for the purpose simply to define the quantities $f(\mathbf{x}, \mathbf{v}, t) d^3x\, d^3v$ as the expected number of particles in the phase-volume element $d^3x\, d^3v$ about the point $(\mathbf{x}, \mathbf{v})$ at time t (defining one such quantity for each type of particle present in the system) and proceed forthwith to an attempted deduction of the explicit dependence of the quantities f upon their arguments for a variety of physically interesting cases. It is probably evident that if the quantities defined above were known, then quantitative estimates of macroscopic observables would be accessible. However, it shall be part of the burden of the present discussion to indicate that the description of the system under consideration cannot be arrived at in quite so straightforward a manner.

In order to illuminate the nature of the difficulty, consider first a somewhat intuitive deduction of an equation for f. Assume for simplicity that there is only one kind of particle, that these particles interact with each other via forces characterized by a "range" which is very small compared to average interparticle distances, and that they are further acted upon by forces generated by external systems. Assume further that these external forces are constant in time. Then if $\mathbf{x}' = \mathbf{x} + \mathbf{v}\, \Delta t$ and $\mathbf{v}' = \mathbf{v} + \mathbf{a}\Delta t$,

$$f(\mathbf{x}', \mathbf{v}', t + \Delta t)\; d^3x'\; d^3v' - f(\mathbf{x}, \mathbf{v}, t)\; d^3x\; d^3v = \frac{\delta_c f}{\Delta t}\, d^3x\; d^3v\; \Delta t \tag{1}$$

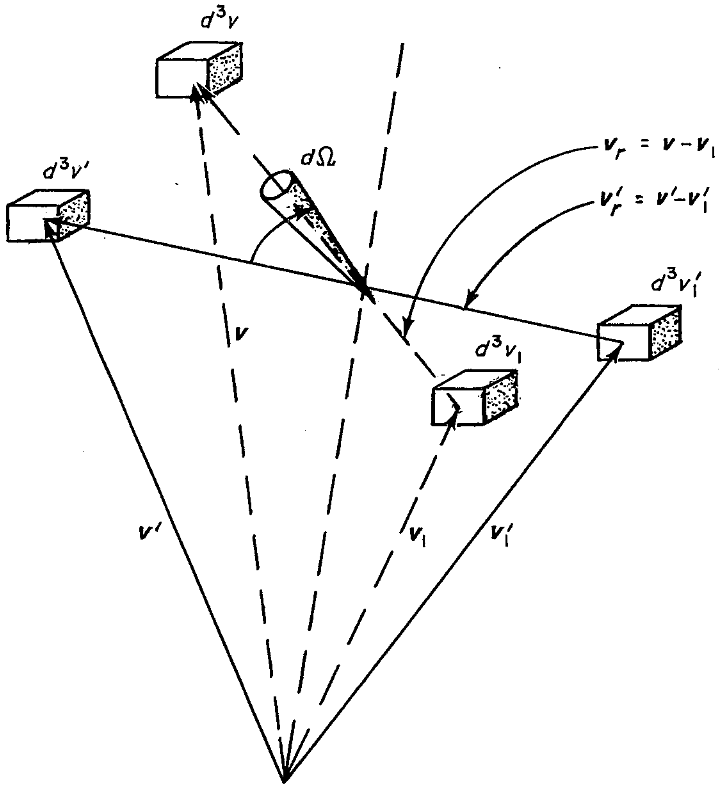
where $[(\delta_c f/\Delta t)\; d^3x\; d^3v]$ is the net change per unit time in the number of particles under consideration due to interactions between them; $\mathbf{a} = \mathbf{F}/m$ where $\mathbf{F}$ is the externally generated force on the particle at the phase point $(\mathbf{x}, \mathbf{v})$; and

$$d^3x'\; d^3v' = \left| J\left(\frac{\mathbf{x}', \mathbf{v}'}{\mathbf{x}, \mathbf{v}}\right) d^3x\; d^3v \right| = \left[1 + \frac{\partial a_j}{\partial v_j}\Delta t + \mathcal{O}(\Delta t^2)\right] d^3x\; d^3v \tag{2}$$

(The summation convention for repeated tensor indices will be assumed throughout.) Dividing the above expression by Δt and taking the limit as $\Delta t \to 0$ yields:

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + \frac{\partial a_j f}{\partial v_j} = \frac{\delta_c f}{\Delta t}\Bigg|_{\Delta t \to 0} \tag{3}$$

In accordance with the explicit assumption that interactions between particles are rare and impulsive events describable in terms of completed collisions, the functional character of the quantity on the right-hand side of the expression above is readily deduced. Sketch 1 demonstrates the kinematics of an elastic collision between particles, with pre-collision velocities $\mathbf{v}'$ and $\mathbf{v}_1'$ going to post-collision velocities $\mathbf{v}$ and $\mathbf{v}_1$, CMCS (center-of-mass coordinate system) scattering angle being Θ.



Sketch 1.

Under these circumstances, the following relationship is obtained conventionally (Ref. 1).

$$\frac{\delta_c f}{\Delta t}\Bigg|_{\Delta t \to 0} = \int_{\mathbf{v}_1,\, \Omega} \left[f(\mathbf{x}, \mathbf{v}', t)\, f(\mathbf{x}, \mathbf{v}_1', t) - f(\mathbf{x}, \mathbf{v}, t)\, f(\mathbf{x}, \mathbf{v}_1, t)\right] \cdot v_r \sigma(v_r, \Theta)\, d\Omega\, d^3 v_1 \tag{4}$$

where $\sigma d\Omega$ is the usual differential scattering cross section in the CMCS. For purposes of subsequent manipulation it is convenient to define a relative scattering frequency by

$$\mathcal{G}(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1)\, d^3v\, d^3v_1 \equiv v_r'\, \sigma(v_r', \Theta)\, d\Omega\, \delta(\mathbf{g}' - \mathbf{g})\, d^3g\, \delta(v_r' - v_r)\, dv_r$$

i.e.,

$$v_r'\, \sigma(v_r', \Theta)\, d\Omega = \int_{v_r, \mathbf{g}} \mathcal{G}(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1)\, d^3v\, d^3v_1 \tag{5}$$

where $\mathbf{g}$ is the center-of-mass velocity. In these terms the equation for f becomes:

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + \frac{\partial a_j f}{\partial v_j} = \int_{\mathbf{v}_1, \mathbf{v}_1', \mathbf{v}'} [f' f_1' - f f_1]\, \mathcal{G}(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')\, d^3v_1\, d^3v_1'\, d^3v' \tag{6}$$

II. SEMI-DEDUCTIVE DERIVATION OF THE NEUTRAL GAS EQUATION

It becomes apparent that the validity of the above arguments (and hence also of the conclusion that f satisfies the conventional Boltzmann equation) is somewhat less than self-evident as soon as the attempt is made to generalize these arguments to apply to plasma-type systems. Because of the long-range character of the Coulomb interaction, it is clear that particle interactions can hardly be completely characterized by "collisions." In fact, the sense in which the "collision" concept is applicable at all to the statistical description of the plasma becomes somewhat obscure.

To illuminate the nature of this problem, it is proposed first to review an axiomatic description of the system, and second to attempt to proceed deductively from the axioms to a tractable, analytical description of the number density f previously defined. This procedure will be initially developed within the context of the problem discussed above and then attempts will be made to generalize to the plasma.

As a mechanical axiom for the system presently under consideration (classical mechanics being adopted throughout, as it is not at all evident at this point that quantum mechanics is either useful or necessary here) it is convenient to adopt the following Hamiltonian for an N-particle system:

$$H = \sum_{\sigma=1}^{N} \frac{(p_j^\sigma)^2}{2m_\sigma} + \sum_{\sigma=1}^{N} \phi_\sigma + \sum_{\substack{\sigma,\ \alpha \\ \sigma < \alpha}} V_{\sigma\alpha} \tag{7}$$

where $\phi_\sigma = \phi(\mathbf{x}^\sigma)$ is the "external" potential at the σth particle whose gradient yields the force on the σth particle due to external systems (this potential may be time-dependent), and $V_{\sigma\alpha} = V(\mathbf{x}^\sigma - \mathbf{x}^\alpha)$ is the potential energy due to the interaction of the σth with the αth particle. The particles are assumed to be structureless point masses. The corresponding canonical equations are

$$\dot{x}_j^\sigma = \frac{\partial H}{\partial p_j^\sigma} = \frac{p_j^\sigma}{m_\sigma}$$

$$\dot{p}_j^\sigma = -\frac{\partial \phi_\sigma}{\partial x_j^\sigma} - \frac{\partial}{\partial x_j^\sigma} \sum_{\substack{\alpha=1 \\ \alpha \neq \sigma}}^{N} V_{\alpha\sigma} = -\frac{\partial H}{\partial x_j^\sigma} \tag{8}$$

A statistical axiom may be introduced in conventional terms (Ref. 2) by defining a frequency in $6N$-dimensional phase space

$$\rho_N(\mathbf{x}^\sigma, \mathbf{p}^\sigma, t) \prod_{\sigma=1}^{N} d^3x^\sigma \, d^3p^\sigma$$

as the probability of finding the phase point $(\mathbf{x}^\sigma, \mathbf{p}^\sigma)$ of the system in the volume element

$$\prod_{\sigma=1}^{N} d^3x^\sigma \, d^3p^\sigma$$

at time t. In order to invest the object ρ_N with satisfactory mathematical properties, the probability referred to is usually conceived in the context of an hypothetical ensemble of macroscopically identical systems. If then $\mathcal{N}$ is the number of systems comprising the ensemble,

$$\int_{\delta V} \mathcal{N} \rho_N \prod_{\sigma=1}^{N} d^3x^\sigma \, d^3p^\sigma \tag{9}$$

is the expected number of members of the ensemble to be found in the phase-volume element δV. In the usual way (Ref. 3) a continuity equation for ρ_N is deduced; thus;

$$\frac{\partial \rho_N}{\partial t} + \sum_{\sigma=1}^{N} \left(\frac{\partial}{\partial x_j^\sigma} \dot{x}_j^\sigma \rho_N + \frac{\partial}{\partial p_j^\sigma} \dot{p}_j^\sigma \rho_N \right) = 0 \tag{10}$$

By virtue of the canonical equations this becomes

$$\frac{\partial \rho_N}{\partial t} + \sum_{\sigma=1}^{N} \left(\frac{\partial \rho_N}{\partial x_j^\sigma} \frac{\partial H}{\partial p_j^\sigma} - \frac{\partial \rho_N}{\partial p_j^\sigma} \frac{\partial H}{\partial x_j^\sigma} \right) = 0 \tag{11}$$

or

$$\frac{\partial \rho_N}{\partial t} + \{ \rho_N, H \} = 0$$

in terms of the conventional Poisson bracket notation.

If now G is an arbitrary function of phase points and time, the ensemble average of G may be defined by

$$\langle G \rangle \equiv \int G \rho_N \prod_{\sigma=1}^{N} d^3x^\sigma \, d^3p^\sigma \tag{12}$$

The equations of motion satisfied by these ensemble averages are readily deduced to be

$$\frac{\partial \langle G \rangle}{\partial t} = \left\langle \frac{\partial G}{\partial t} \right\rangle + \langle \{G, H\} \rangle \tag{13}$$

provided ρ_N vanishes on the surfaces spanning the phase space. (Actually this proviso is only sufficient, not necessary.)

Before proceeding to an attempted deduction of the Boltzmann equation, it is of some interest to digress here and give cursory attention to the microscopic statistical description presently invoked and to its connection with the macroscopic, thermodynamic description of the same system.

Since the thermodynamic characterization of systems first assumes steady-state conditions, consider distributions which satisfy

$$\frac{\partial \rho_N}{\partial t} = \{H, \rho_N\} = 0 \tag{14}$$

Two classes of solutions are immediately suggested:

$$\rho_N = \text{constant, and } \rho_N = \rho_N(H) \tag{15}$$

However, a further restriction to be imposed upon these steady states, in order that they correspond to thermodynamic states, follows from the requirement that weakly interacting classes of systems (say two boxes of ideal gases brought into contact so that they are capable of slowly exchanging energy) must achieve that steady state which is characterized by the same temperature for each system. This suggests the microscopic characterization of such "two-component" systems

$$H = H_A + H_B + H_{AB} \simeq H_A + H_B \tag{16}$$

and the condition for the "equilibrium" distribution function

$$\rho_{N_A + N_B}(H) \simeq \rho_{N_A + N_B}(H_A + H_B) = \rho_{N_A}(H_A)\, \rho_{N_B}(H_B) \tag{17}$$

i.e., that the microscopic character of system A does not affect the equilibrium distribution of system B, and vice versa. Hence, the only property that they can have in common is some macroscopic property.

A solution of the functional equation

$$\rho_{N_A + N_B}(H_A + H_B) = \rho_{N_A}(H_A)\, \rho_{N_B}(H_B) \tag{18}$$

is

$$\rho_N(H) = Q \exp\left(-\frac{H}{\theta}\right) \tag{19}$$

where θ is a parameter independent of the microscopic character of either system, but shared by both. Thus, exhibit $\theta = kT$ where T is temperature in degrees Kelvin and k is a universal thermodynamic constant to be determined experimentally. The normalization constant Q is to be chosen so that

$$\int \rho_N \prod_{\sigma=1}^{N} d^3x^\sigma \, d^3p^\sigma = 1 \tag{20}$$

as is the sign of the exponential, i.e., so that ρ_N be a true frequency.

In order to complete a thermodynamic description of a system, a connection is needed between the internal energy of the system, the work done on it, and the heat added to it. To accomplish this the internal energy u is defined to be $u \equiv \langle H \rangle$, and another average $\overline{S} \equiv -\langle \ln \rho_N \rangle$. It is then a straightforward matter to show, upon variation of $\overline{S}$ and u obtained by varying H and ρ_N, that

$$\theta \delta \overline{S} = \delta u - \langle \delta H \rangle \tag{21}$$

Now the variation in H is to be conceived here as arising from variations in external conditions, such as the locations of restraining walls, or strengths of externally applied electric fields, for example. Thus, a positive δH corresponds to work done on the system and $\delta W = \langle \delta H \rangle$. Thus,

$$\delta u = \theta \delta \overline{S} + \delta W \tag{22}$$

and, further, the "quantity of heat added to the system" is $\delta q = \theta \delta \overline{S} = kT\, \delta \overline{S}$, so that

$$\delta u = kT\, \delta \overline{S} + \delta W \tag{23}$$

Lastly this relation suggests that entropy S be defined by $S = k\overline{S}$. Thus, in summary:

$$\rho = Q \exp\left(-\frac{H}{kT}\right)$$

$$u = \langle H \rangle \tag{24}$$

$$S = -k\langle \ln \rho \rangle$$

The remaining thermodynamic potentials are now readily accessible; e.g.,

$$\begin{aligned} F &= u - TS \\ &= kT \ln Q \end{aligned} \tag{25}$$

A final observation worth noting in passing is that, though the above identification for the entropy seems suitable for the equilibrium state, it is not appropriate for the description of irreversible processes since for $S = -k\langle \ln \rho \rangle$, $dS/dt = 0$, all times rather than $dS/dt \geq 0$, the equality obtaining only for the equilibrium state.

Now the more comprehensive problem, attempting to deduce a tractable description of a many-particle system characterized by short-range interactions capable of describing changes in state of the system as well as the equilibrium state, is rejoined.

For such purposes, the most important ensemble average is the function f_1 defined earlier, and here redefined somewhat more fundamentally as

$$f_1(\mathbf{x}, \mathbf{p}, t) \equiv \left\langle \sum_{\sigma=1}^{N} \delta(\mathbf{x} - \mathbf{x}^{\sigma})\ \delta(\mathbf{p} - \mathbf{p}^{\sigma}) \right\rangle \tag{26}$$

Note that a subscript has been added to f to emphasize its character as a singlet density (expected number of particles "at a point") in contrast to doublet, triplet, etc., densities representing simultaneous expected members at "two points," "three points," etc. As will be seen, the description of the singlet density depends upon the doublet density which is defined to be

$$f_2(\mathbf{x}, \mathbf{p}, \mathbf{x}', \mathbf{p}', t) \equiv \left\langle \sum_{\substack{\sigma,\ \alpha \\ \sigma \neq \alpha}}^{N} \delta(\mathbf{x} - \mathbf{x}^{\sigma})\ \delta(\mathbf{x}' - \mathbf{x}^{\alpha})\ \delta(\mathbf{p} - \mathbf{p}^{\sigma})\ \delta(\mathbf{p}' - \mathbf{p}^{\alpha}) \right\rangle \qquad (27)$$

Note also that f_1 is here exhibited as a function of momenta rather than velocities – a trivial distinction in the absence of velocity-dependent forces.

Letting

$$g_1 \equiv \sum_{\sigma=1}^{N} \delta(\mathbf{x} - \mathbf{x}^{\sigma})\ \delta(\mathbf{p} - \mathbf{p}^{\sigma})$$

the time-dependence of f_1 is expressible as

$$\frac{\partial f_1}{\partial t} = \left\langle \frac{\partial g_1}{\partial t} \right\rangle + \left\langle \{g_1, H\} \right\rangle$$

$$= + \langle \{g_1, H\} \rangle$$

$$= + \langle \{g_1, T\} \rangle + \langle \{g_1, \phi\} \rangle + \langle \{g_1, V\} \rangle \qquad (28)$$

introducing the notation

$$\phi = \sum_{\sigma} \phi_{\sigma}$$

and

$$V = \sum_{\substack{\sigma,\ \alpha \\ \sigma < \alpha}} V_{\sigma \alpha}$$

It is a straightforward task to show that the above averages of Poisson brackets are exhibitable as

$$\left.\begin{aligned} \langle\{g_1,\ T\}\rangle &= -\frac{p_j}{m}\frac{\partial f_1}{\partial x_j} \\ \langle\{g_1,\ \phi\}\rangle &= +\frac{\partial\, \phi(\mathbf{x})}{\partial x_j}\frac{\partial f_1}{\partial p_j} \\ \text{and} \qquad & \\ \langle\{g_1,\ V\}\rangle &= +\int_{\mathbf{x}',\ \mathbf{p}'} d^3x'\ d^3p'\ \frac{\partial V(|\mathbf{x}-\mathbf{x}'|)}{\partial x_j}\ \frac{\partial f_2(\mathbf{x},\ \mathbf{p},\ \mathbf{x}',\ \mathbf{p}',\ t)}{\partial p_j} \end{aligned}\right\} \tag{29}$$

If it is noted that $F_j^e = -\ \partial\, \phi(\mathbf{x})/\partial x_j$ is the force on particles in the vicinity of $\mathbf{x}$ due to external systems, then Eq. (28) may be rewritten as:

$$\frac{\partial f_1}{\partial t} + \frac{p_j}{m}\frac{\partial f_1}{\partial x_j} + F_j^e\,\frac{\partial f_1}{\partial p_j} = \int d^3x'\ d^3p'\ \frac{\partial V(|\mathbf{x}-\mathbf{x}'|)}{\partial x_j}\ \frac{\partial f_2(\mathbf{x},\ \mathbf{p},\ \mathbf{x}',\ \mathbf{p}',\ t)}{\partial p_j} \tag{30}$$

Up to this point no explicit cognizance has been taken of the assumed short-range character of the potential V. Furthermore, it is to be noted that the terms in this relation which are independent of particle interactions are identical to the analogous terms appearing in the previously deduced Boltzmann equation. Presumably some estimate of the validity of the Boltzmann equation should emerge from an investigation of the sense in which the interaction term in the above expression can be replaced by a description of particle interactions characterizable entirely in terms of two-body collisions. That such an investigation should be at least simplified by the assumption of short-range forces (and will initially be pressed for that instance) is probably obvious; this has been carried forward in a number of slightly varying ways employing both the classical and quantum mechanical formalisms (Refs. 4–7). It will be sufficient for the purpose here to attempt this reduction within the classical context.

Let

$$I = \int_{\mathbf{x}', \mathbf{p}'} d^3x' \; d^3p' \; \frac{\partial V(|\mathbf{x} - \mathbf{x}'|)}{\partial x_j} \; \frac{\partial f_2(\mathbf{x}, \mathbf{p}, \mathbf{x}', \mathbf{p}', t)}{\partial p_j} \tag{31}$$

and note that also

$$I = \int_{\mathbf{x}', \mathbf{p}'} d^3x' \; d^3p' \left(\frac{\partial V}{\partial x_j} \frac{\partial f_2}{\partial p_j} + \frac{\partial V}{\partial x'_j} \frac{\partial f_2}{\partial p'_j} \right) \tag{32}$$

the additional term being zero. Now assume that $V(|\mathbf{x} - \mathbf{x}'|) = 0$ for $|\mathbf{x} - \mathbf{x}'| > l$, where $l << (V/N)^{1/3} = L$; i.e., the "interaction distance" is presumed very small compared to the average interparticle spacing. Thus, it is very uncommon to find two particles within an interaction distance of each other; and sufficiently more so for three or more particles to be so spaced that such circumstances will be ignored altogether. Consequently, whenever the integrand in I is not zero, the phase points $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$ may be identified with the actual coordinates in phase space of a pair of particles for the purpose of performing the integration. Thus, labeling $(\mathbf{x}, \mathbf{p}) \to (\mathbf{x}_1, \mathbf{p}_1)$ and $(\mathbf{x}', \mathbf{p}') \to (\mathbf{x}_2, \mathbf{p}_2)$ yields

$$I = \int_{\mathbf{x}_2, \mathbf{p}_2} d^3x_2\, d^3p_2 \left(\frac{\partial V}{\partial x_{1j}} \frac{\partial f_2}{\partial p_{1j}} + \frac{\partial V}{\partial x_{2j}} \frac{\partial f_2}{\partial p_{2j}} \right)$$

$$= \int_{\mathbf{x}_2, \mathbf{p}_2} d^3x_2\, d^3p_2 \left(- \frac{dp_{1j}}{dt} \frac{\partial f_2}{\partial p_{1j}} - \frac{dp_{2j}}{dt} \frac{\partial f_2}{\partial p_{2j}} \right) \tag{33}$$

on the assumption that the change in momentum of Particle 1 is, over the interaction range, entirely due to the forces exerted upon it by Particle 2 and vice versa. If it is now assumed that the variation in f_2 over distances of the order of the collision distance l and over times of the order of the collision time $\tau = l/v_r$ is negligible, this expression may be averaged over τ as follows:

$$\bar{I} \equiv \frac{1}{\tau} \int_t^{t+\tau} I(t')\, dt'$$

$$= - \int_{\mathbf{x}_2, \mathbf{p}_2} d^3x_2\, d^3p_2 \frac{1}{\tau} \int_t^{t+\tau} \left[\frac{dp'_{1j}}{dt'} \frac{\partial f_2(\mathbf{x}_1, \mathbf{p}'_1, \mathbf{x}_1, \mathbf{p}'_2, t)}{\partial p'_{1j}} + \frac{dp'_{2j}}{dt'} \frac{\partial f_2(\mathbf{x}_1, \mathbf{p}'_1, \mathbf{x}_1, \mathbf{p}'_2, t)}{\partial p'_{2j}} \right] dt'$$

$$= - \frac{1}{\tau} \int_t^{t+\tau} \int_{\mathbf{x}_2, \mathbf{p}_2} d^3x_2\, d^3p_2 \left[\frac{dp'_{1j}}{dt'} \frac{\partial f_2(\mathbf{x}_1, \mathbf{p}'_1, \mathbf{x}_1, \mathbf{p}'_2, t)}{\partial p'_{1j}} + \frac{dp'_{2j}}{dt'} \frac{\partial f_2(\mathbf{x}_1, \mathbf{p}'_1, \mathbf{x}_1, \mathbf{p}'_2, t)}{\partial p'_{2j}} \right] dt'$$

$$= \int d^3x_2\, d^3p_2 \frac{1}{\tau} \left\{ f_2[\mathbf{x}_1, \mathbf{p}_1(t), \mathbf{x}_1, \mathbf{p}_2(t), t] - f_2[\mathbf{x}_1, \mathbf{p}_1(t+\tau), \mathbf{x}_1, \mathbf{p}_2(t+\tau), t] \right\} \tag{34}$$

The integration over $\mathbf{x}_2$ now yields simply the collision volume; i.e.,

$$\int_{\mathbf{x}_2} d^3x_2 = l \int_{\Omega} \sigma d\Omega \tag{35}$$

If finally the pre-collision moments are labeled with primes and it is noted that $l/\tau = v_r$, the relative speed of the colliding particles,

$$\overline{I} = \int d^3p_2 \; d\Omega\sigma v_r \; [f_2(\mathbf{x}, \mathbf{p}_1', \mathbf{x}, \mathbf{p}_2', t) - f_2(\mathbf{x}, \mathbf{p}_1, \mathbf{x}, \mathbf{p}_2, t)] \tag{36}$$

The relation for f_1 may now be written as:

$$\frac{\partial \overline{f_1}}{\partial t} + \frac{p_j}{m} \frac{\overline{f_1}}{x_j} + \overline{F_j^e \frac{\partial f_1}{\partial p_j}} = \int d^3p_2 \; d\Omega v_r \sigma [f_2(\mathbf{p}_1', \mathbf{p}_2') - f_2(\mathbf{p}_1, \mathbf{p}_2)] \tag{37}$$

where

$$\overline{f_1} \equiv \frac{1}{\tau} \int_t^{t+\tau} f_1' \; dt'$$

If now the transformation is made from momentum to velocity space, and it is observed that the previous assumptions imply that $\overline{\overline{f}} \simeq \overline{f}$ and it is assumed that F_j^e does not change appreciably over times of order τ and further that $\overline{f_2}(\mathbf{x}, \mathbf{p}_1, \mathbf{x}, \mathbf{p}_2, t) \simeq \overline{f_1}(\mathbf{x}, \mathbf{p}_1, t) \; \overline{f_1}(\mathbf{x}, \mathbf{p}_2, t)$, the Boltzmann equation is obtained in the form deduced earlier.

Clearly quite a number of assumptions underlie the employment of a Boltzmann-type equation for the description of gas-like systems, of which perhaps the most important are:

1. The interactions between particles are entirely in terms of short-range forces, the range being very small compared to average interparticle spacing;

2. The distribution functions do not vary appreciably over distances of the order of the collision distance, and over times of the order of the collision times;

3. The "external" forces do not vary appreciably over times of the order of the collision time, and are weak compared to the interparticle forces experienced by two particles within a "force range" of each other; and

4. The probability of finding Particle 2 within a "force range" of a point $\mathbf{x}$, given that Particle 1 is at $\mathbf{x}$, is largely independent of the probability that Particle 1 be at $\mathbf{x}$, i.e., the assumption implicit in the approximation $f_2 \simeq f_1 f_1$ invoked within the integrand of the interaction integral.

The sense in which any of these assumptions are germane to plasmas is certainly open to question, yet one of the most commonly employed descriptions of the plasma is some variant of a Boltzmann-type equation, or in terms of sets of equations deducible therefrom. Thus, the next logical step is the attempt to deduce some such equation (or equations) appropriate to the description of μ-space distribution functions, $f(\mathbf{x}, \mathbf{p}, t)$, or $f(\mathbf{x}, \mathbf{v}, t)$, suitable for plasma investigations.

III. A LIOUVILLE EQUATION FOR THE PLASMA

The approach to this problem shall be developed along the same general lines as the previous one:

1. Incorporation of reasonably clearly stated dynamical axioms into an appropriate Hamiltonian (again classical) for the physical system under consideration.

2. Formulation of a statistical axiom within the context of a suitable ensemble concept.

3. Attempted deduction of equations describing μ-space distribution functions.

The first task is thus the construction of a suitable Hamiltonian. A complication is encountered immediately when it is compared with the problem previously posed, in that the interactions between particles in the plasma are electromagnetic in character and hence the potentials characterizing the interactions are neither short-range nor velocity-independent. The construction of the Hamiltonian to be employed for illustrative purposes follows closely the arguments presented by Heitler (Ref. 8) and hence shall be presented here only in barest outline.

It is appropriate to adopt, as a starting point for the dynamical description of the system, Maxwell's equations:

$$\left.\begin{aligned} \nabla \times \boldsymbol{H} &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J} \\ \nabla \cdot \boldsymbol{H} &= 0 \\ \nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} &= 0 \\ \nabla \cdot \boldsymbol{E} &= 4\pi Q \end{aligned}\right\} \tag{38}$$

or alternatively in terms of the potentials **A** and ϕ,

$$\left.\begin{aligned}
&\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \\
&\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi Q \\
&\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0
\end{aligned}\right\} \tag{39}$$

where $\mathbf{H} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\nabla\phi - (1/c)(\partial \mathbf{A}/\partial t)$, $Q = \Sigma_\sigma e_\sigma\, \delta(\mathbf{x} - \mathbf{x}^\sigma)$, and $\mathbf{J} = \Sigma_\sigma e_\sigma\, \dot{\mathbf{x}}^\sigma\, \delta(\mathbf{x} - \mathbf{x}^\sigma)$.

It is also appropriate to introduce Lorentz's force law:

$$m_\sigma \ddot{x}^\sigma = e_\sigma \mathbf{E} + \frac{e_\sigma}{c}(\dot{\mathbf{x}}^\sigma \times \mathbf{H})$$

or,

$$m_\sigma \ddot{\mathbf{x}}^\sigma = -e_\sigma \nabla\phi - \frac{e_\sigma}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e_\sigma}{c}[\dot{\mathbf{x}}^\sigma \times (\nabla \times \mathbf{A})] \tag{40}$$

For the moment, consideration is restricted to fully ionized plasmas containing only one type of ion which are not acted upon by "external" electromagnetic fields (or any other external field for that matter) and the particle motions are treated non-relativistically. The first restriction – implying no neutrals – is easily dropped subsequently by simply adjoining the arguments developed in the previous section. Dropping the second restriction requires obvious and trivial generalization. Elimination of the third restriction is also trivial in that external fields need simply be added to the internal fields upon completion of the analysis. However, the fourth restriction (which may be a serious one so far as application of results obtained here to thermonuclear systems is concerned) does not appear to be avoidable within the context of a completely classical discussion.

As a convenient device for the introduction of degrees of freedom appropriate to the characterization of the electromagnetic fields, define:

1. A set of real vector functions $\{\mathbf{X}_n\}$ such that:

   a. $\nabla^2 \mathbf{X}_n + k_n^2 \mathbf{X}_n = 0$

   b. $\mathbf{X}_n$ either vanishes or obeys periodic boundary conditions on the boundaries of the regions of interest;

   c. $\int \mathbf{X}_n \cdot \mathbf{X}_m \, d^3x = \delta_{nm}$

   d. $\nabla \cdot \mathbf{X}_n = 0$

   e. the set is complete in the sense that for an arbitrary vector function $\mathbf{F}$ defined over the domain of definition of the $\mathbf{X}_n$ which satisfies $\nabla \cdot \mathbf{F} = 0$:

$$\int \mathbf{F} \cdot \mathbf{X}_n \, d^3x \neq 0, \text{ some } n$$

2. A set of real scalar functions $\{\psi_n\}$ such that:

   a. $\nabla^2 \psi_n + \lambda_n^2 \psi_n = 0$

   b. same as *b* above,

   c. same as *c* above,

   d. the set is complete with respect to scalar functions in the sense that

$$\int \phi \psi_n \, d^3x \neq 0, \text{ some } n$$

3. As a corollary to (2), a set of real vector functions $\{L_n\}$ by

$$\mathbf{L}_n \equiv \frac{1}{\lambda_n} \nabla \psi_n$$

Interesting properties of this last set are:

a. $\nabla^2 \mathbf{L}_n + \lambda_n^2 \mathbf{L}_n = 0,$

b. $\nabla \times \mathbf{L}_n = 0,$

c. the set is complete with respect to vector functions which satisfy

$$\nabla \times \mathbf{F} = 0,$$

d. $\int \mathbf{L}_n \cdot \mathbf{L}_m \, d^3x = \delta_{nm},$

e. $\int \mathbf{L}_n \cdot \mathbf{X}_m \, d^3x = 0,$ all n and m.

With these mathematical tools now available, the physical problem at hand is rejoined and the vector field $\mathbf{A}$ and the scalar field ϕ are expanded as:

$$\mathbf{A}(\mathbf{x},t) = \sum_n a_n^T(t)\mathbf{X}_n + \sum_n a_n^L(t)\mathbf{L}_n \tag{41}$$

$$\phi(\mathbf{x},\ t) = \sum_n \alpha_n(t)\psi_n \tag{42}$$

Inserting these expansions into the equations for the fields and the particles, the following equations for the field variables (now comprised of a countable set) a_n^L, a_n^T, and α_n and the particle variables $\mathbf{x}^\sigma$ are obtained

$$k_n^2 a_n^T + \frac{1}{c^2}\ddot{a}_n^T = \frac{4\pi}{c}\sum_\sigma e_\sigma \dot{\mathbf{x}}^\sigma \cdot \mathbf{X}_n(\mathbf{x}^\sigma)$$

$$\lambda_n^2 a_n^L + \frac{1}{c^2}\ddot{a}_n^L = \frac{4\pi}{c}\sum_\sigma e_\sigma \dot{\mathbf{x}}^\sigma \cdot \mathbf{L}_n(\mathbf{x}^\sigma) \tag{43}$$

$$\lambda_n^2 \alpha_n + \frac{1}{c^2}\ddot{\alpha}_n = 4\pi \sum_\sigma e_\sigma \psi_n(\mathbf{x}^\sigma)$$

$$\left.\begin{aligned}
&-\lambda_n a_n^L + \frac{1}{c}\dot{\alpha}_n = 0 \\
&m_\sigma \ddot{\mathbf{x}}^\sigma = -e_\sigma \sum_n \left(\alpha_n \lambda_n + \frac{1}{c}\dot{a}_n^L\right) \mathbf{L}_n(\mathbf{x}^\sigma) - \frac{e_\sigma}{c}\sum_n \dot{a}_n^T \mathbf{X}_n(\mathbf{x}^\sigma) \\
&\qquad + \frac{e_\sigma}{c}\sum_n a_n^T \{\dot{\mathbf{x}}^\sigma \times [\nabla \times \mathbf{X}_n(\mathbf{x}^\sigma)]\}
\end{aligned}\right\} \text{(43 Cont'd)}$$

All dependence upon the scalar field variables α_n and longitudinal vector field variables a_n^L may now be eliminated from the dynamical description. Note that these enter into the description of the particle motions via the term

$$-e_\sigma \sum_n \left(\alpha_n \lambda_n + \frac{1}{c}\dot{a}_n^L\right) \mathbf{L}_n(\mathbf{x}^\sigma)$$

Making use of the subsidiary condition

$$\lambda_n a_n^L = \frac{1}{c}\dot{\alpha}_n$$

the equations of motion for the α_n, and the defining relation

$$\mathbf{L}_n = \frac{1}{\lambda_n}\nabla\psi_n$$

it is possible to obtain after some manipulation

$$- e_\sigma \sum_n \left(\alpha_n \lambda_n + \frac{1}{c}\dot{\alpha}_n^L\right) \mathbf{L}_n = - 4\pi e_\sigma \sum_{n,\,\sigma'} \frac{e_{\sigma'}}{\lambda_n^2} \psi_n (\mathbf{x}^{\sigma'}) \nabla_\sigma \psi_n (\mathbf{x}^\sigma)$$

$$= - 4\pi e_\sigma \nabla_\sigma \sum_{\substack{n,\,\sigma' \\ \sigma' \neq \sigma}} e_{\sigma'} \frac{\psi_n(\mathbf{x}^{\sigma'})\psi_n(\mathbf{x}^\sigma)}{\lambda_n^2} - 4\pi e_\sigma^2 \sum_n \frac{\psi_n(\mathbf{x}^\sigma)\nabla_\sigma \psi_n(\mathbf{x}^\sigma)}{\lambda_n^2} \tag{44}$$

The second term in this expression corresponds to an infinite self-energy of a charged particle arising out of its interaction with itself via its own coulomb field and will be dropped from subsequent discussion on the ground that it constitutes an unobservable contribution to the rest-energy of the particle. The first term can be rewritten more conveniently as a consequence of the observation that

$$\nabla_\sigma^2 \sum_n \frac{\psi_n(\mathbf{x}^{\sigma'})\,\psi_n(\mathbf{x}^\sigma)}{\lambda_n^2} = - \sum_n \psi_n(\mathbf{x}^{\sigma'})\,\psi_n(\mathbf{x}^\sigma)$$

$$= - \delta(\mathbf{x}^\sigma - \mathbf{x}^{\sigma'}) \tag{45}$$

But the solution to this equation is well known to be

$$\sum_n \frac{\psi_n(\mathbf{x}^{\sigma'})\,\psi_n(\mathbf{x}^\sigma)}{\lambda_n^2} = \frac{1}{4\pi|\mathbf{x}^\sigma - \mathbf{x}^{\sigma'}|} \tag{46}$$

Hence, finally,

$$- e_\sigma \sum_n \left(\alpha_n \lambda_n + \frac{1}{c} \dot{a}_n^L \right) \mathbf{L}_n = - e_\sigma \nabla_\sigma \sum_{\substack{\sigma' \\ \sigma' \neq \sigma}} \frac{e_{\sigma'}}{\left| \mathbf{x}^\sigma - \mathbf{x}^{\sigma'} \right|} \equiv e_\sigma \mathbf{E}^L(\mathbf{x}^\sigma) \tag{47}$$

Evidently now there is no need for further consideration of the longitudinal field variables, and a complete system of equations for particles plus electromagnetic fields may be listed:

$$\left. \begin{aligned} m_\sigma \ddot{\mathbf{x}}^\sigma &= e_\sigma \mathbf{E}^L(\mathbf{x}^\sigma) - \frac{e_\sigma}{c} \sum_n \dot{a}_n^T \mathbf{X}_n(\mathbf{x}^\sigma) + \frac{e_\sigma}{c} \sum_n a_n^T \left\{ \dot{\mathbf{x}}^\sigma \times \left[\nabla \times \mathbf{X}_n(\mathbf{x}^\sigma) \right] \right\} \\ &\text{and} \\ \ddot{a}_n^T + \omega_n^2 a_n^T &= 4\pi c \sum_\sigma e_\sigma \dot{\mathbf{x}}^\sigma \cdot \mathbf{X}_n(\mathbf{x}^\sigma) \end{aligned} \right\} \tag{48}$$

where $\omega_n = k_n/c$ has been introduced.

These equations give a dynamical description of the system in which the particle degrees of freedom are given explicitly as points in configuration space (the $\mathbf{x}^\sigma$) and the degrees of freedom of the field are characterized by the time-dependent amplitudes a_n (dropping the superscript T which distinguised these quantities as the amplitudes for the transverse oscillators). A Lagrangian which leads to the correct equations of motion for the system may be taken to be

$$\begin{aligned} L = &\sum_\sigma \left[\frac{1}{2} m_\sigma (\dot{\mathbf{x}}^\sigma)^2 - e_\sigma \Phi(\mathbf{x}^\sigma) \right] \\ &+ \frac{1}{2} \sum_n \frac{\dot{a}_n^2 - \omega_n^2 \dot{a}_n^2}{4\pi c^2} \\ &+ \sum_{n,\,\sigma} \frac{e_\sigma}{c} a_n \, \dot{\mathbf{x}}^\sigma \cdot \mathbf{X}_n(\mathbf{x}^\sigma) \end{aligned} \tag{49}$$

where the notation

$$\mathbf{E}^L(\mathbf{x}^\sigma) = -\nabla_\sigma \Phi(\mathbf{x}^\sigma)$$

has been introduced.

The canonical momenta are now readily obtained as

$$p_j^\sigma = \frac{\partial L}{\partial \dot{x}_j^\sigma} = m_\sigma \dot{x}_j^\sigma + \sum_n \frac{e_\sigma}{c} a_n X_{nj}(x^\sigma)$$

$$\pi_n = \frac{\partial L}{\partial \dot{a}_n} = \frac{1}{4\pi c^2} \dot{a}_n \tag{50}$$

It then follows that the Hamiltonian is

$$H \equiv \sum_\sigma \dot{\mathbf{x}}^\sigma \cdot \mathbf{p}^\sigma + \sum_n \dot{a}_n \pi_n - L$$

$$= \sum_\sigma \frac{\left(\mathbf{p}^\sigma - \frac{e_\sigma}{c} \mathbf{A}^T(\mathbf{x}^\sigma)\right)^2}{2m_\sigma} + \sum_{\substack{\sigma, \sigma' \\ \sigma' < \sigma}} \frac{e_\sigma e_{\sigma'}}{|\mathbf{x}^\sigma - \mathbf{x}^{\sigma'}|} + 2\pi c^2 \sum_n \left(\pi_n^2 + \omega_n'^2 a_n^2\right) \tag{51}$$

where $\omega_n' = k_n/4\pi c$, and $\mathbf{A}^T = \sum_n a_n \mathbf{X}_n$

It is a straightforward task to show that again the original dynamical equations are reproduced according to

$$\left.\begin{aligned} \dot{x}_j^\sigma &= \frac{\partial H}{\partial p_j^\sigma} & \qquad \dot{p}_j{}^\sigma &= -\frac{\partial H}{\partial x_j^\sigma} \\ \dot{a}_n &= \frac{\partial H}{\partial \pi_n} & \qquad \dot{\pi}_n &= -\frac{\partial H}{\partial a_n} \end{aligned}\right\} \tag{52}$$

It is this Hamiltonian and these equations of motion that shall tentatively be adopted as an incorporation of the "dynamical axiom" for the system. It should be noted that certain aspects of the fundamental problem of describing a plasma have been investigated in precisely these terms by Brittin (Ref. 9).

The statistical axiom shall be introduced here, exactly as previously, within the context of an ensemble concept. An ensemble frequency is defined in the form of the statement that $\rho(\mathbf{x}^\sigma, \mathbf{p}^\sigma, a_n, \pi_n) \prod_{\sigma=1}^{N} \prod_n d^3x^\sigma\, d^3p^\sigma\, da_n\, d\,\pi_n \equiv$ the probability of finding a member of the ensemble with its phase point $(\mathbf{x}^\sigma, \mathbf{p}^\sigma, a_n, \pi_n)$ in the volume element $\prod_{\sigma=1}^{N} \prod_n d^3x^\sigma d^3p^\sigma\, da_n\, d\pi_n$ at time t. The concept of the phase volume $\prod_n da_n\, d\pi_n$ seems somewhat ambiguous since n runs over an infinite set. However this difficulty is readily circumvented by simply choosing a sufficiently large, but finite, set of oscillator degrees of freedom so that all physically interesting situations can be realized.

The frequency thus defined satisfies the usual Liouville equation, i.e.;

$$\frac{\partial \rho}{\partial t} + \sum_\sigma \frac{\partial H}{\partial p_j^\sigma}\frac{\partial \rho}{\partial x_j^\sigma} - \sum_\sigma \frac{\partial H}{\partial x_j^\sigma}\frac{\partial \rho}{\partial p_j^\sigma} + \sum_n \frac{\partial H}{\partial \pi_n}\frac{\partial \rho}{\partial a_n} - \sum_n \frac{\partial H}{\partial a_n}\frac{\partial \rho}{\partial \pi_n} = 0 \tag{53}$$

For the Hamiltonian indicated above (Eq. 49), this can be exhibited as

$$\frac{\partial \rho}{\partial t} + \sum_{\sigma} \dot{x}_j^{\sigma} \frac{\partial \rho}{\partial x_j^{\sigma}} + \sum_{\sigma} \frac{e_{\sigma}}{m_{\sigma} c} \left(\mathbf{p}^{\sigma} - \frac{e_{\sigma}}{c} \mathbf{A}_T^L\right)_l \frac{\partial A_l^T}{\partial x_j^{\sigma}} \frac{\partial \rho}{\partial p_j^{\sigma}}$$

$$+ \sum_{n} 4\pi c^2 \pi_n \frac{\partial \rho}{\partial a_n} - \sum_{n} 4\pi c^2 \omega_n'^2 a_n \frac{\partial \rho}{\partial \pi_n}$$

$$+ \sum_{\sigma,\, n} \frac{e_{\sigma}}{m_{\sigma} c} \left(\mathbf{p}^{\sigma} - \frac{e_{\sigma}}{c} \mathbf{A}^T\right)_l \frac{\partial A_l^T}{\partial a_n} \frac{\partial \rho}{\partial \pi_n} - \sum_{\sigma} e_{\sigma} \frac{\partial \Phi}{\partial x_j^{\sigma}} \frac{\partial \rho}{\partial p_j^{\sigma}} = 0 \tag{54}$$

It is quite customary to deal with frequencies defined in velocity space rather than momentum space, so define

$$\bar{\rho}(\mathbf{x}^{\sigma}, \dot{\mathbf{x}}^{\sigma}, a_n, \pi_n, t) \prod_{\sigma=1}^{N} \prod_{n} d^3x^{\sigma} d^3\dot{x}^{\sigma} da_n d\pi_n$$

$$\equiv \rho(\mathbf{x}^{\sigma}, \mathbf{p}^{\sigma}, a_n, \pi_n, t) \prod_{\sigma=1}^{N} \prod_{n} d^3x^{\sigma} d^3p^{\sigma} da_n d\pi_n \tag{55}$$

Since the Jacobian of this transformation is a constant,

$$\bar{\rho}(\mathbf{x}^{\sigma}, \dot{\mathbf{x}}^{\sigma}, a_n, \pi_n, t) = (\text{constant})\ \rho\left(\mathbf{x}^{\sigma}, m\dot{\mathbf{x}}^{\sigma} + \frac{e_{\sigma}}{c} \mathbf{A}^T, a_n, \pi_n, t\right) \tag{56}$$

hence,

$$\left.\begin{aligned}
&\frac{\partial \rho}{\partial t} \to \frac{\partial \bar{\rho}}{\partial t} \\
&\frac{\partial \rho}{\partial p_j^\sigma} \to \frac{1}{m_\sigma} \frac{\partial \bar{\rho}}{\partial \dot{x}_j^\sigma} \\
&\frac{\partial \rho}{\partial x_j^\sigma} \to \frac{\partial \bar{\rho}}{\partial x_j^\sigma} - \frac{e_\sigma}{m_\sigma c} \frac{\partial A_k^T}{\partial x_j^\sigma} \frac{\partial \bar{\rho}}{\partial \dot{x}_k^\sigma} \\
&\frac{\partial \rho}{\partial a_n} \to \frac{\partial \bar{\rho}}{\partial a_n} - \sum_\sigma \frac{e_\sigma}{m_\sigma c} \frac{\partial A_j^T}{\partial a_n} \frac{\partial \bar{\rho}}{\partial \dot{x}_j^\sigma} \\
&\frac{\partial \rho}{\partial \pi_n} \to \frac{\partial \bar{\rho}}{\partial \pi_n}
\end{aligned}\right\} \quad (57)$$

The equation for $\bar{\rho}$ may now be written as

$$\frac{\partial \bar{\rho}}{\partial t} + \sum_\sigma \dot{x}_j^\sigma \frac{\partial \bar{\rho}}{\partial x_j^\sigma} + \sum_\sigma \Omega_k^\sigma \frac{\partial \bar{\rho}}{\partial \dot{x}_k^\sigma} + 4\pi c^2 \sum_n \pi_n \frac{\partial \bar{\rho}}{\partial a_n} + \sum_n \Omega_n \frac{\partial \bar{\rho}}{\partial \pi_n} = 0 \quad (58)$$

where

$$\Omega_k^\sigma \equiv \frac{e_\sigma}{m_\sigma c} \dot{x}_j^\sigma \sum_n a_n \left(\frac{\partial X_{nj}^\sigma}{\partial x_k^\sigma} - \frac{\partial X_{nk}}{\partial x_j^\sigma} \right) - \frac{4\pi c e_\sigma}{m_\sigma} \pi_n X_{nk}^\sigma - \frac{e_\sigma}{m_\sigma} \frac{\partial \Phi}{x_k^\sigma} \quad (59)$$

and

$$\Omega_n = \sum_{\sigma} \frac{e_\sigma}{c} \dot{x}_j^\sigma X_{nj}^\sigma - 4\pi c^2 \omega_n'^2 a_n \tag{59 Cont'd}$$

It is to be noted that

$$\frac{\partial \Omega_k^\sigma}{\partial \dot{x}_j^\sigma} = \frac{\partial \Omega_n}{\partial \pi_n} = 0 \tag{60}$$

It is convenient at this point to employ scalar product notation for the ensemble averages, i.e.,

$$\begin{aligned} \langle G \rangle &= (\bar{\rho},\ G) \\ &= (G,\ \bar{\rho}) \equiv \int \bar{\rho} G \prod_{\sigma=1}^{N} \prod_{n} d^3x^\sigma\, d^3\dot{x}^\sigma\, da_n\, d\pi_n \end{aligned} \tag{61}$$

Again interest is attached initially to a description of that ensemble averaged quantity which corresponds to the μ-space distribution function for particles of kind A (there must necessarily be more than one kind of particle present in the plasma since by definition it is electrically neutral over-all):

$$\begin{aligned} f_1^A(\mathbf{x}, \dot{\mathbf{x}}, t) &= \left(\bar{\rho},\ \sum_{\sigma=1}^{N_A} \delta(\mathbf{x} - \mathbf{x}^\sigma)\ \delta(\dot{\mathbf{x}} - \dot{\mathbf{x}}^\sigma) \right) \\ &= (\bar{\rho},\ g_1^A) \end{aligned} \tag{62}$$

The temporal behavior of this distribution function will be seen to depend also upon the following "two-particle distribution functions:"

$$f_2^A(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t) = \left(\bar{\rho}, \sum_{\substack{\sigma, \alpha = 1 \\ \sigma \neq \alpha}}^{N_A} \delta(\mathbf{x} - \mathbf{x}^\sigma)\, \delta(\dot{\mathbf{x}} - \dot{\mathbf{x}}^\sigma)\, \delta(\mathbf{x}' - \mathbf{x}^\alpha)\, \delta(\dot{\mathbf{x}}' - \dot{\mathbf{x}}^\alpha) \right)$$

$$= (\bar{\rho}, g_2^A)$$

and

$$f_2^{AB}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}' t) = \left(\bar{\rho}, \sum_{\sigma = 1}^{N_A} \sum_{\alpha = 1}^{N_B} \delta(\mathbf{x} - \mathbf{x}^\sigma)\, \delta(\dot{\mathbf{x}} - \dot{\mathbf{x}}^\sigma)\, \delta(\mathbf{x}' - \mathbf{x}^\alpha)\, \delta(\dot{\mathbf{x}}' - \dot{\mathbf{x}}^\alpha) \right)$$

$$= (\bar{\rho}, g_2^{AB}) \tag{63}$$

It then follows straightforwardly from the equations of motion for $\bar{\rho}$ that

$$\frac{\partial f_1^A}{\partial t} = \left(\frac{\partial \bar{\rho}}{\partial t}, g_1^A \right) = - \dot{x}_j \frac{\partial f_1^A}{\partial x_j} - \frac{\partial}{\partial \dot{x}_j} \left(\bar{\rho}, \sum_{\sigma = 1}^{N_A} \Omega_j^\sigma\, \delta(\mathbf{x} - \mathbf{x}^\sigma)\, \delta(\dot{\mathbf{x}} - \mathbf{x}^\sigma) \right) \tag{64}$$

Inserting the definition of Ω_j^σ into this expression, additional manipulation produces:

$$\frac{\partial f_1^A}{\partial t} + \dot{x}_j \frac{\partial f_1^A}{\partial x_j} = \frac{e_A}{m_A}\left(\frac{\partial}{\partial x_j}\int d^3x'\, d^3\dot{x}'\, \frac{e_A}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^A(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j}$$

$$+ \frac{e_A}{m_A}\left(\frac{\partial}{\partial x_j}\int d^3x'\, d^3\dot{x}'\, \frac{e_B}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^{AB}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j}$$

$$+ \frac{e_A}{cm_A}\frac{\partial}{\partial \dot{x}_j}\sum_n \sum_{\sigma=1}^{N_A}\left(\bar{\rho},\ \dot{a}_n \mathbf{X}_{nj}^{\sigma}\ \delta(\mathbf{x}-\mathbf{x}^{\sigma})\ \delta(\dot{\mathbf{x}}-\dot{\mathbf{x}}^{\sigma})\right)$$

$$- \frac{e_A}{cm_A}\frac{\partial}{\partial \dot{x}_j}\sum_n \sum_{\sigma=1}^{N_A}$$

$$\left(\bar{\rho},\ a_n\, \dot{x}_l^{\sigma}\left[\frac{\partial X_{nl}^{\sigma}}{\partial x_j^{\sigma}} - \frac{\partial X_{nj}^{\sigma}}{\partial x_l^{\sigma}}\right]\delta(\mathbf{x}-\mathbf{x}^{\sigma})\ \delta(\dot{\mathbf{x}}-\dot{\mathbf{x}}^{\sigma})\right) \tag{65}$$

where the connection $\pi_n = \dot{a}_n/4\pi c^2$ has been used. Now the "field terms" (the last two terms in this relation) may be exhibited in the forms:

$$+ \frac{e_A}{m_A}\frac{\partial}{\partial \dot{x}_j}\left(\bar{\rho},\ g_1^A \sum_n \frac{1}{c}\dot{a}_n X_{nj}\right)$$

$$- \frac{e_A}{cm_A}\frac{\partial}{\partial \dot{x}_j}\left\{\bar{\rho},\ g_1^A\left[\dot{\mathbf{x}}\times\left(\nabla\times\sum_n a_n \mathbf{X}_n\right)\right]_j\right\}$$

where use has been made of the formula

$$\dot{x}_l^\sigma \left(\frac{\partial X_{nl}^\sigma}{\partial x_j^\sigma} - \frac{\partial X_{nj}^\sigma}{\partial x_l^\sigma} \right) = \left[\mathbf{x}'^{\sigma} \times (\nabla_\sigma \times \mathbf{X}_n^\sigma) \right]_j \tag{66}$$

It is recalled that

$$\left. \begin{aligned} E_j^T(\mathbf{x}, t) &= -\frac{1}{c}\frac{\partial}{\partial t} A_j^T(\mathbf{x}, t) = -\frac{1}{c} \sum_n a_n X_{nj}, \\ &\text{and} \\ \boldsymbol{H}(\mathbf{x}, t) &= \nabla \times \mathbf{A} = \nabla \times \mathbf{A}^T = \nabla \times \sum_n a_n \mathbf{X}_n \end{aligned} \right\} \tag{67}$$

Thus the equation for f_1^A may now be written as

$$\begin{aligned} \frac{\partial f_1^A}{\partial t} + \dot{x}_j \frac{\partial f_1^A}{\partial x_j} &= \frac{e_A}{m_A} \int d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e_A}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2^A(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j} \\ &+ \frac{e_A}{m_A} \int d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e_B}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2^{AB}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j} \\ &- \frac{e_A}{m_A} \frac{\partial}{\partial \dot{x}_j} (\overline{\rho}, g_1^A E_j^T) - \frac{e_A}{m_A c} \frac{\partial}{\partial \dot{x}_j} \left(\overline{\rho}, g_1^A \left\{ \dot{\mathbf{x}} \times \boldsymbol{H} \right\}_j \right) \end{aligned} \tag{68}$$

It must be borne in mind that $\mathbf{E}^T$ and $\mathbf{H}$ appearing in the above expression are functions not only of $\mathbf{x}$ and t but of the phase coordinates of all of the particles as well. That this is the case follows from the observation that these quantities satisfy the equations

$$\left.\begin{aligned}
\nabla^2\mathbf{H}(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{x},t) &= -\frac{4\pi}{c}\nabla\times\sum_{\sigma,n} e_\sigma \dot{x}_j^\sigma(t)\, X_{nj}^\sigma\, \mathbf{X}_n(\mathbf{x}) \\
\nabla^2\mathbf{E}^T(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}^T(\mathbf{x},t) &= \frac{4\pi}{c^2}\frac{\partial}{\partial t}\sum_{\sigma,n} e_\sigma \dot{x}_j^\sigma(t)\, X_{nj}^\sigma\, \mathbf{X}_n(\mathbf{x})
\end{aligned}\right\} \quad (69)$$

IV. BOLTZMANN AND MAXWELL EQUATIONS FOR THE PLASMA

The present development of a description of a plasma has, to this point, proceeded deductively from an axiomatic base. However now—as was the case at a comparable stage in the formulation of a description of the neutral gas—deduction gives way to intuition. The procedure is to invoke (more or less arbitrarily) assumptions which are somewhat plausible, somewhat interpretable, but above all sufficient to reduce the exact but contentless relation between diverse functionals obtained above to a Boltzmann-type equation for the one-particle distribution function. The necessity for these assumptions is in no sense implied and the present argument is not to be interpreted as a best, or even an agreed upon, application of the assumptions. The present argument is advanced solely for the purpose of attempting to provide some insight into the nature of the problem.

Rewrite the ensemble averages involving field quantities as follows:

$$\left.\begin{aligned}
&(\overline{\rho},\ g_1^A\ E_j^T) = (\overline{\rho},\ g_1^A)\,(\overline{\rho},\ E_j^T) + [(\overline{\rho},\ g_1^A\,E_j^T) - (\overline{\rho},\ g_1^A)\,(\overline{\rho},\ E_j^T)] \\
&\text{and} \\
&\left(\overline{\rho},\ g_1^A\ \{\dot{\boldsymbol{x}} \times \boldsymbol{H}\}_j\right) = [\dot{\boldsymbol{x}} \times (\overline{\rho},\ g_1^A\ \boldsymbol{H})]_j \\
&\qquad = [\dot{\boldsymbol{x}} \times (\overline{\rho},\ g_1^A)\,(\overline{\rho},\ \boldsymbol{H})]_j + [\dot{\boldsymbol{x}} \times (\overline{\rho},\ g_1^A\ \boldsymbol{H}) - (\overline{\rho},\ g_1^A)\,(\overline{\rho},\ \boldsymbol{H})]_j
\end{aligned}\right\} \quad (70)$$

Then, without any immediate attempt at justification, the assumption can be adopted that the bracketed "correlation terms" may be subsequently ignored. In the sense of this approximation, recalling that $f_1^A \equiv (\overline{\rho},\ g_1^A)$, the following is offered:

$$\frac{\partial f_1^A}{\partial t} + \dot{x}_j \frac{\partial f_1^A}{\partial x_j} + \frac{e_A}{m_A} (\bar{\rho}, E_j^T) \frac{\partial f_1^A}{\partial \dot{x}_j} + \frac{e_A}{m_A c} [\dot{\mathbf{x}} \times (\bar{\rho}, \mathbf{H})] \frac{\partial f_1^A}{\partial \dot{x}_j}$$

$$= \frac{e_A}{m_A} \int d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e_A}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2^A (\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j}$$

$$+ \frac{e_A}{m_A} \int d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e_B}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2^{AB} (\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j} \qquad (71)$$

Next the Coulomb integrals are broken up in the following way: :

$$\int d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2}{\partial \dot{x}_j} = \int_{|\mathbf{x}-\mathbf{x}'| < l} d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2}{\partial \dot{x}_j}$$

$$+ \int_{|\mathbf{x}-\mathbf{x}'| \geq l} d^3x' \, d^3\dot{x}' \left(\frac{\partial}{\partial x_j} \frac{e}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial f_2}{\partial \dot{x}_j} \qquad (72)$$

where the length l is presumed small compared to the mean interparticle spacing in the system of interest. In the second of these integrals make the replacement

$$f_2 (\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t) = f_1 (\mathbf{x}, \dot{\mathbf{x}}, t) \, f_1 (\mathbf{x}'\dot{\mathbf{x}}', t) + (f_2 - f_1 f_1) \qquad (73)$$

and again assume that the correlation term is ignorable. The "greater than" integrals may then be rewritten as

$$\int_{|\mathbf{x}-\mathbf{x}'|\geq l} d^3x'\, d^3\dot{x}' \left(\frac{\partial}{\partial x_j}\frac{e_A}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^A}{\partial \dot{x}_j} + \int_{|\mathbf{x}-\mathbf{x}'|\geq l} d^3x'\, d^3\dot{x}' \left(\frac{\partial}{\partial x_j}\frac{e_B}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^{AB}}{\partial \dot{x}_j}$$

$$\simeq \frac{\partial f_1^A}{\partial \dot{x}_j}\left\{\frac{\partial}{\partial x_j}\left[\int_{|\mathbf{x}-\mathbf{x}'|\geq l} d^3x'\, d^3\dot{x}' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(e_A f_1^A(\mathbf{x}', \dot{\mathbf{x}}', t) + e_B f_1^B(\mathbf{x}', \dot{\mathbf{x}}', t)\right)\right]\right\}$$

$$= -\frac{\partial f_1^A}{\partial \dot{x}_j}\cdot(\overline{\rho}, \tilde{E}_j^L) \tag{74}$$

where the symbol $(\overline{\rho}, \tilde{E}_j^L)$ stands for the longitudinal component of the electric field at the point $\mathbf{x}$ computed as an ensemble average at time t, which however does not count those particles that lie within a volume of radius l about $\mathbf{x}$. The relation now assumes the form

$$\frac{\partial f_1^A}{\partial t} + \dot{x}_j\frac{\partial f_1^A}{\partial x_j} + \frac{e_A}{m_A}(\overline{\rho}, E_j^T + \tilde{E}_j^L)\frac{\partial f_1^A}{\partial \dot{x}_j} + \frac{e_A}{m_A c}[\dot{\mathbf{x}} \times (\overline{\rho}, \mathbf{H})]_j \frac{\partial f_1^A}{\partial \dot{x}_j}$$

$$= \frac{e_A}{m_A}\int_{|\mathbf{x}-\mathbf{x}'|<l} d^3x'\, d^3\dot{x}' \left(\frac{\partial}{\partial x_j}\frac{e_A}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^A(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j}$$

$$+ \frac{e_A}{m_A}\int_{|\mathbf{x}-\mathbf{x}'|<l} d^3x'\, d^3\dot{x}' \left(\frac{\partial}{\partial x_j}\frac{e_B}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\partial f_2^{AB}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}', \dot{\mathbf{x}}', t)}{\partial \dot{x}_j} \tag{75}$$

It is to be noted that the relation as presently constituted is still reversible; i.e., the distribution functions whose arguments are $(\mathbf{x}, -\dot{\mathbf{x}}, -t)$ satisfy the above equation. Hence, the approximation underlying the replacement of averages of products by products of averages does not in itself lead to irreversibility.

The remainder of the argument proceeds in close analogy to that presented above for the "short-range-force case." It is first presumed that if l is chosen sufficiently small compared to the mean interparticle spacing then the probability of finding three or more particles in a volume of $\sim l^3$ is completely negligible. This being the case, appreciable contributions to the truncated Coulomb interaction integrals occur only when there are at most two particles to be considered. It is then further argued that the Coulomb interaction between two particles so closely situated leads to forces large compared to the resultant of all other forces acting upon these particles and, hence, they may be regarded as decoupled from their environment throughout an interaction time given by $\tau = l/v_r$, v_r being the relative speed of the interacting pair. If in the time-averaging coarse-graining procedure all assumptions and approximations pertinent thereto are carried through as suggested for the previous case, one obtains the equation

$$\frac{\partial f_1^A}{\partial t} + v_j \frac{\partial f_1^A}{\partial x_j} + \frac{e_A}{m_A} \mathcal{E}_j \frac{\partial f_1^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \mathcal{H})_j \frac{\partial f_1^A}{\partial v_j}$$

$$= \int_{\mathbf{v}_1, \Omega} [f_1^A(\mathbf{v}') f_1^A(\mathbf{v}_1') - f_1^A(\mathbf{v}) f_1^A(\mathbf{v}_1)] \, v_r \sigma_{AA} d\Omega \, d^3 v_1$$

$$+ \int_{\mathbf{v}_1, \Omega} [f_1^A(\mathbf{v}') f_1^B(\mathbf{v}_1') - f_1^A(\mathbf{v}) f_1^B(\mathbf{v}_1)] \, v_r \sigma_{AB} d\Omega \, d^3 v_1 \tag{76}$$

In this equation certain symbols have been redefined and others introduced according to the following relationships:

$$
\left.
\begin{aligned}
&\dot{\mathbf{x}} \to \mathbf{v} \\
&\frac{1}{\tau}\int_t^{t+\tau} f_1^A\,(\mathbf{x}, \dot{\mathbf{x}}, t')\,dt' \to f_1^A\,(\mathbf{x}, \mathbf{v}, t) \\
&\frac{1}{\tau}\int_t^{t+\tau} (\overline{\rho},\ E_j^T + \widetilde{E}_j^L)\ 'dt' \simeq (\overline{\rho},\ E_j^T + \widetilde{E}_j^L) \to \mathcal{E}_j \\
&\frac{1}{\tau}\int_t^{t+\tau} (\overline{\rho},\ \mathbf{H})\ 'dt' \simeq (\overline{\rho},\ \mathbf{H}) \to \mathcal{H} \\
&\frac{1}{\tau}\int_t^{t+\tau} f_2\,(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}, \dot{\mathbf{x}}', t')\,dt' \simeq f_1\,(\mathbf{x}, \mathbf{v}, t)\, f_1\,(\mathbf{x}, \mathbf{v}', t) \\
&\text{and} \\
&\sigma d\Omega = s ds\ d\phi, \qquad 0 \le s < l \\
&\qquad\ \ = 0, \qquad\qquad l \le s
\end{aligned}
\right\} \quad (77)
$$

It should be noted that the ensemble-averaged electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ satisfy the inhomogeneous Maxwell equations with sources given by ensemble-averaged charge and current densities provided that the source for the longitudinal component of $\mathcal{E}$ does not include particles within a distance l of the point of observation of the field. This implies, as indicated above, that the following is taken for the longitudinal component of $\mathcal{E}$:

$$\mathcal{E}^L = -\nabla \left\{ \int_{|\mathbf{x}-\mathbf{x}'| \geq l} d^3v\, d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \left[e_A f_1^A (\mathbf{x}', \mathbf{v}, t) - e_B f_1^B (\mathbf{x}', \mathbf{v}, t) \right] \right\} \tag{78}$$

However, in most situations in which fields play an important role the contribution to $\mathcal{E}^L$ from an element of volume of order l^3 will be negligible compared to the total contribution when computed in terms of the smoothed functions introduced above; hence, $\mathcal{E}^L$ is usually calculated without regard for the restriction $|\mathbf{x} - \mathbf{x}'| \geq l$ on the region of summation.

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Second Chapter

SOME GENERAL COMMENTS ON THE BOLTZMANN EQUATION AND THE MOMENT EQUATIONS

I. SOME STEADY-STATE SOLUTIONS

The equations arrived at in the first chapter shall hereafter be used as the basic description of systems of the plasma type, and all subsequent analysis shall be predicated upon this description. Thus, it is desirable at this point to restate these equations in detail and indicate some general characteristics, knowledge of which will subsequently prove useful. For this purpose the assumption of a completely ionized, two-component plasma is retained, although it is recognized that the generalization to the description of the partially ionized state of such a system could be accomplished (phenomenologically, at least) quite satisfactorily by incorporating additional collision terms to account for ionization, recombination, the scattering of charged particles by neutrals, the scattering of neutrals by neutrals and, finally, the equation for the distribution function for the neutrals:

$$\frac{\partial f^A}{\partial t} + v_j \frac{\partial f^A}{\partial x_j} + \frac{1}{m_A} F_j \frac{\partial f^A}{\partial v_j} + \frac{e_A}{m_A} E_j \frac{\partial f^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \boldsymbol{H})_j \frac{\partial f^A}{\partial v_j} = I_{AA} + I_{AB} \tag{1}$$

where $\boldsymbol{F}$ is any external, non-electromagnetic force acting upon the particles, $\boldsymbol{E} = \boldsymbol{E}^\circ + \mathcal{E}$, and $\boldsymbol{H} = \boldsymbol{H}^\circ + \mathcal{H}$, $\boldsymbol{E}^\circ$ and $\boldsymbol{H}^\circ$ being external electric and magnetic fields imposed upon the system,

$$\left.\begin{aligned} I_{AA} &\equiv \int_{\mathbf{v}_1, \Omega} \left[f^A(\mathbf{v}')\, f^A(\mathbf{v}_1') - f^A(\mathbf{v})\, f^A(\mathbf{v}_1) \right] v_r \sigma_{AA}\, d\Omega\, d^3 v_1 \\ I_{AB} &\equiv \int_{\mathbf{v}_1, \Omega} \left[f^A(\mathbf{v}')\, f^B(\mathbf{v}_1') - f^A(\mathbf{v})\, f^B(\mathbf{v}_1) \right] v_r \sigma_{AB}\, d\Omega\, d^3 v_1 \end{aligned}\right\} \tag{2}$$

with a completely analogous equation for f^B.

$$\left.\begin{aligned} \nabla \times \mathcal{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathcal{H} &= 0 \\ \nabla \cdot \mathcal{H} &= 0 \\ \nabla \times \mathcal{H} - \frac{1}{c} \frac{\partial}{\partial t} \mathcal{E} &= \frac{4\pi}{c} \left(\int_{\mathbf{v}} e_A \mathbf{v} f^A\, d^3 v + \int_{\mathbf{v}} e_B \mathbf{v} f^B\, d^3 v \right) \end{aligned}\right\} \tag{3}$$

$$\nabla \cdot \mathcal{E} = 4\pi \left(\int_{\mathbf{v}} e_A f^A \, d^3 v + \int_{\mathbf{v}} e_B f^B \, d^3 v \right) \qquad (3) \text{ Cont'd}$$

Alternative ways of exhibiting the collision integrals are:

$$I_{AA} = \int_{\mathbf{v}_1', \mathbf{v}', \mathbf{v}_1} [f^A(\mathbf{v}') f^A(\mathbf{v}_1') - f^A(\mathbf{v}) f^A(\mathbf{v}_1)] \mathcal{G}^n_{AA}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') \, d^3 v' \, d^3 v_1' \, d^3 v_1$$

$$I_{AB} = \int_{\mathbf{v}_1', \mathbf{v}', \mathbf{v}_1} [f^A(\mathbf{v}') f^B(\mathbf{v}_1') - f^A(\mathbf{v}) f^B(\mathbf{v}_1)] \mathcal{G}^n_{AB}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') \, d^3 v_1 \, d^3 v_1' \, d^3 v' \qquad (4)$$

where the $\mathcal{G}$'s have the property

$$\mathcal{G}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') = \mathcal{G}(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1)$$

It is to be noted that an obvious, sufficient condition for the vanishing of the collision integrals is that the various distribution functions in the integrands satisfy the functional equation

$$f' f_1' - f f_1 = 0 \qquad (5)$$

To find a solution to this equation note that it implies

$$\ln f_1' + \ln f' - \ln f_1 - \ln f = 0 \qquad (6)$$

Clearly this will be true if $\ln f$ and $\ln f_1$ are chosen to be linear combinations of scalars whose sums are conserved in a collision. This suggests $\ln f = \alpha + \boldsymbol{\beta} \cdot \mathbf{v} + \gamma m \mathbf{v}^2$, where α, $\boldsymbol{\beta}$, and γ are arbitrary functions of space and time. It then follows that

$$f = \exp(\alpha + \boldsymbol{\beta} \cdot \mathbf{v} + \gamma m \mathbf{v}^2)$$

By an appropriate redefinition of the arbitrary quantities this may be re-expressed as

$$f = n \left(\frac{m}{2\pi\theta}\right)^{\frac{3}{2}} \exp\left[-\frac{m(\mathbf{v} - \mathbf{w})^2}{2\theta}\right] \tag{7}$$

where the sign of the exponential (and of θ) and the normalization have been so chosen that

$$\int_{\mathbf{v}} f \, d^3v = n\,(\mathbf{x}, t)$$

It is evident that the quantities $n(\mathbf{x}, t)$, $\mathbf{w}(\mathbf{x}, t)$, and $\theta(\mathbf{x}, t)$ are immediately interpretable since

1. $\int f \, d^3v = n$, the particle concentration at $\mathbf{x}$ and time t;
2. $\frac{1}{n}\int \mathbf{v} f \, d^3v = \mathbf{w}$, the average velocity of the particles at $\mathbf{x}$ and time t;
3. $\frac{1}{n}\int \frac{1}{2} mv^2 f \, d^3v = \frac{3}{2}\,\theta$, the average kinetic energy of the particles at $\mathbf{x}$ and at time t.

Thus, solutions to the equations $I_{AA} = 0$, $I_{AB} = 0$, are:

$$f^A = n^A \left(\frac{m_A}{2\pi\theta}\right)^{\frac{3}{2}} \exp\left[-\frac{m_A(\mathbf{v} - \mathbf{w})^2}{2\theta}\right]$$

and

$$f^B = n^B \left(\frac{m_B}{2\pi\theta}\right)^{\frac{3}{2}} \exp\left[-\frac{m_B(\mathbf{v} - \mathbf{w})^2}{2\theta}\right] \tag{8}$$

where the two different components must be characterized by the same mean velocity and mean energy, but not necessarily the same concentrations. Observe that these solutions also satisfy

$$\frac{e_A}{m_A}\,\mathcal{E}_j\,\frac{\partial f^A}{\partial v_j} + \frac{e_A}{m_A c}\,(\mathbf{v}\times \boldsymbol{H})_j\,\frac{\partial f^A}{\partial v_j} = I_{AA} + I_{AB}$$

$$\frac{e_B}{m_B}\,\mathcal{E}_j\,\frac{\partial f^B}{\partial v_j} + \frac{e_B}{m_B c}\,(\mathbf{v}\times \boldsymbol{H})_j\,\frac{\partial f^B}{\partial v_j} = I_{BB} + I_{BA} \qquad (9)$$

provided $e_A n^A = e_B n^B$ and the mean velocity $\mathbf{w}$ is parallel to any external magnetic field that may be present. In particular, if thermal equilibrium for the plasma is defined to be the state in which $e_A n^A = e_B n^B$ and $\mathbf{w} = 0$, it then follows that the spatial distribution of the particles is uninfluenced by the presence of externally applied magnetic fields. This observation is perhaps the basis for the occasionally quoted assertion that magnetic confinement of a plasma in thermal equilibrium is impossible.

II. SOME SYMMETRIES OF $\mathcal{G}$

Some properties of the collision integrals which will prove useful at various stages in the subsequent discussion will be noted here. Let $\phi^A(\mathbf{v})$ be an arbitrary function of the velocity of particles of kind A; then consider some conventional manipulations on the scalar product,

$$(\phi^A, I_{AA}) = \int d^3v d^3v' d^3v_1 d^3v_1' \; \phi^A(\mathbf{v}) \; (f^{A\,\prime} f_1^{A\,\prime} - f^A f_1^A) \mathcal{G}_{AA}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1')$$

$$= \int dC \; \phi^A(\mathbf{v}) \; (f^{A\,\prime} f_1^{A\,\prime} - f^A f_1^A) \, \mathcal{G}_{AA} \tag{10}$$

where the "collision volume" in velocity space dC was introduced. Note that the transition "probability" $\mathcal{G}_{AA}$ is characterized by the symmetries

$$\mathcal{G}_{AA}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') = \mathcal{G}_{AA}(\mathbf{v}_1, \mathbf{v}; \mathbf{v}_1', \mathbf{v}')$$

$$= \mathcal{G}_{AA}(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1) = \mathcal{G}_{AA}(\mathbf{v}_1', \mathbf{v}'; \mathbf{v}_1, \mathbf{v}) \tag{11}$$

Thus, if the indicated variable transformations are performed in (ϕ^A, I_{AA}) and terms are collected,

$$(\phi^A, I_{AA}) = \frac{1}{4}\int dC \, (\phi^A + \phi_1^A - \phi^{A\,\prime} - \phi_1^{A\,\prime}) \, (f^{A\,\prime} f_1^{A\,\prime} - f^A f_1^A) \mathcal{G}_{AA} \tag{12}$$

A similar argument based upon the restricted symmetry of $\mathcal{G}_{AB}$, i.e.,

$$\mathcal{G}_{AB}(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') = \mathcal{G}_{AB}(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1) \tag{13}$$

leads to

$$(\phi^A, I_{AB}) = \frac{1}{2}\int dC \; (\phi^A - \phi^{A\,\prime}) \; (f^{A\,\prime} f^{B\,\prime} - f^A f^B) \; \mathcal{G}_{AB} \tag{14}$$

Similar relations also hold for (ϕ^B, I_{BB}) and (ϕ^B, I_{BA}).

For certain special choices of the function ϕ^A these averages are readily interpretable in familiar physical terms. In particular:

1. ϕ^A = constant. Then $(\phi^A, I_{AA}) = (\phi^A, I_{AB}) = 0$, a constant is conserved in collisions.

2. $\phi^A = m_A v_j$. Then $(m_A v_j, I_{AA}) = 0$, the average momentum transferred out of the distribution of particles of kind A by collisions between particles of kind A is zero, but $(m_A v_j, I_{AB}) = P_j^{AB} \neq 0$, the average momentum transferred from particles of kind A to particles of kind B by collisions between such particles does not vanish. Note, however, that $(m_B v_j, I_{BA}) = P_j^{BA} = - P_j^{AB}$.

3. $\phi^A = \frac{1}{2} m_A v^2$. Then (with similar interpretations)

$$(m_A v^2, I_{AA}) = 0$$

$$\left(\frac{1}{2} m_A v^2, I_{AB}\right) = \epsilon^{AB}$$

$$\left(\frac{1}{2} m_B v^2, I_{BA}\right) = \epsilon^{BA} = - \epsilon^{AB}$$

These relations will be used extensively in the development of the various moment equations, as well as in the investigation of Enskog's method of solving the Boltzmann equation by successive approximation.

III. A RELATIVISTIC GENERALIZATION OF THE BOLTZMANN EQUATION

As a brief digression before getting involved in the development of the moment equations, it is amusing to incorporate some slight restriction and redefinition into the phenomenological deduction of the Boltzmann equation presented earlier and obtain it in a form invariant in the sense of special relativity. The restriction is to be comprised in the ignoring of collisions, and the required redefinition is constituted in the consideration of the distribution function as a function of the reduced velocity variables (Refs. 1, 2, 3), $\mathbf{u} = \mathbf{v}/\beta$ and $\beta = [1 - (v/c)^2]^{1/2}$. Electromagnetic interactions only shall be considered, as these are the only common ones that fit smoothly into the scheme of special relativity.

A brief summary of notation follows:

1. $x_\mu \sim (x_j, ict)$ and $x_j \sim (x, y, z)$
2. $d\tau^2 \equiv -(dx_\mu)^2/c^2$ (definition of proper time)
3. $u_\mu \equiv \dfrac{dx_\mu}{d\tau} \sim (v_j/\beta, ic/\beta)$

   where $\beta = [1 - (v/c)^2]^{1/2} = [1 + (u/c)^2]^{-1/2}$ with the notation $u = (u_j^2)^{1/2}$
4. $K_\mu \equiv m \dfrac{du_\mu}{d\tau} \sim \left(\dfrac{1}{\beta} F_j, \dfrac{i}{c} u_j F_j\right)$

   where m is the rest mass and $\mathbf{F}$ is a "force"
5. $A_\mu \sim (A_j, i\phi)$

   where $\mathbf{A}$ and ϕ are the electromagnetic vector and scalar potentials respectively
6. $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x_\mu} - \dfrac{\partial A_\mu}{\partial x_\nu}$

The quantity x_μ^2 must be preserved under transformations between coordinate systems moving with constant velocity with respect to each other; Hence, consider four-dimensional orthogonal transformations such that

$$x'_\mu = a_{\mu\nu} x_\nu \tag{15}$$

where $aa^T = I$, the identity matrix. A typical (and sufficiently general) specific example of such a transformation is obtained by first orienting the two systems so that their z-axes shall remain colinear throughout their relative motion. One then obtains

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-\gamma^2)^{-\frac{1}{2}} & i\gamma(1-\gamma^2)^{-\frac{1}{2}} \\ 0 & 0 & -i\gamma(1-\gamma^2)^{-\frac{1}{2}} & (1-\gamma^2)^{-\frac{1}{2}} \end{pmatrix} \tag{16}$$

where $\gamma = V/c$, V being the magnitude of the relative velocity of the two coordinate systems. The various quantities listed above are to be regarded as tensors in the sense that they transform appropriately under transformations of the kind a (Lorentz transformations).

Define the distribution function according to $f(x_j, u_j, t)\ d^3x\ d^3u$, the expected number of particles to be found in the element of volume d^3x about $\mathbf{x}$ with reduced velocities in d^3u about $\mathbf{u}$ at time t. The variables of position, velocity, and time are to be measured by an observer in his own coordinate system, but the function f is to be so construed that the expected number defined above be the same in all reference frames; i.e., $f\ d^3x\ d^3u$ must be a scalar with respect to Lorentz transformations. It is now a tedious but straightforward matter to show that d^3x/β and $\beta\ d^3u$ are scalars; thus, $d^3x\ d^3u$ is a scalar, and finally f is a scalar.

The deduction of the Boltzmann equation for f proceeds conventionally. Define displacements along particle trajectories by

$$\delta x_j = v_j\,\delta t = \beta u_j\ \delta t$$

$$\delta u_j = \frac{\beta K_j}{m}\ \delta t$$

It is then required that

$$\begin{aligned} f(x_j, u_j, t)\ d^3x\ d^3u &= f(x_j + \delta x_j, u_j + \delta u_j, t + \delta t)\ d^3(x + \delta x)\ d^3(u + \delta u) \\ &= f(x_j + \delta x_j, u_j + \delta u_j, t + \delta t)\ J\left(\frac{x + \delta x}{x}, \frac{u + \delta u}{u}\right) d^3x\ d^3u \end{aligned} \tag{17}$$

Interactions between the particles in $d^3x\ d^3u$ are ignored during the time interval δt; conversely, no other restriction is put upon the origin of the force these particles experience. Noting that the Jacobian has the expansion,

$$J = 1 + \frac{1}{m} \frac{\partial \beta K_j}{\partial u_j} \delta t + \mathcal{O}(\delta t^2) \tag{18}$$

The limit of small δt is obtained by

$$\frac{\partial f}{\partial t} + \beta u_j \frac{\partial f}{\partial x_j} + \frac{\beta K_j}{m} \frac{\partial f}{\partial u_j} + \frac{f}{m} \frac{\partial \beta K_j}{\partial u_j} = 0 \tag{19}$$

This may be conveniently rewritten as

$$u_\mu \frac{\partial f}{\partial x_\mu} + \frac{1}{m\beta} \frac{\partial(\beta K_j f)}{\partial u_j} = 0 \tag{20}$$

Because of the invariance of f and of the operator $u_\mu(\partial/\partial x_\mu)$, the scalar nature of the first term in this expression is self-evident. Not so, however, the second term, though with sufficient patience it may be shown that it is indeed a scalar.

In the instance that the forces on the particles are electromagnetic in origin (as is the case in the plasma) it is convenient to exhibit (from Ref. 4):

$$K_j = \frac{e}{c} F_{j\mu}\, u_\mu \tag{21}$$

The second term may then be written compactly as

$$\frac{e}{m\beta c} F_{j\mu} \frac{\partial}{\partial u_j} \beta u_\mu f = \frac{e}{mc} F_{j\mu} u_\mu \frac{\partial f}{\partial u_j} \tag{22}$$

and the Boltzmann equation as

$$u_{\mu} \frac{\partial f}{\partial x_{\mu}} + \frac{e}{mc} F_{j\mu} u_{\mu} \frac{\partial f}{\partial u_{j}} = 0 \tag{23}$$

Needed to complete the description are the inhomogeneous Maxwell equations, which in the present notation are

$$\left.\begin{aligned} &\frac{\partial F_{\mu\nu}}{\partial x_{\nu}} = \frac{4\pi}{c} e \int_{u} u_{\mu} f \beta \, d^{3}u \\ &\text{and} \\ &\frac{\partial A_{\mu}}{\partial x_{\mu}} = 0 \end{aligned}\right\} \tag{24}$$

This particular formulation of the plasma problem—characterized by Lorentz invariance, self-consistent fields, and the ignoring of collisions—will be given further consideration later in the investigation of the dispersion relation for plasma oscillations.

IV. THE MOMENT EQUATIONS

The next matter of interest is the task of deducing the moment equations. The objective here is twofold: (1) to provide illuminating contact between the microscopic description of matter as developed in the preceding Sections and the macroscopic descriptions commonly employed in disciplines such as hydrodynamics; and (2) to obtain formulations of the plasma problem in which the physical logic of the system is clearly discernible, but which are at the same time amenable to analysis in the context of boundary-value problems. This procedure constitutes the third step in a sequence beginning with a description of matter in the aggregate which is analytically useless but nevertheless incorporates the physical logic in self-evident terms (the Liouville equation); continuing to a less precise description of the physical system which serves as a more useful analytical tool, though it is still too complicated for the investigation of finite systems in general (the Boltzmann equation); and leads to a strict configuration-space description which may hopefully be employed in the investigation of boundary-value situations but which provides a physical tool still more dulled by intuition (the equations of magnetohydrodynamics in this case). Each stage in this process is accomplished by a more or less judicious relinquishment of information in accordance with the observation that there are only a limited number of observables for macroscopic systems.

The information to be dropped at this step is detail about the velocity distribution of the particles in the system. This detail is obscured in the process of deducing from the Boltzmann equation the "equations" describing a limited set of averages over the distribution function in μ-space.

A generic Boltzmann equation for a two-component system is given in the following notation:

$$\frac{\partial f^A}{\partial t} + v_j \frac{\partial f^A}{\partial x_j} + a_j^A \frac{\partial f^A}{\partial v_j} = I_{AA} + I_{AB} \tag{25}$$

where $\mathbf{a}^A$ is to be interpreted as the acceleration (due to all possible force fields whether internally or externally generated) experienced by the A-type particles at the point $(\mathbf{x}, \mathbf{v}, t)$. Let ψ be an arbitrary function of $(\mathbf{x}, \mathbf{v}, t)$, and introduce the notation

$$(\psi, f^A) = (f^A, \psi) = \int_{\mathbf{v}} \psi \, f^A d^3 v \tag{26}$$

Then the equation describing (ψ, f^A) is

$$\frac{\partial(\psi, f^A)}{\partial t} = \left(\frac{\partial \psi}{\partial t}, f^A\right) + \left(\psi, \frac{\partial f^A}{\partial t}\right)$$

$$= \left(\frac{\partial \psi}{\partial t}, f^A\right) - \left(\psi, v_j \frac{\partial f^A}{\partial x_j}\right) - \left(\psi, a_j^A \frac{\partial f^A}{\partial v_j}\right) + (\psi, I_{AA}) + (\psi, I_{AB}) \tag{27}$$

The second and third terms of this expression may be conveniently rewritten:

$$\left(\psi, v_j \frac{\partial f^A}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(\psi v_j, f^A) - \left(v_j \frac{\partial \psi}{\partial x_j}, f^A\right)$$

and

$$\left(\psi, a_j^A \frac{\partial f^A}{\partial v_j}\right) = \left(\psi, \frac{\partial a_j^A f^A}{\partial v_j}\right) = -\left(a_j^A \frac{\partial \psi}{\partial v_j}, f^A\right) \tag{28}$$

on the assumption that a^A has zero-velocity divergence which is valid for all the laws of force that will be considered. Putting these results back into the equation for the generic average, it is found that

$$\frac{\partial(\psi, f^A)}{\partial t} = \left(\frac{\partial \psi}{\partial t}, f^A\right) - \frac{\partial}{\partial x_j}(\psi v_j, f^A) + \left(v_j \frac{\partial \psi}{\partial x_j}, f^A\right) + \left(a_j^A \frac{\partial \psi}{\partial v_j}, f^A\right) + (\psi, I_{AA}) + (\psi, I_{AB}) \tag{29}$$

Five special choices for ψ are selected, which lead to quantities in terms of which the observables of the system are presumably calculable as follows:

1. $\psi = 1; \quad n^A(\mathbf{x}, t) \equiv (1, f^A)$

2, 3, and 4. $\psi = v_k; \quad w_k^A \equiv \dfrac{1}{n^A} (v_k, f^A)$

5. $\psi = \dfrac{1}{3} m_A \, (v_k - w_k^A)^2$

$$\equiv \frac{1}{3} m_A \, (u_k^A)^2$$

$$\theta^A \equiv \frac{1}{n^A}\left(\frac{1}{3} m_A \, \{u_k^A\}^2, f^A\right)$$

Observe that $\mathbf{w}^A$, $\boldsymbol{u}^A$, and θ^A are also functions of position and time. Inserting these choices for ψ into Equation (29) the following system of equations is obtained:

$$\frac{\partial n^A}{\partial t} + \frac{\partial n^A w_j^A}{\partial x_j} = 0$$

and after multiplying by m_A

$$\frac{\partial m_A n^A w_k^A}{\partial t} = - \frac{\partial}{\partial x_j} (m_A u_k^A u_j^A, f^A)$$

$$- \frac{\partial}{\partial x_j} m_A n^A w_k^A w_j^A + (m_A a_k^A, f^A) + P_k^{AB}$$

$$\frac{\partial n^A\theta^A}{\partial t} = -\frac{\partial}{\partial x_j} n^A\theta^A w_j^A - \frac{2}{3}\frac{\partial}{\partial x_j}\left(u_j^A \frac{1}{2} m_A \{u_k^A\}^2, f^A \right)$$

$$- \frac{2}{3}\frac{\partial w_k^A}{\partial x_j} (m_A u_k^A u_j^A, f^A)$$

$$+ \frac{1}{3} m_A \left(a_j^A \frac{\partial}{\partial v_j} \{u_k^A\}^2, f^A \right)$$

$$+ \frac{2}{3} \epsilon^{AB} - \frac{2}{3} w_j^A P_j^{AB} \tag{30}$$

These equations are, of course, simply continuity relations for (1) the particle density, (2) the momentum density, and (3) the energy density. These are not equations in a useful sense, for they contain quantities, $(m_A u_j^A u_k^A, f^A)$ and $(u_j^A \; 1/2\, m_A \{u_k^A\}^2, f^A)$, which are not determinable within the context of this set alone. Thus, either one must deduce the equations these quantities satisfy, or attempt one way or another to represent them as functionals of those quantities for which we already have equations; i.e., n^A, $\mathbf{w}^A$, and θ^A. The former alternative has been given some attention (Ref. 4); however, this leads to still more new quantities for which still more equations must be deduced, etc., *ad infinitum.* Such procedures do not seem suitable for purposes of analysis leading to qualitative understanding, though they may prove useful ultimately in numerical investigations. Hence, for the immediate purpose the second alternative will be adopted; symbols will be chosen to compact the notation, the significance of the quantities will be elucidated in intuitive terms, and finally their functional dependence upon n^A, $\mathbf{w}^A$, and θ^A will be approximately deduced. Thus, the following symbols are introduced:

$$\Psi_{jk}^A = \Psi_{kj}^A = (m_A u_j^A u_k^A, f^A)$$

$$h_j^A = \left(u_j^A \frac{1}{2} m_A \{u_k^A\}^2, f^A \right)$$

Then, noting that

$$\left(a_j \frac{\partial}{\partial v_j} \{u_k^A\}^2, f^A \right) = 2(a_j^A v_j, f^A) - 2w_j^A (a_j^A, f^A) = 0 \tag{31}$$

for accelerations independent of **v**, or of the form $\mathbf{v} \times \boldsymbol{H}$, the continuity statements are rewritten:

$$\frac{\partial n^A}{\partial t} + \frac{\partial n^A w_j^A}{\partial x_j} = 0$$

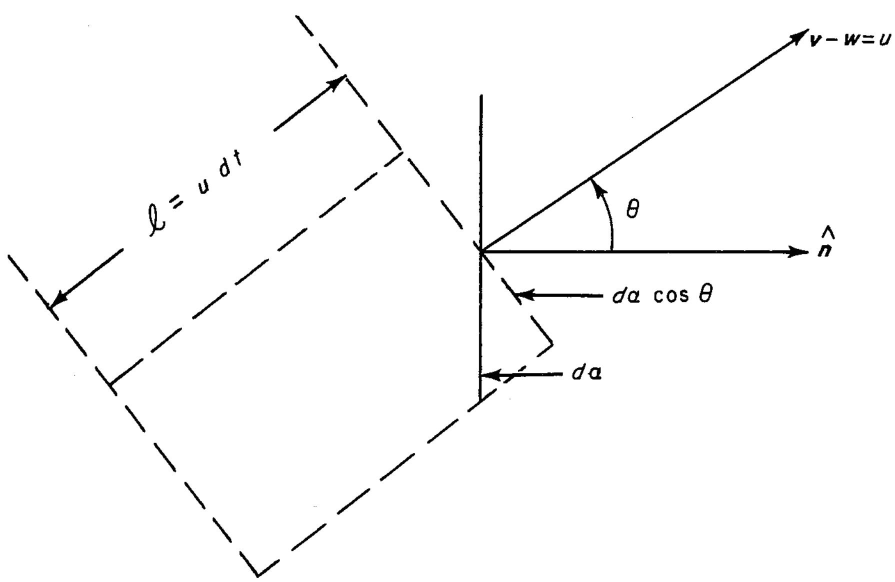
$$\frac{\partial m_A n^A w_k^A}{\partial t} = -\frac{\partial}{\partial x_j}\left[(m_A n^A w_k^A w_j^A, f^A) + \Psi_{jk}^A\right] + (F_k^A, f^A) + P_k^{AB}$$

$$\frac{\partial n^A \theta^A}{\partial t} = -\frac{\partial}{\partial x_j}\left(n^A \theta^A w_j^A + \frac{2}{3} h_j^A\right) - \frac{2}{3}\frac{\partial w_k^A}{\partial x_j}\Psi_{jk}^A + \frac{2}{3}\epsilon^{AB} - \frac{2}{3} w_j^A P_j^{AB} \tag{32}$$

where the following notation has been used:

$$F_k^A = m_A a_k^A$$

The physical significance of the quantities Ψ and h is readily delineated. Let $\phi(\mathbf{v})$ represent some quantity to be associated with particles of velocity **v**; e.g., a component of momentum, or kinetic energy. Then $\phi(u)$ represents the same quantity measured relative to the mean velocity of the gas. Consider now a hypothetical element of area of magnitude da and with orientation $\hat{n}$ to be placed in the gas moving with the mean velocity of the gas (see Sketch 2).



Sketch 2.

The expected number of particles crossing da in a time interval dt going in directions about $\mathbf{u}$ is equal to the number of particles in the box of volume $l da \cos\theta$ having velocities $\mathbf{v}$ in d^3u, where $\mathbf{v} = \mathbf{u} + \mathbf{w}$; i.e.,

$$f d^3 v_l \; da \cos\theta = f d^3 v \; \mathbf{u} \cdot \hat{\mathbf{n}} da \; dt \tag{33}$$

Then the net amount of ϕ that crosses da per second per unit area in direction $\hat{\mathbf{n}}$ is

$$\int \phi d^3 v \; f \mathbf{u} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot (\mathbf{u}\phi, f) \tag{34}$$

If $\hat{\mathbf{n}}$ is oriented along the jth axis of a coordinate system and ϕ chosen to be the kth component of the momentum of the particles measured relative to the mean motion of the gas, then:

$$\hat{\mathbf{n}} \cdot (\mathbf{u}\phi, f) = (m u_j u_k, f) = \Psi_{jk} \tag{35}$$

Thus, Ψ_{jk} is the net force per unit area in the kth direction exerted on a plane perpendicular to the jth axis; i.e., a component of a stress tensor.

Similarily, if ϕ is chosen to be the kinetic energy of the particles measured relative to the mean flow of the gas,

$$\hat{\mathbf{n}} \cdot (\mathbf{u}\phi, f) = (u_j \frac{1}{2} m \; \{u_k\}^2, f) = h_j \tag{36}$$

Since $(1/2)mu^2$ is the kinetic energy of the "random" motion of the particles, $\mathbf{h}$ may be interpreted to be that vector which measures the net flow of heat per unit area per unit time crossing a plane perpendicular to the jth axis in the positive sense and call it the heat-flow vector (or thermal current).

It is evident that, if f is a Maxwell-Boltzmann distribution of the form discussed earlier, then $\mathbf{h} = 0$, and $\Psi_{jk} = \delta_{jk} n\theta$. Further, if $\theta = kT$, then $\Psi_{jk} = \delta_{jk} nkT = \delta_{jk} P$ where P is the gas pressure. It is worth noting in passing that these approximations are commonly employed in order to put the above moment equations in determinate form. A large part of the effort involved shall be devoted to the task of systematically attempting to refine these approximations.

Since the quantities $n^{A,B}$, $w_j^{A,B}$, and $\theta^{A,B}$ are not themselves observables, further manipulations are needed at this point. Quantities which might be expected to be somewhat more accessible to direct measurement are:

1. $\rho = m_A n^A + m_B n^B$, the mass density
2. $N = n^A + n^B$, the number density
3. $Q = e_A n_A + e_B n_B$, the charge density
4. $\Lambda_j = m_A n^A w_j^A + m_B n^B w_j^B$, the momentum density
5. $J_j = e_A n^A w_j^A + e_B n^B w_j^B$, the electrical current density
6. $N\Theta = n^A \theta^A + n^B \theta^B$, the energy (of random motion) density

The equations that these quantities satisfy are obtained by straightforward manipulation from the moment equations presented above. The relevant equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \Lambda_k}{\partial x_k} = 0 \tag{37a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial J_k}{\partial x_k} = 0 \tag{37b}$$

$$\frac{\partial \Lambda_k}{\partial t} = - \frac{\partial}{\partial x_j}\left(\Psi_{jk} + m_A n^A w_j^A w_k^A + m_B n^B w_j^B w_k^B \right) + QE_k + \frac{1}{c}(\boldsymbol{J} \times \boldsymbol{H})_k \tag{37c}$$

$$\frac{\partial J_k}{\partial t} = -\frac{\partial}{\partial x_j}\left(\frac{e_A}{m_A}\Psi^A_{jk} + \frac{e_B}{m_B}\Psi^B_{jk} + e_A n^A w^A_j w^A_k + e_B n^B w^B_j w^B_k\right)$$

$$-\frac{e_A e_B}{m_A m_B}\left[\rho E_k + \frac{1}{c}(\boldsymbol{\Lambda} \times \boldsymbol{H})_k\right]$$

$$+\frac{e_B m_A + e_A m_B}{m_A m_B}\left[QE_k + \frac{1}{c}(\boldsymbol{J} \times \boldsymbol{H})_k\right] + \frac{e_A m_B - e_B m_A}{m_A m_B} P^{AB}_k \tag{37d}$$

$$\frac{\partial N\Theta}{\partial t} = -\frac{2}{3}\frac{\partial}{\partial x_j}\left(h_j + \frac{3}{2}n^A\theta^A w^A_j + \frac{3}{2}n^B\theta^B w^B_j\right) - \frac{2}{3}\frac{\partial w^A_k}{\partial x_j}\Psi^A_{jk}$$

$$-\frac{2}{3}\frac{\partial w^B_k}{\partial x_j}\Psi^B_{jk} - \frac{2}{3}(w^A_j - w^B_j)\, P^{AB}_j \tag{37e}$$

where the notation $\Psi_{jk} = \Psi^A_{jk} + \Psi^B_{jk}$ and $h_j = h^A_j + h^B_j$ is introduced. Again it must be borne in mind that the electromagnetic fields are here thought of as linear combinations of "external" and "internal" fields – the former constituting known parameters and the latter satisfying the inhomogeneous Maxwell equations whose sources are the charge and current densities (Q and $\boldsymbol{J}$) within the plasma.

The above set of equations, within the context of diverse simplifying approximations and assumptions, have found considerable employment in both the illumination of qualitative aspects of plasma behavior (Ref. 5), as well as in detailed investigations of the stability characteristics of certain plasma systems (Refs. 6 and 7).

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Third Chapter

PERTURBATIONS IN PLASMAS

I. SOME GENERAL SOLUTIONS

It was indicated earlier that in order to give meaning to the moment equations it is necessary either to adjoin more equations describing the behavior of the stress tensor and the thermal current or to deduce in some fashion a useful connection between these quantities and n, θ, and w. Clearly, the Boltzmann equation could be solved in a suitably general way (such a connection would be readily available) but in such an instance, the moment equations would not likely be of great interest. In fact, it was precisely because Boltzmann's equation provides such an analytically intractable tool for plasma studies that the moment equations were deduced. Nevertheless, it is possible – and illuminating – to proceed (albeit somewhat deviously) along these lines: to find solutions to the Boltzmann equation for physical situations sufficiently specialized that solutions are reasonably accessible but still sufficiently general that relevant information (which, it is hoped, could be extrapolated to more general physical situations) about the desired connections is forthcoming.

In particular it is felt that an examination of these questions in context of the "perturbation" solutions of the Boltzmann equation is relevant. However, these solutions have considerable intrinsic interest extending considerably beyond the issue of the transport parameters of the hydrodynamic description. They are especially interesting for the light they shed on the tangled and important questions about the response of plasmas to stimulation by electromagnetic radiation and some aspects of intrinsic stability. Thus, the program within this chapter shall consist of a careful formulation of the problem to be solved, reasonably detailed presentation of the technique of solution (Ref. 1), and examination of these solutions for their intrinsic interest. The employment of these results in an investigation of the nature of the transport parameters of the plasma shall be reserved for a later chapter.

For the purpose of formulating the problem of this chapter, the equations which purport to describe the system are exhibited in the following way:

$$\left.\begin{aligned}
&\frac{\partial f^A}{\partial t} + v_j \frac{\partial f^A}{\partial x_j} + \frac{e_A}{m_A} \mathbf{E}^e_j \frac{\partial f^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \mathbf{H}^e)_j \frac{\partial f^A}{\partial v_j} \\
&\qquad + \frac{e_A}{m_A} \mathcal{E}_j \frac{\partial f^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \mathcal{H})_j \frac{\partial f^A}{\partial v_j} = \sum_{A'} I_{AA'} \\
&\nabla \times \mathcal{H} - \frac{1}{c} \frac{\partial}{\partial t} \mathcal{E} = \frac{4\pi}{c} \sum_A e_A \int \mathbf{v} f^A d^3 v \\
&\nabla \times \mathcal{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathcal{H} = 0 \\
&\nabla \cdot \mathcal{H} = 0 \\
&\nabla \cdot \mathcal{E} = 4\pi \sum_A e_A \int f^A d^3 v
\end{aligned}\right\} \qquad (1)$$

The symbols $\mathbf{E}^e$ and $\mathbf{H}^e$ represent electric and magnetic fields completely prescribed by conditions external to the plasma. The presence of neutrals in the system under investigation may presumably be accounted for more or less effectively by the appropriate inclusion of collision terms $I_{AA'}$.

The first basic assumption of the present formulation is that all number densities may be decomposed linearly into two parts, one of which characterizes a small perturbation, i.e.,

$$f^A = f_0^A + f_1^A$$

where $f_1^A/f_0^A << 1$, so that terms $\mathcal{O}(f_1^A)^2$ appearing in the description of the system may be neglected. The second basic assumption adopted for present purposes is that the external electromagnetic fields have a similar decomposition, e.g.,

$$\boldsymbol{E}^e = \boldsymbol{E}_0^e + \boldsymbol{E}_1^e$$

where $|\boldsymbol{E}_1^e|\, f_1^A \sim (f_1^A)^2$, and hence, is also ignorable. Then, assuming that f_0^A is determined by

$$\frac{\partial f_0^A}{\partial t} + v_j \frac{\partial f_0^A}{\partial x_j} + \frac{e_A}{m_A} E_{0j}^e \frac{\partial f_0^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \boldsymbol{H}_0^e)_j \frac{\partial f_0^A}{\partial v_j}$$

$$+ \frac{e_A}{m_A} \mathcal{E}_{0j} \frac{\partial f_0^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \mathcal{H}_0)_j \frac{\partial f_0^A}{\partial v_j} = \sum_{A'} I_{AA'}^{00}, \tag{2}$$

it follows that f_1^A satisfies the equation:

$$\frac{\partial f_1^A}{\partial t} + v_j \frac{\partial f_1^A}{\partial x_j} + \frac{e_A}{m_A} E_{0j}^e \frac{\partial f_1^A}{\partial v_j} + \frac{e_A}{m_A} \mathcal{E}_{0j}^e \frac{\partial f_1^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \boldsymbol{H}_0^e)_j \frac{\partial f_1^A}{\partial v_j} + \frac{e_A}{m_A c} (\mathbf{v} \times \mathcal{H}_0)_j \frac{\partial f_1^A}{\partial v_j}$$

$$= - \frac{e_A}{m_A} \mathcal{E}_{1j} \frac{\partial f_0^A}{\partial v_j} - \frac{e_A}{m_A c} (\mathbf{v} \times \mathcal{H}_1)_j \frac{\partial f_0^A}{\partial v_j} - \frac{e_A}{m_A} E_{1j}^e \frac{\partial f_0^A}{\partial v_j} - \frac{e_A}{m_A c} (\mathbf{v} \times \boldsymbol{H}_1^e)_j \frac{\partial f_0^A}{\partial v_j}$$

$$+ \sum_{A'} (I_{AA'}^{01} + I_{AA'}^{10}) \tag{3}$$

Into these equations has been introduced the following notation:

$$I^{ij}_{AA'} = I_{AA'}\ (f^A_i f^{A'}_j)$$

$$= \int d^3v' d^3v_1' d^3v_1\ [f^A_i(\mathbf{v}')\, f^{A'}_j(\mathbf{v}_1') - f^A_i(\mathbf{v})\, f^{A'}_j(\mathbf{v}_1)]\ \mathcal{G}_{AA'}\ (\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') \quad (4a)$$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 \quad (4b)$$

where, for example,

$$\nabla \cdot \mathcal{E}_0 = 4\pi \sum_A e_A \int_{\mathbf{v}} f^A_0\, d^3v$$

and

$$\nabla \cdot \mathcal{E}_1 = 4\pi \sum_A e_A \int_{\mathbf{v}} f^A_1\, d^3v \quad (4c)$$

Thus, since $\mathcal{E}_1$ and $\mathcal{H}_1$ are linear functions of the f^A_1's, terms of order $|\mathcal{E}_1|\, f^A_1$ and $|\mathcal{H}_1|\, f^A_1$ have been ignored.

The equation for the perturbation f^A_1 is still analytically intractable, however, because of the presence of the terms $I^{ij}_{AA'}$. To obtain adequate simplification without losing track of the effects of collisions completely, approximations are used:

$$\sum_{A'} (I^{10}_{AA'} + I^{01}_{AA'}) \cong -\beta f^A_1 - \beta' f^{A'}_1 \cong -\beta f^A_1 \quad (5)$$

The first of these approximations is illuminated by considering

$$\sum_{A'} I^{10}_{AA'} = \sum_{A'} \int d^3v' d^3v_1' d^3v_1 \; \mathcal{G}_{AA'} \; f_1^A(\mathbf{v}') \; f_0^{A'}(\mathbf{v}_1')$$

$$- \sum_{A'} \int d^3v' d^3v_1' d^3v_1 \; \mathcal{G}_{AA'} \; f_1^A(\mathbf{v}) \; f_0^{A'}(\mathbf{v}_1)$$

$$= \sum_{A'} \int d^3v_1 d\Omega \, v_r \sigma_{AA'} \; f_1^A(\mathbf{v}') \; f_0^{A'}(\mathbf{v}_1')$$

$$- \sum_{A'} \int f_1^A(\mathbf{v}) \quad d^3v_1 d\Omega \, v_r \sigma_{AA'} \; f_0^{A'}(\mathbf{v}_1)$$

$$\cong - f_1^A(\mathbf{v}) \sum_{A'} \int d^3v_1 d\Omega \, v_r \sigma_{AA'} \; f_0^{A'}(\mathbf{v}) = - \beta f_1^A \tag{6}$$

i.e., the "scattering in" collisions from the perturbed distribution to the perturbed distribution are ignored compared to the "scattering out" collisions from the perturbed distribution to "anywhere". Perhaps a more satisfactory statement of the logic of the replacement

$$\sum_{A'} I^{10}_{AA'} \to - \beta f_1^A \tag{7}$$

is: rather than completely ignoring collisions (the principal mechanism of randomization), the unwieldy collision integrals are replaced by a term linear in f_1^A which has the effect of causing f_1^A to decrease with time (on the assumption that f_1^A is "ordered" with respect to f_0^A). The second of the above approximations merely asserts that collisional coupling between the perturbed distributions for different kinds of particles shall be ignored.

The analysis can proceed (at least formally) from this point without introducing additional restrictive assumptions. However, the retention of the quantities $\mathbf{E}_0^e$ and $\mathcal{E}_0$ in the equations for f_1^A serves mainly to complicate the analysis so much that the effect is more to obscure than to illuminate. Similar observations obtain with respect to the presence of the factor $\mathbf{v} \times \mathcal{H}_0$ in the equations for f_1^A. Thus, for present purposes the following restrictions are added: $\mathbf{E}_0^e = 0$, and the set of unperturbed densities f_0^A is such that

$$Q_0 = \sum_A e_A \int f_0^A d^3v = 0$$

$$\mathbf{J}_0 = \sum_A e_A \int \mathbf{v} f_0^A d^3v = 0 \tag{8}$$

so that, consistently, $\mathcal{E}_0 = \mathcal{H}_0 = 0$. Finally, though the above assumptions do not require it, the condition that the f_0^A's be space- and time-independent shall be added. Then, introducing the notation (not to be confused with a different employment of the same symbols encountered earlier),

$$\mathbf{E} = \mathbf{E}_1^e + \mathcal{E}_1 \qquad\qquad \mathbf{H} = \mathbf{H}_1^e + \mathcal{H}_1$$

and $\mathbf{H}_0^e = H_0\hat{\mathbf{h}}$, $\Omega^A = e_A H_0/m_A c$, where $\hat{\mathbf{h}} \cdot \hat{\mathbf{h}} = 1$, the equation for f_1^A may be written as

$$\frac{\partial f_1^A}{\partial t} + v_j \frac{\partial f_1^A}{\partial x_j} + \Omega^A (\mathbf{v} \times \hat{\mathbf{h}})_j \frac{\partial f_1^A}{\partial v_j} + \beta f_1^A = -\frac{e_A}{m_A} E_j \frac{\partial f_0^A}{\partial v_j} - \frac{e_A}{m_A c} (\mathbf{v} \times \mathbf{H})_j \frac{\partial f_0^A}{\partial v_j} \tag{9}$$

To these equations for the number densities must, of course, be adjoined Maxwell's equations for $\mathcal{E}_1$ and $\mathcal{H}_1$, or equivalently, $\mathbf{E}$ and $\mathbf{H}$.

First, consider the equations for the perturbed number densities; then introduce the operators,

$$\mathcal{F} \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int_x d^3x \, e^{-i\boldsymbol{K}\cdot x}$$

and

$$\mathcal{L} \equiv \int_0^\infty dt e^{-st} \tag{10}$$

and the notation,

$$\mathcal{F}\mathcal{L} f_1^A = g^A(\boldsymbol{K}, \mathbf{v}, s) \qquad \mathcal{F} f_1^A(\mathbf{x}, \mathbf{v}, 0) = g_0^A(\boldsymbol{K}, \mathbf{v})$$

$$\mathcal{F}\mathcal{L}\boldsymbol{E} = \boldsymbol{D}(\boldsymbol{K}, s) \qquad \mathcal{F}\boldsymbol{E}(\mathbf{x}, 0) = \boldsymbol{D}_0(\boldsymbol{K})$$

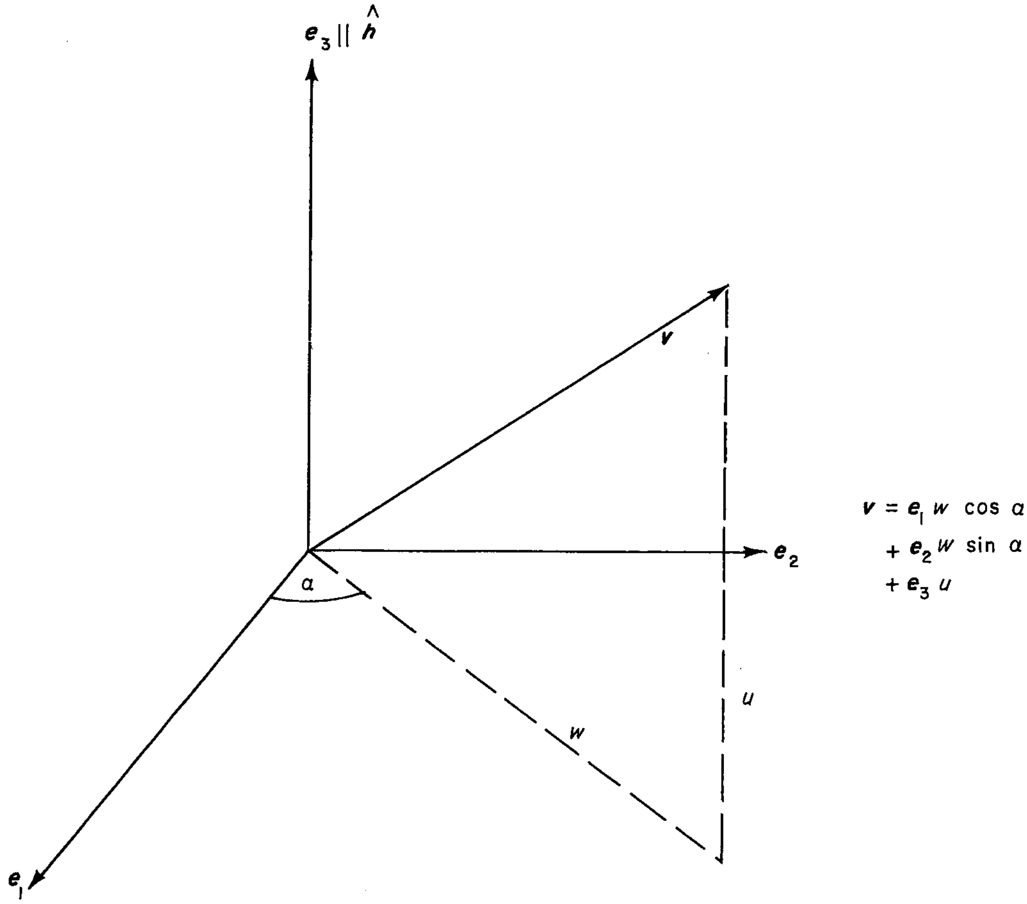
$$\mathcal{F}\mathcal{L}\boldsymbol{H} = \boldsymbol{B}(\boldsymbol{K}, s) \qquad \mathcal{F}\boldsymbol{H}(\mathbf{x}, 0) = \boldsymbol{B}_0(\boldsymbol{K})$$

$$d_j^A = \frac{e_A}{m_A} \frac{\partial f_0^A}{\partial v_j} \tag{11}$$

Then, taking the Laplace and Fourier transforms of the equation for f_1^A, the result is

$$\Omega^A (\mathbf{v} \times \boldsymbol{h})_j \frac{\partial g^A}{\partial v_j} + (s + \beta + i\boldsymbol{K} \cdot \mathbf{v})\, g^A = g_0^A - d_j^A \left[D_j + \frac{1}{c} (\mathbf{v} \times \boldsymbol{B})_j \right] \tag{12}$$

In this equation, $\boldsymbol{D}$ and $\boldsymbol{B}$ are independent of $\mathbf{v}$, whereas g_0^A *and* $\mathbf{d}^A$ are known functions of $\mathbf{v}$; thus, the velocity dependence of g^A is explicitly deducible. To accomplish this, cylindrical coordinates in velocity space, with the axis of the cylinder along $\hat{\boldsymbol{h}}$, are introduced (see Sketch 3).


$e_3 \parallel \hat{h}$
v
e_2
α
w
u
e_1
$v = e_1 w \cos \alpha$
$+ e_2 w \sin \alpha$
$+ e_3 u$

Sketch 3.

In such a coordinate system, $(\mathbf{v} \times \hat{h})_j\, \partial/\partial v_j$ is just the negative of the α-directional derivative; i.e,

$$(\mathbf{v} \times \hat{h})_j \frac{\partial g^A}{\partial v_j} = - \frac{\partial g^A}{\partial \alpha} \tag{13}$$

This equation then becomes

$$\Omega^A \frac{\partial g^A}{\partial \alpha} - (s + \beta + i\boldsymbol{K} \cdot \mathbf{v})\, g^A = d_j^A \left[D_j + \frac{1}{c} (\mathbf{v} \times \boldsymbol{H})_j \right] - g_0^A \tag{14}$$

Define the integrating factor:

$$L_A(\alpha) = \exp \left(\int_c^{\alpha} \frac{s + \beta + i\boldsymbol{K} \cdot \mathbf{v}''}{\Omega^A}\, d\alpha'' \right) \tag{15}$$

Then if $g^A = L_A \psi^A$,

$$\frac{\partial \psi^A}{\partial \alpha} = \frac{L_A^{-1}}{\Omega^A} \left\{ d_j^A \left[D_j + \frac{1}{c} (\mathbf{v} \times \boldsymbol{\beta})_j \right] - g_0^A \right\} \tag{16}$$

Integrating and multiplying the result by $L_A(\alpha)$, a general solution is obtained:

$$g^A = \int_{\chi}^{\alpha} d\alpha'\; I_A\,(\alpha, \alpha')\; \frac{1}{\Omega^A} \left[d_j^{A\,\prime} D_j + \frac{1}{c} (\mathbf{d}^{A\,\prime} \times \mathbf{v}')_j\, B_j - g_0^{A\,\prime} \right] \tag{17}$$

where

$$I_A(\alpha, \alpha') = \exp\left(\int_{\alpha'}^{\alpha} \frac{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}''}{\Omega^A} \, d\alpha''\right)$$

and χ is an arbitrary function of $\boldsymbol{\kappa}$, u, w, and s. The arbitrariness in this solution is largely eliminated by the requirement that g^A be a single-valued function of α, i.e., $g^A(\alpha + 2\pi) = g^A(\alpha)$. Since $\mathbf{v} = \mathbf{e}_1(\cos\alpha)\, w + \mathbf{e}_2(\sin\alpha)\, w + \mathbf{e}_3 u$, it follows that $\mathbf{v}(\alpha + 2\pi) = \mathbf{v}(\alpha)$; hence, any function of $\mathbf{v}$ is invariant under the displacement $\alpha \to \alpha + 2\pi$; hence, the single-valuedness requirement for g^A implies

$$\int_{\chi + 2\pi}^{\alpha} d\alpha' \ldots = \int_{\chi}^{\alpha} d\alpha' \ldots \tag{18}$$

But this is true only for $\chi = \pm\infty$. Since, however,

$$I(\alpha, \alpha') = \exp\left[\frac{s + \beta}{\Omega^A}(\alpha - \alpha') + i\int_{\alpha'}^{\alpha} \frac{\boldsymbol{\kappa} \cdot \mathbf{v}''}{\Omega^A} \, d\alpha''\right]$$

it is seen that the choice $\chi = -\infty$ would imply an exponentially divergent integrand and hence, in general, a meaningless g^A. Thus, the only choice is $\chi = +\infty$. The solution for g^A is now exhibited as

$$g^A = -a^A + b_j^A D_j + r_j^A B_j \tag{19}$$

where

$$\left.\begin{aligned}
a^A &= \int_{\infty}^{\alpha} d\alpha' I(\alpha, \alpha') \left(\frac{g_0^{A\,\prime}}{\Omega^A} \right) \\
b_j^A &= \int_{\infty}^{\alpha} d\alpha' I(\alpha, \alpha') \left(\frac{d_j^{A\,\prime}}{\Omega^A} \right) \\
r_j^A &= \int_{\infty}^{\alpha} d\alpha' I(\alpha, \alpha') \left(\frac{(\mathbf{d}^{A\,\prime} \times \mathbf{v}')_j}{c\Omega^A} \right)
\end{aligned}\right\} \tag{20}$$

The transformations specified above reduce Maxwell's equations to a set of simple algebraic relations,

$$\left.\begin{aligned}
i\boldsymbol{\kappa} \times \boldsymbol{B} - \frac{s}{c} \boldsymbol{D} &= -\frac{1}{c} \boldsymbol{D}_0 + \frac{4\pi}{c} (\mathcal{G}\mathcal{L}\boldsymbol{J}) \\
i\boldsymbol{\kappa} \times \boldsymbol{D} + \frac{s}{c} \boldsymbol{B} &= \frac{1}{c} \boldsymbol{B}_0 \\
\boldsymbol{\kappa} \cdot \boldsymbol{B} &= 0 \\
i\boldsymbol{\kappa} \cdot \boldsymbol{D} &= 4\pi(\mathcal{G}\mathcal{L}Q)
\end{aligned}\right\} \tag{21}$$

Observing that

$$\mathcal{G}\mathcal{L}J_k = -\sum_A e_A \int v_k a^A d^3v + D_j \sum_A e_A \int v_k b_j^A d^3v + B_j \sum_A e_A \int v_k r_j^A d^3v \quad (22a)$$

and

$$\mathcal{G}\mathcal{L}Q = -\sum_A e_A \int a^A d^3v + D_j \sum_A e_A \int b_j^A d^3v + B_j \sum e_A \int r_j^A d^3v \quad (22b)$$

note that the above are linear, inhomogeneous equations for the quantities D_j and B_j. The solutions may be compactly represented in the following notation:

$$D_j = M_{jk}^{-1} X_k$$

and

$$B_j = \frac{1}{s} B_{0j} - \frac{ic}{s} \epsilon_{jlm} \kappa_l M_{mn}^{-1} X_n \quad (23)$$

where ϵ_{jlm} is the completely antisymmetric tensor density with values ± 1 depending on whether the order of the indices is a cyclic or anti-cyclic permutation of (1, 2, 3), and where

$$X_j = sD_{0j} + ic(\boldsymbol{\kappa} \times \mathbf{B}_0)_j + \gamma_j + \frac{1}{s} \rho_{jk} B_{0k}$$

$$M_{jk} = (s^2 + c^2\kappa^2)\delta_{jk} - \beta_{jk} + \frac{ic}{s} \eta_{jk}$$

$$\left.\begin{aligned} \gamma_j &= 4\pi \; s\left(\sum_A e_A \int v_j a^A d^3v + ic^2 k_j \sum_A e_A \int a^A d^3v\right) \\ \beta_{jk} &= -4\pi \; s\left(\sum_A e_A \int v_j b_k^A d^3v + ic^2 k_j \sum_A e_A \int b_k^A d^3v\right) \\ \rho_{jk} &= -4\pi \; s\left(\sum_A e_A \int v_j r_k^A d^3v + ic^2 k_j \sum_A e_A \int r_k^A d^3v\right) \end{aligned}\right\} \quad (24)$$

$$\eta_{jk} = -4\pi \left(s \sum e_A \int v_j \, (\boldsymbol{r}^A \times \boldsymbol{K})_k \, d^3v + ic^2\kappa_j \sum_A e_A \int (\boldsymbol{r}^A \times \boldsymbol{K})_k \, d^3v \right)$$

Note that the matrix M and the vector $\boldsymbol{X}$ are functions solely of initial conditions and system parameters. Of course, knowing $\boldsymbol{D}$ and $\boldsymbol{B}$, g^A may be computed as follows:

$$g^A = -a^A + \frac{1}{s} r_j^A B_{0j} + b_j^A M_{jk}^{-1} X_k - \frac{ic}{s} r_j^A \, \epsilon_{jlm} \kappa_l M_{mn}^{-1} X_n \tag{25}$$

These are essentially the solutions to a similar problem published by Bernstein (Ref. 1). Bernstein, however, assumed that the unperturbed distribution was isotropic in velocity space. This has the consequence that

$$\frac{\partial f_0^A}{\partial v_j} \propto v_j$$

and, hence,

$$\boldsymbol{r}^A = 0$$

so that

$$\rho = \eta = 0 \tag{26}$$

It is desirable to retain the added generality corresponding to anisotropic f_0^A's, as will be seen presently.

Of course, the solutions indicated above are still purely formal, as it is still necessary to carry through the Laplace and Fourier inversions. However, it is frequently possible to obtain relevant physical information about specific systems without inverting completely. Furthermore, many problems are generated in simpler contexts than the one in which the above solutions are set, and if this additional simplification is introduced at this stage of the analysis, the requisite inversion is often greatly facilitated. Thus, these reasonably general results shall be left in their present form and applied directly to obtain specific information about certain subclasses of problems.

Before proceeding to specific cases, it is desirable to present a small calculation, the result of which will be of recurrent use. Because some interesting situations may be characterized by the external magnetic field H_0, either zero or weak, it is convenient to exhibit integrals of the form

$$\mathcal{J} = \int_{\infty}^{\alpha} \frac{d\alpha' I(\alpha, \alpha')\, K(\alpha')}{\Omega^A} \tag{27}$$

as a power series in Ω^A. The function $K(\alpha')$ is to be interpreted as any one of the functions appearing in such integrals. Observing that

$$I(\alpha, \alpha') = -\frac{\Omega^A}{s+\beta+i\boldsymbol{\kappa}\cdot\mathbf{v}'}\frac{\partial I}{\partial \alpha'} \tag{28}$$

and performing successive integrations by parts, the following is obtained:

$$\mathcal{J} = -\frac{K}{s+\beta+i\boldsymbol{\kappa}\cdot\mathbf{v}} + \frac{\Omega^A}{s+\beta+i\boldsymbol{\kappa}\cdot\mathbf{v}}\frac{\partial}{\partial\alpha}\frac{K}{s+\beta+i\boldsymbol{\kappa}\cdot\mathbf{v}} + (\Omega^{A2}) \tag{29}$$

II. "LONGITUDINAL" ELECTRON OSCILLATIONS

The first specific problem to be investigated in some detail is that of longitudinal electron oscillations. The physical and mathematical questions generated by this problem have received the attention of many investigators. For an illuminating survey of most of these investigations consult the review paper by Van Kampen (Ref. 2). Since it is the purpose here mainly to set this particular problem in the context of the "general" solutions, the many other viewpoints and mathematical techniques that have been variously adopted will not be dwelt upon. However, after a cursory examination of the physical implications of the present solutions the problem will be considered briefly from the viewpoint of the hydrodynamical equations and the relativistic Boltzmann equation. Two prime assumptions employed in the specification of this problem are:

1. f_0^A is isotropic in velocity space; hence, $\partial f_0^A/\partial v_j \propto v_j$, and $\mathbf{r}^A = 0$. Thus, it follows that $\rho = \eta = 0$.

2. $\boldsymbol{H}_0 = 0$; hence, $\Omega^A = 0$. It then follows that

$$a^A \to -\frac{g_0^A}{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}}$$

$$b_j^A \to -\frac{\left(\dfrac{e_A}{m_A}\right)}{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}} \frac{\partial f_0^A}{\partial v_j}$$

Because of these assumptions,

$$g^A = \frac{g_0^A}{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}} + b_j^A M_{jk}^{-1} X_k, \quad D_j = M_{jk}^{-1} X_k \tag{30a}$$

where now

$$M_{jk} = (s^2 + c^2 \kappa^2)\, \delta_{jk} - \beta_{jk}$$

$$X_j = sD_{0j} + ic\,(\boldsymbol{\kappa} \times \boldsymbol{B}_0)_j + \gamma_j \tag{30b}$$

The tensor β may now be exhibited as

$$\beta_{jk} = 4\pi \left[s \sum_A e_A \int v_j \frac{\left(\frac{e_A}{m_A}\right)}{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}} \frac{v_k}{v} \frac{\partial f_0^A}{\partial v} d^3v \right.$$

$$\left. + ic^2 \sum_Z e_A \int k_j \frac{\left(\frac{e_A}{m_A}\right)}{s + \beta + i\boldsymbol{\kappa} \cdot \mathbf{v}} \frac{v_k}{v} \frac{\partial f_0^A}{\partial v} d^3v \right] \tag{31}$$

To proceed further with the analysis, orient the coordinate system so that its 3-axis is in the direction of the vector $\boldsymbol{\kappa}$. Then, because of the symmetric limits of integration in velocity space, β is found to be diagonal; i.e.,

$$\beta_{jk} = 4\pi \left(s\delta_{jk} \sum_A \frac{e_A^2}{m_A} \int \frac{v_j^2}{s + \beta + i\kappa v_3} \frac{1}{v} \frac{\partial f_0^A}{\partial v} d^3v \right.$$

$$\left. + ic^2 \delta_{j3}\delta_{k3} \sum_A \frac{e_A^2}{m_A} \int \frac{kv_3}{s + \beta + i\kappa v_3} \frac{1}{v} \frac{\partial f_0^A}{\partial v} d^3v \right) \tag{32}$$

Since β is diagonal, M is diagonal and hence M^{-1} is diagonal. In fact, $M_{jj}^{-1} = 1/M_{jj}$.

A further condition upon the system is:

3. $B_0 = 0$; i.e., there shall be no perturbing external magnetic field at $t = 0$ and the particle perturbations at $t = 0$ shall not constitute a current. Then,

$$X_j = sD_{0j} + \gamma_j \tag{33}$$

Finally, if it is required that the perturbation be accomplished by a mechanical displacement of the particles such that $D_{01} = D_{02} = 0$ (recalling that the 3-axis of the system is along $\boldsymbol{\kappa}$), the following is obtained;

$$i\kappa D_{03} = 4\pi \sum_A e_A \int_v g_0^A d^3v \tag{34a}$$

and, hence,

$$X_j = \delta_{j3}\left(\frac{4\pi s}{i\kappa}\sum_A e_A \int g_0^A d^3v - 4\pi s \sum_A e_A \int \frac{g_0^A v_3 d^3v}{s+\beta+i\kappa v_3}\right.$$

$$\left. - 4\pi i c^2 \kappa \sum_A e_A \int \frac{g_0^A d^3v}{s+\beta+i\kappa v_3}\right) \tag{34b}$$

or, after rearrangement,

$$X_j = \delta_{j3}\frac{4\pi}{i\kappa}(s^2 + c^2\kappa^2 + s\beta)\sum_A e_A \int \frac{d^3v g_0^A}{s+\beta+i\kappa v_3} \tag{34c}$$

Now integrate over the superfluous variables v_1 and v_2, let v stand for v_3 for the remainder of this analysis, and define

$$\bar{g}^A = \int dv_1 dv_2 g^A, \quad \bar{g}_0^A = \int dv_1 dv_2 g_0^A, \quad \text{and } \bar{f}_0^A = \int dv_1 dv_2 f_0^A \tag{35}$$

The results to this point may be summarized as follows:

$$D_3 = \frac{\dfrac{4\pi i}{\kappa} \displaystyle\sum_A e_A \int \frac{dv \bar{g}_0^A}{s + \beta + i\kappa v}}{\dfrac{s^2 + c^2\kappa^2}{s^2 + c^2\kappa^2 + s\beta} - \dfrac{4\pi i}{\kappa} \displaystyle\sum_A \frac{e_A^2}{m_A} \int \frac{dv}{s + \beta + i\kappa v} \frac{\partial \bar{f}_0^A}{\partial v}}$$

and

$$\bar{g}^A = \frac{\bar{g}_0^A}{s + \beta + i\kappa v} + \left(\frac{4\pi i}{\kappa}\right) \frac{\dfrac{e_A/m_A}{s + \beta + i\kappa v} \dfrac{\partial \bar{f}_0^A}{\partial v} \displaystyle\sum_{A'} e_{A'} \int \frac{dv g_0^{A'}}{s + \beta + i\kappa v}}{\dfrac{s^2 + c^2\kappa^2}{s^2 + c^2\kappa^2 + s\beta} - \left(\dfrac{4\pi i}{\kappa}\right) \displaystyle\sum_A \left(\frac{e_A^2}{m_A}\right) \int \frac{dv}{s + \beta + i\kappa v} \frac{\partial \bar{f}_0{}^A}{\partial v}} \tag{36}$$

One of the most readily accessible bits of information contained in these results is a dispersion relation for these oscillations in the long wavelength limit. It is expected that the characteristic frequencies are sufficiently high that only the electrons participate significantly in the motion. Then for many choices of unperturbed states $\bar{f}_0$ and initial perturbations $\bar{g}_0$ (these quantities being interpreted as descriptive of the electrons) the temporal behavior of the perturbed electric field, D_3, would be completely characterized by the roots of

$$(s^2 + c^2\kappa^2) - \frac{4\pi i}{\kappa} (s^2 + c^2\kappa^2 + s\beta) \frac{e^2}{m} \int_{-\infty}^{\infty} \frac{dv}{s + \beta + i\kappa v} \frac{\partial \bar{f}_0}{\partial v} = 0 \tag{37}$$

For purposes of illustration, consider a Maxwell-Boltzmann distribution for $\bar{f}_0$:

$$\bar{f}_0 = n_0 \left(\frac{m}{2\pi kT}\right) \exp\left(-\frac{mv^2}{2kT}\right) \tag{38}$$

In this case the roots (in the limit of small κ) are given, to terms of order κ^3, by

$$(s^2 + c^2\kappa^2) + \frac{\omega_e^2}{(s + \beta)} \frac{s^2 + c^2\kappa^2 + s\beta}{s + \beta} \left[1 - \frac{3kT}{m} \frac{\kappa^2}{(s + \beta)^2}\right] = 0 \tag{39}$$

Further, if terms of the order of (β/ω_e) are neglected, the characteristic equation becomes

$$(s + \beta)^2 + \omega_e^2 - \frac{\kappa^2}{s^2} \omega_e^2 \frac{3kT}{m} = 0 \tag{40}$$

Two limiting cases are of interest here. The first,

$$\kappa \to 0, \quad s = -\beta \pm i\omega_e \tag{41}$$

is the case for very long wavelengths and/or very low temperatures. The system oscillates with angular frequency $\omega_e = (4\pi ne^2/m)^{1/2}$, the oscillations damping out because of collisions.

The second limiting case is as follows:

$$\beta \to 0, \quad s^2 = -\omega_e^2 (1 + 3h^2\kappa^2), \quad h^2 = kT/4\pi ne^2 \tag{42}$$

III. ELECTRON OSCILLATIONS FROM THE MACROSCOPIC POINT OF VIEW

It is of some interest at this point to investigate the question of small perturbations in plasmas from the macroscopic point of view. For this purpose, recall the charge continuity relation,

$$\frac{\partial Q}{\partial t} + \frac{\partial J_k}{\partial x_k} = 0$$

and differentiate it with respect to time; i.e,

$$\frac{\partial^2 Q}{\partial t^2} = - \frac{\partial}{\partial x_k} \frac{\partial J_k}{\partial t} = \frac{\partial^2}{\partial x_j \partial x_k} \left(\frac{e_A}{m_A} \Psi^A_{jk} + \frac{e_B}{m_B} \Psi^B_{jk} + e_A n^A w^A_j w^A_k + e_B n^B w^B_j w^B_k \right)$$

$$+ \frac{e_A e_B}{m_A m_B} \left[\frac{\partial \rho E_k}{\partial x_k} + \frac{1}{c} \frac{\partial}{\partial x_k} (\mathbf{\Lambda} \times \mathbf{H})_k \right] \tag{43}$$

$$+ \frac{e_B m_A + e_A m_B}{m_A m_B} \left[\frac{\partial Q E_k}{\partial x_k} + \frac{1}{c} \frac{\partial}{\partial x_k} (\mathbf{J} \times \mathbf{H})_k \right] + \left(\frac{e_A}{m_A} - \frac{e_B}{m_B} \right) \frac{\partial}{\partial x_k} P^{AB}_k$$

It might be tempting at this point to attempt linearization by resorting to the decomposition of the quantities n, θ, and $\mathbf{w}$ into large and small components, terms quadratic in the small components to be ignored. However, such a procedure would not lift the veil of ignorance from the quantities Ψ_{jk}. Thus, in a fundamental way, the macroscopic viewpoint forces modification of the method of linearization, which, depending on the method adopted, may present subtle but significant implications for the range of validity of conclusions to be drawn. In particular, the approach to linearization to be adopted herein implies a restriction to systems close to equilibrium—hence collision-dominated—which is in marked contrast to the restriction inherent in the preceding analysis from the microscopic viewpoint.

Linearization of the above equation may be achieved by virtue of the following assumptions and approximations:

1. Assume that

$$f^A = n^A \left(\frac{m_A}{2\pi\theta^A} \right)^{\frac{3}{2}} \exp \left[- \frac{m_A}{2\theta^A} (\mathbf{v} - \mathbf{w}^A)^2 \right]$$

and

$$f^B = n^B \left(\frac{m_B}{2\pi\theta^B} \right)^{\frac{3}{2}} \exp \left[- \frac{m_B}{2\theta^B} (\mathbf{v} - \mathbf{w}^B)^2 \right]$$

i.e., the two components of the plasma are separately in local equilibrium. From this assumption,

$$\Psi_{jk}^{A,B} \to \delta_{jk} n^{A,B} \theta^{A,B}$$

and also,

$$h_j^{A,B} \to 0$$

2. Now approximate n, θ, and $\mathbf{w}$ by the usual decomposition into a large and a (quadratically ignorable) small component.

3. Then assume:

(a) n_0^A and n_0^B constant in space and time.

(b) $e_A n_0^A + e_B n_0^B = Q_0 = 0$.

(c) $\mathbf{w}_0^A = \mathbf{w}_0^B = 0$. Hence $\mathbf{J}_0 = \mathbf{\Lambda}_0 = 0$.

(d) $\theta_0^A = \theta_0^B = kT$, a constant.

Of course, (b) and (c) imply that $\mathcal{H}_0 = \mathcal{E}_0 = 0$.

4. Finally assume that there are no external fields imposed upon the system.

Accordingly our equation for Q becomes

$$\frac{\partial^2 Q_1}{\partial t^2} = \nabla^2 \left[\frac{e_A}{m_A} (n_1^A kT + n_0^A \theta_1^A) + \frac{e_B}{m_B} (n_1^B kT + n_0^B \theta_1^B) \right]$$

$$+ \frac{4\pi e_A e_B \rho_0}{m_A m_B} Q_1 + \left(\frac{e_A}{m_A} - \frac{e_B}{m_B} \right) \frac{\partial}{\partial x_k} P_k^{AB} \tag{44}$$

using the following relation:

$$\nabla \cdot \mathcal{E}_1 = 4\pi Q_1 \tag{45}$$

To establish another connection between n_1 and θ_1, the equation of energy conservation is employed. In accordance with the above assumptions, it has become

$$\frac{\partial}{\partial t} (n_0^A \theta_1^A + \theta_0 n_1^A + n_0^B \theta_1^B + \theta_0 n_1^B) = -\frac{5}{3} n_0^A \theta_0 \frac{\partial w_{1j}^A}{\partial x_j} - \frac{5}{3} n_0^B \theta_0 \frac{\partial w_{1j}^B}{\partial x_j} - \frac{2}{3} (w_{1j}^A - w_{1j}^B) P_j^{AB} \tag{46}$$

However, it is to be noted that

$$P_j^{AB} = (mv_j, I_{AB}) \tag{47a}$$

and

$$I_{AB} = I(f^A f^B) = I(f_0^A f_0^B) + I(f_0^A f_1^B + f_1^A f_0^B) \tag{47b}$$

But, since the zero-order distributions are locally in equilibrium with each other,

$$I(f_0^A f_0^B) = 0$$

and the quantity

$$(w_{1j}^A - w_{1j}^B)\ P_j^{AB} \tag{48}$$

is second-order in small quantities, and is to be ignored. Now account is taken of the equation expressing particle conservation, i.e.,

$$\frac{\partial n}{\partial t} + \frac{\partial n w_j}{\partial x_j} = 0 \tag{49}$$

Observe as a result that the energy equation becomes simply

$$(3n_0^A \theta_1^A - 2\theta_0 n_1^A) + (3n_0^B \theta_1^B - 2\theta_0 n_1^B) = \text{constant} = 0 \tag{50}$$

In the sense of the present approximations, this is equivalent to the relation

$$\theta^A\ (n^A)^{-\frac{2}{3}} + \theta^B\ (n^B)^{-\frac{2}{3}} = \text{constant}$$

which, for a gas mixture in local thermal equilibrium ($\theta^A = \theta^B$) is just the law of adiabatic compression.

At this point it is convenient to identify the A-type particles with the electrons in the plasma and the B-type with the ions, and to introduce the assumption that the ions do not respond to the perturbing influences on the system, i.e., $n_1^B = \theta_1^B = 0$. Then, taking into account the adiabatic relation deduced above and the fact that, in accordance with indicated identification, $m_B/m_A \cong 10^4$, the equation for Q_1 becomes

$$\frac{\partial^2 Q_1}{\partial t^2} = \frac{5}{3}\frac{kT}{m_A}\nabla^2 Q_1 - w_e^2 Q_1 + \frac{e_A}{m_A}\frac{\partial}{\partial x_k} P_k^{AB} \tag{51}$$

To proceed further, treat the term representing momentum transfer by collisions in a somewhat cavalier fashion. Again

$$P_k^{AB} = (m_A v_k, I_{AB})$$

and approximate I_{AB} as in the previous section to obtain

$$(m_A v_k, - \beta f^A) = - \beta m_A (v_k, f_1^A) \cong - \beta m_A w_{1k}^A \tag{52}$$

Hence,

$$\frac{e_A}{m_A}\frac{\partial}{\partial x_k} P_k^{AB} \cong - \beta \frac{\partial}{\partial x_k} e_A w_{1k}^A = - \beta \frac{\partial J_{1k}}{\partial x_k} = \beta \frac{\partial Q_1}{\partial t} \tag{53}$$

Consequently the equation for Q_1 becomes

$$\frac{\partial^2 Q_1}{\partial t^2} - \frac{5}{3}\frac{kT}{m_A}\nabla^2 Q_1 + \omega_e^2 Q_1 - \frac{\beta \partial Q_1}{\partial t} = 0 \tag{54}$$

Solutions of the form $e^{i\kappa \cdot x - i\omega t}$ result in the dispersion relation,

$$\omega^2 - i\beta\omega = \omega_e^2 + \frac{5}{3}\frac{kT}{m_A}\kappa^2 \tag{55}$$

If isothermal oscillations were assumed instead of adiabatic oscillations, i.e., $\theta_1^A = 0$ instead of $\theta_1^A = (2/3)(\theta_0/n_0^A)\, n_1^A$, the relation

$$\omega^2 - i\beta\omega = \omega_e^2 + \frac{kT}{m_A}\kappa^2 \tag{56}$$

would be obtained.

One would hesitate to ascribe quantitative significance to the differences between the various results obtained in this section, or between those obtained here when compared with the results of the previous section; however, one should note the extent to which subtle differences in the assumptions employed to define the system to be studied can modify specific answers to specific questions.

IV. TRANSVERSE OSCILLATIONS

It is now appropriate to return to the general solutions of the problem of plasma perturbations obtained earlier, and employ them to investigate the interaction between the plasma and electromagnetic radiation. However, in this case (as in the previous analysis of the longitudinal oscillations) it is immediately apparent that the general solutions are too complex to serve the purpose of simple illustration suitably. Thus, it must suffice here to examine again the specialized case for which there is no external magnetic field and for which it is further assumed that initial distribution in velocity space is isotropic (Ref. 2, 3).

Recall that the above assumptions were sufficient to diagonalize the matrix, M. For present purposes it is convenient to exhibit the equation for the electric field as

$$M_{jk} D_k = X_j \tag{57}$$

or, explicitly,

$$(s^2 + c^2\kappa^2)\, D_j - 4\pi s D_j \sum_A \frac{e_A^2}{M_A} \int \frac{v_j}{s + \beta + i\kappa v_3} \frac{\partial f_0^A}{\partial v_j}\, d^3v$$

$$- 4\pi i c^2 \kappa\, D_3 \sum_A \frac{e_A^2}{M_A} \int \frac{1}{s + \beta + i\kappa v_3} \frac{\partial f_0^A}{\partial v_3}\, d^3v = X_j \tag{58}$$

where the repeated indices in the second term are *not* summed and a coordinate system has been selected with the 3-axis along the direction of $\boldsymbol{\kappa}$. Since concern is specifically given to the propagation of transverse waves, only the components of $\boldsymbol{D}$ perpendicular to $\boldsymbol{\kappa}$, i.e., at $\boldsymbol{\kappa} \times \boldsymbol{D}$, are noted. The solution of the equation that this vector satisfies (after some manipulation on the first integral in the above equation) is the following:

$$(\kappa \times \mathbf{D}) = \frac{\kappa \times \mathbf{X}}{(s^2 + c^2\kappa^2) + 4\pi s \sum_A \frac{e_A^2}{M_A} \int \frac{f_0^A \, d^3v}{s + \beta + i\kappa v_3}} \tag{59}$$

Again, assuming that the numerator is analytic, the temporal behavior of these solutions is determined by the roots of the denominator; i.e.,

$$(s^2 + c^2\kappa^2) + 4\pi s \sum_A \frac{e_A^2}{M_A} \int \frac{f_0^A \, d^3v}{s + \beta + i\kappa v_3} = 0$$

Assuming that only the electrons participate significantly in the motion, it is seen after expanding $(s + \beta \, i\kappa \, v_3)^{-1}$ in a power series in κ and retaining terms only through κ^2 that:

$$s^2 + \frac{s\omega_e^2}{s + \beta} + c^2\kappa^2 \left[1 - \frac{\langle v^2 \rangle}{c^2} \frac{1}{3} \frac{s}{s + \beta} \frac{\omega_e^2}{(s + \beta)^2} \right] = 0 \tag{60}$$

Now ignoring terms $\mathcal{O}(\beta/\omega)$ and $\mathcal{O}(\langle v^2 \rangle / c^2)$,

$$s^2 + c^2\kappa^2 + \omega_e^2 = 0 \tag{61}$$

Observing that here s is purely imaginary and thus replacing $s \to i\omega$, the phase velocity of the wave is found (Ref. 4) to be

$$V^2 = \left(\frac{\omega}{\kappa} \right)^2 = \frac{c^2}{1 - \left(\frac{\omega_e}{\omega} \right)^2} \tag{62}$$

V. ANALYTICAL REDUCTION OF THE GENERAL SOLUTIONS

This chapter closes with a presentation and brief discussion of some manipulative reformulations of the general solutions which lend themselves to further analytical reduction for the ultimate purpose of acquiring greater insight into the nature of the plasma. For this purpose, it is convenient to adopt a compressed notation for the various tensor parameters that enter into the general results. Defining the vector

$$\boldsymbol{\chi} = s\mathbf{v} + ic^2\boldsymbol{\kappa} \tag{63}$$

the following results (in a notation otherwise familiar from the preceding) may be exhibited:

$$\gamma_j = 4\pi \sum_A \frac{e_A}{\Omega^A} \int d^3v\, d\alpha'\, \chi_j\, I_A(\alpha, \alpha')\, g_0^A(\alpha')$$

$$\beta_{jk} = -4\pi \sum_A \frac{e_A}{\Omega^A} \int d^3v\, d\alpha'\, \chi_j\, I_A(\alpha, \alpha')\, d_k^A(\alpha')$$

$$\rho_{jk} = -4\pi \sum_A \frac{e_A}{\Omega^A} \int d^3v\, d\alpha'\, \chi_j\, I_A(\alpha, \alpha')\, (\mathbf{d}^{A'} \times \mathbf{v}')_k$$

$$\eta_{jk} = -4\pi \sum_A \frac{e_A}{\Omega^A} \int d^3v\, d\alpha'\, \chi_j\, I_A(\alpha, \alpha')\, [(\mathbf{d}^{A'} \times \mathbf{v}') \times \boldsymbol{\kappa}]_k \tag{64}$$

An approach which frequently makes possible some rigorous further reduction of these parameters is based upon the relation

$$e^{ix \sin\theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} J_m(x) \tag{65}$$

and permits

$$I_A(\alpha, \alpha') = e^{\left[\frac{s + \beta + iu\,\kappa_{||}}{\Omega^A}(\alpha - \alpha')\right]} \times \sum_{m,m'=-\infty}^{\infty} e^{i\alpha_k(m'-m) + im\alpha - im'\alpha'} J_m(X)\, J_{m'}(X) \tag{66}$$

where $\kappa_{||}$ is the component of κ parallel to $\boldsymbol{H}_0$; $X = w\kappa_{\perp}/\Omega^A$, $\kappa_{\perp}$ being the component of $\boldsymbol{\kappa}$ perpendicular to $\boldsymbol{H}_0$; and α_{κ} is the angular specification of κ in the same cylindrical basis employed for the representation of $\mathbf{v}$. In the event that the initial distributions are independent of α, i.e., have axial symmetry about the direction of the external magnetic field, this representation of I_A renders the (α, α')-integrations trivial, sometimes with interesting results (Ref. 2).

The above approach is most suitable if it is not desirable to make assumptions about the strength of the externally applied magnetic field, $\boldsymbol{H}_0$. Earlier, an expansion of I_A was presented in a power series in Ω^A which has obvious peculiar applicability to "weak field" problems. An outline of an approximating approach valid for strong field problems is given at this point.

Exhibit the function I_A as

$$I_A = e^{\left[\frac{s + \beta}{\Omega^A}(\alpha - \alpha') + i\boldsymbol{\kappa}\cdot\mathbf{v}^A\right]} \tag{67}$$

where

$$\frac{\mathbf{v}''}{\Omega^A} \equiv \int_{\alpha'}^{\alpha} (\mathbf{v}''/\Omega^A)\, d\alpha'' \tag{68}$$

and then take the Fourier–Laplace inverse of the relation

$$M_{jk}D_k = X_j \tag{69}$$

Manipulation and further compression of notation finally gives:

$$\frac{1}{(2\pi)^{\frac{3}{2}}(2\pi i)}\int_{\kappa}\int_{C(s)} d^3\kappa\, ds\, e^{i\kappa\cdot x+st}\,(s^2+c^2\kappa^2)\,D_k$$

$$+\sum_A \frac{e_A}{\Omega^A}\int_{v}\int_{\alpha'} d^3v\, d\alpha' \frac{1}{(2\pi)^{\frac{3}{2}}(2\pi i)}\int_{\kappa}\int_{C(s)} d^3\kappa\, ds$$

$$\times \exp\left[i\boldsymbol{\kappa}\cdot(\mathbf{x}+\mathbf{y}^A)+s\left(t+\frac{\alpha-\alpha'}{\Omega^A}\right)\right]\zeta_{kj}D_j$$

$$=\frac{1}{(2\pi)^{\frac{3}{2}}(2\pi i)}\int_{\kappa}\int_{C(s)} d^3\quad ds\, e^{i\kappa\cdot x+st}\,X_k \tag{70}$$

where the symbol

$$\zeta_{kj}=4\pi\left\{d_k^{A\,\prime}-\frac{ic}{s}\left[(d^{A\,\prime}\times\mathbf{v}')\times\boldsymbol{\kappa}\right]_k\right\}(sv_j+ic^2\kappa_j)\,e^{\beta(\alpha-\alpha')/\Omega^A} \tag{71}$$

has been introduced.

Concentrating attention on the left-hand side of this expression, note that the inversion of the first term is simply

$$\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)E_k \tag{72}$$

Note that the second term on the left-hand side can be modified in such a way as to be appropriate as an inclusion in an integral equation if it is recalled that

$$D_j = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x'} \int_{t'} d^3x' dt' e^{-iK\cdot x' - st'} E_j(\mathbf{x}', t') \tag{73}$$

The tensor Green's function is defined:

$$G^A_{kj}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{(2\pi)^3 (2\pi i)} \int d^3v \, d\alpha' d^3K \, ds \, \zeta^A_{kj} \, e^{iK\cdot(\mathbf{x} - \mathbf{x}' + \mathbf{y}^A)}$$

$$\times \exp\left[s\left(t - t' + \frac{\alpha - \alpha'}{\Omega^A} \right) \right] \tag{74}$$

In terms of the above, the following integro-differential equation for $\boldsymbol{E}$ is obtained:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2 \right) E_k + \sum_A \frac{e_A}{\Omega^A} \int d^3x' \, dt' \, G^A_{kj}(\mathbf{x}, t; \mathbf{x}', t') \, E_j(\mathbf{x}', t') = \mathcal{G}^{-1} \mathcal{L}^{-1} X_k \tag{75}$$

where it is understood that $t \geq t' \geq 0$.

The potential utility of this particular form of Maxwell's equations for the electric field in the presence of plasma-like matter resides in the fact that it is readily converted to a differential equation in which the interaction of the radiation field with matter contributes derivatives of all orders entering something like

$$(\Omega^A)^{-n} \frac{\partial^n E_j}{\partial t^n}$$

and

$$\left(\frac{v}{\Omega^A}\right)^n \frac{\partial^n E_j}{\partial x_k^n}$$

Thus, if the fields do not vary rapidly in time compared to Ω^{-1} or in space compared to (v/Ω), the infinite-order equation may be approximated by an equation of finite order whose validity is presumably the better the larger Ω (or H_0). The procedure for obtaining the indicated differential equation depends, of course, upon a Taylor's expansion of the function $E_j(\mathbf{x}', t')$ in the integrand about the space-time point $(\mathbf{x}, t)$.

Illustrations of one or more of these methods of reducing the general solutions to tractable proportions will be presented in the chapter on the discussion of the transport parameters for plasmas. In general, it does not appear that the full implications of these results have yet been explored.

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Fourth Chapter

ENSKOG'S APPROACH TO THE NEUTRAL GAS PROBLEM

The previous chapter dealt with a class of physical problems in which it was expected that individual particle behavior (collisions) would be considerably less important than collective behavior (interactions between localized charge densities and currents via "fields"). Furthermore, the analysis was restricted to the idealization of systems which, in the unperturbed state, could be regarded as in a steady state in time and spatially uniform (infinite media). For many practical purposes such assumptions and restrictions are overwhelmingly severe. It would be most desirable to have means for generating descriptions of systems which are finite in extent and evolving in time in the gross sense, i.e., systems whose spatial and temporal variations are not construable as small perturbations imposed on underlying uniform, steady states.

Such descriptions, of course, inhere in the "moment equations" potentially, but not actually, until the functional parameters (which, as a consequence of truncating the "equations of moments" at a given order, are undetermined by the truncated system) have been identified as functionals of quantities which are determined by the adopted moment equations. For example, the equations for the number density, mean velocity, and mean energy of the one-component neutral gas acted upon by external forces producing individual particle accelerations $\boldsymbol{a}$ are:

$$\frac{\partial n}{\partial t} + \frac{\partial n w_j}{\partial x_j} = 0 \tag{1a}$$

$$nm \frac{\partial w_j}{\partial t} = - \frac{\partial \Psi_{jk}}{\partial x_k} - nmw_k \frac{\partial w_j}{\partial x_k} + nm \langle a_j \rangle \tag{1b}$$

$$n \frac{\partial \theta}{\partial t} + nw_j \frac{\partial \theta}{\partial x_j} + \frac{2}{3} \frac{\partial h_j}{\partial x_j} + \frac{2}{3} \Psi_{jk} \frac{\partial w_k}{\partial x_j} = \frac{2}{3} nm \langle \boldsymbol{a} \cdot \mathbf{v} \rangle - \frac{2}{3} nm\mathbf{w} \cdot \langle \boldsymbol{a} \rangle \tag{1c}$$

Presumably, knowledge of the quantities n, $\mathbf{w}$, and θ would provide considerable (if not complete) predictive power about the macroscopic behavior of finite gaseous systems varying in time. However, such knowledge is not resident in these relations because of the presence of the unknown functionals, Ψ and h. It was precisely for the purpose of enabling an identification of Ψ and h as functions of n, $\mathbf{w}$, and θ that Enskog formulated his method of a successive approximation "solution" of the Boltzmann equation which underlies these moment equations.[1]

[1] Chapman, S., Cowling, T. G., "The Mathematical Theory of Non-Uniform Gases", Cambridge University Press, England 1952.

Though Enskog's approach to this problem has been exhaustively dealt with, there has been little examination of the relevance of the method or validity of the results deduced therefrom for application to plasma studies. For this reason, a review of Enskog's method is included here in the simplest possible context, e.g., that of the system indicated above. The review is then to be followed by a cursory examination of the possibility of extending the method to the analysis of plasmas.

It is convenient for present purposes to introduce the notation

$$D = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} + a_j \frac{\partial}{\partial v_j}$$

so that the Boltzmann equation from which the above moment equations stem is

$$Df = I\,(f\bar{f}) \tag{2}$$

It is worth recalling that the moment equations correspond to the following integral relations:

$$\int d^3 v Df = \int I\,(f\bar{f})\; d^3 v = 0 \tag{3a}$$

$$\int d^3 v (m\mathbf{u}) Df = \int d^3 v (m\mathbf{u})\; I\,(f\bar{f}) = 0 \tag{3b}$$

$$\int d^3 v \left(\frac{1}{3}\; mu^2\right) Df = \int d^3 v \left(\frac{1}{3}\; mu^2\right) I\,(f\bar{f}) = 0 \tag{3c}$$

together with the definitions

$$n = \int f d^3 v, \qquad \mathbf{w} = \frac{1}{n}\int \mathbf{v} f d^3 v, \qquad \mathbf{u} = \mathbf{v} - \mathbf{w}$$

$$\Psi_{jk} = m \int u_j u_k f d^3 v,$$

$$\theta = \frac{1}{n}\int \frac{1}{3}\; mu^2 f d^3 v \qquad\qquad h_j = \int \frac{1}{2}\; mu^2 u_j f d^3 v \tag{4}$$

The essence of Enskog's idea seems to reside in the following argument: (a) If n, θ, and $\mathbf{w}$ were known for the system under consideration, they would possess adequate predictive power for dealing with most macroscopic observations; (b) the moment equations would be a complete set for the determination of these quantities if Ψ and h were known as functionals of n, θ, and $\mathbf{w}$; (c) the solution to the equation

$$I\,(f\bar{f}) = 0 \tag{5}$$

is a known functional of n, θ, and $\mathbf{w}$ which in no way restricts these quantities, and, (d) thus, it follows that, if the solution to

$$Df - I(f\bar{f}) = 0 \tag{6}$$

can be exhibited as a series

$$f = \sum_n f_n \tag{7}$$

in which $f_n = f_n(f_{n-1} \cdots f_0,\, v)$, then f will be known as a function of f_0–hence of n, θ, and $\mathbf{w}$–and consequently the dependence of Ψ and h on these quantities can be determined from their dependence on f.

The operational achievement of the implications of this argument is facilitated by formally expanding f as

$$f = \sum_{\lambda=0}^{\infty} \zeta^{\lambda-1} f_\lambda \tag{8}$$

where ζ is to be regarded solely as an arbitrary (bookkeeping) parameter. Equations for the terms in the series are then readily deduced, the first three being

$$
\begin{aligned}
&(1) \quad 0 = I\,(f_0\bar{f}_0) \\
&(2) \quad Df_0 = I\,(f_0\bar{f}_1 + f_1\bar{f}_0) \\
&(3) \quad Df_1 = I\,(f_0\bar{f}_2 + f_1\bar{f}_1 + f_2\bar{f}_0)
\end{aligned}
\tag{9}
$$

Clearly the first approximation to f, i.e., f_0, is

$$f_0 = n \left(\frac{m}{2\pi\theta} \right)^{\frac{3}{2}} \exp \left[- \frac{m}{2\theta} (\mathbf{v} - \mathbf{w})^2 \right] \tag{10}$$

However, it is necessary to interpret n, θ, and $\mathbf{w}$ as the actual number density, energy density, and mean velocity of the gas; such an interpretation implies

$$\int d^3v \sum_{\lambda=1}^{\infty} f_\lambda = 0 \qquad \int d^3v \ \mathbf{v} \sum_{\lambda=1}^{\infty} f_\lambda = 0$$

$$\int d^3vu^2 \sum_{\lambda=1}^{\infty} f_\lambda = 0 \tag{11}$$

A sufficient condition for the realization of these constraints is:

$$\int d^3vf_\lambda = \int d^3v \ \mathbf{v}f_\lambda = \int d^3vu^2f_\lambda = 0 \tag{12}$$

for all $\lambda \geq 1$. This condition will be adopted.

Consider now the equation for the second approximation to f

$$Df_0 = I \, (f_0\bar{f}_1 + f_1\bar{f}_0) \tag{13}$$

Let Ψ_α be any one of the quantities 1, v_j, or u^2. Observe that

$$\int \Psi_\alpha Df_0 \ d^3v = 0 \tag{14}$$

independently of the nature of f_1. Furthermore, these five equations provide a complete set of equations for the determination of n, θ, and **w**. But this is hardly possible (in general) without complete knowledge of f. Hence, the equation for the second approximation is, as it is presently constituted, inconsistent with the interpretation given to n, θ, and **w**.

To circumvent this inconsistency, Enskog suggested the following decomposition of the operator D:

$$D = \sum_{\lambda=0}^{\infty} \zeta^{\lambda} \frac{\partial_{\lambda}}{\partial t} + v_j \frac{\partial}{\partial x_j} + a_j \frac{\partial}{\partial v_j} \tag{15}$$

In accordance with this breakup, the equations for the first three approximations to f become,

first (as before):

$$I\,(f_0 \bar{f}_0) = 0$$

second:

$$\frac{\partial_0 f_0}{\partial t} + v_j \frac{\partial f_0}{\partial x_j} + a_j \frac{\partial f_0}{\partial v_j} = I\,(f_0 \bar{f}_1 + f_1 \bar{f}_0) \tag{16}$$

third:

$$\frac{\partial_0 f_1}{\partial t} + \frac{\partial_1 f_0}{\partial t} + v_j \frac{\partial f_1}{\partial x_j} + a_j \frac{\partial f_1}{\partial v_j} = I\,(f_0 \bar{f}_2 + f_1 \bar{f}_1 + f_2 \bar{f}_0)$$

where the significance of the operators $\partial_{\lambda}/\partial t$ has yet to be determined, except that they are differentiation operations of some kind. In particular they are *not* the conventional partial derivatives in time.

Now exhibit $f_1 = f_0\, \Phi_1$. Then, substituting the solution to the first approximation, f_0, into the equation for the second approximation, displays explicitly the fact that ϕ_1 satisfies an inhomogeneous integral equation whose general solution is a linear combination of all solutions of the homogeneous equation plus the particular solutions.

To obtain the solutions of the homogeneous equation, observe first that it is sufficient that ϕ_1 be a summational invariant (one of the ψ_α) in order that it be a solution; that it is necessary that ϕ_1 be one of the ψ_α's may be established as follows: Assume that (a) ϕ_1 is a solution of $I(\phi_1) = 0$, but (b) it is not one of the ψ_α's. Observe that

$$\int \phi_1 I_2(\phi_1)\ d^3v = 0 \tag{24}$$

by virtue of (a). But this integral may be exhibited as

$$-\frac{1}{4}\int d^3v'\, d^3v_1'\, d^3v\ d^3v_1 \left[\phi_1(\mathbf{v}') + \phi_1(\mathbf{v}_1') - \phi_1(\mathbf{v}) - \phi_1(\mathbf{v}_1)\right]^2 f_0(\mathbf{v})\, f_0(\mathbf{v}_1)\, \mathcal{G} \tag{25}$$

Every quantity in the integrand is either positive or zero. But the bracketed quantity is not zero by virtue of (b); f_0 is not zero if there are any particles in the system; and $\mathcal{G}$ does not vanish unless the collision cross section vanishes. Hence,

$$\int \phi_1 I_2(\phi_1)\ d^3v > 0 \tag{26}$$

and it follows that $I_2(\phi_1) \neq 0$;i.e., ϕ_1 thus chosen is not a solution.

It follows from these considerations that the solution of the homogeneous equation may be taken to be of the form

$$\alpha + \boldsymbol{\beta}\cdot\ m\boldsymbol{u} + \gamma \frac{1}{2}\ mu^2 \tag{27}$$

where $\alpha, \boldsymbol{\beta}$, and γ are arbitrary functions of position and time.

In order to obtain the particular solutions, rewrite $I_2(\phi_1)$ in terms of the variable $\boldsymbol{u}$ instead of $\mathbf{v}$; i.e.,

$$I_2(\phi_1) = -\ nu\sigma f_0\, \phi_1 + \int K(\boldsymbol{u}, \boldsymbol{u}')\ \phi_1(\boldsymbol{u}')\ d^3u' \tag{28}$$

If now $\phi_1(\mathbf{u})$ is expanded in spherical harmonics whose arguments are the direction of $\mathbf{u}$, the result is

$$I_2(\phi_1) = -\, nu\sigma f_0 \sum_{l,\, m} \phi^m_{1l}(u)\; Y^m_l(\hat{\mathbf{u}})$$

$$+ \sum_{l,\, m} Y^m_l(\hat{\mathbf{u}}) \int K_l(u, u')\; \phi^m_{1l}(u')\; du' \tag{29}$$

But the inhomogeneous terms in the integral equation are linear combinations of the spherical harmonics Y^m_1 and Y^m_2. Thus, because of the orthogonality of the Y^m_l it is concluded that

$$0 = -\, nu\sigma f_0\; \phi^m_{1l} + \int K_l(u, u')\; \phi^m_{1l}(u')\; du' \tag{30}$$

for $l > 2$. However, this is the homogeneous equation whose solution has already been seen to contain no spherical harmonics of order greater than one. Thus, it may be concluded that the particular solution is a linear combination of tensors in $\mathbf{u}$-space of rank less than, or equal to, two. In fact, because of the presence of the configuration space tensors $\partial ln\theta/\partial x_j$ and $\partial w_k/\partial x_j$ the appropriate linear combination would be expected to be

$$A(\mathbf{x}, u, t)\; u_j \frac{\partial \ln\theta}{\partial x_j} + B(\mathbf{x}, u, t)\left(u_j u_k - \frac{1}{3}\, \delta_{jk}\, u^2\right)\frac{\partial w_k}{\partial x_j} \tag{31}$$

where A and B are to be determined by the requirement of a particular solution, i.e.,

$$f_0\left(\frac{mu^2}{2\theta} - \frac{5}{2}\right) u_j \frac{\partial \ln\theta}{\partial x_j} = I_2(Au_j)\frac{\partial \ln\theta}{\partial x_j},$$

$$\frac{f_0 m}{\theta}\left(u_k u_j - \frac{1}{3}\, \delta_{jk} u^2\right)\frac{\partial w_k}{\partial x_j}$$

$$= I_2\left[B\left(u_k u_j - \frac{1}{3}\, \delta_{jk} u^2\right)\frac{\partial w_k}{\partial x_j}\right] \tag{32}$$

The general solution for ϕ_1 may now be represented as

$$\phi_1 = \alpha + \beta_j m u_j + \gamma \frac{1}{2} m u^2 + A u_j \frac{\partial \ln \theta}{\partial x_j} + B\left(u_k u_j - \frac{1}{3} \delta_{jk} u^2\right) \frac{\partial w_k}{\partial x_j} \tag{33}$$

Finally, it is necessary to guarantee satisfaction of the conditions

$$\int d^3 u \psi_\alpha(u) f_1 = \int d^3 u \psi_\alpha f_0 \phi_1 = 0 \tag{34}$$

The scalar terms in ϕ_1 must be zero, and the vector terms are equivalent; hence, we may take as the second approximation

$$f = f_0 + f_1$$

$$f_1 = f_0 A u_j \frac{\partial \ln \theta}{\partial x_j} + f_0 B \left(u_j u_k - \frac{1}{3} \delta_{jk} u^2\right) \frac{\partial w_k}{\partial x_j} \tag{35}$$

It is of some interest to note that, in terms of the solution developed thus far (reverting to the scalar product notation employed extensively earlier),

$$\left.\begin{aligned} \Psi_{jk} &= \delta_{jk} n\theta + m \frac{\partial w_n}{\partial x_l} \left(u_j u_k, f_0 B \left[u_l u_n - \frac{1}{3} \delta_{ln} u^2\right]\right) \\ h_j &= \frac{1}{3} m \frac{\partial \ln \theta}{\partial x_k} (u^2 u_j, f_0 A u_k) \end{aligned}\right\} \tag{36}$$