# COFINITE CLASSIFYING SPACES FOR LATTICES IN $\mathbb{R}$-RANK ONE SEMISIMPLE LIE GROUPS

by

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# CHAPTER I

# Introduction

In 1956, Milnor showed in his two papers [53], [54] that for every topological group $\Gamma$ there exists a unique (up to homotopy) $\Gamma$-principal bundle $E\Gamma \to B\Gamma$ which is universal in the sense that every $\Gamma$-principal bundle over the base space $B\Gamma$ is induced from it. In other words, the space $E\Gamma$ 'classifies' all $\Gamma$-principal bundle over the base space $B\Gamma$. The base space $B\Gamma$ is called the *classifying space* for the group $\Gamma$. Two main properties of $E\Gamma$ is the following.

1. The space $E\Gamma$ is a $\Gamma$-CW-complex [53, Theorem 3.1], [54, Theorem 3.1]. For the precise definition of a $\Gamma$-CW-complex, see Definition I.2.

2. The space $E\Gamma$ is contractible [53, Lemma 3.6]. In fact, the $\Gamma$-bundle $E\Gamma \to B\Gamma$ is universal if and only if $E\Gamma$ is contractible [22, Theorem 7.5]. Therefore, the space $B\Gamma$ is an Eilenberg–MacLane space, in other words, a $K(\Gamma, 1)$-space, and the cohomology of $\Gamma$ is computed via the chain complex of $B\Gamma$ [18, Proposition 4.1]. For the definition of Eilenberg–MacLand space, see [18, p15].

Milnor proved the existence of the space $E\Gamma$ by constructing of a concrete $\Gamma$-CW-complex model of $E\Gamma$ (equivalently, a CW-complex model of $B\Gamma$). Although this construction works for any topological group, but it always gives an infinite dimensional $\Gamma$-CW-complex. In fact, if $\Gamma$ contains a torsion element, then every model of $E\Gamma$ (or $B\Gamma$) is infinite dimensional. For some special types of groups (c.f. §1.1.3), there exist finite dimensional models of the classifying space.

The notion of the classifying space is generalized to the *classifying spaces for families of subgroups of* $\Gamma$ (c.f. [45, Definition 1.8]). A family $\mathcal{F}$ of subgroups of $\Gamma$ is a collection of subgroups of $\Gamma$ which is closed under conjugation and finite intersections. For example, $\mathcal{F} = \mathcal{TR}$, $\mathcal{FIN}$, or $\mathcal{VCYC}$:

- The family $\mathcal{TR}$ consists of the trivial subgroup $\{Id\}$.
- The family $\mathcal{FIN}$ consists of all finite subgroups of $\Gamma$.
- The family $\mathcal{VCYC}$ consists of all virtually cyclic subgroups of $\Gamma$.

We denote $E_{\mathcal{F}}\Gamma$ the classifying space for the family $\mathcal{F}$ of subgroups of $\Gamma$. From the alternative definition of the classifying spaces (c.f. [45, Remark 2.8]), it follows that the space $E\Gamma$ is the classifying space for the family $\mathcal{TR}$:

$$E\Gamma = E_{\mathcal{TR}}\Gamma.$$

The space $E_{\mathcal{FIN}}\Gamma$ is called the *proper classifying space*, or equivalently, the *classifying space of the proper action* (c.f. Remark I.6(iii)). The space $\underline{E}\Gamma$ is the main object of study in this thesis. Conventionally, we write

$$\underline{E}\Gamma := E_{\mathcal{FIN}}\Gamma, \quad \underline{\underline{E}}\Gamma := E_{\mathcal{VCYC}}\Gamma.$$

The spaces $E\Gamma$, $\underline{E}\Gamma$, and $\underline{\underline{E}}\Gamma$ are studied as important objects in $K$- and $L$-theory (c.f. [23], [31]). For example, they are used in the formulations of many conjectures such as

1. Farrell–Jones conjecture [6, Conjecture 1.2], [23, Conjecture 1.6], [47, Conjecture 2.2];

2. Baum–Connes conjecture [8, Conjecture 3.15], [31, Conjecture 5.1], [47, Conjectures 1.31 and 2.3], [55], [67], [68];
3. Novikov conjecture [31, Conjecture 4.1], [38, Conjecture 6.4], [47, Conjectures 1.51 and 1.52].

The *finiteness* of the proper classifying space is the main interest of this thesis. Often the proper classifying spaces $\underline{E}\Gamma$ carries as much geometric information of the group $\Gamma$ as $E\Gamma$ does. Moreover, in many cases, the proper classifying space $\underline{E}\Gamma$ admits much simpler models than $E\Gamma$ does.

A $\Gamma$-CW-complex $X$ is called *cofinite* if it consists of only finitely many $\Gamma$-equivariant cells, i.e. the orbit space $\Gamma\backslash X$ is a finite CW-complex. Note that a cofinite $\Gamma$-CW-complex is automatically finite dimensional.

The main theorem of this thesis is the following.

**Theorem I.1.** *Let $\Gamma$ be a lattice in a semisimple Lie group $G$ of $\mathbb{R}$-rank one. Fix a maximal compact subgroup $K$ of $G$ and let us denote by $X$ the symmetric space $G/K$. Then there exists a partial compactification $\overline{X}_\Gamma$ of $X$ such that $\overline{X}_\Gamma$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$.*

This chapter is organized as follow. In Section §1.1, we clarify terms used in the main theorem, explain how the main theorem answers a natural question on finite models of proper classifying spaces, and list the known examples of finite dimensional or cofinite models of classifying spaces and proper classifying spaces of various discrete groups. In Section §1.2, we motivate the study of finding cofinite models of proper classifying spaces for lattices. In Section §1.3, we give an idea of the proof of the main theorem. The full proof is given in Chapter IV.

The rest of the thesis is organized as follow. In Chapter II, we review some preliminary facts on symmetric spaces, arithmetic groups, and semisimple Lie groups of rank one. In Chapter III, we give a geometric idea of the proof of the main theorem. We will prove the main theorem for a special case when the discrete group in the main theorem is a Fuchsian lattice. From the natural action of Fuchsian group $\Gamma$ on the upper half plane $\mathbf{H}$, we will construct a partially compactified $\Gamma$-space $\overline{\mathbf{H}}_\Gamma$, and show that the space $\overline{\mathbf{H}}_\Gamma$ is a cofinite model of the proper classifying space $\underline{E}\Gamma$. In Chapter IV, we prove the main theorem. We will use the result of Garland and Raghunathan [24] on fundamental domains of lattices in semisimple Lie groups of $\mathbb{R}$-rank one. Theorem II.78 in §2.4.3 summarizes the result of this reduction theory.

## 1.1 Investigating the main theorem

### 1.1.1 Terminologies

Let us state the main theorem again.

**Theorem.** *Let $\Gamma$ be a lattice in a semisimple Lie group $G$ of $\mathbb{R}$-rank one. Fix a maximal compact subgroup $K$ of $G$ and let us denote by $X$ the symmetric space $G/K$. Then there exists a partial compactification $\overline{X}_\Gamma$ of $X$ such that $\overline{X}_\Gamma$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$.*

The $\mathbb{R}$-rank of semisimple Lie group $G$ is the maximal dimension of $\mathbb{R}$-split tori in $G$. For example, $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, $SO(n,1)$, and $SU(n,1)$ are semisimple (in fact, simple) Lie groups of $\mathbb{R}$-rank one.

A *lattice* $\Gamma$ of a Lie group $G$ is a discrete subgroup whose quotient $\Gamma\backslash G$ has finite volume, measured by the induced measure of the Haar measure of $G$. For example, if a discrete subgroup $\Gamma$ admits a fundamental domain $D$ of $\Gamma$ in $G$ and the volume of $D$ is finite, then $\Gamma$ is a lattice in $G$.

**Definition I.2.** A **Γ-CW-complex** $X$ is a Γ-space with a Γ-fibration

$$X_0 \subset \cdots \subset X_n \subset \cdots \subset X = \bigcup_{i=0}^{\infty} X_i \tag{1.1}$$

such that each $X_n$ is constructed inductively from $X_{n-1}$ by attaching Γ-equivariant cells. In other words, for each $n \geq 1$, there exists a Γ-pushout

$$\begin{array}{ccc} \coprod_{\alpha \in I_n} (\Gamma/H_\alpha \times S_\alpha^{n-1}) & \xrightarrow{q_\alpha^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in I_n} (\Gamma/H_\alpha \times D_\alpha^n) & \xrightarrow{Q_\alpha^n} & X_n \end{array} \tag{1.2}$$

where each $I_n$ is the index set of Γ-equivariant $n$-cells and $H_\alpha$ is the isotropy group of $n$-cell $D_\alpha^n$. The vertical maps are inclusions. For each $n$ and $\alpha \in I_n$, the map $q_\alpha^n$ is a Γ-equivarient homeomorphism and $Q_\alpha^n$ is a homeomorphsm extending $q_\alpha^n$. For $n = 1$, we define $q_\alpha^n$ as the identity map, i.e. $X_0$ is the collection of Γ-equivariant 0-cells.

**Definition I.3.** From the notations in Definition I.2, the maximal $n > 0$ such that $X_i = \emptyset$ for all $i > n$ is called the **dimension** of Γ-CW-complex $X$. If the Γ-CW-complex $X$ is finite dimensional and each index set $I_n$ is finite, then $X$ is called **cofinite**.

**Example I.4.** Let us further assume that Γ be a cocompact Fuchsian group. The Dirichlet fundamental domain $D$ of Γ at $i \in \mathbf{H}$ is compact domain with finitely many vertices in $\mathbf{H}$. The Figure 1.1 below shows a fundamental domain of a *triangle Fuchsian group* (c.f. [66, §2]). The Γ-action on $D$ induces a tessellation of the upper half-plane $\mathbf{H}$. This tessellation induces a Γ-CW-structure of $\mathbf{H}$.

**Definition I.5.** Let $X$ be a Γ-CW-complex and whose Γ-action is proper. We call $X$ a model for the **proper classifying space**, or simply the proper classifying space, if

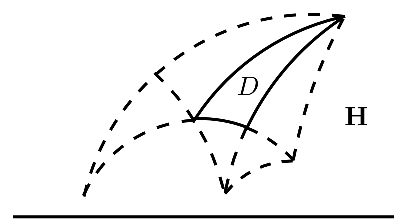


Figure 1.1: Tessellation of **H** by a compact triangular domain

1. for every subgroup $H \subset \Gamma$, the fixed point set $X^H$ is nonempty if and only if $H$ is finite, and
2. for every finite subgroup $H \subset \Gamma$, the fixed point set $X^H$ is contractible.

*Remark* I.6. (i) Another definition of the proper classifying space, equivalent to the Definition I.5 above, is the following: a $\Gamma$-CW-complex $X$ is called the proper classifying space if (1) all isotropy groups are finite, and (2) for every $\Gamma$-CW-complex $Y$ whose isotropy groups are finite, there exists a unique (up to homotopy) $\Gamma$-map $Y \to X$ [45, Definition 1.8]. The equivalence of two definitions is proved in [45, Theorem 1.9].

(ii) A general definition of the classifying space for the family $\mathcal{F}$ of subgroup of $\Gamma$ is the following (c.f. [45, Definition 1.8]). A $\Gamma$-CW-complex $X$ is the classifying space $E_{\mathcal{F}}\Gamma$ for the family $\mathcal{F}$ of subgroups of $\Gamma$ if (1) for every subgroup $H \in \mathcal{F}$, the fixed point set $X^H$ is nonempty if and only if $H \in \mathcal{F}$, and (2) for every $H \in \mathcal{F}$, the fixed point set $X^H$ is nonempty and contractible.

(iii) The space $\underline{E}\Gamma$ in Definition I.5 is also called the **classifying space for the proper action** of $\Gamma$. In general, a proper $\Gamma$-CW-complex $X$ is a $\Gamma$-space such that for every two points $x, y \in X$, there exists neighborhoods $V_x, V_y$ such that the set $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is compact in $G$. It is known that a $\Gamma$-CW-complex is proper if and only if every isotropy group is compact [45, Remark

1.2], [44, Theorem 1.23]. Therefore, the proper classifying space from Definition I.5 is indeed a proper $\Gamma$-CW-complex.

#### 1.1.2 A natural question

Arithmetic subgroups of semisimple Lie groups are examples of lattices [12](French), [51, Theorem 3.2.1], [14, Theorem 7.8]. Let $\Gamma$ be an arithmetic subgroup of semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. The real locus $G := \mathbf{G}(\mathbb{R})$ is a (real) semisimple Lie group containing $\Gamma$, and $\Gamma$ acts canonically on the symmetric space $X = G/K$ where $K$ is a maximal compact subgroup of $G$.

The locally symmetric space $\Gamma\backslash X$ is called *arithmetically defined* if $\Gamma$ is arithmetic. In [11], Borel and Serre constructed a compactification of arithmetically defined locally symmetric space $\Gamma\backslash X$ for torsion-free arithmetic group $\Gamma$, which is called the Borel–Serre compactification and denoted by $\overline{\Gamma\backslash X}^{BS}$. Alternatively, one can construct a partial compactification $\overline{X}^{BS}$ of the symmetric space with the free $\Gamma$-action so that

$$\overline{\Gamma\backslash X}^{BS} = \Gamma\backslash\overline{X}^{BS}.$$

From the contractibility of the Borel–Serre partial compactification, it follows that the space $\overline{X}^{BS}$ is a cofinite model of the classifying space $E\Gamma$ for torsion-free arithmetic group $\Gamma$. In [2, Remark 5.8], Adem and Ruan observed that the space $\overline{X}^{BS}$ is also a cofinite model of the proper classifying space for arithmetic group with torsion. A rigorous proof of this observation is written by Ji in [36, Theorem 3.1]. (The idea of this proof is given in the below of Theorem I.10 in §1.2.2.)

Margulis's arithmeticity theorem [51, Theorem 1] says that every irreducible lattice in a semisimple Lie group whose $\mathbb{R}$-rank is greater than one is arithmetic. In this case, the Borel–Serre partial compactification $\overline{X}^{BS}$ is again a cofinite model of the proper classifying space. In summary, the Borel–Serre partial compactification $\overline{X}^{BS}$ is an cofinite model of the proper classifying space $\underline{E}\Gamma$ when $\Gamma$ is either

1. an arithmetic subgroup of a semisimple Lie group defined over $\mathbb{Q}$, or

2. an irreducible lattice in a semisimple Lie group of higher $\mathbb{R}$-rank.

Meanwhile, there are many non-arithmetic lattices in $\mathbb{R}$-rank one semisimple Lie groups (See examples in §1.1.5). A natural question is the following.

**Question I.7.** *Given an arbitrary lattice $\Gamma$ in semisimple Lie group of $\mathbb{R}$-rank one, is there an analogue of the Borel–Serre partial compactification of the corresponding symmetric space which is a cofinite model of the proper classifiying space $\underline{E}\Gamma$?*

Theorem I.1 gives a positive answer to this question. In the rest of this section, we emphasize the following.

1. There are many explicit models of the classifying spaces of various types of groups. We will list all currently known models in §1.1.3.

2. There is a long history of study of lattices in semimsimple Lie groups of $\mathbb{R}$-rank one (§1.1.5). Especially, the existence of non-arithmetic lattices is not yet fully understood for complex hyperbolic spaces.

#### 1.1.3 Explicit models for the classifying spaces

We first consider finite dimensional models of the universal covering space $E\Gamma$ of the classifying space $B\Gamma$. Milnor's method of constructing a model of the space $E\Gamma$ is strong enough to work for arbitrary topological groups. However, in order to show the weak-contractibility of $E\Gamma$, the space $E\Gamma$ must be infinite dimensional [54, Lemma 2.3 and Theorem 3.1]. In [22], Dold showed that a $\Gamma$-CW-complex $X$ with the free $\Gamma$-action is a model for the classifying space $E\Gamma$ if and only if $X$ is contractible [22, Theorem 7.5]. It is natural to ask for which group $\Gamma$ does there exist a finite dimensional model of the classifying space $E\Gamma$. The following are the known facts about explicit models of $E\Gamma$.

1. For groups with torsion elements, there is no finite dimensional model for the classifying space [18, §2.3] (c.f. [45, Theorem 5.1(i)]).

2. Let $X$ be a simply-connected complete Riemannian manifold of non-positive sectional curvature. If a discrete group $\Gamma$ acts properly and isometrically on $X$, then the space $X$ is a model for the proper classifying space $\underline{E}\Gamma$ [1, Theorem 4.15]. If the group $\Gamma$ is torsion-free, then the space $X$ is a model for the classifying space $E\Gamma$.

3. If a torsion-free group $\Gamma$ contains a free subgroup of finite index, then there exists a 1-dimensional model (i.e. tree) for the classifying space $E\Gamma$. More generally, if $\Gamma$ is finitely generated, then $\Gamma$ admits a 1-dimensional model for the classifying space $E\Gamma$ if and only if the cohomological dimension of $\Gamma$ is less than 2 [64, Theorem 0.1], [65, Theorem B].

4. A group is said to be of type $\mathcal{F}_n$ if there exists a $n$-dimensional model of the classifying space (c.f. [72, §1]). In [5], Bestvina and Brady showed that a group of type $\mathcal{F}_n$ is of type $FP_n$ (see [18, p193] for the definition of $FP_n$).

Next, we consider finite dimensional models of the proper classifying space $\underline{E}\Gamma$. Finite dimensional models of the proper classifying spaces have been known for many types of discrete groups $\Gamma$ even for which contain a torsion element. Some finite dimensional models are, in fact, finite, and we emphasize them by underline.

1. Let $\Gamma$ be a discrete subgroups of a Lie group $G$ with finitely many connected

components. For a maximal compact subgroup $K$ of $G$, the homogeneous space $G/K$ is a model for the proper classifying space $\underline{E}\Gamma$ [45, Theorem 4.4].

2. Let $\Gamma$ be a discrete group acting isometrically on a $CAT(0)$-space $X$ with $\Gamma$-CW-structure. Then $X$ is a model for the proper classifying space $\underline{E}\Gamma$ [46, Theorem 1.1(1)].

3. Let $\Gamma$ be a discrete groups acting continuously and freely on a tree $T$. If the isotropy groups of each point $x \in T$ is compact, then $T$ is a model for the proper classifying space $\underline{E}\Gamma$ [45, Theorem 4.7].

4. Let $\Gamma$ be a $p$-adic algebraic group. Then its associated affine Bruhat–Tits building $\beta(\Gamma)$ is a model for the proper classifying space $\underline{E}\Gamma$ [45, Theorem 4.13].

5. Let $\Gamma$ be a discrete group generated by the set $S$ of $n$ generators. The group $\Gamma$ is called *$\delta$-hyperbolic* if the word-metric $d_S$ satisfies the following inequality: for every four points $x, y, z, w \in \Gamma$,

$$d_S(x, y) + d_S(z, w) \leq \max\{d_S(x, z) + d_S(y, w), d_S(x, w) + d_S(y, z)\} + 2\delta.$$

If $n \geq 16\delta + 8$, then the Rips structure $P_n(G, d_S)$ (c.f. [52, Definition 3]) is a model for the proper classifying space of a $\delta$-hyperbolic group $\Gamma$ [52, Theorem 1] (c.f. [8, §2], [45, §4.7]). For more on hyperbolic groups, see [25], [26].

6. Let $\Gamma$ be an arithmetic subgroup of a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. The group $\Gamma$ acts properly on the real loci $G = \mathbf{G}(\mathbb{R})$ and thus on the corresponding symmetric space $X = G/K$ where $K$ is a maximal compact subgroup of $G$. The Borel–Serre partial compactification $\overline{X}^{BS}$ is a <u>cofinite</u> model for the proper classifying space $\underline{E}\Gamma$ [2, Remark 5.8], [36, Theorem 3.1], [45, §4.8]. For the uniform construction of $\overline{X}^{BS}$, see [15] or §2.3.4 of this thesis.

7. Let $F_n$ be the free group of rank $n$. Let $\Gamma = Out(F_n)$ be the group of outer automorphisms of $F_n$, i.e. $\Gamma = Aut(F_n)/Inn(F_n)$. The group $Out(F_n)$ acts on the outer space $X_n$ (c.f. [20, §0], [71, §1.2]) properly. The outer space $X_n$ is a model of the proper classifying space $\underline{E}\Gamma$, and the spine $K_n$ of $X_n$ is a cofinite model of the proper classifying space $\underline{E}\Gamma$. [43, Theorem 8.1], [74, Theorem 5.1].

8. Let $\Gamma = \mathcal{M}_g^s$ be the mapping class group of an orientable compact surface of genus $g$ with $s$ punctures. Whenever $2g + s > 2$, the Teichmüller space $\mathcal{T}_g^s$ is a model for the proper classifying space $\underline{E}\Gamma$ [41, Theorem 2]. A stronger result is due to Ji and Wolpert: the truncation $\mathcal{T}_g^s(\epsilon)$ of the Teichmüller space $\mathcal{T}_g^s$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$ [37, Theorem 1.3].

### 1.1.4 Applications of cofinite models

As mentioned in the beginning of Chapter I, the proper classifying spaces are used in formulations of many conjectures such as the Farrell–Jones conjecture, the Baum–Connes conjecture, and the integral Novikov conjecture. (For more on the Baum–Connes conjecture, see [55], and for the integral Novikov conjecture, see [68].) The existence of a cofinite model for the classifying spaces has applications on the Novikov conjecture. Yu showed in [75] that if a discrete group $\Gamma$ has finite asymptotic dimension (c.f. [35, §2]) and admits a cofinite classifying space $E\Gamma$, then the Novikov conjecture is true [75, Theorem 1.1].

**Theorem** (Yu)**.** *Let $\Gamma$ be a finitely generated group whose classifying space $E\Gamma$ is a cofinite $\Gamma$-CW-complex. If $\Gamma$ has finite asymptotic dimension as a metric space with word-length metric, then the Novikov conjecture holds for $\Gamma$.*

Another application of the existence of cofinite model for the proper classifying space is to the integral Novikov conjecture. Rosenthal showed in [61] that if a discrete group Γ admits a cofinite classifying space $\underline{E}\Gamma$ with small topology at infinity, then the integral Novikov conjecture is true [61, Theorem 6.1]. Together with a result of Bartels and Rosenthal in [7], Ji showed the following [36, Theorem 1.2].

**Theorem I.8** (Bartels, Ji, Rosenthal)**.** *Let* $\Gamma$ *be a discrete group with finite asymptotic dimension, and admits a cofinite model of the proper classifying space* $\underline{E}\Gamma$*. If for every pair of subgroups* $I \subseteq H$ *of* $\Gamma$*, the fixed point set* $X^I$ *and the quotient* $N_H(I)\backslash X^I$ *are uniformly contractible and of bounded geometry, then the Novikov conjecture is true for* $\Gamma$*.*

### 1.1.5 Lattices in semisimple Lie groups of $\mathbb{R}$-rank one

There are essentially four types of semsimple Lie groups of $\mathbb{R}$-rank one: $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$, and the hyperbolic Cayley surface $F_4^{-20}$ (c.f. [19, §0], [29, Introduction], [56, 8$F$]). We present some examples of lattices in these Lie groups.

#### Lattices in $SO(n,1)$

We first consider the case $n = 2$. The Lie group $SO(2,1)$ is isomorphic to $SL(2,\mathbb{R})$. The Möbius transformation of the group $SL(2,\mathbb{R})$ on the upper half-plane **H** has been studied in analysis, number theory, and geometry since 1800s. By the Gauss-Bonnet theorem, the Dirichlet fundamental domain $D$ for any lattice in $SL(2,\mathbb{R})$ is geometrically finite, i.e. its boundary consists of finitely many geodesic sides (c.f. [39, Theorem 4.1.1]). In the mid 1800s, Poincaré asked the converse question: given a polygon $P$, how do we construct a lattice whose fundamental domain is $P$? The following theorem answers this question. (See [9, Theorem 9.8.4] or [39, Theorem 4.3.2] for the proof of the theorem.)

**Theorem I.9** (Poincaré side pairing theorem)**.** *Let* $P$ *be a geometrically finite polygon such that*

1. *all vertices has angle $\pi/m$ where $m \in \mathbb{N}$, or $m = \infty$ if the vertice is at infinity,*

2. *for each side $s$ there exists an element $\gamma_s \in SL(2, \mathbb{R})$ such that $\gamma_s \cdot s$ is another sides of $P$,*

3. *no side is mapped to itself by $\gamma_s$ above.*

*Then the group generated by $\gamma_s$ is a discrete lattice in $SL(2, \mathbb{R})$ whose fundamental domain is $P$.*

Although it is known that there are infinitely many non-arithmetic lattices in $SO(2, 1)$, there were few examples. Takeuchi proved a criterion for the arithmeticity of triangle Fuchsian groups, and classified all arithmetic Fuchsian lattices of compact and non-compact types [66].

For the $n = 3$ case, the group $SO(3, 1)$ is isomorphic to $SL(2, \mathbb{C})$. The group $SL(2, \mathbb{C})$ acts on the 3-dimensional hyperbolic space $\mathbf{H}^3$. Makarov [50] constructed some examples of non-arithmetic lattices in $SO(n, 1)$ for $3 \leq n \leq 5$, generated by reflections in the sides of triangular prisms in $\mathbf{H}^3$. Later, Maclachlan and Reid constructed tetrahedral non-arithmetic lattices in $SO(3, 1)$ [48]. For more examples of lattices in $SO(n, 1)$ for $6 \leq n \leq 10$, see Ruzmanov's construction in [62].

For higher dimensions, Vinberg introduced the hyperbolic reflection group in $n$-dimensional hyperbolic space [69]. Given a polyheron $P$ bounded by hyperbolic planes with certain property, the group generated by reflections in the hyperplanes 'tiles' the hyperbolic space $\mathbf{H}^n$. Moreover, he stated a criterion of arithmeticity of such reflection groups. However, there is an upper bound on the dimension $n$ where such groups exist [70].

Construction of non-arithmetic lattices in all dimension is achieved by Gromov and Piatetski-Shapiro in [28] . They constructed non-arithmetic lattices in $SO(n, 1)$

for all $n$ by 'hybridating' two arithmetic hyperbolic manifolds (c.f. [3, §6.3.3], [59]).

A criterion for the arithmeticity of lattices in $SL(2,\mathbb{R})$ is established by Weil in [73]. In [66, Theorem 1], Takeuchi proved a criterion of the arithmeticity of triangle Fuchsian groups. In [48, Theorem 8.3.2], Maclachlan and Reid stated a general criterion of the arithmeticity of Kleinian lattices.

**Lattices in $SU(n,1)$**

Relatively few examples of non-arithmetic lattices in $SU(n,1)$ are known. In [57], Mostow constructed examples of non-arithmetic lattices in $SU(2,1)$ using the notion of complex reflection. Later, Deligne and Mostow showed the existence of non-arithmetic lattices in $SU(3,1)$ [21] using the notion of generalized Picard lattices.

**Lattices in $Sp(n,1)$ and the Cayley surface $F_4^{-20}$**

Results on supperrigidity for lattices are established by Corlette in [19] for archimedean case and by Gromov and Schoen in [29] for $p$-adic case. We refer [63], [76], and [27] for more on the theory of rigidity.

## 1.2 Motivations of the main theorem

Let $\Gamma$ be a Fuchsian group, i.e. a discrete subgroup of $SL(2,\mathbb{R})$ acting on the upper half-plane $\mathbf{H}$ by Möbius transformation. The Dirichlet fundamental domain (c.f. Definition II.43) is a 2-dimensional polygon with finitely many vertices whose 1-dimensional sides are geodesic segments (c.f. [39]). The $\Gamma$-translation of the Dirichlet fundamental domain gives a tessellation of the space $\mathbf{H}$, and this tessellation induces a $\Gamma$-CW-structure of $\mathbf{H}$. Similarly, one can generalize this idea of tessellation to Kleinian groups.

In this section, we will investigate the main theorem (Theorem I.1) with examples of Fuchsian groups, Kleinian groups.

### 1.2.1 Constructing cofinite models for arithmetic Fuchsian groups

Let $\Gamma$ be a Fuchsian lattice, i.e. a discrete subgroup of $SL(2,\mathbb{R})$ such that the fundamental domain in the upper half-plane $\mathbf{H}$ has finite volume. If the group $\Gamma$ acts cocompactly on the space $\mathbf{H}$, then the tessellation of $\mathbf{H}$ is cofinite, so $\mathbf{H}$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$. Suppose $\Gamma$ is non-cocompact. A non-cocompact Fuchsian lattices is arithmetic if and only if it is commensurable with the modular group $SL(2,\mathbb{Z})$. For simplicity, we will assume that $\Gamma$ is the modular group $SL(2,\mathbb{Z})$ throughout this section. The Dirichlet fundamental domain $D$ of the Fuchsian group $\Gamma$ is defined as follow.

$$D = \{x \in X \mid d(x, \gamma x_0) \geq d(x, x_0) \text{ for all } \gamma \in \Gamma\}. \tag{1.3}$$

The Dirichlet fundamental domain of the modular group is described in Figure 1.2. To construct a cofinite model of the proper classifying space, we do the following steps.

1. Tesselate the space **H** by translating a fixed Dirichlet fundamental domain of $\Gamma$.

2. Induce a $\Gamma$-CW-structure from the tessellation above. Note that this structure is not cofinite because $\Gamma$ is non-cocompact.

3. Compactify the space **H** to get a cofinite $\Gamma$-CW-structure of **H** and check if the compactified space is the proper classifying space for $\Gamma$.

Figure 1.2 below shows the tessellation of the upper half-plane **H**. Recall that

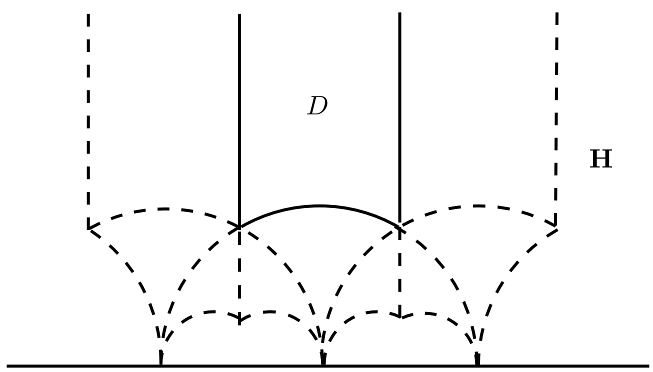


Figure 1.2: Tessellation of **H** by non-compact triangle domain

every cell in a CW-complex is homeomorphic to $D^n$, which is compact. Since the domain $D$ is not compact, this tessellation does not directly give a $\Gamma$-CW-structure on **H**. To obtain a $\Gamma$-CW-structure, we need a further tessellation of $D$ in an equivariant way. That is, the tessellations of two adjacent translations $\gamma_1 \cdot D$ and $\gamma_2 \cdot D$ $(\gamma_1, \gamma_2 \in \Gamma)$ induced from that of $D$ must coincide on the intersection. This yields a $\Gamma$-CW-structure on **H**, but not a cofinite $\Gamma$-CW-structure.

Another way to obtain a $\Gamma$-CW-complex is to compactify the open domain $D$ first, and then do similarly for all translations $\gamma \cdot D$ for $\gamma \in \Gamma$ equivariantly. One method is to assign a point at infinity for each vertex at infinity, as in the Figure 1.3.

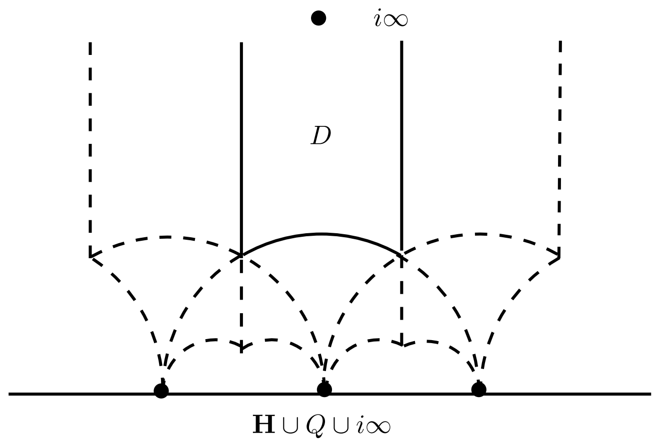


Figure 1.3: The rational Satake compactification of $\mathbf{H}$

The arithmeticity of the modular group $SL(2, \mathbb{Z})$ implies that the set of all vertices at infinity is the union of raional numbers and the point at infinity $i\infty$. The partially compactified space

$$\overline{\mathbf{H}}^S = \mathbf{H} \cup \mathbb{Q} \cup \{i\infty\} \tag{1.4}$$

is then a $\Gamma$-CW-complex. This is called the *(rational) Satake compactification* of $\mathbf{H}$. For every (non-cocompact) arithmetic subgroup $\Gamma$ of $SL(2, \mathbb{R})$, the space $\overline{\mathbf{H}}^S$ is a $\Gamma$-CW-complex. In fact, $\overline{\mathbf{H}}^S$ is a cofinite $\Gamma$-CW-complex.

However, $\overline{\mathbf{H}}^S$ is not a model for the proper classifying space $\underline{E}\Gamma$. The stabilizer of any point at infinity is an infinite group. For example, the stabilizer of $i\infty$ is the subgroup of upper triangle matrices in $SL(2, \mathbb{Z})$. Therefore, the $\Gamma$-action on the boundary of $\overline{\mathbf{H}}^S$ is not proper.

The goal is to compactify the open domain $D$ well such that the proper action of $\Gamma$ on $\mathbf{H}$ extends to the boundary. Instead of assigning a point at infinity for each vertex at infinity, let us assign a line at infinity as in the Figure 1.4.

Note that each geodesic lying in the domain $D$ uniquely determines a point on the line at infinity in the following way: The only semi-infinite geodesics in $D$ are

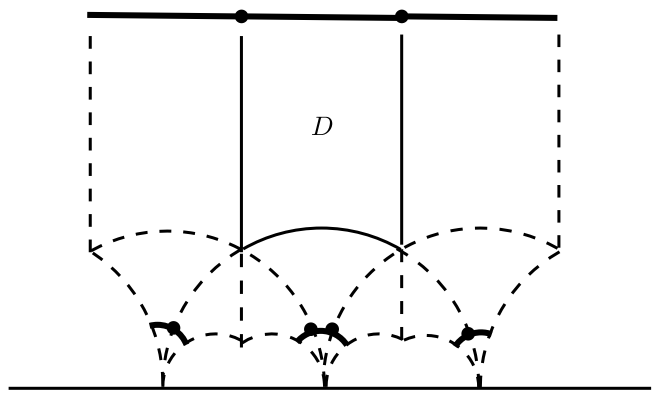


Figure 1.4: Borel–Serre partial compactification of $\mathbf{H}$

vertical geodesics. Each vertical geodesic has fixed horizontal coordinate, and this coordinate corresponds to the point on the line at infinity. Since the action of $\Gamma$ is isometric, every geodesic in $\mathbf{H}$ is mapped to another geodesic in $\mathbf{H}$. Thus the $\Gamma$-action on semi-infinite geodesics extends to the $\Gamma$-action on the points on the lines at infinity.

Later, we will prove that this action is indeed proper. Each boundary component is homeomorphic to $\mathbb{R}$, and attached to a rational points at infinity including $i\infty$. The union

$$\overline{\mathbf{H}}^{BS} = \mathbf{H} \cup \coprod_{\mathbb{Q}\cup\{i\infty\}} \mathbb{R}$$

is called the *Borel–Serre partial compactification of* $\mathbf{H}$. Note that each line segment at infinity connects the common vertices of two adjacent $\Gamma$-translations of $D$. Therefore, the quotient $\Gamma\backslash\overline{\mathbf{H}}^{BS}$ is compact. The space $\overline{\mathbf{H}}^{BS}$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$. In fact, for any arithmetic Fuchsian group, the space $\overline{\mathbf{H}}^{BS}$ is a model of the proper classifying space for that group.

#### 1.2.2 The Borel–Serre partial compactifications for arithmetic Fuchsian groups

Let us investigate the Borel–Serre partial compactification of $\mathbf{H}$ further. Let $\xi$ runs through all rational points at infinity including $i\infty$. For each $\xi$, let us denote $e(P_\xi)$

a line component homeomorphic to $\mathbb{R}$ where $P_\xi$ is the maximal parabolic subgroup of $SL(2,\mathbb{R})$ corresponding to $\xi$. The component $e(P_\xi)$ is called a *rational boundary component*, and will be discussed in detail in §2.3.4.

The interior of $\overline{\mathbf{H}}^{BS}$ is the upper half-plane $\mathbf{H}$, whose topology is the usual topology of $\mathbf{H}$. The topology of each boundary component $e(P_\xi)$ is the Euclidean topology of $\mathbb{R}$. In previous section, we observed that every semi-infinite vertical geodesic ray corresponds to a point on the boundary attached to $i\infty$. In the topology of $\overline{\mathbf{H}}^{BS}$, every unbounded sequence in a vertical geodesic ray pointing out $i\infty$ converges to the point on the boundary component corresponding to that geodesic ray. To help understanding, see Figure 1.5 where the opposite direction of the arrow indicates how a unbounded sequence converges. Likewise, the set $e(P_\xi)$ is parametrized by the geodesics diverging (in terms of the topology of $\mathbf{H}$) to $\xi$. A unbounded sequence lying on a geodesic $\gamma(t)$ diverging (in the topology of $\mathbf{H}$) to $\xi$ converges (tn the topology of $\overline{\mathbf{H}}^{BS}$) to the point in $e(P_\xi)$ corresponding to $\gamma(t)$.

To show that the space $\overline{\mathbf{H}}^{BS}$ is a model for the proper classifying space $\underline{E}\Gamma$, we first show that $\overline{\mathbf{H}}^{BS}$ is contractible. Let $\zeta$ be a point in $e(P_\xi)$ corresponding to the geodesic ray $\gamma(t)$. Then the point $\zeta$ is retracted into the interior $\mathbf{H}$ along the geodesic $\gamma(t)$ as in the Figure 1.5 below.

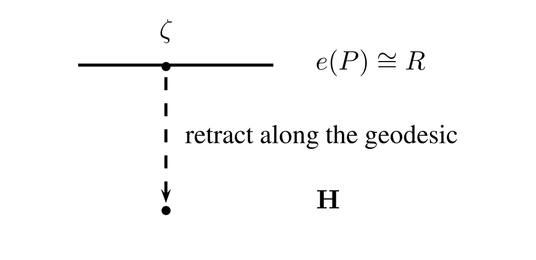


Figure 1.5: The retraction of boundary point in $\overline{\mathbf{H}}^{BS}$

Next, we need to show that for every finite subgroup $H$ of $SL(2,\mathbb{Z})$, the fixed point set $(\overline{\mathbf{H}}^{BS})^H$ is contractible. Elliptic or parabolic elements of $\Gamma$ act on each boundary component without fixed point. In fact, only hyperbolic elements fixes a point on the boundary. However, the discreteness of $\Gamma$ implies that $\Gamma$ does not contain any hyperbolic element (c.f. Lemma IV.16). In fact, no finite subgroup of $SL(2,\mathbb{Z})$ fixes a point at infinity except the trivial subgroup. Thus for any non-trivial finite subgroup $H$ of $\Gamma$,

$$(\overline{\mathbf{H}}^{BS})^H = \mathbf{H}^H. \tag{1.5}$$

Since $\mathbf{H}^H$ is a one-point set, it is contractible.

#### 1.2.3 Contractibility of the general Borel–Serre partial compactification

Let us investigate the contractibility of the Borel–Serre partial compactification in general. Let $\mathbf{G}$ be a semisimple Lie group defined over $\mathbb{Q}$ and $\Gamma$ be a arithmetic subgroup of the real locus $G = \mathbf{G}(\mathbb{R})$. Let us fix a maximal compact subgroup $K$ of $G$ and denote $X = G/K$ the associate symmetric space. As mentioned earlier, the space $\overline{X}^{BS}$ is the proper classifying space for any arithmetic group $\Gamma$ in $G$. The contractibility of the space $\overline{X}^{BS}$ is proved rigorously in the following theorem of Ji.

**Theorem I.10** (Ji [36]). *Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and $G = \mathbf{G}(\mathbb{R})$ be its real locus. For a maximal compact subgroup $K \subset G$, let $X = G/K$ be a symmetric space. For every arithmetic subgroup $\Gamma$ of $G$, the Borel–Serre partial compactification $\overline{X}^{BS}$ is a model for the cofinite classifying space $\underline{E}\Gamma$.*

*Outline of the proof.* The symmetric space $X = G/K$ is a proper $\Gamma$-manifold and the Borel–Serre partial compactification $\overline{X}^{BS}$ is a cocompact $\Gamma$-manifold with corners. Thus there exists a smooth $\Gamma$-equivariant triangulation of $\overline{X}^{BS}$ (c.f. [33]). All isotropy groups are compact subgroups of $\Gamma$. Since $\Gamma$ is discrete, they are finite. For any finite subgroup $H$, the fixed point set is expressed as follow.

$$\overline{X}^H = X^H \cup \coprod_{\mathbf{P}:\ \text{rational}} e(\mathbf{P})^H. \tag{1.6}$$

Suppose $\Gamma$ is not torsion-free and $H$ is non-trivial. The Langlands decomposition of $P = \mathbf{P}(\mathbb{R})$

$$P = N_P \times A_\mathbf{P} \times M_\mathbf{P} \tag{1.7}$$

induces a horopherical decomposition of $X$

$$X = N_p \times A_\mathbf{P} \times X_\mathbf{P} \tag{1.8}$$

where $X_\mathbf{P} = M_\mathbf{P}/(K \cap M_\mathbf{P})$. From Equation (1.6),

$$X^H = Z_{N_P}H \times A_\mathbf{P} \times (X_\mathbf{P})^H, \quad e(\mathbf{P})^H = Z_{N_P}H \times (X_\mathbf{P})^H. \tag{1.9}$$

Thus every point in $e(\mathbf{P})^H$ is contracted into $X^H$ along the geodesic ray parametrized by the $A_\mathbf{P}$-coordinate. Since $X^H$ is a geodesic submanifold of $X$, it is convex and contractible. □

#### 1.2.4 The idea of constructing cofinite models for general Fuchsian lattices

The proof of contractibility of the Borel–Serre partial compactification rely on the choice of parabolic subgroups which determines the boundary components. For example, we choose rational parabolic subgroups for arithmetic subgroups. As observed in §1.1, if $\Gamma$ is an irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank greater than one, the Borel–Serre partial compactification $\overline{X}^{BS}$ is a model for the proper classifying space $\underline{E}\Gamma$ due to Margulis's arithmeticity theorem. Thus we focus on the $\mathbb{R}$-rank one case. As a refinement of Question I.7, we ask the following question.

**Question I.11.** *For a non-arithmetic, non-cocompact lattice $\Gamma$ in a semisimple Lie group $G$ of $\mathbb{R}$-rank one, how can we compactify the symmetric space $X = G/K$,* *or equivalently, how to choose correct parabolic subgroups of $G$, to obtain a cofinite $\Gamma$-CW-complex model for the proper classifying space $\underline{E}\Gamma$?*

Before we attempt to answer this question in full generality, let us consider a special case of Fuchsian lattices. The structure of Dirichlet fundamental domain (1.3) for $\Gamma$ played crucial role to determine boundary components. Let us investigate the structure of Dirichlet fundamental domains for Fuchsian lattices.

1. This fundamental domain is a precise fundamental domain in the sense that for every $\gamma \in \Gamma$ such that $\gamma \neq Id$, the intersection $\gamma \cdot D \cap D$ always lies on the boundary of $D$.

2. Whenever $\Gamma$ is a Fuchsian lattice, a Dirichlet fundamental domain for $\Gamma$ is bounded by only finitely many geodesic sides. In other words, $\Gamma$ is **geometrically finite**.

3. When two adjacent geodesic sides have angle 0, their common vertex lies at a point at infinity, $\mathbb{R} \cup \{i\infty\}$, and such vertex is called a **vertex at infinity**. The geometric finiteness implies there are only finitely many vertices at infinity of a Dirichlet fundamental domain of any Fuchsian lattice.

4. Let $\zeta$ be a vertex at infinity of $D$ and $P_\zeta$ be the $\mathbb{R}$-parabolic subgroup of $SL(2,\mathbb{R})$ which fixes $\zeta$. We say $P_\zeta$ is a **parabolic subgroup corresponding to** $\zeta$. In this case, the subgroup $\Gamma_\zeta = \Gamma \cap P$ is non-trivial, and consists of all parabolic elements fixing the point $\zeta$. Moreover, the group $\Gamma_\zeta$ acts on the boundary component $e(P_\zeta)$ cocompactly. For example, Figure 1.6 below shows the the action of an element $\gamma$ in $\Gamma_\zeta$ on $e(P_\zeta)$ when $\zeta = i\infty$.

Let $\zeta_1, \cdots, \zeta_r$ be all vertices at infinity of $D$ and $P_{\zeta_n}$ $(n = 1, \cdots, r)$ be the $\mathbb{R}$-parabolic subgroup corresponding to $\zeta_n$. Note that $P_{\zeta_n}$ is a $\mathbb{Q}$-parabolic subgroup if

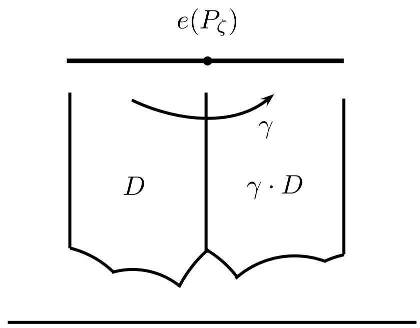


Figure 1.6: Cocompact action on the boundary

and only if $\zeta_n \in \mathbb{Q} \cup \{i\infty\}$. Let $e(P_{\zeta_n})$ be the boundary component corresponding to $P_{\zeta_n}$. Let us consider the union

$$\overline{\mathbf{H}}_\Gamma = \mathbf{H} \cup \coprod_{n=1}^{r} \Gamma \cdot e(P_{\zeta_n}). \tag{1.10}$$

We give a topology on $\overline{\mathbf{H}}_\Gamma$ similar to that of $\overline{\mathbf{H}}^{BS}$. The interior is the upper half-plane $\mathbf{H}$, and every unbounded sequence diverging to $\zeta_n$ along a geodesic $s(t)$, $t \geq 0$, converges to a point in $e(P_{\zeta_n})$ parametrized by $s(t)$. The subgroup $\Gamma_{\zeta_n}$ acts cocompactly on the boundary component $e(P_{\zeta_n})$. Let $I_n$ be the compact fundamental domain of $\Gamma_{\zeta_n}$ in $e(P_{\zeta_n})$ and let us consider the following union.

$$\overline{D} = D \cup \coprod_{n=1}^{r} I_n. \tag{1.11}$$

In Chapter III, we will define the topology and the $\Gamma$-action rigorously, and show that $\overline{D}$ is indeed a compact fundamental domain of $\Gamma$. Thus the space $\overline{\mathbf{H}}_\Gamma$ is a cocompact $\Gamma$-space, and has a cofinite $\Gamma$-CW-structure. With arguments similar to those in the previous section, one can show that the space $\overline{\mathbf{H}}_\Gamma$ is contractible. In fact, the situation is much simpler than the general case. For any finite subgroup $H$ of $\Gamma$, the fixed point set $\overline{\mathbf{H}}_\Gamma^H$ is equal to the fixed point set of the interior $\mathbf{H}^H$. Since $\mathbf{H}^H$ is a geodesic submanifold of $\mathbf{H}$, it is contractible.

#### 1.2.5 The idea of constructing cofinite models for general lattices in semisimple Lie groups of $\mathbb{R}$-rank one

We can generalize the idea of constructing $\overline{\mathbf{H}}_\Gamma$ for Fuchsian lattices to the case of Kleinian lattice, i.e. a discrete lattice subgroup of $SL(2,\mathbb{C})$. In this case, the symmetric space is modeled by the 3-dimensional hyperbolic space $\mathbf{H}^3$,

$$\mathbf{H}^3 = \{z + jy \in \mathbb{C} \times \mathbb{R} \mid z \in \mathbb{C},\, y > 0\}. \tag{1.12}$$

whose group of (orientation preserving) isometries is isomorphic to $PSL(2,\mathbb{C})$. Let $\Gamma \subset SL(2,\mathbb{C})$ be a non-arithmetic lattice and $D$ be the Dirichlet fundamental domain of $\Gamma$ at $j$. The geometric finiteness of $\Gamma$ (c.f. [17]) implies that $D$ is bounded by finitely many geodesic sides. Thus there are only finitely many cusps in $D$. Each cusp contains a geodesic ray which diverges to a point at infinity, which is also called a vertex at infinity. See Figure 1.7 for an example of a vertex at infinity. The action of $SL(2,\mathbb{C})$ extends continuously to $\mathbb{C} \cup \{j\infty\}$. For each vertex at infinity $\zeta$ of $D$, let $P_\zeta$ be the $\mathbb{R}$-parabolic subgroup fixing $\zeta$. The action of $\Gamma \cap P_\zeta$ on $e(P_\zeta)$ is again cocompact. Given the Langlands decomposition $P_\zeta = N_\zeta \times A_\zeta \times M_\zeta$, The boundary component $e(P_\zeta)$ is the nilpotent subgroup $N_\zeta$ which is diffeomorphic to $\mathbb{R}^2$, and the group $\Gamma \cap N_\zeta$ is a uniform lattice in $N_\zeta$.

Let $P$ runs over all $\mathbb{R}$-parabolic subgroup corresponding to all points at infinity which is $\Gamma$-equivalent to a vertex at infinity of $D$. Define

$$\overline{\mathbf{H}^3}_\Gamma = \mathbf{H}^3 \cup \coprod_P \Gamma \cdot e(P). \tag{1.13}$$

The quotient $\Gamma \backslash \overline{\mathbf{H}^3}_\Gamma$ is homeomorphic to the union of $D$ and finitely many compact fundamental domains in the boundary. Thus $\overline{\mathbf{H}^3}_\Gamma$ is a cocompact $\Gamma$-space, and a suitable CW-structure on the quotient gives the $\Gamma$-CW-structure on $\overline{\mathbf{H}^3}_\Gamma$.

In [24], Garland and Raghunathan studied the structure of Dirichlet fundamental

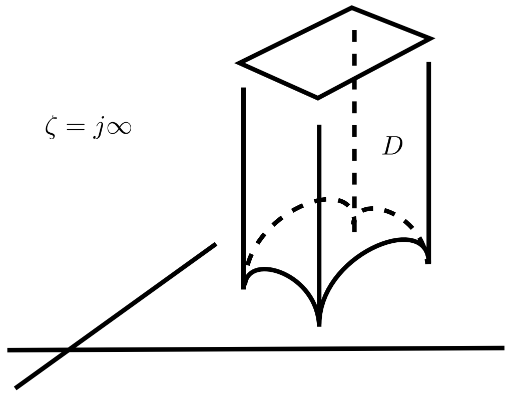


Figure 1.7: A boundary components of $\overline{\mathbf{H}^3}_\Gamma$

domain for lattices in semisimple Lie group of $\mathbb{R}$-rank one. Their main theorem is the following.

**Theorem I.12** (Garland–Raghunathan [24]). *Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one and $K$ be a maximal compact subgroup of $G$. For a lattice $\Gamma$ in $G$ and its Dirichlet fundamental domain $D$ in $X$, there exists a compact subset $C$ of $D$ and finitely many Siegel sets $\mathfrak{S}_{P_n,U_n,t}$ such that the union $\Omega$ defined by*

$$\Omega = C \cup \coprod_{n=1}^{r} \mathfrak{S}_{P_n,U_n,t} \tag{1.14}$$

*is a fundamental set for $\Gamma$. Moreover, the set*

$$\{\gamma \in \Gamma \mid \gamma \cdot \Omega \cap \Omega \neq \emptyset\}$$

*is finite.*

The last condition implies that the quotient $\Gamma\backslash X$ is homeomorphic to the set of $\Gamma$-orbits $\Omega/\sim$. The finite union of Siegel sets in (1.14) implies that there is only finitely many cusps in $D$. The subgroup $\Gamma \cap N_P$ act on $N_P$ cocompactly. Thus, to compactify $D$, we may attach only finitely many compact sets, which are fundamental domains of $\Gamma \cap N_P$ in $e(P)$.

Let $P$ runs through all $\mathbb{R}$-parabolic subgroups which are $\Gamma$-equivalent to one of parabolic subgroups $P_1, \cdots, P_r$. Motivated by two special observations of $\mathbf{H}$ and $\mathbf{H}^3$ (Equation (1.10) and (1.13)), we construct a $\Gamma$-space

$$\overline{X}_\Gamma = X \cup \coprod_P \Gamma \cdot e(P). \tag{1.15}$$

The proof of the main theorem in Chapters III and IV shows that $\overline{X}_\Gamma$ is indeed a cofinite $\underline{E}\Gamma$. In the next section, we will outline this proof, with illustrations of examples in the Fuchsian case.

## 1.3 Idea of the proof of the main theorem

In this section, we present the idea of the proof of Theorem I.1. The proof consists of the following steps. In §1.3.1, we construct a partially compactified space $\overline{X}_\Gamma$ from the symmetric space $X$ by attaching 'rational' (c.f. Remark I.13) boundary components. In §1.3.2, we then show that the proper $\Gamma$-action on $X$ extends continuously to the boundary, and the $\Gamma$-action on the space $\overline{X}_\Gamma$ is again proper. In §1.3.3, we show that $\overline{X}_\Gamma$ is a cofinite $\Gamma$-CW-complex. Lastly, in §1.3.4, we show that $\overline{X}_\Gamma$ is contractible, thus a model for the cofinite proper classifying space $\underline{E}\Gamma$.

### 1.3.1 The construction of the $\Gamma$-space $\overline{X}_\Gamma$

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{R}$ (c.f. Definition II.19 and Definition II.21). We assume that the $\mathbb{R}$-rank of $\mathbf{G}$ is one (See Section 2.2.2 for the $\mathbb{R}$-rank). Then the real locus $G = \mathbf{G}(\mathbb{R})$ is the semisimple Lie group and the corresponding symmetic space $X := G/K$ is of rank one, i.e. the maximal dimension of any flat submanifold is one. Let $\Gamma \subset G$ be a discrete lattice subgroup, i.e. induced from the Haar measure $\mu$ of $G$, the volume of quotient $\Gamma\backslash G$ is finite. Let $P$ denote a $\mathbb{R}$-parabolic subgroup of $G$ and

$$P = N_P \times A_P \times M_P \tag{1.16}$$

be the (real) Langlands decomposition of $P$ which induces the horospherical decomposition of $X$ as follow:

$$X = N_P \times A_P \times X_P. \tag{1.17}$$

Since $G$ is of rank one, all parabolic subgroups are maximal (Proposition II.60). Thus $M_P \subset K$ and the horospherical decomposition (1.17) reduces to

$$X = N_P \times A_P. \tag{1.18}$$

**Definition.** The set $\Delta_\Gamma$ is a collection of all $\mathbb{R}$-parabolic subgroups $P$ such that the intersection $\Gamma \cap N_P$ is a cocompact lattice in $N_P$.

For each $P \in \Delta_\Gamma$, we define a boundary component $e(P)$ as follow.

$$e(P) = N_P. \tag{1.19}$$

*Remark* I.13. In [5], Baily and Borel defined the notion of *rational boundary component* as follow. Let $G$ be a semimsimple Lie group defined over $\mathbb{Q}$ and $X$ be the corresponding symmetric space. For each boundary component $F$ (c.f. [5, §1.5]) of $X$, let

$$N(F) = \{g \in G \mid g \cdot F = F\}, \quad Z(F) = \{g \in G \mid g \cdot f = f,\, f \in F\},$$

and $U(F)$ be the unipotent radical of $N(F)$. Then $F$ is called *rational* if

1. the quotient $U(F)/(U(F) \cap \Gamma)$ is compact, and
2. the image of $\Gamma \cap N(F)$ in $N(F)/Z(F)$ is discrete.

In this notation, the boundary component $e(P)$ is rational. For $F = e(P)$, the group $N(F)$ and $A(F)$ is the parabolic subgroup $P$ and $A_P$ respectively, and $U(F)$ is the unipotent subgroup $N_P$.

Let us define a space $\overline{X}_\Gamma$ as follow.

$$\overline{X}_\Gamma = X \cup \coprod_{P \in \Delta_\Gamma} e(P). \tag{1.20}$$

The topology of $\overline{X}_\Gamma$ is defined by the *convergence class of sequence* (c.f. §2.3.3) generated by the following sequences.

1. every convergent sequence in $X$,
2. every convergent sequence in each boundary component $e(P)$ of $\overline{X}_\Gamma$, and

3. every unbounded sequence $z_j$ in $X$ such that $z_j = (n_j, a_j)$ in terms of the horospherical decomposition of $X$ with respect to some $P \in \Delta_\Gamma$ and $n_j \to n_\infty$ and $a_j^\alpha \to \infty$ for all restricted root $\alpha \in \Phi(P, A_P)$ (c.f. Definition II.15). In this case, $z_j$ converges to $n_\infty$.

A sequence of the third type is illustrated in the following example.

**Example I.14.** Suppose $G = SL(2, \mathbb{R})$ and $X = \mathbf{H}$. Let $P_\infty$ be the standard parabolic subgroup:

$$P_\infty = \left\{ \left( \begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mid a \neq 0,\, b \in \mathbb{R} \right\}. \tag{1.21}$$

The $N_{P_\infty}$-component of the Langland decomposition of $P$ is

$$N_{P_\infty} = \{ \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \mid b \in \mathbb{R} \}.$$

Let $z_j = x_j + iy_j$ be the sequence in $\mathbf{H}$ such that $y_j \to \infty$ and $x_j \to x_\infty \in \mathbb{R}$. That is, the sequence $z_j$ diverges to $i\infty$ (in the topology of $\mathbf{H}$) in the direction of a vertical geodesic ray whose $x$-coordinate is $x_\infty$. In terms of horospherical decomposition of $\mathbf{H}$ with respect to $P_\infty$,

$$z_j = \left( \begin{smallmatrix} 1 & x_j \\ 0 & 1 \end{smallmatrix} \right) \times \begin{pmatrix} \sqrt{y_j} & 0 \\ 0 & 1/\sqrt{y_j} \end{pmatrix} \to \left( \begin{smallmatrix} 1 & x_\infty \\ 0 & 1 \end{smallmatrix} \right) \in N_{P_\infty}.$$

In fact, every unbounded sequence converging to a point in $e(P_\infty)$ is of this form.

We also give an open basis of the topology of $\overline{X}_\Gamma$. Every open set of $\overline{X}_\Gamma$ is generated by the following types of open subsets.

1. Open subsets in $X$, and
2. for an open subset $U$ of a boundary component $e(P)$ for some $P \in \Delta_\Gamma$, the union

$$\mathfrak{S}_{P,U,t} \cup U.$$

We will use these open sets to show that $\overline{X}_\Gamma$ is a manifold with boundary (Proposition IV.26).

### 1.3.2 The Γ-action on $\overline{X}_\Gamma$

Since the group $\Gamma$ is a subgroup of the Lie group $G$, it acts canonically on the symmetric space $X = G/K$:

$$\gamma \cdot gK = \gamma gK, \quad \gamma \in \Gamma, g \in G. \tag{1.22}$$

To extend this action naturally to the boundary, we need to reformulate the $\Gamma$-action in terms of the Iwasawa decomposition of $G$ and the horospherical decomposition of $X$. Let $P$ be a parabolic subgroup in $\Delta_\Gamma$ and $G = N_P \times A_P \times K$ be the Iwasawa decomposition of $G$. Let $(n, a, k)$ be the coordinate of the element $\gamma \in \Gamma$ in this Iwasawa decomposition. Let $(n', a')$ be the horospherical coordinate of the point $z \in X$ with respect to $P$. It is crucial that the parabolic subgroups for the horospherical decomposition is the same as that for the Iwasawa decomposition. In Proposition IV.11, we will show that the action (1.22) is written as follows.

$$(n, a, k) \cdot (n', a') = ({}^{k}(n\, {}^{a}n'), {}^{k}(aa')) \tag{1.23}$$

The upper script on the left means the conjugation, i.e. ${}^{k}n = knk^{-1}$. Also note that the point $\gamma \cdot z$ is written in terms of horospherical coordinate with respect to ${}^{k}P$.

**Example I.15.** Let $\Gamma$ be a Fuchsian group. Then the Iwasawa decomposition of $SL(2, \mathbb{R})$ with respect to the standard parabolic subgroup $\mathbf{P}_\infty$ is obtained Gram-Schmidt process on two column vectors of elements in $SL(2, \mathbb{R})$. Let $\gamma \in \Gamma$ be the element of $SL(2, \mathbb{R})$ of the form

$$\gamma = k \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

where $k \in SO(2)$. Let $z$ be the point in $\mathbf{H}$ such that

$$z = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & (a')^{-1} \end{pmatrix} \cdot i.$$

Then by simple calculation, we obtain

$$\gamma \cdot z = k \begin{pmatrix} 1 & b + a^2 b' \\ 0 & 1 \end{pmatrix} k^{-1} \cdot k \begin{pmatrix} aa' & 0 \\ 0 & (aa')^{-1} \end{pmatrix} k^{-1} \cdot i.$$

The $\Gamma$-action on the boundary component is defined as follow: for the point $n'$ on the boundary $e(P)$ and $\gamma = (n, a, k)$ in $\Gamma$,

$$(n, a, k) \cdot n' = {}^{k}(n^{a} n') \tag{1.24}$$

Again, the point $\gamma \cdot n'$ lies on the boundary component $e({}^{k}P)$.

**Example I.16.** In the Fuchsian case, each $n$, $a$, and $k$ coordinate of $\gamma$ represents parabolic, hyperbolic, and elliptic transformations. In terms of the Iwasawa decomposition with respect to the standard parabolic subgroup $\mathbf{P}_\infty$, $n$ corresponds to a horizontal translation, $a$ to a vertical translation, and $k$ to a rotation centered at $i \in \mathbf{H}$.

In Proposition IV.14 and Proposition IV.17, we will show that the $\Gamma$-action (1.23) and (1.24) on the space $\overline{X}_\Gamma$ is continuous and proper.

### 1.3.3 The cofinite $\Gamma$-CW-structure of $\overline{X}_\Gamma$

Showing that $\overline{X}_\Gamma$ is a cofinite $\Gamma$-CW-complex consists of two steps. First, we show that the quotient $\Gamma\backslash\overline{X}_\Gamma$ is compact. We will use the reduction theory of Garland and Raghunathan (c.f. Theorem I.12). Next, we show that there exists a $\Gamma$-CW-structure on $\overline{X}_\Gamma$. This follows from Illman's theorem on the existence of a $\Gamma$-CW-structure on a subanalytic manifold.

Recall that the fundamental set $\Omega$ for the lattice $\Gamma$ is the finite union

$$\Omega = C \cup \coprod_{n=1}^{r} \mathfrak{S}_{P_n,U_n,t}.$$

Each parabolic subgroup $P_n$ corresponds to a semi-infinite geodesic ray lying entirely on a fixed Dirichlet fundamental domain $D$ in Theorem I.12. In [24], Garland and Raghunathan also proved that for each parabolic subgroup $P_n$ and its Langlands decomposition $P_n = N_{P_n} \times A_{P_n} \times M_{P_n}$, the subgroup $\Gamma \cap N_{P_n}$ is a cocompact lattice in $N_{P_n}$.

We will show in Proposition IV.21 that the converse is true. That is, any $\mathbb{R}$-parabolic subgroup $P$ satisfying that the subgroup $\Gamma \cap N_P$ is a cocompact lattice of $N_P$, there exists an element $\gamma \in \Gamma$ such that ${}^{\gamma}P = P_n$ for some $n = 1, \cdots, r$. In other words,

$$\Delta_\Gamma = \Gamma \cdot \{P_1, \cdots, P_n\}.$$

We then show that the closure of $\Omega$ in $\overline{X}_\Gamma$ is indeed a fundamental set of $\Gamma$.

**Example I.17.** Let $\Gamma$ be a Fuchsian lattice. Figure 1.8 describes the closure $\overline{\Omega}$ in $\overline{\mathbf{H}}_\Gamma$. By geometric finiteness (c.f. [39]), the Dirichlet fundamental domain $D$ is bounded by finitely many geodesic sides. Let $I_n$ $(n = 1, \cdots, r)$ be a compact fundamental domain for $\Gamma \cap N_{P_n}$ in $e(P_n)$. Since each boundary component is homeomorphic (in fact, diffeomorphic) to $\mathbb{R}$, the domain $I_n$ is homeomorphic to a compact interval. Thus the disjoint union $D \cup \coprod_{n=1}^{r} I_n$ is a fundamental domain for $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$.

The next step is to show that $\overline{X}_\Gamma$ is a $\Gamma$-CW-complex. In Proposition IV.26, we will show that $\overline{X}_\Gamma$ is a smooth manifold with boundary. The interior of $\overline{X}_\Gamma$ is the symmetric space $X$ and the boundary is the disjoint union $\coprod_{P \in \Delta_\Gamma} e(P)$. Both $X$ and $\coprod_{P \in \Delta_\Gamma} e(P)$ are smooth $\Gamma$-manifolds.

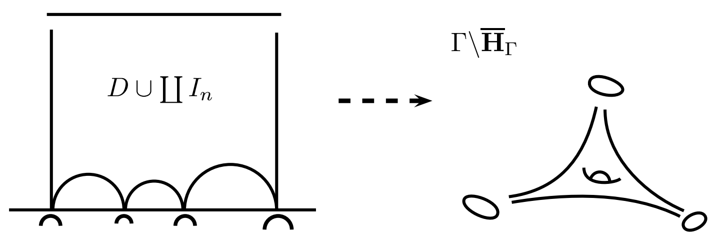


Figure 1.8: Boundary components of $\Gamma\backslash\overline{\mathbf{H}^2}_\Gamma$.

*Remark* I.18. In [60] Raghunathan observed the space $X$ as the interior of a manifold with boundary using the Morse theory. He showed that each cusp neighborhood can be contracted into a relatively compact subset of $X$. Then it follows that there exists a right $K$-invariant Morse function $f : G \to [0, \infty)$ such that its inverse image $f^{-1}[M, \infty)$ with sufficiently large $M \gg 1$ corresponds to the boundary component $\coprod_{P \in \Delta_\Gamma} e(P)$. In this thesis, different method is used. We attach the actual boundary components to $X$, and then show the partially compactified space is a manifold with boundary containing $X$ as the interior.

In [33], Illman showed that any subanalytic (c.f. Definition III.31) proper $\Gamma$-manifold admits a $\Gamma$-CW-structure. (For the complete statement, see Theorem III.32.) By the definition, a manifold with boundary is a subanalytic manifold. As we observed in the previous section, $\overline{X}_\Gamma$ is a proper $\Gamma$-manifold with boundary. Therefore, $\overline{X}_\Gamma$ is a $\Gamma$-CW-complex, and is cofinite.

#### 1.3.4 $\overline{X}_\Gamma$ as model for the proper classifying space

Finally, we conclude that $\overline{X}_\Gamma$ is a model for the proper classifying space $\underline{E}\Gamma$. What remains to show is that for any finite subgroup $H$ of $\Gamma$, the fixed point set $(\overline{X}_\Gamma)^H$ is nonempty and contractible.

We first show that the space $\overline{X}_\Gamma$ itself is contractible. The idea is to construct a homotopy retraction $h_t$ of the fundamental set $\overline{\Omega}$, and extends $h_t$ to the homotopy retraction $H_t$ of the space $\overline{X}_\Gamma$. The homotopy retraction $h_t$ is defined as follow. Each point in the cusp neighborhoods $\overline{\mathfrak{S}}_{P_n,U_n,t_n}$ is retracted along the geodesic direction parametrized by $A_{P_n}$-component. See Figure 1.9 below.

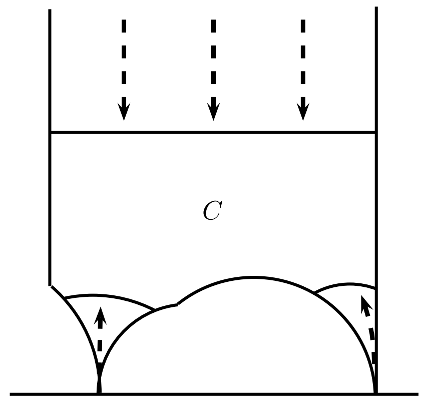


Figure 1.9: The homotopy retraction of $\overline{\Omega}$

By this homotopy retraction, all points on the boundary are retracted into the compact subset $C$ of $\Omega$. The set $C$ is a connected, convex subset of a Dirichlet fundamental domain $D$ of $\Gamma$, thus it is contractible. We then $\Gamma$-equivariantly extend $h_t$ to the homotopy retraction $H_t$ and show that this is well-defined.

Next we show that for any finite subset $H$ of $\Gamma$, the fixed point set $(\overline{X}_\Gamma)^H$ is nonempty and contractible. The fixed point set $(\overline{X}_\Gamma)^H$ nonempty because $H$ fixes at least one point in $X$. In fact, any element of finite order in $\Gamma$ which fixes a point in the boundary component $e(P)$ must lie in the intersection $\Gamma \cap M_P$. Since $M_P$ commutes with $A_P$, such element also fixes the geodesic ray which converges to the fixed point. Therefore, each fixed point in the boundary retracts into the interior $X^H$ along the unique geodesic. Since $X^H$ is a geodesic submanifold, it is contractible.

**Example I.19.** Let us consider the Fuchsian case. If $H$ fixes a point on the boundary $e(P)$, then $H$ cannot contain any parabolic element contained in $P$. This implies $H \cap P = H \cap N_P = \{Id\}$. Since $P$ is an arbitrary parabolic subgroup in $\Delta_\Gamma$, the subgroup $H$ does not fix any point on the boundary.

Finally, from the proper action of $\Gamma$, it follows that that any isotropy subgroup $H \subset \Gamma$ is finite.

# CHAPTER II

## Preliminaries

The construction of cofinite classifying space $\overline{X}_\Gamma$ of the main theorem (Theorem I.1) crucially depends on two results:

1. The uniform construction of the Borel–Serre partial compactification of symmetric spaces (c.f. [16] or see §2.3.4).

2. Garland and Raghunathan's results on the structure of fundamental set of lattices in semsimple Lie group of $\mathbb{R}$-rank one (c.f. [24] or see §2.4.3).

To understand the structure of symmetric space, it is necessary to begin with the structure of semisimple Lie algebra and the Cartan decomposition. In §2.1, we discuss the basic structure of semisimple Lie algebras and how it induces the structure of symmetric spaces. In §2.2, we discuss the structure of semisimple algebraic groups, including notions of parabolic subgroups and their Langlands decompositions. In §2.3, we discuss the notion of arithmetic subgroups of semisimple algebraic groups, which is a natural class of lattices. The uniform construction of Borel–Serre partial compactification is discussed also. Lastly, in §2.4, we discuss the structure of semisimple Lie group of $\mathbb{R}$-rank one. This induces the structure of rank one symmetric spaces. We also review the result of Garland and Raghunathan in [24] on the fundamental set of lattices in semisimple Lie group of $\mathbb{R}$-rank one.

## 2.1 Symmetric spaces and locally symmetric spaces

The structure of symmetric spaces is closely related to the structure of semisimple Lie groups, and also to the structure of semisimple Lie algebras. In section 2.1.1, we discuss the structure of semisimple Lie algebras. In section 2.1.2, we discuss the structure of symmetric spaces and how it is induced from the structure of semisimple Lie algebras. In section 2.1.3, we introduce the Iwasawa decomposition (Definition II.17) of semisimple Lie groups, which is closely related to the structure of parabolic subgroups.

### 2.1.1 Structure of semisimple Lie algebras

Throughout this section, we let $\mathfrak{g}$ denote a finite dimensional real Lie algebra and $G$ the connected Lie group associated to $\mathfrak{g}$. We start with definition of semisimple Lie algebra (Definition II.1) and present some other equivalent definitions of semisimple Lie algebras (Proposition II.2). We then introduce the Cartan involution of a semisimple Lie algebra (Definition II.3) followed by the existence (Proposition II.4) and uniqueness (Proposition II.9) of the Cartan involution.

Let $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the adjoint representation defined by the bracket operation on $\mathfrak{g}$:

$$ad(X)(Y) = [X, Y].$$

Let $Ad : G \to GL(\mathfrak{g})$ be the adjoint representation of $G$ induced from $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(\mathfrak{g}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \end{array} \tag{2.1}$$

We say the Lie algebra $\mathfrak{g}$ is **non-compact** if the image $Ad(G)$ is non-compact. From now on, we assume that $\mathfrak{g}$ is non-compact unless we specify otherwise. Recall that a non-abelian Lie algebra is **simple** if 0 and itself are only ideals. Roughly speaking, a semisimple Lie algebra is a direct sum of simple Lie algebras. More precisely, it is defined as follow.

**Definition II.1.** A Lie algebra $\mathfrak{g}$ is called **semisimple** if there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \tag{2.2}$$

of simple ideals.

**Proposition II.2.** *The following are equivalent.*

1. *The Lie algebra $\mathfrak{g}$ is semisimple.*
2. *The Killing form is non-degenerate.*
3. *Any solvable ideal of $\mathfrak{g}$ is trivial.*

*Proof.* $(1 \Rightarrow 3)$ Every solvable ideal of simple Lie algebra is trivial. The same argument holds for the direct sum of simple Lie algebras. $(2 \Leftrightarrow 3)$ Let $B$ be the Killing form of $\mathfrak{g}$. Let $\mathfrak{a}$ be the ideal of $\mathfrak{g}$ such that for any $Y \in \mathfrak{a}$, the Killing form $B(\mathfrak{g}, Y) = 0$. Since $\mathfrak{a}$ is solvable, it follows that $\mathfrak{a} = 0$. Conversely, let $\mathfrak{b}$ be an abelian idea of $\mathfrak{g}$. For every two vectors $Y, Z \in \mathfrak{b}$, the endomorphism $(ad_Y ad_Z)^2$ maps $\mathfrak{g}$ to 0 surjectively. Thus $\mathfrak{a}$ is nilpotent. The non-degeneracy of $B$ implies that $\mathfrak{a} = 0$. $(2 \Rightarrow 1)$ Since $\mathfrak{g}$ is finite dimensional, for every ideal $\mathfrak{g}_1$ of $\mathfrak{g}$, there exists an ideal $\mathfrak{g}_1^\perp$ which is orthogonal complement to $\mathfrak{g}_1$ with respect to $B$. By induction, the Lie algebra $\mathfrak{g}$ decomposes as in (2.2). $\square$

The Lie group $G$ is called **semisimple** if its Lie algebra $\mathfrak{g}$ is semisimple.

**Definition II.3.** Let $\mathfrak{g}$ is a real Lie algebra. An involution $\theta$ of $\mathfrak{g}$ is called a **Cartan involution** if the bilinear form

$$B_\theta(X,Y) = -B(X,\theta Y) \quad X,Y \in \mathfrak{g} \tag{2.3}$$

is positive definite.

**Proposition II.4.** *Let $\mathfrak{g}$ be a real non-compact semisimple Lie algebra. Then there exists a Cartan involution of $\mathfrak{g}$.*

*Idea of proof.* The proof is found in [30, Theorem 6.3, Theorem 7.1, Proposition 7.4]. We outline this proof in three steps.

**Step 1** The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ admits a *compact real form* $\mathfrak{u}$, i.e. a compact real Lie subalgebra such that $\mathfrak{u} \oplus i\mathfrak{u} = \mathfrak{g}$. The form $\mathfrak{u}$ is constructed as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Phi(\mathfrak{g},\mathfrak{h})$ be the set of nonzero roots. For simplicity, we let $\Phi = \Phi(\mathfrak{g},\mathfrak{h})$. For each root $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote a root space. Choose a vector $X_\alpha \in \mathfrak{g}_\alpha$ such that

(i) $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}$, and

(ii) $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \neq 0$, $\alpha + \beta \notin \Phi$.

Let us write

$$\mathfrak{u} = \bigoplus_{\alpha\in\Phi} \left(\mathbb{R}H_\alpha \oplus \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \mathbb{R}i(X_\alpha + X_{-\alpha})\right). \tag{2.4}$$

Then $\mathfrak{u} \oplus i\mathfrak{u} = \mathfrak{g}$. The restriction of the Killing form of $\mathfrak{g}$ onto $\mathfrak{u}$ is negative definite. Thus $\mathfrak{u}$ is compact (c.f. [30, Proposition 6.6(i)]).

**Step 2** Let $\sigma$ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Then there exists a automorphism $\varphi$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\varphi(\mathfrak{u})$ is $\sigma$-invariant. The map $\varphi$ is defined as follows. Let $\tau$ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{u}$. Since $\mathfrak{u}$ is compact, the Hermitian form $B_\tau(X,Y) = -B(X,\tau Y)$ on $\mathfrak{g}_\mathbb{C}$ is positive definite. The linear transformation $N = \sigma\tau$ is self-adjoint with respect to $B_\tau$. That is,

$$B_\tau(N(X),Y) = B_\tau(X,N(Y)).$$

With suitable basis of $\mathfrak{g}_\mathbb{C}$, we can write the transformation $N$ as the diagonal matrix $\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$. We then define

$$\varphi = \mathrm{diag}(\lambda_1^{1/4},\cdots,\lambda_n^{1/4}).$$

The subspace $\varphi(\mathfrak{u})$ is a compact real form and $\tau' = \varphi\tau\varphi^{-1}$ is the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to $\varphi(\mathfrak{u})$. Since $\tau'\sigma = \sigma\tau'$, $\varphi(\mathfrak{u})$ is $\sigma$-invariant. For convenience, replace $\varphi(\mathfrak{u})$ with $\mathfrak{u}$ and $\tau'$ with $\tau$.

**Step 3** Let $\mathfrak{k} = \mathfrak{g}\cap\mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g}\cap i\mathfrak{u}$. Define a map $\theta : \mathfrak{g}\to\mathfrak{g}$ such that

$$\theta|_\mathfrak{k} = Id, \quad \theta|_\mathfrak{p} = -Id \tag{2.5}$$

Then $\theta$ is a Cartan involution. Note that $B_\theta$ is the restriction of $B_{\tau'}$. Since $B_\tau$ is positive definite, so is $B_\theta$. □

*Remark* II.5. We claim that the subalgebra $\mathfrak{k}$ is a maximal compact Lie algebra of $\mathfrak{g}$. If $\mathfrak{k}$ were not maximal, there exists a maximal compact Lie subalgebra $\mathfrak{k}'$ which contains $\mathfrak{k}$ properly. Assuming that is the case, let $X\in\mathfrak{k}'\cap\mathfrak{p}$ be a nonzero element. For any two vectors $Y,Z\in\mathfrak{g}$, the following holds from the Jacobi identity:

$$B([X,Y],\tau Z) = -B(Y,[X,\tau Z]) = B(Y,[\tau X,\tau' Z]).$$

In other words, $B_\tau(ad(X)(Y),Z) = B_\tau(Y,ad(X)(Z))$. So the eigenvalues of $ad(X)$ are all real and nonzero. Then the one parameter subgroup $\exp(ad(X)t)$ $(t>0)$ lies in the compact subgroup of $Ad(\mathfrak{k}')$. This is a contradiction. Thus $\mathfrak{k}$ is maximal.

**Definition II.6.** Let $\theta$ be a Cartan involution of semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{k}$ be the Lie subalgebras of $\mathfrak{g}$ such that $\theta|_{\mathfrak{k}} = Id$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\theta$. The decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.6}$$

is called a **Cartan decomposition** with respect to $\theta$.

*Remark* II.7. When $\mathfrak{g}$ is a complex semisimple Lie algebra, the conjugation with respect to its compact real form (c.f. Step 1 of the proof of Proposition II.4) is a Cartan involution.

**Example II.8.** Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} b & c \\ c & -b \end{pmatrix} \mid b, c \in \mathbb{R} \right\}. \tag{2.7}$$

The direct sum $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. The associated Cartan involution is the following:

$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mapsto \begin{pmatrix} -x & -z \\ -y & x \end{pmatrix}. \tag{2.8}$$

A Cartan involution is unique up to conjugacy.

**Proposition II.9.** *Let $\theta_1$, $\theta_2$ be two Cartan involutions of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and $\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ be the Cartan decompositions with respect to $\theta_1$, $\theta_2$ respectively. Then there exists a vector $X \in \mathfrak{g}$ such that for $\varphi = \exp(ad(X)) \in GL(\mathfrak{g})$,*

$$\varphi(\mathfrak{k}_1) = \mathfrak{k}_2, \quad \varphi(\mathfrak{p}_2) = \mathfrak{p}_2. \tag{2.9}$$

*Proof.* See [30, Theorem 7.2]. □

From now on, the Cartan involution always means a unique involution up to conjugacy. The Cartan involution of $\mathfrak{g}$ induces an involution of $G$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\theta} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\theta} & G \end{array} \tag{2.10}$$

Such involution of $G$ is also called the *Cartan involution*. When there is no confusion, we denote $\theta$ for the Cartan involution of $G$.

### 2.1.2 Symmetric spaces

Given a semisimple Lie algebra $\mathfrak{g}$, we obtain a symmetric space $X = G/K$. In Definition II.10, we define a symmetric space. Then we show that the structure of $\mathfrak{g}$ induces the structure of $X$ in Proposition II.12. An example of symmetric space is the upper half-plane $\mathbf{H}$ (Example II.13).

**Definition II.10.** A **symmetric space** $X$ is a complete Riemannian manifold such that for each point $x \in X$, the geodesic involution $i_x$ at $x \in X$ is an isometry.

*Remark* II.11. Equivalently, a Reimannian manifold is called symmetric space if its curvature tensor is covariantly constant. In other words, the covariant derivative of the curvature tensor is zero.

**Proposition II.12.** *Let $\mathfrak{g}$ be a real (non-compact) semisimple Lie algebra and $G$ be the connected Lie group of $\mathfrak{g}$. For a maximal compact subgroup $K$ of $G$, the homogeneous space $X = G/K$ is a non-compact symmetric space.*

*Idea of proof.* Let $\theta$ be the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Let $K$ be the connected Lie subgroup of $G$ of $\mathfrak{k}$. The tangent space of $X$ at $x_0 = Id \cdot K$ is isomorphic to $\mathfrak{g}/\mathfrak{k}$. Therefore,

$$\mathfrak{p} \cong T_{x_0}X. \tag{2.11}$$

Since $\mathfrak{g}$ is non-compact, the Killing form $B$ is positive definite and $Ad(K)$-invariant. We define an $Ad(K)$-invariant inner product on $T_{x_0}X$ as follow: for $X, Y \in \mathfrak{p}$,

$$\langle X, Y\rangle_{x_0} = B(X, Y) \tag{2.12}$$

We then define a Riemannian metric $\langle\, ,\, \rangle$ on $X$ as follow. Let $L_g : X \to X$ be the left multiplication of $g$ and $L_g^* : T_{x_0}X \to T_{gx_0}X$ be its derivative. For $X, Y \in T_{g\cdot x_0}X$,

$$\langle X, Y\rangle_{gx_0} = \langle L_{g^{-1}}^* X, L_{g^{-1}}^* Y\rangle_{x_0} \tag{2.13}$$

The geodesic involution $i_{x_0}$ at $x_0$ is defined as follow: for all $g \in G$,

$$i_{x_0}(gx_0) = \theta(g)x_0. \tag{2.14}$$

To show $i_{x_0}$ is an isometry, we need to show $i_{x_0}^*\langle\, ,\, \rangle_{gx_0} = \langle\, ,\, \rangle_{\theta(g)x_0}$. Since is the Killing form $B$ is $\theta$-invariant, so is the metric $\langle\, ,\, \rangle_{x_0}$. Thus it is sufficient to prove $i_{x_0}L_g = L_{\theta(g)}i_{x_0}$. Then $i_{x_0}L_g(hx_0) = i_{x_0}(ghx_0) = \theta(gh)x_0 = \theta(g)i_{x_0}(hx_0)$, and this proves that $X$ is a symmetric space.

Next we show that $X$ is non-compact. Since $X$ is a symmetric space, any point $g \cdot x_0$ lies on a geodesic from $x_0$. So for each $g \in G$, there exists a vector $Y \in \mathfrak{p}$ such that $g \cdot x_0 = \exp(Y) \cdot x_0$. Then $(\exp(-Y)g) \cdot x_0 = x_0$, and this implies $\exp(-Y)g \in K$. Therefore,

$$G = \exp(\mathfrak{p})K. \tag{2.15}$$

Moreover, (2.15) is an isomorphism. Since $B_\theta$ is positive definite and $K$ is maximally compact, we can find an orthonormal basis with respect to $B_\theta$ such that $K \subset O(B_\theta)$. Suppose that $\exp(X)k = \exp(X')k' = gx_0$. Then $Ad(g)Ad(g)^t = Ad(\exp 2X) = Ad(\exp 2X')$. Since $Ad$ is injective on $\exp \mathfrak{p}$, we have $X = X'$, and immediately $k = k'$. Thus the map $(X, k) \mapsto \exp(X)k$ is injective. In fact, $X = G/K \cong \mathfrak{p}$ is a diffeomorphism. Therefore $X$ is non-compact. See [30, Chapter IV] for further detail. □

**Example II.13.** The upper half-plane $\mathbf{H}$ is a symmetric space obtained from the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Cartan involution $\theta$ in Example II.8. The base point $x_0 = i \in \mathbf{H}$ and the geodesic involution at $i$ is

$$z \mapsto -\frac{1}{z}, \quad z \in \mathbf{H}. \tag{2.16}$$

Figure 2.1 is the geometric interpretation of this geodesic involution.

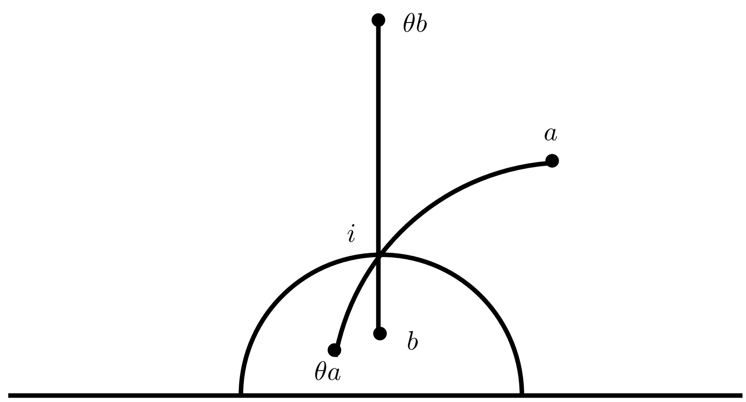


Figure 2.1: Geodesic involution of $\mathbf{H}$ at $i$

**Definition II.14.** Let $X$ be a symmetric space and $Y \subset X$ be a geodesic submanifold. The subspace $Y$ is called **flat** if the induced metric on $Y$ is Euclidean. The **rank** of $X$ is the maximal dimension of any flat submanifold.

### 2.1.3 Semisimple Lie groups

In this section, we introduce the Iwasawa decomposition of $G$. We first define the notion of restricted roots (Definition II.15). We then discuss restricted root space decomposition, followed by the notion of the Iwasawa decomposition (Proposition II.16).

We assume that the Lie group $G$ is semisimple and non-compact, and its Lie algebra admits the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

**Definition II.15.** Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. A linear function $\alpha : \mathfrak{a} \to \mathbb{R}$ is called a **restricted root with respect to $\mathfrak{a}$** if

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid ad(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \tag{2.17}$$

is nonzero. The set of all non-trivial restricted root with respect to $\mathfrak{a}$ is denoted by

$$\Phi(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\} \tag{2.18}$$

The **restricted root decomposition** of $\mathfrak{g}$ is the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha. \tag{2.19}$$

For any two restricted roots $\alpha, \beta$ such that $\alpha \neq \beta$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}. \tag{2.20}$$

Moreover, for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{a})$,

$$\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}. \tag{2.21}$$

Let $\Phi^+$ be the set of all positive restricted roots. Then

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

is a nilpotent Lie subalgebra of $\mathfrak{g}$.

**Proposition II.16.** *Let $N, A, K$ be the Lie subgroup of $G$ corresponding to $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$ respectively. The Lie group $G$ decomposes diffeomorphically into*

$$G = N \times A \times K. \tag{2.22}$$

**Definition II.17.** The decomposition (2.22) of $G$ is called the **Iwasawa decomposition**.

*Idea of proof of Proposition II.16.* (See [30, Theorem 3.4] for a complete proof.) Once we show that $\mathfrak{g}$ decomposes into

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}, \tag{2.23}$$

the proposition follows immediately. Since $\mathfrak{a}$ is abelian and $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \cap \mathfrak{n} = 0$. Let $X \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$. Then $\theta(X) = X$ and $\theta(X) \in \mathfrak{a} + \theta\mathfrak{n}$. The root decomposition of $\mathfrak{g}$ in (2.19) is equivalent to $\mathfrak{a} + \mathfrak{n} + \theta\mathfrak{n}$. Thus $X \in \mathfrak{a}$. Since $X \in \mathfrak{k} \cap \mathfrak{p}$, $X = 0$. Any element of $\mathfrak{g}$ is of the form

$$H + X_0 + \sum_{\alpha \in \Phi(\mathfrak{g},\mathfrak{a}} X_\alpha$$

where $H \in \mathfrak{a}, X_0 \in \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$, and $X_\alpha \in \mathfrak{g}_\alpha$. Then

$$\left( X_0 + \sum_{\alpha \in \Phi^+} (X_{-\alpha} + \theta(X_{-\alpha}) \right) + H + \left( \sum_{\alpha \in \Phi^+} (X_\alpha - \theta(X_{-\alpha})) \right) \in \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

□

*Remark* II.18. For the matrix groups, the Iwasawa decomposition is equivalent to the Gram–Schmidt decomposition.

## 2.2 Algebraic groups

In section 2.2.1, we define notions of algebraic groups, algebraic groups defined over a field, and semisimple algebraic groups. In section 2.2.2, we discuss the parabolic subgroups of an algebraic group defined over $\mathbb{Q}$. In section 2.2.3, we discuss the Lie algebra of parabolic subgroups, which is called *parabolic subalgebras*. In section 2.2.4, we explain the Langlands decomposition of parabolic subgroups, the horospherical decomposition of symmetric space, and the group operation in terms of the Langlands decomposition.

### 2.2.1 Definitions

We define algebraic groups in Definition II.19. Roughly speaking, an algebraic group is a variety which admits a group structure (Remark II.20). We then define the term *algebraic groups defined over a field* in Definition II.21. The main object of study is semisimple algebraic groups, so that its real locus is a semisimple Lie group (Proposition II.22).

**Definition II.19.** A group $\mathbf{G} \subset GL(n, \mathbb{C})$ is called a **(linear) algebraic group** if there exists a collection of polynomials $P_\alpha$ $(\alpha \in I)$ such that

$$\mathbf{G} = \{g \in GL(n, \mathbb{C}) \mid P_\alpha(g) = 0 \text{ for all } \alpha \in I\}. \tag{2.24}$$

*Remark* II.20. In more general settings, an algebraic group is a algebraic variety whose group operations (multiplication and inverse) are morphisms. However, we will not use this general definition of algebraic groups. Note that the group operations of algebraic groups in Definition II.19 are multiplication and inverse of matrices, which are automatically morphisms.

Let $k$ be a field of characteristic 0, such as $\mathbb{Q}$, a number field, $\mathbb{R}$, or $\mathbb{C}$. We say a polynomial is defined over $k$ if all of its coefficients are in $k$.

**Definition II.21.** An algebraic group $\mathbf{G}$ is called **defined over** $k$ if the defining polynomials $P_\alpha$ are defined over $k$.

The **radical** of $\mathbf{G}$, denoted by $\mathbf{R}(\mathbf{G})$, is the maximal connected normal solvable subgroup of $\mathbf{G}$. We say $\mathbf{G}$ is **semisimple** if $\mathbf{R}(\mathbf{G}) = 0$. Let $\mathbf{G}(k) = \mathbf{G} \cap GL(n, k)$.

**Proposition II.22.** *For semisimple algebraic group* $\mathbf{G}$*, the real locus* $G = \mathbf{G}(\mathbb{R})$ *is a semisimple Lie group.*

*Proof.* This follows immediately from Proposition II.2. $\square$

#### 2.2.2 Parabolic subgroups

Throughout this section, we assume that $\mathbf{G}$ is an algebraic group defined over $\mathbb{Q}$. Let $k$ be a field of characteristic zero such as $\mathbb{Q}$, number fields, $\mathbb{R}$, or $\mathbb{C}$. In Definition II.23, we define parabolic subgroups of $\mathbf{G}$ defined over $k$. In the following Proposition II.24, we discuss the structure of a particular parabolic subgroup $\mathbf{P}_0$ which is minimal. We then introduce the notion of standard parabolic subgroup with respect to $\mathbf{P}_0$. In Proposition II.26, we prove that every parabolic subgroup is conjugate to a standard parabolic subgroup.

**Definition II.23.** Let $\mathbf{G}$ be an algebraic group defined over a field $k$.

1. An algebraic subgroup $\mathbf{P}$ of $\mathbf{G}$ is called **parabolic subgroup** if the quotient $\mathbf{G}/\mathbf{P}$ is compact. If $\mathbf{P}$ is defined over a subfield $k'$ of $k$, then $\mathbf{P}$ is called a $k'$**-parabolic subgroup**.

2. A $k$**-split torus** (simply, $k$**-torus**) $\mathbf{T}$ in $\mathbf{G}$ is an algebraic subgroup defined over $k$ such that there exists a $k$-isomorphism $\mathbf{T} \cong GL(1, \mathbb{C})^n$ where the positive integer $n$ is the dimension of $\mathbf{T}$.

3. The $k$**-rank** of $\mathbf{G}$ is the maximal dimension of $k$-split torus in $\mathbf{G}$.

For simplicity, we assume that $k = k' = \mathbb{Q}$. A minimal $\mathbb{Q}$-parabolic subgroup is written as the product of a nilpotent subgroup and the centralizer of a maximal $\mathbb{Q}$-split torus. We first set some notations.

- Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. The Lie algebra $\mathfrak{s}$ of $\mathbf{S}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ (c.f. [32, §15.3]).
- Let $\Phi(\mathfrak{g}, \mathfrak{s})$ (respectively, $\Phi^+(\mathfrak{g}, \mathfrak{s})$) be the set of (respectively, positive) roots on $\mathfrak{g}$ with respect to $\mathfrak{s}$.
- Let
$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{s})} \mathfrak{g}_\alpha$$
and $\mathbf{N}$ be the Lie subgroup of $\mathbf{G}$ corresponding to $\mathfrak{n}$.
- Let $\mathbf{Z}(\mathbf{S})$ be the centralizer of a maximal $\mathbb{Q}$-torus $\mathbf{S}$ in an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$.

**Proposition II.24.** *We keep the notations above. The subgroup* $\mathbf{P}_0$ *defined by*

$$\mathbf{P}_0 = \mathbf{N} \cdot \mathbf{Z}(\mathbf{S}) \tag{2.25}$$

*is a minimal* $\mathbb{Q}$*-parabolic subgroup of* $\mathbf{G}$*.*

*Proof.* Since the torus $\mathbf{S}$ is $\mathbb{Q}$-split, all roots in $\Phi(\mathfrak{g}, \mathfrak{s})$, considered as characters on $\mathbf{G}$, are defined over $\mathbb{Q}$. So algebraic groups $\mathbf{N}$ and $\mathbf{Z}(\mathbf{S})$ are both defined over $\mathbb{Q}$. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$. The Lie subalgebra $\mathfrak{q}_0$ corresponding to the parabolic subgroup $\mathbf{P}_0$ is then written as

$$\mathfrak{q}_0 = \mathfrak{n} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{n} + \mathfrak{a} + \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \tag{2.26}$$

where $\mathfrak{a} = \mathfrak{s} \cap \mathfrak{p}$. By Iwasawa decomposition (2.23) of $\mathfrak{g}$, the Lie algebra of $\mathbf{G}/\mathbf{P}_0$ is isomorphic to $\mathfrak{k}/\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. Since the $ad(\mathfrak{k})$-representation of $\mathfrak{p}$ is faithful, $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = 0$. This implies that $\mathbf{G}/\mathbf{P}_0$ is embedded into a maximal compact subgroup of $\mathbf{G}$. Thus the subgroup $\mathbf{P}_0$ is minimal and $\mathbb{Q}$-parabolic. $\square$

Every minimal $\mathbb{Q}$-parabolic subgroup containing a minimal $\mathbb{Q}$-torus $\mathbf{S}$ is of the form $\mathbf{N}\cdot\mathbf{Z}(\mathbf{S})$ with respect to some suitable ordering of roots. Thus the Weyl group

$$ {}_{\mathbb{Q}}W(\mathbf{G},\mathbf{S}) = \mathbf{N}(\mathbf{S})/\mathbf{Z}(\mathbf{S}) \tag{2.27}$$

acts transitively on the set of all minimal $\mathbf{Q}$-parabolic subgroup containing $\mathbf{S}$. Let $\Delta(\mathfrak{g},\mathfrak{s})$ be the set of all simple roots, and $I \subset \Delta(\mathfrak{g},\mathfrak{s})$ be a proper subset. Define $\mathbf{S}_I = \bigcap_{\alpha\in I}\ker\alpha$. Let

$$ \mathbf{P}_I = \mathbf{N}\cdot\mathbf{Z}(\mathbf{S}_I). \tag{2.28}$$

Since $\mathbf{S}_I$ is defined over $\mathbb{Q}$, the subgroup $\mathbf{P}_I$ is a $\mathbb{Q}$-parabolic subgroup containing $\mathbf{P}_0$.

**Definition II.25.** The subgroup $\mathbf{P}_I$ (including $\mathbf{P}_0$) is called the **standard $\mathbb{Q}$-parabolic subgroup**.

Every parabolic subgroup containing a maximal $\mathbb{Q}$-torus is standard (c.f. [42, Lemma 7.74]). Moreover, we have the following.

**Proposition II.26.** *Every $\mathbb{Q}$-parabolic subgroup is conjugate to a standard $\mathbb{Q}$-parabolic subgroup.*

*Proof.* Let $\mathbf{P}$ be a minimal $\mathbb{Q}$-parabolic subgroup, not necessarily containing a maximal $\mathbb{Q}$-torus $\mathbf{S}$. Since the embedding of $\mathbf{G}/\mathbf{P} \hookrightarrow \mathbf{G}$ is maximally compact, there exists a maximally compact subalgebra $\mathfrak{k}'$ of $\mathfrak{g}$ such that the Lie algebra of $\mathbf{G}/\mathbf{P}$ is isomorphic to $\mathfrak{k}'$. Since $\mathfrak{g}$ is semisimple, any two maximal compact subalgebras are conjugate. Thus $\mathbf{P}$ is conjugate to $\mathbf{P}_0$ in Proposition II.24. Similarly, every $\mathbb{Q}$-parabolic subgroup containing $\mathbf{P}$ is conjugate to a standard parabolic subgroup. $\square$

One can replace the rational field $\mathbb{Q}$ with the real field $\mathbb{R}$, and follow the same argument above to obtain the structure of standard $\mathbb{R}$-parabolic subgroups.

### 2.2.3 Parabolic subalgebra

Parabolic subgroups can be viewed as the normalizer of *parabolic subalgebra*. In Definition II.27, we define a special minimal parabolic subalgebras. Similar to standard parabolic subgroups, we introduce the notion of standard parabolic subalgebras. In Proposition II.29, we introduce the decomposition of $G = P \cdot K$ as the product of a parabolic subgroup and the maximal compact subgroup $K$.

**Definition II.27.** Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{s}$ be a Cartan subalgebra. Denote $\mathfrak{n}_0 = \sum_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{s})} \mathfrak{g}_\alpha$, $\mathfrak{a}_0 = \mathfrak{s} \cap \mathfrak{p}$, and $\mathfrak{m}_0 = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$. The subalgebra of the form

$$\mathfrak{q}_0 = \mathfrak{n}_0 + \mathfrak{a}_0 + \mathfrak{m}_0 \tag{2.29}$$

is called a **minimal parabolic subalgebra**.

Note that a minimal parabolic subalgebra $\mathfrak{q}_0$ is the Lie algebra of a minimal parabolic subgroup $\mathbf{P}_0$. Any subalgebra containing $\mathfrak{q}_0$ is called a *standard parabolic subalgebra*, and is parametrized by proper subset $I$ of simple roots. Let $\mathfrak{a}_I = \cap_{\alpha\in I} \ker\alpha$ and $\mathfrak{a}^I$ be the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$. Let $\Phi^I$ be the set of all roots that are linear combination of elements in $I$. Define

$$\mathfrak{n}_I = \sum_{\alpha\in\Phi^+-\Phi^I} \mathfrak{g}_\alpha, \quad \mathfrak{m}_I = \mathfrak{m}_0 \oplus \mathfrak{a}^I \oplus \sum_{\alpha\in\Phi^I} \mathfrak{g}_\alpha.$$

**Definition II.28.** The parabolic subalgebra of the form

$$\mathfrak{q}_I = \mathfrak{n}_I + \mathfrak{a}_I + \mathfrak{m}_I \tag{2.30}$$

is called a **standard parabolic subalgebra**.

The uniqueness of the Cartan decomposition (upto conjugacy) implies that any parabolic subalgebra is conjugate to a standard parabolic subalgebra. Note that the decomposition (2.29) and (2.30) of standard parabolic subalgebra depends on the choice of a Cartan decomposition. We emphasize that any parabolic subgroups acts transitively on $X$.

**Proposition II.29.** *Let $P$ be a parabolic subgroup whose Lie algebra is a parabolic subalgebra $\mathfrak{q}$. Then $P$ is the normalizer of $\mathfrak{q}$ and $G = P \cdot K$ where $K$ is a maximal compact subgroup fixed by the Cartan involution.*

*Proof.* See [42, Proposition 7.83]. □

### 2.2.4 Langlands decomposition

We discuss the *Langlands decomposition* of parabolic subgroups and related properties. Starting from the description of *Levi quotient* (2.31), we define the Langlands decomposition of parabolic subgroup in (2.32). In Example II.34, we observe the Langlands decomposition of the minimal parabolic subgroup $\mathbf{P}_0$ (c.f. Proposition II.24). We define the action of a parabolic subgroup on the Langlands decomposition of itself in (2.35). The Langlands decomposition implies the horospherical decomposition (2.36) of symmetric space. We revisit the decomposition $G = P \cdot K$ in Proposition II.36 (c.f. Proposition II.29). Using the Langlands decomposition of $P$, we define a group multiplication in (2.40) and show that this is well-define in Proposition II.37.

**Notation II.30.** Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{Q}$ and $\mathbf{H}$ be an algebraic subgroup of $\mathbf{G}$. If the structure of $\mathbf{H}$ depends on a $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, we denote $\mathbf{H}_\mathbf{P}$ instead of $\mathbf{H}$. If $\mathbf{H}_\mathbf{P}$ is defined over $\mathbb{Q}$, then the real locus $\mathbf{H}_\mathbf{P}(\mathbb{R}) = \mathbf{H}_\mathbf{P} \cap \mathbf{G}(\mathbb{R})$

of $\mathbf{H_P}$ is denoted by

$$H_P = \mathbf{H_P}(\mathbb{R}).$$

If $\mathbf{H_P}$ is defined over $\mathbb{R}$, but its structure still depends on the $\mathbb{Q}$-parabolic subgroup $P$, then we denote

$$H_{\mathbf{P}} = \mathbf{H_P}(\mathbb{R}).$$

Let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$ and $\mathbf{N_P}$ be the unipotent radical (i.e. the maximal unipotent normal subgroup) of $\mathbf{P}$.

**Definition II.31.** The quotient $\mathbf{L_P} = \mathbf{N_P}\backslash\mathbf{P}$ is called **Levi quotient**.

The Levi quotient $\mathbf{L_P}$ is defined over $\mathbb{Q}$. Let $\mathbf{S_P}$ be a maximal $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$, and $A_{\mathbf{P}}$ be the identity component of $S_P$. Let $M_{\mathbf{P}}$ be the complement of $A_{\mathbf{P}}$ in $L_P$, i.e.

$$L_P = A_{\mathbf{P}} \times M_{\mathbf{P}}. \tag{2.31}$$

We want to decompose $P$ into the product of $N_P$ and $L_P$. Let $\theta$ be a Cartan involution of $\mathbf{G}$. Choose a lift $\iota_\theta : \mathbf{L_P} \to \mathbf{P}$ such that the image $\iota_\theta(L_P)$ is $\theta$-stable in $P$. Then $P = N_P \times \iota_\theta(A_{\mathbf{P}}) \times \iota_\theta(M_{\mathbf{P}})$. When there is no confusion, we drop $\iota$ and write

$$P = N_P \times A_{\mathbf{P}} \times M_{\mathbf{P}}. \tag{2.32}$$

**Definition II.32.** The decomposition (2.32) is called the **Langlands decomposition of P**.

Let $\mathfrak{q}$ be the parabolic subalgebra of $\mathbf{P}$. The decomposition (2.30) of $\mathfrak{q}$ is the same as (2.32), and therefore it is also called the Langlands decomposition of $\mathfrak{q}$.

Let $P$ be a $\mathbb{R}$-parabolic subgroup. Let $N_P$ be the unipotent radical of $P$ and $L_P = N_P\backslash P$ be the Levi quotient. The maximal torus $S_P$ in $L_P$ is not necessarily the real locus of $\mathbb{Q}$-split torus. Let $A_P, M_P$ be the identity component of $S_P$ and the complement of $A_P$ so that $L_P = A_P \times M_P$. With the suitable $\theta$-stable lift, we write

$$P = N_P \times A_P \times M_P. \tag{2.33}$$

This is also called the **Langlands decomposition of** $P$. Since a $\mathbb{Q}$-parabolic subgroup is $\mathbb{R}$-parabolic, for $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, we have two Langlands decompositions: one is from (2.32), and the other is from (2.33) where $\mathbf{P}$ is viewed as a $\mathbb{R}$-parabolic subgroup.

**Notation II.33.** The decomposition (2.32) is called a **rational Langlands decomposition** whereas the decomposition (2.33) is called a **real Langlands decomposition**.

**Example II.34** (The Langlands decomposition of standard parabolic subgroups)**.** Let $\mathbf{P}_0$ be a minimal $\mathbb{Q}$-parabolic subgroup in Proposition II.24. Since $\mathbf{N}$ is nilpotent and $\mathbf{Z}(\mathbf{S})$ normalizes $\mathbf{N}$, the group $\mathbf{N}$ is the unipotent radical of $\mathbf{P}_0$. The group $\mathbf{Z}(\mathbf{S})$ is a lift of the Levi quotient of $\mathbf{P}_0$. Since the Lie algebra of $\mathbf{Z}(\mathbf{S})$ is $\mathfrak{a}_0 + \mathfrak{m}_0$, the group $\mathbf{Z}(\mathbf{S})$ is the $\theta$-stable lift. The Lie subgroup $A_0$ of $\mathfrak{a}_0$ is the identity component of $S$. Let $\mathbf{M_P} = \cap_{\xi \in X(\mathbf{L_P})} \ker \xi^2$ where $X(\mathbf{L_P})$ is the group of all $\mathbb{Q}$-morphisms from $\mathbf{L_P}$ to $GL(1, \mathbb{C})$. Then the real locus $M_P = M_P(\mathbb{R})$ is the complement $M_\mathbf{P}$ of $A_\mathbf{P}$ in $L_P$, and is the Lie subgroup of $\mathfrak{m}_0$. Thus the Langlands decomposition of $\mathbf{P}_0$ is

$$P_0 = N \times A_0 \times M_P. \tag{2.34}$$

The $\theta$-stable condition of the lift $\iota_\theta$ is crucial to get the Langlands decomposition. From the Langlands decomposition of parabolic subalgebra, the Langlands decomposition of parabolic subgroup is a diffeomorphism. Moreover, the Langlands decomposition is $P$-equivariant map. Let $P = N \times A \times M$ be the Langlands decomposition of $P$. The group multiplication of $P$ is written as follow:

$$(n, a, m) \cdot (n', a', m') = (n{}^{am}n', aa', mm'). \tag{2.35}$$

where the left-superscript always means the conjugation, i.e. ${}^{am}n' = (am)n'(am)^{-1}$. Let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$. From Proposition II.29, the group $P$ acts transitively on $X = G/K$. Thus the (rational) Langlands decomposition of $P$ induces the decomposition of $X$:

$$X = N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}} \text{ where } X_{\mathbf{P}} = M_{\mathbf{P}} \cap K. \tag{2.36}$$

**Definition II.35.** This decomposition is called the **rational horosperical decomposition of** $X$. For $\mathbb{R}$-parabolic subgroup $P$, the decomposition

$$X = N_P \times A_P \times X_P \text{ where } X_P = M_P \cap K. \tag{2.37}$$

is called the **real horospherical decomposition of** $X$.

**Proposition II.36.** *For every $\mathbb{R}$-parabolic subgroup $P$ of $G$, the group $G$ decomposes as follow:*

$$G = P \cdot K. \tag{2.38}$$

*Proof.* From the Iwasawa decomposition (c.f. Proposition II.16 and Definition II.17), it follows that

$$G = N \cdot A \cdot K. \tag{2.39}$$

Since $P \cap K = M_P \cap K$,

$$G = (N \times A \times M_P) \cdot K = P \cdot K.$$

This completes the proof. □

Let us write an element $g \in G$ as $g = nam \cdot k$ where $n \in N_P$, $a \in A_P$, $m \in M_P$, and $k \in K$. For two element $g = nam \cdot k$ and $g_1 = n_1a_1m_1 \cdot k_1$, let us define the decomposition of $gg_1$ as follow.

$$(nam \cdot k) \cdot (n_1a_1m_1 \cdot k_1) = n^{am}n_1aa_1mm_1 \cdot kk_1. \tag{2.40}$$

**Proposition II.37.** *The equation* (2.40) *is the group multiplication.*

*Proof.* We check the three conditions of the definition of groups: existence of identity element, associativity, and existence of inverse element.

**Identity** The identity element is $Id$, which is obvious.

**Associativity** We need to show that

$$((nam\cdot k)\cdot(n_1a_1m_1\cdot k_1))\cdot(n_2a_2m_2)\cdot k_2 = (nam\cdot k)\cdot((n_1a_1m_1\cdot k_1)\cdot(n_2a_2m_2\cdot k_2)). \tag{2.41}$$

The left hand side of (2.41) is

$$(n^{am}n_1aa_1mm_1 \cdot kk_1) \cdot (n_2a_2m_2 \cdot k_2) = n^{am}{n_1}^{aa_1mm_1}n_2aa_1a_2mm_1m_2 \cdot kk_1k_2. \tag{2.42}$$

The right hand side of (2.41) is

$$(nam \cdot k) \cdot ({n_1}^{a_1m_1}n_2a_1a_2m_1m_2 \cdot k_1k_2) = n^{am}({n_1}^{a_1m_1}n_2)aa_1a_2mm_1m_2 \cdot kk_1k_2, \tag{2.43}$$

which are the same.

**Inverse** The inverse of $nam \cdot k$ is

$$(nam \cdot k)^{-1} = {}^{(am)^{-1}}n^{-1}a^{-1}m^{-1} \cdot k^{-1} \tag{2.44}$$

□

*Remark* II.38. The $M_P$ lies in the centralizer of $A_P$. Thus every element in $M_P$ commutes with $A_P$.

## 2.3 Arithmetic subgroups

In section 2.3.1, we give a definition of arithmetic subgroups. In section 2.3.2, we discuss the reduction theory on the action of arithmetic subgroups. In the preliminary section 2.3.3, we discuss the convergence class of sequences, and then in section 2.3.4 we discuss the Borel–Serre compactifications for arithmetic subgroups.

### 2.3.1 Definitions

We give a brief definition of arithmetic subgroup (Definition II.39) and examples (Example II.40). Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{Q}$ and $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathrm{GL}(N, \mathbb{Z})$.

**Definition II.39.** A subgroup $\Gamma$ of $\mathbf{G}(\mathbb{R})$ is called **arithmetic** if $\Gamma$ and $\mathbf{G}(\mathbb{Z})$ are commensurable, i.e. the intersection $\Gamma \cap \mathbf{G}(\mathbb{Z})$ is a subgroup of finite index in both $\mathbf{G}(\mathbb{Z})$ and $\Gamma$.

**Example II.40.** A Fuchsian group $\mathrm{PSL}(2, \mathbb{Z})$ is a arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})$. For $N > 1$, the congruence subgroups $\Gamma(N)$ is also arithmetic:

$$\Gamma(N) = \{g \in \mathrm{PSL}(2, \mathbb{Z}) \mid g \equiv I \pmod{N}\} \tag{2.45}$$

One can extend the definition of arithmetic subgroups by extending field of definition. Let $k$ be a totally real number field and $\mathbf{G}$ be a linear algebraic group defined over $k$. Then a discrete subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ is called *arithmetic* if $\Gamma$ is commensurable with $\mathbf{G}(\mathcal{O}_k)$. However, this does not enlarge the class of arithmetic subgroup. To understand this, we introduce the *restriction of scalar*. Note that $k$ is a $\mathbb{Q}$-vector space under suitable basis. The left multiplication induces the embedding $k \hookrightarrow \mathrm{GL}(r, \mathbb{C})$. This embedding is defined over $\mathbb{Q}$ in the sense that one can find a linear algebraic group $\mathbf{G}' \subset \mathrm{GL}(r, \mathbb{C})$ defined over $\mathbb{Q}$ such that $\mathbf{G}'(\mathbb{Q})$ isomorphic to

$k$. This induces the embedding of $\mathbf{G}$ into $\mathrm{GL}(rN, \mathbb{C})$ as follow: for each $g \in G$, let $g_{ij}$ be the $i, j$-entry of $g$. We map each $g_{ij}$ into $\mathrm{GL}(r, \mathbb{C})$ by embedding of $k$. More precisely, let

$$\mathbf{H} = \{g_{ij} \in \mathrm{GL}(rN, \mathbb{C}) \mid P_\alpha(g_{ij}) = 0,\ g_{ij} \in \mathbf{G}'(\mathbb{Q})\}\,. \tag{2.46}$$

Then $\mathbf{H}(\mathbb{Z}) = \mathbf{G}(\mathcal{O}_k)$.

**Definition II.41.** The algebraic group $\mathbf{H}$ defined in (2.46) is called the **restriction of scalar** of $\mathbf{G}$ onto $k$, and denoted by $\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}$.

Throughout this section, we assume that $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$ and $\Gamma$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$.

#### 2.3.2 The Siegel sets and reduction theory

In this section, we discuss the reduction theory for arithmetic subgroups $\Gamma$. We first define a *fundamental set* in Definition II.46. We then state Proposition II.47 on homeomorphism between the $\Gamma$-orbit $\Gamma\backslash X$ and the $\Gamma$-equivalent image $\Omega/\sim$ of fundamental set. The Siegel set is defined in Definition II.48. We then mention the reduction theory for arithmetic subgroup due to Borel in Proposition II.49 and Proposition II.50.

Let $\Gamma$ be a discrete group acting properly on $X$.

**Definition II.42.** A **fundamental domain** for $\Gamma$ is an open subset $D \subset X$ such that $X = \Gamma \cdot \overline{D}$ and each $\Gamma$-orbit intersects at most one point in the interior of $D$.

An example of fundamental domain for metric space is *Dirichlet fundamental domain.*

**Definition II.43.** Let $X$ be a metric space and $\Gamma$ acts properly and isometrically on $X$. The **Dirichlet fundamental domain at** $x_0 \in X$ is the set

$$D(x_0, \Gamma) = \{x \in X \mid d(x, x_0) < d(x, \gamma \cdot x_0) \text{ for all } \gamma \in \Gamma, \gamma \neq e\} \tag{2.47}$$

**Definition II.44.** A subset $D$ of $\Gamma$-space $X$ is called **locally finite** if for every compact subset $C$ of $X$, the subset $\Gamma'$ defined by

$$\Gamma' = \{\gamma \in \Gamma \mid \gamma \cdot C \cap D \neq \emptyset\}$$

is finite.

**Proposition II.45.** *If $X$ is a symmetric space of constant curvature, then the Dirichlet fundamental domain is locally finite.*

In the reduction theory of arithmetic subgroups, the notion of fundamental set is commonly used.

**Definition II.46.** A open subset $\Omega \subset X$ is called **fundamental set** for $\Gamma$ if $X = \Gamma \cdot \Omega$ and the set

$$\{\gamma \in \Gamma \mid \gamma \cdot \Omega \cap \Omega \neq \emptyset\} \tag{2.48}$$

is finite.

**Proposition II.47.** *Let $\Omega$ be a locally finite fundamental set for $\Gamma$ in $X$. Then the surjective map $\Omega \to \Gamma \backslash X$ induces the homeomorphism $\Gamma \backslash \Omega \cong \Gamma \backslash X$.*

The reduction theory for arithmetic subgroups is established by using Siegel sets. Let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic subgroup and $P = N_P \times A_{\mathbf{P}} \times M_{\mathbf{P}}$ be the (rational) Langlands decomposition. Let $X = G/K$ and $X = N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ be the rational horospherical decomposition. We denote $\Phi(P, A_{\mathbf{P}})$ for the set of all restricted roots (c.f. Definition II.15) whose elements are viewed as characters on $A_{\mathbf{P}}$.

**Definition II.48.** Let $U \subset N_P, V \subset X_{\mathbf{P}}$ be bounded open sets and $A_{\mathbf{P},t} = \{a \in A_{\mathbf{P}} \mid a^\alpha > 1 \text{ for all } \alpha \in \Phi(P, A_{\mathbf{P}})\}$. The subset $\mathfrak{S}_{P,t,U,V}$ of the form

$$\mathfrak{S}_{P,t,U,V} = U \times A_{P,t} \times V \tag{2.49}$$

is called a **Siegel set**.

The $\mathbb{Q}$-*rank* of $\mathbf{G}$ is the maximal dimension of $\mathbb{Q}$-split torus in $\mathbf{G}$. It is proved by Borel and Harish-Chandra and independently by Mostow and Tamagawa that the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive if and only if $\mathbf{G}(\mathbb{Q})$ does contain some nontrivial unipotent element [13, Theorem 3] [58, Theorem in Chapter II, p461]. The $\mathbb{Q}$-rank of $\mathbf{G}$ is zero if and only if the arithmetic subgroups are cocompact in $\mathbf{G}(\mathbb{R})$ (c.f. [10, Theorem 2.16]). Thus we assume that $\mathbb{Q}$-rank of $\mathbf{G}$ is always positive. The following reduction theory is due to A. Borel.

**Proposition II.49.** *The following holds.*

1. *There are finitely many $\Gamma$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$.*
2. *Let $\mathbf{P}_1, \cdots, \mathbf{P}_r$ be representatives of such $\Gamma$-conjugacy classes. Then there exists a Siegel set $\mathfrak{S}_{P_n,U_n,t_n,V_n}$ for each $i = 1, \cdots, r$ such that the union $\Omega = \bigcup_{n=1}^r \mathfrak{S}_{P_n,U_n,t_n,V_n}$ is a fundamental set. Moreover, for any $g \in \mathbf{G}(\mathbb{Q})$,*

$$\{\gamma \in \Gamma \mid \gamma \cdot \Omega \cap g \cdot \Omega \neq \emptyset\} \tag{2.50}$$

*is finite.*

*Proof.* See [16, Proposition 3.2.19]. □

The condition (2.50) is called the **Siegel finiteness property**. For arithmetic subgroup, the Siegel finiteness property is equivalent to (2.48), which is often called a weak Siegel finiteness condition. More precise reduction theory for arithmetic subgroups is the following.

**Proposition II.50.** *Let $\mathbf{P}_1, \cdots, \mathbf{P}_r$ be $\mathbb{Q}$-parabolic subgroups from Proposition II.49. There exists a compact subset $C \subset X$ and Siegel sets $\mathfrak{S}_{\mathbf{P}_i, U_i, t_i, V_i}$ $(i = 1, \cdots, r)$ such that*

1. *each $\mathfrak{S}_{\mathbf{P}_i, U_i, t_i, V_i}$ is mapped injectively into $\Gamma \backslash X$ under the projection $\pi : X \to \Gamma \backslash X$,*
2. *the image of $U_i \times V_i$ in $(\Gamma \cap P_i) \backslash N_{P_i} \times X_{\mathbf{P}_i}$ is compact, and*
3. *$\Gamma \backslash X$ is homeomorphic to the disjoint union of $C$ and Siegel sets:*

$$\Gamma \backslash X \cong C \cup \coprod_{i=1}^{r} \pi \left( \mathfrak{S}_{\mathbf{P}_i, U_i, t_i, V_i} \right) \tag{2.51}$$

Next, we introduce the notion of horoball.

**Definition II.51.** Let $\mathbf{P}$ be a parabolic subgroup of semisimple Lie group $G$ defined over $\mathbb{Q}$. The **horoball** $S_{P,t}$ at $P$ is the set

$$S_{P,t} = N_P \times A_{P,t} \times X_P.$$

**Example II.52.** In the upper half-plane $\mathbf{H}$, the horoball at $P_\infty$ is the set of the form

$$\{x + iy \mid x \in \mathbb{R},\, y \geq t\}$$

for some $t > 0$. For a parabolic subgroup $P \neq P_\infty$, the horoball is a closed disc tangent to the real $\mathbb{R}$ at the point $\zeta$ fixed by $P$.

**Proposition II.53.** *Let $C$ be a compact subset of $X$. For every parabolic subgroup $P$, there exists a sufficiently large $T \gg 1$ such that*

$$S_{P,T} \cap C = \emptyset.$$

### 2.3.3 Convergence class of sequence

Before we introduce the Borel–Serre partial compactification, we shortly discuss methods of defining a topology. More detailed discussion is in [15, §1.8].

**Definition II.54.** For $A \in \mathcal{P}(X)$, a map $A \mapsto \overline{A}$ is called a **closure operator** on $X$ if it satisfies the following.

1. For the empty set $\emptyset$, $\overline{\emptyset} = \emptyset$
2. For every two subsets $A, B \subset X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. For every subset $A \subset X$, $A \subset \overline{A}$
4. For every subset $A \subset X$, $\overline{\overline{A}} = \overline{A}$

**Proposition II.55.** *Let $A \mapsto \overline{A}$ be an closure operator on $X$. Then there exists a topology on $X$ whose closed sets are $\overline{A}$ for all $A \in \mathcal{P}(X)$ and the closure of $A$ is $\overline{A}$.*

Another equivalent way of defining a topological space is by *convergence class of sequence.* Let $\mathcal{C}$ be a collection of pairs $(\{y_j\}_{j\in\mathbb{N}}, y_\infty)$ of an infinite sequence and a point in $X$. We denote $y_j \xrightarrow{\mathcal{C}} y_\infty$ ($y_j \overset{\mathcal{C}}{\nrightarrow} y_\infty$, respectively) if $(\{y_j\}_{j\in\mathbb{N}}, y_\infty) \in \mathcal{C}$ (if $(\{y_j\}_{j\in\mathbb{N}}, y_\infty) \notin \mathcal{C}$, respectively).

**Definition II.56.** The collection $\mathcal{C}$ is called a **convergence class of sequence** of $X$ if it satisfies the following.

1. If $y_j = y$ for all $j$, then $y_j \xrightarrow{\mathcal{C}} y$.
2. If $y_j \xrightarrow{\mathcal{C}} y_\infty$, then $y'_j \xrightarrow{\mathcal{C}} y_\infty$ for any subsequence $\{y'_j\}_{j\in\mathbb{N}}$, .
3. If $y_j \overset{\mathcal{C}}{\nrightarrow} y_\infty$, then there is a subsequence $\{y'_j\}_{j\in\mathbb{N}}$ such that $y''_j \overset{\mathcal{C}}{\nrightarrow} y_\infty$ for any further subsequence $\{y''_j\}_{j\in\mathbb{N}}$.

4. Let $\{y_{j,k}\}_{j,k\in\mathbb{N}}$ be a double sequence such that $y_{j,k} \overset{\mathcal{C}}{\to} y_{\infty,k}$ and $y_{\infty,k} \overset{\mathcal{C}}{\to} y_{\infty,\infty}$ for each $k$. Then there exists an increasing function $m : \mathbb{N} \to \mathbb{N}$ such that $y_{j,m(j)} \overset{\mathcal{C}}{\to} y_{\infty,\infty}$.

**Proposition II.57.** *Let $\mathcal{C}$ be a convergence class of sequence of $X$. For any subset $A$ of $X$, define*

$$\overline{A} = \{y \in X \mid \text{ there exists a sequence } \{y_j\} \text{ in } A \text{ such that } y_j \overset{\mathcal{C}}{\to} y\}. \tag{2.52}$$

*Then $A \to \overline{A}$ is a closure operator.*

*Proof.* See [34, §6]. □

*Remark* II.58. From Proposition II.55, the convergence class of sequence $\mathcal{C}$ also defines a topology on $X$.

**Proposition II.59.** *Suppose $X$ is a topological space with the topology $\mathcal{T}$. Then a sequence $y_j$ converges if and only if $y_j$ belongs to the collection $\mathcal{C}$. Moreover, a subset $C$ of $X$ is compact if and only if every sequence $y_j$ in $C$ has a convergent subsequence.*

*Proof.* See [15, Proposition I.8.13] and [40]. □

#### 2.3.4 The Borel–Serre partial compactification

The Borel–Serre partial compactification $\overline{X}^{BS}$ is the compactification of locally symmetric space $\Gamma\backslash X$ constructed by *geodesic action* associated with all $\mathbb{Q}$-parabolic subgroups $\mathbf{P}$. Let $X = N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ be the rational horospherical decomposition of $X$. For every element $a \in A_{\mathbf{P}}$, we have $a \cdot (n', a', z') = ({}^a n', aa', z')$. This $A_{\mathbf{P}}$-action on $X$ is called the *geodesic action*. The geodesic action is equivariant with respect to $P$-action (2.35). Note that the geodesic action depends not only on the choice of $A_{\mathbf{P}}$ but also on the choice of $\mathbf{P}$. We will give a uniform construction of $\overline{X}^{BS}$

by attaching boundary components whose topology is given by convergence class of sequence. Define

$$e(\mathbf{P}) = N_P \times X_{\mathbf{P}}. \tag{2.53}$$

Let $\mathbf{P} \subset \mathbf{Q}$ be two $\mathbb{Q}$-parabolic subgroups. Note that $\mathbf{M_P}$ (c.f. Example II.34) is defined over $\mathbb{Q}$. There is a unique $\mathbb{Q}$-parabolic subgroup $\mathbf{P'}$ of $\mathbf{M_P}$ such that

$$N_P = N_Q N_{P'}, \quad M_{\mathbf{P}} = M_{\mathbf{P'}}, \quad A_{\mathbf{P}} = A_{\mathbf{Q}} A_{\mathbf{P'}}. \tag{2.54}$$

This implies that $X_{\mathbf{Q}} = N_{P'} \times A_{\mathbf{P'}} \times X_{\mathbf{P}}$, and $e(\mathbf{Q}) = N_P \times A_{\mathbf{P'}} \times X_{\mathbf{P}}$. Therefore, $e(\mathbf{P}) \subset e(\mathbf{Q})$. Let $\mathbf{P}$ run over all $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$. The (rational) Borel–Serre partial compactification is

$$_{\mathbb{Q}}\overline{X}^{BS} = X \cup \coprod_{\text{rational } \mathbf{P}} e(\mathbf{P}) \tag{2.55}$$

with the convergence class of sequence generated by obvious ones and two special convergence sequences below:

1. An unbounded sequence $y_j \in X$ converges to a boundary point $(n_\infty, z_\infty) \in e(\mathbf{P})$ if and only if in terms of the rational horospherical decomposition of $X$ with respect to $\mathbf{P}$, $y_j = (n_j, a_j, z_j)$ and $n_j \to n_\infty, z_j \to z_\infty$, and $a_j^\alpha \to \infty$ for all $\alpha \in \Phi(P, A_{\mathbf{P}})$.

2. For $\mathbf{P} \subset \mathbf{Q}$, a sequence $y_j \in e(\mathbf{Q})$ converges to a point $(n_\infty, z_\infty) \in e(\mathbf{P})$ if and only if the coordinates of $y_j = (n_j, a_j, z_j) \in N_P \times A_{\mathbf{P'}} \times X_{\mathbf{P}}$ satisfy that $n_j \to n_\infty, z_j \to z_\infty$, and $a_j^\alpha \to \infty$ for all $\alpha \in \Phi(P', A_{\mathbf{P'}})$.

## 2.4 The rank one semisimple Lie groups

Throughout this section, we assume that $G$ is a semisimple Lie group of $\mathbb{R}$-rank one, and $K$ is a fixed maximal compact subgroup of $G$. We let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition.

In section 2.4.1, we discuss the structure of the rank one symmetric space $X = G/K$. In section 2.4.2, we discuss the geodesic compactification of $X$. In section 2.4.3, we outline the results of Garland and Raghunathan [24] on the fundamental set of any lattice in $G$. In the last section 2.4.4, we give an example of the upper-half plane $\mathbf{H}$.

### 2.4.1 Basic structure

The following proposition characterizes a structure of rank one semisimple Lie groups.

**Proposition II.60.** *Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one. Then every minimal $\mathbb{R}$-parabolic subgroup of $G$ is maximal.*

*Proof.* Let $\mathfrak{q}$ be a Lie algebra of a minimal parabolic subgroup $P$. From Definition II.27 and Proposition II.26,

$$\mathfrak{q} = \mathfrak{n} + \mathfrak{a} + \mathfrak{m}.$$

Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition. The Lie algebra $\mathfrak{a}$ is a one-dimensional abelian Lie subalgebra in $\mathfrak{p}$. Since the $\mathbb{R}$-rank of $G$ is one, $\mathfrak{a}$ is maximal. Thus $\mathfrak{q}$ is a maximal parabolic subalgebra. Therefore, $P$ is maximal. $\square$

**Corollary II.61.** *Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one, and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. There is a one-to-one correspondence between the projective set $\mathfrak{p}/\mathbb{R}^+$ and the set of all minimal parabolic subgroups of $G$.*

*Proof.* A minimal parabolic subgroups are determined uniquely by the maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Since $\mathfrak{a}$ is one-dimensional, there exists a vector $Y \in p$ such that $\mathfrak{a} = \mathbb{R} \cdot Y$. Since such $Y$ is unique up to positive scalar, the statement follows. □

**Notation II.62.** For a vector $Y \in \mathfrak{p}$, denote $P_Y$ the minimal parabolic subgroup corresponding to $Y$. The Langlands decomposition of $P_Y$ is denoted by

$$P_Y = N_Y \times A_Y \times M_Y.$$

**Proposition II.63.** *We keep the notations from Notation II.62 above. The Lie group $A_Y$ is isomorphic to $\mathbb{R}^+$.*

*Proof.* From the definition of Langlands decomposition (2.33), the Lie group $A_Y$ is the connected component of the abelian Lie group whose Lie algebra $\mathfrak{a}_Y$ is isomorphic to $\mathbb{R}$. Thus $A_Y \cong \mathbb{R}^+$. □

**Proposition II.64.** *Let $P$ be a $\mathbb{R}$-parabolic subgroup of a semisimple Lie group $G$ of $\mathbb{R}$-rank one, and $P = N_P \times A_P \times M_P$ be the Langlands decomposition of $P$. Then $M_P \subset K$.*

*Proof.* Let $\mathfrak{q} = \mathfrak{n} + \mathfrak{a} + \mathfrak{m}$ be the parabolic subalgebra of $P$. Note that the subalgebra $\mathfrak{m}$ is the Lie algebra of $M_P$. Since $P$ is minimal, it follows that $\mathfrak{m} \subset \mathfrak{k}$. Therefore, the $M_P \subset K$. □

**Corollary II.65.** *Let $X = G/K$ be the symmetric space corresponding to a Riemannian symmetric pair $(G, K)$ where $G$ is a semisimple Lie group of $\mathbb{R}$-rank one. Then the horospherical decomposition of $X$ with respect to $P$ is*

$$X = N_P \times A_P.$$

*Proof.* This follows from the definition of the horospherical decomposition $X = N_P \times A_P \times X_P$ where $X_P = M_P/(M_P \cap K)$ and Proposition II.64. □

**Proposition II.66.** *We keep the notations from Definition II.15. There exists a unique root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_Y)$ such that $\alpha(Y) > 0$ and*

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_Y) \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}. \tag{2.56}$$

*Proof.* See [24, §0] and [42, Proposition 6.40]. □

*Remark* II.67. Let $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_Y) = \mathfrak{m}_Y \oplus \mathfrak{a}_Y$ where $\mathfrak{m}_Y = \mathfrak{k} \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_Y$, and $\mathfrak{n}_Y = \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}$. Denote $N_Y$, $A_Y$, and $M_Y$ the Lie subgroup corresponding to $\mathfrak{n}_Y$, $\mathfrak{a}_Y$, and $\mathfrak{m}_Y$ respectively. Then the decomposition

$$N_Y \times A_Y \times M_Y$$

is the Langlands decomposition of $P_Y$ in Notation II.62.

**Proposition II.68.** *Let $k \in K$, and $\alpha \in \Phi^+(P, A_P)$. Then the character $\beta \in \Phi^+({}^kP, A_{^kP})$ satisfies*

$$\beta(a) = \alpha({}^{k^{-1}}a).$$

*Proof.* This follows from the isomorphism $k : P \to {}^kP$ and $k : A_P \to {}^kA_P = A_{^kP}$. The isomorphisms follows from Proposition II.26. □

**Notation II.69.** We denote $\beta = {}^k\alpha$.

#### 2.4.2 The geodesic compactification $\overline{X}(\infty)$

In this section, we mean a *geodesic* by a (globally) distance minimizing (smooth) curve $s : [0, \infty) \to X$ with the unit speed $\|s\| = 1$.

**Definition II.70.** Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition and $\|\cdot\|$ be the norm on $\mathfrak{p}$ induced from positive definite quadratic form $B_\theta$ (c.f. Definition II.3). The subset $\mathfrak{p}^1$ of $\mathfrak{p}$ is defined by

$$\mathfrak{p}^1 = \{Y \in \mathfrak{p} \mid \|Y\|_{\mathfrak{p}} = 1\}.$$

Let us fix a point $x_0 = Id \cdot K$ in $X$. Every geodesic $s(t)$ passing through $x_0$ is written as follow: for some vector $Y \in \mathfrak{p}^1$,

$$s(t) = \exp(tY) \cdot x_0 \tag{2.57}$$

Then $\mathfrak{p}^1$ is in one-to-one correspondence with all geodesics passing through $x_0$.

**Definition II.71.** Two geodesics $s_1(t)$ and $s_2(t)$ are **equivalent** if the distance $d(s_1(t), s_2(t))$ is bounded for all $t \geq 0$. For each geodesic $s$, the equivalent class containing $s$ is denoted by $[s]$.

**Definition II.72.** The set of all equivalent classes of geodesics in $X$ is denoted by $X(\infty)$. Each element in $X(\infty)$ is called a **point at infinity**.

The topology of $X(\infty)$ is induced from the topology of $\mathfrak{p}^1$, i.e. a sequence $[s_j]$ converges to $[s_\infty]$ if and only if the sequence of vectors $Y_j \in \mathfrak{p}^1$ such that $s_j(t) = \exp(tY_j) \cdot x_0$ converges to $Y_\infty$ where $s_\infty(t) = \exp(tY_\infty) \cdot x_0$. Since $\mathfrak{p}^1$ homeomorphic to a unit sphere $\mathbb{S}^n$ where $n$ is the dimension of $X$, the space $X(\infty)$ is compact. Let $\overline{X}(\infty)$ be the union of $X$ and $X(\infty)$:

$$\overline{X}(\infty) = X \cup X(\infty).$$

We claim that the space $\overline{X}(\infty)$ is compact. For every unbounded sequence $y_j$ in $X$, there exist a positive real $t_j > 0$ and a vector $Y_j \in \mathfrak{p}^1$ such that $y_j = \exp(t_j Y_j) \cdot x_0$. Since $y_j$ is unbounded and $X(\infty)$ is compact, it follows that

$$t_j \to \infty, \quad Y_j \to Y_\infty.$$

Thus $y_j$ converges to the point at infinity $[s_\infty]$ where $s_\infty$ is a geodesic of the form $s_\infty = \exp(tY_\infty) \cdot x_0$.

**Definition II.73.** The space $\overline{X}(\infty) = X \cup X(\infty)$ is called the geodesic compactification of $X$.

The $G$-action on the symmetric space $X$ extends to the boundary $\overline{X}(\infty)$ as follow. Let $s : [0, \infty) \to X$ be a geodesic in $X$. Then

$$g \cdot [s] = [g \cdot s].$$

**Proposition II.74.** *The stabilizer $G_{[s]}$ of $[s]$ is a parabolic subgroup of $G$.*

*Proof.* Since the $\Gamma$-action on $X$ is transitive, the $\Gamma$-action on $X(\infty)$ is also transitive. Therefore, $G/G_{[s]}$ is homeomorphic to $X(\infty)$, which is compact. $\square$

**Corollary II.75.** *Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one. There exists a one-to-one correspondence between the set of all $\mathbb{R}$-parabolic subgroups of $G$, the projective space $\mathfrak{p}^1$, and the boundary $X(\infty)$:*

$$\mathfrak{p}^1 \longleftrightarrow \{\textit{all } \mathbb{R}\textit{-parabolic subgroups}\} \longleftrightarrow X(\infty).$$

*Proof.* The one-to-one correspondence

$$\{\text{all } \mathbb{R}\text{-parabolic subgroups}\} \longleftrightarrow \mathfrak{p}^1$$

follows from Corollary II.61. Proposition II.74 implies the one-to-one correspondence

$$\{\text{all } \mathbb{R}\text{-parabolic subgroups}\} \longleftrightarrow X(\infty).$$

This completes the proof. $\square$

**Proposition II.76.** *For every two minimal parabolic subgroup $P$ and $P'$ such that $P \neq P'$, there exists an element $k \in K$ such that ${}^kP = P'$. Moreover, the subset $K'$*

$$K' = \{k \in K \mid {}^kP = P'\}$$

*of $K$ is finite.*

*Proof.* The existence of the element $k$ follows from the proof of Proposition II.26. We prove that $K'$ is finite by contradiction. Suppose $K'$ is infinite, thus contains an infinite sequence of element $k_j \in K$. Note that for every $j \geq 1$, the element $k_j(k_{j+1})^{-1}$ lies in the normalizer $N_G(P)$ of $P$. Since $N_G(P) = P$, it follows that $k_j(k_{j+1})^{-1} \in P \cap K = M_P$. Thus, by passing to a subsequence, we may assume that $k_j(k_{j+1})^{-1}$ converges to $k_\infty \in M_P$. Then $(k_\infty)^{-1}k_j(k_{j+1})^{-1}$ converges to $Id$. Note that the $K$-action on the set of parabolic subgroups is induced from the $ad_{\mathfrak{k}}$ representation on $\mathfrak{p}$. Under this representation, the subalgebra $\mathfrak{k}$ acts on $\mathfrak{p}$ faithfully. Thus the sequence $(k_\infty)^{-1}k_j(k_{j+1})^{-1}$ cannot converges to $Id$. This is a contradiction. □

**Proposition II.77.** *Let $P$ and $Q$ be a minimal parabolic subgroup of semisimple Lie group $G$ of $\mathbb{R}$-rank one. For every $t > 0$, there exists a sufficiently large $T \gg 1$ such that horoballs at $P$ and $Q$ are disjoint:*

$$S_{P,t} \cap S_{Q,T} = \emptyset.$$

*Proof.* Since $P \cap Q = \{Id\}$, no sequence in $S_{Q,T}$ converges to a point at infinity corresponding to $P$. □

### 2.4.3 Results of Garland and Raghunathan

The main result of Garland and Raghunathan in [24] is the following.

**Theorem II.78.** *Let $\Gamma$ be a non-uniform lattice in a semisimple Lie group $G$ of $\mathbb{R}$-rank one and $X$ be the symmetric space corresponding to a Riemannian symmetric pair $(G, K)$. Then there exists a compact subset $C \subset X$, a finite subset $\Sigma \in G$, and a Siegel set $\mathfrak{S}_{P,U,t}$ for sufficiently large $t > 0$ such that the domain $\Omega$ defined by*

$$\Omega = C \cup \bigcup_{g \in \Sigma} g \cdot \mathfrak{S}_{P,U,t} \tag{2.58}$$

*is a fundamental set of* $\Gamma$ *in* $X$. *Moreover, the domain* $\Omega$ *is a coarse fundamental domain, i.e. the subset* $\Gamma_0$ *of* $\Gamma$ *defined by*

$$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma \cdot \Omega \cap \Omega \neq \emptyset\} \tag{2.59}$$

*is finite.*

**Definition II.79.** A fundamental domain $\Omega$ is called a **coarse fundamental domain** if the set

$$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma \cdot \Omega \cap \Omega \neq \emptyset\}$$

is finite.

**Proposition II.80.** *Let* $\Gamma$ *be a discrete group acting properly on* $X$. *Let* $\pi : X \to \Gamma\backslash X$ *be the natural projection and* $\Omega$ *be a coarse fundamental domain for* $\Gamma$. *Then the following homeomorphism hold if and only if* $\Omega$ *is locally finite.*

$$\Gamma\backslash\Omega \cong \Gamma\backslash X. \tag{2.60}$$

**Corollary II.81.** *Under the projection* $\pi : X \to \Gamma\backslash X$, *the image* $\pi(\Omega)$ *is homeomorphic to the quotient space* $\Gamma\backslash X$.

*Proof.* Since $\Omega$ is a fundamental domain for $\Gamma$, the map $\pi|_\Omega$ is surjective. Since $\Omega$ is the union of a compact set and finitely many Siegel sets, it is locally finite. Thus $\Omega$ is coarse and locally finite fundament set for $\Gamma$. This completes the proof. □

The proof of Theorem II.78 depends on the next two propositions (Proposition II.83 and Proposition II.84).

We first define a notion of *ray*.

**Definition II.82.** A vector $Y \in \mathfrak{p}^1$ is called a **ray with respect to** $D$ if the geodesic $\exp(tY) \cdot x_0 \in D$ $(t \geq 0)$ lies in $D$.

The following proposition is proved by Garland and Raghunathan [24, Theorem 0.7].

**Proposition II.83.** *Let* $Y \in \mathfrak{p}^1$ *be a ray. Then the subgroup* $\Gamma \cap N_Y \subset N_Y$ *is a cocompact lattice.*

*Idea of a proof.* The proof consists of five steps.

**Step 1** Let $Y$ be a ray. We claim that $\Gamma \cap N_Y$ contains a nontrivial element. There exists a sequence of unipotent element $\gamma_j \in \Gamma \cap N_Y$ such that for $q = \exp(Y)$,

$$q^j \gamma_j q^{-j} \to Id. \tag{2.61}$$

Moreover one can choose such $\gamma_j$'s from $\Gamma \cap N_Y$.

**Step 2** Let $\rho \in \Gamma \cap N_Y$ be a nontrivial element. Let $G_\rho$, $N_\rho$, and $M_\rho$ be centralizer of $\rho$ in $G$, unipotent radical of $G_\rho$, and the reductive complement of $N_\rho$ in $G_\rho$ respectively. The group $N_\rho$ is the centralizer of $\rho$ in $N_Y$. The kernel of the natural action of $M_\rho$ on $N_\rho$ lies on the center of $G$. Without loss of generality, we can assume that $G$ has trivial center[1]. Thus the action of $M_\rho$ on $N_\rho$ is faithful.

**Step 3** Let $P_Y = N_Y \times A_Y \times M_Y$ be the Langlands decomposition of $P_Y$. The group $G_\rho$ lies in $N_Y \cdot M_Y$. Moreover, there exists a compact subgroup $M$ of $M_Y$ such that $G_\rho$ decomposes into semi-direct product

$$G_\rho = N_\rho \rtimes M. \tag{2.62}$$

**Step 4** Now we introduce the following lemma due to Auslander [4]:

[1] As a semisimple Lie group, $G$ has finite center

**Lemma** (Auslander)**.** *Let $N$ be a connected, simply connected nilpotent Lie group. Let $M$ be a compact subgroup of $Aut(N)$ which acts faithfully on $N$. Let $G = N \rtimes M$ and $\Gamma$ be a uniform lattice in $G$. Then $\Gamma \cap N$ is a cocompact lattice in $N$.*

Let $\Gamma_\rho$ be the centralizer of $\rho$ in $\Gamma$. We apply this lemma with $G = G_\rho$, $N = N_\rho$, $\Gamma = \Gamma_\rho$, and $M$. We the get a cocompact lattice $\Gamma_\rho \cap N_\rho$ in $N_\rho$.

**Step 5** The last step is to show that there exists an element in $\Gamma \cap N_Y$ which is central in $\Gamma$. The element $\rho$ either central, or $\Gamma \cap N_Y$ does not contain any element which is central in $N_Y$. However, there the subgroup $\Gamma \cap N_Y$ does contain a central element. Therefore we can replace $\rho$ by a central element so that $\Gamma_\rho = \Gamma$ and $N_\rho = N_Y$.

This finishes the proof. □

The next proposition explains the finiteness of the set $\Sigma$ of $G$ in Theorem II.78.

**Proposition II.84.** *Let $\Gamma$ be a lattice. Then there exist only finitely many rays.*

Since the boundary $X(\infty)$ is compact, Proposition II.84 follows immediately from the next lemma.

**Lemma II.85.** *Let $Y$, $Y'$ be rays. There exists a positive constant $\epsilon > 0$ such that $\|Y - Y'\| > \epsilon$.*

*Idea of proof.* Let $G = N_Y \times A_Y \times K$ be the Iwasawa decomposition of $G$ with respect to $P_Y$. For each element $g \in G$, let us define projections $\mathbf{n} : G \to N_Y$, $\mathbf{a} : G \to A_Y$, and $\mathbf{k} : G \to K$ so that

$$g = \mathbf{n}(g)\mathbf{a}(g)\mathbf{k}(g). \tag{2.63}$$

Let $f_Y : G \to \mathbb{R}$ be a smooth function defined as follow.

$$\mathbf{a}(\exp(tY')) = \exp(f_{Y'}(t)Y) \tag{2.64}$$

The following are true.

1. $f_Y(t) \to -\infty$ as $t \to \infty$.

2. There exists a sufficiently small constant $\epsilon > 0$ such that if $\|Y - Y'\| < \epsilon$, then there exists a negative constant $t_\epsilon < 0$ such that $f_Y(t_\epsilon) = 0$.

3. Let $t_\epsilon(Y)$ be a constant defined by

   $$t_\epsilon(Y) = \sup\{t_\epsilon < 0 \mid f_{Y'}(t_\epsilon) = 0\}.$$

   If $Y' \to Y$, then $t_\epsilon(Y) \to -\infty$.

4. For sufficiently large $M > 0$, there exists small constant $\epsilon > 0$ such that for every $Y'$ satisfying $\|Y - Y'\| < \beta$,

   $$d((\mathbf{n}(\exp(t_\epsilon(Y')Y, Id), x_0) \geq M. \tag{2.65}$$

5. Since $Y$ is a ray, from Proposition II.83, the subgroup $\Gamma \cap N_Y$ is a cocompact lattice in $N_Y$. So we can find a compact subset $\omega$ of $N_Y$ such that

   $$N_Y = \omega \cdot (\Gamma \cap N_Y).$$

   Thus for every element $n \in \omega$ and constant $M > 0$,

   $$d((n, Id), x_0) < M.$$

Now we prove the lemma by contradiction. Suppose there exists a ray $Y'$ such that $\|Y - Y'\| < \epsilon$. From (2.65), there exists an element $n \in N_Y$ such that $(n, Id)$ lies on the geodesic $\exp(tY') \cdot x_0$ and

$$d((n, Id), x_0) \geq M.$$

Let us choose $\gamma \in \Gamma \cap N_Y$ so that $\gamma \cdot n \in \omega$. Then

$$d((n, Id), x_0) \geq M > d((\gamma \cdot n, Id), x_0) = d(\gamma \cdot (n, Id), x_0).$$

From the definition of Dirichlet fundamental domain, the point $(n, Id)$ does not lie in $D$. Thus $Y'$ is not a ray. This is a contradiction. □

The fundamental set $\Omega$ in Theorem II.78 is written alternatively as follows. Let $Y_1, \cdots, Y_n$ be the rays with respect to $D$ and $P_1, \cdots, P_n$ be the corresponding parabolic subgroups of $G$. Then there exists a compact subset $C$ of $D$ and Siegel sets $\mathfrak{S}_{P_n,U_n,t_n}$ such that

$$\Omega = C \cup \bigcup_{n=1}^{r} \mathfrak{S}_{P_n,U_n,t_n}.$$

We emphasize the following.

1. The finiteness of rays implies that there are only finitely many cusp neighborhoods of a Dirichlet fundamental domain. For the case of Fuchsian lattices, this is implied by the geometric finiteness.

2. The cusp neighborhood of $D$ is fully covered by Siegel sets. Thus every unbounded sequence in $\Omega$ will eventually (by taking further subsequence) belong to one of Siegel sets.

#### 2.4.4 An example: the upper half-plane H

The upper half-plane $\mathbf{H}$ is a set of points in $\mathbb{C}$ whose imaginary parts are positive. Let $z = x + iy$ be a point in $\mathbf{H}$. A metric $ds$ on $\mathbf{H}$ is defined by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The pair $(\mathbf{H}, ds)$ is a Riemannian manifold with constant sectional curvature $-1$. The Möbius transformation of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbf{H}$ is defined by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}.$$

Every isometry of $\mathbf{H}$ arises as a Möbius transformation.

**Proposition II.86.** *The group of orientation preserving isometries* $Isom^{\circ}(\mathbf{H})$ *is isomorphic to* $PSL(2,\mathbb{R})$.

The stabilizer of the point $i \in \mathbf{H}$ is $PSO(2)$. Thus we may identify $\mathbf{H}$ as the quotient space as follow.

$$\mathbf{H} = \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2).$$

The geodesic compactification $\overline{\mathbf{H}}(\infty)$ of $\mathbf{H}$ is the union

$$\overline{\mathbf{H}}(\infty) = \mathbf{H} \cup \mathbb{R} \cup \{i\infty\}.$$

It is immediately follows from the disk model of $\mathbf{H}$ that the space $\overline{\mathbf{H}}(\infty)$ is compact.

**Definition II.87.** The Cayley transformation $\iota : \mathbf{H} \to D$ is the map

$$\iota(z) = \frac{z-i}{-iz+1}.$$

The image $\iota(\mathbf{H})$ is the open disk $D^1$. The Cayley transformation extends to the geodesic compactification $\overline{\mathbf{H}}(\infty)$. The image $\iota(\overline{\mathbf{H}}(\infty))$ is then the closed disk $\overline{D^1}$, which is compact.

The stabilizer $P_\infty$ of the point at infinity $i\infty$ in $SL(2,\mathbb{R})$ is the group of upper triangle matrices. The Langlands decomposition $P_\infty = N_\infty \times A_\infty \times M_\infty$ is given by

$$N_\infty = \left\{ \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \mid b \in \mathbb{R} \right\},$$

$$A_\infty = \left\{ \left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) \mid a > 0 \right\},$$

$$M_\infty = \{\pm Id\}.$$

The horospherical decomposition of $\mathbf{H}$ with respect to $P_\infty$ is the map

$$N_\infty \times A_\infty \to \mathbf{H}$$

$$\left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \times \left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) \mapsto b + ia^2$$

Let $\mathfrak{n}_\infty$, $\mathfrak{a}_\infty$, and $\mathfrak{m}_\infty$ be the Lie subalgebras of $N_\infty$, $A_\infty$, and $M_\infty$ respectively. Then

$$\mathfrak{n}_\infty = \left\{ \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \mid b \in \mathbb{R} \right\},$$

$$\mathfrak{a}_\infty = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \mid a \in \mathbb{R} \right\},$$

$$\mathfrak{m}_\infty = 0$$

Thus the standard parabolic subalgebra $\mathfrak{p}_\infty$ is

$$\mathfrak{p}_\infty = \mathfrak{n}_\infty \oplus \mathfrak{a}_\infty.$$

The set of restricted roots $\Phi(,\mathfrak{a}_\infty)$ consists of one root $\alpha_\infty$ defined by

$$\alpha_\infty \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) = 2a.$$

In other words, the restricted root $\alpha_\infty$ in $\Phi(P_\infty, A_\infty)$ is the map

$$\alpha_\infty \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) = a^2.$$

A discrete subgroup of $SL(2,\mathbb{R})$ is called a *Fuchsian group.* Each element of Fuchsian group is called

1. *elliptic* if it fixes an interior point,
2. *hyperbolic* if it fixes two points at infinity, and does not fix any point in the interior, and
3. *parabolic* if it fixes a unique point at infinity.

**Proposition II.88.** *Every element of $SL(2,\mathbb{R})$ is either elliptic, hyperbolic, or parabolic.*

**Proposition II.89.** *Any Fuchsian subgroup $\Gamma$ of $SL(2,\mathbb{R})$ does not contain both hyperbolic element and parabolic element which share the same fixed point at infinity.*

*Proof.* We observe a following special case. Let $p = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), e = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ such that $a < 1$. Then

$$e^n p e^{-n} = \left(\begin{smallmatrix} 1 & ba^{2n} \\ 0 & 1 \end{smallmatrix}\right) \to Id$$

as $n \to \infty$. □

The standard parabolic subgroup $P_\infty$ is defined over $\mathbb{Q}$. Every $\mathbb{Q}$-parabolic subgroup $P$ is obtained from $P_\infty$ by the conjugation of elements in $SL(2, \mathbb{Q})$. As a set, the Borel–Serre partial compactification $\overline{\mathbf{H}}^{BS}$ of $\mathbf{H}$ is the union of the upper half-plane $\mathbf{H}$ and the boundary components $N_P$ for all $\mathbb{Q}$-parabolic subgroups:

$$\overline{\mathbf{H}}^{BS} = \mathbf{H} \cup \coprod_{\text{rational } P} N_P.$$

# CHAPTER III

## Proper classifying spaces of the Fuchsian lattices

A Fuchsian group is a discrete subgroup of $SL(2, \mathbb{R})$, which acts on the upper half-plane $\mathbf{H}$ by Möbius transformation. The main goal of this chapter is the following: given a Fuchsian group $\Gamma$ acting on $\mathbf{H}$ as a lattice, we construct a partial compactification $\overline{\mathbf{H}}_\Gamma$ of $\mathbf{H}$ and show that it is a cofinite $\Gamma$-CW-complex model for the proper classifying space $\underline{E}\Gamma$. This proves the main theorem (Theorem I.1) for the case of Fuchsian lattices. Each section of this chapter is summarized as follow.

1. In §3.1, we construct a topological space $\overline{\mathbf{H}}_\Gamma$ from the upper half-plane $\mathbf{H}$ by attaching boundary components corresponding to some parabolic subgroups of $SL(2, \mathbb{R})$. The topology of $\overline{\mathbf{H}}_\Gamma$ is first defined by the convergence class of sequences, and then its open basis is described.

2. In §3.2, we define the $\Gamma$-action on the boundary of the space $\overline{\mathbf{H}}_\Gamma$. Together with the Möbius transformation of $\Gamma$ on the interior $\mathbf{H}$, we show that the $\Gamma$-action on $\overline{\mathbf{H}}_\Gamma$ is continuous and proper.

3. In §3.3, we describe the fundamental domain of $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$ using the Dirichlet fundamental domain of $\Gamma$ in $\mathbf{H}$. We then show that the $\Gamma$-action on $\overline{\mathbf{H}}_\Gamma$.

4. In §3.4, we prove that the space $\overline{\mathbf{H}}_\Gamma$ is a manifold with boundary. Using Illman's results on the existence of $\Gamma$-CW-structure on subanalytic proper $\Gamma$-manifolds,

we prove that there exists a $\Gamma$-CW-structure on the space $\overline{\mathbf{H}}_\Gamma$. From the co-compactness of the $\Gamma$-action, it follows that $\overline{\mathbf{H}}_\Gamma$ is a cofinite $\Gamma$-CW-complex.

5. Lastly, in §3.5, we prove the main theorem (Theorem I.1) for the Fuchsian lattices.

To explain geometry of the space $\overline{\mathbf{H}}_\Gamma$, we describe the cell structure of the quotient $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$. Let $D \subset \mathbf{H}$ be a Dirichlet fundamental domain for $\Gamma$ at the base point $i \in \mathbf{H}$. Since $\Gamma$ is a lattice subgroup of $SL(2,\mathbb{R})$, there are only finitely many vertices at infinity of $D$. Let $\xi_1, \cdots, \xi_r$ be such vertices, and $P_1, \cdots, P_r$ be the corresponding minimal parabolic subgroups of $SL(2,\mathbb{R})$. For each parabolic subgroup $P_n$, let

$$P_n = N_{P_n} \times A_{P_n} \times M_{P_n}$$

be the Langlands decomposition of $P_n$ (c.f. §2.2.4). The stabilizer $\Gamma_{\xi_n}$ of $\xi_n$ in $\Gamma$ is non-trivial subgroup and

$$\Gamma_{\xi_n} = \Gamma \cap N_{P_n}.$$

Since $N_{P_n}$ is diffeomorphic to $\mathbb{R}$ as a Lie group, the subgroup $\Gamma \cap N_{P_n}$ acts cocompactly on $N_{P_n}$.

The Dirichlet fundamental domain $D$ is expressed as the union

$$D = C \cup \bigcup_{n=1}^{r} \mathfrak{S}_{P_n, I_n, t_n}$$

where $I_n$ is a bounded closed subset of $N_{P_n}$. The subset $I_n$ is a fundamental domain for $\Gamma \cap N_{P_n}$ in $N_{P_n}$ (Proposition III.25). It follows from the topology of $\overline{\mathbf{H}}_\Gamma$ (c.f. Proposition III.6) that the closure $\overline{D}$ of $D$ in the space $\overline{\mathbf{H}}_\Gamma$ is the union

$$\overline{D} = D \cup \coprod_{n=1}^{r} I_n.$$

The domain $\overline{D}$ is a fundamental domain for $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$ (Proposition III.26).

For example, if the quotient $\Gamma\backslash\mathbf{H}$ is a Riemann surface with $r$ punctures, the quotient space $\Gamma\backslash\overline{\overline{\mathbf{H}}}_\Gamma$ is obtained from the Riemann surface $\Gamma\backslash\mathbf{H}$ by attaching a compact boundary which is homeomorphic to a circle $\mathbb{S}^1$ (See Figure 3.1 below).

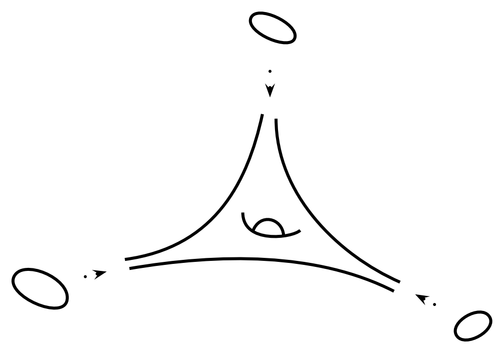

Figure 3.1: Partial Compactification of a Riemann surface with three punctures

## 3.1 The construction of the space $\overline{\mathbf{H}}_\Gamma$

The goal of this section is to define the space $\overline{\mathbf{H}}_\Gamma$ from the upper half-plane $\mathbf{H}$ for Fuchsian lattice $\Gamma$. In section §3.5, we will show that the space $\overline{\mathbf{H}}_\Gamma$ is the cofinite model for the proper classifying space $\underline{E}\Gamma$. In Definition III.5, we define a space $\overline{\mathbf{H}}_\Gamma$ from the upper half-plane by attaching boundary components corresponding to certain parabolic subgroups of $SL(2,\mathbb{R})$. We then endow a topology on $\overline{\mathbf{H}}_\Gamma$ in Proposition III.6 and classify all closed sets in that topology (Corollary IV.5). Using this classification, in Proposition III.12, we explicitly describe the elements of the open basis which generates the same topology of $\overline{\mathbf{H}}_\Gamma$. The description of open basis will be used to show that the space $\overline{\mathbf{H}}_\Gamma$ is a manifold with boundary in section §3.4.

**Definition III.1.** Let $P$ be a parabolic subgroup of $SL(2,\mathbb{R})$ and $P = N_P \times A_P \times M_P$ be the Langlands decomposition of $P$ where $N_P$ is the nilpotent normal subgroup of $P$. The **boundary component** $e(P)$ **corresponding to** $P$ is the space defined by

$$e(P) = N_P \tag{3.1}$$

(c.f. §2.3.4 Definition 2.53).

**Definition III.2.** The set $\Delta_\Gamma$ is a collection of minimal parabolic subgroups $P$ of $SL(2,\mathbb{R})$ such that $\Gamma \cap N_P$ is a lattice in $N_P$.

*Remark* III.3. Recall that for every parabolic subgroup $P$ of $SL(2,\mathbb{R})$, the nilpotent subgroup $N_P$ is isomorphic to $\mathbb{R}$ (c.f. §2.4.4). Thus $\Gamma \cap N_P$ being a lattice in $N_P$ implies that $\Gamma \cap N_P$ is a uniform lattice in $N_P$.

**Proposition III.4.** *For every parabolic subgroup $P$ in $\Delta_\Gamma$, the following holds:*

$$\Gamma \cap P = \Gamma \cap N_P.$$

*Proof.* We first show that the statement holds for $P = P_\infty$. Since $\Gamma \cap N_P \neq \emptyset$, there exists an element $\gamma$ in $\Gamma$ such that

$$\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \, b \neq 0.$$

We prove that $\Gamma \cap P_\infty = \Gamma \cap N_{P_\infty}$ by contradiction. Suppose there exists $\gamma_1 \in \Gamma \cap P_\infty \backslash N_{P_\infty}$ such that

$$\gamma_1 = \left(\begin{smallmatrix} a & b_1 \\ 0 & a^{-1} \end{smallmatrix}\right), \, a \neq 1.$$

Without loss of generality, assume $a < 1$. Then for every integer $n > 0$,

$$\gamma_1^n \gamma \gamma_1^{-n} = \left(\begin{smallmatrix} 1 & a^{2n}b \\ 0 & 1 \end{smallmatrix}\right).$$

Thus $\gamma_1^n \gamma \gamma_1^{-n} \to Id$. This is a contradiction because $\Gamma$ is discrete. Let $P$ be a parabolic subgroup in $\Delta_\Gamma$ and $k$ be an element in $SO(2)$ such that ${}^kP = P_\infty$. Since $\Gamma \cap P \neq \emptyset$, it follows that

$$\begin{aligned} {}^k\Gamma \cap {}^kP &= {}^k\Gamma \cap P_\infty \\ &= {}^k\Gamma \cap N_{P_\infty} \end{aligned}$$

By taking the conjugation of $k^{-1}$ on both sides, we obtain $\Gamma \cap P = \Gamma \cap N_P$. □

**Definition III.5.** The set $\overline{\mathbf{H}}_\Gamma$ is the union of the upper half-plane and the boundary components corresponding to all parabolic subgroups in the set $\Delta_\Gamma$:

$$\overline{\mathbf{H}}_\Gamma = \mathbf{H} \cup \coprod_{P \in \Delta_\Gamma} e(P). \tag{3.2}$$

The union of boundary components $\coprod_{P \in \Delta_\Gamma} e(P)$ is called the **boundary** of $\overline{\mathbf{H}}_\Gamma$. The space $\mathbf{H}$ is called the **interior** of $\overline{\mathbf{H}}_\Gamma$.

In the next proposition (Proposition III.6), we define a topology on the set $\overline{\mathbf{H}}_\Gamma$ in terms of the convergence class of sequences (c.f. Definition II.56), followed by the Corollary IV.5, which classifies all closed subsets of $\overline{\mathbf{H}}_\Gamma$ in that topology.

**Proposition III.6.** *Let $\mathcal{T}$ be the topology on $\overline{\mathbf{H}}_\Gamma$ defined by the convergence class of sequence $\mathcal{C}$ consisting of all combinations of the following types of convergent sequences:*

**Type S1** *Let $y_j$ be a sequence in the upper half-plane $\mathbf{H}$ which converges to $y_\infty \in \mathbf{H}$ with respect to the topology of $\mathbf{H}$. Then $y_j \stackrel{\mathcal{C}}{\to} y_\infty$.*

**Type S2** *Let $y_j$ be a sequence in a boundary component $e(P)$ corresponding to a parabolic subgroup $P \in \Delta_\Gamma$ which converges to $y_\infty \in e(P)$ with respect to the topology of $e(P)$. Then $y_j \stackrel{\mathcal{C}}{\to} y_\infty$.*

**Type S3** *Let $y_j$ be a unbounded sequence in the upper half-plane $\mathbf{H}$. If there exists a parabolic subgroup $P \in \Delta_\Gamma$ such that*

- *$y_j = (n_j, a_j)$ in terms of horospherical decomposition of $\mathbf{H}$ with respect to $P$,*
- *$n_j \to y_\infty$ with respect to the topology of $e(P)$, and*
- *$a_j^\alpha \to \infty$ where $\alpha$ is the restricted root in $\Phi(P, A_P)$ (c.f. Definition II.15),*

*then $y_j \stackrel{\mathcal{C}}{\to} y_\infty \in e(P)$.*

*Then the following types of subsets of $\overline{\mathbf{H}}_\Gamma$ are closed in $\mathcal{T}$.*

**Type C1** *All subsets in $\mathbf{H}$ which are closed and bounded with respect to the topology of $\mathbf{H}$.*

**Type C2** *Every disjoint union*

$$\coprod_{P \in \Delta_\Gamma} C_P$$

*where each $C_P$ is a closed subset of $e(P)$.*

**Type C3** *Every subset* $\overline{\mathfrak{S}}_{P,C_P,t}$ *defined as follow. For a closed subset* $C_P$ *of a boundary component* $e(P)$ *and* $\overline{A}_{P,t} = \{a \in A_P \mid a^\alpha \geq t, \alpha \in \Phi(P, A_P)\}$,

$$\overline{\mathfrak{S}}_{P,C_P,t} := (C_P \times \overline{A}_{P,t}) \cup C_P.$$

*Proof.* A subset $A$ is closed if and only if its closure $\overline{A}$ is $A$ itself. The closure $\overline{A}$ is defined by (c.f. Proposition II.57)

$$\overline{A} = \{y_\infty \in \overline{\mathbf{H}}_\Gamma \mid y_j \xrightarrow{\mathcal{C}} y_\infty,\ \{y_j\}_{j=1}^\infty \subset A\}.$$

Since every constant sequence always converges (Condition 1 in Definition II.56), $A \subset \overline{A}$. We will show that for subsets of types **C1**, **C2**, and **C3**, the converse inclusion $\overline{A} \subset A$ holds.

1. Let $A$ be a subset of the type **C1**, i.e. $A$ is a closed and bounded subset of $\mathbf{H}$. Suppose $y_j \in A$ and $y_j \xrightarrow{\mathcal{C}} y_\infty$. Since $A$ is bounded, $y_\infty \in \mathbf{H}$. Thus $y_j$ converges to $y_\infty$ in the topology of $\mathbf{H}$. Since $A$ is closed in $\mathbf{H}$, $y_\infty \in A$. Therefore, $\overline{A} \subset A$.

2. Let $A$ be a subset of the type **C2**, i.e. $A = \coprod_{P \in \Delta_\Gamma} C_P$ where each $C_P$ is a closed subset of the boundary component $e(P)$. We first show that each subsets $C_P$ is closed in $\overline{\mathbf{H}}_\Gamma$. Then it follows that

$$\overline{A} = \overline{\coprod_{P \in \Delta_\Gamma} C_P} = \coprod_{P \in \Delta_\Gamma} \overline{C_P} = A.$$

   Suppose $\{y_j\}_{j=1}^\infty \subset C_P$ and $y_j \xrightarrow{\mathcal{C}} y_\infty$. Then the sequence $y_j$ is of type **S2**. So the limit $y_\infty$ belongs to $e(P)$. Since $C_P$ is closed in $e(P)$, $y_\infty \in C_P$. Thus $C_P$ is closed in $\overline{\mathbf{H}}_\Gamma$.

3. Let $A$ be a subset of the type **C3**, i.e. $A = \overline{\mathfrak{S}}_{P,C_P,t}$ for some parabolic subgroup $P$ in $\Delta_\Gamma$ and a closed subset $C_P$ of $e(P)$. Suppose $\{y_j\}_{j=1}^\infty \subset A$ and $y_j \xrightarrow{\mathcal{C}} y_\infty$. By passing to a subsequence (if necessary), the (sub)sequence $\{y_j\}_{j=1}^\infty$ is contained either in (a) the interior $\mathbf{H}$, or (b) the boundary component $e(P)$.

(a) Since $\{y_j\}_{j=1}^{\infty} \subset \overline{\mathfrak{S}}_{P,C_P,t}$, $y_j \in C_P \times \overline{A}_{P,t}$. Note that Since $C_P \times \overline{A}_{P,t}$ is closed in $\mathbf{H}$. If $y_j$ is bounded, then $y_j$ is of type **S1**. It follows that $y_\infty \in \mathbf{H}$, and thus $y_\infty \in C_P \times \overline{A}_{P,t}$. If $y_j$ is unbounded, then $y_j$ is of type **S3**, Thus $y_\infty \in C_P$.

(b) Since $\{y_j\}_{j=1}^{\infty} \subset C_P$, the sequence $y_j$ is of type **S2**. Therefore, $y_\infty \in C_P$.

In either case, we have shown that $y_\infty \in A$.

Thus all three types of subsets are closed in $\overline{\mathbf{H}}_\Gamma$. □

**Corollary III.7.** *With respect to the topology $\mathcal{T}$ on the set $\overline{\mathbf{H}}_\Gamma$ defined in Proposition III.6, every closed subset of $\overline{\mathbf{H}}_\Gamma$ is obtained by a combination of finite union and infinite intersection of closed subsets of the types **C1**, **C2**, and **C3**.*

*Proof.* Let $A$ be a closed subset of $\overline{\mathbf{H}}_\Gamma$. Then the closure $\overline{A}$ equal to $A$. This implies that for every sequence $y_j \in A$ such that $y_j \xrightarrow{\mathcal{C}} y_\infty$, the limit $y_\infty$ belongs to $A$. Suppose every convergent sequence $y_j \xrightarrow{\mathcal{C}} y_\infty$ has a subsequence $y'_j$ such that $y'_j = y_\infty$. Then $A$ is a discrete subset of $\overline{\mathbf{H}}_\Gamma$. Thus we assume that $A$ is not discrete and there is no constant subsequence $y'_j$ of every convergent sequence $y_j$ in $A$. By replacing with its further subsequence, we may assume that a sequence $y'_j$ is either of the type **S1**, **S2**, or **S3** in Proposition III.6. For each $n = 1, \cdots, 3$, let $A_n$ be the set

$$A_n = \{y_\infty \mid y'_j \xrightarrow{\mathcal{C}} y_\infty \text{ where } y'_j \text{ is of type } \mathbf{Sn}\}.$$

From Proposition III.6, each $A_n$ is a closed subset in $\overline{\mathbf{H}}_\Gamma$. Since

$$A = A_1 \cup A_2 \cup A_3,$$

we are done. □

*Remark* III.8. Let $C_P$ be a bounded, closed, and connected subset of a boundary component $e(P)$ whose interior $Int(C_P)$ is non-empty. Then the subset of the type

**C3** in Proposition III.6 is the closure of the following Siegel set (c.f. §2.3.2 Definition II.48):

$$\mathfrak{S}_{P,Int(C_P),t} = Int(C_P) \times A_{P,t}.$$

From now on, we always assume that the space $\overline{\mathbf{H}}_\Gamma$ in the topology $\mathcal{T}$ defined in Proposition III.6. We will observe that the space $\overline{\mathbf{H}}_\Gamma$ is a manifold with boundary in Section §3.4. To do so, we describe the topology of $\overline{\mathbf{H}}_\Gamma$ in terms of open basis in the Proposition III.12. In the next two lemmas, we observe two special types of open subsets in $\overline{\mathbf{H}}_\Gamma$.

**Lemma III.9.** *Every open subset of* $\mathbf{H}$ *is open in* $\overline{\mathbf{H}}_\Gamma$*.*

*Proof.* Let $U$ be an open subset of $\mathbf{H}$. The complement of $U$ in $\overline{\mathbf{H}}_\Gamma$ is the union of complement of $U$ in $\mathbf{H}$ and the boundary of $\overline{\mathbf{H}}_\Gamma$:

$$U^c = (\mathbf{H} - U) \cup \coprod_{P \in \Delta_\Gamma} e(P).$$

Since each $\mathbf{H} - U$ and $\coprod_{P \in \Delta_\Gamma} e(P)$ is closed, $U^c$ is closed. □

**Lemma III.10.** *Let $U$ be an open subset of a boundary component $e(P)$ for some parabolic subgroup $P$ in $\Delta_\Gamma$. For a positive real number $t$, let us define a subset $\mathfrak{S}_{P,U,t} = U \times A_{P,t}$. Then the union*

$$\mathfrak{S}_{P,U,t} \cup U$$

*is open in* $\overline{\mathbf{H}}_\Gamma$*.*

*Proof.* The complement of $\mathfrak{S}_{P,U,t} \cup U$ is the union of following closed subsets:

$$(\mathfrak{S}_{P,U,t} \cup U)^c = (\mathbf{H} - \mathfrak{S}_{P,U,t}) \cup (e(P) - U) \cup \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} e(Q).$$

From Proposition III.6, each component is closed in $\overline{\mathbf{H}}_\Gamma$. □

*Remark* III.11. The subset $\mathfrak{S}_{P,U,t}$ in Lemma III.10 is a Siegel set in $\mathbf{H}$ only if $U$ is bounded. The only condition on the set $U$ is that $U$ is a open subset of the boundary component $e(P)$. If $U$ is $e(P)$ itself, then $\mathfrak{S}_{P,U,t}$ is a horoball at $P$ (c.f. Definition II.51).

**Proposition III.12.** *In the topology $\mathcal{T}$ of $\overline{\mathbf{H}}_\Gamma$ defined in Proposition III.6, the open basis of $\overline{\mathbf{H}}_\Gamma$ consists of the following subsets:*

**Type O1** *All open subsets in the upper half-plane $\mathbf{H}$.*

**Type O2** *Every subset of the form*

$$\mathfrak{S}_{P,U,t} \cup U$$

*where $U$ is an open subset of a boundary component $e(P)$ corresponding to a parabolic subgroup $P$ in $\Delta_\Gamma$, $t$ is a positive real number, and $\mathfrak{S}_{P,U,t} = U \times A_{P,t}$.*

*Proof.* By Lemma III.9 and Lemma III.10, subsets of types **O1** and **O2** are open. By Corollary IV.5, the topology of $\overline{\mathbf{H}}_\Gamma$ is equivalent to the topology on $\overline{\mathbf{H}}_\Gamma$ generated by closed basis consists of subsets of the types **C1**, **C2**, and **C3**. We will show that the complements of such closed subsets are expressed as the union of open subsets of the types **O1** and **O2**. Thus the collection of all open subsets of these types generates (with respect to the axioms of open sets) the same topology on $\overline{\mathbf{H}}_\Gamma$.

1. Let $C$ be a closed subset of the type **C1**, i.e. $C$ is a bounded and closed subset of $\mathbf{H}$. Its complement $C^c$ in $\overline{\mathbf{H}}_\Gamma$ is expressed as follow.

$$C^c = (\mathbf{H} - C) \cup \coprod_{P \in \Delta_\Gamma} e(P).$$

   The set $\mathbf{H} - C$ is open. For sufficiently large $T \gg 1$, the horoball $S_{P,T}$ (c.f. Definition II.51 and Example II.52) is disjoint from $C$ for all parabolic subgroups

$P \in \Delta_\Gamma$. Since

$$\coprod_{P\in\Delta_\Gamma} e(P) \subset \bigcup_{P\in\Delta_\Gamma} \left(S_{P,T} \cup N_P\right),$$

the complement $C^c$ is expressed as follow:

$$C^c = (\mathbf{H} - C) \cup \bigcup_{P\in\Delta_\Gamma} \left(S_{P,T} \cup N_P\right).$$

The set $\mathbf{H} - C$ and the union $S_{P,T} \cup N_P$ are open subset of the type **O1** and **O2** respectively.

2. Let $C$ be a closed subset of the type **C2**, i.e.

$$C = \coprod_{P\in\Delta_\Gamma} C_P$$

where each $C_P$ is a closed subset of a boundary component $e(P)$. Let $U_P = e(P) - C_P$. The complement $C^c$ is expressed as follows:

$$C^c = \mathbf{H} \cup \coprod_{P\in\Delta_\Gamma} U_P.$$

For an arbitrary positive real number $t$,

$$\coprod_{P\in\Delta_\Gamma} U_P \subset \bigcup_{P\in\Delta_\Gamma} \left(\mathfrak{S}_{P,U_P,t} \cup U_P\right).$$

(Note that if $C_P = \emptyset$, then $\mathfrak{S}_{P,U_P,t}$ is the horoball $S_{P,t}$.) Thus the complement $C^c$ is also expressed as follow:

$$C^c = \mathbf{H} \cup \bigcup_{P\in\Delta_\Gamma} \left(\mathfrak{S}_{P,U_P,t} \cup U_P\right).$$

The set $\mathbf{H}$ and the unions $\mathfrak{S}_{P,U_P,t} \cup U_P$ are open subsets of the types **O1** and **O2** respectively.

3. Let $C$ be a closed subset of the type **C3**, i.e.

$$C = \left(C_P \times \overline{A}_{P,t}\right) \cup C_P$$

for a parabolic subgroup $P$ in $\Delta_\Gamma$ and a closed subset $C_P$ of a boundary component $e(P)$. The complement $C^c$ is expressed as follow:

$$C^c = \big(\mathbf{H} - (C_P \times \overline{A}_{P,t})\big) \cup (e(P) - C_P) \cup \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} e(Q).$$

In terms of the horospherical decomposition of $\mathbf{H}$ with respect to $P$, the subset $C_P \times \overline{A}_{P,t}$ is closed in $\mathbf{H}$. Thus the subset $\mathbf{H} - (C_P \times \overline{A}_{P,t})$ is open in $\mathbf{H}$. Next, for sufficiently large $T \gg 1$, every horoball $S_{Q,T}$ at a parabolic subgroup $Q \neq P$ in $\Delta_\Gamma$ is disjoint from $C_P \times \overline{A}_{P,t}$ (Figure 3.2 below).

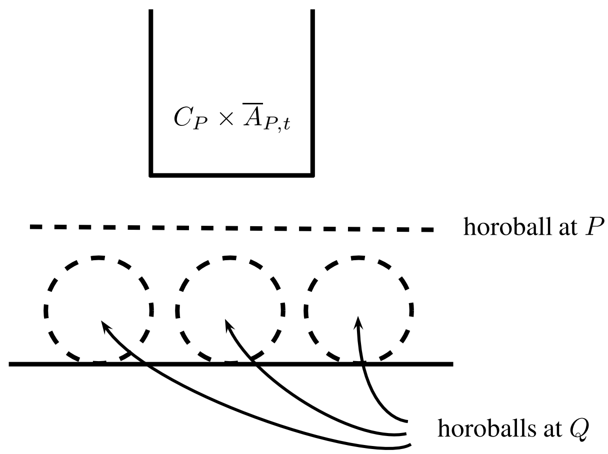


Figure 3.2: The disjoint horoballs in $\mathbf{H}$

Thus we have

$$\coprod_{Q \in \Delta_\Gamma \setminus \{P\}} S_{Q,T} \subset \mathbf{H} - (C_P \times \overline{A}_{P,t}).$$

Therefore, the complement $C^c$ is alternatively expressed as follow:

$$C^c = (\mathbf{H} - (C_P \times \overline{A}_{P,t})) \cup (\mathfrak{S}_{P,U_P,t} \cup U_P) \cup \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} (S_{Q,T} \cup e(Q))\,.$$

The subset $\mathbf{H} - (C_P \times \overline{A}_{P,t})$ is an open set of the type **O1** and the subset $\mathfrak{S}_{P,U_P,t} \cup U_P$ and $S_{Q,T} \cup N_Q$ are open sets of the type **O2**.

This completes the proof. □

**Corollary III.13.** *In the topology $\mathcal{T}$ of $\overline{\mathbf{H}}_\Gamma$ defined in Proposition III.6, the interior (the boundary, respectively) of $\overline{\mathbf{H}}_\Gamma$ is the upper half-plane $\mathbf{H}$ (the disjoint union $\coprod_{P\in\Delta_\Gamma} e(P)$, respectively).*

*Proof.* Since $\mathbf{H}$ is open, it is enough to show that every point in the boundary (as a name we defined in Definition III.5) $\coprod_{P\in\Delta_\Gamma} e(P)$ does not admit any open neighborhood contained in the boundary. However, every open neighborhood is either of type **O1** or **O2**, and neither is contained in the boundary. Thus $\coprod_{P\in\Delta_\Gamma} e(P)$ is indeed the boundary of $\overline{\mathbf{H}}_\Gamma$. □

*Remark* III.14. Every open neighborhood of a point on the boundary is of the type **O2**. This type of neighborhoods will be used in Proposition III.29 of Section §3.4 to show that the space $\overline{\mathbf{H}}_\Gamma$ is a manifold with boundary.

In the next section, we define a Γ-action on the space $\overline{\mathbf{H}}_\Gamma$ and show that the action is proper.

### 3.2 The $\Gamma$-action on $\overline{\mathbf{H}}_\Gamma$

In Definition III.15 and Definition III.16, we define the action of the group $\Gamma$ on the interior $\mathbf{H}$ and boundary $\coprod_{P\in\Delta_\Gamma} e(P)$ of $\overline{\mathbf{H}}_\Gamma$ respectively. In Proposition III.20, we prove that this action extends the Möbius transformation of the group $\Gamma$ on the upper half-plane $\mathbf{H}$. In Proposition III.23, we show that the $\Gamma$-action is proper.

**Definition III.15.** The **Möbius transformation** of an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the group $SL(2,\mathbb{R})$ on the upper-half plane $\mathbf{H}$ is defined by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}, \quad z \in \mathbf{H}.$$

**Definition III.16.** Let $\Gamma$ be a Fuchsian lattice. The action of $\Gamma$ on the boundary $\coprod_{P\in\Delta_\Gamma} e(P)$ (Definition III.5) of $\overline{\mathbf{H}}_\Gamma$ is defined as follow: For a point $\zeta \in e(P)$ and an element $\gamma \in \Gamma$ with the coordinate $(n, a, k)$ in terms of Iwasawa decomposition of $N_P \times A_P \times SO(2)$ of $SL(2,\mathbb{R})$,

$$\gamma \cdot \zeta = {}^k(n^a\zeta).$$

**Lemma III.17.** *The $\Gamma$-action on the boundary of $\overline{\mathbf{H}}_\Gamma$ is well-defined.*

*Proof.* The subgroup $\Gamma \cap {}^kN_P$ is a lattice in ${}^kN_P$ if and only if $\Gamma \cap N_P$ is a lattice in $N_P$. In other words, ${}^kP \in \Delta_\Gamma$ if and only if $P \in \Delta_\Gamma$. Thus $e({}^kP) = {}^ke(P)$ and the $\Gamma$-action on the boundary is well-defined. □

**Example III.18.** Suppose $\zeta = \left(\begin{smallmatrix} 1 & \kappa \\ 0 & 1 \end{smallmatrix}\right) \in e(P_\infty)$ and $\gamma = k\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ where $k \in SO(2)$. Then

$$\gamma \cdot \zeta = k\left(\begin{smallmatrix} 1 & b+\kappa \\ 0 & 1 \end{smallmatrix}\right)k^{-1} \in e({}^kP_\infty).$$

In the next proposition (Proposition III.20), we show that the $\Gamma$-action on the boundary is continuously extended from the Möbius transformation on the interior.

We first observe the Möbius transformation in terms of horospherical decomposition of the upper-half plane $\mathbf{H}$.

**Lemma III.19.** *Let $z$ be a point in $\mathbf{H}$ and $\gamma$ be an element in $\Gamma$. Suppose $(n_1, a_1)$ is the horospherical coordinate of $z$ with respect to a parabolic subgroup $P$ and $(n, a, k)$ is the coordinate of $\gamma$ in terms of Iwasawa decomposition $N_P \times A_P \times SO(2)$ of $SL(2, \mathbb{R})$. Then the horoshperical coordinate of the Möbius transformation $\gamma \cdot z$ with respect to ${}^kP$ is*

$$({}^k(n^a n_1), {}^k(aa_1)) \in N_{{}^kP} \times A_{{}^kP}.$$

*Proof.* We first show that the statement holds for $\gamma = k \in SO(2)$. We then prove the special case when $P = P_\infty$. Lastly, we prove the lemma for general case.

**Step 1** Suppose $\gamma = k \in SO(2)$. From the diffeomorphism

$$N_P \times A_P \to \mathbf{H}; \quad (n_1, a_1) \mapsto (n_1 a_1) \cdot i,$$

it follows that $k \cdot z = k \cdot ((n_1 a_1) \cdot i)$. Since every element in $SO(2)$ fixes $i$, we have

$$k \cdot ((n_1 a_1) \cdot i) = ({}^k n_1 {}^k a_1) \cdot i.$$

Thus the horospherical coordinates of $k \cdot z$ with respect to ${}^kP$ is

$$({}^k n_1, {}^k a_1) \in N_{{}^kP} \times A_{{}^kP}.$$

**Step 2** Suppose $P = P_\infty$. Let

$$\left( \left( \begin{smallmatrix} 1 & \kappa \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{smallmatrix} \right) \right), \quad \left( \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right), k \right)$$

be the horospherical coordinates of $z$ and the coordinates of $\gamma$ in terms of the Iwasawa decomposition $N_P \times A_P \times SO(2)$ of $SL(2, \mathbb{R})$. Note that

$$z = \kappa + i\eta^2 \in \mathbf{H}, \quad \gamma = k \left( \begin{smallmatrix} a & ba^{-1} \\ 0 & a^{-1} \end{smallmatrix} \right) \in SL(2, \mathbb{R}).$$

The Möbius tranformation $\gamma \cdot z$ is

$$\gamma \cdot z = k \cdot (a^2\kappa + b + i(a\eta)^2).$$

Thus the horospherical coordinate of $\gamma \cdot z$ with respect to ${}^kP$ is

$$\left({}^k\begin{pmatrix} 1 & a^2\kappa+b \\ 0 & 1 \end{pmatrix}, {}^k\begin{pmatrix} a\eta & 0 \\ 0 & (a\eta)^{-1} \end{pmatrix}\right).$$

It follows from a simple calculation that

$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, k\right) \cdot \left(\begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}\right) = \left({}^k\begin{pmatrix} 1 & a^2\kappa+b \\ 0 & 1 \end{pmatrix}, {}^k\begin{pmatrix} a\eta & 0 \\ 0 & (a\eta)^{-1} \end{pmatrix}\right).$$

**Step 3** Now we consider an arbitrary parabolic subgroup $P$. Since every parabolic subgroup is conjugate to the standard parabolic subgroup $P_\infty$, there exists $h \in SO(2)$ such that $P_\infty = {}^hP$. Let us assume that the coordinates of $z$ and $\gamma$ are given as in the statement of the lemma. Since $\gamma = kna = (kh^{-1\,h}n^{h}a) \cdot h$, it follows that

$$\gamma \cdot z = (kh^{-1\,h}n^{h}a) \cdot (h \cdot z).$$

From **Step 1**, the horospherical coordinates of $h \cdot z$ with respect to $P_\infty$ is

$$({}^hn_1, {}^ha_1).$$

The coordinates of $kh^{-1\,h}n^{h}a$ in terms of the Iwasawa decomposition $N_{P_\infty} \times A_{P_\infty} \times SO(2)$ of $SL(2,\mathbb{R})$ is

$$({}^hn, {}^ha, kh^{-1}).$$

It follows from **Step 2** that the coordinates of $(kh^{-1\,h}n^{h}a) \cdot (h \cdot z)$ is

$$\begin{aligned}(kh^{-1\,h}n^{h}a) \cdot (h \cdot z) &= ({}^{kh^{-1}}({}^hn^{{}^ha}({}^hn_1), {}^{kh^{-1}}({}^ha^ha_1)) \\ &= ({}^k(n^an_1), {}^k(aa_1)).\end{aligned}$$

The last equality follows from a simple calculation.

This completes the proof. □

**Proposition III.20.** *Let a Fuchsian lattice* $\Gamma$ *acts on the space* $\overline{\mathbf{H}}_\Gamma$ *as follow:*

- *the Möbius transformation on the interior* $\mathbf{H}$ *in Definition III.15, and*
- *the action on the boundary* $\coprod_{P\in\Delta_\Gamma} e(P)$ *in Definition III.16.*

*Then the* $\Gamma$*-action is continuous, i.e. for every convergent sequence* $y_j \to y_\infty$ *in* $\overline{\mathbf{H}}_\Gamma$ *and every element* $\gamma \in \Gamma$*,*

$$\gamma \cdot y_j \to \gamma \cdot y_\infty.$$

*Proof.* Since the topology of $\overline{\mathbf{H}}_\Gamma$ is essentially defined by the convergent sequences of types **S1**, **S2**, and **S3** (c.f. Proposition III.6), it is enough to check that for those types of sequence, the statement holds.

1. Let $y_j$ be a convergent sequence of the type **S1**, i.e. a convergent sequence in the interior $\mathbf{H}$. Since the Möbius transformation is continuous, we are done.

2. Let $y_j$ be a convergent sequence of the type **S2**, i.e. a convergent sequence in a boundary component $e(P)$ for some parabolic subgroup $P$ in $\Delta_\Gamma$. Since action on the boundary (Definition III.16) is defined by the matrix multiplication, it is continuous.

3. Let $y_j$ be a convergent sequence of the type **S3**, i.e. $y_j = (n_j, a_j)$ in terms of horospherical decomposition of $\mathbf{H}$ with respect to a parabolic subgroup $P$ in $\Delta$ such that $n_j \to y_\infty \in e(P)$ and $a_j^\alpha \to \infty$ for all restricted roots $\alpha \in \Phi(P, A_P)$ (c.f. Definition III.5). Let $(n, a, k)$ be the coordinate of an element $\gamma \in \Gamma$ in terms of the Iwasawa decomposition $N_P \times A_P \times SO(2)$ of $SL(2, \mathbb{R})$. From Lemma III.19, the coordinates of the Möbius transformation $\gamma \cdot y_j$ is

$$\gamma \cdot y_j = ({}^k(n^a n_j), {}^k(a a_j)).$$

Since $n_j \to y_j$, it follows that ${}^k(n^a n_j) \to {}^k(n^a y_\infty)$. For every restricted root $\beta$ in $\Phi({}^kP, A_{{}^kP})$, there exists a restricted root $\alpha$ in $\Phi(P, A_P)$ such that $\beta = {}^k\alpha$ (c.f. Proposition II.68 and Notation II.69). Thus ${}^k(aa_j)^\beta = (aa_j)^\alpha = a^\alpha a_j^\alpha \to \infty$. Therefore, $\gamma \cdot y_j \to {}^k(n^a y_\infty) = \gamma \cdot y_\infty$.

This completes the proof. □

*Remark* III.21. For simplicity, we denote $\gamma = (n, a, k) \in N_P \times A_P \times SO(2)$ and $z = (n_1, a_1) \in N_P \times A_P$ for element $\gamma \in \Gamma$ and $z \in \mathbf{H}$ respectively.

*Remark* III.22. Let $P$ be a parabolic subgroup in $\Delta_\Gamma$. Note that the action of every element in $\Gamma \cap P$ stabilizes the boundary component $e(P)$. If an element $\gamma \in \Gamma$ has the coordinate $\gamma = (n, a, k) \in N_P \times A_P \times K$ where $k \neq Id$, then $\gamma$ induces a homeomorphism $\gamma : e(P) \to e({}^kP)$:

$$\begin{array}{ccc} e(P) & \xrightarrow{\gamma} & e({}^kP) \\ & \searrow^{(n,a,Id)} & \uparrow k \\ & & e(P). \end{array}$$

In the next proposition (Proposition III.23), we show that the $\Gamma$-action defined in Proposition III.20 is proper.

**Proposition III.23.** *Let a Fuchsian lattice $\Gamma$ acts on the space $\overline{\mathbf{H}}_\Gamma$ as follow:*

- *the Möbius transformation on the interior $\mathbf{H}$ in Definition III.15, and*
- *the action on the boundary $\coprod_{P \in \Delta_\Gamma} e(P)$ in Definition III.16.*

*Then the $\Gamma$-action is proper, i.e. for every compact subset $C$ of $\overline{\mathbf{H}}_\Gamma$, the subset $\Gamma'$ of $\Gamma$, defined by*

$$\Gamma' = \{\gamma \in \Gamma \mid \gamma \cdot C \cap C \neq \emptyset\},$$

*is finite.*

*Proof.* From Proposition III.6 and Corollary IV.5, every compact subset $C$ is essentially of the type **C1**, **C2**, or **C3**. We claim that for each cases, the set $\Gamma'$ is finite.

**Case 1** Suppose that the set $C$ is of the type **C1**, i.e. $C$ is a compact subset of $\mathbf{H}$. Since the lattice $\Gamma$ is discrete subgroup of $SL(2,\mathbb{R})$, the $\Gamma$-action on $\mathbf{H}$ is proper, thus the set $\Gamma'$ is finite.

**Case 2** Suppose that the set $C$ is of the type **C2**, i.e. $C$ is a finite disjoint union of closed subsets of $e(P)$, say

$$C = \coprod_{P \subset \Delta} C_P$$

where $\Delta$ is a finite subset of $\Delta_\Gamma$. Let $\Gamma'_{P,Q} = \{\gamma \in \Gamma' \mid \gamma \cdot C_P \cap C_Q \neq \emptyset\}$ so that

$$\Gamma' = \bigcup_{P,Q \in \Delta} \Gamma'_{P,Q}.$$

We will show that each subset $\Gamma'_{P,Q}$ is finite.

1. Suppose $P = Q$. Then $\Gamma'_{P,Q} \subset \Gamma \cap P = \Gamma \cap N_P$ (c.f. Proposition III.4). The group $\Gamma \cap N_P$ acts on $e(P)$ cocompactly by definition of $\Delta_\Gamma$ (c.f. Definition III.2 and Remark III.3). Thus the set $\Gamma'_{P,Q}$ is finite.

2. Suppose $P \neq Q$. For every element $\gamma \in \Gamma_{P,Q}$, let $\gamma = (n, a, k) \in N_P \times A_P \times SO(2)$. Since ${}^kP = Q$, there are only finitely many choices of the coordinate $k$. For each choice of $k$, we have

$$(n, a, Id) \cdot C_P \cap k^{-1} \cdot C_Q \neq \emptyset.$$

   Since $k^{-1} \cdot C_Q$ and $C_P$ are compact subsets in $e(P)$, there are only finitely many choice of the points $(n, a, Id)$.

This proves the second case.

**Case 3** Lastly, suppose that $C$ is of the type **C3**, i.e.

$$C = \overline{\mathfrak{S}}_{P,C_P,t}$$

where $C_P$ is a compact subset of $e(P)$. We will prove that the set $\Gamma'$ is finite by contradiction. Suppose there exists an infinite sequence $\gamma_j \in \Gamma'$ such that $\gamma_j \neq \gamma_i$ for all $i \neq j \geq 1$. For each $\gamma_j$, there exists an element $z_j \in C$ such that $\gamma_j \cdot z_j \in C$. Let us denote $z'_j = \gamma_j \cdot z_j$. If the set $\{z_j \mid j \geq 1\}$ is finite, then by passing to a subsequence, we may assume that $z_j = z$ for all $j$. Then the sequence $\{z'_j\}_1^\infty$ consists of infinitely many points in $C$, thus it admits an accumulation point in $C$. This contradicts to the discreteness of $\Gamma$. So let us assume that the set $\{z_j \mid j \geq 1\}$ is infinite, more specifically, $z_j \neq z_i$ for all $i \neq j \geq 1$. Two infinite sequences, $\{z_j\}_1^\infty$ and $\{z'_j\}_1^\infty$, are contained in a compact set $C$, thus there exist accumulation points $z_\infty$ and $z'_\infty$ such that

$$z_j \to z_\infty, \quad z'_j \to z'_\infty.$$

From Proposition III.6, each sequence is essentially of the type **S1**, **S2**, or **S3**. For each case, we will show a contradiction.

1. If either $z_j$ or $z'_j$ is of the type **S2**, then so is the other, because no point in the boundary can mapped into the interior by the $\Gamma$-action. In this case, the set $\Gamma'$ being infinite contradicts to the finiteness result from **Case 2**.
2. Suppose both $z_j$ and $z'_j$ are of the type **S1**. This contradicts to the **Case 1**. Also, we can directly show a contradiction as follow: for every small $\epsilon > 0$, there exists sufficiently large $N \gg 1$ such that

$$d(z_j, z_\infty), d(z'_j, z'_\infty) < \epsilon \quad \text{for all } j > N.$$

Then $d(z'_\infty, \gamma_j \cdot z_\infty) < 2\epsilon$ for all $j > N$. Thus $\gamma_j \cdot z_\infty \to z'_\infty$ which contradicts to the discreteness of $\Gamma$.

3. Suppose $z_j$ and $z'_j$ are of the types **S1** and **S3** respectively. (The proof of the remaining case follows from interchanging the roles of $z_j$ and $z'_j$ below.) Without loss of generality, we assume that

$$d(z'_j, z'_{j+1}) = N_j, \quad d(z_j, z_\infty) = \epsilon_j$$

such that $N_j < N_{j+1}$, $\epsilon_j > \epsilon_{j+1}$, and $N_j \to \infty$, $\epsilon_j \to 0$ as $j \to \infty$. By the triangle inequality, it follows that

$$d(\gamma_j \cdot z_\infty, z'_j) = \epsilon_j$$

and

$$d(\gamma_j \cdot z_\infty, \gamma_{j+1} \cdot z_\infty) > N_j - \epsilon_j - \epsilon_{j+1}.$$

Thus $\gamma_j \cdot z_\infty$ is a convergent sequence of the type **S3**. Then for almost all $j \geq 1$, elements $\gamma_j$ belong to $\Gamma \cap P = \Gamma \cap N_P$. The cocompactness of the lattice $\Gamma \cap N_P$ implies that $\{\gamma_j \mid j \geq 1\}$ is finite.

This completes the proof. □

### 3.3 Cocompactness of the $\Gamma$-action on $\overline{\mathbf{H}}_\Gamma$

In this section, we will show that the $\Gamma$-space $\overline{\mathbf{H}}_\Gamma$ is cocompact, i.e. the quotient space $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$ is compact. Using the Dirichlet fundamental domain for $\Gamma$, we will construct a fundamental domain for $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$ by attaching boundaries components. In Proposition III.24, we show that every boundary component of $\overline{\mathbf{H}}_\Gamma$ corresponds to a point at infinity in $\mathbf{H}(\infty)$ which is $\Gamma$-equivalent to a vertex at infinity of a Dirichlet fundamental domain, and vice versa. In Proposiiton III.25, we observe that the boundary components of $D$ is a fundamental domain of the boundary component of $\overline{\mathbf{H}}_\Gamma$. Using above propositions, in Proposition III.26, we show that $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$ is compact.

We first observe the geometric property of the boundary components.

**Proposition III.24.** *Let $D$ be a Dirichlet fundamental domain for $\Gamma$ in $\mathbf{H}$ at some base point. For each minimal parabolic subgroup $P$ of $SL(2,\mathbb{R})$, let $\xi_P$ be the points at infinity in $\mathbf{H}(\infty)$ corresponding to $P$ (c.f. Corollary II.75). Then the parabolic subgroup $P$ belong to the set $\Delta_\Gamma$ if and only if the point at infinity $\xi_P$ is a $\Gamma$-equivalent to a vertex at infinity of the domain $D$.*

*Proof.* We first show that for every vertex at infinity $\xi$ of $D$, the corresponding parabolic subgroup $P_\xi$ belongs to the set $\Delta_\Gamma$. We then show that the same statement holds for every point at infinity $\Gamma$-equivalent to $\xi$. Finally, we prove that if a parabolic subgroup $P$ belongs to the set $\Delta_\Gamma$, then the corresponding point at infinity $\xi_P$ is $\Gamma$-equivalent to a vertex at infinity of $D$.

Let $\xi$ be a point at infinity of $D$. Then there exists a parabolic element $\gamma$ in $\Gamma\cap P$. Thus $\Gamma\cap N_P$ is non-empty.

Suppose that a point at infinity $\xi'$ is $\Gamma$-equivalent to $\xi$, i.e. $\xi' = \gamma\cdot\xi$ for some $\gamma\in\Gamma$. Let $P$ and $P'$ be the minimal parabolic subgroup of $SL(2,\mathbb{R})$ corresponding to $\xi$ and $\xi'$ respectively. Let $k$ be an element in $SO(2)$ such that ${}^kP = P'$. Since $k : P \to {}^kP$ is a diffeomorphism, the diagram

$$\begin{array}{ccc} P & \xrightarrow{k} & P' \\ \uparrow & & \uparrow \\ \Gamma \cap N_P & \xrightarrow{k} & \Gamma \cap N_{P'} \end{array}$$

commutes, so $\Gamma \cap N_P \neq \emptyset$.

Suppose that a parabolic subgroup $P$ belong to $\Delta_\Gamma$. Then there exists a parabolic element in $\Gamma \cap N_P$ fixing $\xi_P$. We will show that $\xi_P$ is $\Gamma$-equivalent to a vertex at infinity of $D$ for the special case when $P = P_\infty$ and $\xi = i\infty$. Assuming this is true, let $P$ be a general parabolic subgroup in $\Delta_\Gamma$. There exists an element $k \in SO(2)$ such that ${}^kP = P_\infty$. Then $k \cdot \xi_P = i\infty$ is ${}^k\Gamma$-equivalent to a vertex at infinity of $k \cdot D$. Therefore, $\xi_P$ is $\Gamma$-equivalent to a vertex at infinity of $D$.

Let us assume that the standard parabolic subgroup $P_\infty$ is in $\Delta_\Gamma$. Let $s : [0, \infty) \to \mathbf{H}$ be a geodesic segment defined by

$$s(t) = t + i.$$

If there exists an element $\gamma$ in $\Gamma$ such that $s(t) \in \gamma \cdot D$ for all $t > N \gg 1$, then we are done. Suppose there exists a infinite sequence of increasing numbers $t_j > 0$ and a sequence of elements $\gamma_j \in \Gamma$ such that

$$\gamma_j \neq \gamma_j \text{ for all } i \neq j \geq 1 \text{ and } s(t_j) \in \gamma_j \cdot D.$$

For each $j \geq 1$, let $s_j : [0, \infty) \to \mathbf{H}$ be a geodesic segment lying in $\gamma_j \cdot D$ such that

$$s_j(0) = s(t_j) \text{ and } s_j(t) \to v_j \text{ as } t \to \infty$$

where $v_j$ is a vertex at infinity of $\gamma_j \cdot D$. Since $v_j \in \mathbb{R}$, by passing to a subsequence if necessary, the sequence $v_j$ is either bounded or diverges to infinity.

1. Suppose $v_j$ is bounded, i.e. $v_j \to v_\infty \in \mathbb{R}$. The geodesic $s_j$ converges to the geodesic $t \mapsto v_\infty + \exp(t)$. Thus for a point $z = v_\infty + i$ in $\mathbf{H}$ and a neighborhood $U$ of $z$, the intersection $s_j[0,\infty) \cap U$ is non-empty for almost all $j \geq 1$. This contradicts to the local finiteness of the Dirichlet fundamental domain.

2. Suppose $v_j$ diverges to infinity. Let us assume

$$\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$$

   for some $b > 0$. For sufficiently large $j \gg 1$, the absolute value $|v_j|$ is greater than $b$. Then the $\gamma$-translates of the geodesic segment $s_j$ eventually intersects another geodesic segment. This contradicts to the definition of fundamental domain.

This completes the proof. □

In the next proposition (Proposition III.26), we prove that the closure of Dirichlet fundamental domain is a fundamental domain of $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$.

**Proposition III.25.** *Let $D$ be a Dirichlet fundamental domain of $\Gamma$, $\xi_n$ ($n = 1, \cdots, r$) be the vertices at infinity of $D$, and $P_n$ be the minimal parabolic subgroups corresponding to $\xi_n$'s. Then the set $I_n$ defined by*

$$I_n = \left\{ y_\infty \in e(P_n) \mid y_j \stackrel{\mathcal{C}}{\to} y_\infty,\ y_j \in D \right\}.$$

*is a closed and locally finite fundamental domain of $\Gamma \cap N_{P_n}$ in $N_{P_n}$.*

*Proof.* We first prove that each $I_n$ is closed. Suppose $y_j \in e(P_n)$ and $y_j \stackrel{\mathcal{C}}{\to} y_\infty$ For each $j \geq 1$, there exists a sequence $y_{i,j} \in D$ such that

$$y_{i,j} \stackrel{\mathcal{C}}{\to} y_{\infty,j} = y_j.$$

By the condition 4 in Definition II.56, there exists an increasing function $j : \mathbb{N} \to \mathbb{N}$ such that

$$y_{i,j(i)} \xrightarrow{\mathcal{C}} y_\infty.$$

Therefore $y_\infty \in I_n$.

We next prove that for each $n = 1, \cdots, r$, the following holds:

$$(\Gamma \cap N_{P_n}) \cdot I_n = e(P_n).$$

Let $y_\infty \in e(P_n)$ and $y_j \in \mathbf{H}$ be a convergent sequence of the type **S3** such that $y_j \xrightarrow{\mathcal{C}} y_\infty$. Let $y_j = (n_j, a_j) \in N_{P_n} \times A_{P_n}$. For every bounded neighborhood $U_{P_n}$ of $y_\infty$ in $e(P_n)$, there exists a sufficiently large $N \gg 1$ such that $n_j \in U_{P_n}$ for all $j > N$. Let $U_{P_n}$ be sufficiently small and $t \gg 1$ so that the Siegel set $\mathfrak{S}_{P_n, U_{P_n}, t}$ is properly contained in a $(\Gamma \cap N_{P_n})$-translates of $D$:

$$\mathfrak{S}_{P_n, U_{P_n}, t} \subset \gamma \cdot D, \, \gamma \in \Gamma \cap N_{P_n}.$$

Thus $y_\infty \in \gamma \cdot I_n$.

Lastly we show that for each $n = 1, \cdot, r$, the set $I_n$ is locally finite. Let $U_{P_n}$ be an open subset of $e(P_n)$ and $U$ be an open subset of $\overline{\mathbf{H}}_\Gamma$ of the type **O2**, i.e.

$$U = \mathfrak{S}_{P_n, U_{P_n}, t} \cup U_{P_n}.$$

By the local finiteness of $D$, the open set $U$ intersects finitely many translates of $D$. Since $U_{P_n}$ is properly contained in $U$, the open subset $U_{P_n}$ also intersect finitely many translates of $I_n$. $\square$

Let $D$ and $I_n$ denote a Fundamental domain and its boundaries defined in Lemma III.25 above. The closure $\overline{D}$ of the Dirichlet fundamental domain $D$ is the union

$$\overline{D} = D \cup \coprod_{n=1}^{r} I_n.$$

In the next proposition (Proposition III.26), we show that the set $\overline{D}$ is a fundamental domain of $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$.

**Proposition III.26.** *Let $D$ be a Dirichlet fundamental domain of $\Gamma$ in $\mathbf{H}$. Then the closure $\overline{D}$ of $D$ in $\overline{\mathbf{H}}_\Gamma$ is a locally finite fundamental domain of $\Gamma$ in $\overline{\mathbf{H}}_\Gamma$.*

*Proof.* We first show that

$$\Gamma \cdot \overline{D} = \overline{\mathbf{H}}_\Gamma.$$

By Proposition III.24 and Proposition III.25,

$$\Gamma \cdot \coprod_{n=1}^{r} I_n = \coprod_{P \in \Delta_\Gamma} e(P).$$

The local finiteness follows from Proposition III.25. □

**Corollary III.27.** *The quotient $\Gamma \backslash \overline{\mathbf{H}}_\Gamma$ is compact*

*Proof.* Let $\sim$ be an equivalent relation on $\overline{\mathbf{H}}_\Gamma$ such that

$$z_1 \sim z_2 \text{ if there exists } \gamma \in \Gamma \text{ such that } \gamma \cdot z_1 = z_2.$$

For every point $z$ in $\overline{\mathbf{H}}_\Gamma$, Let $[z]$ be an equivalence class of points containing $z$. Then the map $D \to \Gamma \backslash \overline{\mathbf{H}}_\Gamma$ defined by

$$z \mapsto [z]$$

is surjective. To complete the proof, we will show that $\overline{D}$ is compact.

Since the group $\Gamma$ is a lattice, the domain $D$ is geometrically finite. Every vertex at infinity of the domain $D$ is a vertex of two parallel geodesic sides $s_1$ and $s_2$ of $D$. Thus there exists a compact subset $C$ of $D$ such that

$$\overline{D} = C \cup \bigcup_{n=1}^{r} \overline{\mathfrak{S}}_{P_n, I_n, t_n}.$$

where $\mathfrak{S}_{P_n, I_n, t_n}$ is a closure of a Siegel set contained in $D$. Since $\mathfrak{S}_{P_n, I_n, t_n}$ is a closed set of the type **C3** and the subset $I_n$ in $e(P_n)$ is compact, the set $\mathfrak{S}_{P_n, I_n, t_n}$ is compact in $\overline{\mathbf{H}}_\Gamma$. Therefore, $\overline{D}$ is compact. □

## 3.4 The $\Gamma$-CW-structure of $\overline{\mathbf{H}}_\Gamma$

In this section, we prove that there exists a $\Gamma$-CW-structure on the space $\overline{\mathbf{H}}_\Gamma$. In Proposition III.29, we show that the space $\overline{\mathbf{H}}_\Gamma$ is a smooth manifold with boundary. Using the result of Illman (Theorem III.32) on the existence of $\Gamma$-CW-structure on subanalytic manifold (c.f. Definition III.31), we prove that there exists a $\Gamma$-CW-structure on $\overline{\mathbf{H}}_\Gamma$ (Corollary III.35). Every Dirichlet fundamental domain of Fuchsian lattice is a Polygon with finitely many vertices. In Example III.36 and Example III.37, we show concrete pictures of CW-structure on the quotient $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$.

**Lemma III.28.** *For each minimal parabolic subgroup $P$ of $SL(2,\mathbb{R})$, let $\varphi_P : N_P \to \mathbb{R}$ be the homeomorphism and $\alpha_P$ be the unique positive restricted root in $\Phi^+(P, A_P)$. Let $\psi : \overline{\mathbf{H}}_\Gamma \to \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the map defined as follow:*

$$\psi(z) = \begin{cases} \left(\varphi_P(n), \frac{1}{\alpha_P(a)}\right) & \text{if } z = (n,a) \in N_P \times A_P \\ (\varphi_P(z), 0) & \text{if } z \in e(P) \end{cases}$$

*Then for each open basis $V$ of the space $\overline{\mathbf{H}}_\Gamma$, the restriction $\psi|_V : V \to \mathbb{R} \times \mathbb{R}_{\geq 0}$ is a homeomorphism onto its image.*

*Proof.* Let $V$ be an open subset of the type **O1**. Since $\varphi_P, \alpha_P$ are homeomorphisms, the map $\psi|_V$ is a homeomorphism. Let $V$ be an open subset of the type **O2**, i.e. $V = \mathfrak{S}_{P,U,t} \cup U$ for some open subset $U$ of $e(P)$. Let $y_j = (n_j, a_j) \in N_P \times A_P$ be a convergent sequence in $V$ of the type **S3** such that $y_j \to y_\infty$. Then

$$\psi|_V(y_j) = (\varphi_P(n_j), \frac{1}{\alpha_P(a_j)}) \to (\varphi_P(y_\infty), 0) = \psi|_V(y_\infty).$$

Thus the map $\psi|_V$ is continuous. The image of $\psi|_V$ is $\varphi_P(U) \times [0, \frac{1}{t})$, and is open in $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Thus the map $\psi|_V$ is homeomorphic onto its image. □

**Proposition III.29.** *The space $\overline{\mathbf{H}}_\Gamma$ is a smooth manifold with boundary.*

*Proof.* For every open neighborhood $V$ of $\overline{\mathbf{H}}_\Gamma$, let $\psi|_V$ be the local chart on $V$. We will show that for two open neighborhoods $V_1$ and $V_2$ such that $V_1 \cap V_2 \neq \emptyset$, the transition map $(\psi|_{V_2})^{-1} \circ \psi|_{V_1} : V_1 \cap V_2 \to V_1 \cap V_2$ is a diffeomorphism. We next show that transition maps are diffeomorphic. Let $V_1$ and $V_2$ be open neighborhoods of the type either **O1** or **O2**. Thus there are essentially four cases to consider (Figure 3.3).

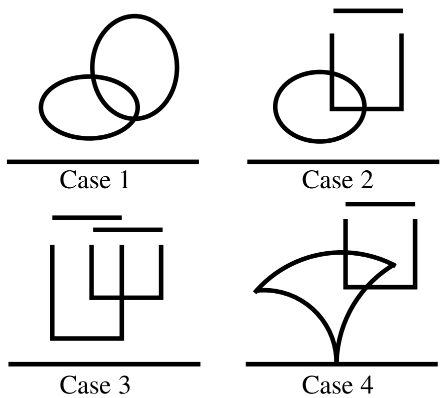


Figure 3.3: The intersections of open basis of $\overline{\mathbf{H}}_\Gamma$

**Case 1** Suppose both $V_1$ and $V_2$ are of the type **O1**. Then the transition map is the identity map, thus it is a diffeomorphism.

**Case 2** Suppose $V_1$ and $V_2$ are of the type **O1** and **O2** respectively. Then the intersection $V_1 \cap V_2$ is open subset of the type **O1**, so it reduces to the **Case 1**.

**Case 3** Suppose that $V_1$ and $V_2$ are open subsets of the type **O2** with respect to the common parabolic subgroup $P$, i.e. $V_1 = \mathfrak{S}_{P,U_1,t_1} \cup U_1$ and $V_2 = \mathfrak{S}_{P,U_2,t_2} \cup U_2$ for open subsets $U_1$ and $U_2$ of the boundary component $e(P)$. Then two maps $\psi|_{V_1}$ and $\psi|_{V_2}$ are the same on the domain $V_1 \cap V_2$. Thus the transition map is identity.

**Case 4** Suppose that $V_1$ and $V_2$ are open subsets of the type **O2** such that $V_1 = \mathfrak{S}_{P_1,U_1,t_1}$ and $V_2 = \mathfrak{S}_{P_2,U_2,t_2}$ where $P_1 \neq P_2$. We will first prove for the special case when

$$P_1 = P_\infty = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times,\, b \in \mathbb{R}\},\ \text{and}$$

$$P_2 = P_0 = \left\{\left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times,\, b \in \mathbb{R}\right\}.$$

We then observe that for arbitrary two parabolic subgroups $P_1, P_2$, there exists $g \in \mathrm{SL}(2,\mathbb{R})$ such that ${}^gP_1 = P_\infty$ and ${}^gP_2 = P_0$. Since $\mathrm{SL}(2,\mathbb{R})$-action on $\mathbf{H}$ is diffeomorphic, it reduces to the special case above.

Since $e(P_0)$ and $e(P_\infty)$ are disjoint, the intersection $V_1 \cap V_2$ is an open subset in $\mathbf{H}$. The homeomorphisms $\psi|_{V_1}$ and $\psi|_{V_2}$ are given by

$$\psi|_{V_1}\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right) = (b, 1/a^2),\ \text{and}$$

$$\psi|_{V_2}\left(\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right) = (-b, a^2).$$

Thus the transition map $(\psi|_{V_2})^{-1} \circ \psi|_{V_1}$ is

$$(x, y) \mapsto (-x, y^{-1}).$$

Therefore, it is a diffeomorphism.

This completes the proof. □

In the next proposition (Proposition III.30), we prove that the $\Gamma$-action on the space $\overline{\mathbf{H}}_\Gamma$ is smooth with respect to the smooth structure of $\overline{\mathbf{H}}_\Gamma$. This will lead to the existence of $\Gamma$-CW-structure on the space using Illman's result on [33].

**Proposition III.30.** *The $\Gamma$-action on $\overline{\mathbf{H}}_\Gamma$ is smooth.*

*Proof.* The $\Gamma$-action is smooth on the interior. We need to show that $\Gamma$-action is smooth the boundary. Let $V = \mathfrak{S}_{P,U,t} \cup U$ be an open set of the type **O2** and $\psi|_V$ be the homeomorphism from Lemma III.28. For each $\gamma \in \Gamma$, let $f_\gamma : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}\times\mathbb{R}_{\geq 0}$ be the map is define by the following diagram:

$$\begin{array}{ccc} \mathbb{R}\times\mathbb{R}_{\geq 0} & \xrightarrow{f_\gamma} & \mathbb{R}\times\mathbb{R}_{\geq 0} \\ {\scriptstyle \psi_V}\uparrow & & \uparrow{\scriptstyle \psi_{\gamma\cdot V}} \\ V & \xrightarrow{\gamma} & \gamma\cdot V \end{array}$$

For every element $\gamma$ in $\Gamma$, let $\gamma = (n, a, k) \in N_P \times A_P \times K$. Since $\gamma = k\cdot n\cdot a$, it follows that $f_\gamma = f_k \circ f_n \circ f_a$. We will show that $f_k$, $f_a$, and $f_n$ are diffeomorphic.

- For each $n \in N_P$, the map $f_n$ is a translation, i.e.

$$f_n(x, y) = (\varphi_P(n) + x, y).$$

- For each $a \in A_P$, the map $f_n$ is a dilation, i.e.

$$f_a(x, y) = (x, \frac{y}{a_P^\alpha}).$$

- For each $k \in K$, the map $f_k$ is a rotation at the base point $i \in \mathbf{H}$, i.e.

$$\psi^{-1} \circ f_k(x, y) = k \cdot \psi^{-1}(x, y).$$

This completes the proof. $\square$

The existence of the $\Gamma$-CW-structure on the space $\overline{\mathbf{H}}_\Gamma$ follows from the existence of $\Gamma$-CW-structure on a $\Gamma$-subanalytic manifold in a smooth $\Gamma$-manifold. We first define the notion of subanalytic submanifold.

**Definition III.31.** Let $M$ be a smooth manifold. For each open neighborhood $U$ in $M$, let us define $\mathcal{J}(U)$ be the smallest collection of all subsets of $M$ containing the subsets of the form

$$\{x \in M \mid f(x) \geq 0\}$$

for all smooth functions $f$ on $U$, and which is closed under finite unions, finite intersections, and the complement. A submanifold $N$ of $M$ is called **subanalytic** if every point $x \in M$ has an open neighborhood $U$ such that $N \cap U$ belongs to $\mathcal{J}(U)$.

The following is the result of Illman [33].

**Theorem III.32.** *Let $M$ be a smooth $\Gamma$-manifold and $N$ be a subanalytic $\Gamma$-submanifold in $M$. Then there exists a unique $\Gamma$-CW-complex structure on $N$.*

*Proof.* A smooth manifold (i.e. real analytic manifold) is a subanalytic by Definition III.31 (simply replace $f \geq 0$ by $f > 0$). Then the theorem follows from [33, Theorem 6.4] and [33, Theorem 3.8]. □

**Proposition III.33.** *Let $N$ be a manifold with boundary. Then there exists a smooth manifold $M$ and an embedding $N \hookrightarrow M$ such that $N$ is a subanalytic submanifold of $M$*

*Proof.* Let $\partial N$ be the boundary of $N$. Let $N'$ be the copy of $N$, i.e. there exist a homeomorphism $\varphi : N \to N'$. Define a relation $\sim$ on the disjoint union $N \cup N'$ such that for two element $x$, $x' \in N \cup N'$,

$$x \sim x' \text{ if and only if } x \in N,\ x' \in N' \text{ and } \varphi(x) = x'.$$

Then the space

$$M = (N \cup N')/\sim$$

is a manifold (without boundary). □

*Remark* III.34. The manifold $M$ in Proposition III.33 is called the **double** of $N$.

**Corollary III.35.** *There exists a $\Gamma$-CW-structure on $\overline{\mathbf{H}}_\Gamma$.*

*Proof.* Let $N$ in Proposition III.33 be the space $\overline{\mathbf{H}}_\Gamma$. Since the structure of $\overline{\mathbf{H}}_\Gamma$ is smooth, the manifold $M$ in Proposition III.33 is a smooth manifold with boundary. Theorem III.32 then completes the proof. □

Here are two examples of actual CW-structure of the quotient space $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$.

**Example III.36.** Let $D$ be a Dirichlet fundamental domain of a Fuchsian group. The Figure 3.4 shows an example of $D$.

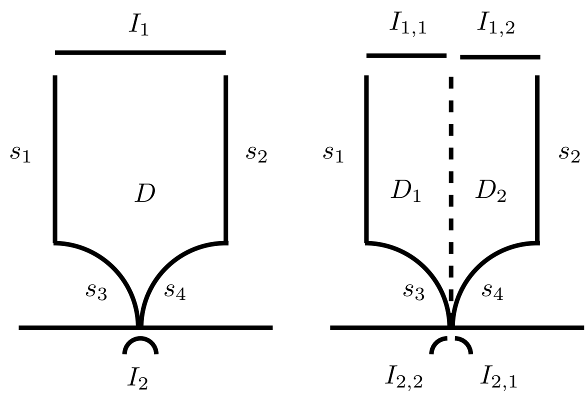


Figure 3.4: A Dirichlet fundamental domain and its CW-structure

The geodesic side $s_1$ is glued to $s_2$ and $s_3$ is glued to $s_4$. In order to get a proper CW-structure, let us divide each $I_1$ and $I_2$ at middle, and add a separating geodesic $s_5$ in $D$ so that $\overline{D}$ is divided into two 2-cells

$$\overline{D}_1 = D_1 \cup I_{1,1} \cup s_1 \cup s_3 \cup s_5, \overline{D}_2 = D_2 \cup I_{1,2} \cup s_2 \cup s_4 \cup s_5.$$

In fact, this gives the simplicial structure of $\overline{D} \sim$ and thus induces $\Gamma$-equivariant simplicial structure of $\overline{\mathbf{H}}_\Gamma$.

**Example III.37.** Another example of Dirichlet fundamental domain of $\Gamma$ is Figure 3.5.

The geodesic side $t_1$ is glued to $t_2$ and $t_3$ is glued to $t_4$. The boundary component $I_1$ is mapped to loop, while the boundary components $I_2$ and $I_3$ are first glued them-

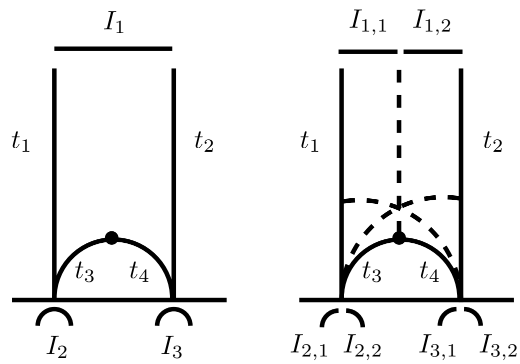


Figure 3.5: A subdivision of fundamental domain

selves point-wise, and then the end points are glued. In order to get a subdivision of $\overline{D}$ which induces a CW-structure of $\overline{D}/\sim$, we divide each boundary component $I_1, I_2, I_3$ into two pieces and then subdivide the interior $D$ as in the Figure 3.5.

## 3.5 Proof of the main theorem for the Fuchsian lattices

So far, we observed the following:

- The construction of the space $\overline{\mathbf{H}}_\Gamma$ (§3.1).
- The proper $\Gamma$-action on the space $\overline{\mathbf{H}}_\Gamma$ (§3.2).
- The compactness of the quotient space $\Gamma\backslash\overline{\mathbf{H}}_\Gamma$ (Corollary III.27 in §3.3).
- The existence of $\Gamma$-CW-structure on the space $\overline{\mathbf{H}}_\Gamma$ (Corollary III.35 in §3.4).

In this section, we prove the main theorem (Theorem I.1) for the Fuchsian lattice $\Gamma$. First, we need following two lemmas.

**Lemma III.38.** *For each point $\zeta$ in the boundary component $e(P)$ corresponding to a parabolic subgroup $P \in \Delta_\Gamma$, the stabilizer $\Gamma_\zeta$,*

$$\Gamma_\zeta = \{\gamma \in \Gamma \mid \gamma \cdot \zeta = \zeta\}$$

*is a trivial group.*

*Proof.* If an element $\gamma$ belongs to $\Gamma_\zeta$, then $\gamma$ belongs to the intersection $\Gamma \cap P = \Gamma \cap N_P$. Thus $\gamma = (n, Id, Id)$ in terms of Iwasawa decomposition. From the $\Gamma$-action on the boundary (c.f. Definition III.16), it follows that $n = Id$. □

**Lemma III.39.** *The space $\overline{\mathbf{H}}_\Gamma$ is contractible.*

*Proof.* We will show that there exists a homotopy retraction of $\overline{\mathbf{H}}_\Gamma$ onto a contractible image.

Let $P_1, \cdots, P_r$ be the minimal parabolic subgroups corresponding to the vertices at infinity of $D$. For each $n = 1, \cdots, r$, we can choose a sufficiently large $t_n \gg 1$ such that

$$\mathfrak{S}_{P_n, I_n, t_n} \cap \mathfrak{S}_{P_m, I_m, t_m} = \emptyset, \quad n \neq m.$$

For each $n = 1, \cdots, r$ and $t \in (0, 1]$, let $\mathfrak{S}_n(t)$ be a set defined by

$$\mathfrak{S}_n(t) = \mathfrak{S}_{P_n, I_n, t_n/t},$$

and for the simple positive root $\alpha_n \in \Phi^+(P_n, A_{P_n})$, let $a_{n,t}$ be an element in $A_{P_n}$ such that

$$\alpha_n(a_{n,t}) = t_n/t.$$

We define a map $h : [0, 1] \times \overline{D} \to \overline{D}$ as follow: for $t \in (0, 1]$,

$$h_t(z) = \begin{cases} z & \text{if } z \in \overline{D} - \bigcup_{n=1}^r \overline{\mathfrak{S}_n(t)} \\ (n_1, a_{n,t}) & \text{if } z \in \mathfrak{S}_n(t) \text{ and } z = (n_1, a_1) \in N_{P_n} \times A_{P_n} \\ (z, a_{n,t}) & \text{otherwise, i.e. } z \in I_n \end{cases}$$

$$h_0(z) = z$$

See Figure 3.7 for geometric interpretation of $h_t$.

We will show that $h_t$ is continuous, i.e. $h_t \to Id$ as $t \to 0$ point-wise.

1. Suppose a point $z$ lies in $D$. Then there exists a sufficiently small positive number $\epsilon \ll 1$ such that $z \in \overline{D} - \bigcup_{n=1}^r \overline{\mathfrak{S}_n(\epsilon)}$. Thus $h_t(z) = z$ for all $t < \epsilon$, so $h_t(z) \to z$ as $t \to 0$.

2. Suppose $z \in I_n$ for some $n = 1, \cdots, r$. Since $h_t(z) = (z, a_{n,t})$ and $\alpha_n(a_{n,t}) = t_n/t \to \infty$, it follows that $h_t(z) \to z$.

The map $h_t$ is a homotopy retraction. We note that the $N_P$-component of points in each Siegel set remains constant, and the retraction is along the $A_P$-component. In other words, each point in the Siegel sets $S_n(\epsilon)$ $(n = 1, \cdots, r)$ is retracted along a geodesic.

We extend the map $h_t$ to a homotopy retraction on the entire space $\overline{\mathbf{H}}_\Gamma$. Let us define a map $H_t$ ($t \in [0,1]$) on the space $\overline{\mathbf{H}}$ as follow.

$$H_t(z) = \gamma^{-1}h_t(\gamma \cdot z) \text{ where } \gamma \cdot z \in \overline{D}.$$

The map $H_t$ is well-defined. For a point $z$ in $\overline{\mathbf{H}}_\Gamma$, suppose there exists two elements $\gamma_1$ and $\gamma_2$ in $\Gamma$ such that both $\gamma_1 \cdot z$ and $\gamma_2 \cdot z$ lie on the fundamental domain $\overline{D}$ but $\gamma_1 \cdot z \neq \gamma_2 \cdot z$. (See Figure 3.6 below) By the homotopy $h_t$, each $\gamma_j \cdot z$ ($j = 1, 2$) is retracted to the point $h_t(\gamma_j \cdot z)$ along the geodesic path, say $s_j$, connecting $\gamma_j \cdot z$ and $h_t(\gamma_j \cdot z)$. Two geodesics $s_1$ and $s_2$ share a common point at infinity at one end, corresponding to the parabolic subgroup $P$. Thus the geodesics $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ also have the common point at infinity at one end. Moreover, both $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ pass through the point $z$. Thus two geodesics $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ coincide. Since $d(h_t(\gamma_1 \cdot z), \gamma_1 \cdot z) = d(h_t(\gamma_2 \cdot z), \gamma_2 \cdot z)$, it follows that $\gamma_1^{-1} \cdot h_t(\gamma_1 \cdot z) = \gamma_1^{-1} \cdot h_t(\gamma_1 \cdot z)$.

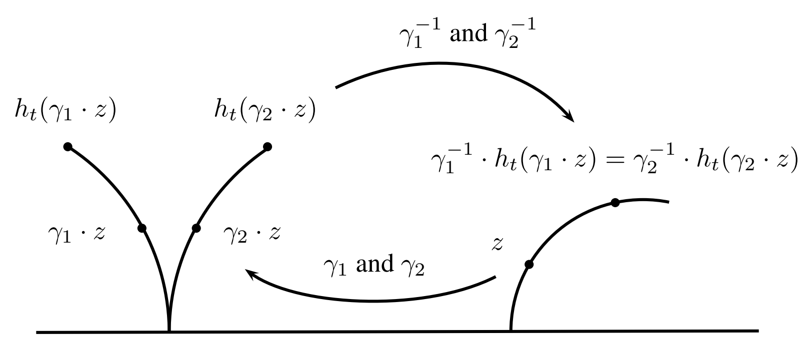


Figure 3.6: Well-defined homotopy retraction $H_t$

For a point $\zeta$ in $e(P)$ for some parabolic subgroup $P \in \Delta_\Gamma$, we apply the similar argument above. Suppose there exist two elements $\gamma_1$ and $\gamma_2$ of $\Gamma$ such that $\gamma_1 \cdot \zeta \in I_{n_1}$ and $\gamma_2 \cdot \zeta \in I_{n_2}$ for some $1 \leq n_1, n_2 \leq r$ and $\gamma_1 \cdot \zeta \neq \gamma_2 \cdot \zeta$. Although the points $\gamma_j \cdot \zeta$ ($j = 1, 2$) do not lie on any geodesic, one may consider an infinite geodesic path, say $s_j$, which passes through $h_t(\gamma_j \cdot \zeta)$ and diverges (in the topology of $\overline{X}_\Gamma$) to $\gamma_j \cdot \zeta$ where all points lying between $h_t(\gamma_j \cdot \zeta)$ and $\gamma_j \cdot \zeta$ retracts along the geodesic $s_j$ under the homotopy $h_t$. Therefore, such geodesics $s_1$ and $s_2$ are unique. Since two geodesic $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ diverge to the common point $\zeta$, they are the same. Since the $A_P$-components of $h_t(\gamma_1 \cdot \zeta)$ and $h_t(\gamma_2 \cdot \zeta)$ have the same evaluation at their roots, we have $\gamma_1^{-1} \cdot h_t(\gamma_1 \cdot \zeta) = \gamma_2^{-1} \cdot h_t(\gamma_2 \cdot \zeta)$.

To summarize, the map $H_t$ is a homotopy retraction of $\overline{X}_\Gamma$ onto the union $\Gamma \cdot \overline{D}$. Therefore, the the space $\overline{\overline{\mathbf{H}}}_\Gamma$ is contractible. □

**Example III.40.** Let $D$ be a Dirichlet fundamental domain of a Fuchsian lattice $\Gamma$ with three vertices at infinity as below. There exists a compact subset $C$ in $D$ such that the complement is the disjoint union of the closures of three Siegel sets $S_1(1)$, $S_2(1)$, and $S_3(1)$. As $t \to 0$, each subset $S_n(t)$ converges futher to the boundary $I_n$.

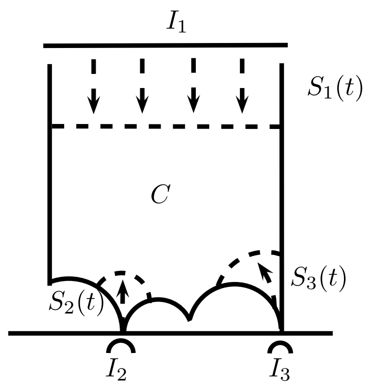


Figure 3.7: The homotopic retraction of $\overline{D}$ with $h_t$

The next theorem (Theorem III.41) proves the main theorem.

**Theorem III.41.** *The space* $\overline{\overline{\mathbf{H}}}_\Gamma$ *is a cofinite* $\Gamma$*-CW-complex model for the proper classifying space* $\underline{E}\Gamma$*.*

*Proof.* By Corollary III.35, there exists a unique $\Gamma$-CW-structure on the space $\overline{\overline{\mathbf{H}}}_\Gamma$. Since the $\Gamma$-action is cocompact (Corollary III.27), the $\Gamma$-CW-structure on $\overline{\overline{\mathbf{H}}}_\Gamma$ is cofinite.

Let $H$ be a nontrivial finite subgroup of $\Gamma$. By Lemma III.38, the fixed point set $\overline{\mathbf{H}}_\Gamma^H$ lies in $\mathbf{H}$, i.e.

$$\overline{\mathbf{H}}_\Gamma^H = \mathbf{H}^H.$$

Thus it is a geodesic submanifold and is convex and contractible. For a trivial subgroup of $\Gamma$, the fixed point set in the entire space $\overline{\mathbf{H}}_\Gamma$, which is contractible by Lemma III.39.

Since the $\Gamma$-action is proper, every isotropy subgroup $H$ of $\Gamma$ is finite. $\square$

# CHAPTER IV

## Proper classifying spaces for lattices in semisimple Lie groups of $\mathbb{R}$-rank one

In this chapter, we prove the main theorem (Theorem I.1). Throughout the chapter, we denote $G$ for a semisimple Lie group of $\mathbb{R}$-rank one, and $\Gamma$ for a discrete lattice subgroup of $G$. We fix a maximal compact subgroup $K$ of $G$ and the homogeneous space $X = G/K$ which is a symmetric space of rank one. The $\Gamma$-action on $X$ is proper and the volume of the quotient space $\Gamma\backslash X$ is finite.

Each section of Chapter IV is summerized as follow.

1. In Section §4.1, we define a topological space $\overline{X}_\Gamma$ by attaching boundary components to the symmetric space $X$.

2. In Section §4.2, we define a $\Gamma$-action on the space $\overline{X}_\Gamma$ and show that this action is continuous and proper.

3. In Section §4.3, we prove that the $\Gamma$-action is cocompact. We also describe a fundamental set of $\Gamma$ in $\overline{X}_\Gamma$ using the result of Garland and Raghunathan in [24].

4. In Section §4.4, we prove that $\overline{X}_\Gamma$ is a $\Gamma$-CW-complex. We will use Illman's result (Theorem III.32) in [33].

5. Finally, in Section §4.5, we prove the main theorem.

A geometric example of the rank one symmetric space is the 3-dimensional upper half-space $\mathbf{H}^3$:

$$\mathbf{H^3} = \{z + yj \mid z \in \mathbb{C},\, y > 0\}, \quad ds^2 = \frac{dz^2 + dy^2}{y^2}.$$

The group of (orientation preserving) isometries of $\mathbf{H}^3$ is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. The action of $PSL(2, \mathbb{C})$ is given by the Möbius transformation

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (z + yj) = (a(z + yj) + b)(c(z + yj) + d)^{-1}$$

where $z + yj$ is considered as an element of the standard quaternion algebra $\mathbb{H}$. The lattice $\Gamma$ is then the Kleinian lattice in $SL(2, \mathbb{C})$. In particular, arithmetic Kleinian groups are lattices in $SL(2, \mathbb{C})$. See [49] for more on arithmetic Kleinian groups.

Another good example of symmetric space $X$ is the quotient space $SO(n, 1)/SO(n)$. The group $SO(n, 1)$ is the group of linear transformations in $SL(n + 1, \mathbb{R})$ which preserves the quadratic form of signature $(n, 1)$. For example, a quadratic form $q : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$q(v) = {}^t v \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} v$$

is a quadratic form of signature $(n, 1)$ where $I_{n-1}$ is the $(n - 1) \times (n - 1)$ identity matrix. In this case, the standard parabolic subgroup is given by

$$P_\infty = \left\{ \begin{pmatrix} t & * & * \\ 0 & M & * \\ 0 & 0 & 1/t \end{pmatrix} \mid t > 0,\, M \in SO(n - 1) \right\}.$$

The Langlands decomposition of $P_\infty = N_\infty \times A_\infty \times M_\infty$ is then

$$\begin{pmatrix} 1 & * & * \\ 0 & I_{n-1} & * \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/t \end{pmatrix} \times SO(n-1)$$

## 4.1 The construction of the space $\overline{X}_\Gamma$

In this section, we define a space $\overline{X}_\Gamma$ and introduce a topology of this space. In section §4.5, the main theorem is proved by showing that the space $\overline{X}_\Gamma$ is a cofinite model for the proper classifying space $\underline{E}\Gamma$. A short outline of this section is the following. In Definition IV.3, we define $\overline{X}_\Gamma$ as a set. In Proposition IV.4, we introduce a topology on $\overline{X}_\Gamma$ and three types of closed subsets of $\overline{X}_\Gamma$ with respect to that topology. In Proposition IV.8, we give an alternative description of this topology via open basis which consists of two types of open subsets of $\overline{X}_\Gamma$. The open basis of $\overline{X}_\Gamma$ will be used in section §4.4 to show that the space $\overline{X}_\Gamma$ is a manifold with boundary.

**Definition IV.1.** Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one. Let $P$ be a parabolic subgroup of $G$ and $P = N_P \times A_P \times M_P$ be the Langlands decomposition of $P$ where $N_P$ is the nilpotent normal subgroup of $P$. The **boundary component** $e(P)$ **corresponding to** $P$ is the space defined by

$$e(P) = N_P \tag{4.1}$$

(c.f. §2.3.4 Definition 2.53).

**Definition IV.2.** Let $G$ be a semisimple Lie group of $\mathbb{R}$-rank one and $\Gamma$ be a lattice subgroup of $G$. The set $\Delta_\Gamma$ is a collection of minimal parabolic subgroups $P$ of $G$ such that $\Gamma \cap N_P$ is a uniform lattice in $N_P$.

**Definition IV.3.** Assume that the groups $G$, $P$, $K$, and the symmetric space $X$ are those defined in Definition IV.1, and the lattice $\Gamma$ in Definition IV.2. The set $\overline{X}_\Gamma$ is defined by

$$\overline{X}_\Gamma = X \cup \coprod_{P \in \Delta_\Gamma} e(P) \tag{4.2}$$

the union of the symmetric space $X$ defined by the disjoint union of the symmetric space $X$ and the boundary components corresponding to all parabolic subgroups in the set $\Delta_\Gamma$. The symmetric space $X$ and the disjoint union of boundary components $\coprod_{P\in\Delta_\Gamma} e(P)$ are called **interior** and **boundary** of $\overline{X}_\Gamma$ respectively.

In the next proposition, we define a topology on the set $\overline{X}_\Gamma$ in terms of the convergence class of sequences. The statement and proof are similar to Proposition III.6 in §3.1.

**Proposition IV.4.** *Let $\overline{X}_\Gamma$ be the set defined in Definition IV.3. Let $\mathcal{T}$ be the topology on $\overline{X}_\Gamma$ defined by the convergence class of sequences $\mathcal{C}$ consisting of all combinations of the following types of convergent sequences:*

**Type *S1*** *Let $y_j \in X$ be a convergent sequence in the symmetric space $X$ such that $y_j \to y_\infty \in X$. Then $y_j \xrightarrow{\mathcal{C}} y_\infty$.*

**Type *S2*** *Let $y_j \in e(P)$ be a convergent sequence in a boundary component $e(P)$ for some parabolic subgroup $P$ in $\Delta_\Gamma$ such that $y_j \to y_\infty \in e(P)$. Then $y_j \xrightarrow{\mathcal{C}} y_\infty$.*

**Type *S3*** *Let $y_j$ be a unbounded sequence in the symmetric space $X$. If there exists a parabolic subgroup $P \in \Delta_\Gamma$ such that*

- *$y_j = (n_j, a_j)$ in terms of horospherical decomposition of $X$ with respect to $P$,*
- *$n_j \to y_\infty$ with respect to the topology of $e(P)$, and*
- *$a_j^\alpha \to \infty$ where $\alpha$ is the restricted root in $\Phi(P, A_P)$ (c.f. Definition II.15),*

*then $y_j \xrightarrow{\mathcal{C}} y_\infty \in e(P)$.*

*Then the following types of subsets of $\overline{X}_\Gamma$ are closed in $\mathcal{T}$.*

**Type *C1*** *All subsets in $X$ which are closed and bounded (i.e. compact) with respect to the topology of* $\mathbf{H}$.

**Type *C2*** *Every disjoint union*

$$\coprod_{P \in \Delta_\Gamma} C_P$$

*where each $C_P$ is a closed subset of $e(P)$.*

**Type *C3*** *Every subset $\overline{\mathfrak{S}}_{P,C_P,t}$ defined as follow. For a closed subset $C_P$ of a boundary component $e(P)$ and $\overline{A}_{P,t} = \{a \in A_P \mid a^\alpha \geq t, \alpha \in \Phi(P, A_P)\}$,*

$$\overline{\mathfrak{S}}_{P,C_P,t} := (C_P \times \overline{A}_{P,t}) \cup C_P.$$

*Proof.* A subset $A$ is closed if and only if its closure $\overline{A}$ is $A$ itself. The closure $\overline{A}$ is defined by (c.f. Proposition II.57)

$$\overline{A} = \{y_\infty \in \overline{\mathbf{H}}_\Gamma \mid y_j \overset{\mathcal{C}}{\to} y_\infty,\ \{y_j\}_{j=1}^\infty \subset A\}.$$

Since every constant sequence always converges (Condition 1 in Definition II.56), $A \subset \overline{A}$. We will show that for subsets of types ***C1***, ***C2***, and ***C3***, the converse inclusion $\overline{A} \subset A$ holds.

1. Let $A$ be a subset of the type ***C1***, i.e. $A$ is a closed and bounded subset of $X$. Suppose $y_j \in A$ and $y_j \overset{\mathcal{C}}{\to} y_\infty$. Since $A$ is bounded, $y_\infty \in X$. Thus $y_j$ converges to $y_\infty$ with respect to the topology of $X$. Since $A$ is closed in $X$, $y_\infty \in A$. Therefore, $\overline{A} \subset A$.

2. Let $A$ be a subset of the type ***C2***, i.e. $A = \coprod_{P \in \Delta_\Gamma} C_P$ where each $C_P$ is a closed subset of the boundary component $e(P)$. We first show that each subset $C_P$ is closed in $\overline{X}_\Gamma$. Then it follows that

$$\overline{A} = \overline{\coprod_{P \in \Delta_\Gamma} C_P} = \coprod_{P \in \Delta_\Gamma} \overline{C_P} = A.$$

Suppose $\{y_j\}_{j=1}^{\infty} \subset C_P$ and $y_j \stackrel{\mathcal{C}}{\to} y_\infty$. Then the sequence $y_j$ is of type ***S2***. So the limit $y_\infty$ belongs to $e(P)$. Since $C_P$ is closed in $e(P)$, $y_\infty \in C_P$. Thus $C_P$ is closed in $\overline{X}_\Gamma$.

3. Let $A$ be a subset of the type ***C3***, i.e. $A = \overline{\mathfrak{S}}_{P,C_P,t}$ for some parabolic subgroup $P$ in $\Delta_\Gamma$ and a closed subset $C_P$ of $e(P)$. Suppose $\{y_j\}_{j=1}^{\infty} \subset A$ and $y_j \stackrel{\mathcal{C}}{\to} y_\infty$. By passing to a subsequence (if necessary), the (sub)sequence $\{y_j\}_{j=1}^{\infty}$ is contained either in (a) the interior $X$, or (b) the boundary component $e(P)$.

   (a) Since $\{y_j\}_{j=1}^{\infty} \subset \overline{\mathfrak{S}}_{P,C_P,t}$, $y_j \in C_P \times \overline{A}_{P,t}$. Note that Since $C_P \times \overline{A}_{P,t}$ is closed in $X$. If $y_j$ is bounded, then $y_j$ is of type ***S1***. It follows that $y_\infty \in X$, and thus $y_\infty \in C_P \times \overline{A}_{P,t}$. If $y_j$ is unbounded, then $y_j$ is of type ***S3***, Thus $y_\infty \in C_P$.

   (b) Since $\{y_j\}_{j=1}^{\infty} \subset C_P$, the sequence $y_j$ is of type ***S2***. Therefore, $y_\infty \in C_P$.

   In either case, we have shown that $y_\infty \in A$.

Thus all three types of subsets are closed in $\overline{X}_\Gamma$. □

The following corollary classifies all closed sets in $\overline{X}_\Gamma$.

**Corollary IV.5.** *With respect to the topology $\mathcal{T}$ on the set $\overline{X}_\Gamma$ defined in Proposition IV.4, every closed subset of $\overline{X}_\Gamma$ is obtained by finite union of closed subsets of the types **C1**, **C2**, and **C3**.*

*Proof.* Let $A$ be a closed subset of $\overline{X}_\Gamma$. Then the closure $\overline{A}$ equal to $A$. This implies that for every sequence $y_j \in A$ such that $y_j \stackrel{\mathcal{C}}{\to} y_\infty$, the limit $y_\infty$ belongs to $A$. Suppose every convergent sequence $y_j \stackrel{\mathcal{C}}{\to} y_\infty$ has a subsequence $y'_j$ such that $y'_j = y_\infty$. Then $A$ is a discrete subset of $\overline{X}_\Gamma$. Thus we assume that $A$ is not discrete, thus there is no constant subsequence $y'_j$ of every convergent sequence $y_j$ in $A$. By replacing with its further subsequence, we may assume that a sequence $y'_j$ is either of the type ***S1***, ***S2***, or ***S3***. For each $n = 1, \cdots, 3$, let $A_n$ be the set

$$A_n = \{y_\infty \mid y'_j \overset{\mathcal{C}}{\to} y_\infty \text{ where } y'_j \text{ is of type } \boldsymbol{Sn}\}.$$

From Proposition IV.4, each $A_n$ is a closed subset in $\overline{X}_\Gamma$. Since

$$A = A_1 \cup A_2 \cup A_3,$$

we are done. □

From now on we assume that the set $\overline{X}_\Gamma$ is always a space with the topology $\mathcal{T}$ in Proposition IV.4. In the next two lemmas, we show that there are two special types of open subsets in $\overline{X}_\Gamma$. In Proposition IV.7, we prove that such open subsets generate the same topology of $\overline{X}_\Gamma$.

**Lemma IV.6.** *Every open set in the symmetric space $X$ is open in $\overline{X}_\Gamma$.*

*Proof.* Let $U$ be an open subset of $X$. The complement of $U$ in $\overline{X}_\Gamma$ is

$$U^c = (\mathbf{H} - U) \cup \coprod_{P \in \Delta_\Gamma} e(P).$$

Since each $\mathbf{H} - U$ and $\coprod_{P \in \Delta_\Gamma} e(P)$ is closed, $U^c$ is closed. □

**Lemma IV.7.** *For every open subset $U$ in a boundary component $e(P)$ corresponding to a parabolic subgroup $P$ in $\Delta_\Gamma$, the union $\mathfrak{S}_{P,U,t} \cup U$ is open in $\overline{X}_\Gamma$.*

*Proof.* The complement of $\mathfrak{S}_{P,U,t} \cup U$ is

$$(X - \mathfrak{S}_{P,U,t}) \cup (e(P) - U) \cup \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} e(Q).$$

Since each subset $X - \mathfrak{S}_{P,U,t}$, $e(P) - U$, and $e(Q)$ are closed, we are done. □

**Proposition IV.8.** *The open basis of $\overline{X}_\Gamma$ consists of following types of subsets:*

**Type *O1*** *All open sets in the topology of $X$.*

**Type *O2*** *Every subset of the form*

$$\mathfrak{S}_{P,U,t} \cup U$$

*where $U$ is an open subset of a boundary component $e(P)$ corresponding to a parabolic subgroup $P$ in $\Delta_\Gamma$, $t$ is a positive real number, and $\mathfrak{S}_{P,U,t} = U \times A_{P,t}$.*

*Proof.* It is enough to prove that the complement of closed subsets of the types ***C1***, ***C2***, and ***C3*** are finite unions of open subsets of above types.

1. Let $C$ be a closed subset of the type ***C1***, i.e. $C$ is a bounded and closed subset of $X$. The complement $C^c$ of $C$ is

$$(X - C) \cup \coprod_{P \in \Delta_\Gamma} e(P).$$

First, the set $X - C$ is open. Next, since $C$ is bounded, for each parabolic subgroup $P$ in $\Delta_\Gamma$, there exists a sufficiently large $T_P \gg 1$ such that the horoball $S_{P,T_P}$ is disjoint from $C$. Thus $C^c$ is expressed as follow.

$$C^c = (X - C) \cup \bigcup_{P \in \Delta_\Gamma} \left(S_{P,T_P} \cup N_P\right).$$

The subset $X - C$ and $S_{P,T_P} \cup N_P$ are open subsets of $\overline{X}_\Gamma$ of types ***O1*** and ***O1*** respectively.

2. Let $C$ be a closed subset of the type ***C2***, i.e.

$$C = \coprod_{P \in \Delta_\Gamma} C_P$$

where each $C_P$ is a closed subset of a boundary component $e(P)$. Let $U_P = e(P) - C_P$. The complement $C^c$ is

$$X \cup \coprod_{P \in \Delta_\Gamma} U_P.$$

For an arbitrary positive real number $t$,

$$C^c = \mathbf{H} \cup \bigcup_{P \in \Delta_\Gamma} \left( \mathfrak{S}_{P,U_P,t} \cup U_P \right).$$

The set $X$ and the unions $\mathfrak{S}_{P,U_P,t} \cup U_P$ are open subsets of the types ***O1*** and ***O2*** respectively.

3. Let $C$ be a closed subset of the type **C3**, i.e.

$$C = \left( C_P \times \overline{A}_{P,t} \right) \cup C_P.$$

The complement of $C$ is

$$\left( X - (C_P \times \overline{A}_{P,t} \right) \cup (e(P) - C_P) \cup \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} e(Q).$$

For each parabolic subgroup $Q \neq P$ in $\Delta_\Gamma$, let $T_Q \gg 1$ be a sufficiently large number such that the horoball $S_{Q,T_Q}$ is disjoint from $\mathfrak{S}_{P,C_P,t}$. Let $U_P = e(P) - C_P$. Then the complement $C^c$ is expressed as follow:

$$C^c = (X - \mathfrak{S}_{P,C_P,t}) \cup \mathfrak{S}_{P,U_P,t} \cup U_P \cup \left( \coprod_{Q \in \Delta_\Gamma \setminus \{P\}} S_{Q,T_Q} \cup N_Q \right).$$

Since the subset $X - \mathfrak{S}_{P,C_P,t}$ is of the type ***O1*** and unions $S_{P,U_P,t} \cup U_P$ and $S_{Q,T_Q} \cup N_Q$ are open subsets of the type ***O2***.

□

Similar to Corollary III.13 in §3.1, the boundary $\coprod_{P \in \Delta_\Gamma}$ and the interior $X$ are indeed the boundary and interior of $\overline{X}_\Gamma$ with respect to the topology $\mathcal{T}$.

**Corollary IV.9.** *With respect to the topology $\mathcal{T}$ on the set $\overline{X}_\Gamma$ defined in Proposition III.6, the interior and the boundary of $\overline{\mathbf{H}}_\Gamma$ is the symmetric space $X$ and the disjoint union $\coprod_{P \in \Delta_\Gamma} e(P)$ respectively.*

*Remark* IV.10. We will use the open basis of $\overline{X}_\Gamma$ to prove that the space $\overline{X}_\Gamma$ is a smooth manifold with boundary (Proposition IV.26).

In the next section, we define a $\Gamma$-action on the space $\overline{X}_\Gamma$ and show that this action is proper.

## 4.2 The Γ-action on $\overline{X}_\Gamma$

In this section, we will define the action of the lattice Γ on the space $\overline{X}_\Gamma$, and show that this action is continuous and proper. In Definition IV.11 and Definition IV.13, we define the Γ-action on the interior and boundary of $\overline{X}_\Gamma$ respectively using the horospherical coordinates. In Proposition IV.14, we show that this action is continuous, i.e. the Γ-action on the interior extends continuously to the boundary. In Proposition IV.17, we prove that the Γ-action is proper.

The action of the group Γ on the symmetric space $X$ is canonically defined by the left multiplication of $G$. This action is described in terms of horospherical decomposition as follow.

**Proposition IV.11.** *Let $P$ be a parabolic subgroup in the set $\Delta_\Gamma$, $X = N_P \times A_P$ be the horospherical decomposition of $X$ with respect to $P$, and $G = N_P \times A_P \times K$ be the Iwasawa decomposition of $G$ with respect to $P$. For every point $z = (n_1, a_1) \in N_P \times A_P$ in $X$ and every element $\gamma = (n, a, k) \in N_P \times A_P \times K$ in Γ, the action $\gamma \cdot z$ is given by*

$$(n, a, k) \cdot (n_1, a_1) = ({}^k(n^a n_1), {}^k(aa_1))$$

*where the right hand-side is the coordinate of $\gamma \cdot z$ in terms of the horospherical decomposition of $X$ with respect to ${}^kP$.*

*Proof.* Let $(n_1, a_1, k_1)$ be the element of $G$. Under the canonical projection $G \to X$, the point $(n_1, a_1, k_1)$ is mapped to $(n_1, a_1)$. Under the isomorphism $N_P \times A_P \times K \to G$

$$(n, a, k) \mapsto nak,$$

(c.f. Proposition II.37), we have

$$(n, a, k) \cdot (n_1, a_1, k_1) = (n^{ak} n_1, a^k a_1, kk_1).$$

Thus the coordinates of $\gamma \cdot z$ in terms of the horospherical decomposition of $X$ with respect to $P$ is

$$(n, a, k) \cdot (n_1, a_1) = (n^{ak} n_1, a^k a_1).$$

By the conjugation of $k$, we convert the coordinate system to the horospherical decomposition of $X$ with respect to ${}^kP$. We then have

$$(n, a, k) \cdot (n_1, a_1) = ({}^k(n^a n_1), {}^k(aa_1)).$$

□

*Remark* IV.12. The reason for the horospherical decomposition of $X$ with respect to ${}^kP$ is that we want the action to be continuously extended to the boundary (c.f. Proposition IV.14).

The action of the lattice $\Gamma$ on the boundary is defined as follow.

**Definition IV.13.** Let $P$ be a parabolic subgroup in the set $\Delta_\Gamma$ and $G = N_P \times A_P \times K$ be the Iwasawa decomposition of $G$ with respect to $P$. For every point $\zeta$ in the boundary component $e(P)$ and every element $\gamma = (n, a, k) \in N_P \times A_P \times K$ in $\Gamma$, the action of $\gamma \cdot \zeta$ is defined by

$$(n, a, k) \cdot \zeta = {}^k(n^a \zeta).$$

**Proposition IV.14.** *Let the lattice* $\Gamma$ *acts on the space* $\overline{X}_\Gamma$ *as in Definition IV.11 and Definition IV.13. Then the* $\Gamma$*-action is continuous.*

*Proof.* Let $y_j = (n_j, a_j) \in N_P \times A_P$ be a convergent sequence of the type ***S3*** such that $y_j \to y_\infty \in e(P)$. Let $\gamma$ be an element in $\Gamma$ such that $\gamma = (n, a, k)$. Then

$$\gamma \cdot y_j = ({}^k(n^a n_j), {}^k(aa_j)).$$

By Proposition II.68, every positive restricted root $\beta$ in $\Phi^+({}^kP, A_{{}^kP})$ is of the form $\beta = {}^k\alpha$ where $\alpha \in \Phi^+(P, A_P)$. Therefore,

$$({}^k(aa_j))^\beta = (aa_j)^\alpha = a^\alpha a_j^\alpha \to \infty.$$

So $y_j$ converges to ${}^k(n^a y_\infty) = \gamma \cdot y_\infty$. $\square$

The following lemma will be used in the proofs of Proposition IV.17 and Proposition IV.21.

**Lemma IV.15.** *Let $y_j \in X$ be a convergent sequence of the type **S3** in $\overline{X}_\Gamma$. Then there is no sequence of elements $\gamma_j \in \Gamma$ such that $y'_j = \gamma_j \cdot y_j \in X$ is a convergent sequence of the type **S1**.*

*Proof.* We prove by contradiction. Suppose $y'_j \to y'_\infty \in X$. Since $\Gamma$ is discrete, there exists $\epsilon > 0$ such that $d(\gamma \cdot y'_\infty, y'_\infty) > \epsilon$ for all $\gamma \in \Gamma$. Thus

$$\begin{aligned} 2d(y'_\infty, y'_j) + d(\gamma \cdot y'_j, y'_j) &= d(\gamma \cdot y'_\infty, \gamma \cdot y'_j) + d(\gamma \cdot y'_j, y'_j) + d(y'_j, y'_\infty) \\ &\geq d(\gamma \cdot y'_\infty, y'_\infty) > \epsilon > 0. \end{aligned}$$

For a positive $\epsilon' > 0$ such that $\epsilon' < \epsilon$, we can choose a sufficiently large $N \gg 1$ such that

$$2d(y'_\infty, y'_j) < \epsilon' < \epsilon \text{ for all } j > N.$$

Then for every nontrivial $\gamma \in \Gamma$ and $j > N$,

$$d(\gamma \cdot y'_j, y'_j) > \epsilon - \epsilon' > 0.$$

Note that both $\epsilon$ and $\epsilon'$ do not depend on the choice of $\gamma$. Let $\gamma$ be a nontrivial element $\gamma \in \Gamma \cap N_P$ and $\gamma'_j = \gamma_j \gamma \gamma_j^{-1}$. Since $d(\gamma \cdot y_j, y_j) \to 0$, it follows that

$$d(\gamma \cdot y_j, y_j) = d(\gamma_j \gamma \cdot y_j, y'_j) = d(\gamma'_j \cdot y'_j, y'_j) \to 0$$

This is a contradiction. $\square$

From above lemma, it follows that

**Lemma IV.16.** *For every parabolic subgroup* $P \in \Delta_\Gamma$*, the intersection* $\Gamma \cap A_P$ *is trivial.*

*Proof.* We will give two versions of the proof.

Suppose $\Gamma \cap A_P$ is not trivial. If it is a finite subgroup of $A_P$, then it is a finite subgroup of the Lie group $\mathbb{R}^+$, which is a contradiction. Therefore, $\Gamma \cap A_P$ must be infinite. Let $Y \in \mathfrak{p}^1$ be the vector corresponding to the parabolic subgroup $P$ and choose an element $\gamma \in \Gamma \cap A_P$ such that

$$\gamma = \exp(t_0 Y), \quad t_0 > 1.$$

For a fixed point $x_0$ in $X$ and for all positive integer $j$, the sequence $y_j \gamma^j \cdot x_0$ is unbounded and lying on the geodesic $\exp(tY) \cdot x_0$. Thus it is a convergence sequence of the type ***S3*** in $\overline{X}_\Gamma$. However, the sequence $y'_j$ from Lemma IV.15 is the constant $x_0$, which is a convergent sequence of the type ***S1***. This is a contradiction.

Another proof is the following. Let $\gamma_0$ be an element in $\Gamma \cap N_P$ such that $\gamma \neq Id$. Let us assume $\Gamma \cap A_P$ is non-trivial and $\gamma$ be a non-trivial element in it. There exists a non-zero vector $H$ in the positive Weyl chamber $\in \mathfrak{a}_P^+$ such that

$$\gamma = \exp(H).$$

Since the Lie algebra $\mathfrak{n}_P$ of $N_P$ is

$$\mathfrak{n}_P = \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_P)} \mathfrak{g}_\alpha,$$

we can write

$$\gamma_0 = \exp(\sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_P)} X_\alpha).$$

Then the conjugate action of $\gamma^{-n}$ on $\gamma_0$ for $n \in \mathbb{N}$ is written as

$$\gamma^{-n}\gamma_0\gamma^n = \exp(ad_{-nH} \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a}_P)} X_\alpha) = \exp(\sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a}_P)} e^{-n\alpha(H)} X_\alpha).$$

Therefore, as $n \to \infty$, the sequence $\gamma^{-n}\gamma_0\gamma^n \to Id$ which contradicts the discreteness of the group $\Gamma$.

□

**Proposition IV.17.** *Let the lattice $\Gamma$ acts on the space $\overline{X}_\Gamma$ as in Definition IV.11 and Definition IV.13. The $\Gamma$-action on $\overline{X}_\Gamma$ is proper.*

*Proof.* Let $C$ be a compact subset of $\overline{X}_\Gamma$ and $\Gamma'$ be the set defined by

$$\Gamma' = \{\gamma \in \Gamma \mid \gamma \cdot C \cap C \neq \emptyset\}.$$

We will show that the set $\Gamma'$ is finite for the cases when $C$ is of the type ***C1***, ***C2***, and ***C3***.

**Case 1** Suppose $C$ of the type ***C1***. Since the $\Gamma$-action on a symmetric space $X$ is proper, we are done.

**Case 2** Suppose that the set $C$ is of the type ***C2***, i.e. $C$ is a finite disjoint union of closed subsets of $e(P)$, say

$$C = \coprod_{P \subset \Delta} C_P$$

where $\Delta$ is a finite subset of $\Delta_\Gamma$. Let $\Gamma'_{P,Q} = \{\gamma \in \Gamma' \mid \gamma \cdot C_P \cap C_Q \neq \emptyset\}$ so that

$$\Gamma' = \bigcup_{P,Q \in \Delta} \Gamma'_{P,Q}.$$

We will show that each subset $\Gamma'_{P,Q}$ is finite.

1. Suppose $P = Q$. Then every element $\gamma$ in $\Gamma'_{P,P}$ is of the form

$$\gamma = (n, a, Id).$$

Since the group $\Gamma \cap N_P$ acts on $e(P)$ cocompactly, there are only finitely many choices of the coordinate $n$. From Lemma IV.16, the coordinate $a$ is constant for each $n$. Thus the set $\Gamma'_{P,P}$ is finite.

2. Suppose $P \neq Q$ and the coordinates of an element $\gamma$ in $\Gamma_{P,Q}$ is given by

$$\gamma = (n, a, k) \in N_P \times A_P \times K.$$

Since ${}^{k_\gamma}P = Q$, there are only finitely many choices of the coordinate $k$. For each choice of the coordinate $k$, we have

$$(n, a, Id) \cdot C_P \cap k^{-1} \cdot C_Q \neq \emptyset.$$

Since $k^{-1} \cdot C_Q$ and $C_P$ are compact subsets in $e(P)$, there are only finitely many choice of the coordinates $n$ and $a$.

This proves the second case.

**Case 3** Lastly, suppose that $C$ is of the type ***C3***, i.e.

$$C = \overline{\mathfrak{S}}_{P,C_P,t}$$

where $C_P$ is a compact subset of $e(P)$. We will prove that the set $\Gamma'$ is finite by contradiction. Suppose there exists a infinite sequence $\gamma_j \in \Gamma'$ such that $\gamma_j \neq \gamma_i$ for all $i \neq j \geq 1$. For each $\gamma_j$, there exists an element $z_j \in C$ such that $\gamma_j \cdot z_j \in C$. Let us denote $z'_j = \gamma_j \cdot z_j$. If the set $\{z_j \mid j \geq 1\}$ is finite, then by passing to a subsequence, we may assume that $z_j = z$ for all $j$. Then the sequence $\{z'_j\}_1^\infty$ consists of infinitely points in $C$, thus it admits an accumulation point in $C$. This contradicts to the discreteness of $\Gamma$. So let us assume that the set $\{z_j \mid j \geq 1\}$ is infinite, more specifically, $z_j \neq z_i$ for all $i \neq j \geq 1$. Two infinite sequences, $\{z_j\}_1^\infty$ and $\{z'_j\}_1^\infty$, are contained in a compact set $C$, thus there exist accumulation points $z_\infty$ and $z'_\infty$ such that

$$z_j \to z_\infty, \quad z'_j \to z'_\infty.$$

From Proposition IV.4, each sequence is essentially of the type ***S1***, ***S2***, or ***S3***. For each case, we will show a contradiction.

1. If either $z_j$ or $z'_j$ is of the type ***S2***, then so is the other, because no point in the boundary can mapped into the interior by the $\Gamma$-action. In this case, the set $\Gamma'$ being infinite contradicts to the finiteness result from **Case 2**.

2. Suppose both $z_j$ and $z'_j$ are of the type ***S1***. This contradicts to the **Case 1**.

3. Suppose $z_j$ and $z'_j$ are of the types ***S1*** and ***S3*** respectively. Without loss of generality, we assume that

   $$d(z'_j, z'_{j+1}) = N_j, \quad d(z_j, z_\infty) = \epsilon_j$$

   such that $N_j < N_{j+1}$, $\epsilon_j > \epsilon_{j+1}$, and $N_j \to \infty$, $\epsilon_j \to 0$ as $j \to \infty$. By the triangle inequality, it follows that

   $$d(\gamma_j \cdot z_\infty, z'_j) = \epsilon_j$$

   and

   $$d(\gamma_j \cdot z_\infty, \gamma_{j+1} \cdot z_\infty) > N_j - \epsilon_j - \epsilon_{j+1}.$$

   Thus $\gamma_j \cdot z_\infty$ is a convergent sequence of the type ***S3***. Then for almost all $j \geq 1$, elements $\gamma_j$ belong to $\Gamma \cap P$. The cocompactness of the lattice $\Gamma \cap N_P$ implies that the set of $N_P$-coordinates of $\gamma_j$'s are finite. From Lemma IV.16, it follows that for each $N_P$-coordinates of $\gamma_j$, the $A_P$-coordinates are constant. Thus the set $\{\gamma_j \mid j \geq 1\}$ is finite.

This completes the proof. □

## 4.3 Cocompactness of the $\Gamma$-action on $\overline{X}_\Gamma$

In this section, we show that the $\Gamma$-action on the space $\overline{X}_\Gamma$ is cocompact, i.e. the quotient space $\Gamma\backslash\overline{X}_\Gamma$ is compact. In the propositions IV.19 and IV.21, we observe a one-to-one correspondence between the set $\Delta_\Gamma$ and the vectors which are $\Gamma$-equivalent to a ray (Definition IV.18) with respect to a Dirichlet fundamental domain for $\Gamma$. We then use the results (Theorem IV.22) of Garland and Raghunathan in [24] on fundamental set $\Omega$ of $\Gamma$ in $X$ to prove that the closure $\overline{\Omega}$ is a fundamental set of $\Gamma$ in $\overline{X}_\Gamma$ (Proposition IV.23). Then the cocompactness of the $\Gamma$-action follows immediately (Corollary IV.24).

**Definition IV.18.** Let $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of $G$ and $K$ respectively, and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$. Let $D$ be the Dirichlet fundamental domain for $\Gamma$ at the base point $x_0 \in X$. A vector $Y \in \mathfrak{p}$ is called a **ray with respect to** $D$ if the geodesic segment $\gamma(t) = \exp(tY) \cdot x_0$ $(t \geq 0)$ lies in the domain $D$

From Corollary II.75 in §2.4.2, each ray $Y$ corresponds to a unique minimal parabolic subgroup $P_Y$. The following proposition is proved by Garland and Raghunathan.

**Proposition IV.19.** *For every ray $Y$ with respect to $D$ and corresponding parabolic subgroup $P_Y$, the subgroup $\Gamma \cap N_Y$ of $N_Y$ is a cocompact lattice.*

*Proof.* See [24, Theorem 0.7]. □

**Corollary IV.20.** *Let $Y$ be a vector in $\mathfrak{p}^1$ which is $\Gamma$-equivalent to a ray with respect to a Dirichlet fundamental domain $D$. Then the minimal parabolic subgroup $P_Y$ corresponding to $Y$ belongs to the set $\Delta_\Gamma$.*

*Proof.* Let $\gamma$ be an element in $\Gamma$ such that ${}^\gamma Y$ is a ray with respect to $D$. Then the following diagram holds:

$$\begin{array}{ccc} \Gamma \cap N_Y & \hookrightarrow & N_Y \\ \downarrow \gamma & & \downarrow \gamma \\ \Gamma \cap {}^{\gamma}N_Y & \hookrightarrow & {}^{\gamma}N_Y \end{array}$$

Since the subgroup $\Gamma \cap {}^{\gamma}N_Y$ of ${}^{\gamma}N_Y$ is a cocompact lattice, so is the subgroup $\Gamma \cap N_Y$ of $N_Y$. This completes the proof. □

In the next proposition (Proposition IV.21), we prove that the converse is true. That is, for every parabolic subgroup $P$ in the set $\Delta_\Gamma$, the corresponding vector $Y_P$ in $\mathfrak{p}^1$ is a ray with respect to a Dirichlet fundamental domain.

**Proposition IV.21.** *Let $D$ be the Dirichlet fundamental domain for $\Gamma$ at the base point $x_0 \in X$. For every parabolic subgroup $P$ of $G$ in the set $\Delta_\Gamma$, the corresponding vector $Z_P$ in $\mathfrak{p}^1$ is $\Gamma$-equivalent to a ray with respect to $D$.*

*Proof.* Let $t_j$ be an increasing sequence of positive numbers and $y_j$ be a sequence in $X$ defined by

$$y_j = \exp(t_j Z) \cdot x_0.$$

The sequence $y_j$ lies on the the geodesic diverging to the point at infinity $\zeta_Z \in X(\infty)$ corresponding to $Z$. If there exists a sufficiently large $N \gg 1$ and an element $\gamma \in \Gamma$ such that $y_j \in \gamma \cdot D$ for all $j > N$, then $Z$ is a ray with respect to $\gamma \cdot D$. Suppose this fails, i.e. there exists an infinite sequence of elements $\gamma_j$ in $\Gamma$ such that $y_j \in \gamma_j \cdot D$. In other words, $y'_j = \gamma_j^{-1} \cdot y_j \in D$. By Lemma IV.15, the sequence $y'_j$ is unbounded. By passing to its subsequence, we may assume that the sequence $y'_j$ lies on a Siegel set $\mathfrak{S}_{P,U,t}$ (c.f. Theorem IV.22 below). Thus $y'_j$ is a convergent sequence of the type ***S3***. Note that the sequence $y_j$ is also a convergent sequence of the type ***S3***. For each element $\gamma_j$, let $\gamma_j = (n_j, a_j, k_j) \in N_P \times A_P \times K$. From Proposition II.76, the set of the coordinates for $k_j$ is finite. From the cocompactness of $\Gamma \cap N_P$, the set of the coordinates for $n_j$ is finite. From Lemma IV.16, the set of the coordinates for $a_j$ is finite. Thus $\gamma_j$ stabilizes, and we are done. □

We recall the main theorem of Garland and Raghunathan's paper [24] (c.f. Theorem II.78 and [24, Theorem 0.6]).

**Theorem IV.22.** *Let $\Gamma$ be a lattice subgroup of semisimple Lie group $G$ of $\mathbb{R}$-rank one. Let $D$ be a Dirichlet fundamental domain for $\Gamma$ in the symmetric space $X = G/K$.*

1. *There are only finitely many rays with respect to $D$.*
2. *There exists a compact subset $C$ of $D$ and finitely many Siegel sets $\mathfrak{S}_{P_n,U_n,t_n}$ ($n = 1, \cdots, r$) where each $P_n$ is a parabolic subgroup corresponding to a ray with respect to $D$ such that the union*

$$\Omega = C \cup \bigcup_{n=1}^{r} \mathfrak{S}_{P_n,U_n,t_n}$$

*is a locally finite fundamental set for $\Gamma$ in $X$.*

In the next proposition (Proposition IV.23), we will show that the closure $\overline{\Omega}$ of $\Omega$ in $\overline{X}_\Gamma$ is a fundamental set for the lattice $\Gamma$ acting on $\overline{X}_\Gamma$. We first observe that the closure $\overline{\Omega}$ is the union

$$\overline{\Omega} = C \cup \bigcup_{n=1}^{r} \overline{\mathfrak{S}}_{P_n,U_n,t_n}$$

where each $\overline{\mathfrak{S}}_{P_n,U_n,t_n}$ is a closed subset of the type ***C3***:

$$\overline{\mathfrak{S}}_{P_n,U_n,t_n} = \left(\overline{U}_n \times \overline{A}_{P_n,t_n}\right) \cup \overline{U}_n.$$

**Proposition IV.23.** *The closure $\overline{\Omega}$ of $\Omega$ in Theorem IV.22 is a locally finite fundamental set of $\Gamma$ in $\overline{X}_\Gamma$.*

*Proof.* Since $\Omega$ is a fundamental set of the $\Gamma$-action on $X$, each closure $\overline{U}_n$ of $U_n$ is a compact fundamental set for the $\Gamma \cap N_{P_n}$-action on the boundary component $e(P_n)$. It follows from Proposition IV.19, Corollary IV.20, and Proposition IV.21 that

$$\Gamma \cdot \bigcup_{n=1}^{r} \overline{\mathfrak{S}}_{P_n, U_n, t_n} = \coprod_{P \in \Delta_\Gamma} e(P).$$

The local finiteness of $\overline{\Omega}$ immediately follows from the local finiteness of compact domains and the Siegel set. □

**Corollary IV.24.** *The space* $\Gamma \backslash \overline{X}_\Gamma$ *is compact.*

*Proof.* This follows from the surjective homeomorphism $\overline{\Omega} \to \Gamma \backslash \overline{X}_\Gamma$. □

### 4.4 Γ-CW-structure of $\overline{X}_\Gamma$.

In Proposition IV.26, we show that $\overline{X}_\Gamma$ is a manifold with boundary. Furthermore, we prove that $\overline{X}_\Gamma$ is smooth in Proposition IV.27. Using Theorem III.32 due to Illman [33], we show that $\overline{X}_\Gamma$ admits a Γ-CW-structure (Proposition IV.29).

**Definition IV.25.** A topological space $M$ is called a **manifold with boundary** if for every point $x \in M$ has an open neighborhood $U$ which is homeomorphic to an open subset in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. The positive integer $n$ is the dimension of $M$

**Proposition IV.26.** *The space $\overline{X}_\Gamma$ is a manifold with boundary.*

*Proof.* For each minimal parabolic subgroup $P$ of $G$, let $\varphi_P : N_P \to \mathbb{R}^n$ be the homeomorphism and $\alpha_P$ be the positive simple root in $\Phi^+(P, A_P)$. Let $\psi : \overline{X}_\Gamma \to \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the map defined as follow:

$$\psi(z) = \begin{cases} \left(\varphi_P(n), \frac{1}{\alpha_P(a)}\right) & \text{if } z = (n, a) \in N_P \times A_P \\ (\varphi_P(z), 0) & \text{if } z \in e(P) \end{cases}$$

We claim that for each open basis $V$ of the space $\overline{X}_\Gamma$, the restriction $\psi|_V : V \to \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is a homeomorphism onto its image (c.f. Lemma III.28).

Since the map $\varphi_P$ and $\alpha_P$ are homeomorphic onto its image, it follows that the map $\psi|_V$ is the identity map if the subset $V$ is of the type ***O1***. Let $V$ be an open subset of the type ***O2***, i.e. $V = \mathfrak{S}_{P,U,t} \cup U$ for some open subset $U$ of $e(P)$. Let $y_j = (n_j, a_j) \in N_P \times A_P$ be a convergent sequence in $V$ of the type ***S3*** such that $y_j \to y_\infty$. Then

$$\psi|_V(y_j) = (\varphi_P(n_j), \frac{1}{\alpha_P(a_j)}) \to (\varphi_P(y_\infty), 0) = \psi|_V(y_\infty).$$

Thus the map $\psi|_V$ is continuous. The image of $\psi|_V$ is $\varphi_P(U) \times [0, \frac{1}{t})$, and is open in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Thus the map $\psi|_V$ is homeomorphic onto its image. □

In the next proposition (Proposition IV.27), we show that the space $\overline{X}_\Gamma$ is a smooth manifold with boundary.

**Proposition IV.27.** *The space $\overline{X}_\Gamma$ is a smooth manifold with boundary.*

*Proof.* We need to show that for two open neighborhoods $V_1$ and $V_2$ such that $V_1 \cap V_2 \neq \emptyset$, the transition map $(\psi|_{V_2})^{-1} \circ \psi|_{V_1} : V_1 \cap V_2 \to V_1 \cap V_2$ is a diffeomorphism. Let $V_1$ and $V_2$ be open neighborhoods of the type either ***O1*** or ***O2***. Thus there are essentially four cases to consider (Figure 3.3).

**Case 1** Suppose both $V_1$ and $V_2$ are of the type ***O1***. Then the transition map is the identity map, thus it is a diffeomorphism.

**Case 2** Suppose $V_1$ and $V_2$ are of the type ***O1*** and ***O2*** respectively. Then the intersection $V_1 \cap V_2$ is open subset of the type ***O1***, so it reduces to the **Case 1**.

**Case 3** Suppose that $V_1$ and $V_2$ are open subsets of the type ***O2*** with respect to the common parabolic subgroup $P$, i.e. $V_1 = \mathfrak{S}_{P,U_1,t_1} \cup U_1$ and $V_2 = \mathfrak{S}_{P,U_2,t_2} \cup U_2$ for open subsets $U_1$ and $U_2$ of the boundary component $e(P)$. Then two maps $\psi|_{V_1}$ and $\psi|_{V_2}$ are the same on the domain $V_1 \cap V_2$. Thus the transition map is identity.

**Case 4** Suppose that $V_1$ and $V_2$ are open subsets of the type **O2** such that $V_1 = \mathfrak{S}_{P_1,U_1,t_1}$ and $V_2 = \mathfrak{S}_{P_2,U_2,t_2}$ where $P_1 \neq P_2$. Let $k$ be an elemen in $K$ such that $P_2 = {}^kP_1$. Let $f : N_{P_1} \to N_{P_2}$ and $g : A_{P_1} \to A_{P_2}$ be the maps defined as

$$f(n) = {}^kn, \quad g(a) = {}^ka.$$

Then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad k \quad} & X \\ \downarrow \cong & & \downarrow \cong \\ N_{P_1} \times A_{P_1} & \xrightarrow{f \times g} & N_{P_2} \times A_{P_2} \end{array}$$

Note that the transition map $\psi|_{V_2} \circ (\psi|_{V_1})^{-1}$ is defined by the following diagram:

$$\begin{array}{ccc} N_{P_1} \times A_{P_1} & \xrightarrow{f \times g} & N_{P_2} \times A_{P_2} \\ \downarrow{\scriptstyle \psi|_{V_1}} & & \downarrow{\scriptstyle \psi|_{V_2}} \\ \mathbb{R}^n \times \mathbb{R}_{\geq 0} & \longrightarrow & \mathbb{R}^n \times \mathbb{R}_{\geq 0} \end{array}$$

Since the map $k : X \to X$ induced from the $G$-action is smooth, it follows that the map $\psi|_{V_2} \circ (\psi|_{V_1})^{-1}$ is smooth.

This completes the proof. $\square$

*Remark* IV.28. In [60], Raghunathan already observed the structure of smooth manifold with boundary of $\overline{X}_\Gamma$. He showed that there exists a smooth map $f : G \to \mathbb{R}_{\geq 0}$ such that

1. $f(g \cdot k) = f(g)$ for all $k \in K$, $g \in G$,
2. $f(\gamma \cdot g) = f(g)$ for all $\gamma \in \Gamma$, $g \in G$, and
3. there exists a compact subset $E \subset G$ such that $f$ does not have any critical point outside $\Gamma \cdot E$.

From the Morse theory, there exists a sufficiently large $N > 0$ such that $f(G) \subset [0, N]$. Thus $G$ is a open submanifold of a manifold with boundary whose boundary component corresponds to the set $f^{-1}(N)$. Taking a quotient from the right by the subgroup $K$, we obtain the desired submanifold structure on the symmetric space $X$.

In the next proposition (Proposition IV.29), we will show that $\overline{X}_\Gamma$ admits a $\Gamma$-CW-structure. Recall that a **subanalytic manifold** $N$ is a topological space with closed embedding $l : N \to M$ such that every point $x \in N$ has an open neighborhood $U$ which belongs to $\mathcal{J}(U)$ (c.f. Definition III.31). We also recall that a manifold with boundary is a subanalytic submanifold of a smooth manifold (Proposition III.33).

**Proposition IV.29.** *The space $\overline{X}_\Gamma$ admits a $\Gamma$-CW-structure.*

*Proof.* From Proposition IV.26 and Proposition III.33, the space $\overline{X}_\Gamma$ is a subanalytic manifold. The double (c.f. Remark III.34) of $\overline{X}_\Gamma$ is a smooth $\Gamma$-manifold by Proposition IV.27. It follows from Theorem III.32 that $M$ has a $\Gamma$-CW-structure. $\square$

## 4.5 Proof of the main theorem

In this section, we prove the main theorem (Theorem I.1). Recall that a $\Gamma$-CW-complex $X$ is a model for the proper classifying space $\underline{E}\Gamma$ if

1. every isotropy group is finite, and
2. for each finite subgroup $H \subset \Gamma$, the fixed point set $\overline{X}_\Gamma^H$ is contractible.

**Lemma IV.30.** *The space $\overline{X}_\Gamma$ is contractible.*

*Proof.* From Theorem IV.22, there exists a compact subset $C$ of a Dirichlet fundamental domain $D$ and finitely many Siegel sets $\mathfrak{S}_{P_n,U_n,t_n}$ such that

$$\Omega = C \cup \bigcup_{n=1}^{r} \mathfrak{S}_{P_n,U_n,t_n}$$

is a fundamental set for $\Gamma$. From Proposition IV.23, the closure $\overline{\Omega}$ of $\Omega$ is a fundamental set for $\Gamma$ in $\overline{X}_\Gamma$. We will show that there exists a homotopy retraction of $\overline{\Omega}$ onto the compact set $C$. Since $C$ is convex and compact, it is contractible.

For $t \in (0, 1]$ and each $n = 1, \cdots, r$, let $a_{n,t}$ be an element in $A_{P_n}$ such that

$$\alpha(P_n)(a_{n,t}) = t_n/t$$

where $\alpha_{P_n}$ is the positive simple root in $\Phi^+(P, A_P)$. Since $A_{P_n} \cong \mathbb{R}^+$ (c.f. Proposition II.63), such $a_t$ is unique. Let us define a map $h_t : \overline{\Omega} \to \overline{\Omega}$ as follow.

$$h_t(z) = \begin{cases} z & \text{if } z \in C \\ (n_1, a_{n,t}) & \text{if } z \in \mathfrak{S}_{P_n,U_n,t_n/t} \text{ for some } n = 1, \cdots, r \\ & \text{and } z = (n_1, a_1) \in N_{P_n} \times A_{P_n} \\ (z, a_{n,t}) & \text{otherwise, i.e. } z \in U_n \end{cases}$$

$$h_0(z) = z. \tag{4.3}$$

See Figure 3.7 for geometric description. To show that $h_t$ is continuous, it is sufficient to show that $h_t \to Id$ as $t \to 0$. We will show the limit holds point-wise.

1. Suppose a point $z$ lies in $\Omega$. Then there exists a sufficiently small positive $t \ll 1$ such that $z \notin \bigcup_{n=1}^{r} \overline{\mathfrak{S}}_{P_n,U_n,t_n/t}$. Thus $h_t(z) = z$ for all $t < \epsilon$, so $h_t(z) \to z$ as $t \to 0$.

2. Suppose $z \in U_n$ for some $n = 1, \cdots, r$. Since $h_t(z) = (z, a_{n,t})$ and $\alpha_n(a_t) = t_n/t \to \infty$, it follows that $h_t(z) \to z$.

Thus the map $h_t$ is a homotopy retraction of $\overline{D}$ onto the compact subset $C$. We will extend $h_t$ to a homotopy retraction of the entire space $\overline{X}_\Gamma$. Let us define a map $H_t : X \times [0,1] \to X$ as follow: for every point $z \in \overline{X}_\Gamma$ and an element $\gamma \in \Gamma$ such that $\gamma \cdot z \in \overline{\Omega}$,

$$H_t(z) = \gamma^{-1} \cdot h_t(\gamma \cdot z). \tag{4.4}$$

We will show that $H_t$ is well-defined. Once $H_t$ is well-defined, it follows immediately that the map $H_t$ is a homotopy retraction of the space $\overline{X}_\Gamma$ onto the union $\Gamma \cdot C$, which is contractible.

Let $z$ be a point in $X$. Suppose there exist two elements $\gamma_1$ and $\gamma_2$ of $\Gamma$ such that $\gamma_1 \cdot z \neq \gamma_2 \cdot z$ and both $\gamma_1 \cdot z$ and $\gamma_2 \cdot z$ lie on the same Siegel set $\mathfrak{S}_{P_n,U_n,t_n/t}$. For $j = 1, 2$, let $s_j$ be the geodesic connecting $h_t(\gamma_j \cdot z)$ and $\gamma_j \cdot z$. Two geodesics $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ pass through the common point $z$. Moreover, in the topology of geodesic compactification $\overline{X}$, they converges to the same point at infinity. Thus two geodesics $s_1$ and $s_2$ are the same, since the rank of $X$ is one. Therefore, $\gamma_1^{-1} \cdot h_t(\gamma_1 \cdot z) = \gamma_2^{-1} \cdot h_t(\gamma_2 \cdot z)$.

Let $z$ be a point in the boundary component $e(P)$ for some parabolic subgroup $P$ in $\Delta_\Gamma$. Suppose $\gamma_1$ and $\gamma_2$ be the elements of $\Gamma$ such that $\gamma_1 \cdot z \neq \gamma \cdot z$. Two points $\gamma_1 \cdot z$ and $\gamma \cdot z$ are lying on the boundary of $\overline{\Omega}$, but not necessarily on the same boundary component. For each $j = 1, 2$, let $s_j$ be a geodesic path properly contained in a Siegel set such that it passes through the point $h_t(\gamma_j \cdot z)$ and converges (in the topology of $\overline{X}_\Gamma$) to a point $\gamma_j \cdot z$ on the boundary. Note that two geodesic $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ converge to the same point $z$ on the boundary, so $\gamma_1^{-1} \cdot s_1$ and $\gamma_2^{-1} \cdot s_2$ are the same. Therefore, $\gamma_1^{-1} \cdot h_t(\gamma_1 \cdot z) = \gamma_2^{-1} \cdot h_t(\gamma_2 \cdot z)$. □

We prove the main theorem of the thesis.

**Theorem** (The main theorem)**.** *Let* $\Gamma$ *be a discrete subgroup of a semisimple Lie group* $G$ *of* $\mathbb{R}$*-rank one. Then the space* $\overline{X}_\Gamma$ *is a cofinite* $\Gamma$*-CW-complex model for the proper classifying space* $\underline{E}\Gamma$*.*

*Proof.* From Proposition IV.29, the space $\overline{X}_\Gamma$ is a $\Gamma$-CW-complex. From Corollary IV.24, such $\Gamma$-CW-structure is cofinite.

Let $H$ be a non-trivial finite subgroup of $\Gamma$ and $\gamma \in H$ be an element of fintie order. If $H$ fixes a point in $e(P)$, then $H \subset \Gamma \cap P$. Let $(n, a, m)$ be the coordinates of $\gamma$ in the Langlands decomposition of $P$. Since $\Gamma \cap N_P$ acts on $N_P$ a lattice, $n$ must be identity. If $a$ is nontrivial, the order of $a$ is not finite, which contradicts the assumption that the order of $\gamma$ is finite. Thus $\Gamma \cap P = \Gamma \cap M_P$. That is, every point in the set $e(P)^H$ is fixed by an element of $M_P$. Since $M_P$ commutes with $A_P$, the element $\gamma$ also fixes all points on geodesic rays converging to points in $e(P)^H$. Thus the points at boundary in $(\overline{X}_\Gamma)^H$ retracts along these geodesics into the geodesic submanifold $X^H$. Since $X^H$ is contractible, so is $(\overline{X}_\Gamma)^H$. For the trivial subgroup of $\Gamma$, the fixed point set $\overline{X}_\Gamma$ is contractible by Lemma IV.30.

From the proper action of $\Gamma$, every isotropy group is finite. □

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