TORIC PROJECTIVE BUNDLES

by

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To Antonio and Nury

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CHAPTER I

Introduction

Toric projective bundles arise as projectivizations of toric vector bundles. These varieties are not toric in general, however, they are endowed with a torus action and they have a well-understood combinatorial description. In addition, these rational varieties enjoy some of the finiteness properties of Mori dream spaces, such as the finite generation of their nef and Mori cones (see [HMP10, Remark 2.5]). Toric vector bundles were classified by A. Klyachko in [Kly90], in terms of certain filtrations of a suitable vector space (see Theorem II.4), and they have been the focus of some recent activity, e.g. [Gon09], [Gon10], [GHPS10], [HMP10], [Pay09], [Pay08]. In [HMP10], M. Hering, M. Mustaţă and S. Payne raised one of the main questions regarding the geometric structure of their projectivizations. Namely, whether their Cox rings are indeed finitely generated. In this dissertation we study some invariants of these projectivized toric vector bundles such as their global Okounkov bodies, their Cox rings and their cones of pseudoeffective divisors. In the first of our main results we associate to each rank two toric vector bundle a flag of torus invariant subvarieties on its projectivization, and we describe the corresponding global Okounkov body in terms of the combinatorial data of the toric variety on the base and the data in the Klyachko filtrations of the toric vector bundle. Later on, we present two proofs of the finite generation of the Cox rings of projectivizations of rank two toric vector bundles, which use different techniques from those used by J. Hausen and H. Süß in [HS10] in their solutions to this finite generation question in the rank two case. In one case our approach is direct and combinatorial; in the other, the argument relies on our description of the global Okounkov bodies of these varieties. We conclude by presenting my joint work with M. Hering, S. Payne and H. Süß, in which we give negative answers to the finite generation of the Cox rings and pseudoeffective cones of projectivizations of higher rank toric vector bundles. Our counterexamples have two flavors, some are very general in their moduli spaces, and some are determined by the combinatorial data of the toric varieties such as their cotangent bundles.

Mori dream spaces were introduced by Hu and Keel in [HK00] as a class of varieties with interesting features from the point of view of Mori theory. For instance, their pseudoeffective and nef cones are both polyhedral, and the Mori program can be carried out for any pseudoeffective divisor on these varieties. Additionally, their pseudoeffective cones can be decomposed into finitely many closed convex chambers that are in correspondence with the birational contractions of X having $\mathbf{Q}$-factorial image (see Proposition 1.11 [HK00]). A Mori dream space can be defined as a normal projective $\mathbf{Q}$-factorial variety X, that satisfies $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$ and has a finitely generated Cox ring (see Definition II.13). One basic example is that of toric varieties, in which the Cox ring is a polynomial ring in finitely many variables (see [Cox95]). Thus, a projective simplicial toric variety is a Mori dream space. Another example is given by the (log) Fano varieties, which were recently proven to be Mori dream spaces (see [BCHM08]). For one more example, if a vector bundle over a toric variety splits as a sum of line bundles, then its projectivization admits a toric variety structure (see [Oda78, §7]), and therefore it is a Mori dream space. Some references for recent work on Cox rings of particular Mori dream spaces are [BP04], [Cas08], [Gon09], [Gon10], [GHPS10], [HS10], [Ott10] and [STV07].

In his work on log-concavity of multiplicities, e.g. [Ok96], [Ok03], A. Okounkov introduced a procedure to associate convex bodies to linear systems on projective varieties. This construction was systematically studied by R. Lazarsfeld and M. Mustaţă in the case of big line bundles in [LM08]. The construction of these *Okounkov bodies* depends on a fixed flag of subvarieties and produces a convex compact set for each Cartier divisor on a projective variety. The Okounkov body of a divisor encodes asymptotic invariants of the divisor's linear system, and it is determined solely by the divisor's numerical equivalence class. Moreover, these bodies vary as fibers of a linear map defined on a closed convex cone as one moves in the space of numerical equivalence classes of divisors on the variety. As a consequence, one can expect to obtain results about line bundles by applying methods from convex geometry to the study of these Okounkov bodies.

Let us consider an n-dimensional projective variety X over an algebraically closed field, endowed with a flag $X_\bullet \colon X = X_n \supseteq \cdots \supseteq X_0 = \{pt\}$, where X_i is an i-dimensional subvariety that is nonsingular at the point X_0. In [LM08], Lazarsfeld and Mustaţă established the following:

(a) For each big rational numerical divisor class ξ on X, Okounkov's construction yields a convex compact set $\Delta(\xi)$ in $\mathbf{R}^n$, now called the *Okounkov body* of ξ, whose Euclidean volume satisfies

$$\mathrm{vol}_{\mathbf{R}^n}\big(\Delta(\xi)\big) \;=\; \frac{1}{n!}\cdot \mathrm{vol}_X(\xi).$$

The quantity $\mathrm{vol}_X(\xi)$ on the right is the *volume* of the rational class ξ, which is defined by extending the definition of the volume of an integral Cartier divisor D on

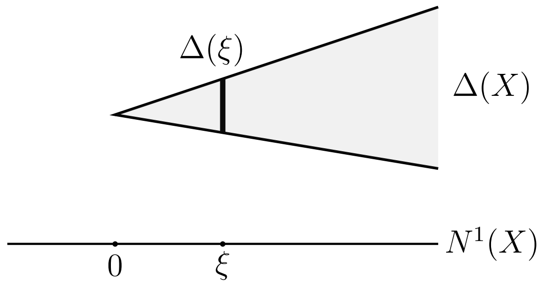

Figure 1.1: The global Okounkov body.

X, namely,

$$\mathrm{vol}_X(D) \ =_{\mathrm{def}} \ \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

We recall that the volume is an interesting invariant of big divisors which plays an important role in several recent developments in higher dimensional geometry. For basic properties of volumes we refer to [Laz04].

(b) There exists a closed convex cone $\Delta(X) \subseteq \mathbf{R}^n \times N^1(X)_{\mathbf{R}}$ characterized by the property that in the diagram

$$\begin{tikzcd} \Delta(X) \arrow[rr, hook] \arrow[dr] & & \mathbf{R}^n \times N^1(X)_{\mathbf{R}} \arrow[dl] \\ & N^1(X)_{\mathbf{R}}, & \end{tikzcd}$$

the fiber $\Delta(X)_\xi \subseteq \mathbf{R}^n \times \{\xi\} = \mathbf{R}^n$ of $\Delta(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta(\xi)$. This is illustrated schematically in Figure 1.1. $\Delta(X)$ is called the *global Okounkov body* of X.

Lazarsfeld and Mustaţă have used this theory to reprove and generalize results about volumes of divisors, including Fujita's Approximation Theorem. Using (**b**), they also give alternative proofs of properties of the volume function $\mathrm{vol}_X\colon \mathrm{Big}(X) \to \mathbf{R}$, defined on the set of big classes of $\mathbf{R}$-divisors. For example, it follows that vol_X

is of class $\mathcal{C}^1$ and satisfies the log-concavity relation

$$\mathrm{vol}_X(\xi+\xi')^{1/n} \;\geq\; \mathrm{vol}_X(\xi)^{1/n}+\mathrm{vol}_X(\xi')^{1/n},$$

for any two big classes $\xi, \xi' \in N^1(X)_{\mathbf{R}}$. It is worth mentioning that in [KK08], K. Kaveh and A. Khovanskii use a similar procedure to associate convex bodies to finite dimensional subspaces of the function field $\mathrm{K}(X)$ of a variety X. In their work, they use the bridge between convex and algebraic geometry provided by their construction to obtain results in both areas.

The explicit description of Okounkov bodies in concrete examples can be rather difficult. One easy case is that of smooth projective toric varieties. If D is an invariant divisor on such a variety, and if the flag consists of invariant subvarieties, then it is shown in [LM08] that the Okounkov body of D is the polytope P_D that one usually associates to D in toric geometry, up to a suitable translation.

As we mentioned, the goal of this dissertation is to study the Cox rings, the pseudoeffective cones and the global Okounkov bodies of projectivized toric vector bundles. We now summarize our main results and their organization in this dissertation.

In Chapter III, we present a description of the Okounkov bodies of all divisors on the projectivization $\mathbf{P}(\mathcal{E})$ of a rank two toric vector bundle $\mathcal{E}$ on a smooth projective toric variety X. For the reader's convenience, Klyachko's classification of toric vector bundles is reviewed in Chapter II, where we also include a description of the filtrations corresponding to tensor products and Schur functors applied to toric vector bundles. As we will see, these filtrations can be used to compute the sections of all line bundles on $\mathbf{P}(\mathcal{E})$. For our main result concerning Okounkov bodies (see Theorem III.10 and Remark III.11), we restrict to the case of rank two toric vector bundles, where the Klyachko filtrations are considerably simpler. Using the data from the filtrations, we construct a flag of torus invariant subvarieties on $\mathbf{P}(\mathcal{E})$ and produce finitely many linear inequalities defining the global Okounkov body of $\mathbf{P}(\mathcal{E})$ with respect to this flag. In particular, we see that this is a rational polyhedral cone. This description of $\Delta(\mathbf{P}(\mathcal{E}))$ will be used in Chapter IV to give a proof of the finite generation of the Cox ring of $\mathbf{P}(\mathcal{E})$.

In Chapter IV, we present two distinct proofs of the finite generation of the Cox rings of projectivized rank two toric vector bundles. While preparing the published versions of these two proofs, Milena Hering kindly brought to my attention the article [HS10] of J. Hausen and H. Süß, where they prove this finite generation using a different approach to ours, and also point out that it is possible to give yet another argument based on the main result in the paper [Kn93] of F. Knop. Our first proof (see Theorem IV.2) arises as an application of our description of the global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$ of a projectivized rank two toric vector bundle obtained in Theorem III.10. Here, we prove that an appropriate Veronese subalgebra of the Cox ring of $\mathbf{P}(\mathcal{E})$ is isomorphic to the semigroup algebra obtained from the semigroup of lattice points with sufficiently divisible coordinates in the global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$, which is finitely generated using Theorem III.10. For our second proof (see Theorem IV.6), we consider a finer grading on $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$ induced by the torus action. Next, we describe the graded pieces and the multiplication map in terms of the data appearing in the Klyachko filtrations of $\mathcal{E}$, and we obtain the finite generation by exhibiting a finite generator set of a Veronese subalgebra of $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$.

In the final chapter, I present the results of my joint work [GHPS10], studying the Cox rings and pseudoeffective cones of projectivizations of toric vector bundles of higher rank, with M. Hering, S. Payne and H. Süß. Here we consider the projectivizations of a special class of toric vector bundles that includes cotangent bundles, up to a twist. The definition of these special toric vector bundles is meant to guarantee the existence of a T-invariant open subset of their projectivizations having a geometric quotient by the torus action that is isomorphic to a blow up of projective space at finitely many points of our choice. In this case, results of Hausen and Süß allow us to present the Cox rings of the projectivizations of this class of toric vector bundles as polynomial rings over the Cox rings of those blow ups. Using results of Mukai, A.-M. Castravet and J. Tevelev, and B. Totaro concerning blow ups of projective spaces at finite collections of points, we obtain many new examples of Mori dream spaces as well as many examples of projectivized toric vector bundles whose Cox rings and pseudoeffective cones are not finitely generated. In particular, we will see that for each $d \geq 3$ there exist d-dimensional smooth projective toric varieties whose projectivized cotangent bundles are not Mori dream spaces.

CHAPTER II

Preliminaries

In this chapter, we review the construction of Okounkov bodies and some basic facts about Cox rings and Mori dream spaces. We also review Klyachko's classification of toric vector bundles, and we describe the Klyachko filtrations associated to tensor products and Schur functors applied to toric vector bundles. Unless explicitly stated otherwise, the notation introduced here will be used throughout this dissertation. All our varieties are assumed to be defined over a fixed algebraically closed field k. By a *divisor* on a variety Z we always mean a Cartier divisor on Z. We denote the group of numerical equivalence classes of divisors on Z by $N^1(Z)$, and we denote the spaces $N^1(Z) \otimes \mathbf{Q}$ and $N^1(Z) \otimes \mathbf{R}$ by $N^1(Z)_{\mathbf{Q}}$ and $N^1(Z)_{\mathbf{R}}$, respectively.

By a *line bundle* on a variety Z, we mean a locally free sheaf of rank one on Z. We follow the convention that the *geometric vector bundle* associated to the locally free sheaf $\mathcal{F}$ is the variety $\mathbf{V}(\mathcal{F}) = \mathbf{Spec} \bigoplus_{m \geq 0} Sym^m \mathcal{F}^{\vee}$, whose sheaf of sections is $\mathcal{F}$. Also, by the fiber of $\mathcal{F}$ over a point $z \in Z$, we mean the fiber over z of the projection $f\colon \mathbf{V}(\mathcal{F}) \to Z$. Lastly, by the *projectivization* $\mathbf{P}(\mathcal{F})$ of $\mathcal{F}$, we mean the projective bundle $\mathbf{Proj} \bigoplus_{m \geq 0} Sym^m \mathcal{F}$ over Z. This bundle is endowed with a projection $\pi\colon \mathbf{P}(\mathcal{F}) \to Z$ and an invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ (see II.7 in [Har77]).

2.1 Okounkov bodies

Let us consider a normal l-dimensional variety Z with a fixed flag $Y_\bullet\colon Z = Y_l \supseteq \cdots \supseteq Y_0$, where each Y_i is an i-dimensional normal subvariety that is nonsingular at the point Y_0. Given a big divisor D on Z, we will describe a procedure to assign a compact convex set with nonempty interior $\Delta_{Y_\bullet}(D)$ in $\mathbf{R}^l$ to D. First, given any divisor F on Z and a nonzero section $s = s_l \in H^0(Y_l, \mathcal{O}_{Y_l}(F))$, we can associate an l-tuple of nonnegative integers $\nu_{Y_\bullet,F}(s) = (\nu_1(s), \ldots, \nu_l(s))$ to s as follows. By restricting to a neighborhood of Y_0 we can assume that each Y_i is smooth. We define $\nu_1(s)$ to be the vanishing order $\mathrm{ord}_{Y_{l-1}}(s)$ of s along Y_{l-1}. Then, s determines a section $\widetilde{s_l} \in H^0(Y_l, \mathcal{O}_{Y_l}(F) \otimes \mathcal{O}_{Y_l}(-\nu_1(s)Y_{l-1}))$ that does not vanish along Y_{l-1}. By restricting, we get a nonzero section $s_{l-1} \in H^0(Y_{l-1}, \mathcal{O}_{Y_l}(F)|_{Y_{l-1}} \otimes \mathcal{O}_{Y_l}(-\nu_1(s)Y_{l-1})|_{Y_{l-1}})$, and we iterate this procedure. More precisely, assume that we have defined nonnegative integers $\nu_1(s), \ldots, \nu_h(s)$, and nonzero sections $s_l \in H^0(Y_l, \mathcal{O}_{Y_l}(F)), \ldots, s_{l-h} \in H^0(Y_{l-h}, \mathcal{O}_{Y_l}(F)|_{Y_{l-h}} \otimes \bigotimes_{i=1}^{h} \mathcal{O}_{Y_{l-i+1}}(-\nu_i(s)Y_{l-i})|_{Y_{l-h}})$, for some nonnegative integer $h < l$. We define $\nu_{h+1}(s)$ as the vanishing order $\mathrm{ord}_{Y_{l-h-1}}(s_{l-h})$ of s_{l-h} along Y_{l-h-1}; then, s_{l-h} determines a section

$$\widetilde{s_{l-h}} \in H^0(Y_{l-h}, \mathcal{O}_{Y_l}(F)|_{Y_{l-h}} \otimes \bigotimes_{i=1}^{h+1} \mathcal{O}_{Y_{l-i+1}}(-\nu_i(s)Y_{l-i})|_{Y_{l-h}})$$

that does not vanish along Y_{l-h-1}; and finally, by restricting, we get a nonzero section

$$s_{l-h-1} \in H^0(Y_{l-h-1}, \mathcal{O}_{Y_l}(F)|_{Y_{l-h-1}} \otimes \bigotimes_{i=1}^{h+1} \mathcal{O}_{Y_{l-i+1}}(-\nu_i(s)Y_{l-i})|_{Y_{l-h-1}}).$$

We repeat this procedure until we obtain nonnegative integers $\nu_1(s), \ldots, \nu_l(s)$. This construction gives us a function

$$\begin{array}{rccc} \nu_{Y_\bullet} = \nu_{Y_\bullet,F}\colon & H^0(Z, \mathcal{O}_Z(F)) \smallsetminus \{0\} & \longrightarrow & \mathbf{Z}^l \\ & s & \longmapsto & (\nu_1(s), \ldots, \nu_l(s)). \end{array}$$

We denote the image of $\nu_{Y_\bullet,F}$ by either $\nu(F)$ or $\nu(\mathcal{O}_Z(F))$. The function $\nu_{Y_\bullet}$ satisfies the following valuation-like properties:

- For any nonzero sections $s_1, s_2 \in H^0(Z, \mathcal{O}_Z(F))$, we have that $\nu_{Y_\bullet,F}(s_1+s_2) \geq_{lex} \min_{\geq_{lex}}\{\nu_{Y_\bullet,F}(s_1), \nu_{Y_\bullet,F}(s_2)\}$, where $\geq_{lex}$ denotes the lexicographic order in $\mathbf{Z}^l$.
- For any divisors F_1 and F_2 in Z, and nonzero sections $s_1 \in H^0(Z, \mathcal{O}_Z(F_1))$ and $s_2 \in H^0(Z, \mathcal{O}_Z(F_2))$, we have $\nu_{Y_\bullet,F_1+F_2}(s_1 \otimes s_2) = \nu_{Y_\bullet,F_1}(s_1) + \nu_{Y_\bullet,F_2}(s_2)$.

Remark II.1. If W is a finite dimensional subspace of $H^0(Z, \mathcal{O}_Z(F))$, then the number of vectors arising as images under $\nu_{Y_\bullet}$ of nonzero sections in W is equal to the dimension of W. For example, when Z is complete, $\nu(F)$ is a finite set with cardinality $\dim_k H^0(Z, \mathcal{O}_Z(F))$. A more general statement is proven by Lazarsfeld and Mustaţă as Lemma 1.3 in [LM08].

Finally, $\Delta_{Y_\bullet}(F) = \Delta_{Y_\bullet}(\mathcal{O}_Z(F))$ is defined to be the following closed convex hull in $\mathbf{R}^l$:

$$\Delta_{Y_\bullet}(F) = \overline{\mathrm{Conv}\Big(\bigcup_{m\in\mathbf{Z}^+} \frac{1}{m}\nu(mF)\Big)}.$$

We will denote the set $\Delta_{Y_\bullet}(F)$ simply by either $\Delta(F)$ or $\Delta(\mathcal{O}_Z(F))$ whenever the corresponding flag is understood. In [LM08], Lazarsfeld and Mustaţă proved that when Z is a projective variety and D is a big divisor, $\Delta(D)$ is a compact convex subset of $\mathbf{R}^l$ with nonempty interior, i.e. a *convex body*. In this case, $\Delta(D)$ is called the *Okounkov body* of D.

Since $\Delta_{Y_\bullet}(mF) = m\Delta_{Y_\bullet}(F)$ for any divisor F and any $m \in \mathbf{Z}^+$, this definition extends in a natural way to associate an Okounkov body to any big divisor with rational coefficients. As it turns out, the outcome depends only on the numerical equivalence class of the divisor. We refer to **(a)** and **(b)** in the Introduction for some of the main properties of this construction, including the existence of the *global*

Okounkov body of a projective variety Z. This global Okounkov body is a closed convex cone $\Delta(Z) \subseteq \mathbf{R}^l \times N^1(Z)_{\mathbf{R}}$ characterized by the property that the fiber of the second projection over any big class $D \in N^1(Z)_{\mathbf{Q}}$ is the Okounkov body $\Delta(D)$. For proofs of these assertions, as well as of **(a)** and **(b)**, we refer to [LM08].

Example II.2. Let $Z = \mathbf{P}^l$ with homogeneous coordinates $z_0, \ldots, z_l$. Let $Y_\bullet$ be the flag of linear subspaces defined by $Y_i = \{z_1 = \cdots = z_{l-i} = 0\}$ for each i. If $|D|$ is the linear system of hypersurfaces of degree m, then $\nu_{Y_\bullet, D}$ is the lexicographic valuation determined on monomials of degree m by

$$\nu_{Y_\bullet}(z_0^{\alpha_0} \cdots z_l^{\alpha_l}) = (\alpha_1, \ldots, \alpha_l),$$

and the Okounkov body $\Delta(D)$ is the simplex

$$\Delta(D) = \{(\lambda_1, \ldots, \lambda_l) \in \mathbf{R}^l \mid \lambda_1 \geq 0, \ldots, \lambda_l \geq 0, \sum_{i=1}^{l} \lambda_i \leq m\}.$$

2.2 Toric vector bundles and Klyachko filtrations

Let X be an n-dimensional toric variety corresponding to a fan Δ in the lattice N. We denote the algebraic torus acting on X by T, and the character lattice $\operatorname{Hom}(N, \mathbf{Z})$ of T by M. Thus, $T = \operatorname{Spec} k[M] = \operatorname{Spec} k[\,\chi^u \mid u \in M\,]$ and X has an open covering given by the affine toric varieties $U_\sigma = \operatorname{Spec} k[\,\sigma^\vee \cap M\,]$ corresponding to the cones $\sigma \in \Delta$, where for each such cone $\sigma^\vee$. We denote the rays in Δ by $\rho_1, \ldots, \rho_d$. For each ray ρ_j, we denote its primitive lattice generator by v_j and its associated codimension one torus invariant subvariety by D_j. Let t_0 denote the unit element of the torus. For a detailed treatment of toric varieties we refer to [Ful93].

If T acts on a vector space V in such a way that each element of V belongs to a finite dimensional T-invariant subspace, we get a decomposition $V = \bigoplus_{u \in M} V_u$, where $V_u =_{def} \{v \in V \mid tv = \chi^u(t)v \text{ for each } t \in T\}$. The spaces V_u and their elements are called isotypical summands and isotypical elements, respectively. This motivates the use of the following terminology. If T acts on the space of sections of a vector bundle on some variety, we say that a section s is *T-isotypical* if there exists $u \in M$ such that $ts = \chi^u(t)s$ for each $t \in T$. Likewise, if T acts on a variety, we say that a rational function f on the variety is *T-isotypical* if there exists $u \in M$ such that $tf = \chi^u(t)f$ for each $t \in T$, i.e. the domain $\operatorname{dom}(f)$ of f is T-invariant and $(tf)(z) =_{def} f(t^{-1}z) = \chi^u(t)f(z)$ for each $z \in \operatorname{dom}(f)$ and each $t \in T$. When T acts algebraically on an affine variety Z, the induced action of T on $H^0(Z, \mathcal{O}_Z)$ satisfies the above finiteness condition, and we get a decomposition $H^0(Z, \mathcal{O}_Z) = \bigoplus_{u \in M} H^0(Z, \mathcal{O}_Z)_u$ as before (see I.6.3 in [LP97]).

A *toric vector bundle* on the toric variety X is a locally free sheaf $\mathcal{E}$ together with an action of the torus T on the variety $\mathbf{V}(\mathcal{E})$, such that the projection $f\colon \mathbf{V}(\mathcal{E}) \to X$ is equivariant and T acts linearly on the fibers of f. In general, if $\mathcal{E}$ is a toric vector bundle, $\mathbf{V}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E})$ need not be toric varieties. Given any T-invariant open subset U of X, there is an induced action of T on $H^0(U, \mathcal{E})$, defined by the equation

$$(t \cdot s)(x) =_{\text{def}} t(s(t^{-1}x)),$$

for any $s \in H^0(U, \mathcal{E})$, $t \in T$ and $x \in X$. This action induces a direct sum decomposition

$$H^0(U, \mathcal{E}) = \bigoplus_{u \in M} H^0(U, \mathcal{E})_u,$$

where $H^0(U, \mathcal{E})_u = \{s \in H^0(U, \mathcal{E}) \mid t \cdot s = \chi^u(t)s \text{ for each } t \in T\}$, as before.

Example II.3. For each torus invariant Cartier divisor D on a toric variety X, the line bundle $\mathcal{O}_X(D)$ has a natural toric vector bundle structure. For each $w \in M$, the isotypical summands in the decomposition of $H^0(U_\sigma, \mathcal{O}_X(\operatorname{div} \chi^w))$ over U_σ are given by

$$H^0(U_\sigma, \mathcal{O}_X(\operatorname{div}\chi^w))_u = \begin{cases} k\chi^{-u} & \text{if } w-u \in \sigma^\vee \cap M, \\ 0 & \text{otherwise,} \end{cases}$$

for each cone $\sigma \in \Delta$ and each $u \in M$.

A toric vector bundle over an affine toric variety is equivariantly isomorphic to a direct sum of toric line bundles (see Proposition 2.2 in [Pay08]). Every line bundle $\mathcal{L}$ on X admits a T-equivariant structure, and choosing one such structure is equivalent to choosing a torus invariant divisor D such that $\mathcal{L} \cong \mathcal{O}_X(D)$. The classification of toric vector bundles of higher rank is considerably more complicated.

Let E be the fiber over t_0 of the toric vector bundle $\mathcal{E}$ on X, and for any T-invariant open subset U, let $ev_{t_0}\colon H^0(U,\mathcal{E}) \to E$ be the evaluation map at t_0. For each ray $\rho_j \in \Delta$ and each $u \in M$, the evaluation map ev_{t_0} gives an inclusion $H^0(U_{\rho_j}, \mathcal{E})_u \hookrightarrow E$. If $u, u' \in M$ satisfy $\langle u, v_j\rangle \geq \langle u', v_j\rangle$, then

$$\operatorname{Im}(H^0(U_{\rho_j}, \mathcal{E})_u \hookrightarrow E) \subseteq \operatorname{Im}(H^0(U_{\rho_j}, \mathcal{E})_{u'} \hookrightarrow E).$$

Therefore the images of these maps depend only on $\langle u, v_j\rangle$, or equivalently, only on the class of u in $M/\rho_j^\perp \cap M \cong \mathbf{Z}$. Hence, we may denote the image of the map $H^0(U_{\rho_j}, \mathcal{E})_u \hookrightarrow E$ simply by $\mathcal{E}^{\rho_j}(\langle u, v_j\rangle)$. Note that for each $u \in M$ the image of the evaluation map $H^0(X, \mathcal{E})_u \hookrightarrow E$ is equal to $\mathcal{E}^{\rho_1}(\langle u, v_1\rangle) \cap \cdots \cap \mathcal{E}^{\rho_d}(\langle u, v_d\rangle) \subseteq E$. The ordered collection of finite dimensional vector subspaces $\mathcal{E}^{\rho_j} =_{def} \{\mathcal{E}^{\rho_j}(i) \mid i \in \mathbf{Z}\}$ gives a decreasing filtration of E. The filtrations $\{\mathcal{E}^{\rho_j} \mid j = 1, \ldots, d\}$ are called the *Klyachko filtrations* associated to $\mathcal{E}$. For each $\sigma \in \Delta$, by equivariantly trivializing $\mathcal{E}$ over the affine open subset U_σ of X, one can show that there exists a decomposition $E = \bigoplus_{\bar{u} \in M/\sigma^\perp \cap M} E_{\bar{u}}$, such that $\mathcal{E}^\rho(i) = \sum_{\langle \bar{u}, v_\rho\rangle \geq i} E_{\bar{u}}$, for each ray $\rho \subseteq \sigma$ and each $i \in \mathbf{Z}$. Klyachko proved in [Kly90] that the vector space E together with these filtrations, satisfying the above compatibility condition, completely describe $\mathcal{E}$. More precisely,

Theorem II.4 (Klyachko)**.** *The category of toric vector bundles on the toric variety X is equivalent to the category of finite dimensional k-vector spaces E with collections of decreasing filtrations $\{\mathcal{E}^\rho(i) \mid i \in \mathbf{Z}\}$, indexed by the rays of Δ, satisfying the following compatibility condition: For each cone $\sigma \in \Delta$, there is a decomposition $E = \bigoplus_{\bar{u} \in M/\sigma^\perp \cap M} E_{\bar{u}}$ such that*

$$\mathcal{E}^\rho(i) = \sum_{\langle \bar{u}, v_\rho \rangle \geq i} E_{\bar{u}},$$

for every ray $\rho \subseteq \sigma$ and every $i \in \mathbf{Z}$.

Example II.5. Let $m_1 D_1 + \cdots + m_d D_d$ be a torus invariant Cartier divisor on X, for some $m_1, \ldots, m_d \in \mathbf{Z}$. Denote by $\mathcal{L}$ the line bundle $\mathcal{O}_X(m_1 D_1 + \cdots + m_d D_d)$ on X. The Klyachko filtrations for $\mathcal{L}$ are given by

$$\mathcal{L}^{\rho_j}(i) = \begin{cases} k & \text{if } i \leq m_j, \\ 0 & \text{if } i > m_j, \end{cases}$$

for each ray $\rho_j \in \Delta$.

Example II.6. The projective plane $\mathbf{P}^2$ can be represented as the toric variety associated to the fan in $N \otimes \mathbf{R} = \mathbf{R}^2$ with maximal cones $\sigma_1 = \langle v_2, v_3 \rangle$, $\sigma_2 = \langle v_3, v_1 \rangle$ and $\sigma_3 = \langle v_1, v_2 \rangle$, where $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (-1, -1)$. The tangent bundle $T_{\mathbf{P}^2}$ of $\mathbf{P}^2$ is naturally a toric vector bundle on $\mathbf{P}^2$. This bundle can be equivariantly trivialized as

$$T_{\mathbf{P}^2}|_{U_{\sigma_1}} = \mathcal{O}_{\mathbf{P}^2}(D_2)|_{U_{\sigma_1}} \oplus \mathcal{O}_{\mathbf{P}^2}(D_3)|_{U_{\sigma_1}}, \qquad T_{\mathbf{P}^2}|_{U_{\sigma_2}} = \mathcal{O}_{\mathbf{P}^2}(D_3)|_{U_{\sigma_2}} \oplus \mathcal{O}_{\mathbf{P}^2}(D_1)|_{U_{\sigma_2}},$$

$$T_{\mathbf{P}^2}|_{U_{\sigma_3}} = \mathcal{O}_{\mathbf{P}^2}(D_1)|_{U_{\sigma_3}} \oplus \mathcal{O}_{\mathbf{P}^2}(D_2)|_{U_{\sigma_3}}.$$

It follows that the Klyachko filtrations associated to $T_{\mathbf{P}^2}$ are

$$T^{\rho_1}_{\mathbf{P}^2}(i) = \begin{cases} E & \text{if } i \leq 0, \\ V_1 & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases} \quad T^{\rho_2}_{\mathbf{P}^2}(i) = \begin{cases} E & \text{if } i \leq 0, \\ V_2 & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases} \quad T^{\rho_3}_{\mathbf{P}^2}(i) = \begin{cases} E & \text{if } i \leq 0, \\ V_3 & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

where V_1, V_2 and V_3 are distinct one-dimensional subspaces of the fiber E of $T_{\mathbf{P}^2}$ over t_0.

For any toric vector bundle $\mathcal{E}$ over X of rank at least two, we have an isomorphism $N^1(X) \oplus \mathbf{Z} = \operatorname{Pic} X \oplus \mathbf{Z} \cong \operatorname{Pic} \mathbf{P}(\mathcal{E}) = N^1(\mathbf{P}(\mathcal{E}))$, which is induced by $(\mathcal{L}, m) \mapsto \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{L}$, where π is the projection map $\pi \colon \mathbf{P}(\mathcal{E}) \to X$.

2.3 The Klyachko filtrations for tensor products and Schur functors

As we reviewed in §II.4, Klyachko proved that the category of toric vector bundles on a toric variety X is equivalent to the category of finite dimensional vector spaces endowed with a collection of filtrations that satisfy a certain compatibility condition. Klyachko's result allows us to carry out some explicit computations in this category, including the description of the space of sections of a toric vector bundle over any invariant open subset of X. Throughout this section X denotes an arbitrary toric variety.

Using the notation introduced in Sections 2.1 and 2.2, each line bundle on $\mathbf{P}(\mathcal{E})$ is isomorphic to a line bundle of the form $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*(\mathcal{O}_X(D))$ for some T-invariant Cartier divisor D on X. For such an isomorphic representative we have a toric vector bundle structure on $\pi_*\big(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{O}_X(D)\big) = (Sym^m\mathcal{E}) \otimes \mathcal{O}_X(D)$, and we have

$$H^0\left(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*(\mathcal{O}_X(D))\right) = H^0\left(X, (Sym^m\mathcal{E}) \otimes \mathcal{O}_X(D)\right).$$

From this, we see that the Klyachko filtrations associated to tensor products and symmetric powers of toric vector bundles can be used to describe the spaces of global sections of line bundles on projectivized toric vector bundles. The goal of this section is to provide appropriate descriptions of these filtrations for toric vector bundles of arbitrary rank. We present the filtrations for tensor products in Lemma II.7 and its Corollary II.8. The filtrations for symmetric powers are given in Corollary II.11 to Lemma II.10. More generally, in that lemma we describe the filtrations for any Schur functor, e.g. symmetric and wedge products.

We start by presenting the filtrations for tensor products.

Lemma II.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be toric vector bundles on the toric variety X. Then the Klyachko filtrations for their tensor product $\mathcal{E} \otimes \mathcal{F}$ are given by*

$$\big(\mathcal{E} \otimes \mathcal{F}\big)^{\rho}(i) = \sum_{i_1+i_2=i} \mathcal{E}^{\rho}(i_1) \otimes \mathcal{F}^{\rho}(i_2),$$

for each ray $\rho \in \Delta$ and each $i \in \mathbf{Z}$.

Proof. Since the filtration corresponding to the ray ρ only depends on U_ρ, it suffices to consider the case when $X = U_\rho$ for some ray $\rho \in \Delta$. Hence we can assume that $\mathcal{E}$ and $\mathcal{F}$ equivariantly trivialize as

$$\mathcal{E} = \mathcal{O}_X(d_1 D_\rho) \oplus \mathcal{O}_X(d_2 D_\rho) \oplus \cdots \oplus \mathcal{O}_X(d_r D_\rho)$$

$$\mathcal{F} = \mathcal{O}_X(e_1 D_\rho) \oplus \mathcal{O}_X(e_2 D_\rho) \oplus \cdots \oplus \mathcal{O}_X(e_s D_\rho)$$

for some $d_1, \ldots, d_r, e_1, \ldots, e_s \in \mathbf{Z}$. Now we note that

$$\big(\mathcal{O}_X(d_{j_1} D_\rho) \otimes \mathcal{O}_X(e_{j_2} D_\rho)\big)^{\rho}(i) = \sum_{i_1+i_2=i} \mathcal{O}_X(d_{j_1} D_\rho)^{\rho}(i_1) \otimes \mathcal{O}_X(e_{j_2} D_\rho)^{\rho}(i_2)$$

for each $i \in \mathbf{Z}$ and each $j_1 \in \{1, \ldots, r\}$ and $j_2 \in \{1, \ldots, s\}$. Since the Klyachko filtrations for a direct sum are the direct sums of the filtrations for the summands, the result follows. $\square$

Corollary II.8. *Let $\mathcal{E}_1, \dots, \mathcal{E}_s$ be toric vector bundles on the toric variety X. Then the Klyachko filtrations for their tensor product $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_s$ are given by*

$$\big(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_s\big)^\rho(i) = \sum_{i_1+\cdots+i_s=i} \mathcal{E}_1^\rho(i_1) \otimes \cdots \otimes \mathcal{E}_s^\rho(i_s),$$

for each $i \in \mathbf{Z}$ and each ray $\rho \in \Delta$.

Proof. The conclusion follows from the previous lemma by induction on s. □

Example II.9. Let $\mathcal{E}$ be a toric vector bundle on the toric variety X, and let $D = m_1D_1+\cdots+m_dD_d$ be a torus invariant Cartier divisor on X, for some $m_1, \dots, m_d \in \mathbf{Z}$. Let us denote the fiber over t_0 of the line bundle $\mathcal{O}_X(D)$ by G. From the previous lemma and Example II.5, it follows that the Klyachko filtrations of $\mathcal{E} \otimes \mathcal{O}_X(D)$ are given by

$$(\mathcal{E} \otimes \mathcal{O}_X(D))^{\rho_j}(i) = \mathcal{E}^{\rho_j}(i - m_j) \otimes G,$$

for each $i \in \mathbf{Z}$ and each ray $\rho_j \in \Delta$.

In the following lemma we describe the Klyachko filtrations for the toric vector bundle obtained by applying a Schur functor to another toric vector bundle. As a corollary, we state the case of symmetric products, which will be used in §3.2. For the definition and basic properties of Schur functors we refer to §6 in [FH91].

Lemma II.10. *Let $\mathcal{E}$ be a toric vector bundle on the toric variety X, and let S_λ be the Schur functor associated to a Young tableau λ with m entries. Then the Klyachko filtrations for $S_\lambda\mathcal{E}$ are given by*

$$\big(S_\lambda\mathcal{E}\big)^\rho(i) = \sum_{i_1+\cdots+i_m=i} \operatorname{Im}\big(\mathcal{E}^\rho(i_1) \otimes \cdots \otimes \mathcal{E}^\rho(i_m) \longrightarrow S_\lambda(E)\big),$$

for each ray $\rho \in \Delta$ and each $i \in \mathbf{Z}$.

Proof. Since $S_\lambda\mathcal{E}$ is a quotient of $\mathcal{E}^{\otimes m}$, it follows that $(S_\lambda\mathcal{E})^\rho(i)$ is the image of $(\mathcal{E}^{\otimes m})^\rho(i)$ under the natural map $E^{\otimes m} \to S_\lambda(E)$, for each ray ρ and each $i \in \mathbf{Z}$. Now, the result follows at once from Corollary II.8. □

Corollary II.11. *Let $\mathcal{E}$ be a toric vector bundle on the toric variety X. Then for each $m \in \mathbf{Z}^+$, the Klyachko filtrations for $Sym^m\mathcal{E}$ are given by*

$$\big(Sym^m\mathcal{E}\big)^\rho(i) = \sum_{i_1+\cdots+i_m=i} \mathrm{Im}\,\big(\mathcal{E}^\rho(i_1)\otimes\cdots\otimes\mathcal{E}^\rho(i_m) \longrightarrow Sym^m E\big),$$

for each ray $\rho \in \Delta$ and each $i \in \mathbf{Z}$.

Proof. This is a particular case of the previous lemma. □

Example II.12. Let $D = m_1D_1 + \cdots + m_dD_d$ be a torus invariant Cartier divisor on the toric variety X, and let us denote the fiber over t_0 of the line bundle $\mathcal{O}_X(D)$ by G. The Klyachko filtration associated to a rank two toric vector bundle $\mathcal{E}$ on X corresponding to a ray ρ_j has one of the following two forms:

$$\mathcal{E}^{\rho_j}(i) = \begin{cases} E & \text{if } i \le a_j, \\ 0 & \text{if } i > a_j, \end{cases} \qquad \mathcal{E}^{\rho_j}(i) = \begin{cases} E & \text{if } i \le a_j, \\ V & \text{if } a_j < i \le b_j, \\ 0 & \text{if } i > b_j, \end{cases}$$

where V is some one-dimensional subspace of the fiber E of $\mathcal{E}$ over t_0, and a_j and b_j are some integers. For each positive integer m the corresponding Klyachko filtration associated to $(Sym^m\mathcal{E})\otimes\mathcal{O}_X(D)$ has respectively one of the forms:

$$((Sym^m\mathcal{E})\otimes\mathcal{O}_X(D))^{\rho_j}(i) = \begin{cases} (Sym^m E)\otimes G & \text{if } i \le a_jm + m_j, \\ 0 & \text{if } i > a_jm + m_j, \end{cases}$$

$$((Sym^m\mathcal{E})\otimes\mathcal{O}_X(D))^{\rho_j}(i)=\begin{cases}(Sym^m E)\otimes G & \text{if } i\le a_jm+m_j,\\ Sym^m_E\big(V^{\left\lceil\frac{i-a_jm-m_j}{b_j-a_j}\right\rceil}\big)\otimes G & \text{if } a_jm{+}m_j{<}i{\le}b_jm{+}m_j,\\ 0 & \text{if } i>b_jm+m_j,\end{cases}$$

where $\lceil\ \rceil$ denotes the ceiling function, and $Sym^m_E(V^c)=_{def}\mathrm{Im}(V^{\otimes c}\otimes E^{\otimes(m-c)}\to Sym^mE)$ for each integer $0\le c\le m$. This convenient notation will be generalized in the next chapter (see Notation III.2).

2.4 Cox rings

2.4.1 An invitation to Mori dream spaces

Mori dream spaces were defined by Hu and Keel in the paper [HK00]. Their essential feature is the finite generation of their total coordinate ring or Cox ring (see Definition II.13), and some of their most remarkable properties come from an interpretation of their Mori theory in terms of variation of geometric invariant theory (GIT). In a Mori dream space the cones of pseudoeffective, movable and nef divisors are rational polyhedral. Moreover, the pseudoeffective cone can be decomposed into finitely many rational polyhedral chambers called *Mori chambers*, one of them being the nef cone, and such that the movable cone is a union of some of these Mori chambers. The different equivalent interpretations of these Mori chambers constitutes a particularly beautiful picture in algebraic geometry.

Given a line bundle $\mathcal{L}$ on a variety X, for each $n\in\mathbf{Z}$ there is a rational map $\phi_n\colon X\dashrightarrow\mathbf{P}(H^0(X,\mathcal{L}^n))$ given by the linear system $|\mathcal{L}^n|$. For sufficiently divisible n, these maps stabilize to a rational map $\phi_{\mathcal{L}}$ called Iitaka fibration of X (see §2.1 in [Laz04]). Two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined to be *Mori equivalent* if they give rise to equivalent Iitaka fibrations $\phi_{\mathcal{L}_1}$ and $\phi_{\mathcal{L}_2}$. The relation of Mori equivalence can be extended in a natural way to $\mathrm{Pic}(X)_{\mathbf{Q}}$. On a Mori dream space the Mori chambers are precisely the closures of the open equivalence classes under this relation.

Mori chambers can also be interpreted in terms of variation of GIT. Let G be a reductive algebraic group acting on a normal variety X. Two ample G-linearized line bundles are defined to be *GIT-equivalent* if they have the same set of semistable points on X. Recall that in GIT one associates to an ample G-linearized line bundle on X, a good categorical quotient of the action of G on the G-invariant open subset of $\mathcal{L}$-semistable points of X, and moreover one gets a geometric quotient over the G-invariant open subset of $\mathcal{L}$-stable points of X. The relation of GIT-equivalence is the same as the relation of producing the same GIT quotient. The set of ample G-linearized line bundles on X forms a cone in a finite dimensional vector space. If X is a Mori dream space, after choosing line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_m$ that form a basis of the Neron-Severi space $\mathrm{N}^1(X)$, one gets an action of the torus T with character lattice $M = \mathbf{Z} \cdot \mathcal{L}_1 \oplus \cdots \oplus \mathbf{Z} \cdot \mathcal{L}_m \subseteq \mathrm{N}^1(X)$ on the Cox ring $\mathrm{Cox}(X)$ of X and on the variety $\mathrm{Spec}(\mathrm{Cox}(X))$. In the case of Mori dream spaces the T-ample cone for this action and the space containing it, can be identified with the cone of pseudoeffective divisors and the space $\mathrm{N}^1(X)$. The Mori chambers are precisely the closures of the open GIT-equivalence classes for this action.

The Mori chambers on a Mori dream space X are in correspondence with the birational contractions of X having $\mathbf{Q}$-factorial image, and those inside the movable cone correspond to those contractions that are isomorphic to X in codimension 1, i.e. *small modifications* of X. If $X_1, \ldots, X_l$ are these distinct small modifications of X, their Neron-Severi spaces can be identified in a natural way, and their nef cones are exactly the Mori chambers contained in the movable cone of X. Additionally, with the notation of the previous paragraph, the GIT quotient of $\mathrm{Spec}\,\mathrm{Cox}(X)$ by the action of T, given by the trivial line bundle on $\mathrm{Spec}(\mathrm{Cox}(X))$ linearized by a character of T in the interior of the cone $\mathrm{Nef}(X_i)$ recovers the variety X_i for each $1 \leq i \leq l$.

This deep understanding of the birational geometry of X permits to run the minimal model program for any movable divisor on X. This is, one can find a birational model of X where the divisor becomes nef, and moreover semiample. In this setting the flips and divisorial contractions required, correspond to moving from a cone to an adjacent cone in the fan decomposition of the movable cone induced by the Mori chambers. Similarly, there is a version of the minimal model program that can be run for effective divisors with an analogous interpretation in terms of convex geometry.

2.4.2 Definitions and basic results

Let us state the definition of the algebras that we will call Cox rings of varieties in this dissertation, and let us briefly compare it with the definition introduced by Hu and Keel in [HK00].

Definition II.13. Let X be a variety such that $\mathrm{Pic}(X)_{\mathbf{Q}}$ is finite dimensional. Any k-algebra of the form

$$\mathrm{Cox}\big(X, (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s)\big) =_{def} \bigoplus_{(m_1, m_2, \ldots, m_s) \in \mathbf{Z}^s} H^0(X, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_s^{\otimes m_s})$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ are line bundles on X whose classes span $\mathrm{Pic}(X)_{\mathbf{Q}}$, will be called a *Cox ring* of X.

Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ be line bundles on X whose classes span $\mathrm{Pic}(X)_{\mathbf{Q}}$, and let $\mathcal{L}$ be a line bundle on X. If we fix an isomorphism between $\mathcal{L}^{\otimes a}$ and a line bundle of the form $\mathcal{L}_1^{\otimes a_1} \otimes \mathcal{L}_2^{\otimes a_2} \otimes \cdots \otimes \mathcal{L}_s^{\otimes a_s}$, for some $a \in \mathbf{Z}^+$ and some $a_1, \ldots, a_s \in \mathbf{Z}$, then we get induced isomorphisms between $H^0(X, \mathcal{L}_1^{\otimes (m_1 - a_1 m)} \otimes \mathcal{L}_2^{\otimes (m_2 - a_2 m)} \otimes \cdots \otimes \mathcal{L}_s^{\otimes (m_s - a_s m)} \otimes$

$\mathcal{L}^{\otimes am})$ and $H^0(X, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_s^{\otimes m_s})$ for each tuple $(m_1, m_2, \ldots, m_s, m) \in \mathbf{Z}^{s+1}$. These identifications induce an isomorphism of $\mathbf{Z}^{s+1}$-graded algebras between $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s, \mathcal{L}^{\otimes a})\big)$ and the localization $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s)\big)[x]_x$, where x is an indeterminate, and where $H^0(X, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_s^{\otimes m_s})\, x^m$ is the homogeneous component of degree $(m_1 - a_1 m, m_2 - a_2 m, \ldots, m_s - a_s m, m) \in \mathbf{Z}^{s+1}$, for each $(m_1, m_2, \ldots, m_s, m) \in \mathbf{Z}^{s+1}$. Then the finite generation of $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s)\big)$ is equivalent to the finite generation of $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s, \mathcal{L}^{\otimes a})\big)$. Now, a basic result on Veronese subalgebras of graded algebras (see [SYJ10, Lemma 1.4]) gives us that the finite generation of $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s, \mathcal{L}^{\otimes a})\big)$ is equivalent to the finite generation of $\mathrm{Cox}\big(X, (\mathcal{L}_1, \ldots, \mathcal{L}_s, \mathcal{L})\big)$. From this it follows that in the setting of Definition II.13 the finite generation of any Cox ring of X is equivalent to the finite generation of every Cox ring of X. In [HK00], Hu and Keel introduced the Cox ring of any projective variety X that satisfies $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$. Their definition is similar to Definition II.13, but they require the line bundles $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ to satisfy some extra conditions. Namely, the classes of $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ are required to form a basis for $\mathrm{Pic}(X)_{\mathbf{Q}}$ and their affine hull must contain the pseudoeffective cone of X. For projective varieties that satisfy $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$, our previous discussion implies that a particular (equivalently every) Cox ring of X in the sense of Hu and Keel is finitely generated if and only if a particular (equivalently every) Cox ring of X in the sense of Definition II.13 is finitely generated. Since our interest lies in the finite generation of these k-algebras, and this is independent of the one we choose, we follow the convention of calling any Cox ring of a given variety *the Cox ring* of the variety. And since this finite generation does not depend on which of these two definitions of Cox ring we use, we will use Definition II.13 since it applies in the case of the projectivization of a toric vector bundle over an arbitrary toric variety.

As we mentioned in the Introduction, toric varieties and log Fano varieties are examples of varieties with finitely generated Cox rings. In particular, for $n \leq 6$, the moduli space $\overline{M_{0,n}}$ of stable, rational curves with n marked points has a finitely generated Cox ring, since in those cases it is log Fano (see [KM96]). In the case of $\overline{M_{0,6}}$, a proof of the finite generation of the associated Cox ring by exhibiting generators can be found in [Cas08]. Additionally, we remark that whether the Cox ring of $\overline{M_{0,n}}$ is finitely generated for $n \geq 7$ is an important open problem. On the other hand, we can get some examples of varieties with nonfinitely generated Cox rings by using the results of Mukai, Castravet and Tevelev and Totaro that we describe in what follows. These results will be fundamental for our work in Chapter V.

A collection of d points $q_1, \ldots, q_d$ in the projective space $\mathbf{P}^{r-1}$ is in *Cremona general position* if they are in linear general position (i.e. for each $i \leq r$, there are no i of these points contained in a linear subspace of dimension $i-2$), and in addition this condition remains true for the images of these points when we apply a finite sequence of standard Cremona transformations each one centered at a collection of r of the partial images of the collection of d points (see §2 in [Tot08]). We recall that the standard Cremona transformation of $\mathbf{P}^{r-1}$ centered at an ordered collection of r points in linear general position is the rational map given in the coordinates induced by the r points by $[x_1 : \cdots : x_r] \mapsto [\frac{1}{x_1} : \cdots : \frac{1}{x_r}]$, where this expression is well defined. We also recall that this rational map can be extended to an automorphism of the blow up of $\mathbf{P}^{r-1}$ at the corresponding coordinate points. One can easily check that for collections of points in projective space, very general position implies Cremona general position.

One of the main results in the paper [Muk04] of Mukai can be rephrased by saying that the Cox ring of the blow up of $\mathbf{P}^{r-1}$ at d points in Cremona general position is not finitely generated if $d \geq r+2+\frac{4}{r-2}$. Castravet and Tevelev prove a more general version of this result and its converse in [CT06]. We now state the particular case of their result that we will use in Chapter V. Intuitively, it will say that we can think of the quantity $r+2+\frac{4}{r-2}$ as a threshold for the behavior of the blow up of $\mathbf{P}^{r-1}$ at d points in very general position regarding the finite generation of its Cox ring and semigroup of effective divisors.

Theorem II.14 (Castravet-Tevelev)**.** *For $d > r \geq 2$, let X be the blow up of P^{r-1} at d points in very general position. The following statements are equivalent:*

(a) The Cox ring of X is a finitely generated algebra.

(b) The semigroup of effective divisors of X is finitely generated.

(c) $\frac{1}{r}+\frac{1}{d-r} > \frac{1}{2}$.

Proof. This is a particular case of Theorem 1.3 in [CT06]. □

While it is a general fact that (a) implies (b), and the proof that (c) implies (a) only uses that the given points are in linear general position, the proof that (b) implies (c) in [CT06] uses that the points are in very general position. Indeed, this is used to deduce that they are in Cremona general position and then proceed to apply the results of Mukai in [Muk04]. In Chapter V we will construct examples over more general fields, so we want to avoid restrictions on the cardinality and the characteristic of the ground field. In his paper on Hilbert's 14th problem over finite fields [Tot08], Totaro provided configurations of points in various projective spaces that are in Cremona general position. In addition, he constructs configurations of points that in spite of not satisfying that condition, still give rise to a variety with a nonfinitely generated Cox ring when they are blown up. We refer to Totaro's paper for this interesting collection of examples, but we state here a particular one that we will use in Chapter V when constructing examples of smooth projective toric varieties whose projectivized cotangent bundles are not Mori dream spaces.

Theorem II.15. *Let k be a field of characteristic different from 2 and 3. Let $q_1 \ldots, q_9$ be the points in $\mathbf{P}^2$ corresponding to the normal hyperplanes to the columns of the following matrix*

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then the Cox ring of the blow up of $\mathbf{P}^2$ along the points $q_1, \ldots, q_9$ is not finitely generated.

Proof. This is part of the conclusion of Corollary 5.1 to Theorem 5.2 in [Tot08]. □

On the other hand, when the points $q_1, \ldots, q_d$ are not in general position, the variety $\mathrm{Bl}_{\{q_1,\ldots,q_d\}} \mathbf{P}^{r-1}$ can be a Mori dream space even if $\frac{1}{r} + \frac{1}{d-r} \leq \frac{1}{2}$. For instance if $q_1, \ldots, q_d$ lie in a rational normal curve, then $\mathrm{Bl}_{\{q_1,\ldots,q_d\}} \mathbf{P}^{r-1}$ is a Mori dream space [CT06, Theorem 1.2]. Also, if $q_1, \ldots, q_d$ are collinear then $\mathrm{Bl}_{\{q_1,\ldots,q_d\}} \mathbf{P}^{r-1}$ is a rational variety with a torus action with orbits of codimension one , and hence $\mathrm{Bl}_{\{q_1,\ldots,q_d\}} \mathbf{P}^{r-1}$ is a Mori dream space [EKW04], [HS10], [Ott10] (c.f. Lemma II.17).

Our main tool to study the Cox rings of projectivized toric vector bundles of higher ranks will be the following result of Hausen and Süß that gives presentations of the Cox rings for some varieties with torus actions over the Cox rings of appropriate quotients.

Proposition II.16. *Let X be a smooth variety with an action of a torus T. Let us assume that $D_1, \ldots, D_h$ are the prime divisors of X that have positive dimensional* *generic stabilizers. Also, suppose that the action of T on $X \setminus (D_1 \cup \cdots \cup D_h)$ is free with geometric quotient equal to a smooth variety Y, and that the class group of Y is torsion free. Then the Cox ring of X is isomorphic to a polynomial ring in h variables over the Cox ring of Y.*

Proof. This is the special case of [HS10, Theorem 1.1] where X is smooth, the T-action on the complement of $D_1 \cup \cdots \cup D_h$ is free, and the geometric quotient Y is separated, with torsion free class group. $\square$

The following lemma relates the Cox ring of the blow up of projective space at finitely many points lying on a hyperplane and the Cox ring of the blow up of the hyperplane at these points. By applying the lemma repeatedly, we also get an analogous result for blow ups at points lying on any linear subvariety of projective space. This result appears as Lemma 3.5 in [GHPS10], although instances of it can be found in [HT04, Example 1.8]. This lemma will be used in our proof of Theorem V.16.

Lemma II.17. *Let S be a finite set of points contained in a hyperplane H in $\mathbf{P}^n$, and assume $n > 2$. Then the Cox ring of $\mathrm{Bl}_S \mathbf{P}^n$ is isomorphic to a polynomial ring in one variable over the Cox ring of $\mathrm{Bl}_S H$.*

Proof. Choose coordinates on $\mathbf{P}^n$ so that H is a coordinate hyperplane, and let $\mathbf{G}_m = k^*$ act by scaling on the coordinate that cuts out H. The action of $\mathbf{G}_m$ lifts to an action on $\mathrm{Bl}_S \mathbf{P}^n$, and we let Y be the locus of fixed points of this action. Then $\mathbf{G}_m$ acts freely on $\mathrm{Bl}_S \mathbf{P}^d \setminus Y$, with quotient $\mathrm{Bl}_S H$. The strict transform of H is the only divisor contained in Y, so the lemma follows by applying Proposition II.16. $\square$

We also mention the useful fact that if the blow up X' of a variety X at a smooth point x has a finitely generated Cox ring, then X itself has a finitely generated

Cox ring. This follows for instance from [CT06, Proposition 3.1], where the authors prove the following more general fact: If F denotes the exceptional divisor and $x_F \in H^0(X,F) \subseteq \mathrm{Cox}(X)$ denotes the corresponding section then $\mathrm{Cox}(X)_{x_F}$ is isomorphic to $\mathrm{Cox}(X')[x]_x$, where x is an indeterminate.

We conclude this chapter with the following lemma describing the cone of effective divisors of projectivized toric vector bundles which we will use in Chapter V when constructing examples of such varieties with a nonfinitely generated pseudoeffective cone. This result is a version in our setting of the well known fact that an effective divisor on a normal variety with a torus action is linearly equivalent to an effective torus invariant divisor.

Lemma II.18. *Let $\mathcal{E}$ be a toric vector bundle. The effective cone $\mathrm{Eff}(\mathbf{P}(\mathcal{E}))$ of its projectivization $\mathbf{P}(\mathcal{E})$ is given by*

$$\mathrm{Eff}(\mathbf{P}(\mathcal{E})) = \sum_{1\leq i\leq d} \mathbf{Z}_{\geq 0}\cdot \pi^{-1}(D_i) \ + \sum_{\substack{C\subseteq \mathbf{P}(E)\subseteq\mathbf{P}(\mathcal{E})\\ C \text{ prime divisor}}} \mathbf{Z}_{\geq 0}\cdot \overline{T\cdot C}$$

Proof. Given a divisor D on $\mathbf{P}(\mathcal{E})$ such that $H^0(\mathbf{P}(\mathcal{E}),\mathcal{O}(D))\neq 0$, choose any hyperplane $H\subseteq \mathbf{P}(\mathcal{E}|_{t_0})$. Clearly,

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(\,\overline{T\cdot H}\,) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)\otimes \mathcal{O}_X(a_1\pi^{-1}(D_1)+\cdots+a_d\pi^{-1}(D_d)),$$

for some $a_1,\dots,a_d\in\mathbf{Z}$. Then,

$$\begin{aligned}\mathcal{O}_{\mathbf{P}(\mathcal{E})}(D) &\cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(b)\otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(b_1\pi^{-1}(D_1)+\cdots+b_d\pi^{-1}(D_d))\\ &\cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(b\,\overline{T\cdot H}+c_1\pi^{-1}(D_1)+\cdots+c_d\pi^{-1}(D_d))\end{aligned}$$

for some $b\in\mathbf{Z}_{\geq 0}$ and some $b_1,\dots,b_d,c_1,\dots,c_d\in\mathbf{Z}$. The space of global sections V of the T-invariant divisor $\overline{T\cdot H}+c_1\pi^{-1}(D_1)+\cdots+c_d\pi^{-1}(D_d)$ decomposes as a direct sum of T-eigenspaces. Let $\phi\in V\subseteq K(\mathbf{P}(\mathcal{E}))$ be a nonzero section in one of those T-eigenspaces. Hence, D is linearly equivalent to the T-invariant effective divisor

$b\,\overline{T \cdot H} + c_1\pi^{-1}(D_1) + \cdots + c_d\pi^{-1}(D_d) + \operatorname{div}\phi$. Since $\mathbf{P}(\mathcal{E}) \smallsetminus \mathbf{P}(\mathcal{E}|_T) = \bigcup_{1\le i\le d} D_i$, the conclusion follows by noticing that any T-invariant prime divisor on $\mathbf{P}(\mathcal{E}|_T)$ has the form $T \cdot C$ for some prime divisor $C \subseteq \mathbf{P}(E) = \mathbf{P}(\mathcal{E}|_{t_0}) \subseteq \mathbf{P}(\mathcal{E}|_T)$. $\square$

CHAPTER III

Okounkov bodies of projectivized rank two toric vector bundles

In this chapter we give an explicit description of the global Okounkov body of a projectivized rank two toric vector bundle over a smooth projective toric variety, with respect to a T-invariant flag obtained by completing to a full flag the pull back of a T-invariant flag on the base. In particular we prove that this global Okounkov body is a rational polyhedral cone, and we provide inequalities that define it in terms of the data in the Klyachko filtrations of the toric vector bundle and the combinatorial data of the toric variety on the base (see Theorem III.10 and Remark III.11). We conclude the chapter with some examples that illustrate our main result. In Chapter IV, this description of the global Okounkov body will be used to obtain a proof of the finite generation of the Cox rings of projectivized rank two toric vector bundles.

3.1 Vanishing orders on $\mathbf{P}(\mathcal{E})$

The description of the global Okounkov body of a projective variety Z, with respect to a flag $Y_\bullet$, involves identifying the image of the map $\nu_{Y_\bullet}\colon H^0(Z,\mathcal{L})\smallsetminus\{0\} \longrightarrow \mathbf{Z}^{\dim Z}$ for each line bundle $\mathcal{L}$ on Z. In this section we study these images for a suitable flag $Y_\bullet$, when Z is the projectivization $\mathbf{P}(\mathcal{E})$ of a rank two toric vector bundle $\mathcal{E}$ on a smooth projective toric variety X. First, in §3.1.1 we introduce a flag of torus invariant subvarieties $Y_\bullet$ in $\mathbf{P}(\mathcal{E})$, essentially by pulling back a flag of invariant subvarieties from X. Next, in Definition III.3 we present a collection of sections $\mathcal{W}_\mathcal{L}$ for each line bundle $\mathcal{L}$ on $\mathbf{P}(\mathcal{E})$. We consider these sections since we can compute their images under $\nu_{Y_\bullet}$ using the formulas in Lemma III.4, and because they map onto $\nu_{Y_\bullet}(H^0(\mathbf{P}(\mathcal{E}),\mathcal{L}) \smallsetminus \{0\})$, as we prove in Proposition III.8. In passing, we prove that after choosing an isomorphic representative of $\mathcal{L}$ so as to have an induced torus action on $H^0(\mathbf{P}(\mathcal{E}),\mathcal{L})$, the isotypical sections with respect to this action also map onto the image of $\nu_{Y_\bullet}$. Throughout we use the notation introduced in Sections 2.1 and 2.2, and additionally assume that the toric variety X is smooth and projective.

3.1.1 A flag of invariant subvarieties in a projectivized rank two toric vector bundle

Given a toric vector bundle $\mathcal{E}$ of rank two, we construct a flag of smooth T-invariant subvarieties $Y_\bullet \colon \mathbf{P}(\mathcal{E}) = Y_{n+1} \supseteq \ldots \supseteq Y_0$ in $\mathbf{P}(\mathcal{E})$, as follows. Let $X_\bullet \colon X = X_n \supseteq \ldots \supseteq X_0$ be a flag in X, where each X_i is an i-dimensional T-invariant subvariety. By reordering the rays in Δ if necessary, we can assume that $X_{n-i} = \bigcap_{j=1}^{i} D_j$ for each $i \in \{1,\ldots,n\}$. Note that this implies that the rays $\rho_1,\ldots,\rho_n$ span a maximal cone τ in Δ.

Let u_1 and u_2 in M be such that we can equivariantly trivialize $\mathcal{E}$ over U_τ as $\mathcal{E}|_{U_\tau} \cong \mathcal{O}_X(\operatorname{div}\chi^{u_1})|_{U_\tau} \oplus \mathcal{O}_X(\operatorname{div}\chi^{u_2})|_{U_\tau}$. The lexicographic order $\geq_{lex}$ in $\mathbf{Z}^n$ induces an order $\geq_{lex}$ in M, via the isomorphism $M \cong \mathbf{Z}^n$ induced by $v_1,\ldots,v_n$. By reordering u_1 and u_2 if necessary, we can assume that $u_1 \geq_{lex} u_2$. In other words, either $u_1 = u_2$, or the first nonzero number in the list $\langle u_1 - u_2, v_1\rangle,\ldots,\langle u_1 - u_2, v_n\rangle$ is positive.

For each $i \in \{1,\ldots,n+1\}$, we define $Y_i = \mathbf{P}(\mathcal{E}|_{X_{i-1}}) = \pi^{-1}(X_{i-1}) \subseteq \mathbf{P}(\mathcal{E})$. To define Y_0, note that the isomorphism $\mathcal{E}|_{U_\tau} \cong \mathcal{O}_X(\operatorname{div}\chi^{u_1})|_{U_\tau} \oplus \mathcal{O}_X(\operatorname{div}\chi^{u_2})|_{U_\tau}$ induces an isomorphism $Y_1 \cong \mathbf{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0})$. Hence, we get an isomorphism $\mu \colon Y_1 \to$

$\mathbf{P}^1$ between Y_1 and the projective space $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0})$ with homogeneous coordinates x, y. We take Y_0 to be the point in Y_1 corresponding under μ to the point $(0:1)$, defined by the ideal (x) in $\mathbf{P}^1$. Note that the flag $Y_\bullet \colon \mathbf{P}(\mathcal{E}) = Y_{n+1} \supseteq \ldots \supseteq Y_0$ in $\mathbf{P}(\mathcal{E})$ consists of smooth T-invariant subvarieties.

3.1.2 Computing vanishing orders

In this subsection we introduce the collection of T-isotypical sections $\mathcal{W}_{\mathcal{L}}$ and compute their vanishing vectors. We continue working in the setting of §3.1.1.

Notation III.1. We denote by E_1 and E_2 the fibers over t_0 of $\mathcal{O}_X(\operatorname{div}\chi^{u_1})$ and $\mathcal{O}_X(\operatorname{div}\chi^{u_2})$. We identify E_1 and E_2 with subspaces of the fiber E of $\mathcal{E}$ over t_0 in the natural way. We denote by $L_1, \ldots, L_p$ the distinct one-dimensional subspaces of E that are different from E_1, but are equal to $\mathcal{E}^\rho(i)$ for some ray $\rho \in \Delta$ and some $i \in \mathbf{Z}$. We fix once and for all a one-dimensional subspace L of E, different from each of the subspaces $E_1, L_1, \ldots, L_p$ of E. This is done just as an alternative to *ad hoc* choices at different points in our discussion.

Notation III.2. Let $V_1, \ldots, V_l$ be subspaces of a vector space V. For any nonnegative integers $m, \alpha_1, \ldots, \alpha_l$, we define the notation $Sym_V^m(V_1^{\alpha_1}, V_2^{\alpha_2}, \ldots, V_l^{\alpha_l})$ to represent either the subspace of $Sym^m V$ equal to the image of the composition of the natural maps

$$V_1^{\otimes\alpha_1} \otimes V_2^{\otimes\alpha_2} \otimes \cdots \otimes V_l^{\otimes\alpha_l} \otimes V^{\otimes(m-\sum_{i=1}^l \alpha_i)} \longrightarrow V^{\otimes m} \longrightarrow Sym^m V,$$

if $m \geq \sum_{i=1}^l \alpha_i$, or the subspace 0 of $Sym^m V$, otherwise.

On the toric variety X, the map defined by $(m_{n+1}, \ldots, m_d) \mapsto \sum_{i=n+1}^d m_i D_i$ induces an isomorphism between $\mathbf{Z}^{d-n}$ and $\operatorname{Pic} X = N^1(X)$. Hence, each line bundle $\mathcal{L}$ on $\mathbf{P}(\mathcal{E})$ is isomorphic to a unique line bundle of the form $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_i D_i)$.

Definition III.3. Let $\mathcal{L}$ be the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)\otimes\pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$ on $\mathbf{P}(\mathcal{E})$, where $m, m_{n+1},\dots,m_d\in\mathbf{Z}$. Let us consider the torus action on $H^0\big(\mathbf{P}(\mathcal{E}),\mathcal{L}\big) = H^0\big(X,(Sym^m\mathcal{E})\otimes\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)\big)$, induced by the T-equivariant structure on $(Sym^m\mathcal{E})\otimes\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$. Let G be the fiber over t_0 of $\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$. We define the following subsets of $H^0\big(\mathbf{P}(\mathcal{E}),\mathcal{L}\big)$:

$$\mathcal{V}_{\mathcal{L}} = \big\{s\in H^0\big(\mathbf{P}(\mathcal{E}),\mathcal{L}\big)\mid s\in H^0\big(\mathbf{P}(\mathcal{E}),\mathcal{L}\big)_u\smallsetminus\{0\},\text{ for some } u\in M\big\}$$

$$\mathcal{W}_{\mathcal{L}} = \big\{s\in\mathcal{V}_{\mathcal{L}}\mid s(t_0)\text{ lies in the subspace } Sym^m_E(E_1^{\alpha_0},L_1^{\alpha_1},\dots,L_p^{\alpha_p},L^{\alpha})\otimes G$$
$$\text{of } Sym^mE,\text{ for some }\alpha_0,\dots,\alpha_p,\alpha\in\mathbf{Z}_{\geq0},\text{ satisfying }\sum_{j=0}^p\alpha_j+\alpha=m\big\}.$$

In the following lemma we give some formulas for the vanishing vector $\nu_{Y_\bullet}(s)$ of a section $s\in\mathcal{W}_{\mathcal{L}}$.

Lemma III.4. *Let $\mathcal{L}$ be the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)\otimes\pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$ on $\mathbf{P}(\mathcal{E})$, for some $m,m_{n+1},\dots,m_d\in\mathbf{Z}$. Let G be the fiber over t_0 of $\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$. Let s be a nonzero section in $H^0\big(\mathbf{P}(\mathcal{E}),\mathcal{L}\big)_u = H^0\big(X,(Sym^m\mathcal{E})\otimes\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)\big)_u$, for some $u\in M$, and let $\nu_{Y_\bullet}(s)=(\nu_1,\dots,\nu_{n+1})\in\mathbf{Z}^{n+1}$. Then:*

(a) *For each $j\in\{1,\dots,n\}$ we have $\nu_j=\langle\nu_{n+1}u_1+(m-\nu_{n+1})u_2-u,v_j\rangle$.*

(b) *If $s(t_0)$ lies in the subspace $Sym^m_E(E_1^{\alpha_0},L_1^{\alpha_1},\dots,L_p^{\alpha_p},L^{\alpha})\otimes G$ of $(Sym^mE)\otimes G$, for some $\alpha_0,\dots,\alpha_p,\alpha\in\mathbf{Z}_{\geq0}$ such that $\sum_{i=0}^p\alpha_i+\alpha=m$, then $\nu_{n+1}=\alpha_0$.*

Proof.

(a) The vector $\nu_{Y_\bullet}(s)$ can be computed in any neighborhood of Y_0 in $\mathbf{P}(\mathcal{E})$. Hence we can assume that $X=U_\tau$, that $\mathcal{E}=\mathcal{O}_X(\operatorname{div}\chi^{u_1})\oplus\mathcal{O}_X(\operatorname{div}\chi^{u_2})$, and that s is a section in $H^0\big(X,Sym^m\mathcal{E}\big)_u$. Note that $Sym^m\mathcal{E}=\bigoplus_{i=0}^m\mathcal{O}_X(\operatorname{div}\chi^{(m-i)u_1+iu_2})$, and so s corresponds to the section

$$\big(c_0\chi^{-u},\dots,c_i\chi^{-u},\dots,c_m\chi^{-u}\big)\in\bigoplus_{i=0}^m H^0\big(X,\mathcal{O}_X(\operatorname{div}\chi^{(m-i)u_1+iu_2})\big)_u = H^0\big(X,Sym^m\mathcal{E}\big)_u,$$

for some $c_0, \ldots, c_m \in k$. Let us denote $\mathcal{O}_X \oplus \mathcal{O}_X$ by $\mathcal{E}'$. By combining the natural isomorphisms in each component, we get an isomorphism $\mathcal{E} = \mathcal{O}_X(\operatorname{div} \chi^{u_1}) \oplus \mathcal{O}_X(\operatorname{div} \chi^{u_2}) \cong \mathcal{O}_X \oplus \mathcal{O}_X = \mathcal{E}'$. This isomorphism induces a commutative diagram,

$$\begin{tikzcd}
\mathbf{P}(\mathcal{E}) = \mathbf{P}\big(\mathcal{O}_X(\operatorname{div}\chi^{u_1}) \oplus \mathcal{O}_X(\operatorname{div}\chi^{u_2})\big) \arrow[r, "\varphi"] \arrow[dr] & \mathbf{P}\big(\mathcal{O}_X \oplus \mathcal{O}_X\big) = \mathbf{P}(\mathcal{E}') \arrow[d] \\
 & X
\end{tikzcd}$$

where the map φ is an isomorphism. Let $Y'_\bullet \colon Y'_{n+1} \supseteq Y'_n \supseteq \cdots \supseteq Y'_0$ be the T-invariant flag in $\mathbf{P}\big(\mathcal{E}'\big)$, as defined in §3.1.1. Note that the T-invariant flags in $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E}')$ correspond to each other under φ. On $Y'_i = \mathbf{P}(\mathcal{E}'|_{X_{i-1}})$, let us denote $\mathcal{O}_{\mathbf{P}(\mathcal{E}'|_{X_{i-1}})}(m)$ by $\mathcal{O}_{Y'_i}(m)$ for each $i \in \{1, \ldots, n+1\}$. Under the isomorphism φ, s corresponds to the section

$$\begin{aligned}
s' = s'_{n+1} &= \big(c_0\chi^{mu_1-u}, \ldots, c_i\chi^{(m-i)u_1+iu_2-u}, \ldots, c_m\chi^{mu_2-u}\big) \in \bigoplus_{i=0}^{m} H^0\big(X, \mathcal{O}_X\big) \\
&= H^0\big(X, Sym^m(\mathcal{E}')\big) = H^0\big(\mathbf{P}(\mathcal{E}'), \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(m)\big) = H^0\big(Y'_{n+1}, \mathcal{O}_{Y'_{n+1}}(m)\big).
\end{aligned}$$

Note that $(\nu_1, \ldots, \nu_{n+1}) = \nu_{Y_\bullet}(s) = \nu_{Y'_\bullet}(s')$. Let us set $h = \max\{i \mid 0 \leq i \leq m \text{ and } c_i \neq 0\}$, and let $v_1^*, \ldots, v_n^*$ be the basis of M dual to the basis $v_1, \ldots, v_n$ of N.

It is straightforward to see that when we follow the procedure to compute $\nu_{Y'_\bullet}(s')$ outlined in §2.1, for each $0 \leq l \leq n$, the section obtained in the step when we restrict to Y'_{n+1-l} corresponds to the section

$$\begin{aligned}
s'_{n+1-l} = \big(&c_0\chi^{mu_1-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}}, \ldots, c_i\chi^{(m-i)u_1+iu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}}, \ldots, \\
&c_m\chi^{mu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}}\big) \in \bigoplus_{i=0}^{m} H^0\big(X_{n-l}, \mathcal{O}_{X_{n-l}}\big) \\
&= H^0\big(X_{n-l}, Sym^m(\mathcal{E}'|_{X_{n-l}})\big) = H^0\big(\mathbf{P}(\mathcal{E}'|_{X_{n-l}}), \mathcal{O}_{\mathbf{P}(\mathcal{E}'|_{X_{n-l}})}(m)\big) \\
&= H^0\big(Y'_{n+1-l}, \mathcal{O}_{Y'_{n+1-l}}(m)\big),
\end{aligned} \tag{3.1}$$

under the natural identification.

Assume now that for some $0 \le l < n$, we have proven that $\nu_j = \langle (m-h)u_1 + hu_2 - u, v_j\rangle$, for each $j \in \{1,\dots,l\}$. Note that for each $a \in \mathbf{Z}_{\ge 0}$ we have the following commutative diagram

$$\begin{array}{ccc} H^0\big(Y'_{n+1-l}, \mathcal{O}_{Y'_{n+1-l}}(m) \otimes \mathcal{O}_{Y'_{n+1-l}}(-aY'_{n-l})\big) & \xrightarrow{\varphi_a} & H^0\big(Y_{n+1-l}, \mathcal{O}_{Y'_{n+1-l}}(m)\big) \\ \| & & \| \\ H^0\big(X_{n-l}, Sym^m(\mathcal{E}'|_{X_{n-l}}) \otimes \mathcal{O}_{X_{n-l}}(-aX_{n-l-1})\big) & \longrightarrow & H^0\big(X_{n-l}, Sym^m(\mathcal{E}'|_{X_{n-l}})\big), \end{array}$$

and denote the map in its top row by φ_a. Next, we note that

(3.2)

$$\begin{aligned} \nu_{l+1} &= \max\{a \in \mathbf{Z}_{\ge 0} \mid s'_{n+1-l} \in \operatorname{Im}(\varphi_a)\} \\ &= \max\{a \in \mathbf{Z}_{\ge 0} \mid \text{For each } i = 0,\dots,m\, c_i\chi^{(m-i)u_1+iu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}} \in \\ &\qquad \operatorname{Im}(H^0(X_{n-l}, \mathcal{O}_{X_{n-l}}(-aX_{n-l-1})) \hookrightarrow H^0(X_{n-l}, \mathcal{O}_{X_{n-l}}))\} \\ &= \max\{a \in \mathbf{Z}_{\ge 0} \mid a \le \langle (m-i)u_1 + iu_2 - u, v_{l+1}\rangle \text{ for each } i \text{ such that} \\ &\qquad c_i\chi^{(m-i)u_1+iu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}} \ne 0\}. \end{aligned}$$

We also note that $c_h\chi^{(m-h)u_1+hu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}} \ne 0$. Now, if $\langle u_1 - u_2, v_{l+1}\rangle < 0$, then there exists $q \in \{1,\dots,l\}$, such that $\langle u_1 - u_2, v_q\rangle > 0$. In this case, for each $i \in \{0,\dots,h-1\}$, it follows that $c_i\chi^{(m-i)u_1+iu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-q}} = 0$. Hence, either $\langle u_1 - u_2, v_{l+1}\rangle \ge 0$, or $c_i\chi^{(m-i)u_1+iu_2-u-\sum_{j=1}^{l}\nu_j v_j^*}|_{X_{n-l}} = 0$ for each $i \in \{0,\dots,h-1\}$. In either case, it follows from (3.2) that $\nu_{l+1} = \langle (m-h)u_1 + hu_2 - u, v_{l+1}\rangle$. Therefore we can iterate this procedure, and in this way we obtain that for each $j \in \{1,\dots,n\}$,

$$\nu_j = \langle (m-h)u_1 + hu_2 - u, v_j\rangle.$$

Now, ν_{n+1} is equal to the vanishing order along Y'_0 of the section $s'_1 \in H^0(Y'_1, \mathcal{O}_{Y'_1}(m))$ described in (3.1) for $l = n$. We have a natural isomorphism $\mu\colon Y'_1 \to \mathbf{P}^1$ between $Y'_1 = \mathbf{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0})$ and the projective space $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0})$ with homogeneous coordinates x, y. Recall that under μ, Y'_0 corresponds to the point $(0:1)$, defined by the ideal (x) in $\mathbf{P}^1$. Under μ, $\mathcal{O}_{Y_1'}(m)$ corresponds to $\mathcal{O}_{\mathbf{P}^1}(m)$. Depending on whether or not $u_1 = u_2$, there are two possibilities for the section in $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m))$ that corresponds to s_1'. Namely, s_1' corresponds to $c_h x^{m-h} y^h$ if $u_1 \neq u_2$, and it corresponds to $\sum_{i=0}^m c_i x^{m-i} y^i$ if $u_1 = u_2$. In either case, we obtain that $\nu_{n+1} = m - h$, and then part (a) is proven.

(b) As in part (a), we can reduce to the case when $X = U_\tau$, $\mathcal{E} = \mathcal{O}_X(\operatorname{div}\chi^{u_1}) \oplus \mathcal{O}_X(\operatorname{div}\chi^{u_2})$, $s \in H^0\big(X, Sym^m\mathcal{E}\big)_u$, and $s(t_0)$ lies in the subspace $Sym_E^m(E_1^{\alpha_0}, L_1^{\alpha_1}, \dots, L_p^{\alpha_p}, L^\alpha)$ of $Sym^m E$. Let $x, y \in E$ be such that $E_1 = kx$ and $E_2 = ky$. Then x and y form a basis for E, and $x^{m-i}y^i$ for $i = 0, \dots, m$ form a basis for $Sym^m E$. Let $\beta_1, \dots, \beta_p, \beta \in k$ be such that $L = k(\beta x + y)$ and $L_i = k(\beta_i x + y)$ for each $i \in \{1, \dots, p\}$. For $c_0, \dots, c_m \in k$, defined as in part (a), we proved that $\max\{i \mid 0 \leq i \leq m \text{ and } c_i \neq 0\} = m - \nu_{n+1}$. On the one hand, we see that the image of s at t_0 is $s(t_0) = \sum_{i=0}^m c_i x^{m-i} y^i \in Sym^m E$. On the other hand,

$$s(t_0) \in Sym_E^m(E_1^{\alpha_0}, L_1^{\alpha_1}, \dots, L_p^{\alpha_p}, L^\alpha) = k\Big(\sum_{i=0}^{m-\alpha_0-1} \beta_i' x^{m-i} y^i + x^{\alpha_0} y^{m-\alpha_0}\Big),$$

for some $\beta_0', \dots, \beta_{m-\alpha_0-1}' \in k$. From this it follows that $\nu_{n+1} = \alpha_0$ as desired. $\square$

3.1.3 The image of $\nu_{Y_\bullet}$

In this subsection we prove that $\nu_{Y_\bullet}$ maps the collection of T-isotypical sections $\mathcal{W}_\mathcal{L}$ onto $\nu_{Y_\bullet}(H^0(\mathbf{P}(\mathcal{E}), \mathcal{L}) \smallsetminus \{0\})$. We continue working in the setting of §3.1.1-3.1.2.

We start by proving that if a line bundle $\mathcal{L}$ and a flag $Y_\bullet$ on an affine variety Z are suitably compatible with the action of a torus T on Z, then the nonzero T-isotypical sections of $\mathcal{L}$ map onto the image of $\nu_{Y_\bullet}$.

Lemma III.5. *Let Z be an affine variety with an algebraic action of a torus T, and a flag $Y_\bullet\colon Z = Y_l \supseteq Y_{l-1} \supseteq \dots \supseteq Y_0$, where each Y_i is a normal i-dimensional T-invariant subvariety. Assume that for each $i \in \{1, \dots, l\}$, there is a T-isotypical ra-*

tional function h_i on Y_i, such that $Y_{i-1} = \operatorname{div} h_i$. Let g be a T-isotypical rational function on Z and let $s_1, \ldots, s_q \in H^0(Z, \mathcal{O}_Z(\operatorname{div} g))$ be nonzero T-isotypical sections corresponding to distinct characters of T. Then $\nu_{Y_\bullet}(s_1 + \cdots + s_q) \in \{\nu_{Y_\bullet}(s_1), \ldots, \nu_{Y_\bullet}(s_q)\}$.

Proof. We proceed by induction on the dimension of Z. Let $Z_\bullet \colon Y_{l-1} = Z_{l-1} \supseteq Z_{l-2} \supseteq \ldots \supseteq Z_0$ be the flag of normal T-invariant subvarieties in Y_{l-1} defined by $Z_i = Y_i$, for each i. Using the natural isomorphism $H^0(Z, O_Z(\operatorname{div} g)) \cong H^0(Z, \mathcal{O}_Z)$, we can reduce to the case when $g = 1$ and the sections are identified with regular functions. Let $s = s_1 + \cdots + s_q$. For each $a \in \mathbf{Z}_{\geq 0}$ the natural inclusion map $\varphi_a \colon H^0(Y_l, \mathcal{O}_{Y_l}(-aY_{l-1})) \hookrightarrow H^0(Y_l, \mathcal{O}_Z)$ is compatible with the decomposition of these spaces into T-isotypical summands. It follows that

$$\nu_1(s) = \operatorname{ord}_{Y_{l-1}}(s) = \min\{\operatorname{ord}_{Y_{l-1}}(s_1), \ldots, \operatorname{ord}_{Y_{l-1}}(s_q)\} = \min\{\nu_1(s_1), \ldots, \nu_1(s_q)\}.$$

If we reorder the sections so that $\nu_1(s) = \nu_1(s_i)$ for $1 \leq i \leq e$, and $\nu_1(s) < \nu_1(s_i)$ for $e+1 \leq i \leq q$, for some $e \in \{1, \ldots, q\}$, then $(h_l^{-\nu_1(s)} s)|_{Y_{l-1}} = \sum_{i=1}^{e} (h_l^{-\nu_1(s)} s_i)|_{Y_{l-1}}$ and using the induction hypothesis we get

$$\begin{aligned} \nu_{Y_\bullet}(s) = \big(\nu_1(s), \nu_{Z_\bullet}\big((h_l^{-\nu_1(s)} s)|_{Y_{l-1}}\big)\big) &\in \big\{\big(\nu_1(s), \nu_{Z_\bullet}\big((h_l^{-\nu_1(s)} s_i)|_{Y_{l-1}}\big)\big) \mid i = 1, \ldots, e\big\} \\ &= \{\nu_{Y_\bullet}(s_1), \ldots, \nu_{Y_\bullet}(s_e)\} \subseteq \{\nu_{Y_\bullet}(s_1), \ldots, \nu_{Y_\bullet}(s_q)\}. \end{aligned}$$

□

Recall that any line bundle on $\mathbf{P}(\mathcal{E})$ is isomorphic to a unique line bundle of the form $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^* \mathcal{O}_X(\sum_{i=n+1}^{d} m_i D_i)$. In the following proposition, we prove that for a line bundle on $\mathbf{P}(\mathcal{E})$ of that form, the T-isotypical sections map onto the image of $\nu_{Y_\bullet}$.

Proposition III.6. *Let $\mathcal{L}$ be the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^* \mathcal{O}_X(\sum_{i=n+1}^{d} m_i D_i)$ on $\mathbf{P}(\mathcal{E})$ for some $m, m_1, \ldots, m_d \in \mathbf{Z}$, and let s be a nonzero global section of $\mathcal{L}$.*

Let $s_1, \ldots, s_q \in H^0(\mathbf{P}(\mathcal{E}), \mathcal{L})$ *be the unique nonzero* T*-isotypical sections corresponding to distinct characters of* T *such that* $s = s_1 + \cdots + s_q$. *Then* $\nu_{Y_\bullet}(s) \in \{\nu_{Y_\bullet}(s_1), \ldots, \nu_{Y_\bullet}(s_q)\}$.

Proof. It is enough to consider the case $X = U_\tau$. There is a natural choice of coordinates $X = \operatorname{Spec} k[x_1, \ldots, x_n] = \mathbf{A}^n$ and $\mathbf{P}(\mathcal{E}) = \operatorname{Spec} k[x_1, \ldots, x_n] \times \operatorname{Proj} k[x, y] = \mathbf{A}^n \times \mathbf{P}^1$, which is induced by the ordering of the rays of τ and the trivialization of $\mathcal{E}$ over U_τ. In these coordinates we have that $X_i = \{(x_1, \ldots, x_n) \in \mathbf{A}^n \mid x_j = 0 \text{ for } 1 \leq j \leq n - i\}$, and $Y_{i+1} = X_i \times \mathbf{P}^1$ for $0 \leq i \leq n$ and $Y_0 = X_0 \times \{(0:1)\}$. We also have that $T = \operatorname{Spec} k[x_1, \ldots, x_n]_{x_1 \cdots x_n} = (k^*)^n$ acts on $\mathbf{A}^n$ by componentwise multiplication, and that an element $t = (t_1, \ldots, t_n) \in T$ acts on $P = \big((x_1, \ldots, x_n), (x : y)\big) \in \mathbf{A}^n \times \mathbf{P}^1$ by $tP = \big((t_1x_1, \ldots, t_nx_n), (t_1^{\langle u_1, v_1\rangle} \cdots t_n^{\langle u_1, v_n\rangle} x : t_1^{\langle u_2, v_1\rangle} \cdots t_n^{\langle u_2, v_n\rangle} y)\big)$. If $U \subseteq \mathbf{P}^1$ is the complement of $(1:0)$, it is enough to prove that in the T-invariant affine open set $Z = \mathbf{A}^n \times U$ the restriction of the flag $Y_\bullet$ and the line bundle $\mathcal{L}$ satisfy the hypotheses of Lemma III.5. We show that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_Z = \mathcal{O}_Z(\operatorname{div} g)$ for some T-isotypical rational function g on Z, since from this all the assertions follow at once. The surjective map $\mathcal{E} \to \mathcal{O}_X(\operatorname{div} \chi^{u_2})$ corresponds to a geometric section θ of the projection $\pi \colon \mathbf{P}(\mathcal{E}) \to X$, i.e. a morphism $\theta \colon X \to \mathbf{P}(\mathcal{E})$ such that $\pi \circ \theta = id_X$. If we set $X' = \theta(X)$, then $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{X'}) = \mathcal{O}_X(\operatorname{div} \chi^{u_2})$ and we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X') \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{X'} \longrightarrow 0. \tag{3.3}$$

By Grauert's theorem (see III.12.9 in [Har77]) $R^1\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X')) = 0$. Then applying π_* to (3.3) gives $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X')) = \mathcal{O}_X(\operatorname{div} \chi^{u_1})$. Since $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X')$ has degree zero along the fibers of π,

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X') = \pi^*\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-X')) = \pi^*\mathcal{O}_X(\operatorname{div} \chi^{u_1}).$$

And since in local coordinates $X' = \mathbf{A}^n \times (0:1)$, we can take $g = (x/y)\pi^*(\chi^{u_1})$. $\square$

The following lemma will be used in the proofs of Proposition III.8 and Theorem III.10.

Lemma III.7. *Let $V_1, \dots, V_l$ be distinct one-dimensional subspaces of a two-dimensional vector space V. Let $m, \alpha_1, \dots, \alpha_l$ be nonnegative integers. Then*

$$Sym^m_V(V_1^{\alpha_1}) \cap Sym^m_V(V_2^{\alpha_2}) \cap \cdots \cap Sym^m_V(V_l^{\alpha_l}) = Sym^m_V(V_1^{\alpha_1}, V_2^{\alpha_2}, \dots, V_l^{\alpha_l}).$$

Furthermore, this subspace of $Sym^m V$ is nonzero precisely when $m \geq \sum_{i=1}^l \alpha_i$, and in that case its dimension is $m + 1 - \sum_{i=1}^l \alpha_i$.

Proof. We fix an isomorphism of k-algebras between $\oplus_{h\geq 0} Sym^h V$ and the polynomial ring in two variables $k[x,y]$. The subspaces $V_1, \dots, V_l$ of V correspond to the linear spans of some distinct linear forms $f_1, \dots, f_l$. The subspaces $\cap_{i=1}^l Sym^m_V(V_i^{\alpha_i})$ and $Sym^m_V(V_1^{\alpha_1}, \dots, V_l^{\alpha_l})$ of $Sym^m V$ both correspond to the homogeneous polynomials of degree m divisible by $f_1^{\alpha_1} \cdots f_l^{\alpha_l}$. From this observation the conclusion follows. $\square$

In the next proposition we prove that for every line bundle $\mathcal{L}$ on $\mathbf{P}(\mathcal{E})$, in order to find the image of $\nu_{Y_\bullet}$, we can restrict our attention to the sections in $\mathcal{W}_\mathcal{L}$.

Proposition III.8. *Let $\mathcal{L}$ be the line bundle $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_i D_i)$ on $\mathbf{P}(\mathcal{E})$, for some $m, m_{n+1}, \dots, m_d \in \mathbf{Z}$. Then we have the following equality of subsets of $\mathbf{Z}^{n+1}$:*

$$\nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big) \smallsetminus \{0\}) = \nu_{Y_\bullet}(\mathcal{V}_\mathcal{L}) = \nu_{Y_\bullet}(\mathcal{W}_\mathcal{L}).$$

Proof. From their definitions, we have that $\nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big) \smallsetminus \{0\}) \supseteq \nu_{Y_\bullet}(\mathcal{V}_\mathcal{L}) \supseteq \nu_{Y_\bullet}(\mathcal{W}_\mathcal{L})$. The sets $\nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big) \smallsetminus \{0\})$ and $\nu_{Y_\bullet}(\mathcal{V}_\mathcal{L})$ are equal by Proposition III.6. Let us consider $\nu_{Y_\bullet}(s) \in \nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big) \smallsetminus \{0\}) = \nu_{Y_\bullet}(\mathcal{V}_\mathcal{L})$. We can assume that

$s \in H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u$, for some $u \in M$. By Remark II.1, the set $\nu_{Y_\bullet}(H^0(\mathbf{P}(\mathcal{E}), \mathcal{L})_u \smallsetminus \{0\})$ is finite with cardinality $\dim_k H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u$. Let us denote the fiber over t_0 of $\mathcal{O}_X(\sum_{i=n+1}^d m_i D_i)$ by G. From Example II.12 and Lemma III.7 we see that there exist $\alpha_0, \ldots, \alpha_p \in \mathbf{Z}_{\geq 0}$ such that

$$\begin{aligned}
\operatorname{Im}\Big(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u = H^0\big(X, \pi_*\mathcal{L}\big)_u \hookrightarrow (Sym^m E) \otimes G\Big) \\
&= (\pi_*\mathcal{L})^{\rho_1}(\langle u, v_1\rangle) \cap (\pi_*\mathcal{L})^{\rho_2}(\langle u, v_2\rangle) \cap \cdots \cap (\pi_*\mathcal{L})^{\rho_d}(\langle u, v_d\rangle) \\
&= (Sym^m_E(E_1^{\alpha_0}) \otimes G) \cap (Sym^m_E(L_1^{\alpha_1}) \otimes G) \cap \cdots \cap (Sym^m_E(L_p^{\alpha_p}) \otimes G) \\
&= Sym^m_E(E_1^{\alpha_0}, L_1^{\alpha_1}, L_2^{\alpha_2}, \ldots, L_p^{\alpha_p}) \otimes G.
\end{aligned}$$

Using again Lemma III.7 we see that $\alpha =_{def} m - \sum_{i=0}^p \alpha_i$ is a nonnegative integer and $\alpha + 1 = \dim_k H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u$. For each $j \in \{0, \ldots, \alpha\}$, let $s_j \in H^0\big(X, \pi_*\mathcal{L}\big)_u \smallsetminus \{0\} = H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u \smallsetminus \{0\}$ be such that

$$s_j(t_0) \in Sym^m_E(E_1^{\alpha_0+j}, L_1^{\alpha_1}, L_2^{\alpha_2}, \ldots, L_p^{\alpha_p}, L^{\alpha-j}) \otimes G.$$

From Lemma III.4 it follows that $\nu_{Y_\bullet}(s_0), \ldots, \nu_{Y_\bullet}(s_\alpha) \in \nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u \smallsetminus \{0\})$ are pairwise distinct, so

$$\nu_{Y_\bullet}(s) \in \nu_{Y_\bullet}(H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{L}\big)_u \smallsetminus \{0\}) = \{\nu_{Y_\bullet}(s_0), \ldots, \nu_{Y_\bullet}(s_\alpha)\} \subseteq \nu_{Y_\bullet}(\mathcal{W}_{\mathcal{L}}),$$

and this completes the proof of the proposition. $\square$

3.2 The global Okounkov body of $\mathbf{P}(\mathcal{E})$

In this section we describe the global Okounkov body of the projectivization of a rank two toric vector bundle over a smooth projective toric variety, with respect to the flag of invariant subvarieties constructed in §3.1.1. We introduce the relevant terminology in §3.2.1, and we prove our result describing this global Okounkov body in terms of linear inequalities in §3.2.2. Throughout this section we use the notation and constructions introduced in §2.1, §2.2 and §3.1.1-3.1.2.

3.2.1 Supporting hyperplanes of the global Okounkov body of $\mathbf{P}(\mathcal{E})$

Let $\mathcal{E}$ be a toric vector bundle of rank two over the smooth projective toric variety X. Let $u_1, u_2 \in M$, with $u_1 \geq_{lex} u_2$, be as defined in §3.1.1, and let the subspaces $E_1, L_1, \ldots, L_p$ of the fiber E of $\mathcal{E}$ over t_0 be as defined in §3.1.2. Let us classify the filtrations $\{\mathcal{E}^{\rho_j} \mid j = 1, \ldots, d\}$ associated to $\mathcal{E}$ by defining

$$A =_{def} \{j \in \{1, \ldots, d\} \mid \dim_k \mathcal{E}^{\rho_j}(i) \neq 1 \text{ for all } i \in \mathbf{Z}\}$$

$$B =_{def} \{j \in \{1, \ldots, d\} \mid \mathcal{E}^{\rho_j}(i) = E_1 \text{ for some } i \in \mathbf{Z}\}$$

$$C_h =_{def} \{j \in \{1, \ldots, d\} \mid \mathcal{E}^{\rho_j}(i) = L_h \text{ for some } i \in \mathbf{Z}\}$$

for each $h \in \{1, \ldots, p\}$. And let us define a nonempty set $J \subseteq \{1, \ldots, d\}$ to be *admissible* if it has one of the following three forms:

- $J = \{j\}$ for some $j \in A$.
- $J = \{j\}$ for some $j \in B$.
- $J = \{j_1, \ldots, j_l\}$ for some $j_1, \ldots, j_l \in \{1, \ldots, d\}$ such that there exist distinct indices $i_1, \ldots, i_l \in \{1, \ldots, p\}$ with $j_h \in C_{i_h}$, for each $h \in \{1, \ldots, l\}$.

Note that each admissible subset of $\{1, \ldots, d\}$ is contained in exactly one of the sets A, B and $C =_{def} \cup_{i=1}^{p} C_i$. For each ray $\rho_j \in \Delta$ we define integers a_j and b_j as follows. Let $a_j = \max\{i \in \mathbf{Z} \mid \mathcal{E}^{\rho_j}(i) = E\}$, and let

$$b_j = \begin{cases} a_j + 1 & \text{if } j \in A, \\ \max\{i \in \mathbf{Z} \mid \dim_k \mathcal{E}^{\rho_j}(i) = 1\} & \text{if } j \in B \cup C. \end{cases}$$

Example III.9. In the case of $T_{\mathbf{P}^2}$, the tangent bundle of the projective plane (see Example II.6), if we take the T-invariant flag $\mathbf{P}^2 \supseteq D_1 \supseteq D_1 \cap D_2$ in $\mathbf{P}^2$, we get $\tau = \sigma_3$, $u_1 = (1, 0)$ and $u_2 = (0, 1)$. We get $a_1 = a_2 = a_3 = 0$ and $b_1 = b_2 = b_3 = 1$.

We also get $E_1 = V_1$, $L_1 = V_2$ and $L_2 = V_3$, and then $A = \{\emptyset\}$, $B = \{1\}$ and $C = \{2, 3\}$. Hence, in this case the admissible subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$ and $\{2, 3\}$.

The isomorphisms $N^1(\mathbf{P}(\mathcal{E})) \cong N^1(X) \oplus \mathbf{Z}$ and $N^1(X) \cong \mathbf{Z}^{d-n}$, described in §2.2 and §3.1.2, induce an isomorphism between $N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}}$ and $\mathbf{R}^{d-n+1}$, which we use to identify these spaces hereafter. Likewise, we identify $\mathbf{R}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}}$ with $\mathbf{R}^{d+2}$, with coordinates $(x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w)$. Let $v_1^*, \ldots, v_n^*$ be the basis of $M_{\mathbf{R}} =_{def} M \otimes \mathbf{R}$ dual to the basis $v_1, \ldots, v_n$ of $N_{\mathbf{R}} =_{def} N \otimes \mathbf{R}$. Note that we have an isomorphism

$$\begin{array}{rccc} \psi\colon & \mathbf{R}^{d+2} & \longrightarrow & M_{\mathbf{R}} \times \mathbf{R}^{d-n+2} \\ & (x_1, \ldots, x_n, x_{n+1}, w_{n+1}, \ldots, w_d, w) & \longmapsto & \big(-\sum_{i=1}^{n} x_i v_i^* + x_{n+1}u_1 + (w - x_{n+1})u_2, \\ & & & x_{n+1}, w_{n+1}, \ldots, w_d, w\big) \end{array}$$

For each $j \in \{1, \ldots, d\}$ we define the linear function:

$$\begin{array}{rccc} \gamma_{\mathcal{E},j}\colon & M_{\mathbf{R}} \times \mathbf{R}^{d-n+2} & \longrightarrow & \mathbf{R} \\ & (u, x_{n+1}, w_{n+1}, \ldots, w_d, w) & \longmapsto & \frac{\langle u, v_j\rangle - a_j w - w_j}{b_j - a_j} \end{array}$$

for any $u \in M_{\mathbf{R}}$, and any $x_{n+1}, w_{n+1}, \ldots, w_d, w \in \mathbf{R}$, and where $w_j = 0$ for each $j \leq n$. We will denote this function simply by γ_j, when no confusion is likely to arise. Finally, for each admissible set $J \subseteq \{1, \ldots, d\}$, we define the linear function $I_J\colon \mathbf{R}^{d+2} \to \mathbf{R}$ by declaring its value at $P = (x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \mathbf{R}^{d+2}$ to be:

$$I_J(P) = \begin{cases} \gamma_j \circ \psi(P), & \text{if } J = \{j\} \subseteq A, \\ \gamma_j \circ \psi(P) - x_{n+1}, & \text{if } J = \{j\} \subseteq B, \\ \sum_{j \in J} \gamma_j \circ \psi(P) - w + x_{n+1}, & \text{if } J \subseteq C. \end{cases}$$

For notational convenience, we define for each admissible set $J \subseteq \{1, \ldots, d\}$ the linear function $I'_J\colon M_{\mathbf{R}} \times \mathbf{R}^{d-n+2} \to \mathbf{R}$ to be $I'_J = I_J \circ \psi^{-1}$.

3.2.2 The global Okounkov body of $\mathbf{P}(\mathcal{E})$

Theorem III.10. *Let $\mathcal{E}$ be a toric vector bundle of rank two on the smooth projective toric variety X. The global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$ of $\mathbf{P}(\mathcal{E})$ is the rational polyhedral cone in $\mathbf{R}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}} \simeq \mathbf{R}^{d+2}$ given by*

$$\begin{aligned}\boldsymbol{\Delta} = \Big\{&(x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \mathbf{R}^{d+2} \mid w \geq x_{n+1} \geq 0 \text{ and} \\ &I_J(x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \leq 0 \text{ for all admissible } J \subseteq \{1, \ldots, d\}\Big\}.\end{aligned}$$

Proof. From the characterization of the global Okounkov body in terms of its fibers over big classes in $N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}$, it suffices to show the following stronger assertion: For every class $\mathcal{L} \in N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}$, the fiber $\boldsymbol{\Delta}_{\mathcal{L}}$ of $\boldsymbol{\Delta}$ over $\mathcal{L}$ is equal to $\Delta(\mathcal{L}) \times \{\mathcal{L}\}$.

To prove the assertion, we consider a class $\mathcal{L} \in N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}$. Note that $\boldsymbol{\Delta}_{\mathcal{L}^m} = m\boldsymbol{\Delta}_{\mathcal{L}}$ and $\Delta(\mathcal{L}^m) \times \{\mathcal{L}^m\} = m(\Delta(\mathcal{L}) \times \{\mathcal{L}\})$, for each $m \in \mathbf{Z}^+$. Hence, we can assume that $\mathcal{L} \in N^1(\mathbf{P}(\mathcal{E}))$. Let $w_{n+1}, \ldots, w_d, w \in \mathbf{Z}$ be the unique integers such that

$$\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(w) \otimes \pi^* \mathcal{O}_X\Big(\sum_{i=n+1}^{d} w_i D_i\Big).$$

For notational convenience, we set $w_i = 0$ for each $i \in \{1, \ldots, n\}$. We first show that $\Delta(\mathcal{L}) \times \{\mathcal{L}\} \subseteq \boldsymbol{\Delta}_{\mathcal{L}}$. For this, it is enough to see that the set

$$\Delta(\mathcal{L}) \times \{\mathcal{L}\} = \overline{\operatorname{Conv}\Big(\bigcup_{m \in \mathbf{Z}^+} \frac{1}{m}\nu\big(\mathcal{L}^m\big)\Big)} \times (w_{n+1}, \ldots, w_d, w)$$

is contained in $\boldsymbol{\Delta}$. Since $\boldsymbol{\Delta}$ is closed and convex, it suffices show that

$$\big(\frac{1}{m}\nu(\mathcal{L}^m)\big) \times (w_{n+1}, \ldots, w_d, w) \subseteq \boldsymbol{\Delta},$$

for each $m \in \mathbf{Z}^+$. Furthermore, since $\boldsymbol{\Delta}$ is a cone, it is enough to prove that this inclusion holds when $m = 1$. With this in mind, we consider

$$P = (x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \nu(\mathcal{L}) \times (w_{n+1}, \ldots, w_d, w).$$

Note that the existence of P implies that $w \geq 0$. Let $Q = \psi(P)$, i.e.

$$Q = (-\sum_{i=1}^{n} x_i v_i^* + x_{n+1}u_1 + (w - x_{n+1})u_2, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in M_{\mathbf{R}} \times \mathbf{R}^{d-n+2}.$$

By replacing $\mathcal{L}$ with a suitable tensor power, we can assume that $\gamma_j(Q) \in \mathbf{Z}$ for all $j \in \{1, \ldots, d\}$. By the projection formula we have $\pi_*\mathcal{L} = (Sym^w\mathcal{E}) \otimes \mathcal{O}_X(\sum_{i=1}^{d} w_i D_i)$. If we denote the fiber of $\mathcal{O}_X(\sum_{i=1}^{d} w_i D_i)$ over the unit of the torus by G, then by Example II.9 we get

$$(\pi_*\mathcal{L})^{\rho_j}(i) = (Sym^w\mathcal{E})^{\rho_j}(i - w_j) \otimes G,$$

for all $j \in \{1, \ldots, d\}$ and all $i \in \mathbf{Z}$. Since $(x_1, \ldots, x_{n+1}) \in \nu(\mathcal{L})$, Proposition III.8 implies that there exist $u \in M$ and a nonzero section $s \in H^0(\mathbf{P}(\mathcal{E}), \mathcal{L})_u = H^0(X, \pi_*\mathcal{L})_u$ such that $\nu_{Y_\bullet}(s) = (x_1, \ldots, x_{n+1})$, and such that

$$s(t_0) \in V =_{def} Sym^w_E(E_1^{\alpha_0}, L_1^{\alpha_1}, \ldots, L_p^{\alpha_p}, L^{\alpha}) \otimes G,$$

for some one-dimensional subspace L of E different from $E_1, L_1, \ldots, L_p$ and for some $\alpha_0, \ldots, \alpha_p, \alpha \in \mathbf{Z}_{\geq 0}$ with $\sum_{i=0}^{p} \alpha_i + \alpha = w$. Note now that

$$0 \neq V \subseteq \operatorname{Im}\left(H^0\big(X, \pi_*\mathcal{L}\big)_u \hookrightarrow (Sym^w E) \otimes G\right) = \bigcap_{i=1}^{d} (\pi_*\mathcal{L})^{\rho_i}(\langle u, v_i \rangle) \subseteq (\pi_*\mathcal{L})^{\rho_j}(\langle u, v_j \rangle),$$

for each $j \in \{1, \ldots, d\}$. By Lemma III.4, we have that $\alpha_0 = x_{n+1}$ and $x_i = \langle -u + \alpha_0 u_1 + (w - \alpha_0)u_2, v_j \rangle$ for each $i \in \{1, \ldots, n\}$. In particular, we see that $Q = (u, x_{n+1}, w_{n+1}, \ldots, w_d, w)$. From $\alpha_0 = x_{n+1}$ we get

$$w \geq x_{n+1} \geq 0. \tag{3.4}$$

Hence, we are reduced to proving that $I_J(P) \leq 0$ for each admissible set $J \subseteq \{1, \ldots, d\}$, or equivalently, to proving that $I'_J(Q) \leq 0$ for every such J.

Let us consider an admissible set $J \subseteq \{1, \ldots, d\}$. Then either $J = \{j\}$ for some $j \in A$,

$J = \{j\}$ for some $j \in B$, or $J = \{j_1, \ldots, j_l\}$ for some $j_1, \ldots, j_l \in C$ such that there exist distinct indices $i_1, \ldots, i_l \in \{1, \ldots, p\}$, with $j_h \in C_{i_h}$ for each $h \in \{1, \ldots, l\}$. In the first case $0 \neq V \subseteq (\pi_* \mathcal{L})^{\rho_j}(\langle u, v_j \rangle)$ gives $\langle u, v_j \rangle \leq a_j w + w_j$, and then

$$I'_J(Q) \leq 0. \tag{3.5}$$

In the second case $0 \neq V \subseteq (\pi_* \mathcal{L})^{\rho_j}(\langle u, v_j \rangle) = Sym^w_E(E_1^{max\{0,\gamma_j(Q)\}}) \otimes G$, and this implies

$$0 \neq V = V \cap (Sym^w_E(E_1^{max\{0,\gamma_j(Q)\}}) \otimes G) \subseteq (Sym^w_E(E_1^{max\{0,\gamma_j(Q)\}}) \otimes G) \cap$$
$$(Sym^w_E(L_1^{\alpha_1}) \otimes G) \cap \cdots \cap (Sym^w_E(L_p^{\alpha_p}) \otimes G) \cap (Sym^w_E(L^{\alpha}) \otimes G),$$

and then from Lemma III.7 we get $\gamma_j(Q) \leq max\{0, \gamma_j(Q)\} \leq w - \sum_{i=1}^p \alpha_i - \alpha = x_{n+1}$, so

$$I'_J(Q) \leq 0. \tag{3.6}$$

In the third case, we have $\gamma_{j_h}(Q) \leq \alpha_{j_h}$ for each $h \in \{1, \ldots, l\}$. Otherwise we would have $0 \neq V \subseteq (\pi_* \mathcal{L})^{\rho_{j_h}}(\langle u, v_{j_h} \rangle) = Sym^w_E(L_{i_h}^{\gamma_{j_h}(Q)}) \otimes G$, and from Lemma III.7 we would get

$$V = V \cap (Sym^w_E(L_{i_h}^{\gamma_{j_h}(Q)}) \otimes G) = (Sym^w_E(E_1^{\alpha_0}) \otimes G) \cap (Sym^w_E(L_1^{\alpha_1}) \otimes G) \cap \cdots \cap$$
$$(Sym^w_E(L_{i_h-1}^{\alpha_{i_h-1}}) \otimes G) \cap (Sym^w_E(L_{i_h}^{\gamma_{j_h}(Q)}) \otimes G) \cap (Sym^w_E(L_{i_h+1}^{\alpha_{i_h+1}}) \otimes G) \cap \cdots \cap$$
$$(Sym^w_E(L_p^{\alpha_p}) \otimes G) \cap (Sym^w_E(L^{\alpha}) \otimes G) = 0,$$

which is a contradiction. By adding these inequalities over $j_h \in J$, we get

$$\sum_{j \in J} \gamma_j(Q) \leq \sum_{h=1}^{l} \alpha_{j_h} \leq \sum_{i=1}^{p} \alpha_i = w - \alpha_0 - \alpha \leq w - x_{n+1},$$

and therefore

$$I'_J(Q) \leq 0. \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7), it follows that $P \in \mathbf{\Delta}$.

As we have completed the proof of $\Delta(\mathcal{L}) \times \{\mathcal{L}\} \subseteq \mathbf{\Delta}_{\mathcal{L}}$, we now prove that $\mathbf{\Delta}_{\mathcal{L}} \subseteq \Delta(\mathcal{L}) \times \{\mathcal{L}\}$. For this, we note that

$$\mathbf{\Delta}_{\mathcal{L}} = \overline{\mathbf{\Delta}_{\mathcal{L}} \cap (\mathbf{Q}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}})},$$

since $\mathbf{\Delta}_{\mathcal{L}}$ is defined by rational linear inequalities. Thus, it suffices to show that

$$\mathbf{\Delta}_{\mathcal{L}} \cap \left(\mathbf{Q}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}\right) \subseteq \Delta(\mathcal{L}) \times \{\mathcal{L}\}.$$

To prove this, let us consider

$$P = (x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \mathbf{\Delta}_{\mathcal{L}} \cap \left(\mathbf{Q}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}\right),$$

and define Q to be $\psi(P)$, i.e.

$$Q = (-\sum_{i=1}^{n} x_i v_i^* + x_{n+1}u_1 + (w - x_{n+1})u_2, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in M_{\mathbf{Q}} \times \mathbf{Q} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}.$$

By replacing $\mathcal{L}$ with a suitable tensor power, we can assume that $Q \in M \times \mathbf{Z} \times N^1(\mathbf{P}(\mathcal{E}))$ and $\gamma_j(Q) \in \mathbf{Z}$ for each $j \in \{1, \ldots, d\}$. We have $w \geq x_{n+1} \geq 0$, and $I'_J(Q) = I_J(P) \leq 0$ for each admissible set $J \subseteq \{1, \ldots, d\}$. Let us define

$$u = -\sum_{i=1}^{n} x_i v_i^* + x_{n+1}u_1 + (w - x_{n+1})u_2 \in M,$$

and note that $Q = (u, x_{n+1}, w_{n+1}, \ldots, w_d, w)$. We will show the existence of a nonzero section $s \in H^0(\mathbf{P}(\mathcal{E}), \mathcal{L})_u = H^0(X, \pi_*\mathcal{L})_u$, satisfying $\nu_\bullet(s) = (x_1, \ldots, x_{n+1})$, which will give us

$$P \in \nu(\mathcal{L}) \times (w_{n+1}, \ldots, w_d, w) \subseteq \Delta(\mathcal{L}) \times \{\mathcal{L}\}.$$

By the projection formula we have $\pi_*\mathcal{L} = (Sym^w\mathcal{E}) \otimes \mathcal{O}_X(\sum_{i=1}^d w_i D_i)$. If we denote the fiber of $\mathcal{O}_X(\sum_{i=1}^d w_i D_i)$ over the unit of the torus by G, then by Example II.9 we get

$$(\pi_*\mathcal{L})^{\rho_j}(i) = (Sym^w\mathcal{E})^{\rho_j}(i - w_j) \otimes G,$$

for all $j \in \{1, \ldots, d\}$ and all $i \in \mathbf{Z}$. For each $i \in \{1, \ldots, p\}$, define $\alpha_i \in \mathbf{Z}_{\geq 0}$ by

$$\alpha_i =_{def} max\left(\{0\} \cup \{\gamma_j(Q) \mid j \in C_i\}\right).$$

We claim that $\alpha =_{def} w - x_{n+1} - \sum_{i=1}^{p} \alpha_i$ is a nonnegative integer. Indeed, this is clear if $\alpha_i = 0$ for each $i \in \{1, \ldots, p\}$. On the other hand, if $\alpha_i \neq 0$ for some $i \in \{1, \ldots, p\}$, let $i_1, \ldots, i_l \in \{1, \ldots, p\}$ be the distinct indices such that for $i \in \{1, \ldots, p\}$, $\alpha_i \neq 0$ if and only if $i \in \{i_1, \ldots, i_l\}$. For each $h \in \{1, \ldots, l\}$, let us choose $j_h \in \{1, \ldots, d\}$ such that $j_h \in C_{i_h}$ and $\alpha_{i_h} = \gamma_{j_h}(Q)$. Then the set $J = \{j_1 \ldots, j_l\}$ is admissible, and we have $I'_J(Q) \leq 0$. Hence

$$\sum_{i=1}^{p} \alpha_i = \sum_{h=1}^{l} \alpha_{i_h} = \sum_{j \in J} \gamma_j(Q) \leq w - x_{n+1}.$$

In either case, it follows that α is a nonnegative integer. Let L be a one-dimensional subspace of E different from $E_1, L_1, \ldots, L_p$. From Lemma III.7, we see that

$$V =_{def} Sym_E^w(E_1^{x_{n+1}}, L_1^{\alpha_1}, \ldots, L_p^{\alpha_p}, L^{\alpha}) \otimes G$$

is a one-dimensional subspace of $(Sym^w E) \otimes G$. We now prove that $V \subseteq (\pi_* \mathcal{L})^{\rho_j}(\langle u, v_j \rangle)$ for each $j \in \{1, \ldots, d\}$, considering separately the cases $j \in A$, $j \in B$ and $j \in C$. If $j \in A$, then $J = \{j\}$ is admissible, and $I'_J(Q) \leq 0$. This gives $\langle u, v_j \rangle \leq a_j w + w_j$, and therefore

$$V \subseteq (Sym^w E) \otimes G = (\pi_* \mathcal{L})^{\rho_j}(\langle u, v_j \rangle).$$

If $j \in B$, then $J = \{j\}$ is admissible, and $I'_J(Q) \leq 0$. This gives

$$\langle u, v_j \rangle \leq a_j w + w_j + (b_j - a_j) x_{n+1},$$

and therefore

$$V \subseteq Sym_E^w(E_1^{x_{n+1}}) \otimes G = (\pi_* \mathcal{L})^{\rho_j}(a_j w + w_j + (b_j - a_j) x_{n+1}) \subseteq (\pi_* \mathcal{L})^{\rho_j}(\langle u, v_j \rangle).$$

If $j \in C$, then there exists $i \in \{1, \ldots, p\}$ such that $j \in C_i$, and

$$V \subseteq Sym^w_E(L_i^{\alpha_i}) \otimes G \subseteq Sym^w_E(L_i^{max\{0,\gamma_j(Q)\}}) \otimes G = (\pi_*\mathcal{L})^{\rho_j}(\langle u, v_j\rangle).$$

Therefore

$$V \subseteq \bigcap_{j=1}^{d} (\pi_*\mathcal{L})^{\rho_j}(\langle u, v_j\rangle) = \mathrm{Im}\left(H^0\big(X, \pi_*\mathcal{L}\big)_u \longrightarrow (Sym^w E) \otimes G\right).$$

We can now choose a nonzero section $s \in H^0(\mathbf{P}(\mathcal{E}), \mathcal{L})_u = H^0(X, \pi_*\mathcal{L})_u$ such that $s(t_0) \in V$. By Lemma III.4, the section s satisfies

$$\nu_{Y_\bullet}(s) = \big(\langle -u + x_{n+1}u_1 + (w - x_{n+1})u_2, v_1\rangle, \ldots,$$
$$\langle -u + x_{n+1}u_1 + (w - x_{n+1})u_2, v_n\rangle, x_{n+1}\big) = (x_1, \ldots, x_{n+1}).$$

Thus $P = \nu_{Y_\bullet}(s) \times (w_{n+1}, \ldots, w_d, w) \in \Delta(\mathcal{L}) \times \{\mathcal{L}\}$. It follows that $\mathbf{\Delta}_{\mathcal{L}} \subseteq \Delta(\mathcal{L}) \times \{\mathcal{L}\}$, and this completes the proof. $\square$

Remark III.11. Explicitly, the inequalities that define the global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$ of $\mathbf{P}(\mathcal{E})$ given in Theorem III.10 are $w \geq x_{n+1} \geq 0$ together with:

For each $j \in A$,

$$\sum_{i=1}^{n} \langle v_i^*, v_j\rangle x_i + \langle u_2 - u_1, v_j\rangle x_{n+1} + (a_j - \langle u_2, v_j\rangle)w + w_j \geq 0,$$

for each $j \in B$,

$$\sum_{i=1}^{n} \langle v_i^*, v_j\rangle x_i + (\langle u_2 - u_1, v_j\rangle + b_j - a_j)x_{n+1} + (a_j - \langle u_2, v_j\rangle)w + w_j \geq 0,$$

and for each admissible set $J \subseteq C$,

$$\sum_{j \in J} \frac{1}{b_j - a_j} \left[\sum_{i=1}^{n} \langle v_i^*, v_j\rangle x_i + \langle u_2 - u_1, v_j\rangle x_{n+1} + (a_j - \langle u_2, v_j\rangle)w + w_j\right] + w - x_{n+1} \geq 0.$$

3.3 Examples

We recall that the explicit description of Okounkov bodies in concrete examples can be rather difficult. The main result of this chapter, Theorem III.10, allows us

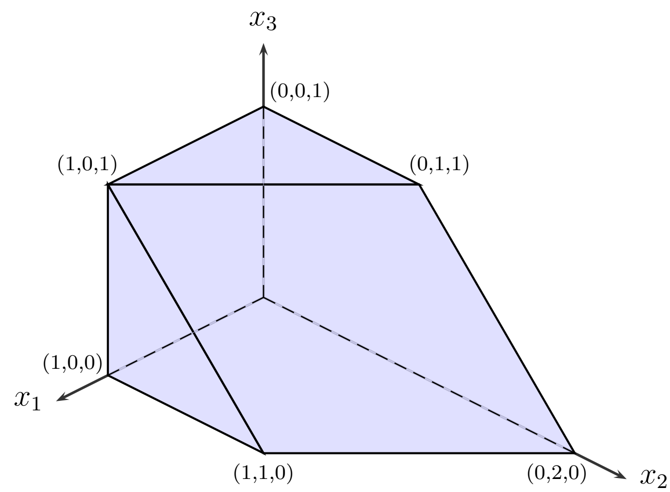

Figure 3.1: The Okounkov body of $\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)$.

to explicitly compute the Okounkov bodies of all line bundles on projectivizations of rank two toric vector bundles over smooth projective toric varieties, with respect to the flag from §3.1.1, by substituting combinatorial data into the inequalities given in Remark III.11. In this section we present some examples to illustrate this theorem.

Example III.12. We consider $T_{\mathbf{P}^2}$, the tangent bundle of the projective plane (see Example II.6). From Remark III.11 (see Example III.9), we get inequalities for the Okounkov body of each line bundle on $\mathbf{P}(T_{\mathbf{P}^2})$. For instance, by setting $w = 1$ and $w_j = 0$ for each j, we deduce that the Okounkov body $\Delta(\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1))$ is defined inside $\mathbf{R}^3$ by the inequalities:

$$\begin{array}{ccc} 1 \geq x_3, & x_3 \geq 0, & x_1 \geq 0, \\ x_2 \geq 0, & 2 \geq x_1 + x_2 + x_3, & 1 \geq x_1 \end{array}$$

(see Figure 3.1). In particular, we see that the volume of $\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)$ is

$$\operatorname{vol}_{\mathbf{P}(T_{\mathbf{P}^2})}\big(\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)\big) = \operatorname{vol}_{\mathbf{R}^3}\big(\Delta(\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1))\big) \cdot 3! = 6.$$

In the next example we see that our description gives the expected answer for line bundles that are pulled back from the base.

Example III.13. The inequalities for the Okounkov body of a line bundle on $\mathbf{P}(\mathcal{E})$ of the form $\pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)$ are $x_{n+1}=0$ and

$$\sum_{i=1}^n \langle v_i^*, v_j\rangle x_i + m_j \geq 0,$$

for each $j \in \{1,\dots,d\}$. Furthermore, from the description of the Okounkov body of a toric line bundle on X given in [LM08, Proposition 6.1], we see that

$$\Delta_{Y_\bullet}\big(\pi^*\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)\big) = P_{\sum_{i=n+1}^d m_iD_i} \times \{0\} = \Delta_{X_\bullet}\big(\mathcal{O}_X(\sum_{i=n+1}^d m_iD_i)\big) \times \{0\}.$$

In the next example, we see that our description gives the expected answer when the toric vector bundle equivariantly splits.

Example III.14. When $\mathcal{E}$ equivariantly splits as the sum of two toric line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, the variety $\mathbf{P}(\mathcal{E})$ is a toric variety. The subvarieties in our flag in $\mathbf{P}(\mathcal{E})$ from §3.1.1 are also invariant with respect to the torus T' of $\mathbf{P}(\mathcal{E})$. Hence, in this case we have two descriptions of the Okounkov bodies of line bundles on $\mathbf{P}(\mathcal{E})$ with respect to this flag of invariant subvarieties, namely, the one given by our theorem, and the one given by Lazarsfeld and Mustaţă in [LM08] in the case of toric varieties. It is good to see that these two descriptions agree, as expected.

Let h_1 and h_2 be the piecewise linear functions associated to $\mathcal{L}_1$ and $\mathcal{L}_2$ (see 3.4 in [Ful93]). In particular, $\mathcal{L}_1$ and $\mathcal{L}_2$ correspond to the T-invariant divisors $-\sum_{j=1}^d h_1(v_j)D_j$ and $-\sum_{j=1}^d h_2(v_j)D_j$. Let $\Phi\colon N_{\mathbf{R}} \to N_{\mathbf{R}} \times \mathbf{R}$ be the piecewise linear map defined by $\Phi(v) = (v, h_1(v) - h_2(v))$. For each cone $\sigma \in \Delta$, let σ^+ and σ^- be the cones in $N_{\mathbf{R}} \times \mathbf{R}$ spanned by $\Phi(\sigma)$ and $(0,1)$, and by $\Phi(\sigma)$ and $(0,-1)$, respectively. Let $\widetilde{\Delta}$ be the fan in $N_{\mathbf{R}} \times \mathbf{R}$ consisting of the faces of σ^+ and σ^- for all $\sigma \in \Delta$. The toric variety associated to the fan $\widetilde{\Delta}$ is isomorphic to $\mathbf{P}(\mathcal{E})$ (see §7 in [Oda78]). The rays of $\widetilde{\Delta}$ are $\rho^+ = \mathbf{R}_{\geq 0}\cdot(0,1)$, $\rho^- = \mathbf{R}_{\geq 0}\cdot(0,-1)$ and $\widetilde{\rho_j} = \Phi(\rho_j)$

for each $j \in \{1, \ldots, d\}$. Let us denote the corresponding T'-invariant divisors by D^+, D^- and $\widetilde{D_j}$ for each $j \in \{1, \ldots, d\}$, respectively. The flag in $\mathbf{P}(\mathcal{E})$ given by our construction from §3.1.1 is

$$Y_\bullet \colon \mathbf{P}(\mathcal{E}) \supseteq \widetilde{D_1} \supseteq \widetilde{D_1} \cap \widetilde{D_2} \supseteq \cdots \supseteq \widetilde{D_1} \cap \widetilde{D_2} \cap \cdots \cap \widetilde{D_n} \supseteq \widetilde{D_1} \cap \widetilde{D_2} \cap \cdots \cap \widetilde{D_n} \cap D^-.$$

Note that $\Phi(v_1), \ldots, \Phi(v_n)$ and $(0, -1)$ span the maximal cone $\tau^- \in \widetilde{\Delta}$. Let us change the reference ordered basis in $N_{\mathbf{R}} \times \mathbf{R}$ to $\{\Phi(v_1), \ldots, \Phi(v_n), (0, -1)\}$. In these new coordinates the rays are given by $\rho^+ = \mathbf{R}_{\geq 0} \cdot (0, -1)$, $\rho^- = \mathbf{R}_{\geq 0} \cdot (0, 1)$ and $\widetilde{\rho_j} = \mathbf{R}_{\geq 0} \cdot (v_j, h_2(v_j) - h_1(v_j) + \langle u_2, v_j \rangle - \langle u_1, v_j \rangle)$ for each $j \in \{1, \ldots, d\}$. Using the argument in the proof of Proposition III.6, we see that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(D^+) \otimes \pi^*\mathcal{L}_2 = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(D^-) \otimes \pi^*\mathcal{L}_1$. We set $D = D^+ + \sum_{j=1}^{d}(-h_2(v_j) - \langle u_2, v_j \rangle)\widetilde{D_j}$, and note that the T'-invariant divisor D satisfies $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(D)$ and $D|_{U_{\tau^-}} = 0$.

Let us consider a line bundle $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^{d} m_i D_i)$ on $\mathbf{P}(\mathcal{E})$. Let us identify the dual of $N_{\mathbf{R}} \times \mathbf{R}$ with $\mathbf{R}^{n+1}$ by identifying the ordered basis $\{\Phi(v_1), \ldots, \Phi(v_n), (0, -1)\}$ of $N_{\mathbf{R}} \times \mathbf{R}$ with the coordinates $x_1, \ldots, x_{n+1}$ on $\mathbf{R}^{n+1}$. We set $m_i = 0$ for each $i \in \{1, \ldots, n\}$. On the one hand, the description in [LM08, Proposition 6.1] says that with this identification $\Delta_{Y_\bullet}(\mathcal{L})$ is the polytope $P_{mD + \sum_{i=n+1}^{d} m_i \widetilde{D_i}}$. This polytope is defined as a subset of $\mathbf{R}^{n+1}$ by the inequalities

$$\begin{cases} x_{n+1} \geq 0, \;\; m \geq x_{n+1}, \\ \sum_{i=1}^{n} \langle v_i^*, v_j \rangle x_i + (h_2(v_j) - h_1(v_j) + \langle u_2, v_j \rangle - \langle u_1, v_j \rangle) x_{n+1} - m h_2(v_j) \\ \qquad\qquad\qquad -m\langle u_2, v_j \rangle + m_j \geq 0, \qquad\qquad \text{for each } j \in \{1, \ldots, d\}. \end{cases}$$

On the other hand, the admissible subsets of $\{1, \ldots, d\}$ associated to a toric vector bundle that equivariantly splits are exactly the singletons. From Remark III.11, our inequalities for $\Delta_{Y_\bullet}(\mathcal{L})$ are

$$\begin{cases} m \geq x_{n+1} \geq 0, \\ \sum_{i=1}^{n} \langle v_i^*, v_j \rangle x_i + \langle u_2 - u_1, v_j \rangle x_{n+1} + (h_2(v_j) - h_1(v_j)) x_{n+1} - m h_2(v_j) \\ \qquad\qquad\qquad -m\langle u_2, v_j \rangle + m_j \geq 0, \qquad \text{for each } j \in \{1, \dots, d\}. \end{cases}$$

Therefore, the two descriptions of the Okounkov body $\Delta_{Y_\bullet}(\mathcal{L})$ coincide.

CHAPTER IV

The Cox ring of a projectivized rank two toric vector bundle

4.1 Introduction

In this chapter we present two distinct proofs of the finite generation of the Cox rings of projectivized rank two toric vector bundles. These two arguments were obtained in our study of this problem for arbitrary ranks (c.f. Chapter V). There are yet two more proofs of this finite generation that can be found in the literature (see [HS10]). In their work [HS10], J. Hausen and H. Süß provide a different argument for this finite generation and point out that one can obtain yet another argument based on the main theorem in the paper [Kn93] of F. Knop. The proof in [HS10] is based on some interesting results where the authors find presentations of Cox rings of varieties with torus actions over Cox rings of some associated quotient prevarieties, and in turn they find presentations of those Cox rings over the Cox rings of some other varieties associated to the aforementioned quotient prevarieties which they call separations. In the additional argument suggested by Hausen and Süß, starting from a variety X in a certain class of varieties that includes projectivizations of rank two toric vector bundles, one constructs a unirational T-variety of complexity one that has the Cox ring of X as its function field. We recall that a *T-variety* is a normal variety with an action of a torus, and that its *complexity* is defined as the smallest codimension of a T-orbit. From this the conclusion follows from the finite generation as algebras of the function fields of T-varieties of complexity one, which was proved by Knop in [Kn93].

Our first proof arises as an application of our description of the global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$ of a projectivized rank two toric vector bundle $\mathcal{E}$, with respect to our flag of T-invariant subvarieties from 3.1.1 as a rational polyhedral cone in a finite dimensional real vector space. The finite generation will follow after we prove that there is a Veronese subalgebra of a Cox ring of $\mathbf{P}(\mathcal{E})$ that is isomorphic to the semigroup algebra associated to the semigroup of lattice points with sufficiently divisible coordinates in $\Delta(\mathbf{P}(\mathcal{E}))$, which is clearly finitely generated.

For our second proof (see Theorem IV.6), we consider a particular set of generators of the group $\mathrm{Pic}(\mathbf{P}(\mathcal{E}))$, and describe a finite set of generators for the associated Cox ring. For this, we will refine the grading of $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$ using the induced torus action on the global sections of invariant T-divisors. Next, we describe the graded pieces and the multiplication map in terms of the data appearing in the Klyachko filtrations of $\mathcal{E}$. In this case, we obtain the finite generation by exhibiting a finite generator set of a Veronese subalgebra of the Cox ring of $\mathbf{P}(\mathcal{E})$.

In this chapter we use the notation for toric varieties and toric vector bundles introduced in Section 2.2 unless we state otherwise. We conclude the introduction to this chapter with the following remark that shows that the question of finite generation of Cox rings of projectivized toric vector bundles over arbitrary toric varieties can be reduced to the case when the base is smooth.

Remark IV.1. Given a toric variety X there exists a toric resolution of singularities, i.e. there exists a smooth toric variety X' and a proper birational toric morphism $f\colon X' \to X$. Given a toric vector bundle $\mathcal{E}$ on X, the induced map $f'\colon \mathbf{P}(f^*\mathcal{E}) \to$

$\mathbf{P}(\mathcal{E})$ is also proper and birational. In this case $f^*\mathcal{E}$ is a toric vector bundle on X' and the finite generation of a Cox ring of $\mathbf{P}(f^*\mathcal{E})$ implies the finite generation of any Cox ring of $\mathbf{P}(\mathcal{E})$. To see this we consider line bundles $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ on $\mathbf{P}(\mathcal{E})$ whose classes span $\mathrm{Pic}(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}$ and line bundles $\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{s'}$ on $\mathbf{P}(f^*\mathcal{E})$ whose classes span $\mathrm{Pic}(\mathbf{P}(f^*\mathcal{E}))_{\mathbf{Q}}$. The finite generation of the algebra

$$\mathrm{Cox}\big(\mathbf{P}(f^*\mathcal{E}), (\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{s'}, f'^*\mathcal{L}_1, f'^*\mathcal{L}_2, \ldots, f'^*\mathcal{L}_s)\big)$$

implies the finite generation of the subalgebra

$$\bigoplus_{(m_1,\ldots,m_s)\in\mathbf{Z}^s} H^0(\mathbf{P}(f^*\mathcal{E}), f'^*\mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes f'^*\mathcal{L}_s^{\otimes m_s}) = \mathrm{Cox}\big(\mathbf{P}(\mathcal{E}), (\mathcal{L}_1, \ldots, \mathcal{L}_s)\big),$$

since $0 \times \mathbf{Z}^s$ is a finitely generated saturated subsemigroup of $\mathbf{Z}^{s'} \times \mathbf{Z}^s$ (see Lemma 4.8 in [ELMNP05]). Therefore, to prove the finite generation of the Cox rings of projectivizations of rank two toric vector bundles over arbitrary toric varieties it is enough to consider the case when the base is a smooth toric variety.

4.2 The finite generation of the Cox ring of $\mathbf{P}(\mathcal{E})$ using its Okounkov body

In the next theorem we show that the projectivization of a rank two toric vector bundle over a simplicial projective toric variety is a Mori dream space (c.f. [HS10] and Theorem IV.6 for alternative arguments).

Theorem IV.2. *Any Cox ring of the projectivization $\mathbf{P}(\mathcal{E})$ of a rank two toric vector bundle $\mathcal{E}$ over the projective simplicial toric variety X is finitely generated and $\mathbf{P}(\mathcal{E})$ is a Mori dream space.*

Proof. Any simplicial toric variety is $\mathbf{Q}$-factorial, and a projective bundle over a $\mathbf{Q}$-factorial variety is again $\mathbf{Q}$-factorial, hence the we are reduced to prove the finite generation of any Cox ring of $\mathbf{P}(\mathcal{E})$ in the sense of Hu and Keel. By Remark IV.1 we can assume that X is smooth. Let us prove that the semigroup defined by

$$S = \Big\{(x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \mathbf{R}^{d+2} = \mathbf{R}^{n+1} \times N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}} \mid \\ \text{There exists } s \in H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(w) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^{d} w_i D_i)\big) \\ \text{such that } \nu_{Y_\bullet}(s) = (x_1, \ldots, x_{n+1})\Big\}$$

is finitely generated. Since $S \subseteq \mathbf{Z}^{d+2}$, it is enough to prove that the semigroup $S \cap (c \cdot \mathbf{Z}^{d+2})$ is finitely generated, where $c = \operatorname{lcm}\{b_j - a_j \mid j = 1, 2, \ldots, d\}$. And for this it suffices to prove that $S \cap (c \cdot \mathbf{Z}^{d+2}) = \Delta(\mathbf{P}(\mathcal{E})) \cap (c \cdot \mathbf{Z}^{d+2})$, since $\Delta(\mathbf{P}(\mathcal{E}))$ is a rational polyhedral cone. From the definition of $\Delta(\mathbf{P}(\mathcal{E}))$, we have that $S \cap (c \cdot \mathbf{Z}^{d+2}) \subseteq \Delta(\mathbf{P}(\mathcal{E})) \cap (c \cdot \mathbf{Z}^{d+2})$. Let $(x_1, \ldots, x_{n+1}, w_{n+1}, \ldots, w_d, w) \in \Delta(\mathbf{P}(\mathcal{E})) \cap (c \cdot \mathbf{Z}^{d+2})$. Proceeding exactly as in the second part of the proof of Theorem III.10, it follows that there exists a nonzero section $s \in H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(w) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^{d} w_i D_i)\big)$ that satisfies $\nu_{Y_\bullet}(s) = (x_1, \ldots, x_{n+1})$. Therefore the semigroup S is finitely generated as we claimed. Now we prove that the Cox ring of $\mathbf{P}(\mathcal{E})$ associated to the line bundles $\pi^*\mathcal{O}_X(D_{n+1}), \ldots, \pi^*\mathcal{O}_X(D_d)$ and $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$ on $\mathbf{P}(\mathcal{E})$ is finitely generated. This Cox ring is equal to

$$R = \bigoplus_{(m_{n+1}, \ldots, m_d, m) \in \mathbf{Z}^{d-n+1}} R_{(m_{n+1}, \ldots, m_d, m)},$$

where for each $(m_{n+1}, \ldots, m_d, m) \in \mathbf{Z}^{d-n+1}$,

$$R_{(m_{n+1}, \ldots, m_d, m)} =_{def} H^0(X, (Sym^m \mathcal{E}) \otimes \mathcal{O}_X(m_{n+1} D_{n+1}) \otimes \cdots \otimes \mathcal{O}_X(m_d D_d)).$$

Let $\{g_1, g_2, \ldots, g_l\}$ be generators of S. For each $j \in \{1, 2, \ldots, l\}$, there exist $m_{n+1}^{(j)}$, $\ldots, m_d^{(j)}, m^{(j)} \in \mathbf{Z}$ and a nonzero section

$$s_j \in H^0\big(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m^{(j)}) \otimes \pi^*\mathcal{O}_X(\sum_{i=n+1}^{d} m_i^{(j)} D_i)\big)$$

such that $g_l = (\nu_{Y_\bullet}(s_j), m^{(j)}_{n+1}, \ldots, m^{(j)}_d, m^{(j)})$. For each $(m_{n+1}, \ldots, m_d, m) \in \mathbf{Z}^{d-n+1}$, since $g_1, g_2, \ldots, g_l$ generate S, it follows that

$$\nu_{Y_\bullet}\big((k[s_1, s_2, \ldots, s_l] \cap R_{(m_{n+1},\ldots,m_d,m)}) \smallsetminus \{0\}\big) = \nu_{Y_\bullet}\big(R_{(m_{n+1},\ldots,m_d,m)} \smallsetminus \{0\}\big).$$

By Remark II.1 the finite dimensional vector spaces $k[s_1, s_2, \ldots, s_l] \cap R_{(m_{n+1},\ldots,m_d,m)}$ and $R_{(m_{n+1},\ldots,m_d,m)}$ have the same dimension, and thus they are equal. Therefore $R = k[s_1, s_2, \ldots, s_l]$ and this completes the proof. $\square$

4.3 A combinatorial approach to the finite generation of the Cox ring of $\mathbf{P}(\mathcal{E})$

As before, we use Remark IV.1 to reduce the question of finite generation of the Cox rings of projectivized toric vector bundles of a given rank over arbitrary toric varieties, to the case when the base is smooth. Let X be a smooth toric variety and let $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ be the projectivization of the rank two toric vector bundle $\mathcal{E}$ over X. The classes of the line bundles $\mathcal{O}_X(D_1), \mathcal{O}_X(D_2), \ldots, \mathcal{O}_X(D_d)$ span $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$. Since $\mathrm{Pic}(\mathbf{P}(\mathcal{E})) = \mathrm{Pic}(X) \oplus \mathbf{Z}\cdot[\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)]$, the classes of the line bundles $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1), \pi^*\mathcal{O}_X(D_1), \pi^*\mathcal{O}_X(D_2), \ldots, \pi^*\mathcal{O}_X(D_d)$ span $\mathrm{Pic}(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}} = N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{Q}}$. Therefore the following algebra is a Cox ring of $\mathbf{P}(\mathcal{E})$ in the sense of Definition II.13:

$$C \;=\; \bigoplus_{(m,m_1,\ldots,m_d)\in\mathbf{Z}^{d+1}} H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \pi^*\mathcal{O}_X(m_1D_1) \otimes \cdots \otimes \pi^*\mathcal{O}_X(m_dD_d)).$$

By the projection formula, C is isomorphic to the algebra

$$R \;=\; \bigoplus_{(m,m_1,\ldots,m_d)\in\mathbf{Z}^{d+1}} H^0(X, Sym^m\mathcal{E} \otimes \mathcal{O}_X(m_1D_1) \otimes \cdots \otimes \mathcal{O}_X(m_dD_d)),$$

where $Sym^m\mathcal{E} = 0$ for $m < 0$. For each $\underline{\mathbf{m}} = (m_1, \ldots, m_d) \in \mathbf{Z}^d$, let us denote the toric line bundle $\mathcal{O}_X(m_1D_1) \otimes \cdots \otimes \mathcal{O}_X(m_dD_d)$ on X by $\mathcal{L}^{\underline{\mathbf{m}}}$, and the fiber of $\mathcal{L}^{\underline{\mathbf{m}}}$

over t_0 by $L_{\underline{\mathbf{m}}}$. Note that for each $m, m' \in \mathbf{Z}$, each $\underline{\mathbf{m}}, \underline{\mathbf{m}}' \in \mathbf{Z}^d$ and each $u, u' \in M$, the product of $H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u$ and $H^0(X, Sym^{m'}\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}'})_{u'}$ in the algebra R is contained in $H^0(X, Sym^{m+m'}\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}+\underline{\mathbf{m}}'})_{u+u'}$. Therefore we get a finer grading for R given by

$$R \;=\; \bigoplus_{(u,m,\underline{\mathbf{m}})\in M\times\mathbf{Z}\times\mathbf{Z}^d} R_{(u,m,\underline{\mathbf{m}})} \;=\; \bigoplus_{(u,m,\underline{\mathbf{m}})\in M\times\mathbf{Z}\times\mathbf{Z}^d} H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u.$$

Note that each of the homogeneous components $R_{(u,m,\underline{\mathbf{m}})} = H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u$ is a finite dimensional k-vector space, as each of them is isomorphic to a subspace of E via ev_{t_0}. For each $l \in \mathbf{Z}^+$, let $R^{(l)}$ be the Veronese subalgebra of R given by

$$(4.1)\quad R^{(l)} \;=\; \bigoplus_{(u,m,\underline{\mathbf{m}})\in M\times\mathbf{Z}\times\mathbf{Z}^d} R_{(lu,lm,l\underline{\mathbf{m}})} \;=\; \bigoplus_{(u,m,\underline{\mathbf{m}})\in M\times\mathbf{Z}\times\mathbf{Z}^d} H^0(X, Sym^{lm}\mathcal{E}\otimes\mathcal{L}^{l\underline{\mathbf{m}}})_{lu}.$$

Since R is a domain and $H^0(X, \mathcal{O}_X)$ is a finitely generated k-algebra, it follows from general considerations that the finite generation of R is equivalent to the finite generation of $R^{(l)}$ for any $l \in \mathbf{Z}^+$ (see [SYJ10, Lemma 1.4]).

4.3.1 Preliminary lemmas

Let $W_1, W_2, \ldots, W_l$ be subspaces of a vector space W. For every collection of nonnegative integers $m, c_1, c_2, \ldots, c_l$, we denote by $Sym^m_W(W_1^{c_1}, W_2^{c_2}, \cdots, W_l^{c_l})$ the subspace of $Sym^m W$ equal to the image of the composition of the natural maps

$$W_1^{\otimes c_1} \otimes W_2^{\otimes c_2} \otimes \cdots \otimes W_l^{\otimes c_l} \otimes W^{\otimes(m-\sum_{i=1}^l c_i)} \longrightarrow W^{\otimes m} \longrightarrow Sym^m W,$$

if $m \geq \sum_{i=1}^l c_i$, or the subspace 0 of $Sym^m W$, otherwise.

Lemma IV.3. *Let $W_1, W_2, \ldots, W_q$ be distinct subspaces of a vector space W. Let $m, m', c_1, c_2, \ldots, c_q, c'_1, c'_2, \ldots, c'_q$ be nonnegative integers. If $\sum_{i=1}^q c_i \leq m$ and $\sum_{i=1}^q c'_i \leq m'$, then*

$$\mu\Big(Sym^m_W\big(W_1^{c_1}, \ldots, W_q^{c_q}\big) \otimes Sym^{m'}_W\big(W_1^{c'_1}, \ldots, W_q^{c'_q}\big)\Big) = Sym^{m+m'}_W\big(W_1^{c_1+c'_1}, \ldots, W_q^{c_q+c'_q}\big),$$

where $\mu\colon Sym^m W \otimes Sym^{m'} W \to Sym^{m+m'} W$ *is the multiplication map.*

Proof. The conclusion follows at once from the commutativity of the diagram

$$\begin{tikzcd}
\Big((\bigotimes_i W_i^{\otimes c_i}) \otimes W^{\otimes(m-\sum c_i)}\Big) \otimes \Big((\bigotimes_i W_i^{\otimes c_i'}) \otimes W^{\otimes(m'-\sum c_i')}\Big) \arrow[r] \arrow[d, "\wr"] & W^{\otimes m} \otimes W^{\otimes m'} \arrow[r] & Sym^m W \otimes Sym^{m'} W \arrow[d, "\mu"] \\
(\bigotimes_i W_i^{\otimes(c_i+c_i')}) \otimes W^{\otimes((m+m')-\sum(c_i+c_i'))} \arrow[r] & W^{\otimes(m+m')} \arrow[r] & Sym^{m+m'} W.
\end{tikzcd}$$

□

Let $\mathcal{E}$ be a toric vector bundle of rank two on X. As before, let E be the fiber of $\mathcal{E}$ over the unit element t_0 of the torus, and let $\mathcal{E}^{\rho_1}, \mathcal{E}^{\rho_2}, \ldots, \mathcal{E}^{\rho_d}$ be the Klyachko filtrations associated to $\mathcal{E}$. Let $V_1, V_2, \ldots, V_p$ be the distinct one-dimensional subspaces of E that appear in the Klyachko filtrations of $\mathcal{E}$, i.e. each V_l is equal to $\mathcal{E}^{\rho_j}(i)$ for some $j \in \{1, 2, \ldots, d\}$ and some $i \in \mathbf{Z}$. We now define the subsets $A_0, A_1, \ldots, A_p$ of $\{1, 2, \ldots, d\}$, which intuitively classify the filtrations according to their one-dimensional subspace, as follows. For each $l \in \{1, 2, \ldots, p\}$, we define

$$A_l = \{j \in \{1, 2, \ldots, d\} \mid \mathcal{E}^{\rho_j}(i) = V_l \text{ for some } i \in \mathbf{Z}\},$$

and we also define

$$A_0 = \{1, 2, \ldots, d\} \smallsetminus \bigcup_{1 \le l \le p} A_l.$$

For each $j \in \{1, 2, \ldots, d\}$ let $l(j)$ be the unique element of $\{0, 1, 2, \ldots, p\}$ such that $j \in A_{l(j)}$. We next introduce the following collection of possibly empty subsets of $\{1, 2, \ldots, d\}$:

$$\mathcal{J} = \{J \mid J \subseteq \bigcup_{1 \le l \le p} A_l, \text{ and } J \cap A_l \text{ has at most one element, for each } l = 1, 2, \ldots, p\}.$$

Note that $\mathcal{J}$ is a finite set. For each $j \in \{1, 2, \ldots, d\}$, we define the integers a_j and

b_j by $a_j = \max\{i \in \mathbf{Z} \mid \mathcal{E}^{\rho_j}(i) = E\}$ and

$$b_j = \begin{cases} \max\{i \in \mathbf{Z} \mid \dim_k \mathcal{E}^{\rho_j}(i) = 1\} & \text{if } j \in \bigcup_{1 \leq l \leq p} A_l \\ a_j + 1 & \text{if } j \in A_0. \end{cases}$$

To each filtration $\mathcal{E}^{\rho_j}$ we associate the linear functional λ_j on $M_{\mathbf{R}} \times \mathbf{R}^{d+1}$ defined by

$$\begin{array}{cccc} \lambda_j\colon & M_{\mathbf{R}} \times \mathbf{R}^{d+1} & \longrightarrow & \mathbf{R} \\ & (u, m, m_1, \ldots, m_d) & \longmapsto & \frac{\langle u, v_j \rangle - a_j m - m_j}{b_j - a_j} \end{array}$$

for any $u \in M_{\mathbf{R}}$ and any $m, m_1, m_2, \ldots, m_d \in \mathbf{R}$.

Remark IV.4. To motivate the preceding definitions, we note that for each $j \in \{1, 2, \ldots, d\}$, and for each $m \in \mathbf{Z}_{\geq 0}$, $\underline{\mathbf{m}} \in \mathbf{Z}^d$ and $u \in M$, we have

$$((Sym^m \mathcal{E}) \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u, v_j \rangle) = Sym^m_E\big((\mathcal{E}^{\rho_j}(b_j))^{\max\{0, \lceil \lambda_j(u, m, \underline{\mathbf{m}}) \rceil\}}\big) \otimes L_{\underline{\mathbf{m}}},$$

where $\lceil x \rceil = \min\{l \in \mathbf{Z} \mid l \geq x\}$ for any $x \in \mathbf{R}$. This equality is a reformulation of Example II.12, and it can also be verified through direct computation.

We now define a rational polyhedral cone Q_J in $M_{\mathbf{R}} \times \mathbf{R}^{d+1}$ for each $J \in \mathcal{J}$, as follows. Given such a set $J = \{j_1, j_2, \ldots, j_q\}$, the cone Q_J is defined as the set of elements $(x, w, w_1, \ldots, w_d)$ satisfying the linear inequalities

$$w \geq 0, \tag{4.2}$$

$$\sum_{h=1}^{q} \lambda_{j_h}(x, w, w_1, \ldots, w_d) \leq w, \tag{4.3}$$

$$\lambda_j(x, w, w_1, \ldots, w_d) \leq 0 \text{ for each } j \in \{1, 2, \ldots, d\} \setminus \bigcup_{1 \leq h \leq q} A_{l(j_h)}, \tag{4.4}$$

$$\lambda_{j_h}(x, w, w_1, \ldots, w_d) \geq 0 \text{ for each } h \in \{1, 2, \ldots, q\}, \tag{4.5}$$

$$\lambda_{j_h}(x, w, w_1, \ldots, w_d) \geq \lambda_j(x, w, w_1, \ldots, w_d) \text{ for each } h \in \{1, 2, \ldots, q\} \text{ and each } j \in A_{l(j_h)}, \tag{4.6}$$

where $(x, w, w_1, \ldots, w_d)$ are the coordinates in $M_{\mathbf{R}} \times \mathbf{R}^{d+1}$. Intuitively, the cones Q_J were defined precisely to satisfy the conclusions of the following lemma.

Lemma IV.5. *For each $g = (u, m, \underline{\mathbf{m}}) \in M \times \mathbf{Z} \times \mathbf{Z}^d$ we have:*

(a) *If $(u, m, \underline{\mathbf{m}})$ belongs to the cone Q_J, for some $J = \{j_1, j_2, \ldots, j_q\} \in \mathcal{J}$, then*

$$ev_{t_0}\left(H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u\right) = Sym^m_E\left(V_{l(j_1)}^{\lceil \lambda_{j_1}(g)\rceil}, V_{l(j_2)}^{\lceil \lambda_{j_2}(g)\rceil}, \ldots, V_{l(j_q)}^{\lceil \lambda_{j_q}(g)\rceil}\right) \otimes \; L_{\underline{\mathbf{m}}}.$$

(b) *If $H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u \neq 0$, then $(u, m, \underline{\mathbf{m}})$ belongs to the cone Q_J for some $J \in \mathcal{J}$.*

(c) *Assume that $(u, m, \underline{\mathbf{m}}) \in c\,(M \times \mathbf{Z} \times \mathbf{Z}^d)$ belongs to the cone Q_J, for some $J = \{j_1, j_2, \ldots, j_q\} \in \mathcal{J}$, where $c = \operatorname{lcm}\{b_j - a_j \mid j = 1, 2, \ldots, d\}$. Then $H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u \neq 0$.*

Proof.

(a) From the definition of the Klyachko filtrations of $Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}}$, we have that

$$ev_{t_0}\left(H^0(X, Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u\right) = \bigcap_{1 \leq j \leq d} (Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u, v_j\rangle).$$

For each $j \in \{1, 2, \ldots, d\} \smallsetminus \bigcup_{1 \leq h \leq q} A_{l(j_h)}$, the inequality $\lambda_j(g) \leq 0$ implies that $(Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u, v_j\rangle) = Sym^m E \otimes L_{\underline{\mathbf{m}}}$. For each $h \in \{1, 2, \ldots, q\}$, the inequality $\lambda_{j_h}(g) \geq 0$ implies that $(Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_{j_h}}(\langle u, v_{j_h}\rangle) = Sym^m_E(V_{l(j_h)}^{\lceil \lambda_{j_h}(g)\rceil}) \otimes L_{\underline{\mathbf{m}}}$. Similarly, for each $h \in \{1, 2, \ldots, q\}$ and each $j \in A_{l(j_h)}$ the inequality $\lambda_{j_h}(g) \geq \lambda_j(g)$ implies that

$$(Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u, v_j\rangle) \supseteq (Sym^m\mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})^{\rho_{j_h}}(\langle u, v_{j_h}\rangle).$$

Therefore,

$$
\begin{aligned}
&ev_{t_0}\left(H^0(X, Sym^m\mathcal{E}\otimes\mathcal{L}^{\underline{\mathbf{m}}})_u\right)\\
&\quad=\bigcap_{0\le l\le p}\bigcap_{j\in A_l}(Sym^m\mathcal{E}\otimes\mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u,v_j\rangle)=\bigcap_{1\le h\le q}\bigcap_{j\in A_{l(j_h)}}(Sym^m\mathcal{E}\otimes\mathcal{L}^{\underline{\mathbf{m}}})^{\rho_j}(\langle u,v_j\rangle)\\
&\qquad=\bigcap_{1\le h\le q}(Sym^m\mathcal{E}\otimes\mathcal{L}^{\underline{\mathbf{m}}})^{\rho_{j_h}}(\langle u,v_{j_h}\rangle)=\bigcap_{1\le h\le q}Sym^m_E(V_{l(j_h)}^{\lceil\lambda_{j_h}(g)\rceil})\otimes L_{\underline{\mathbf{m}}}\\
&\qquad\quad=Sym^m_E\left(V_{l(j_1)}^{\lceil\lambda_{j_1}(g)\rceil},V_{l(j_2)}^{\lceil\lambda_{j_2}(g)\rceil},\ldots,V_{l(j_q)}^{\lceil\lambda_{j_q}(g)\rceil}\right)\otimes L_{\underline{\mathbf{m}}}.
\end{aligned}
$$

(b) We define

$$I=\Big\{\,l\in\{1,2,\ldots,p\}\ \mid\ \max\{\lambda_j(u,m,\underline{\mathbf{m}})\mid j\in A_l\}\ge 0\,\Big\}.$$

We can assume that $I\ne\phi$, since otherwise $(u,m,\underline{\mathbf{m}})$ belongs to Q_J for $J=\phi\in\mathcal{J}$. Let $l_1,l_2,\ldots,l_q$ be the distinct elements of I, and for each $h\in\{1,2,\ldots,q\}$ let us choose $j_h\in A_{l_h}$ such that $\lambda_{j_h}(u,m,\underline{\mathbf{m}})=\max\{\lambda_j(u,m,\underline{\mathbf{m}})\ \mid j\in A_{l_h}\}$. Clearly, the set $J=_{def}\{j_h\mid h=1,2,\ldots,q\}$ belongs to $\mathcal{J}$. Then $(u,m,\underline{\mathbf{m}})$ satisfies (4.2) and (4.3) since

$$\bigcap_{1\le h\le q}Sym^m_E\big(V_{l(j_h)}^{\lceil\lambda_{j_h}(u,m,\underline{\mathbf{m}})\rceil}\big)\otimes L_{\underline{\mathbf{m}}}=ev_{t_0}\left(H^0(X,Sym^m\mathcal{E}\otimes\mathcal{L}^{\underline{\mathbf{m}}})_u\right)\ne 0,$$

and it satisfies (4.4)-(4.6) by the definitions of I and J. Thus, $(u,m,\underline{\mathbf{m}})\in Q_J$.

(c) By (a), it suffices to show that $Sym^m_E\big(V_{l(j_1)}^{\lambda_{j_1}(u,m,\underline{\mathbf{m}})},V_{l(j_2)}^{\lambda_{j_2}(u,m,\underline{\mathbf{m}})},\ldots,V_{l(j_q)}^{\lambda_{j_q}(u,m,\underline{\mathbf{m}})}\big)$ is nonzero, which is true by (4.3) and Lemma III.7. □

4.3.2 The finite generation of the Cox ring of $\mathbf{P}(\mathcal{E})$

In the next theorem we prove that any Cox ring of $\mathbf{P}(\mathcal{E})$, in the sense of Definition II.13, is finitely generated. As a corollary we obtain that $\mathbf{P}(\mathcal{E})$ is a Mori dream space as defined by Hu and Keel in [HK00] (c.f. [Kn93], [HS10] and Theorem IV.2), if the toric variety X is projective and Δ is simplicial (i.e. each cone in Δ is spanned by as many vectors as its dimension). Throughout this section we use the notation from §2.2 and the notation previously introduced in this chapter, in particular the definitions from §4.3.1 will play an important role.

Theorem IV.6. *Any Cox ring of the projectivization* $\mathbf{P}(\mathcal{E})$ *of a rank two toric vector bundle* $\mathcal{E}$ *over an arbitrary toric variety* X *is finitely generated as a* k*-algebra.*

Proof. By Remark IV.1 we can assume that X is smooth. It suffices to find a finite set of generators for the k-algebra $R^{(c)}$ from (4.1), where $c = \operatorname{lcm}\{b_j - a_j \mid j = 1, 2, \ldots, d\}$. For each set $J \in \mathcal{J}$, let $G_J \subseteq c\,(M \times \mathbf{Z}^{d+1})$ be a finite set of generators for the semigroup $Q_J \cap c\,(M \times \mathbf{Z}^{d+1})$. For each $g \in M \times \mathbf{Z}^{d+1}$, let β_g be a k-basis of R_g. We claim that the finite set

$$\beta =_{def} \bigcup_{J \in \mathcal{J}} \bigcup_{g \in G_J} \beta_g$$

generates $R^{(c)}$ as a k-algebra. In order to prove the claim, consider $(u, m, \underline{\mathbf{m}}) \in c\,(M \times \mathbf{Z} \times \mathbf{Z}^d)$ such that $H^0(X, Sym^m \mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u \neq 0$. By Lemma IV.5 (a) and (b), there exist $J = \{j_1, j_2, \ldots, j_q\} \in \mathcal{J}$, such that $(u, m, \underline{\mathbf{m}}) \in Q_J$ and

$$ev_{t_0}\left(H^0(X, Sym^m \mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}})_u\right) = Sym^m_E\left(V_{l(j_1)}^{\lambda_{j_1}(u,m,\underline{\mathbf{m}})}, V_{l(j_2)}^{\lambda_{j_2}(u,m,\underline{\mathbf{m}})}, \ldots, V_{l(j_q)}^{\lambda_{j_q}(u,m,\underline{\mathbf{m}})}\right) \otimes L_{\underline{\mathbf{m}}}.$$

Now, fix an expression $(u, m, \underline{\mathbf{m}}) = \sum_{g \in G_J} c_g g$, where $c_g \in \mathbf{Z}_{\geq 0}$ for each $g \in G_J$. Let $g = (u_g, m_g, \underline{\mathbf{m}}_g)$ for each $g \in G_J$, be the corresponding coordinates in $M \times \mathbf{Z} \times \mathbf{Z}^d$. From Lemma IV.5 (a) applied in the cone Q_J, we get that for each $g \in G_J$,

$$ev_{t_0}\left(H^0(X, Sym^{m_g} \mathcal{E} \otimes \mathcal{L}^{\underline{\mathbf{m}}_g})_{u_g}\right) = Sym^{m_g}_E\left(V_{l(j_1)}^{\lambda_{j_1}(g)}, V_{l(j_2)}^{\lambda_{j_2}(g)}, \ldots, V_{l(j_q)}^{\lambda_{j_q}(g)}\right) \otimes L_{\underline{\mathbf{m}}_g}.$$

Since $\sum_{g \in G_J} c_g \lambda_{j_h}(g) = \lambda_{j_h}(u, m, \underline{\mathbf{m}})$ for each $h \in \{1, 2, \ldots, q\}$, it follows by Lemma IV.5 (c) and Lemma IV.3 that in the commutative diagram

$$\begin{tikzcd}
\bigotimes_{g\in G_J} (R_g)^{\otimes c_g} \arrow[r] \arrow[d] & R_{(u,m,\underline{\mathbf{m}})} \arrow[d, hook, "ev_{t_0}"] \\
\bigotimes_{g\in G_J} (Sym^{m_g} E \otimes L_{\underline{\mathbf{m}}_g})^{\otimes c_g} \arrow[r] & Sym^m E \otimes L_{\underline{\mathbf{m}}}
\end{tikzcd}$$

the images of $\bigotimes_{g\in G_J} (R_g)^{\otimes c_g}$ and $R_{(u,m,\underline{\mathbf{m}})}$ in $Sym^m E \otimes L_{\underline{\mathbf{m}}}$ coincide. The injectivity of ev_{t_0} implies that $\bigotimes_{g\in G_J} (R_g)^{\otimes c_g}$ surjects onto $R_{(u,m,\underline{\mathbf{m}})}$, and this completes the proof. □

Corollary IV.7. *The projectivization* $\mathbf{P}(\mathcal{E})$ *of a rank two toric vector bundle* $\mathcal{E}$ *over the projective simplicial toric variety* X *is a Mori dream space.*

Proof. The finite generation of any Cox ring of $\mathbf{P}(\mathcal{E})$ in the sense of Hu and Keel is a consequence of Theorem IV.6. Any simplicial toric variety is $\mathbf{Q}$-factorial, and a projective bundle over a $\mathbf{Q}$-factorial variety is again $\mathbf{Q}$-factorial, hence the additional conditions follow at once from the hypotheses. □

In spite of having these several distinct approaches giving a positive answer to the finite generation of the Cox rings of projectivized rank two toric vector bundles (namely, the argument in [HS10], the proof based on the main result of [Kn93] as pointed out in [HS10], and our proofs in Theorem IV.2 and Theorem IV.6), they all failed to extend to the higher rank case. Now we understand that the situation for higher ranks is quite different. In the next chapter I present my joint work with Hering, Payne and Süß, that gives negative answers to the finite generation question for Cox rings and pseudoeffective cones of for projectivized toric vector bundles of higher rank.

CHAPTER V

Cox rings and pseudoeffective cones in higher ranks

In this chapter we give negative answers to the questions of finite generation of the Cox rings and pseudoeffective cones of projectivized toric vector bundles (see Question 7.2 in [HMP10]). All results included in this chapter were obtained in collaboration with M. Hering, S. Payne and H. Süß, see [GHPS10]. Our examples and counterexamples will arise by studying a particular family of toric vector bundles that includes cotangent bundles, up to a twist. This family will consist of those toric vector bundles whose Klyachko filtrations satisfy a condition that we will call $(\star)$ (see Definition V.1). Our motivation for this definition comes from two independent collections of results. One the one hand we are motivated by the results of Hausen and Süß in [HS10] that under suitable conditions provide a presentation of the Cox ring of a variety with a torus action as an algebra over the Cox ring of another variety closely related to a certain Chow quotient prevariety associated to the given action. For our work in this chapter the special case of their result presented in Proposition II.16 will suffice. On the other hand, we are motivated by results of Mukai in [Muk04], Castravet and Tevelev in [CT06], and Totaro in [Tot08], regarding the finite and nonfinite generation of the Cox rings and pseudoeffective cones of blow ups of projective space at finite collections of points. Our goal will be to prove that the projectivizations of toric vector bundles satisfying $(\star)$ contain a T-invariant open subset where the restricted action is free, and such that this restricted action has as geometric quotient equal to a projective space blown up at a finite collection of points. Moreover, we will be able to vary the collection of points being blown up on projective space by varying the initial data in the Klyachko filtrations of the toric vector bundle while still preserving the condition $(\star)$. Ultimately, using the results of Hausen and Süß, we will obtain a presentation of the Cox ring of our projectivized toric vector bundle as a polynomial ring over a blow up of projective space at finitely many points. Therefore we get a complete understanding of the Cox rings associated to projectivizations of toric vector bundles whose filtrations satisfy the condition $(\star)$, and in this way we obtain many new examples of both finite generation and nonfinite generation of these algebras. In addition, in some examples we will use the quotient map that we mentioned before to identify the Neron-Severi vector spaces of the projectivized toric vector bundle and the blow up of projective space, and we show that in those cases their pseudoeffective cones also get identified. Again, using the results of Mukai, Castravet-Tevelev, and Totaro that we mentioned before, we will also get examples of projectivized toric vector bundles where the pseudoeffective cone is not finitely generated. This second approach also gives a negative answer to the question about the finite generation of Cox rings, nonetheless the first approach additionally provides presentations of the Cox rings of many new varieties which is a fact of independent interest.

5.1 Preliminary Lemmas

In this Chapter we use the notation for toric varieties and toric vector bundles introduced in Section 2.2. Then, $X = X(\Delta)$ will denote an n-dimensional toric variety with d torus invariant divisors and $\mathcal{E}$ will denote a toric vector bundle of rank $r \geq 3$ over X.

Definition V.1. Let $\{\mathcal{E}^{\rho_j} \mid j = 1, \dots, d\}$ be a collection of decreasing filtrations of a vector space E, each of them indexed by the integer numbers. We say that the filtrations satisfy the condition $(\star)$ if they have the form

$$\mathcal{E}^{\rho_j}(i) = \begin{cases} E & \text{for } i \leq 0, \\ E_j & \text{for } i = 1, \\ 0 & \text{for } i > 1, \end{cases} \tag{5.1}$$

where each E_j is either 0 or a codimension one subspace of E, and all of the nonzero E_j are distinct.

Now we prove that filtrations of this form define a toric vector bundle, i.e. satisfy Klyachko's condition, exactly when for each maximal cone of the fan the hyperplanes that occur in the filtrations associated to its rays intersect transversely.

Lemma V.2. *Let $\{\mathcal{E}^{\rho_j} \mid j = 1, \dots, d\}$ be a collection of filtrations as in Definition V.1 that satisfy the condition $(\star)$. Then, over a smooth complete nondegenerate toric variety X these filtrations satisfy Klyachko's compatibility condition if and only if, for each maximal cone $\sigma \in \Delta$, the nonzero hyperplanes E_j such that $\rho_j \subseteq \sigma$ meet transversely.*

Proof. Let σ be a maximal cone in Δ. If the filtrations satisfy Klyachko's compatibility condition, then there is a splitting $E = G_1 \oplus \cdots \oplus G_r$ such that if $\rho_j \subseteq \sigma$ and E_j is nonzero, then E_j is the sum of $r-1$ of the G_i. Since the nonzero E_j are distinct by hypothesis, after renumbering all the rays we can assume that

$$E_j = G_1 \oplus \cdots \oplus \widehat{G}_j \oplus \cdots \oplus G_r.$$

For each nonzero E_j such that $\rho_j \subseteq \sigma$, and therefore they intersect transversely.

Conversely, let $\sigma \in \Delta$ be any maximal cone. Since the nonzero E_j such that $\rho_j \subseteq \sigma$ intersect transversely, after renumbering all rays in Δ if necessary, we can choose an ordered basis of E so that the one-dimensional coordinate subspaces $L_1, \ldots, L_r$ satisfy $E_j = L_1 \oplus \cdots \oplus \widehat{L}_j \oplus \cdots \oplus L_r$, for each nonzero E_j such that $\rho_j \subseteq \sigma$. Since $X(\Delta)$ is smooth, we can choose $u_1, \ldots, u_r \in M$, so that for each $\rho_j \subseteq \sigma$, the product $\langle u_i, v_j \rangle$ is equal to one if E_j is nonzero and $i \neq j$, or equal zero otherwise. Then the characters $u_1, \ldots, u_r$ and the decomposition $E = L_1 \oplus \cdots \oplus L_r$ satisfy Klyachko's compatibility condition. $\square$

Remark V.3. Since at most r hyperplanes can meet transversely in a vector space of dimension r, for a collection of filtrations satisfying $(\star)$ to also satisfy Klyachko compatibility condition over a smooth nondegenerate toric variety $X(\Delta)$, it is necessary that for each cone in the fan Δ at most r of the E_j associated to the rays of the cone can be different from zero. In particular, if all the E_j are nonzero, in order for the filtrations to define a toric vector bundle over $X(\Delta)$, it is necessary that $r \geq n$. In this same setting, if the hyperplanes E_j are chosen in general position the condition $r \geq n$ is also sufficient.

Lemma V.4. *Let X be an arbitrary toric variety and let $\mathcal{E}$ be a toric vector bundle over X satisfying $(\star)$, or more generally satisfying that* $\mathrm{ev}_{t_0} : H^0(X, \mathcal{E})_0 \to E$ *is surjective. Then the isomorphism* $\mathrm{ev}_{t_0}^{-1}$ *canonically induces a rational map $\psi \colon \mathbf{P}(\mathcal{E}) \dashrightarrow \mathbf{P}(E)$. Moreover ψ is constant on T-orbits and it is a retraction of the inclusion $\mathbf{P}(E) \hookrightarrow \mathbf{P}(\mathcal{E}|_{U_\rho})$.*

Proof. For each T-invariant affine open subset U of X, the evaluation map ev_{t_0} gives an isomorphism $\phi_U = \mathrm{ev}_{t_0} \colon H^0(U, \mathcal{E})_0 \to E$. For each such U, the composition of

ϕ_U^{-1} with the inclusion $H^0(X,\mathcal{E})_0 \hookrightarrow H^0(X,\mathcal{E})$, induces a rational map

$$\psi_U\colon \mathbf{P}(\mathcal{E}|_U) = \operatorname{Proj}_U \operatorname{Sym}\mathcal{E}|_U \dashrightarrow \operatorname{Proj}\operatorname{Sym} E = \mathbf{P}(E).$$

The rational maps $\psi_\rho := \psi_{U_\rho}$ can be glued to a rational map $\psi\colon \mathbf{P}(\mathcal{E}) \dashrightarrow \mathbf{P}(E)$, since they agree with ψ_T over some nonempty open subset. Fix a ray ρ and fix an equivariant trivialization $\mathcal{E}|_{U_\rho} \cong \mathcal{O}_X(\operatorname{div}\chi^{u_1})|_{U_\rho} \oplus \cdots \oplus \mathcal{O}_X(\operatorname{div}\chi^{u_r})|_{U_\rho}$ for some $u_1,\ldots,u_r \in M$. We have

$$\begin{aligned} H^0(U_\rho,\mathcal{E}) &= \chi^{-u_1}\, k[\rho^\vee \cap M] \oplus \cdots \oplus \chi^{-u_r}\, k[\rho^\vee \cap M] \\ &= x_1\, k[\rho^\vee \cap M] \oplus \cdots \oplus x_r\, k[\rho^\vee \cap M] \end{aligned}$$

where $x_i = (0,\ldots,\chi^{-u_i},\ldots,0) \in H^0(U_\rho,\mathcal{E})$ for each $i = 1,\ldots,r$. The local trivialization induces isomorphisms $\mathbf{P}(\mathcal{E}|_{U_\rho}) = \operatorname{Proj} k[\,\rho^\vee \cap M\,][x_1,\ldots,x_r]$ and $\mathbf{P}(E) = \operatorname{Proj} k[y_1,\ldots,y_r]$, and in this coordinates ψ_ρ corresponds to the algebra homomorphism

$$\begin{array}{cccc} \psi_\rho^*\colon & k[y_1,\ldots,y_r] & \longrightarrow & k[\,\rho^\vee \cap M\,][x_1,\ldots,x_r] \\ & y_i & \longmapsto & \chi^{u_i} x_i. \end{array} \tag{5.2}$$

This description implies that ψ_ρ is constant on T-orbits since each $\chi^{u_i}x_i$ is T-invariant, and that ψ_ρ is a retraction of the inclusion $\mathbf{P}(E) \hookrightarrow \mathbf{P}(\mathcal{E}|_{U_\rho})$, as this last map corresponds to the algebra homomorphism

$$\begin{array}{ccccl} \psi_\rho^*\colon & k[\,\rho^\vee \cap M\,][x_1,\ldots,x_r] & \longrightarrow & k[y_1,\ldots,y_r] & \\ & \chi^w & \longmapsto & 1 & \text{for all } w \in \rho^\vee \cap M, \\ & x_i & \longmapsto & y_i. & \end{array}$$

□

Notation V.5. When we have a toric vector bundle $\mathcal{E}$ associated to some filtrations satisfying $(\star)$ over the toric variety $X = X(\Delta)$, we will renumber the rays $\rho_1,\ldots,\rho_d$ of

Δ so that the E_j are distinct hyperplanes for $1 \leq j \leq s$ and are zero for $s+1 \leq j \leq d$. For each $1 \leq j \leq s$ the hyperplane E_j of E corresponds to a point in the projective space $\mathbf{P}(E)$ that we will denote by p_j. We will denote by S the set whose elements are the distinct points $p_1, \ldots, p_s$. In addition, for each $1 \leq j \leq s$ we will denote by F_j each of the exceptional divisors corresponding to p_j in the blow ups $\mathrm{Bl}_{p_j} \mathbf{P}(E)$ and $\mathrm{Bl}_S \mathbf{P}(E)$.

To simplify the next part of our discussion, we assume that the fan Δ of the toric variety X has rays as maximal cones. This does not change the Cox ring of X and under the assumption of smoothness it will preserve the Picard group and the semigroup of effective divisors. Now let us consider a toric vector bundle $\mathcal{E}$ over X satisfying $(\star)$. For each ray $\rho \in \Delta$ such that the $\mathcal{E}^\rho(1) = E_\rho$ is nonzero, fix an equivariant trivialization $\mathcal{E}|_{U_\rho} \cong \mathcal{O}_X(\operatorname{div} \chi^{u_1})|_{U_\rho} \oplus \cdots \oplus \mathcal{O}_X(\operatorname{div} \chi^{u_r})|_{U_\rho}$ for some $u_1, \ldots, u_r \in M$. Since $\mathcal{E}$ satisfies $(\star)$, we can reorder $u_1, \ldots, u_r$ and assume that $\langle u_r, v_\rho \rangle = 0$ and

$$\langle u_1, v_\rho \rangle = \cdots = \langle u_{r-1}, v_\rho \rangle = 1.$$

Let us define the subvariety

$$Z_\rho = \mathbf{P}\left((\mathcal{O}_X(\operatorname{div} \chi^{u_1}) \oplus \cdots \oplus \mathcal{O}_X(\operatorname{div} \chi^{u_{r-1}}))|_{U_\rho \cap D_\rho} \right) \subseteq \mathbf{P}(\mathcal{E}|_{U_\rho \cap D_\rho}) \subseteq \mathbf{P}(\mathcal{E}|_{U_\rho}).$$

Form our work it will follow that Z_ρ is independent of the choice of equivariant trivialization. Also, if $p_\rho \in \mathbf{P}(E)$ is the point corresponding to the hyperplane $\mathcal{E}^\rho(1) = E_\rho$ of E, we define $W_\rho = \overline{T \cdot p_\rho} \subseteq \mathbf{P}(\mathcal{E})$, that is, W_ρ is the closure of the T-orbit of $p_\rho \in \mathbf{P}(E) \subseteq \mathbf{P}(\mathcal{E})$. Finally, we define $Z = \bigcup_\rho Z_\rho \subseteq \mathbf{P}(\mathcal{E})$ and $W = \bigcup_\rho W_\rho \subseteq \mathbf{P}(\mathcal{E})$. Note that the closed subset $Z \cup W$ has codimension at least two in $\mathbf{P}(\mathcal{E})$. The motivation behind these definitions is that Z is the indeterminacy locus of the map ψ in the Lemma V.4 and after we remove W the inverse image of the ideal sheaf of $\{p_\rho \mid \mathcal{E}^\rho(1) \neq 0\}$ is locally principal.

Lemma V.6. *Let X be a toric variety corresponding to a fan whose maximal cones are rays and let $\mathcal{E}$ be a toric vector bundle over X satisfying ($\star$). Then the rational map ψ from Lemma V.4 is a morphism on the open subset $\mathbf{P}(\mathcal{E}) \smallsetminus Z$ and there exists a unique surjective morphism $\theta : \mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W) \to \mathrm{Bl}_S \mathbf{P}(E)$ that factors the restriction of ψ to $\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W)$ through the blow up $\mathrm{Bl}_S \mathbf{P}(E)$.*

Proof. Let ρ be a ray in Δ and let us consider the equivariant trivialization of $\mathcal{E}|_{U_\rho}$ that we fixed earlier, and local coordinates $\mathbf{P}(\mathcal{E}|_{U_\rho}) = \operatorname{Proj} k[\,\rho^\vee \cap M\,][x_1, \ldots, x_r]$ and $\mathbf{P}(E) = \operatorname{Proj} k[y_1, \ldots, y_r]$ as in Lemma V.4. Since $\mathcal{E}$ satisfies ($\star$) the subspace $\mathcal{E}^\rho(1) = E_\rho$ in filtration $\mathcal{E}^\rho$ is either zero or a hyperplane of E, and we will now consider these two cases separately. In the first case, we have that $\chi^{u_1}, \ldots, \chi^{u_r}$ are units in $k[\,\rho^\vee \cap M\,]$ and that $\mathbf{P}(\mathcal{E}|_{U_\rho})$ is T-equivariantly isomorphic to $U_\rho \times \mathbf{P}(E) = \operatorname{Proj} k[\,\rho^\vee \cap M\,][y_1, \ldots, y_r]$ with ψ_ρ corresponding to the projection. In particular, ψ is a morphism over $\mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus Z_\rho = \mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus Z$. We define $B_\rho = \mathbf{P}(E) \smallsetminus S \subseteq \mathrm{Bl}_S \mathbf{P}(E)$. We have that $\psi_\rho^{-1}(B_\rho) = \mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus (Z \cup W)$. In this case, it is clear that $\psi| : \mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus (Z \cup W) \to B_\rho$ factors through the blow up $\mathrm{Bl}_S \mathbf{P}(E)$ via a unique morphism that we call $\theta_\rho : \mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus (Z \cup W) \to B_\rho$. We note that θ_ρ is surjective and that $\theta_\rho^{-1}(B_\rho) = \mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus (Z \cup W)$. In the second case, i.e. when the filtration $\mathcal{E}^\rho$ contains a hyperplane, let us again choose coordinates $\mathbf{P}(\mathcal{E}|_{U_\rho}) = \operatorname{Proj} k[\,\rho^\vee \cap M\,][x_1, \ldots, x_r]$ and $\mathbf{P}(E) = \operatorname{Proj} k[y_1, \ldots, y_r]$ as in Lemma V.4. We will denote $W_\rho \cap \mathbf{P}(\mathcal{E}|_{U\rho})$ by W'_ρ. In these coordinates $Z_\rho = V(\chi^{u_1} x_1, \ldots, \chi^{u_r} x_r)$ and $W'_\rho = V(x_1, \ldots, x_{r-1})$. Since $\mathcal{E}$ satisfies the ($\star$) condition, it follows that $Z_\rho = V(\chi^{u_1}, x_r)$. The rational map ψ_ρ is surjective, and from (5.2) its domain of definition is $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus Z_\rho = \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus Z$. Then ψ is a morphism over $\mathbf{P}(\mathcal{E}) \smallsetminus Z$. Again from (5.2), the inverse image under ψ_ρ of the ideal sheaf of $p_\rho \in \mathbf{P}(E)$ is locally principal on $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W'_\rho)$, therefore by the universal property of blow ups there exists a unique morphism $\theta_\rho : \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W'_\rho) \to \mathrm{Bl}_{p_\rho} \mathbf{P}(E)$ such that $\pi_\rho \circ \theta_\rho = \psi_\rho|_{\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W'_\rho)}$, where $\pi_\rho : \mathrm{Bl}_{p_\rho} \mathbf{P}(E) \to \mathbf{P}(E)$ is the blow up morphism. Let $B_\rho = \mathrm{Bl}_{p_\rho} \mathbf{P}(E) \smallsetminus \{p_{\rho'} \mid \rho' \text{ ray of } \Delta,\ \rho' \neq \rho \text{ and } E_{\rho'} \neq 0\}$. Note that $\theta_\rho^{-1}(p_\rho) = \pi^{-1}(D_\rho) \cap (\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho))$, and also that for each ray $\rho' \in \Delta$, different from ρ, and such that $E_{\rho'}$ is nonzero we have

$$\theta_\rho^{-1}(p_{\rho'}) = W_{\rho'} \cap (\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)) = \overline{T \cdot p_{\rho'}} \cap (\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)).$$

Therefore $\theta_\rho^{-1}(B_\rho) = \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z \cup W)$. Now we consider the maps $\{\theta_\rho|_{\theta_\rho^{-1}(B_\rho)}\}$ as rational maps from $\mathbf{P}(\mathcal{E})$ to $\mathrm{Bl}_S \mathbf{P}(E)$. These rational maps agree on the common open subset $\pi^{-1}(T)$, since they clearly agree after composing them with the blow up map $\mathrm{Bl}_S \mathbf{P}(E) \to \mathbf{P}(E)$. Therefore they can be glued to a surjective morphism $\theta : \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z \cup W) \to \mathrm{Bl}_S \mathbf{P}(E)$ that factors the restriction of ψ through $\mathrm{Bl}_S \mathbf{P}(E)$ as required. $\square$

In the next lemma we collect some properties of the morphism $\theta : \mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W) \to \mathrm{Bl}_S \mathbf{P}(E)$ from Lemma V.6 that follow directly from its construction. This lemma will be used in Section 5.3 to relate the pseudoeffective cones of $\mathbf{P}(\mathcal{E})$ and $\mathrm{Bl}_S \mathbf{P}(E)$.

Lemma V.7. *The morphism $\theta : \mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W) \to \mathrm{Bl}_S \mathbf{P}(E)$ from Lemma V.6 satisfies the following properties:*

(a) $\theta^*(F_i) = \pi^{-1}(D_i) \cap (\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W))$, for each $1 \leq i \leq s$.

(b) $\theta^* p^*(H) = \mathcal{O}(1)$, where H is the hyperplane class in $\mathbf{P}(E)$.

(c) $\theta^*(\widetilde{C}) = \overline{T \cdot C}$, for any prime divisor C on $\mathbf{P}(E) = \mathbf{P}(\mathcal{E}|_{t_0}) \subseteq \mathbf{P}(\mathcal{E})$ with strict transform $\widetilde{C}$ on $\mathrm{Bl}_S \mathbf{P}(E)$.

Proof. Property (a) follows from the construction of θ, as we have seen that the inverse image under $\psi|$ of the ideal sheaf of the point $p_i \in \mathbf{P}(E)$ is the ideal sheaf of $\pi^{-1}(D_i) \cap (\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W))$ in $\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W)$. Property (b) follows from the functoriality of the Proj functor. Let C and $\widetilde{C}$ be as in (c). For each $1 \leq i \leq s$, since the divisor $\pi^{-1}(D_i) \cap (\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W))$ maps onto F_i, then $\pi^{-1}(D_i) \cap (\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W))$ has coefficient zero in $\theta^*(\widetilde{C})$. Since $\theta^*(\widetilde{C}|_{\pi^{-1}T}) = \psi^*(C|_{\pi^{-1}T})$, we just need to show that $\psi_T^*(C) = T \cdot C$. As in the construction of the map ψ in Lemma V.4, choose an equivariant trivialization $\mathcal{E}|_T \cong \mathcal{O}_X(\operatorname{div} \chi^{u_1})|_T \oplus \cdots \oplus \mathcal{O}_X(\operatorname{div} \chi^{u_r})|_T$ for some $u_1, \ldots, u_r \in M$. We have that $\chi^{u_1}, \ldots, \chi^{u_r}$ are units in $k[\,M\,]$ and it follows that $\mathbf{P}(\mathcal{E}|_T)$ is T-equivariantly isomorphic to $T \times \mathbf{P}(E)$ with ψ_T corresponding to the projection. From this the conclusion of (c) follows. $\square$

Let X be a toric variety corresponding to a fan whose maximal cones are rays and let $\mathcal{E}$ be a toric vector bundle over X satisfying $(\star)$. Let U be the T-invariant subset of $\mathbf{P}(\mathcal{E})$ obtained by removing the proper closed subsets Z, W and the preimages under π of the divisors $D_{s+1}, \ldots, D_d$ of X, i.e.

$$U = \mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W \cup \bigcup_{s+1 \leq i \leq d} \pi^{-1}(D_i))$$

Now we achieve our first goal by proving that the action of T on U is free and that a projective space blown up at a finite collection of points is a geometric quotient for this restricted action.

Lemma V.8 (Geometric Quotient)**.** *The action of T on U is free and the morphism $\theta|_U : U \to \operatorname{Bl}_S \mathbf{P}(E)$ is a geometric quotient for this action.*

Proof. Let ρ be one of the rays $\rho_i \in \Delta$, for some $1 \leq i \leq s$. We continue using the notion introduced in the proofs of the Lemmas V.4 and V.6. We claim that the map θ_ρ is the geometric quotient of the action of T on $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)$. To see this we first show that θ_ρ is a toric map. The variety $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)$ admits a toric variety structure given as follows. Let $e_1, \ldots, e_{r-1}$ be the canonical basis of $\mathbf{R}^{r-1}$, and let $e_r = -e_1 - \cdots - e_{r-1}$ and $\widetilde{v_\rho} = (v_\rho, e_1 + \cdots + e_{r-1}) \in N \times \mathbf{R}^{r-1}$. Then $\mathbf{P}(\mathcal{E}|_{U\rho})$ is isomorphic to the toric variety associated to the fan in $N \times \mathbf{R}^{r-1}$ whose maximal cones are the cones spanned by $\widetilde{v_\rho}$ together with any $r-1$ of the vectors $(0, e_1), \ldots, (0, e_r)$ (see §7 in [Oda78]). Note that both Z_ρ and W_ρ are invariant subvarieties for the toric variety structure of $\mathbf{P}(\mathcal{E}|_{U\rho})$. The fan Σ_ρ for the toric variety $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)$ is the fan in $N \times \mathbf{R}^{r-1}$ obtained from the fan of $\mathbf{P}(\mathcal{E}|_{U\rho})$ by removing the cones containing either both of the vectors $\widetilde{v_\rho}$ and $(0, e_r)$ or all of the vectors $(0, e_1), \ldots, (0, e_{r-1})$. Let Λ_ρ be the complete fan in $\mathbf{R}^{r-1}$ whose maximal cones are those cones spanned by any $r-1$ of the vectors $e_1, \ldots, e_r$. The toric variety associated to the fan Λ_ρ is isomorphic to $\mathbf{P}^{r-1}$. We get a toric structure on $\mathbf{P}(E) = \operatorname{Proj} k[y_1, \ldots, y_r]$ by identifying it with this toric variety via the unique isomorphism between them that extends the isomorphism $\operatorname{Spec} k[\frac{y_1}{y_r}, \ldots, \frac{y_{r-1}}{y_r}] \to \operatorname{Spec} k[\chi^{e_1^*}, \ldots, \chi^{e_{r-1}^*}]$, induced by the unique algebra map satisfying $\chi^{e_i^*} \mapsto \frac{y_i}{y_r}$ for each $i = 1, \ldots, r-1$. In this way, since the ideal of p_ρ in this coordinates is $(y_1, \ldots, y_{r-1})$, the variety $\operatorname{Bl}_{p_\rho} \mathbf{P}(E)$ is identified with the toric variety associated to the fan Λ'_ρ in $\mathbf{R}^{r-1}$ obtained from Λ_ρ by the star subdivision corresponding to the vector $(1, \ldots, 1)$. It is now easy to see that the action of T on $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)$ is free and that the map θ_ρ is the toric map $\mathbf{P}(\mathcal{E}|_{U_\rho}) \smallsetminus (Z_\rho \cup W_\rho) = X(\Sigma_\rho) \to X(\Lambda'_\rho) = \operatorname{Bl}_{p_\rho} \mathbf{P}(E)$ induced by the projection $N \times \mathbf{R}^{r-1} \to \mathbf{R}^{r-1}$. By Proposition 3.2 in [AHN99] this map is the geometric quotient of the action of the torus T on $\mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z_\rho \cup W_\rho)$. Consider now the open set B_ρ of $\operatorname{Bl}_S \mathbf{P}(E)$ defined in Lemma V.6 and satisfying that $\theta_\rho^{-1}(B_\rho) = \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z \cup W)$. Therefore we get a geometric quotient

$$\theta| = \theta_\rho| : \mathbf{P}(\mathcal{E}|_{U\rho}) \smallsetminus (Z \cup W) \longrightarrow B_\rho.$$

We note that for $\rho \in \{\rho_1, \dots, \rho_s\}$, the open subsets B_ρ cover the variety $\mathrm{Bl}_S \mathbf{P}(E)$ when put together, and the open subsets $\mathbf{P}(\mathcal{E}|_{U\rho})$ cover U when put together. Since the freeness of an action is a local property and the property of a map being a geometric quotient is local on the base, the desired conclusions follow. $\square$

5.2 Cox rings of projectivized toric vector bundles

In this section we use our lemmas from Section 5.1 to construct examples of toric vector bundles of higher rank whose projectivizations have finitely and nonfinitely generated Cox rings. We start with a result that gives presentations for the Cox rings of projectivized toric vector bundles that satisfy the condition ($\star$).

Theorem V.9. *Let $\mathcal{E}$ be a toric vector bundle over the smooth toric variety X satisfying the condition ($\star$). Then the Cox ring of $\mathbf{P}(\mathcal{E})$ is isomorphic to a polynomial ring in $d - s$ variables over the Cox ring of $\mathrm{Bl}_S \mathbf{P}(E)$.*

Proof. Codimension two modifications to a smooth variety do not change its Cox ring. Thus, we can assume that the fan of X has rays as its maximal cones. Likewise, we can canonically identify the Cox rings of $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W)$, for the closed sets Z and W that we introduced before. For each $s + 1 \leq i \leq d$ the divisor $\pi^{-1}(D_i)$ of $\mathbf{P}(\mathcal{E})$ is stabilized pointwise by the one parameter subgroup of T-corresponding to $v_j \in N$. By Lemma V.8 the action of T on $U = \mathbf{P}(\mathcal{E}) \smallsetminus (Z \cup W \cup \pi^{-1}(D_{s+1}) \cup \cdots \cup \pi^{-1}(D_d))$ is free and there exists a geometric quotient morphism $\theta| : U \to \mathrm{Bl}_S \mathbf{P}(E)$. The desired relation between the Cox rings now follows from Proposition II.16. $\square$

The following theorem gives a negative answer to the question of finite generation for the Cox rings of projectivized toric vector bundles (see Question 7.2 in [HMP10]).

Theorem V.10. *Suppose that k is uncountable, $d > r \geq n$, and $\frac{1}{r} + \frac{1}{d-r} \leq \frac{1}{2}$. Then* *there is a nonsplit toric vector bundle $\mathcal{E}$ of rank r on $X(\Delta)$ such that the Cox ring of the projectivization $\mathbf{P}(\mathcal{E})$ is not finitely generated.*

Proof. We can take $\mathcal{E}$ as in Theorem V.9 such that the Cox ring of $\mathrm{Bl}_S \mathbf{P}(E)$ is not finitely generated [Muk04], and then the conclusion follows from that theorem. □

Remark V.11. As pointed out by Hering, Mustaţă and Payne in [HMP10], the finite generation of the Cox rings of all projectivized toric vector bundles is equivalent to the finite generation of the section rings of the canonical line bundles $\mathcal{O}(1)$ for all projectivized toric vector bundles (see Question 7.1 and Remark 7.3 in [HMP10]). Therefore, Theorem V.10 also implies the existence of a toric vector bundle $\mathcal{E}$ so that the section ring $\bigoplus_{m\in\mathbf{Z}_{\geq 0}} H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}(m)) \cong \bigoplus_{m\in\mathbf{Z}_{\geq 0}} H^0(X, \mathrm{Sym}^m \mathcal{E})$ of the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(\mathcal{E})$ is not finitely generated. If $\mathcal{F}$ is a toric vector bundle and $\mathcal{L}_1, \ldots, \mathcal{L}_h$ are line bundles on X that generate $\mathrm{Pic}(X)$ as a group, then the finite generation of the Cox ring of $\mathbf{P}(\mathcal{F})$ is equivalent to the finite generation of the section ring of the line bundle $\mathcal{O}(1)$ on the projectivization of the toric vector bundle $\mathcal{E} = \mathcal{F} \oplus \mathcal{L}_1 \cdots \oplus \mathcal{L}_h$. By choosing $\mathcal{F}$ such that the Cox ring of its projectivization is not finitely generated we get the desired example.

Remark V.12. We notice that the Theorem V.10 holds when the Klyachko filtrations of the toric vector bundle contain sufficiently many nontrivial subspaces in very general position. In the following corollary to Theorem V.9, we show that Theorem V.10 is sharp in the sense that if $\frac{1}{r} + \frac{1}{d-r} > \frac{1}{2}$ and the nontrivial subspaces are in general position, then the projectivization of any bundle of this form is a Mori dream space.

Corollary V.13. *Suppose $\mathcal{E}$ is given by filtrations satisfying ($\star$) with the hyperplanes E_i in general position. If $\frac{1}{r} + \frac{1}{d-r} > \frac{1}{2}$ then $\mathbf{P}(\mathcal{E})$ is a Mori dream space.*

Proof. Suppose $\frac{1}{r} + \frac{1}{d-r} > \frac{1}{2}$. Then the blow up of $\mathbf{P}^{r-1}$ at d points in general position is a Mori dream space [CT06, Theorem 1.3], and then so is the blow up $\mathrm{Bl}_s \mathbf{P}^{r-1}$ of $\mathbf{P}^{r-1}$ at s points in general position, where s is the number of rays ρ_j such that E_j is nonzero. The corollary then follows immediately from Theorem V.9, which says that the Cox ring of $\mathbf{P}(\mathcal{E})$ is finitely generated over the Cox ring of $\mathrm{Bl}_s \mathbf{P}^{r-1}$. □

5.3 Pseudoeffective cones of projectivized toric vector bundles

In this section we use our lemmas from Section 5.1 to construct examples of toric vector bundles whose projectivizations have finitely and nonfinitely generated Cox rings and pseudoeffective cones. We start with a result that relates the effective cone of a projectivized toric vector bundles that satisfies the condition $(\star)$ to the effective cone of the associated blow up of projective space that we have been considering.

Let $X = X(\Delta)$ be a smooth toric variety and let $\mathcal{E}$ be a toric vector bundle over X that satisfies the condition $(\star)$. In Section 5.1 we constructed a canonical rational map $\psi : \mathbf{P}(\mathcal{E}) \dashrightarrow \mathbf{P}(E)$, and proved the existence of a unique rational map $\theta : \mathbf{P}(\mathcal{E}) \dashrightarrow \mathrm{Bl}_S \mathbf{P}(E)$ that factors ψ through $\mathrm{Bl}_S \mathbf{P}(E)$. The map θ is a morphism on the complement of a closed set of codimension at least two, and it is surjective as a map from its domain to $\mathrm{Bl}_S \mathbf{P}(E)$. Therefore, via pullback, θ induces a group homomorphism

$$\theta^* : \mathrm{Pic}(\mathrm{Bl}_S \mathbf{P}(E)) \to \mathrm{Pic}(\mathrm{Domain}(\theta)) = \mathrm{Pic}(\mathbf{P}(\mathcal{E})).$$

Theorem V.14. *Let $X = X(\Delta)$ be a smooth toric variety and let $\mathcal{E}$ be a toric vector bundle over X that satisfies the condition $(\star)$. Then the effective cone of $\mathbf{P}(\mathcal{E})$ is generated by the image under θ^* of the effective cone of $\mathrm{Bl}_S \mathbf{P}(E)$ together with the classes of the divisors $\pi^{-1}(D_i)$ such that E_i is zero, i.e.*

$$\mathrm{Eff}(\mathbf{P}(\mathcal{E})) = \theta^*(\mathrm{Eff}\,\mathrm{Bl}_S \mathbf{P}(E)) + \sum_{s+1 \le i \le d} \mathbf{Z}_{\ge 0} \cdot \pi^{-1}(D_i)$$

Proof. We can assume that the maximal cones of the fan Δ are rays. Since the map θ is surjective it pulls back effective divisors to effective divisors, thus

$$\theta^*(\operatorname{Eff}\operatorname{Bl}_S \mathbf{P}(E)) + \sum_{s+1\leq i\leq d} \mathbf{Z}\cdot \pi^{-1}(D_i) \subseteq \operatorname{Eff}(\mathbf{P}(\mathcal{E})).$$

By Lemma II.18, the cone $\operatorname{Eff}(\mathbf{P}(\mathcal{E}))$ is generated by $\pi^{-1}(D_1),\ldots,\pi^{-1}(D_d)$ and by the closures of the T-orbits of prime divisors of $\mathbf{P}(E)$ considered as subvarieties of of $\mathbf{P}(\mathcal{E})$ via the identification $\mathbf{P}(E) = \mathbf{P}(\mathcal{E}|_{t_0})$. Therefore, parts (a) and (b) of Lemma V.7 imply that

$$\operatorname{Eff}(\mathbf{P}(\mathcal{E})) \subseteq \theta^*(\operatorname{Eff}\operatorname{Bl}_S \mathbf{P}(E)) + \sum_{s+1\leq i\leq d} \mathbf{Z}\cdot \pi^{-1}(D_i),$$

and the result follows. $\square$

In the next theorem we construct examples of projectivized toric vector bundles whose pseudoeffective cone and Cox ring are not finitely generated. We show that such toric vector bundles exist over any smooth toric variety whose fan has sufficiently many rays and has a cone containing each ray or its negative. For instance, one can construct smooth projective toric varieties whose fan has arbitrarily many rays and satisfy this condition on cones through sequences of iterated blowups of $(\mathbf{P}^1)^n$.

Theorem V.15. *Suppose that k is uncountable, $d-n > r \geq n$, and $\frac{1}{r} + \frac{1}{d-n-r} \leq \frac{1}{2}$, and assume that there is some cone $\sigma \in \Delta$ such that every ray of Δ is contained in either σ or $-\sigma$. Then there is a nonsplit toric vector bundle $\mathcal{E}$ of rank r on $X(\Delta)$ such that the pseudoeffective cone and the Cox ring of $\mathbf{P}(\mathcal{E})$ are not finitely generated.*

Proof. We can assume that the cone σ is maximal and that it is spanned by the rays $\rho_{d-n+1},\ldots,\rho_d$. Choose a collection of filtrations satisfying $(\star)$, such that $E_{d-n+1},\ldots,$ E_d are zero and such that $E_1,\ldots,E_{d-n}$ are hyperplanes on E. Moreover, we choose the hyperplanes $E_1,\ldots,E_{d-n}$ in very general position, so that we can assume that these filtrations satisfy the Klyachko compatibility condition (see Lemma V.2), and that the blow up $\mathrm{Bl}_S \mathbf{P}(E)$ of $\mathbf{P}(E)$ at the collection of points $S = p_1, \ldots, p_{d-n}$ has a nonfinitely generated pseudoeffective cone (see Theorem II.14). The Neron-Severi space $N^1(\mathrm{Bl}_S \mathbf{P}(E))_{\mathbf{R}} = \mathrm{Pic}(\mathrm{Bl}_S \mathbf{P}(E))_{\mathbf{R}}$ has a basis given by the pullback of the hyperplane class in $\mathbf{P}(E)$ and the classes $F_1, \ldots, F_{d-n}$ of the exceptional divisors, and the Neron-Severi space $N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}} = \mathrm{Pic}(\mathrm{Bl}_S \mathbf{P}(\mathcal{E}))_{\mathbf{R}}$ has a basis given by the class of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and the classes of the divisors $\pi^{-1}(D_1), \ldots, \pi^{-1}(D_{d-n})$. By Lemma V.7, it follows that the map $\theta^* : N^1(\mathrm{Bl}_S \mathbf{P}(E))_{\mathbf{R}} \to N^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}}$ induced by the map θ^* from Theorem V.14 is an isomorphism. Moreover, the form of the fan Δ ensures that each divisor $D_{d-n+1}, \ldots, D_d$ is an effective combination of the divisors $D_1, \ldots, D_{d-n}$, and hence that each divisor $\pi^{-1}(D_{d-n+1}), \ldots, \pi^{-1}(D_d)$ is an effective combination of the divisors $\pi^{-1}(D_1), \ldots, \pi^{-1}(D_{d-n})$. Theorem V.14 together with property (a) from Lemma V.7, imply that

$$\begin{aligned}
\mathrm{Eff}(\mathbf{P}(\mathcal{E})) &= \theta^*(\mathrm{Eff}\,\mathrm{Bl}_S \mathbf{P}(E)) + \sum_{d-n+1 \leq i \leq d} \mathbf{Z} \cdot \pi^{-1}(D_i) \\
&\subseteq \theta^*(\mathrm{Eff}\,\mathrm{Bl}_S \mathbf{P}(E)) + \sum_{1 \leq i \leq d-n} \mathbf{Z} \cdot \pi^{-1}(D_i) \\
&\subseteq \theta^*(\mathrm{Eff}\,\mathrm{Bl}_S \mathbf{P}(E)) \subseteq \mathrm{Eff}(\mathbf{P}(\mathcal{E})).
\end{aligned}$$

Hence, the isomorphism θ^* identifies both the semigroup of effective divisors and the pseudoeffective cones of $\mathrm{Bl}_S \mathbf{P}(E)$ and $\mathbf{P}(\mathcal{E})$. It follows that the pseudoeffective cone and Cox ring of $\mathbf{P}(E)$ are not finitely generated as desired. $\square$

The constructions in our proofs of Theorem V.10 and Theorem V.15 involve the choice of toric vector bundles that are very general in their moduli spaces, which in principle leaves the finite generation questions open over more general fields and for toric vector bundles that are determined by the combinatorial data of the fan. In this direction, we know that the projectivization of the tangent bundle has a finitely generated Cox ring as it was proved in [HS10, Theorem 5.8]. Nonetheless, cotangent bundles behave quite differently. In the final result of this dissertation, we show that even over more general fields there are smooth projective toric varieties whose projectivized cotangent bundle is not a Mori dream space.

Theorem V.16. *Suppose that $d \geq 3$ and the characteristic of k is not two or three. Then there exists a smooth projective toric variety $X(\Delta)$ of dimension d over k such that the Cox ring of the projectivized cotangent bundle on $X(\Delta)$ is not finitely generated.*

Proof of Theorem V.16. We construct the required examples inductively, starting in dimension $d = 3$. The vectors $v_1 = (0,0,1)$, $v_2 = (0,1,0)$, $v_3 = (1,1,1)$ and $v_4 = (-1,-2,-2)$ span the rays of a unique smooth projective fan in $\mathbf{R}^3$. We sequentially perform the star subdivisions of fans, associated to the rays spanned by $v_5 = v_1 + v_3 = (1,1,2)$, $v_6 = v_1 + v_4 + v_5 = (0,-1,1)$, $v_7 = v_3 + v_4 + v_5 = (1,0,1)$, $v_8 = v_4 + v_5 + v_7 = (1,-1,1)$, $v_9 = v_1 + v_4 = (-1,-2,-1)$, $v_{10} = v_1 + v_2 + v_9 = (-1,-1,0)$, $v_{11} = v_1 + v_{10} = (-1,-1,1)$, $v_{12} = v_2 + v_{11} = (-1,0,1)$, $v_{13} = v_2 + v_{12} = v_2 + v_{12} = (-1,1,1)$ and $v_{14} = v_1 + v_2 = (0,1,1)$, which correspond to successive blow ups along smooth invariant subvarieties. Then, the vectors $v_1, \ldots, v_{14}$ span the rays of a smooth projective fan Δ in $\mathbf{R}^3$, which clearly does not contain any pair of opposite rays. Let $X(\Delta)$ be the corresponding toric variety over the given field k. Let $\mathcal{E}$ be the tensor product of the cotangent bundle $\Omega_{X(\Delta)}$ and the anticanonical class $-K_{X(\Delta)}$ of $X(\Delta)$. The filtrations corresponding to $\mathcal{E}$ are given by

$$\mathcal{E}^\rho(j) = \begin{cases} M \otimes_{\mathbf{Z}} k & \text{for } j \leq 0, \\ \rho^\perp & \text{for } j = 1, \\ 0 & \text{for } j > 1, \end{cases}$$

for each ray $\rho \in \Delta$. Since Δ contains no pair of opposite rays, the hyperplanes $\rho^\perp$ are distinct, and this collection of filtrations satisfies $(\star)$. By Theorem V.9, the Cox ring of $\mathbf{P}(\Omega_{X(\Delta)}) \cong \mathbf{P}(\mathcal{E})$ is isomorphic to the Cox ring of $\mathbf{P}^2 = \mathbf{P}(M \otimes_{\mathbf{Z}} k)$ blown up at the collection of points corresponding to the collection $\{\rho^\perp \mid \rho \text{ ray of } \Delta\}$ of hyperplanes of $M \otimes_{\mathbf{Z}} k$. By Theorem II.15, the blow up of $\mathbf{P}^2$ at the nine points corresponding to $v_{11}^\perp, v_{12}^\perp, v_{13}^\perp, v_6^\perp, v_1^\perp, v_{14}^\perp, v_8^\perp, v_7^\perp$ and $v_3^\perp$, is not a Mori dream space, and therefore the Cox ring of $\mathbf{P}(\Omega_{X(\Delta)})$ is not finitely generated.

Now, given a smooth projective fan Δ' in $\mathbf{R}^d$ such that the Cox ring of $\mathbf{P}(\Omega_{X(\Delta')})$ is not finitely generated, there is a smooth projective fan Δ in $\mathbf{R}^{d+1} = \mathbf{R}^d \times \mathbf{R}$ with rays spanned by $(1, \ldots, 1, 1)$, $(1, \ldots, 1, -1)$, and $(v, 0)$ for each primitive generator v of a ray in Δ'. The points $(v, 0)^\perp$, for each primitive generator v of a ray in Δ', lie on a hyperplane H in $\mathbf{P}^d$. We let S' be the set of points in H corresponding to $\{\rho'^\perp \mid \rho' \text{ ray of } \Delta'\}$, and we let S be the set of points in $\mathbf{P}^d$ corresponding to $\{\rho^\perp \mid \rho \text{ ray of } \Delta\}$. As before, by Theorem V.9 the Cox ring of $\mathrm{Bl}_{S'} H$ is isomorphic to the Cox ring of $\mathbf{P}(\Omega_{X(\Delta')})$, so it is not finitely generated. Then, by Lemma II.17 we conclude that the Cox ring of $\mathrm{Bl}_S \mathbf{P}^d$ is not finitely generated. By applying Theorem V.9 again, we have that the Cox ring of $\mathbf{P}(\Omega_{X(\Delta)})$ is isomorphic to the Cox ring of $\mathrm{Bl}_S \mathbf{P}^d$, and therefore it is not finitely generated. $\square$

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