Higher Derivative Corrections, Consistent Truncations, and IIB Supergravity

by

Phillip George Szepietowski

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Physics)
in The University of Michigan
2011

Doctoral Committee:

Associate Professor James T. Liu, Chair
Professor Igor Kriz
Professor Keith Riles
Associate Professor Finn Larsen
Assistant Professor Aaron Thomas Pierce

To my parents.

ACKNOWLEDGEMENTS

First and foremost I would like to thank my thesis advisor, Jim Liu, for his guidance and support throughout my graduate career. His physical insights and detailed explanations have helped to shape me as a physicist and scientist in general.

Secondly, I would like to extend gratitude to my collaborators: Sera Cremonini, Kentaro Hanaki, and Zhichen Zhao; without whom my graduate career would have been much less fruitful. I am particularly grateful for the encouragement and many engaging conversations with Sera Cremonini.

Furthermore, I would like to thank professors Leopoldo Pando Zayas, Finn Larsen, Henriette Elvang, Aaron Pierce, Gordon Kane, James Wells, and Ratindranath Akhoury for the physics that I have learned from them both through conversation and in various courses.

I also wish to thank my undergraduate advisor Keith Riles for taking me in as a young researcher and giving me a glimpse of the world of experimental gravitational wave physics.

Thanks to all of the graduate students in the Physics Department at Michigan. In particular I wish to acknowledge interactions, both physics and recreational, with Ibrahima Bah, Alejandra Castro, Tim Cohen, Alberto Faraggi, Gourab Ghoshal, Sunghoon Jung, Brian Karrer, Eric Kuflik, Sam McDermott, Ross O'Connell, Cheng Peng, Daniel Phalen, Ryo Saotome, Brooks Thomas as well as those mentioned previously.

Finally, I would especially like to thank all my friends and family for their unending and unconditional support. I love you all.

TABLE OF CONTENTS

LIST OF APPENDICES

Appendix

CHAPTER I

Introduction

String theory was originally formulated to describe the observed spectrum of higher spin mesons. While this initial description of mesonic excitations as string-like objects eventually failed, the realization that string theory actually contained in its description a spin-2 excitation which can be identified with the graviton led the way for many important works attempting to use string theory as a UV complete description of the world – leading to many important discoveries in physics as well as mathematics along the way. While this program is still a work-in-progress, an astounding feature has become apparent which relates string theory to quantum field theories in a very fundamental way. Relying on the properties of D-branes – extended objects in superstring theory on which open strings end – a duality became apparent relating string theory in the presence of D-branes to the field theories residing on the D-brane world-volumes. The discovery, which was brought to the forefront of modern physics by the construction due to Maldacena of the AdS/CFT correspondence [138], and was made precise in [96, 175], has led to many surprising and interesting relations between quantum field theories and classical theories of gravity in an anti de-Sitter background. The relation in its most useful regime relates classical string theory to the strong coupling regime of certain QCD-like quantum field theories, thus in a sense reviving the original motivations for constructing string theory.

In its original and most rigorously studied formulation, the AdS/CFT correspondence relates type IIB superstring theory in an $AdS_5 \times S^5$ spacetime, which arises as the near-stack limit of N D-branes, to $\mathcal{N} = 4$, $SU(N)$ super Yang-Mills (SYM) theory in four-dimensions. Much work over the past 15 years in the high energy physics community has focussed on investigating this correspondence, both by performing tests of the duality and by formulating generalizations and constructing further examples of dual systems. For reviews see [3, 64] as well as many others.

This correspondence may seem surprising so let us make a quick necessary check on this proposed duality. If the duality is true, it must be the case that the global symmetries on either side are the same. First, on the string theory side, the background geometry is $AdS_5 \times S^5$ which has isometry group given by $SO(4,2) \times SO(6)$; this is straightforwardly seen because S^5 and AdS_5 are naturally seen as maximally symmetric solutions embedded in $\mathbb{R}^6$ with euclidean and $(-,-,+,+,+,+)$ signature, respectively. In the gauge theory, $\mathcal{N} = 4$ SYM is a super-conformal theory in four-dimensions and conveniently $SO(4,2)$ is precisely the conformal group in four-dimensions. Also, being an extended supersymmetric theory the R-symmetry group of the field theory is given by $SU(4)$ which is isomorphic to $SO(6)$. Therefore we can identify the conformal symmetry and R-symmetry in the gauge theory with the symmetry groups of AdS_5 and S^5, respectively. So indeed the global symmetries on each side of the correspondence match. In fact the statement above extends to the full symmetry group. Including fermionic generators, the group of spacetime isometries is extended to $PSU(2,2|4)$ which is also the full super-conformal algebra in four-dimensions.

To put the duality more precisely, the correspondence equates the full string theory partition function in the $AdS_5 \times S^5$ background to the partition function of the dual quantum field theory. The correspondence relies on making the following identification of parameters,

$$\frac{L^4}{\alpha'^2} = 4\pi g_s N, \qquad 4\pi g_s = g_{YM}^2. \tag{I.1}$$

Here L is the radius of the AdS_5 and the S^5, α' is the square of the string length, ℓ_s, (or the inverse of the string tension), $g_s = e^{\phi}$ is the string coupling constant, and g_{YM} is the Yang-Mills coupling constant. Note that L, α', and g_s are all defined in the string theory and g_{YM} is a quantity in the SYM theory. The parameter N is defined on each side of the correspondence: as the number of D3-branes in the string theory setup and the rank of the gauge group in the field theory. The utility of the correspondence takes full effect in a particular limiting case in which we take $N \gg 1$ while holding the 't Hooft coupling, $\lambda = g_{YM}^2 N$, fixed. This ensures that we can treat the string theory perturbatively since we must have g_s small for λ to be fixed in the large-N limit. Furthermore, in order to retain computability in the string theory it is convenient to take λ large as well; this corresponds to taking the classical approximation which is seen as follows. Rewriting (I.1) as $L/\ell_s = \lambda^{1/4}$, we see that the large λ limit ensures that the AdS-radius, L, is much larger than the string length, ℓ_s. In this sense a supergravity approximation is justified. Therefore, in its simplest form, we have a duality which maps string theory in the small g_s and α' limit to a gauge theory in the large N and strong coupling, λ, limit. It is thus a weak/strong duality, where strong coupling in the gauge theory is described by weak coupling in the string theory. In this limit the duality has been tested many, many times and the expectation is that the duality holds to all orders. However, supporting evidence of the duality outside of the regime described above is still a rather ripe area for exploration and the applicability of the duality outside this regime is technically still an open question. It is therefore useful to explore probes of the duality which venture away from the large-N and large-λ limits.

Most studies of the AdS/CFT correspondence have focussed both on the canonical example of $AdS_5 \times S^5$ (or various other simple cases such as M-theory on $AdS_4 \times S^7$) and on the parameter regime described above. The focus of the present thesis is in understanding deviations from both of these regimes. First, we will focus on the inclusion of what are called higher derivative corrections to the bulk action. These take the form of four-derivative (or higher) terms which correspond to either finite λ or finite N corrections to the setup described above. In this sense these higher derivative actions can be viewed as effective theories descending from string theory. The other aspect which we will probe in this thesis concerns modifications of the internal manifold, both by replacing the five-sphere with a less symmetric space and by deforming the differential structure by allowing for deformations of the internal metric by fields which depend on the space-time coordinates. Before moving on to the results proper, we first give an overview of the types of systems which will be analyzed in this thesis and highlight some of the main results.

1.1 Higher derivative Lagrangians as effective theories

The bosonic sector of the low-energy effective action of type II string theories and M-theory is described by a metric coupled to various p-form field strengths. When reduced to lower dimensions these theories generically lead to matter coupled supergravity theories. In particular, focussing only on the gravitational sector, the low-energy five-dimensional lagrangian typically takes the following effective form:

$$\mathcal{L} = R + \sum_{m=1} \zeta_m (\mathcal{R})^{m+1}, \tag{I.2}$$

where $(\mathcal{R})^{m+1}$ represent terms involving $2(m+1)$ derivatives acting on the metric and ζ_m is a coupling constant of dimension $(length)^{2m}$. In this work we will focus on the lowest corrections to the Einstein-Hilbert term, so $m = 1$, which corresponds to curvature-squared corrections. These can be thought of in two ways, depending on the ten-dimensional origin of these terms. From the effective string theory action point of view they can be thought of as $\zeta_1 \sim \alpha' \sim \ell_s^2$ corrections, or from an alternative perspective they can correspond to finite N corrections, such that $\zeta_1 \sim L^2/N$. It has been known for some time that, by expanding the tree-level string scattering amplitude in powers of momenta, the first corrections to the IIB action show up at order α'^3, or fourth-order in the Riemann curvature [94, 92, 74]. As such it seems likely that these Riemann-squared terms enter as the latter type of correction – i.e. they are $1/N$ corrections. This expectation has been verified by relating the coefficient of these terms to the R-currect and Weyl anomalies in the gauge theory via holographic anomaly matching as will be discussed in more detail in the relevant section.

1.2 Consistent truncations of Kaluza-Klein theories

The low energy limits of type IIA/B string theory and M-theory correspond to IIA/B ten-dimensional supergravity and eleven-dimensional supergravity respectively. To make connection with realistic physics it is therefore prudent to reduce these theories to an effective lower dimensional theory, a process known as Kaluza-Klein reduction. In this procedure one decomposes the full $(D+1)$-dimensional space-time manifold as a product of a $(d+1)$-dimensional space-time manifold and a $(D-d)$-dimensional compact internal manifold. The fields in the full theory are then decomposed as products of $(d+1)$ dimensional fields and a complete set of fields on the internal manifold. The complete set is usually obtained by writing the internal dependence of the fields as a sum over harmonic functions and forms on the internal space.

The classic example of this program is the decomposition of a gravitational theory in $(D+1)$-dimensions on a circle. Consider the Einstein-Hilbert action in $(D+1)$-dimensions,

$$\mathcal{S} = \frac{1}{16\pi G_{D+1}} \int d^{D+1}x \sqrt{-g}\, R. \tag{I.3}$$

Following the above discussion we take the compact circle direction to correspond to $y \equiv$

x^{D+1} and decompose the metric as follows,

$$ds^2 = G_{MN}dx^M dx^N = e^{2a\Phi}g_{\mu\nu}dx^\mu dx^\nu + e^{2b\Phi}(dy + A_\mu dx^\mu)^2, \tag{I.4}$$

where the fields $g_{\mu\nu}$, A_μ, and Φ depend on all coordinates.

Since we are decomposing the fields on a circle we can simply expand the y-dependence of these fields on a complete set of functions on the circle; this is similar to a Fourier decomposition. Explicitly we have,

$$X(x^\mu, y) = \sum_{n=-\infty}^{\infty} X_n(x^\mu)e^{2\pi i n y}, \tag{I.5}$$

where $X(x^\mu, y)$ represents any one of the fields defined above and $X_n(x^\mu)$ are essentially Fourier components. Thus far, the reduction is completely general. We can then think of the Fourier components

$$X_n(x^\mu) = \int_0^1 X(x^\mu, y)e^{-2\pi i n y}dy \tag{I.6}$$

as representing fields in the lower dimensional theory.

Returning to the theory of interest, it turns out that the components of the $(D+1)$-dimensional Einstein equation reduce to an infinite tower of equations describing the $n = 0$ massless sector as well as higher modes with masses proportional to n^2/R^2. If we take the limit $R \to 0$ the massive modes become infinitely massive so we can remove them from our spectrum. Essentially, it would take an infinite amount of energy to excite these modes so that they become non-dynamical and it is consistent to set them to zero. We thus arrive at a theory consisting of massless fields $g_{\mu\nu}$, A_μ, and Φ – in one lower dimension. The truncated theory thus contains simply a massless graviton, a massless $U(1)$ gauge field and a massless scalar, complete with D-dimensional diffeomorphism and $U(1)$ gauge invariance, both descending from diffeomorphism invariance in $D+1$-dimensions. So we have conveniently removed two potential issues in one step: by making the radius small we have simultaneously removed an infinite set of massive modes from the theory, while also making the extra dimension very small, roughly unobservable.

There is an important point to make here; in the above we have implicitly performed two steps. First, by performing the circle reduction we are able to separate the fields into two sectors corresponding to their charge under the $U(1)$ gauge symmetry of the massless gauge field; the massless fields – corresponding to the zero modes on the circle – are the $U(1)$ neutral sector, and the massive fields are the charged sector. At the classical level it is consistent to set the charged fields to zero. This can be seen as follows: the charged fields must occur in a $U(1)$ neutral combination in the Lagrangian and so must enter at least quadratically in any coupling – of note, in any terms containing a neutral field. Therefore, at least classically, it is fully consistent to set the fields in the charged sector to zero and truncate to the massless sector of the theory while fully satisfying the equations of motion. The second step we have performed is the limiting procedure $R \to 0$. This has the further effect of completely decoupling the massive – $U(1)$ charged – fields in the effective field theory at scales smaller than $1/R$ as they can be integrated out of the spectrum. In this sense, the truncated theory should be well behaved even at the quantum level. We will see that this decoupling is not so easily achieved in more complicated reductions.

The first step described in the preceding paragraph is the simplest and most trivial example of the concept of a consistent truncation. A consistent truncation is a reduction of the Lagrangian to a lower dimensional Lagrangian system such that the equations of motion derived from the lower dimensional theory solve the higher dimensional system of equations. Any solution of the truncated theory is necessarily a solution to the full higher dimensional theory. The simplest way to achieve this is to restrict the fields to simply not allow for dependence on the internal coordinates. For manifolds with a group structure, this corresponds to a truncation to the singlet representations of the group or an appropriate subgroup.

A less trivial example of this process is that of reducing IIB supergravity on a five-sphere. The field content of IIB supergravity is described in detail in section 4.2. The general Kaluza-Klein reduction of this theory is quite complicated. However, the entire Kaluza-Klein spectrum has been worked out to linearized order for the case where the five-dimensional space is assumed to be $AdS_5 \times S^5$ in [124, 104].

Although it is quite a bit more complicated, it is consistent to perform the first step detailed above to this case and truncate the Kaluza-Klein reduction to appropriate singlet (uncharged) modes on the five-sphere. The simplest truncation here is to complete singlets on the five-sphere; this is the equivalent of the circle reduction above – you only keep modes which have no internal dependence at all. In this thesis we will demonstrate a slightly more non-trivial truncation which is appropriate for any Sasaki-Einstein manifold, the five-sphere being a special case.

Due to the non-zero curvature of the Sasaki-Einstein manifold these truncations are slightly different in spirit from the simple circle reduction described above. For the circle reduction we can set the radius, R, of the circle to be very small. This had the effect of giving very large masses to all of the higher Kaluza-Klein modes and so at energies far below the scale $1/R$ these modes will not be important in the effective theory and so are completely decoupled. However, for the reduction of IIB on the five-sphere (or a generic Sasaki-Einstein five-manifold) even the lowest KK modes are not massless, but have a mass proportional to the inverse-radius $1/L$ of the five-sphere. Furthermore, recalling that this theory is an effective theory descending from string theory, taking the limit $EL \to 0$, where E is the energy scale of perturbations, is essentially taking $L/\sqrt{\alpha'} \to 0$ so we actually move away from the regime where the supergravity approximation is valid and stringy/quantum corrections become important. Thus, a simple $L \to 0$ limit cannot be performed to decouple the higher KK modes.

We will present a consistent truncation of IIB on a Sasaki-Einstein five-manifold. This class includes the five-sphere as a special case. A $(2d+1)$-dimensional Sasaki-Einstein manifold is defined such that the cone metric over it is Kahler and Ricci-flat, i.e. it is a Calabi-Yau-(d+1) manifold with metric given by

$$ds^2(CY_6) = dr^2 + r^2 ds^{@}(SE_5). \tag{I.7}$$

Additionally, many Sasaki-Einstein manifolds can be generically thought of as a $U(1)$ fiber over a complex d-dimensional Kahler-Einstein base space. As described in section 4.2 these Sasaki-Einstein manifolds possess an $SU(2)$ structure group which is inherited from the Kahler-Einstein base. Decomposing the IIB field content on singlets of this $SU(2)$ structure group ensures the consistency of the truncation. The details we leave for the bulk of the thesis, but the main point is that the $SU(2)$ structure defines a set of differential forms which form a closed set under the appropriate multiplication operations and under differential operations – pragmatically this is what guarantees consistency.

For the special case of the five-sphere, the truncation can be seen most easily from a group theory perspective. We first note that the isometry group of S^5 is $SO(6) \cong SU(4)$. The truncation presented herein follows by restricting to singlets on a transitively acting subgroup of $SU(4)$. In this case it corresponds to $SU(3) \subset SU(4)$. Any $SU(4)$ representation can be decomposed into representations of $SU(3) \times U(1)$; the truncation retains only singlets of the $SU(3)$ factor in this decomposition. Note that this global symmetry of the five-sphere is unrelated to the $SU(2)$ structure described above which is related to the differential structure and is universal to all Sasaki-Einstein manifolds.

The truncations presented in this thesis will thus satisfy the first criterion we described in the circle reduction – solutions to the truncated theory will correspond to classical solutions of the full higher dimensional theory. However, we will not determine the stability of these solutions as we are not able to systematically remove the other modes from the theory in an effective field theory sense. It is thought that truncations which preserve some amount of supersymmetry will in fact be stable to perturbations. However, a careful check of stability should be performed. An analysis of the stability of similar truncations of eleven-dimensional supergravity has been performed in [21] verifying the stability of various supersymmetric truncations while also demonstrating that some non-supersymmetric truncations are actually unstable to perturbations – specifically, certain scalar perturbations have a mass below the Breitenlohner-Freedman bound [23, 24] which governs the stability of scalar fields in an AdS background.

The generic reduction of IIB supergravity on S^5 corresponds to an $\mathcal{N} = 8$ maximally supersymmetric theory in five-dimensions. The truncations discussed herein break $\mathcal{N} = 8$ generically to $\mathcal{N} = 4$ and for the special cases we will consider to $\mathcal{N} = 2$. The dual field theory thus also has reduced supersymmetry and corresponds to certain $\mathcal{N} = 1$ SYM theories. In this sense these truncations can be thought of as further examples of AdS/CFT with reduced supersymmetry. Furthermore, these systems have been of recent interest in constructing holographic duals to certain condensed matter systems by allowing for nontrivial scalar condensates and also more drastic deformations which result in four and five-dimensional geometries with anisotropic scaling.

1.3 Looking forward – gauge/gravity duality as a general principle

The ideas involving the AdS/CFT correspondence described above pertain to very specific cases within a more general paradigm which has come to prominence in the past few years. The idea that a theory of gravity and a gauge theory in one lower dimension are really two descriptions of the same physics seems to be very robust. Many intriguing results have been established linking phenomena in both descriptions to each other – holographic descriptions involving aspects of superconductors and the quark-gluon plasma have been developed along with more technical results involving gravity duals to quiver gauge theories to name a few examples. While these ideas have been motivated by and are on their firmest footing within string theory, many of the phenomenological results are independent of explicit string theory constructions. This seems to lead one to conclude that certain strong coupling regimes of particular gauge theories have a completely equivalent description as a theory of gravity with appropriate matter content, independent of string theory. Even given this universality, it is of importance to study both avenues.

From one perspective the phenomenological approach to gauge/gravity duality is very enticing and will be the most likely route to a description of "real world" physical systems. It also provides a glimpse towards a potential understanding of quantum gravity in the following sense. If they truly are equivalent theories, the quantum description of physics in these geometries should be given simply by the dual gauge theory – understanding this equivalence precisely will hopefully lead to many insights into the correct quantum description of gravity. On the other hand, string theory has provided many insights into gauge/gravity duality and will continue to be the most robust tool in these investigations. A true understanding of the nature of these dualities and ultimately of quantum gravity will most likely require a description within string theory.

The remainder of this thesis is organized as follows. In chapter II we discuss various technical issues regarding higher derivative corrections to the minimal gauged supergravity action. In particular, working in a perturbative framework we construct R-charged black hole solutions and discuss the effects of field redefinitions of the metric and gravi-photon. Following this we discuss the modification of the Gibbons-Hawking term due to higher derivative terms in the action. The Gibbons-Hawking term is required in order to retain a well defined variational principle, and in general higher derivative terms will spoil this. We will see that, by working perturbatively – in the sense that the coefficients of the higher derivative terms are taken to be small – a well defined Gibbons-Hawking term can be constructed. We end this chapter with a discussion on the mass-charge ratio of some extremal black holes.

Chapter III involves an analysis of a particular higher derivative action. This action was originally constructed in [107] and is the supersymmetric completion of the $A \wedge Tr(R \wedge R)$ term in $\mathcal{N} = 2$ five-dimensional supergravity coupled to an arbitrary number of vector multiplets. We eliminate the vector multiplets and reduce this action to minimal $\mathcal{N} = 2$ supergravity in five-dimensions. Again working perturbatively, we construct black hole solutions and discuss corrections to the so-called superstar, a solution to five-dimensional supergravity which contains a naked singularity. However, in the perturbative analysis we present, we are unable to determine if the higher derivative terms resolve the singularity. We end chapter III with a computation of the shear-viscosity to entropy density ratio (η/s) in the gauge theory plasma dual to a finite-chemical potential solution of this higher derivative theory. This quantity has received much attention recently due to its universal behavior at the two-derivative level and the proposed lower bound $\eta/s \geq 1/4\pi$, termed the KSS (for Kovtun, Son and Starinets) bound [125]. We find, in agreement with previous studies, that the addition of higher derivative terms can generically violate this bound and that the addition of chemical potential only enhances the effect.

Chapter IV is devoted to the construction of consistent truncations of IIB supergravity on five dimensional squashed-deformed Sasaki-Einstein manifolds. These truncations are novel, in the fact that they admit massive modes to be included in the truncation. Utilizing the $SU(2)$ structure of the Sasaki-Einstein manifold, the reduction follows straightforwardly and is guaranteed to be consistent. We begin by presenting the bosonic reduction, highlighting the truncation to $\mathcal{N} = 2$ supergravity coupled to a massive vector multiplet and a universal hypermultiplet. These reductions include an AdS solution and we discuss the spectrum of excitations about the AdS vacuum; this allows us to assemble the excitations into the appropriate AdS supermultiplets. Furthermore, we demostrate the consistency of various further truncations. We then present the reduction of the fermionic sector of IIB. We again are able to organize the spectrum in terms of AdS supermultiplets. The supersymmetry variations of the fermions are determined and shown to be consistent with the supermultiplet structure. The fermion equations of motion are also determined, and from these we construct the Lagrangian for the full truncated theory. Finally, we present a particular truncation which contains only a single hypermultiplet, along with the supergravity multiplet. This truncation is particularly interesting as it is the supersymmetric completion of a bosonic system which describes a holographic superconductor.

This thesis is based on the following papers, all of which were completed in collaboration with (some combination of) Jim Liu, Sera Cremonini, Kentaro Hanaki and Zhichen Zhao:

- [132] – J. T. Liu, P. Szepietowski, "Higher derivative corrections to R-charged AdS(5) black holes and field redefinitions," Phys. Rev. **D79**, 084042 (2009)

- [54] – S. Cremonini, K. Hanaki, J. T. Liu, P. Szepietowski, "Black holes in five-dimensional gauged supergravity with higher derivatives," JHEP **0912**, 045 (2009)

- [55] – S. Cremonini, K. Hanaki, J. T. Liu, P. Szepietowski, "Higher derivative effects on eta/s at finite chemical potential," Phys. Rev. **D80**, 025002 (2009)

- [56] – S. Cremonini, J. T. Liu, P. Szepietowski, "Higher Derivative Corrections to R-charged Black Holes: Boundary Counterterms and the Mass-Charge Relation," JHEP **1003**, 042 (2010)

- [133] – J. T. Liu, P. Szepietowski, Z. Zhao, "Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds," Phys. Rev. **D81**, 124028 (2010)

- [134] – J. T. Liu, P. Szepietowski, Z. Zhao, "Supersymmetric massive truncations of IIb supergravity on Sasaki-Einstein manifolds," Phys. Rev. **D82**, 124022 (2010).

CHAPTER II

Addition of higher derivative terms to the gravitational action - field redefinitions and boundary terms

In this chapter we discuss two technical aspects involving theories which include higher derivative terms. First, a detailed analysis of the effects of field redefinitions on charged black hole solutions is presented. Second, we provide a discussion on modifications to the Gibbons-Hawking surface term when higher derivative corrections are included. We end with a discussion on the black hole thermodynamics and relations to the weak gravity conjecture [8]. This chapter is based on work published in [132, 56] in collaboration with Jim Liu and Sera Cremonini.

2.1 Field Redefinitions and Higher Derivative Terms

Higher derivative corrections to the Einstein-Hilbert action have received much notice in recent years, as such terms naturally show up in the α' expansion of effective actions derived from string theory. In general, the first non-trivial terms arise at the four derivative level, corresponding to curvature-squared corrections to classical Einstein theory of the form

$$e^{-1}\delta\mathcal{L} = \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{II.1}$$

where the coefficients α_1, α_2 and α_3 are determined by the underlying theory. It was suggested in [176] that the natural form of such terms would be given by the Gauss-Bonnet combination

$$e^{-1}\delta\mathcal{L}_{\rm GB} = \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}), \tag{II.2}$$

as this is the unique combination that avoids introducing ghosts in the effective theory. It was subsequently argued, however, that in the absence of an off-shell formulation such as string field theory, the α_1 and α_2 coefficients are physically indeterminate as they may be eliminated by an on-shell field redefinition of the form $g_{\mu\nu} \to g_{\mu\nu} + aRg_{\mu\nu} + bR_{\mu\nu}$. In this sense, only the Riemann-squared term parameterized by α_3 carries physical information from the underlying string theory.

The form of the higher derivative corrections are further constrained by supersymmetry. Explicit computations for the uncompactified closed superstring indicate that the first corrections enter at the R^4 order [94, 92, 74]. This is a feature of maximal supersymmetry, as curvature-squared terms are present in, for example, the uncompactified heterotic theory [93, 140]. An alternate route to obtaining supersymmetric higher derivative corrections is to make use of supersymmetry itself to construct higher derivative invariants that may show up in the action. This was applied in the heterotic supergravity by supersymmetrizing the Lorentz Chern-Simons form responsible for the modified Bianchi identity $dH = \alpha' \mathrm{Tr}\,(F \wedge F - R \wedge R)$ [16]; the result agrees with the explicit calculations, once field redefinitions are properly taken into account [49]. More recently, the supersymmetric completion of the $A \wedge \mathrm{Tr}\, R \wedge R$ term in five-dimensional $\mathcal{N} = 2$ supergravity (coupled to a number of vector multiplets) was obtained in [107]. This result has led to new progress in the study of black hole entropy and precision microstate counting in five dimensions (see *e.g.* [45] and references therein).

The supersymmetric four-derivative terms given in [107] were obtained using conformal supergravity methods. Thus it should be no surprise that they involve the square of the five-dimensional Weyl tensor [107]

$$
\begin{aligned}
e^{-1}\delta\mathcal{L}_{\rm sugra} &= \tfrac{c_I}{24}[\tfrac{1}{8}M^I C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \cdots] \\
&= \tfrac{c_I}{24}[\tfrac{1}{8}M^I(\tfrac{1}{6}R^2 - \tfrac{4}{3}R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \cdots], \qquad \text{(II.3)}
\end{aligned}
$$

as opposed to the Gauss-Bonnet combination, (II.2). In principle, an appropriate field redefinition may be performed to bring this into the Gauss-Bonnet form. However, this is usually not done, as it would obscure the overall supersymmetric structure of the theory. Thus in practice two somewhat complementary approaches have been taken to investigating the curvature-squared corrections to the Einstein-Hilbert action. The first, which applies whether the underlying theory is supersymmetric or not, is to use a parameterized action of the form (II.1), with special emphasis on the Gauss-Bonnet combination. The second is to focus directly on supergravity theory, and hence to use explicitly supersymmetric higher-derivative actions of the form (II.3). In principle, these two approaches are related by appropriate field redefinitions. However, in practice this is complicated by the fact that additional matter fields (*e.g.* $\mathcal{N}=2$ vector multiplets) as well as auxiliary fields may be present, thus making any field redefinition highly non-trivial.

In this section, we investigate and clarify some of the issues surrounding field redefinitions in the presence of additional fields. In particular, we take the bosonic sector of five-dimensional $\mathcal{N}=2$ gauged supergravity and extend it with four-derivative terms built from the Riemann tensor $R_{\mu\nu\rho\sigma}$ as well as the graviphoton field-strength tensor $F_{\mu\nu}$. Although we introduce eight such terms, we demonstrate that only four independent combinations remain physical once field redefinitions are taken into account. To be explicit, we construct the higher-derivative corrections to the spherically symmetric R-charged[1] AdS$_5$ black holes of [12, 13], working to linear order in the higher-derivative terms, and then investigate the effect of field redefinitions on these black hole solutions.

[1] Here R-charge refers to the charge under the $U(1)$ gravi-photon.

To some extent, our solutions generalize the Gauss-Bonnet black holes originally constructed in [22, 172] and extended to Einstein-Maxwell theory in [173] and, with the inclusion of Born-Infeld terms, in [174]. One advantage that the Gauss-Bonnet combination has over the generic form of (II.1) is that it leaves the graviton propagator unmodified, and also yields a modified Einstein equation involving at most second derivatives of the metric. With an appropriate metric ansatz, the resulting Gauss-Bonnet black holes are then obtained by solving a simple quadratic equation. Furthermore, this feature of the Gauss-Bonnet term leads to a good boundary variation and natural generalization of the Gibbons-Hawking surface term [143]. This is a primary reason behind the popularity of applying Gauss-Bonnet (and more generally Lovelock) extensions to braneworld physics (see *e.g.* [48]).

Our interest in studying the higher order corrections to R-charged AdS_5 black holes is also motivated by our desire to explore finite 't Hooft coupling corrections in AdS/CFT. Using the relation $\alpha' = L^2/\sqrt{\lambda}$, we see that each additional factor of $\alpha' R_{\mu\nu\rho\sigma}$ in the string effective action gives rise to a $1/\sqrt{\lambda}$ factor in the strong coupling expansion of the dual gauge theory. Since supersymmetry ensures that the leading correction terms in IIB theory are of order α'^3, this indicates that the $\mathcal{N} = 4$ super-Yang Mills theory dual to $\mathrm{AdS}_5 \times S^5$ will first receive such corrections at the $\lambda^{-3/2}$ order. The effect of these finite 't Hooft coupling corrections on both the thermodynamics [100, 155] and hydrodynamics [33, 35, 14, 31, 32, 30] of the $\mathcal{N} = 4$ plasma have received much attention in the context of extrapolations between the strong and weak coupling limits of the $\mathcal{N} = 4$ theory.

In principle, it would be greatly desirable to extend the finite coupling analysis to $\mathcal{N} = 1$ gauge theories dual to $\mathrm{AdS}_5 \times Y^5$ where Y^5 is Sasaki-Einstein. This is of particular interest in resolving conjectures on the nature of the shear viscosity bound η/s [156, 126, 33, 125, 122, 27, 26]. One difficulty in doing so, however, lies in the fact that the higher derivative corrections involving the Ramond-Ramond five-form have not yet been fully explored (but see [154]). While it may be argued that these terms will not contribute in the maximally supersymmetric case, there is no reason to expect this to continue to hold for the reduced supersymmetric backgrounds dual to $\mathcal{N} = 1$ super-Yang Mills. For this reason, recent investigations of the shear viscosity [122, 27, 26] (and drag force [71, 170]) have assumed a parameterized set of curvature-squared corrections of the form indicated above in (II.1). Our present construction of higher-derivative corrected R-charged black holes allows for a generalization of the finite coupling shear viscosity calculation to backgrounds dual to turning on a chemical potential [15].

We start with the two-derivative bosonic action of $\mathcal{N} = 2$ gauged supergravity, and in Section 2 we introduce a parameterized set of four derivative terms involving both curvature and graviphoton field strengths. Then, in Section 3, we obtain the linearized corrections to the spherically symmetric R-charged AdS$_5$ black holes. As one of the aims of this chapter is to clarify the use of field redefinitions, we take a closer look at this in Section 4. Finally, we conclude with a discussion of our results in Section 5.

2.1.1 The higher-derivative theory

Our starting point is the bosonic sector of pure $\mathcal{N} = 2$ gauged supergravity in five dimensions. The theory is described by a metric, $g_{\mu\nu}$, and a $U(1)$ gauge field, A_μ, termed the gravi-photon with Lagrangian given by

$$e^{-1}\mathcal{L}_0 = R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + 12g^2 + \tfrac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda. \tag{II.4}$$

Although the Chern-Simons term is important from a supergravity point of view, it will not play any role in the electrically charged solutions that are investigated below.

In general, higher-derivative corrections to $\mathcal{L}_0$ may be expanded in the number of derivatives. We are mainly interested in the first non-trivial corrections, which arise at the four-derivative level. In a pure gravity theory, this would correspond to the addition of R^2

terms to the Lagrangian. However, for the Einstein-Maxwell system, we may also consider higher-order terms in the Maxwell field, such as F^4 and RF^2 terms. We thus introduce the higher-derivative Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{R^2} + \mathcal{L}_{F^4} + \mathcal{L}_{RF^2}, \tag{II.5}$$

where $\mathcal{L}_0$ is given in (II.4), while the additional terms are

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2} &= \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \\ e^{-1}\mathcal{L}_{F^4} &= \beta_1 (F_{\mu\nu}F^{\mu\nu})^2 + \beta_2 F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu, \\ e^{-1}\mathcal{L}_{RF^2} &= \gamma_1 R F_{\mu\nu}F^{\mu\nu} + \gamma_2 R_{\mu\nu}F^{\mu\rho}F_\rho{}^\nu + \gamma_3 R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \end{aligned} \tag{II.6}$$

Note that we have not considered terms such as $F_{\mu\nu}\Box F^{\mu\nu}$ that would in principle enter at the same order. Although we are not complete in this regard, the terms that enter in $\mathcal{L}_{F^4}$ are nevertheless sufficient for capturing the expansion of the Born-Infeld action.

Equations of Motion

Both the Maxwell and Einstein equations pick up corrections from the higher-derivative terms in (II.5). The modified Maxwell equation is straightforward

$$\begin{aligned} \nabla_\mu F^{\mu\nu} + \tfrac{1}{4\sqrt{3}}\epsilon^{\nu\rho\lambda\sigma\delta}F_{\rho\lambda}F_{\sigma\delta} &= \nabla_\mu\big(8\beta_1 F^2 F^{\mu\nu} - 8\beta_2 F^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu} \\ &\qquad + 4\gamma_1 R F^{\mu\nu} + 4\gamma_2 (R^{[\mu}{}_\lambda F^{\nu]\lambda}) + 4\gamma_3 R^{\mu\nu\lambda\sigma}F_{\lambda\sigma}\big). \end{aligned} \tag{II.7}$$

The Einstein equation is somewhat cumbersome, but can be expressed in Ricci form as

$$
\begin{aligned}
R_{\mu\nu} + 4g^2 g_{\mu\nu} - \tfrac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda + \tfrac{1}{12}g_{\mu\nu}F^2 ={}& \\
&(2\alpha_1 + \alpha_2 + 2\alpha_3)\nabla_\mu\nabla_\nu R - (\alpha_2 + 4\alpha_3)\Box R_{\mu\nu} \\
&-2\alpha_1 R R_{\mu\nu} + 4\alpha_3 R_{\mu\lambda}R_\nu{}^\lambda - 2(\alpha_2 + 2\alpha_3)R_{\mu\lambda\nu\sigma}R^{\lambda\sigma} - 2\alpha_3 R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} \\
&+\tfrac{1}{3}g_{\mu\nu}[(2\alpha_1 + \alpha_2 + 2\alpha_3)\Box R + \alpha_1 R^2 + \alpha_2 R^2_{\lambda\sigma} + \alpha_3 R^2_{\rho\lambda\sigma\delta}] \\
&-4\beta_1 F^2 F_{\mu\lambda}F_\nu{}^\lambda - 4\beta_2 F_{\mu\rho}F^{\rho\lambda}F_{\lambda\sigma}F^\sigma{}_\nu + g_{\mu\nu}[\beta_1(F^2)^2 + \beta_2 F^4] \\
&+\gamma_1(\nabla_\mu\nabla_\nu F^2 - R_{\mu\nu}F^2 - 2RF_{\mu\lambda}F_\nu{}^\lambda) \\
&+\gamma_2(-\nabla_\lambda\nabla_{(\mu}F_{\nu)\rho}F^{\lambda\rho} + \tfrac{1}{2}\Box F_{\mu\lambda}F_\nu{}^\lambda + 2R_{(\mu}{}^\lambda F_{\nu)}{}^\rho F_{\lambda\rho} + R_{\lambda\sigma}F_\mu{}^\lambda F_\nu{}^\sigma) \\
&-\gamma_3(2\nabla^\lambda\nabla^\sigma F_{\mu\lambda}F_{\nu\sigma} + 3R_{\mu\rho\lambda\sigma}F_\nu{}^\rho F^{\lambda\sigma}) \\
&+\tfrac{1}{3}g_{\mu\nu}[(\gamma_1 - \tfrac{1}{2}\gamma_2)\Box F^2 + 2\gamma_3\nabla_\lambda\nabla_\sigma F^{\lambda\rho}F^\sigma{}_\rho \\
&\qquad +2\gamma_1 RF^2 - 2\gamma_2 R_{\lambda\sigma}F^{\lambda\rho}F^\sigma{}_\rho + 2\gamma_3 R^{\rho\lambda\sigma\delta}F_{\rho\lambda}F_{\sigma\delta}].
\end{aligned}
\tag{II.8}
$$

Since we are mainly interested in obtaining corrections *linear* in the parameters (α_1, α_2, α_3, β_1, β_2, γ_1, γ_2, γ_3) of the higher derivative terms, we may substitute the lowest order equations of motion, given by setting the left-hand-sides of (II.7) and (II.8) to zero, into the right-hand-side of (II.8) to obtain a slightly simpler form of the Einstein equation

$$
\begin{aligned}
R_{\mu\nu} + 4g^2 g_{\mu\nu} - \tfrac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda + \tfrac{1}{12}g_{\mu\nu}F^2 = \\
& 4g^2(5\alpha_1 + \alpha_2 - 2\alpha_3 + 10\gamma_1 - 2\gamma_2)F_{\mu\lambda}F_\nu{}^\lambda \\
& -2\alpha_3 R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - (\alpha_2 + 2\alpha_3 - \gamma_2)R_{\mu\lambda\nu\sigma}F^{\lambda\rho}F^\sigma{}_\rho - 3\gamma_3 R_{(\mu}{}^{\rho\lambda\sigma}F_{\nu)\rho}F_{\lambda\sigma} \\
& +\tfrac{1}{12}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 12\gamma_1 - 3\gamma_2)\nabla_\mu\nabla_\nu F^2 - \tfrac{1}{2}(\alpha_2 + 4\alpha_3 - \gamma_2)\Box F_{\mu\lambda}F_\nu{}^\lambda \\
& -2\gamma_3\nabla^\lambda\nabla^\sigma F_{\mu\lambda}F_{\nu\sigma} - \tfrac{1}{12}(\alpha_1 - \alpha_2 + 2\alpha_3 + 48\beta_1 + 8\gamma_1 + 2\gamma_2)F^2 F_{\mu\lambda}F_\nu{}^\lambda \\
& +(\alpha_3 - 4\beta_2 + \gamma_2)F_{\mu\rho}F^{\rho\lambda}F_{\lambda\sigma}F^\sigma{}_\nu \\
& +\tfrac{1}{3}g_{\mu\nu}[-16g^4(5\alpha_1 + \alpha_2) - \tfrac{2}{3}g^2(17\alpha_1 + 7\alpha_2 + 42\gamma_1 - 12\gamma_2)F^2 \\
& \qquad +\tfrac{1}{6}(\alpha_1 + 2\alpha_2 + 7\alpha_3 + 6\gamma_1 - 3\gamma_2 + 3\gamma_3)\Box F^2 \\
& \qquad +\tfrac{1}{144}(7\alpha_1 - 13\alpha_2 + 432\beta_1 + 60\gamma_1 + 24\gamma_2)(F^2)^2 \\
& \qquad +\tfrac{1}{4}(\alpha_2 + 12\beta_2 - 4\gamma_2)F^4 + \alpha_3 R^2_{\rho\lambda\sigma\delta} + 2\gamma_3 R_{\rho\lambda\sigma\delta}F^{\rho\lambda}F^{\sigma\delta}] \\
& +\cdots. \qquad \text{(II.9)}
\end{aligned}
$$

This is valid to first order in the four-derivative corrections.

Numerous previous studies higher-derivative corrections in five dimensions have concentrated on the purely gravitational sector of the theory. In this case, the first order Einstein equation simplifies to

$$R_{\mu\nu} + 4g^2 g_{\mu\nu} = -2\alpha_3 R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} + \tfrac{1}{3}g_{\mu\nu}[-16g^4(5\alpha_1 + \alpha_2) + \alpha_3 R^2_{\rho\lambda\sigma\delta}]. \qquad \text{(II.10)}$$

Working to this same order, we may define an effective cosmological constant

$$g^2_{\text{eff}} = g^2[1 + \tfrac{2}{3}(10\alpha_1 + 2\alpha_2 + \alpha_3)g^2], \qquad \text{(II.11)}$$

so that

$$R_{\mu\nu} + 4g^2_{\text{eff}}g_{\mu\nu} = \alpha_3(-2C_{\mu\rho\lambda\sigma}C_\nu{}^{\rho\lambda\sigma} + \tfrac{1}{3}g_{\mu\nu}C^2_{\rho\lambda\sigma\delta}), \qquad \text{(II.12)}$$

where we made the substitution $R_{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma} - g^2(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) + \cdots$ which is a consequence of the zeroth order Einstein equation, $R_{\mu\nu} = -4g^2 g_{\mu\nu} + \cdots$. We see that the coefficients α_1 and α_2 of R^2 and $R_{\mu\nu}^2$, respectively, do not enter at linear order, so long as we use the effective cosmological constant given by g_{eff}. This is related to the fact that these two terms may be removed by a field redefinition of the form $g_{\mu\nu} \to g_{\mu\nu} + ag_{\mu\nu}R + bR_{\mu\nu}$ with appropriate constants a and b.

Although neutral black hole solutions may be obtained directly from (II.12), we are mainly interested in R-charged solutions which may be obtained from the full equations (II.7) and (II.9). We turn to this in the next section.

2.1.2 R-charged black holes

The two-derivative Lagrangian, (II.4), admits a well-known two-parameter family of static, stationary AdS_5 black hole solutions, given by [12, 13]

$$\begin{aligned} ds^2 &= -H^{-2} f dt^2 + H(f^{-1}dr^2 + r^2 d\Omega_3^2), \\ A &= \sqrt{3}\coth\beta\left(\frac{1}{H} - 1\right) dt, \end{aligned} \tag{II.13}$$

where the functions H and f are

$$\begin{aligned} f &= 1 - \frac{\mu}{r^2} + g^2 r^2 H^3, \\ H &= 1 + \frac{\mu \sinh^2\beta}{r^2}. \end{aligned} \tag{II.14}$$

The parameter μ is a non-extremality parameter, while β is related to the electric charge of the black hole. The extremal (BPS) limit is obtained by taking $\mu \to 0$ and $\beta \to \infty$ with $Q \equiv \mu \sinh^2\beta$ fixed, so that $f = 1 + g^2 r^2 H^3$ with $H = 1 + Q/r^2$. These extremal solutions are naked singularities, and may be interpreted as 'superstars' [147]. In the absence of higher-derivative corrections, the BPS solutions may be smoothed out by turning on angular momentum to form true black holes [106, 105, 52, 128]

The first order solution

We wish to find the first order corrections to the R-charged black hole solution given by (II.13). To do so, we treat the coefficients (α_1, α_2, ..., γ_3) of the four-derivative terms in (II.6) as small parameters, and make the ansatz

$$\begin{aligned} ds^2 &= -H^{-2} f dt^2 + H(f^{-1} dr^2 + r^2 d\Omega_3^2), \\ A &= \sqrt{3} \coth\beta \left(\frac{1+a_1}{H} - 1 \right) dt, \end{aligned} \tag{II.15}$$

where

$$\begin{aligned} f &= 1 - \frac{\mu}{r^2} + g^2 r^2 H^3 + f_1, \\ H &= 1 + \frac{\mu \sinh^2\beta}{r^2} + h_1. \end{aligned} \tag{II.16}$$

Here, we treat h_1, f_1 and a_1 as small corrections, and will solve for them to linear order in the parameters of the higher-derivative Lagrangian. Note that this ansatz was designed so that the zeroth order equations are automatically satisfied in the absence of h_1, f_1 and a_1.

Even after linearization in the small parameters, the individual equations of motion, (II.7) and (II.9), yield complicated coupled equations for the first order corrections. However, the use of certain symmetries of these equations yields tractable equations. In particular, the difference between the tt and rr components of the Einstein equation, $R^t_t - R^r_r$, gives a second order equation involving only h_1, which is easily solved. The solution for h_1 can then be inserted into the Maxwell equation, (II.7), to obtain a solution for a_1. Finally, the remaining components of the Einstein equation can be solved for f_1, thus yielding the full solution. The result is

$$h_1 = \frac{\mu^2 \sinh^2 2\beta}{6H_0^2 r^6}\big(7\alpha_1 + 5\alpha_2 + 13\alpha_3 + 42\gamma_1 - 12\gamma_2 + 12\gamma_3\big),$$

(II.17)

$$\begin{aligned} a_1 = \frac{\mu^2 \sinh^2 2\beta}{6H_0^3 r^6}\Big[&\big(7\alpha_1 + 5\alpha_2 + 13\alpha_3 + 42\gamma_1 - 12\gamma_2 - 12\gamma_3 \tanh^2\beta\big) \\ &+\frac{\mu \sinh^2\beta}{2r^2}\big(7\alpha_1 + 5\alpha_2 + 13\alpha_3 \\ &+24(6\beta_1 + 3\beta_2 + 2\gamma_1 - \gamma_2 + \gamma_3(1+\operatorname{sech}^2\beta))\big)\Big], \end{aligned} \tag{II.18}$$

$$\begin{aligned} f_1 = {}& \tfrac{2}{3} g^4 \big(10\alpha_1 + 2\alpha_2 + \alpha_3\big) r^2 H_0^3 \\ &+\frac{g^2\mu^2 \sinh^2 2\beta}{r^4}\big(10\alpha_1 - \alpha_2 - 13\alpha_3 + 20\gamma_1 - \gamma_2 - 6\gamma_3\big) \\ &+\frac{\mu^2}{r^6 H_0}\big[\sinh^2 2\beta\big(3\alpha_1 - \alpha_3 + 18\gamma_1 - 3\gamma_2\big) + 2\alpha_3\big] \\ &-\frac{\mu^3 \sinh^2 2\beta \cosh^2 2\beta}{2r^8 H_0^2}\big(5\alpha_1 + \alpha_2 + \alpha_3 + 30\gamma_1 - 6\gamma_2\big) \\ &+\frac{\mu^4 \sinh^4 2\beta}{96 r^{10} H_0^3}\big(47\alpha_1 + 13\alpha_2 + 17\alpha_3 - 144\beta_1 - 72\beta_2 + 276\gamma_1 - 48\gamma_2 - 24\gamma_3\big), \end{aligned}$$

(II.19)

where $H_0 = 1 + \mu \sinh^2\beta/r^2$ is the zeroth order solution for H. (Since h_1, a_1 and f_1 are already linear in the parameters of the higher order corrections, we may use H and H_0 interchangeably in the above expressions.) Note that the first line in f_1 reproduces the shift of the cosmological constant $g^2 \to g_{\text{eff}}^2$ given in (II.79). This allows us to write

$$f = 1 - \frac{\mu}{r^2} + g_{\text{eff}}^2 r^2 H^3 + \bar{f}_1, \tag{II.20}$$

where $\bar{f}_1$ is given by the remaining terms in (II.19).

In obtaining the above solution, we have imposed the boundary conditions that h_1 and a_1 both fall off faster than $1/r^2$ as $r \to \infty$ so that the R-charge is not modified from its zeroth order value. For f_1, the boundary condition is taken as (II.20), with $\bar{f}_1$ falling off faster than $1/r^2$.

2.1.3 Field Redefinitions

As given in (II.6), we have parameterized the four-derivative terms in the Lagrangian in terms of the eight coefficients (α_1, α_2, ..., γ_3). However, not all of these coefficients are physical. This is because some of the terms in the higher derivative Lagrangian can be removed by field redefinition.

To proceed, we consider transformations of the form

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} + a(R+20g^2)g_{\mu\nu} + b(R_{\mu\nu}+4g^2 g_{\mu\nu}) + cF_{\mu\lambda}F^{\lambda}{}_{\nu} + dF^2 g_{\mu\nu}, \\ A_\mu &\rightarrow (1+g^2(25a+5b-12c+60d))A_\mu. \end{aligned} \tag{II.21}$$

Note that the first two terms in the metric shift incorporate the cosmological constant; this corresponds to the zeroth order Einstein equation in the absence of gauge excitations. While this shift by the cosmological constant is not strictly speaking necessary in performing the field redefinition, we nevertheless find it convenient, as this avoids a shift in the effective cosmological constant g_{eff} after the field redefinition. In addition, the scaling of the gauge field is chosen so that it will remain canonically normalized after the shift of the metric. The result of this combined transformation is to shift the original Lagrangian (II.5) into

$$\begin{aligned} e^{-1}\mathcal{L} = \big(1+12g^2(5a+b)\big)\Big[& R - \tfrac{1}{4}F_{\mu\nu}^2 + 12g^2\left(1-2g^2(5a+b)\right) \\ & + \tfrac{1}{12\sqrt{3}}\left(1+3g^2(5a+b-12c+60d)\right)\epsilon^{\mu\nu\rho\lambda\sigma}F_{\mu\nu}F_{\rho\lambda}A_\sigma \\ & + \left(\alpha_1+\tfrac{1}{2}(3a+b)\right)R^2 + (\alpha_2-b)R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & + \left(\beta_1+\tfrac{1}{8}(c-d)\right)(F_{\mu\nu}F^{\mu\nu})^2 + (\beta_2-\tfrac{1}{2}c)F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\sigma}F^{\sigma}{}_{\mu} \\ & + \left(\gamma_1-\tfrac{1}{8}(a+b+4c-12d)\right)RF^2 \\ & + \left(\gamma_2-\tfrac{1}{2}(b+2c)\right)R_{\mu\nu}F^{\mu\rho}F_{\rho}{}^{\nu} + \gamma_3 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\Big], \end{aligned} \tag{II.22}$$

where, as usual, we only work to linear order in the shift parameters (a, b, c, d).

Up to an overall rescaling, this new Lagrangian can almost be brought back to the original form, provided we shift the various coefficients as follows:

$$\begin{aligned}
&g^2 \to g^2\left(1+2g^2(5a+b)\right), && && \\
&\alpha_1 \to \alpha_1 - \tfrac{1}{2}(3a+b), && \alpha_2 \to \alpha_2 + b, && \alpha_3 \to \alpha_3, \\
&\beta_1 \to \beta_1 - \tfrac{1}{8}(c-d), && \beta_2 \to \beta_2 + \tfrac{1}{2}c, && \\
&\gamma_1 \to \gamma_1 + \tfrac{1}{8}(a+b+4c-12d), && \gamma_2 \to \gamma_2 + \tfrac{1}{2}(b+2c), && \gamma_3 \to \gamma_3.
\end{aligned} \tag{II.23}$$

One difference remains, however, and that is the coefficient of the $F \wedge F \wedge A$ Chern-Simons term. This suggests that, when considering higher derivative corrections in gauged supergravity, there is in fact a preferred field redefinition frame where this Chern-Simons term remains uncorrected. (Such a preferred frame also shows up when considering the supersymmetric completion of the mixed $\mathrm{Tr}\, R \wedge R \wedge A$ term [107].) This $F \wedge F \wedge A$ term is unimportant, however, for the spherically symmetric R-charged black holes considered above in Section 3.

Ignoring the $F \wedge F \wedge A$ term, the freedom to perform field redefinitions of the form (II.21) indicates that at most four of the eight coefficients of the higher derivative terms will be physical. Clearly α_3 and γ_3 are physical, as they cannot be removed by the transformation of (II.23). The additional two physical coefficients can be taken to be a linear combination of

$$\hat{\beta}_1 \equiv \beta_1 + \tfrac{1}{144}(\alpha_1 - 7\alpha_2) + \tfrac{1}{12}(\gamma_1 + \gamma_2) \qquad \text{and} \qquad \hat{\beta}_2 \equiv \beta_2 + \tfrac{1}{4}\alpha_2 - \tfrac{1}{2}\gamma_2. \tag{II.24}$$

In addition, although g^2 is shifted by the field redefinition, the physical cosmological constant, g_{eff}^2, as defined in (II.79), remains invariant.

The use of field redefinitions allows us to rewrite the four-derivative Lagrangian in various forms. A common choice would be to use the Gauss-Bonnet combination $R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\lambda\sigma}^2$ for the curvature-squared terms. This system has been extensively studied in the absence of higher-derivative gauge field corrections, and has the feature that it admits *exact* spherically symmetric black hole solutions, both without [22, 172] and with [173] R-charge. An alternate choice, perhaps more natural from a supersymmetric point of view [107], is the Weyl-squared combination $C^2_{\mu\nu\lambda\sigma} = \frac{1}{6}R^2 - \frac{4}{3}R^2_{\mu\nu} + R^2_{\mu\nu\lambda\sigma}$. Either one of these choices would fix two of the coefficients (*i.e.* α_1 and α_2 in terms of α_3). The additional freedom to perform field redefinitions may then be used to eliminate the mixed RF^2 and $R_{\mu\nu}F^{\mu\lambda}F^{\lambda\nu}$ terms parameterized by γ_1 and γ_2.

Field redefinitions and the first order solution

Given the above field redefinition, it is instructive to examine its effect on the first order black hole solution of (II.17), (II.18) and (II.19). In this case, it is straightforward to see that the coefficient shift of (II.23) results in

$$\begin{aligned}
h_1 \quad &\to \quad \tilde{h}_1 = h_1 + \frac{\mu^2 \sinh^2 2\beta}{8H_0^2 r^6}(-7a + b + 12c - 84d),\\
a_1 \quad &\to \quad \tilde{a}_1 = a_1 + \frac{\mu^2 \sinh^2 2\beta}{8H_0^3 r^6}\left[(-7a + b + 12c - 84d) - \frac{3\mu \sinh^2\beta}{r^2}(a + b - 4c + 12d)\right],\\
f_1 \quad &\to \quad \tilde{f}_1 = f_1 - 2g^4(5a+b)r^2H_0^3 - \frac{g^2\mu^2\sinh^2 2\beta}{2r^4}(25a + 8b - 18c + 60d)\\
&\qquad + \frac{3\mu^2\sinh^2 2\beta}{8r^6 H_0}\left[-2(3a + b - 8c + 36d) + \frac{\mu\cosh^2 2\beta}{r^2 H_0}(5a + b - 12c + 60d)\right.\\
&\qquad \left. - \frac{\mu^2\sinh^2 2\beta}{r^4 H_0^2}(a - 2c + 12d)\right].
\end{aligned} \tag{II.25}$$

At first, this result may appear somewhat surprising. After all, this field redefinition is supposed to be 'unphysical', and yet the form of the solution has changed. The resolution of this puzzle lies in the fact that the we have shifted the metric by terms that are not necessarily proportional to the lowest order equations of motion. (While we have taken care to incorporate the cosmological constant in (II.21), we have omitted the gauge field stress tensor in the shift.) In this sense, while the original and shifted metrics both solve the equations of motion, they nevertheless correspond to physically distinct solutions. The field redefinition of (II.21) is then more naturally thought of as a mapping between solutions.

More explicitly, we note that the shift of the metric given in (II.21) takes the black hole solution away from the form of the initial ansatz given by (II.15). In particular, shifting the metric by (II.21) and using the zeroth order solution gives

$$
\begin{aligned}
g_{tt} &\rightarrow \tilde{g}_{tt} = g_{tt}\left[1 - \frac{\mu^2 \sinh^2 2\beta}{2r^6 H_0^3}(a + 2b - 6c + 12d)\right],\\
g_{rr} &\rightarrow \tilde{g}_{rr} = g_{rr}\left[1 - \frac{\mu^2 \sinh^2 2\beta}{2r^6 H_0^3}(a + 2b - 6c + 12d)\right],\\
g_{\alpha\beta} &\rightarrow \tilde{g}_{\alpha\beta} = g_{\alpha\beta}\left[1 - \frac{\mu^2 \sinh^2 2\beta}{2r^6 H_0^3}(a - b + 12d)\right],
\end{aligned}
\tag{II.26}
$$

where α and β refer to coordinates on S^3. It is now possible to see that a coordinate transformation $r \rightarrow \tilde{r}$ is necessary in order to restore the canonical form of the shifted metric. By identifying

$$
\begin{aligned}
d\tilde{s}^2 &= \tilde{g}_{tt}dt^2 + \tilde{g}_{rr}dr^2 + \tilde{g}_{\theta\theta}d\Omega_3^2\\
&= -\tilde{H}^{-2}\tilde{f}dt^2 + \tilde{H}(\tilde{f}^{-1}d\tilde{r}^2 + \tilde{r}^2 d\Omega_3^2),
\end{aligned}
\tag{II.27}
$$

we end up with expressions for $\tilde{H}$ and $\tilde{f}$

$$
\tilde{H} = \frac{\tilde{g}_{\theta\theta}}{\tilde{r}^2}, \qquad \tilde{f} = -\tilde{g}_{tt}\tilde{g}_{\theta\theta}^2\tilde{r}^4,
\tag{II.28}
$$

as well as a differential equation relating $\tilde{r}^2$ with r^2

$$
\frac{d(\tilde{r}^2)}{d(r^2)} = \frac{\tilde{g}_{tt}\tilde{g}_{rr}\tilde{g}_{\theta\theta}}{r^2}.
\tag{II.29}
$$

Note that, in defining the angular coordinate θ, we have taken the metric on the unit S^3 to be of the form $d\Omega_3^2 = d\theta^2 + \cdots$. The equation for $\tilde{r}^2$ is easily solved, and yields the relation

$$
\tilde{r}^2 = r^2\left[1 + \frac{3\mu^2 \sinh^2 2\beta}{8r^6 H_0^2}(3a + 3b - 12c + 36d)\right],
\tag{II.30}
$$

where we have set a possible integration constant to zero to preserve the $r \rightarrow \infty$ asymptotics.

We are now able to explicitly compute the shifted metric functions $\tilde{h}_1$ and $\tilde{f}_1$ as well as the shifted gauge potential $\tilde{a}_1$. For $\tilde{h}_1$, we use the definition

$$\tilde{H} = 1 + \frac{\mu \sinh^2 \beta}{\tilde{r}^2} + \tilde{h}_1, \tag{II.31}$$

along with (II.28) and (II.30) to obtain

$$\tilde{h}_1 = h_1 + \frac{\mu^2 \sinh^2 2\beta}{8 H_0^2 r^6}(-7a + b + 12c - 84d), \tag{II.32}$$

which is in perfect agreement with (II.25). For $\tilde{f}_1$, on the other hand, we find

$$\begin{aligned} \tilde{f}_1 \;=\; & f_1 - 2g^4(5a+b)r^2 H_0^3 - \frac{3g^2\mu^2 \sinh^2 2\beta}{2r^4}(b - 2c) \\ & + \frac{3\mu^2 \sinh^2 2\beta}{8r^6 H_0}\Bigg[-2(3a + b - 8c + 36d) + \frac{\mu \cosh 2\beta}{r^2 H_0}(5a + b - 12c + 60d) \\ & \qquad - \frac{\mu^2 \sinh^2 2\beta}{r^4 H_0^2}(a - 2c + 12d)\Bigg]. \end{aligned} \tag{II.33}$$

Note that we have defined $\tilde{f}_1$ by

$$\tilde{f} = 1 - \frac{\mu}{\tilde{r}^2} + \tilde{g}^2 \tilde{r}^2 \tilde{H}^3 + \tilde{f}_1, \tag{II.34}$$

where $\tilde{g}^2 = g^2(1 + 2g^2(5a + b))$ is the shifted cosmological constant given in (II.23).

Comparison of (II.33) with (II.25) clearly demonstrates a difference in the $\mathcal{O}(g^2)$ term. The origin of this difference is somewhat subtle, and is related to the choice of boundary conditions for the shifted and unshifted solutions. To see this, we recall that the gauge potential A_μ is also shifted by the field redefinition (II.21) so that it maintains canonical normalization. The implication of this shift on the black hole solution is that

$$A_t \to \big(1 + g^2(25a + 5b - 12c + 60d)\big) A_t, \tag{II.35}$$

where

$$A_t = \sqrt{3} \coth \beta \left(\frac{1 + a_1}{H} - 1 \right), \qquad H = 1 + \frac{\mu \sinh^2 \beta}{r^2} + h_1. \tag{II.36}$$

In order to rescale the potential without adding any $\mathcal{O}(1/r^2)$ terms to H_0, h_1 or a_1, we must instead shift the two parameters μ and β of the black hole according to

$$\coth\beta \to \coth\beta\big(1 + g^2(25a + 5b - 12c + 60d)\big), \qquad \mu\sinh^2\beta \to \mu\sinh^2\beta. \tag{II.37}$$

This corresponds to a rescaling of the nonextremality parameter μ

$$\mu \to \tilde{\mu} = \mu(1 + 2g^2\cosh^2\beta(25a + 5b - 12c + 60d)). \tag{II.38}$$

In this case, the shifted metric function $\tilde{f}$, given in (II.34), ought to more properly be written as

$$\tilde{f} = 1 - \frac{\tilde{\mu}}{\tilde{r}^2} + \tilde{g}^2\tilde{r}^2\tilde{H}^3 + \hat{f}_1, \tag{II.39}$$

where

$$\begin{aligned} \hat{f}_1 &= \tilde{f}_1 + \frac{2g^2\mu\cosh^2\beta}{r^2}(25a + 5b - 12c + 60d) \\ &= f_1 + \lambda\frac{H_0}{r^2} - 2g^4(5a + b)r^2H_0^3 - \frac{g^2\mu^2\sinh^2 2\beta}{2r^4}(25a + 8b - 18c + 60d) + \cdots . \end{aligned} \tag{II.40}$$

This now agrees with $\tilde{f}_1$ of (II.25) up to a solution $\lambda H_0/r^2$ to the homogeneous differential equation for f_1, where

$$\lambda = 2g^2\mu\cosh^2\beta(25a + 5b - 12c + 60d). \tag{II.41}$$

This is a modification of the $\mathcal{O}(1/r^2)$ term in f_1, which, however, is subdominant in f, as the leading behavior of f is given by $f \sim g_{\text{eff}}^2 r^2$ for an asymptotically Anti-de Sitter background.

Finally, we may follow the effect of the field redefinition (II.21) on the gauge potential term a_1. Given the μ and β rescaling of (II.37), we obtain

$$\tilde{a}_1 = (1 + a_1)\frac{\tilde{H}}{H} - 1. \tag{II.42}$$

Working out the right hand side of this expression, we find that it agrees with (II.25). We have thus seen that the first order solution for the spherically symmetric R-charged black hole indeed transforms as expected under field redefinitions.

2.1.4 Discussion

While we have considered general field redefinitions given by four parameters (a, b, c, d), a preferred subset of this would be to shift the metric by the full zeroth order equation of motion

$$R_{\mu\nu} + 4g^2 g_{\mu\nu} - \tfrac{1}{2}F_{\mu\lambda}F_{\nu}{}^{\lambda} + \tfrac{1}{12}g_{\mu\nu}F^2. \tag{II.43}$$

In the above notation, this corresponds to taking

$$c = \tfrac{1}{2}b, \qquad d = -\tfrac{1}{12}(a-b). \tag{II.44}$$

In this case, we may redefine the coefficients of the higher derivative terms according to

$$\begin{aligned}
\beta_1 &= \hat{\beta}_1 - \tfrac{1}{12}(\hat{\gamma}_1 + \hat{\gamma}_2) + \tfrac{1}{144}(\alpha_1 - 7\alpha_2),\\
\beta_2 &= \hat{\beta}_2 + \tfrac{1}{2}\hat{\gamma}_2 + \tfrac{1}{4}\alpha_2,\\
\gamma_1 &= \hat{\gamma}_1 - \tfrac{1}{6}(\alpha_1 - \alpha_2),\\
\gamma_2 &= \hat{\gamma}_2 + \alpha_2,
\end{aligned} \tag{II.45}$$

so that the set (α_3, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\gamma}_1$, $\hat{\gamma}_2$, γ_3) are invariant under the restricted field redefinitions. Note that $\hat{\beta}_1$ and $\hat{\beta}_2$ are the physical coefficients previously defined in (II.24).

It is illuminating to rewrite the higher derivative Lagrangian (II.5) in terms of the new parameters. Ignoring the Chern-Simons term, the result is

$$\begin{aligned}
e^{-1}\mathcal{L} = \big(1 - 8g^2(5\alpha_1 + \alpha_2)\big)\Big[&R - \tfrac{1}{4}\hat{F}^2 + 12g_{\mathrm{eff}}^2 + \alpha_1\mathcal{E}^2 + \alpha_2\mathcal{E}_{\mu\nu}^2 + \alpha_3\big(R_{\mu\nu\lambda\sigma}^2 - 8g^4\big)\\
&+\hat{\beta}_1(\hat{F}^2)^2 + \hat{\beta}_2\hat{F}^4 + \hat{\gamma}_1\mathcal{E}\hat{F}^2 - \hat{\gamma}_2\mathcal{E}_{\mu\nu}\hat{F}^{\mu\sigma}\hat{F}^{\nu}{}_{\sigma} + \gamma_3 R^{\mu\nu\lambda\sigma}\hat{F}_{\mu\nu}\hat{F}_{\lambda\sigma}\Big],
\end{aligned} \tag{II.46}$$

where

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} + 4g^2 g_{\mu\nu} - \tfrac{1}{2}\hat{F}_{\mu\lambda}\hat{F}_{\nu}{}^{\lambda} + \tfrac{1}{12} g_{\mu\nu}\hat{F}^2, \qquad \mathcal{E} = \mathcal{E}^{\mu}{}_{\mu} \tag{II.47}$$

is the zeroth order equation of motion. Note that we have worked to linear order in pulling out the overall factor $1 - 8g^2(5\alpha_1 + \alpha_2)$ renormalizing Newton's constant. Furthermore, $\hat{F} = d\hat{A}$ is a rescaled field strength defined by

$$\hat{A}_\mu = \left[1 + 8g^2\left(\tfrac{1}{3}(5\alpha_1 + \alpha_2) + 5\hat{\gamma}_1 - \hat{\gamma}_2\right)\right] A_\mu, \tag{II.48}$$

so that $\hat{A}_\mu$ remains invariant under the field redefinition of (II.21). The structure of (II.46) now clearly demonstrates that, of the four-derivative terms, only those parameterized by $(\alpha_3,\ \hat{\beta}_1,\ \hat{\beta}_2,\ \gamma_3)$ are physical, as the remaining terms are manifestly proportional to the zeroth order equation of motion.

In principle, the choice of field redefinitions allows us to go back and forth between the Gauss-Bonnet and Weyl-squared parameterizations of the higher-derivative terms in the Lagrangian. In this sense, it is perhaps not a complete surprise to see that in some cases both parameterizations yield the same results for the entropy of BPS black holes [103, 44, 61], even though the bare Gauss-Bonnet correction is not supersymmetric in itself. (Of course, the bare Weyl-squared term is not supersymmetric by itself either.) What this suggests is that the Riemann-squared term parameterized by α_3 plays a crucial and perhaps dominant role in the geometry of higher-derivative black holes, and that the additional matter and auxiliary field terms may contribute only indirectly through their effects on the geometry, at least in the BPS case where there is additional symmetry at the horizon.

Finally, given the general higher-derivative corrected R-charged black holes, it would be interesting to study their thermodynamics and hydrodynamics. One outcome of this study ought to be a clear identification of physical versus unphysical parameters of the theory.

In particular, in the parameterization of (II.46), we would expect all dependence on (α_1, α_2, $\hat{\gamma}_1$, $\hat{\gamma}_2$) to drop out of the thermodynamical quantities. One difficulty in exploring the higher-derivative theory is that some care must be taken in generalizing the Gibbons-Hawking surface term. This is because the general (*i.e.* non Gauss-Bonnet) combination of R^2 terms leads to higher than second-derivative terms in the equations of motion, and hence necessitates specifying additional boundary data [143]. As demonstrated in [35], one way around this is to perturb in the higher-derivative terms and to demand that the undesired boundary variations vanish when the lowest-order equations of motion are imposed. The goal of the next sections is to apply this procedure to the gravitational sector of the general parameterized four-derivative Lagrangian with a goal of exploring higher-derivative black hole thermodynamics using holographic renormalization.

2.2 Gibbons-Hawking Terms

As mentioned, in theories that are maximally supersymmetric (e.g. IIB theory in ten dimensions), the first corrections do not enter until $\alpha'^3 R^4$ order. However, generically one would expect the first non-trivial terms to appear at curvature-squared level. This has motivated numerous recent holographic studies with R^2 terms parameterized by

$$e^{-1}\delta\mathcal{L} = \alpha_1 R^2 + \alpha_2 R_{\mu\nu}^2 + \alpha_3 R_{\mu\nu\rho\sigma}^2\,. \tag{II.49}$$

In the absence of matter fields, the Einstein equation takes the form $R_{\mu\nu} = -(d-1)g^2 g_{\mu\nu}$, where $g = 1/L$ is the inverse AdS radius. As a result, the α_1 and α_2 terms in (II.49) may be shifted away by an on-shell field redefinition of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda_1[R_{\mu\nu} + (d-1)g^2 g_{\mu\nu}] + \lambda_2 g_{\mu\nu}[R + d(d-1)g^2], \tag{II.50}$$

for appropriate choices of λ_1 and λ_2. In particular, such a field redefinition allows (II.49) to be replaced by the well-known Gauss-Bonnet combination

$$e^{-1}\mathcal{L}_{GB} = \alpha_3(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2), \tag{II.51}$$

which is the unique curvature-squared combination that nevertheless yields equations of motion that are no higher than second derivative in the metric.

Many of the positive features of the Gauss-Bonnet combination, including exact Gauss-Bonnet black hole solutions, have been exploited in recent investigations of AdS/CFT hydrodynamics [27, 26]. However, it is important to realize that the α_1 and α_2 terms in (II.49) are not always unphysical once matter fields are turned on. For example, in an Einstein-Maxwell theory, shifting away the α_1 and α_2 terms in (II.49) would at the same time generate new mixed terms of the form RF^2 and $R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda}$. This is especially relevant in studies of R-charged backgrounds in five-dimensional gauged supergravity, where the natural curvature-square correction arises as the Weyl-tensor squared, as opposed to the Gauss-Bonnet combination [107, 54].

Perturbative approach to higher-derivative terms

The purpose of this section is to revisit the holographic renormalization of R-squared AdS gravity and to demonstrate the systematic construction of both generalized Gibbons-Hawking surface terms and local boundary counterterms in theories with higher derivatives. It is well known that higher derivative theories generically lead to unpleasant features such as ghosts and additional propagating degrees of freedom. However, since the theories we are interested in arise from the low energy limit of string theory, it is only consistent to treat the higher derivative terms perturbatively, as part of the α' expansion. In this way, these terms will not generate additional ghost modes, and thus will not drastically alter the dynamics of the lowest order two-derivative theory.

As an example of what we mean by the perturbative treatment of higher derivative terms, consider a toy model of a simple harmonic oscillator with a four-derivative addition [162]

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - \tfrac{1}{2}\alpha(\ddot{x}^2 - \omega^2\dot{x}^2). \tag{II.52}$$

The resulting equation of motion is

$$(1+\alpha\omega^2)\ddot{x}+\omega^2x^2+\alpha x^{(4)}=0, \tag{II.53}$$

and has solution

$$x(t)=A_1e^{i\omega t}+A_2e^{-i\omega t}+A_3e^{it/\sqrt{\alpha}}+A_4e^{-it/\sqrt{\alpha}}\,. \tag{II.54}$$

The first two terms are conventional, while the last two arise because of the higher derivative nature of the model. This demonstrates that additional degrees of freedom are present in this theory, and in particular it is no longer sufficient to specify only two boundary conditions when constructing the Green's function. This is also clear when considering the variation of the action

$$\delta S=-\int_{t_1}^{t_2}[\mathrm{EOM}]dt+\Big[((1+\alpha\omega^2)\dot{x}+\alpha\dddot{x})\delta x-\alpha\ddot{x}\delta\dot{x}\Big]_{t_1}^{t_2}. \tag{II.55}$$

In order to have a well-defined variational principle, we must hold both x and $\dot{x}$ fixed at the endpoints of the time interval.

In general, for finite non-zero α, there is no possibility of avoiding the complications of the higher-derivative theory. However, it is instructive to consider the limit $\alpha\to 0$. In this case, it is clear that the second solution, with frequency $1/\sqrt{\alpha}$, is not perturbatively connected to the $\alpha=0$ theory. Assuming the toy Lagrangian (II.52) arises from an $\mathcal{O}(\alpha)$ expansion of a more complete theory, it is then clear that the second solution would never have appeared in the full theory, and thus must be discarded for perturbative consistency. A simple way of arriving at the perturbative solution is to rewrite the equation of motion (II.53) as

$$\ddot{x}+\omega^2x^2=-\alpha\frac{d^2}{dt^2}(\ddot{x}+\omega^2x), \tag{II.56}$$

We may then substitute in the lowest order equation of motion to obtain $\ddot{x}+\omega^2x^2=\mathcal{O}(\alpha^2)$, and in general iterate to any arbitrary order of α (our choice of shifting the kinetic term in (II.52) leads to vanishing perturbative corrections in α, but in general they could be present).

While perturbative solutions to the equation of motion are routinely investigated, it is often equally important to construct a well-defined variational principle at the perturbative level. Looking at the toy model, the difficulty here arises from the $-\alpha\ddot{x}\delta\dot{x}$ surface variation in (II.55). In general, no surface term exists that can remove the dependence on $\delta\dot{x}$ on the boundary (after all, this is a four derivative theory). However, at the perturbative level, we may use the lowest order equation of motion to rewrite $-\alpha\ddot{x}\delta\dot{x} = \alpha\omega^2 x\delta\dot{x} + \mathcal{O}(\alpha^2)$. This variation can then be canceled at $\mathcal{O}(\alpha)$ by adding a surface term of the form

$$S_{\text{surface}} = \Big[-\alpha\omega^2 x\dot{x}\Big]_{t_1}^{t_2}. \tag{II.57}$$

In principle, this can be continued order by order in α.

Using this toy model, we have motivated the fact that there is a consistent perturbative treatment of higher derivative gravitational theories arising out of string theory. In particular, the gravitational analog of (II.57) is a generalized Gibbons-Hawking surface term, and this was constructed in a particular case in [35] when examining the effect of the IIB R^4 term on the shear viscosity to entropy density ratio η/s in $\mathcal{N}=4$ super-Yang-Mills theory. The construction in [35] was based on scalar channel fluctuations, and hence focused on an effective scalar field theory. Our present aim is to extend this construction to the full gravity theory, and hence to demonstrate that (perturbative) holographic renormalization of higher derivative gravity theories is indeed consistent.

Allowing for a gauge field, we focus on the holographic renormalization of d-dimensional Einstein-Maxwell theory with generic curvature-squared corrections given by

$$e^{-1}\mathcal{L} = R - \tfrac{1}{4}F^2 + (d-1)(d-2)g^2 + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}^2 + \alpha_3 R_{\mu\nu\rho\sigma}^2. \tag{II.58}$$

The bulk action from this Lagrangian must be supplemented by a set of surface terms, whose goal is to ensure that the variational principle is well defined. In fact, when defined on a space with boundary, the two-derivative Einstein-Hilbert action itself requires the addition of the Gibbons-Hawking surface term to cancel boundary variations which would otherwise spoil the variational principle. The presence of higher derivative corrections leads to additional boundary terms which need to be canceled, and therefore requires the inclusion of an appropriate generalization of the Gibbons-Hawking term.

For particular combinations of curvature corrections, the so-called Lovelock theories where the equations of motion involve no higher than second derivatives of the metric – which include the Gauss-Bonnet combination as a special case – proper boundary terms have already been constructed [167, 143]. However, for more general corrections, we must treat the corrections perturbatively, and only in this case does the construction of a generalized Gibbons-Hawking term become feasible[2]. We demonstrate below how this is done, and furthermore construct the set of local counterterms removing the leading divergences from the action. This generalizes the case of Gauss-Bonnet gravity, for which all the counterterms needed to regularize the action were constructed in [29, 9, 28, 130].

R-charged black holes and the mass-charge relation

For an application of the counterterm corrected action, we will look at R-charged black hole thermodynamics. In fact, one of the driving forces behind the studies of AdS/CFT at finite temperature has been the close resemblance of the laws of black hole physics with those of standard thermodynamics. To extract thermodynamic quantities from black hole backgrounds one typically evaluates the on-shell action I and the boundary stress tensor, given by

$$T^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ab}} , \tag{II.59}$$

[2]A similar construction has also been done for $F(R)$ theories of gravity in [70] and also for more general higher derivative theories in [150, 153, 152, 60].

where h_{ab} denotes the boundary metric. The on-shell value of the gravitational action may then be identified with the thermodynamic potential Ω according to $I = \beta\Omega$, where in the grand canonical ensemble

$$\Omega = E - TS - Q_I \Phi^I \, . \tag{II.60}$$

Here Q_I are a set of conserved R-charges and Φ^I their respective potentials. Holographic renormalization ensures that both Ω and E are finite in the above expression.

Below we will perturbatively construct the d-dimensional spherically symmetric R-charged black hole solutions to the R-squared theory (II.58) and study their thermodynamic properties using the holographically renormalized action. Extracting the higher curvature effects on the black hole mass will also allow us to discuss the weak gravity conjecture in the context of AdS black holes. In fact, according to the conjecture, the linear mass-charge relation for extremal (not necessarily SUSY) black holes cannot be exact, but should receive corrections as the charge decreases. For extremal R-charged black-holes, we find a deviation from the leading relation $m = q$ of the form

$$\frac{m}{q} = \left(\frac{m}{q}\right)_0 \left[1 - \frac{1}{r_+^2}\Big(\alpha_1 f_1(r_+) + \alpha_2 f_2(r_+) + \alpha_3 f_3(r_+)\Big)\right] , \tag{II.61}$$

where r_+ is the horizon radius, and the $f_i(r_+)$ are all positive functions. Thus, m/q will necessarily decrease when all the couplings α_i are positive. Clearly, it is still possible for the ratio to decrease if some of the α_i are negative, and in this respect it is important to be able to determine the precise form of the couplings from UV physics.

A feature which we would like to emphasize is that the deviation from the $m = q$ relation seems to be tied to the correction to some of the transport coefficients which have been computed holographically in the context of the quark gluon plasma. In particular, the sign of the correction to the shear viscosity to entropy ratio η/s has received a lot of attention, precisely because curvature-squared terms have been shown to lead to a violation of the KSS bound [126]. For the examples that have been studied thus far, the sign of the higher derivative couplings responsible for the bound violation is precisely the same as that needed by the weak gravity conjecture [8]. For instance, for the special case of Weyl-squared corrections, where $\alpha_1 = \frac{1}{6}\alpha$, $\alpha_2 = -\frac{4}{3}\alpha$, $\alpha_3 = \alpha$, the mass-charge relation becomes

$$\frac{m}{q} = \left(\frac{m}{q}\right)_0 \left[1 - \alpha \frac{f(r_+)}{r_+^2}\right], \tag{II.62}$$

where the function $f(r_+)$ is positive, while the expression for η/s takes the form

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \alpha\, g(Q)\right], \tag{II.63}$$

where $g(Q)$ is a non-negative function of the R-charge.

The outline of the rest of this chapter is as follows. Section 2.2.1 is dedicated to the construction of the perturbative generalization of the Gibbons-Hawking surface term for the R^2 action (II.58). Following this, in Section 2.2.2 we present the local counterterms needed to render this action finite in dimensions $d \leq 7$. We then present the R-charged black hole solution in Section 2.2.3 and explore their thermodynamics in Section 2.2.4. Finally in section 2.2.5 we discuss the implications of the mass to charge ratio for the weak gravity conjecture.

2.2.1 Generalizing the Gibbons-Hawking surface term

Before considering the higher derivative gravitational action, it is worth recalling that the ordinary Einstein-Hilbert action

$$S_{\text{bulk}} = -\frac{1}{2\kappa_d^2}\int_{\mathcal{M}} d^d x\sqrt{-g}R \tag{II.64}$$

contains explicitly second derivatives of the metric $g_{\mu\nu}$. Thus, on a space with a boundary, variation with respect to the metric yields, in addition to the standard $\delta g_{\mu\nu}$ factors, terms involving the normal derivative of the metric. In order to have a well-defined variational principle where the metric, but not its derivative, is held fixed at the boundary, the Einstein-Hilbert action must be supplemented by the Gibbons-Hawking surface term

$$S_{\rm GH} = -\frac{1}{\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}K\,. \tag{II.65}$$

Here K denotes the trace of the extrinsic curvature tensor, $K_{\mu\nu} = \nabla_{(\mu}n_{\nu)}$, where n_μ specifies the normal direction to the boundary surface, and h_{ab} is the boundary metric. With the inclusion of the Gibbons-Hawking term, the unwanted normal derivative terms are canceled, and the variational principle is well-defined.

We now consider the addition of curvature-squared terms, and take the bulk action to be of the form

$$S_{\rm bulk} = -\frac{1}{2\kappa_d^2}\int_{\mathcal{M}} d^dx\sqrt{-g}\left[R-\frac{1}{4}F^2+(d-1)(d-2)g^2+\alpha_1 R^2+\alpha_2 R_{\mu\nu}^2+\alpha_3 R_{\mu\nu\rho\sigma}^2\right]. \tag{II.66}$$

In general, this four-derivative action gives rise to higher order equations of motion. However, for the special choice of coefficients $\alpha_1 = \alpha_3$ and $\alpha_2 = -4\alpha_3$, the higher derivative terms combine to form the well-known Gauss-Bonnet term $R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2$, which is the unique combination that gives rise to equations of motion involving no higher than second derivatives of the metric. This motivates us to rewrite (II.66) in the equivalent form

$$\begin{aligned} S_{\rm bulk} = -\frac{1}{2\kappa_d^2}\int_{\mathcal{M}} d^dx\sqrt{-g}\Big[R - \frac{1}{4}F^2 + (d-1)(d-2)g^2 \\ +\tilde\alpha_1 R^2 + \tilde\alpha_2 R_{\mu\nu}^2 + \alpha_3(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2)\Big], \end{aligned} \tag{II.67}$$

where

$$\tilde\alpha_1 = \alpha_1 - \alpha_3, \qquad \tilde\alpha_2 = \alpha_2 + 4\alpha_3. \tag{II.68}$$

For the special case of Gauss-Bonnet gravity, where $\tilde\alpha_1 = \tilde\alpha_2 = 0$, the Gibbons-Hawking surface term can be generalized [167, 143], and takes the form

$$\begin{aligned} S_{\rm GH}^{\text{Gauss-Bonnet}} &= -\frac{1}{\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\,\alpha_3\Big[-\frac{2}{3}K^3 + 2KK_{ab}K^{ab} - \frac{4}{3}K_{ab}K^{bc}K_c{}^a \\ &\qquad -4(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}h_{ab})K^{ab}\Big], \end{aligned} \tag{II.69}$$

where $\mathcal{R}_{ab}$ is the boundary Ricci tensor. However, no equivalent term exists for $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ non-vanishing, because in this case the equations of motion are of higher order, and in general it is no longer sufficient to specify only the metric (and not derivatives) on the boundary.

This issue is unavoidable whenever we are faced with higher order equations of motion. However, we are really only interested in viewing the higher order terms as corrections to the two-derivative action. In this case, we only need to develop a perturbative expansion where the higher derivative terms do not generate their own dynamics, but instead contribute merely correction terms, thus effectively maintaining a two-derivative equation of motion. In this case, it should be possible to write down an effective Gibbons-Hawking term, not just for the Gauss-Bonnet combination, but also for the R^2 and $R_{\mu\nu}^2$ terms in the action. This has been done for R^2 corrections in $d = 5$ by introducing auxiliary fields [60]. However, one can avoid the complications involved in utilizing auxiliary fields by working directly with the perturbative expansion.

To see how this may be done, we begin with the observation that the ordinary Gibbons-Hawking term (II.65) is designed to cancel the appropriate part of the variation of the Einstein-Hilbert term, namely $\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$. With this in mind, consider the variation

$$
\begin{aligned}
\delta[R + \tilde{\alpha}_1 R^2 + \tilde{\alpha}_2 R_{\mu\nu}^2] &= \delta R + 2\tilde{\alpha}_1 R\delta R + 2\tilde{\alpha}_2 (R^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\rho}R^{\mu}{}_{\sigma}\delta g^{\rho\sigma}) \\
&= (g^{\mu\nu} + 2\tilde{\alpha}_1 R g^{\mu\nu} + 2\tilde{\alpha}_2 R^{\mu\nu})\delta R_{\mu\nu} \\
&\quad + (R_{\mu\nu} + 2\tilde{\alpha}_1 R R_{\mu\nu} + 2\tilde{\alpha}_2 R_{\mu\rho} R_{\nu}{}^{\rho})\delta g^{\mu\nu}.
\end{aligned} \tag{II.70}
$$

Substituting in the lowest order equation

$$
R_{\mu\nu} = -(d-1)g^2 g_{\mu\nu} + \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{2(d-2)}g_{\mu\nu}F^2\right) + \mathcal{O}(\alpha_i) \tag{II.71}
$$

results in

$$\delta[R+\tilde{\alpha}_1 R^2+\tilde{\alpha}_2 R_{\mu\nu}^2] = (1-2\tilde{\alpha}_1 g^2 d(d-1)-2\tilde{\alpha}_2 g^2(d-1))g^{\mu\nu}\delta R_{\mu\nu} + \frac{1}{2(d-2)}(\tilde{\alpha}_1(d-4)-\tilde{\alpha}_2)F^2 g^{\mu\nu}\delta R_{\mu\nu}+\tilde{\alpha}_2 F^{\mu\lambda}F^{\nu}{}_{\lambda}\delta R_{\mu\nu}+\cdots, \tag{II.72}$$

where we have ignored higher order terms as well as terms not related to the variation $\delta R_{\mu\nu}$.

For the terms in (II.72) involving simply $g^{\mu\nu}\delta R_{\mu\nu}$, it is straightforward to generalize the usual Gibbons-Hawking term, (II.65), to obtain a corresponding surface term canceling the variation of the normal derivative of the metric

$$S^1_{\mathrm{GH}} = -\frac{1}{\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\Big[(1-2\tilde{\alpha}_1 g^2 d(d-1)-2\tilde{\alpha}_2 g^2(d-1))K + \frac{1}{2(d-2)}(\tilde{\alpha}_1(d-4)-\tilde{\alpha}_2)KF^2\Big]. \tag{II.73}$$

However, the last term in (II.72) is a not as straightforward to deal with, and the variation $\delta R_{\mu\nu}$ must be computed explicitly. We find,

$$\begin{aligned}
&\int_{\mathcal{M}} d^dx\sqrt{-g}F^{\mu\lambda}F^{\nu}{}_{\lambda}\delta R_{\mu\nu}\\
&\quad= \int_{\mathcal{M}} d^dx\sqrt{-g}F^{\mu\lambda}F^{\nu}{}_{\lambda}\left(\nabla_\sigma\delta\Gamma^{\sigma}_{\mu\nu}-\nabla_\mu\delta\Gamma^{\sigma}_{\nu\sigma}\right)\\
&\quad= \frac{1}{2}\int_{\mathcal{M}} d^dx\sqrt{-g}F^{\mu\lambda}F^{\nu}{}_{\lambda}\left(2n^\rho\nabla_{(\mu}\delta g_{\nu)\rho}-n^\rho\nabla_\rho\delta g_{\mu\nu}-n_\mu g^{\rho\sigma}\nabla_\nu\delta g_{\rho\sigma}\right)\\
&\quad= \frac{1}{2}\int_{\mathcal{M}} d^dx\sqrt{-g}\,[\mathrm{bulk}]+\frac{1}{2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\big(-h^a{}_c h^b{}_d F^{c\lambda}F^d{}_\lambda\\
&\qquad -h^{ab}n_\mu F^{\mu\lambda}n_\nu F^\nu{}_\lambda\big)n^\rho\nabla_\rho\delta g_{ab}+\cdots,
\end{aligned} \tag{II.74}$$

where in the last line we have kept only the terms on the boundary coming from integration by parts and including normal derivatives of the metric. The proper Gibbons-Hawking boundary term associated with this variation is then simply:

$$S^2_{\mathrm{GH}} = -\frac{1}{\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\frac{\tilde{\alpha}_2}{2}\left(Kn_\mu F^{\mu\lambda}n_\nu F^\nu{}_\lambda+K_{ab}F^{a\lambda}F^b{}_\lambda\right). \tag{II.75}$$

It is now clear that the full effective Gibbons-Hawking term generalizing (II.65) is just the sum of (II.73) and (II.75), which handles the $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ terms, and (II.69), which takes care of the α_3 Gauss-Bonnet combination:

$$
\begin{aligned}
S_{\mathrm{GH}} \;=\; & -\frac{1}{\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\Big[\big(1-2\tilde{\alpha}_1 g^2 d(d-1)-2\tilde{\alpha}_2 g^2(d-1)\big)K \\
& +\frac{1}{2(d-2)}\big(\tilde{\alpha}_1(d-4)-\tilde{\alpha}_2\big)KF^2+\frac{\tilde{\alpha}_2}{2}\left(Kn_\mu F^{\mu\lambda}n_\nu F^\nu{}_\lambda+K_{ab}F^{a\lambda}F^b{}_\lambda\right) \\
& -2\alpha_3\Big(\frac{1}{3}K^3-KK_{ab}K^{ab}+\frac{2}{3}K_{ab}K^{bc}K_c{}^a+2(\mathcal{R}_{ab}-\frac{1}{2}\mathcal{R}h_{ab})K^{ab}\Big)\Big]. \qquad \text{(II.76)}
\end{aligned}
$$

We note that the Gibbons-Hawking term now involves the gauge field strength evaluated on the boundary. Variation of S_{GH} then results in δF terms on the boundary, thus complicating the variational principle for the potential A_μ. This can in principle be avoided by working in the canonical ensemble, where the charge is held fixed, and which corresponds to taking $\delta(n_\mu F^{\mu a})=0$ instead of $\delta A_\mu=0$ on the boundary. A natural way to do this is to add a Hawking-Ross boundary term of the form $\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\,n_\mu F^{\mu a}A_a$ to cancel the boundary term which arises from the variation of the gauge kinetic term in the bulk action [111]. However, for our present purposes, all terms involving the field strength in (II.76) are actually subdominant and, in fact, vanish for all of the thermodynamic quantities discussed below. Therefore, we will chose to work in the grand-canonical ensemble without adding the Hawking-Ross term.

2.2.2 Boundary Counterterms

It is well known that the gravitational action (II.66) evaluated on the background solution is divergent. The divergences can be removed, however, using the method of holographic renormalization, which involves introducing appropriate boundary counterterms S_{ct} so that the full action

$$\Gamma = S_{\mathrm{bulk}} + S_{\mathrm{GH}} - S_{\mathrm{ct}}, \qquad \text{(II.77)}$$

remains finite on-shell. This method has become quite standard in the framework of AdS/CFT, since the boundary counterterms have a natural interpretation as conventional field theory counterterms in the dual CFT.

Along with counterterms to remove divergences, one is also free to add an arbitrary number of finite counterterms. While such terms shift the values of the action and boundary stress tensor, they are natural from the CFT point of view, since they correspond to the freedom to change renormalization prescriptions. Their inclusion has played a key role, for example, in resolving the puzzle of the unusual mass/charge relation $M \sim \frac{3}{2}\mu + Q - \frac{1}{3}g^2 Q^2$ observed in [38] for single R-charged black holes in AdS_5, in apparent conflict with the BPS bound $M \geq Q$, saturated in this case when $\mu = 0$. With the addition of an appropriate finite counterterm, the expected linear relation $M \sim \frac{3}{2}\mu + 3Q$ may be restored [129]. The finite counterterms are also necessary for maintaining diffeomorphism invariance in the renormalized theory, and may be unambiguously generated using the Hamilton-Jacobi approach to boundary counterterms.

In order to explore the appropriate counterterm structure needed to regulate the action (II.66), we first note that it admits a vacuum AdS solution with

$$R_{\mu\nu} = -(d-1) g_{\mathrm{eff}}^2 g_{\mu\nu}, \tag{II.78}$$

where

$$g_{\mathrm{eff}}^2 = g^2 \left(1 + \tilde{\alpha}_1 g^2 \frac{d(d-1)(d-4)}{d-2} + \tilde{\alpha}_2 g^2 \frac{(d-1)(d-4)}{d-2} + \alpha_3 g^2 (d-3)(d-4) \right) \tag{II.79}$$

is the shifted inverse AdS radius. Writing the vacuum AdS metric as

$$ds^2 = -(k + g_{\mathrm{eff}}^2 r^2) dt^2 + \frac{dr^2}{k + g_{\mathrm{eff}}^2 r^2} + r^2 d\Omega_{d-2,k}^2, \tag{II.80}$$

it is easy to see that $\sqrt{-g} \sim r^{d-2}$, and hence that the leading divergence of the on-shell goes as r_0^{d-1} where r_0 is an appropriate cutoff.

The counterterm action for the theory (II.66) may be expanded in powers of the inverse metric $h^{ab} \sim 1/r_0^2$:

(II.81) $$S_{\rm ct} = \frac{1}{2\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\,[A + B\mathcal{R} + C_1\mathcal{R}^2 + C_2\mathcal{R}_{ab}^2 + C_3\mathcal{R}_{abcd}^2 + \cdots]\,.$$

Note that we have ignored possible counterterms built out of $F_{\mu\nu}$ since in the configurations we are interested in the gauge field vanishes sufficiently rapidly at the boundary so that it will not contribute to any potential counterterms. The A and B coefficients are chosen to cancel the r_0^{d-1} and r_0^{d-3} power law divergences, respectively, while the C_i terms will cancel the r_0^{d-5} divergence. Note, however, that at lowest order the asymptotic Einstein condition $R_{\mu\nu} = -(d-1)g^2 g_{\mu\nu}$ along with the boundary symmetry implied by (II.80) ensures that the boundary curvature satisfies the algebraic relation $\mathcal{R}^2 = (d-2)\mathcal{R}_{ab}^2$. Furthermore, isotropy of the transverse space relates $\mathcal{R}_{abcd}^2$ to the other boundary curvature squared quantities as well. What this means is that divergence cancellation by itself is insufficient to fix the relative factors among the C_i coefficients.

An elegant way around this ambiguity in fixing the C_i coefficients is to use the Hamilton-Jacobi method to obtain the counterterms. In particular, this was done in [130] to generate the counterterms for the Gauss-Bonnet component of the action proportional to α_3. (These counterterms were previously constructed in [29, 9, 28] using more direct methods.) In order to determine the $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ dependent counterterms, we may take a shortcut and note that they may be absorbed by a field redefinition in the asymptotic limit. In this case, their only effect is to rescale the usual counterterms for the two-derivative theory, which is proportional to the combination

(II.82) $$\mathcal{R}_{ab}^2 - \frac{d-1}{4(d-2)}\mathcal{R}^2$$

at curvature squared order. At the linear level, we combine the various ingredients to obtain

$$
\begin{aligned}
S_{\mathrm{ct}} \;=\; \frac{1}{2\kappa_d^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\Bigg[&2g(d-2)\bigg(1-\frac{1}{2}\tilde{\alpha}_1 g^2\frac{d(d-1)(3d-4)}{d-2}\\
&\qquad -\frac{1}{2}\tilde{\alpha}_2 g^2\frac{(d-1)(3d-4)}{d-2}-\frac{1}{6}\alpha_3 g^2(d-3)(d-4)\bigg)\\
&+\frac{1}{g(d-3)}\bigg(1-\frac{1}{2}\tilde{\alpha}_1 g^2\frac{d(d-1)(5d-12)}{d-2}\\
&\qquad -\frac{1}{2}\tilde{\alpha}_2 g^2\frac{(d-1)(5d-12)}{d-2}+\frac{3}{2}\alpha_3 g^2(d-3)(d-4)\bigg)\mathcal{R}\\
&+\frac{1}{g^3(d-3)^2(d-5)}\bigg(1-\frac{1}{2}\tilde{\alpha}_1 g^2\frac{d(d-1)(7d-20)}{d-2}-\frac{1}{2}\tilde{\alpha}_2 g^2\frac{(d-1)(7d-20)}{d-2}\\
&\qquad -\frac{7}{2}\alpha_3 g^2(d-3)(d-4)\bigg)\left(\mathcal{R}_{ab}^2-\frac{d-1}{4(d-2)}\mathcal{R}^2\right)\\
&+\frac{\alpha_3}{g(d-5)}\left(\mathcal{R}^2-4\mathcal{R}_{ab}^2+\mathcal{R}_{abcd}^2\right)+\cdots\Bigg]. \qquad \text{(II.83)}
\end{aligned}
$$

We have only explicitly worked out the counterterms up to $\mathcal{O}(r_0^{d-5})$. This is sufficient to cancel divergences for $d \leq 7$, but is insufficient for removing finite terms that spoil diffeomorphism invariance in $d = 7$. Hence our results are explicit only for $d < 7$, although the counterterm action can be extended to arbitrary dimension if desired.

2.2.3 The R^2 corrected black hole solution

The full theory we are interested in is determined by the bulk action (II.66) along with the generalized Gibbons-Hawking term (II.76) and counterterm action (II.83). We now turn to the construction of R^2 corrected spherically symmetric black hole solutions to this system. Since we are working to linear order in α_i, we may substitute the lowest order equations of motion wherever possible into the higher curvature terms. We find that the

Einstein equation takes the form

$$
\begin{aligned}
&R_{\mu\nu} + (d-1)g^2 g_{\mu\nu} + \frac{1}{4(d-2)}F^2 g_{\mu\nu} - \frac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda = \\
&\quad \Bigg[-g^4(\alpha_1 d+\alpha_2)\frac{(d-4)(d-1)^2}{d-2} - g^2\big(\alpha_1(d^2-8)+\alpha_2(3d-8)\big)\frac{(d-1)}{2(d-2)^2}F^2 \\
&\qquad +\big(\alpha_1(d-4)(3d-8)-\alpha_2(5d-12)\big)\frac{1}{16(d-2)^3}(F^2)^2 + \frac{\alpha_2}{4(d-2)}F_{\gamma\lambda}F^{\lambda\sigma}F_{\sigma\rho}F^{\rho\gamma} \\
&\qquad +\big(\alpha_1(d-4)+\alpha_2(d-3)+\alpha_3(3d-8)\big)\frac{1}{2(d-2)^2}\nabla_\lambda\nabla^\lambda F^2 + \frac{\alpha_3}{d-2}R^2_{\gamma\rho\lambda\sigma}\Bigg]g_{\mu\nu} \\
&\qquad +g^2(\alpha_1 d+\alpha_2-2\alpha_3)(d-1)F_{\mu\lambda}F_\nu{}^\lambda + \alpha_3 F_{\mu\lambda}F^{\lambda\sigma}F_{\sigma\rho}F^\rho{}_\nu \\
&\quad -\big(\alpha_1(d-4)-\alpha_2+2\alpha_3\big)\frac{1}{4(d-2)}F^2 F_{\mu\lambda}F_\nu{}^\lambda - 2\alpha_3 R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} \\
&\quad -(\alpha_2+2\alpha_3)R_{\mu\rho\nu\lambda}F^{\rho\sigma}F^\lambda{}_\sigma - \frac{1}{2}(\alpha_2+4\alpha_3)\nabla_\lambda\nabla^\lambda(F_{\mu\lambda}F_\nu{}^\lambda) \\
&\quad +(2\alpha_1+\alpha_2+2\alpha_3)\frac{(d-4)}{4(d-2)}\nabla_\mu\nabla_\nu F^2\,, \qquad \text{(II.84)}
\end{aligned}
$$

while the Maxwell equation is simply

$$\nabla^\mu F_{\mu\nu} = 0. \qquad \text{(II.85)}$$

The presence of F^4 terms in the Einstein equation indicates that we will end up with metric terms up to $\mathcal{O}(Q^4)$ where Q is the electric charge.

We now take the spherically symmetric metric ansatz

$$ds^2 = -f_1(r)\,dt^2 + \frac{1}{f_2(r)}\,dr^2 + r^2 d\Omega^2_{d-2,k}\,, \qquad \text{(II.86)}$$

where $k = 1, 0, -1$ specifies the curvature of the transverse space. Inserting this into the

Einstein equations yields the solution to linear order in the α_i:

$$\begin{aligned}
f_1(r) &= k + g_{\text{eff}}^2 r^2 - \frac{\mu}{r^{d-3}} \\
&\quad + \big(1 + 2g^2(\tilde\alpha_1 d(d-1) - \tilde\alpha_2(d^2-6d+7) + \alpha_3(d-3)(d-4))\big)\frac{Q^2}{2(d-2)(d-3)r^{2(d-3)}} \\
&\quad + \frac{kQ^2}{r^{2d-4}}\left(2\tilde\alpha_1\frac{(d-4)}{(d-2)^2} - \tilde\alpha_2\frac{d^2-6d+10}{(d-2)^2}\right) + \alpha_3(d-3)(d-4)\frac{\mu^2}{r^{2d-4}} \\
&\quad - \frac{\mu Q^2}{r^{3d-7}}\left(\tilde\alpha_1\frac{(d-1)(d-4)}{(d-2)^2} - \frac{\tilde\alpha_2}{(d-2)^2} + \alpha_3\frac{(d-4)}{(d-2)}\right) \\
&\quad + \frac{Q^4}{4r^{4d-10}}\left(\tilde\alpha_1\frac{(d-4)(11d^2-45d+44)}{(d-2)^3(d-3)(3d-7)} + \tilde\alpha_2\frac{4d^3-33d^2+83d-64}{(d-2)^3(d-3)(3d-7)}\right. \\
&\qquad\qquad \left. + \alpha_3\frac{(d-4)}{(d-2)^2(d-3)}\right) \qquad \text{(II.87)} \\
f_2(r) &= \left(1 - 2\gamma\frac{Q^2}{r^{2d-4}}\right) f_1(r)\,,
\end{aligned}$$

where g_{eff} is defined in (II.79) and

$$\gamma = \tilde\alpha_1\frac{(2d-3)(d-4)}{(d-2)^2} + \tilde\alpha_2\frac{d^2-5d+5}{(d-2)^2}. \tag{II.88}$$

The gauge field is given by

$$A_t = \frac{Q}{(d-3)r^{d-3}} + \gamma\frac{Q^3}{(3d-7)r^{3d-7}}, \tag{II.89}$$

up to a possible constant.

Other than k, the black hole depends on two parameters: μ, which is related to the mass, and Q, which is essentially the electric charge. Note that the mass parameter μ is shifted from the conventional Gauss-Bonnet black hole mass parameter by a constant proportional to α_3. In particular, the Gauss-Bonnet theory ($\tilde\alpha_1 = \tilde\alpha_2 = 0$) admits an exact solution with a corresponding mass parameter $\hat\mu$ of the form

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega_{d-2,k}^2, \tag{II.90}$$

where [22, 172, 173, 39]

$$f = k + \frac{r^2}{2\tilde\alpha_3}\left[1 \mp \sqrt{1 + 4\tilde\alpha_3\left(\frac{\hat\mu}{r^{d-1}} - g^2 - \frac{Q^2}{2(d-2)(d-3)r^{2(d-2)}}\right)}\,\right], \tag{II.91}$$

and $\tilde{\alpha}_3 = \alpha_3(d-3)(d-4)$. Taking the 'negative' branch of (II.91), which is the only one that admits a perturbative expansion, we find to linear order in α_3

$$\begin{aligned} f &= k + g_{\text{eff}}^2 r^2 - (1+2g^2\tilde{\alpha}_3)\frac{\hat{\mu}}{r^{d-3}} + (1+2g^2\tilde{\alpha}_3)\frac{Q^2}{2(d-2)(d-3)r^{2(d-3)}} \\ &\quad + \tilde{\alpha}_3\frac{\hat{\mu}^2}{r^{2(d-2)}} - \frac{\tilde{\alpha}_3\hat{\mu}Q^2}{(d-2)(d-3)r^{3d-7}} + \frac{\tilde{\alpha}_3 Q^4}{4(d-2)^2(d-3)^2 r^{2(2d-5)}}, \end{aligned} \tag{II.92}$$

where in this case $g_{\text{eff}}^2 = g^2(1+g^2\tilde{\alpha}_3)$. Comparing this with (II.87) demonstrates the relation

$$\mu = \hat{\mu}(1+2g^2\tilde{\alpha}_3) = \hat{\mu}\big(1+2g^2\alpha_3(d-3)(d-4)\big). \tag{II.93}$$

Note also that for $Q = 0$ the dependence of the solution (II.87) on $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ is indirect through the shift in g_{eff}. This is related to the fact that these contributions may be removed at linear order through a field redefinition. However, with nonzero charge, a field redefinition of the form $g_{\mu\nu} \to g_{\mu\nu} + aRg_{\mu\nu} + bR_{\mu\nu}$ can in principle remove the R^2 and $R_{\mu\nu}^2$ terms in the action but also generates RF^2 and $R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda}$ terms, implying that the coefficients $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ remain physical [132].

2.2.4 Thermodynamics

Given the holographically renormalized action, it is straightforward to study the thermodynamics of the R-charged black holes. We begin with the temperature, which is given by the surface gravity of the black hole, or equivalently by the requirement of the absence of a conical singularity at the horizon of the Euclideanized black hole. The relevant part of the Euclideanized metric has the form

$$ds^2 = f_1(r)d\tau^2 + \frac{dr^2}{f_2(r)}, \tag{II.94}$$

where both f_1 and f_2 have a zero at the outer horizon, $f_1(r_+) = f_2(r_+) = 0$. In this case, the temperature is given by

$$T = \frac{1}{4\pi}\left[\sqrt{f_1'(r)f_2'(r)}\right]_{r=r_+}. \tag{II.95}$$

For f_1 and f_2 given in (II.87), we find:

$$
\begin{aligned}
T &= \frac{1}{4\pi r_+}\Bigg[(d-3)\frac{\mu}{r_+^{d-3}} + 2g_{\rm eff}^2 r_+^2 - \frac{Q^2}{(d-2)r_+^{2d-6}} - 2\alpha_3(d-2)(d-3)(d-4)\frac{\mu^2}{r_+^{2d-4}} \\
&\quad -\frac{g^2Q^2}{r_+^{2d-6}}\bigg(2\tilde\alpha_1\frac{(d^4-36d^2+107d-84)}{(d-2)^2(d-3)} - 2\tilde\alpha_2\frac{(d^4-14d^3+65d^2-121d+77)}{(d-2)^2(d-3)} \\
&\qquad +2\alpha_3\frac{(d-3)(d-4)}{(d-2)}\bigg) + \frac{k\,Q^2}{r_+^{2d-4}}\left(-12\tilde\alpha_1\frac{(d-4)}{(d-3)} + \tilde\alpha_2\frac{2(d-4)(d-5)}{(d-3)}\right) \\
&\quad +\frac{\mu\,Q^2}{r_+^{3d-7}}\left(\tilde\alpha_1\frac{(d-4)(d^2+6d-15)}{(d-2)(d-3)} - \tilde\alpha_2\frac{(d^3-13d^2+49d-53)}{(d-2)(d-3)} + \alpha_3\frac{(d-4)(3d-7)}{(d-2)}\right) \\
&\quad -\frac{Q^4}{r_+^{4d-10}}\bigg(\tilde\alpha_1\frac{(d-4)(10d^3-49d^2+84d-57)}{2(d-2)^2(d-3)^2(3d-7)} + \tilde\alpha_2\frac{(2d^4-14d^3+31d^2-32d+25)}{2(d-2)^2(d-3)^2(3d-7)} \\
&\qquad +\alpha_3\frac{(2d-5)(d-4)}{2(d-2)^2(d-3)}\bigg)\Bigg]. \qquad \text{(II.96)}
\end{aligned}
$$

While this expression is written in terms of the parameters μ, r_+ and Q, they are not all independent. In particular, μ may be written in terms of r_+ and Q through the horizon condition $f_1(r_+)=0$ (although μ enters quadratically in (II.87), it is only necessary to obtain μ to first order in the α_i).

The entropy can be obtained by using Wald's formula

$$
S = -2\pi\int_{\rm horizon} E^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}d^{d-2}x, \qquad \text{(II.97)}
$$

where

$$
E^{\mu\nu\rho\sigma} = \frac{\delta S_{\rm bulk}}{\delta R_{\mu\nu\rho\sigma}}\bigg|_{g_{\mu\nu}\ \text{fixed}}, \qquad \text{(II.98)}
$$

and $\epsilon_{\mu\nu}$ is the binormal to the horizon. For the action (II.66), we have

$$
\begin{aligned}
E^{\mu\nu\rho\sigma} &= -\frac{1}{2\kappa_d^2}\sqrt{-g}\Big[\tfrac12(1+\alpha_1 R)(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\
&\qquad +\tfrac12\alpha_2(g^{\mu\rho}R^{\nu\sigma}+g^{\nu\sigma}R^{\mu\rho}-g^{\mu\sigma}R^{\nu\rho}-g^{\nu\rho}R^{\mu\sigma}) + 2\alpha_3 R^{\mu\nu\rho\sigma}\Big], \qquad \text{(II.99)}
\end{aligned}
$$

in which case we find the entropy to be

$$
\begin{aligned}
S &= \frac{2\pi\omega_{d-2,k}}{\kappa_d^2}r_+^{d-2}\Bigg[1-2\tilde\alpha_1 g^2 d(d-1) - 2\tilde\alpha_2 g^2(d-1) + 2\alpha_3(d-2)(d-3)\frac{k}{r_+^2} \\
&\qquad -\frac{Q^2}{r_+^{2d-4}}\left(\tilde\alpha_1\frac{d-4}{d-2}+\tilde\alpha_2\frac{d-3}{d-2}\right)\Bigg]. \qquad \text{(II.100)}
\end{aligned}
$$

Here $\omega_{d-2,k}$ denotes the area of the transverse space given by $d\Omega_{d-2,k}$.

The next ingredient we are interested in is the energy, which can be extracted from the time-time component of the boundary stress tensor,

$$
\begin{aligned}
T_{ab} &= \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} \\
&= \frac{1}{2\kappa_d^2}\Bigg[2\big(1-2\tilde{\alpha}_1 g^2 d(d-1)-2\tilde{\alpha}_2 g^2(d-1)\big)(K_{ab}-Kh_{ab}) \\
&\qquad +\left(\tilde{\alpha}_1\frac{(d-4)}{d-2}-\frac{\tilde{\alpha}_2}{d-2}\right)\left(F^2K_{ab}+2KF_{\lambda a}F^{\lambda}{}_b-\frac{1}{2}KF^2\right) \\
&\qquad +\tilde{\alpha}_2\left(K_{ab}h_{cd}n_\mu F^{\mu c}n_\nu F^{\nu d}+Kn_\mu F^{\mu}{}_a n_\nu F^{\nu}{}_b-\frac{1}{2}Kh_{cd}n_\mu F^{\mu c}n_\nu F^{\nu d}h_{ab}\right) \\
&\qquad +\tilde{\alpha}_2\left(K_{cd}F^c{}_aF^d{}_b-\frac{1}{2}K_{cd}F^{c\lambda}F^d{}_\lambda h_{ab}\right)\Bigg]+T^{GB}_{ab}+T^{CT}_{ab}\,,
\end{aligned}
$$

(II.101)

giving us the refreshingly simple expression

$$
E=\frac{\omega_{d-2}}{2\kappa_d^2}(d-2)\mu\left(1-2\tilde{\alpha}_1g^2d(d-1)-2\tilde{\alpha}_2g^2(d-1)-2\alpha_3g^2(d-3)(d-4)\right)\,, \tag{II.102}
$$

which we expect to be valid in arbitrary dimension d. Notice that in the absence of higher derivative corrections this expression reproduces the familiar result $E\sim\mu$ found in [47]. This also matches the Gauss-Bonnet black hole mass [146, 39] in the case $\tilde{\alpha}_1=\tilde{\alpha}_2=0$, and agrees with [60], with arbitrary α_i coefficients (note that we have removed the k^2 dependent 'Casimir energy' by the addition of finite counterterms, which was not done in [60]).

The final quantity we are interested in finding is the thermodynamic potential, which can be obtained by evaluating the complete on-shell action:

$$
\beta\Omega=S_{\rm bulk}+S_{\rm GH}+S_{\rm ct}. \tag{II.103}
$$

Computing this explicitly we find the renormalized free energy:

$$
\begin{aligned}
\Omega \;=\;& \frac{\omega_{d-2,k}}{2\kappa_d^2}\Big[\mu\big(1-2\tilde\alpha_1 g^2 d(d-1)-2\tilde\alpha_2 g^2(d-1)-2\alpha_3 g^2(d-2)(d-3)\big)\\
&-2g^2 r_+^{d-1}\left(1-\tilde\alpha_1 g^2\frac{d^2(d-1)}{d-2}-\tilde\alpha_2 g^2\frac{d(d-1)}{d-2}-\alpha_3 g^2 d(d-3)\right)\\
&-\frac{Q^2}{(d-2)(d-3)r_+^{d-3}}+\frac{kQ^2}{r_+^{d-1}}\left(4\tilde\alpha_1\frac{(d-1)(d-4)}{(d-2)}+\tilde\alpha_2\frac{d(d-4)}{(d-2)}\right)\\
&+\frac{g^2Q^2}{r_+^{d-3}}\left(2\tilde\alpha_1\frac{(d-1)(d-4)(2d-3)}{(d-2)^2}+\tilde\alpha_2\frac{d^3-4d^2+6}{(d-2)^2}-2\alpha_3\frac{(d-4)}{(d-2)}\right)\\
&+\frac{\mu Q^2}{r_+^{2d-4}}\left(-4\tilde\alpha_1\frac{(d-1)(d-4)}{(d-2)}-\tilde\alpha_2\frac{d(d-4)}{(d-2)}+2\alpha_3\frac{(2d-5)}{(d-2)}\right)-2\alpha_3(d-2)(d-3)\frac{\mu^2}{r_+^{d-1}}\\
&+\frac{Q^4}{2r_+^{3d-7}}\left(\tilde\alpha_1\frac{(d-4)(12d^2-45d+43)}{(d-2)^2(d-3)(3d-7)}+\tilde\alpha_2\frac{(3d^3-23d^2+53d-39)}{(d-2)^2(d-3)(3d-7)}\right.\\
&\left.\left.-\alpha_3\frac{(3d-8)}{(d-2)^2(d-3)}\right)\right],
\end{aligned}
\tag{II.104}
$$

where we again recall that μ is a redundant parameter, and may be rewritten in terms of r_+ and Q.

Since the d-dimensional expressions are rather unwieldy, we have checked our calculations by verifying that the thermodynamic potential and energy satisfy

$$\Omega = E - TS - \mathcal{Q}\Phi, \tag{II.105}$$

and the first law,

$$dE = TdS + \Phi d\mathcal{Q}\,. \tag{II.106}$$

Here Φ is the chemical potential, defined as the difference in the potential between the horizon and spatial infinity,

$$\Phi(r_+) = A_t(r\to\infty) - A_t(r=r_+)\,, \tag{II.107}$$

and $\mathcal{Q} = (\omega_{d-2}/2\kappa_d^2)Q$ is the normalized electric charge which is unmodified by the higher derivative terms.

Finally, we note that a subtlety arises when applying the above thermodynamic expressions in an AdS/CFT context. For the R-charged black hole solution, we have chosen a parameterization of the background which is asymptotic to vacuum AdS given by (II.80). Taking $r \to \infty$, this has the form

$$ds^2 \sim -g_{\text{eff}}^2 r^2 dt^2 + r^2 d\Omega_{d-2,k}^2 + \frac{dr^2}{g_{\text{eff}}^2 r^2}. \tag{II.108}$$

Working on the Poincaré patch ($k = 0$), the natural spatial coordinates are written in terms of the zeroth order AdS radius, so that

$$\begin{aligned} ds^2 &\sim -g_{\text{eff}}^2 r^2 dt^2 + g^2 r^2 d\vec{x}^{\,2} + \frac{dr^2}{g_{\text{eff}}^2 r^2} \\ &\sim g^2 r^2 \left(-\frac{g_{\text{eff}}^2}{g^2} dt^2 + d\vec{x}^{\,2} \right) + \frac{dr^2}{g_{\text{eff}}^2 r^2}. \end{aligned} \tag{II.109}$$

The boundary CFT metric thus has a redshift factor

$$\lambda = \frac{g_{\text{eff}}}{g}, \tag{II.110}$$

which may be removed by rescaling asymptotic time

$$t \to t' = \lambda t\,. \tag{II.111}$$

Thus, in the CFT, all thermodynamic quantities in this section ought to be rescaled via

$$\{E, T, \Phi, \Omega\} \to \frac{1}{\lambda} \{E, T, \Phi, \Omega\}\,. \tag{II.112}$$

We will only perform the scaling explicitly for the energy, since it is the quantity which plays a key role in the discussion of the mass to charge relation.

2.2.5 The weak gravity conjecture and M/Q

It is not surprising that the relation between the mass m and the charge q of extremal black hole solutions is modified in the presence of curvature corrections. In light of the weak gravity conjecture, which emerged from the ideas explored in [169] and later refined in [8], it is interesting to examine the precise dependence of the mass on the R-charge for the solutions we have constructed above.

One of the key points emphasized in [169] is the fact that string theory, or any theory of quantum gravity, puts constraints on low energy physics, so that not every (consistent) effective field theory can in fact be UV completed. Thus, the *landscape* of "good" theories – those which are compatible with quantum gravity – is much smaller than the actual *swampland* of all effective field theories which do not have a UV completion. Building on the simple observation that "gravity is the weakest force," the authors of [8] conjectured that there should always be elementary objects whose mass to charge ratio is smaller than the corresponding one for macroscopic extremal black holes. The presence of such objects would then provide a decay channel for extremal black holes, alleviating the problem of remnants. Thus, according to the weak gravity conjecture, the mass/charge relation $m = q$ for extremal black holes cannot be exact, but must instead receive corrections as the charge q decreases. Furthermore, the deviation from the extremal limit is expected to become more pronounced as the charge becomes smaller.

An analysis of higher derivative corrections to the mass/charge ratio of four-dimensional, *asymptotically flat* black holes was performed in [121]. In the examples where the sign of the correction to m/q could be verified from UV physics, it was found to be negative, in agreement with the claims of [8]. Similar results appeared more recently [91] in the context of d-dimensional black holes with two electric charges, which are solutions corresponding to fundamental strings with generic momentum and winding on an internal circle. While the weak gravity conjecture was originally phrased in terms of four-dimensional, asymptotically flat black holes, it is worth exploring its analog in the context of extremal black holes in AdS. In particular, there have been suggestions in the literature that the correction to m/q might be somehow tied to the correction to the shear viscosity to entropy density ratio η/s (as well as to the charge conductivity) [122, 144, 55]. When discussing the effects of higher derivatives on various transport coefficients, the authors of [144] included an analysis of m/q

for five-dimensional R-charged black holes, and their results were in qualitative agreement with those of [121].

Given our analysis in this paper, we may extend some of these studies to R-charged solutions in d-dimensions. As we will see, our results will be similar to those already found in [121] and [144]. Moreover, we emphasize that in five dimensions the deviation from the linear extremal mass-charge relation predicted by the weak gravity conjecture seems to be intimately tied to the corrections observed in some of the hydrodynamic calculations in $\mathrm{AdS}_5/\mathrm{CFT}_4$. Such a connection could be a consequence of gravity constraining the set of allowed dual CFTs.

In Section 2.2.4 we extracted the energy of the corrected R-charge solutions from the boundary stress tensor. In this case, the mass to charge ratio is given simply by

$$\frac{m}{q} = \frac{1}{\lambda}\frac{E}{\mathcal{Q}}, \tag{II.113}$$

where the energy E is given in (II.102), but must be rescaled by the redshift factor λ introduced in (II.110) to ensure proper boundary asymptotics. Recall that the normalized charge $\mathcal{Q}$ is given by $\mathcal{Q} = (\omega_{d-2}/2\kappa_d^2)Q$. Since we are interested in m/q for extremal black holes, we make use of the extremality condition $T = 0$ as well as the horizon condition $f(r_+) = 0$.

Although we ultimately want to consider black holes in AdS, we start by setting $g = 0$ and $k = 1$ in order to examine m/q for asymptotically flat solutions with a spherical horizon, as was done in [121]. We find

$$\begin{aligned} \frac{m}{q} &= \left(\frac{m}{q}\right)_0 \Bigg(1 - \frac{\alpha_1}{r_+^2}\frac{(d-3)^2(d-4)^2}{2(d-2)(3d-7)} - \frac{\alpha_2}{r_+^2}\frac{(d-3)^2(2d^2-11d+16)}{2(d-2)(3d-7)} \\ &\qquad - \frac{\alpha_3}{r_+^2}\frac{(d-3)(2d^3-16d^2+45d-44)}{(d-2)(3d-7)}\Bigg), \end{aligned} \tag{II.114}$$

where

$$\left(\frac{m}{q}\right)_0 = \sqrt{\frac{2(d-2)}{d-3}} \tag{II.115}$$

is the uncorrected mass to charge ratio. Note first of all that, independent of the number of dimensions, the correction is always negative whenever the α_i's are positive. Furthermore, as one can easily check by trading r_+ dependence for Q dependence, the $1/r_+^2$ factor in front of the higher derivative corrections implies that the deviation from the linear relation $m \sim q$ is enhanced as the charge decreases. This was precisely one of the predictions of the weak gravity conjecture, and was also observed in [121]. Of course to say anything more about the precise form of the correction, one needs to determine the couplings.

The expressions corresponding to spherical horizon black holes in AdS are significantly more complicated. Here we quote the result in $d = 5$, and relegate the $d = 4$ and $d = 6$ cases to the appendix, since they are qualitatively the same:

$$\begin{aligned}\left(\frac{m}{q}\right)_{d=5} &= \left(\frac{m}{q}\right)_{0,d=5}\left(1-\alpha_1\frac{(816\beta^3+1024\beta^2+300\beta+1)}{6r_+^2(1+2\beta)(2+3\beta)}\right.\\ &\left.-\alpha_2\frac{(336\beta^3+392\beta^2+132\beta+11)}{6r_+^2(1+2\beta)(2+3\beta)}-\alpha_3\frac{(564\beta^3+586\beta^2+216\beta+31)}{6r_+^2(1+2\beta)(2+3\beta)}\right),\end{aligned} \tag{II.116}$$

where $\beta = g^2 r_+^2$ and

$$\left(\frac{m}{q}\right)_{0,d=5} = \frac{\sqrt{3}(2+3\beta)}{2\sqrt{1+2\beta}}. \tag{II.117}$$

As in the asymptotically flat case, the corrections are sensitive to the sign of the couplings, and will necessarily push the solution below the extremal limit when all the α_i are positive. Of course, if some of the couplings are negative the various terms can conspire to yield a positive correction to the mass to charge ratio. However, if the weak gravity conjecture holds, we would expect that, in an effective theory that is consistent with gravity in the UV, the couplings would be constrained in such a way as to lower m/q. Again, this underlines the importance of obtaining the higher derivative couplings from UV physics.

In the asymptotically Minkowski case we observed that m/q became smaller as the charge decreased, since the overall $1/r_+^2$ factor decreases monotonically as r_+ increases. Here the AdS black hole situation is similar only as long as r_+ does not become too large.

When $r_+ \sim 1/g$, the coefficient of the α_3 term reaches a minimum and starts growing as r_+ increases. This effect was already noticed in [144] and is intrinsic to the AdS geometry – it reflects the fact that the size of the black hole is becoming of the same order as the AdS radius.

One of the results of the investigations of the hydrodynamic regime of four-dimensional SCFTs has been the universality [33] of the shear viscosity to entropy ratio, $\eta/s = 1/4\pi$ in the leading supergravity approximation. Studies of R^4 corrections [35, 30, 145] increased the ratio, and seemed to favor the existence of a new bound in nature, $\eta/s \geq 1/4\pi$, the celebrated KSS bound. However, with the inclusion of curvature-squared corrections the bound has been shown to be violated by $1/N$ effects on the CFT side [122, 37, 144, 55]. The size of the violation is related to the two central charges a, c of the dual four-dimensional CFT. Holographic Weyl anomaly matching demonstrates that the coefficient of the R^2 terms in the action is proportional to $(c-a)/c$, and it is precisely the quantity $c-a$ which controls the strength of the correction to η/s, with $c-a>0$ necessarily giving violation of the bound.

Until recently, all the available CFT examples with a known gravity dual corresponded to $c-a>0$, so that violating the η/s bound seemed to be the rule rather than the exception. However, a large class of four dimensional $\mathcal{N}=2$ CFTs was constructed recently in [80], and shown in [81] to contain examples with $c-a<0$ and a known dual gravitational description. These are quiver gauge theories which can be viewed as arising from M5 branes wrapping a Riemann surface. Furthermore, one can add non-compact branes that intersect the surface at points (punctures on the Riemann surface). In the large N limit, these yield a large class of AdS_5 compactifications of M-theory with four-dimensional $\mathcal{N}=2$ supersymmetry, some of which correspond to $c-a<0$.

In light of these constructions, the requirement of $c-a>0$ which seemingly arises from the weak gravity conjecture is rather puzzling. Ideally, we may have expected the gravity duals to restrict the set of allowed CFTs, effectively placing the ones with $c - a < 0$ into the swampland. However, such a statement would have to be reconciled with the results of [81], which found no such sign restrictions. Still, it is remarkable that the issue of the sign of $c - a$ arises not only in the computation of transport coefficients, but also in the context of the weak gravity conjecture. We illustrate this connection with a simple example.

To make contact with the AdS/CFT work on transport coefficients, we take $d = 5$ and consider the Weyl-tensor-squared corrected action

(II.118)
$$S_{\text{bulk}} = -\frac{1}{2\kappa_5^2}\int_{\mathcal{M}} d^5x\sqrt{-g}\left[R - \frac{1}{4}F^2 + 12g^2 + \alpha\left(\frac{1}{6}R^2 - \frac{4}{3}R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right].$$

This choice is motivated by the general form of the supersymmetric higher derivative action that was used in [55] to obtain the corrections to η/s in $\mathcal{N} = 1$ SCFT. In fact, η/s for (II.118) can be read off from [55], and takes the form

$$\frac{\eta}{s} = \frac{1}{4\pi}[1 - 4\alpha(2 - q)], \tag{II.119}$$

where $0 \leq q \leq 2$, and q is the R-charge in the notation of [55]. The main feature to point out is that, since $(2 - q)$ is non-negative, the condition $\alpha > 0$ (or alternatively $c - a > 0$) always leads to violation of the η/s bound, and also guarantees that the entropy increases. But $\alpha > 0$ is also the requirement needed to satisfy the weak gravity conjecture. In fact, for the Weyl squared correction in $d = 5$, our result for m/q reduces to:

$$\left(\frac{m}{q}\right)_{d=5} = \left(\frac{m}{q}\right)_0\left(1 - \alpha\frac{168\beta^3 + 156\beta^2 + 60\beta + 11}{4\,r_+^2\,(1 + 2\beta)(2 + 3\beta)}\right). \tag{II.120}$$

While here we have focused on five dimensions, these features are generic in other dimensions as well (as can be inferred from the m/q expressions in the appendix).

For a less trivial example in $d = 5$ we can look at the most general four-derivative action describing R-charged solutions, which has been studied in [144, 55], and can be reduced –

via appropriate field redefinitions – to the simple form:

(II.121)

$$e^{-1}\delta\mathcal{L} = c_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + c_3(F^2)^2 + c_4 F^4 + c_5\epsilon^{\mu\nu\rho\sigma\lambda}A_\mu R_{\nu\rho\alpha\beta}R_{\sigma\lambda}{}^{\alpha\beta}\,.$$

The effect of such terms on the shear viscosity to entropy density ratio can be read off from [144, 55], and for the special case where the terms are constrained by supersymmetry (so that all the c_i's are related to each other), one finds:

$$\frac{\eta}{s} = \frac{1}{4\pi}\Big[1 - c_1\, g(Q)\Big]\,, \tag{II.122}$$

where $g(Q)$ is a non-negative function of R-charge. The mass to charge relation for this case has been worked out in [144] and exhibits the same behavior we found in the simpler Weyl-tensor-squared case:

$$\left(\frac{m}{q}\right)_{d=5} = \left(\frac{m}{q}\right)_0 \left(1 - c_1\, f(r_+)\right)\,, \tag{II.123}$$

where again $f(r_+)$ is always positive. While the precise form of the corrections to m/q and η/s is different, the behavior required by the weak gravity conjecture (in this case $c_1 > 0$) is again correlated with the violation of the viscosity to entropy bound.

The correlation between the behavior of η/s and the corrections to m/q is intriguing. It hints, at least in the five-dimensional context, at a close connection between the sign of $c-a$ and possible fundamental constraints on the gravitational side of the duality. However, in this case one would need to understand the role played by the strongly coupled theories investigated in [81], which allow for negative $c-a$. We should also point out that studies of causality in the CFT [26, 36] as well as the requirement of positive energy measurements in collider experiments [117, 116] (also note the work of [161]) have resulted in bounds on the central charges a and c, but so far have not lead to any restrictions on the actual sign of $c-a$. Nevertheless, theories with $c-a<0$ would naively seem to be in conflict with the weak gravity conjecture, and thus may be expected to possess unusual features. We note that these ideas have already been explored in several contexts. For example, [1] have identified consistency conditions for effective field theories with a UV completion, based on the idea that the signs of certain higher dimensional operators must be strictly positive. Such arguments, however, still need to be fully extended to generic gravitational settings.

Having a geometrical understanding of the origin of the higher derivative couplings – and of their sign in particular – would also be valuable. For example, for the case of ungauged $\mathcal{N} = 2$, $d = 5$ supergravity (obtained by reducing $d = 11$ supergravity on a CY_3), the coupling of the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term can be shown to be related to the second Chern class of the CY_3, which is known to be positive. For the case of $\mathcal{N} = 2$, $d = 5$ gauged supergravity (which is needed to discuss black holes in AdS), the compactification manifold would be a five-dimensional Sasaki-Einstein manifold, and the geometric origin of the higher derivative couplings is less clear. While there is work [78, 79] relating geometric data of generic supersymmetric AdS_5 solutions of type IIB supergravity to the central charges a, c of the dual CFTs, so far this applies only to the leading supergravity approximation, where $a = c = \mathcal{O}(N^2)$. Thus, it would be valuable to generalize these constructions to accommodate finite N corrections to the central charges. Whether through geometric data, or through consistency arguments on the field theory side, a better understanding of the signs of the higher derivative gravitational couplings is needed. This is especially relevant if we want to achieve a deeper insight into the weak gravity conjecture, and how it is tied to seemingly unrelated quantities such as hydrodynamic transport coefficients.

CHAPTER III

A supersymmmetric higher derivative lagrangian and η/s

In this chapter we present a supersymmetric higher derivative extension of the minimal gauged supergravity Lagrangian and relate the coefficients of the higher derivative terms to gauge theory parameters through an anomaly matching procedure. We compute black hole solutions and discuss their thermodynamics. Finally we present a calculation of the shear viscosity to entropy density ratio for charged black holes in this theory, making comments relevant to the conjectured KSS bound [125]. This chapter is based on work published in [54, 55] in collaboration with Sera Cremonini, Kentaro Hanaki and Jim Liu.

3.1 Introduction

In this chapter, we investigate black holes in higher-derivative corrected five-dimensional $\mathcal{N} = 2$ gauged supergravity. Our motivation is two-fold. Firstly, we are interested in exploring the nature of stringy corrections to supergravity and in particular whether such higher-order corrections may smooth out singular horizons of small black holes. Secondly, five-dimensional gauged supergravity is a natural context in which to explore AdS/CFT, and black holes are important thermal backgrounds for this duality. By working out these gravity corrections, we may learn more about finite-coupling as well as $1/N$ effects in the dual $\mathcal{N} = 1$ super-Yang-Mills theory.

Because of the reduced supersymmetry, we expect the first corrections to $\mathcal{N} = 2$ gauged supergravity to occur at R^2 order. For this reason, we will limit our focus on four-derivative terms in the effective supergravity action. While in principle these terms may be derived directly from string theory, doing so would involve specific choices of string compactifications down to five dimensions as well as the potential need to work out contributions from the Ramond-Ramond sector. To avoid these issues, we instead make use of supersymmetry, and in particular the result of [107], which worked out the supersymmetric completion of the $A \wedge \operatorname{Tr} R \wedge R$ term in $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets using the superconformal tensor calculus methods developed in [127, 17, 76, 18].

Although we are not aware of an actual uniqueness proof, we expect the four-derivative terms constructed in [107] to be uniquely determined by supersymmetry (modulo field redefinitions). The ungauged story is rather elegant, and may be tied to M-theory compactified on a Calabi-Yau three-fold. In this case the higher derivative corrections are given by

$$e^{-1}\delta\mathcal{L} = \tfrac{1}{24}c_{2I}\left[\tfrac{1}{16}\epsilon_{\mu\nu\rho\lambda\sigma}A^{I\,\mu}R^{\nu\rho\alpha\beta}R^{\lambda\sigma}{}_{\alpha\beta} + \cdots\right], \tag{III.1}$$

where the ellipses denote the supersymmetric completion of the $A \wedge \operatorname{Tr} R \wedge R$ Chern-Simons term. Comparing this term with the Calabi-Yau reduction of the M5-brane anomaly term demonstrates that the coefficients c_{2I} are related to the second Chern class on the Calabi-Yau manifold. The higher-derivative corrected action has recently been applied to the study of five-dimensional black holes in string theory (see *e.g.* [45] and references therein).

While much has already been made of the higher-derivative corrections to ungauged supergravity, here we are mainly interested in the gauged supergravity case and resulting applications to AdS/CFT. In this case, the natural setup would be to take IIB string theory compactified on $\mathrm{AdS}_5 \times Y^5$ where Y^5 is Sasaki-Einstein, which is dual to $\mathcal{N} = 1$ super-Yang-Mills theory in four dimensions. While the four-derivative terms worked out in [107] apply equally well to both gauged and ungauged supergravity, in this case their stringy origin is less clear. As we will show, however, the c_{2I} coefficients governing the four-derivative terms may be related to gauge theory data using holographic anomaly matching.

Before constructing the R-charged black holes in the higher-derivative corrected theory, we first integrate out the auxiliary fields of the off-shell formulation, yielding an on-shell supergravity action. Throughout this paper, we furthermore work in the truncation to minimal supergravity involving only the graviton multiplet $(g_{\mu\nu}, A_\mu, \psi_\mu)$. While this on-shell action is implicit in the work of [107], we find it useful to have it written out explicitly, as it facilitates comparison with other recent results. This is especially of interest in providing a more rigorous supergravity understanding of the R^2 corrections to shear viscosity [122, 27, 26] and drag force [71, 170].

3.2 Higher Derivative Gauged Supergravity

In this section we investigate five-dimensional $\mathcal{N} = 2$ supergravity with the inclusion of (stringy) higher-derivative corrections. We are mainly interested in the case of gauged supergravity, which is the natural setting for the AdS/CFT setup. Because of the reduced amount of supersymmetry, we expect the first corrections to this theory to occur at R^2 order. For this reason, we will limit ourselves to four-derivative terms in the effective supergravity action.

The conventional on-shell formulation of minimal $\mathcal{N} = 2$ gauged supergravity is given in terms of the graviton multiplet $(g_{\mu\nu}, A_\mu, \psi^i_\mu)$ where ψ^i_μ is a symplectic-Majorana spinor with $i = 1, 2$ labeling the doublet of SU(2). The bosonic two-derivative Lagrangian takes the form

$$e^{-1}\mathcal{L}_0 = -R - \tfrac{1}{4}F^2_{\mu\nu} + \tfrac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\lambda\sigma}F_{\mu\nu}F_{\rho\lambda}A_\sigma + 12g^2, \tag{III.2}$$

where g is the coupling constant of the gauged R-symmetry, and where we have followed the sign conventions of [107][1]. We are, of course, interested in obtaining four-derivative

[1] We take $[\nabla_\mu, \nabla_\nu]v^\sigma = R_{\mu\nu\rho}{}^{\sigma}\, v^\rho$ and $R_{ab} = R_{ac}{}^{c}{}_{b}$.

corrections to the above Lagrangian that are consistent with supersymmetry. Along with purely gravitational corrections of the form (II.49), other possible four-derivative terms include F^4, mixed RF^2 and parity violating ones. Given the large number of such terms, it would appear to be a daunting task to work out the appropriate supersymmetric combinations. Fortunately, however, it is possible to make use of manifest supersymmetry in the form of superconformal tensor calculus to construct supersymmetric R^2 terms. (See *e.g.* [141] for a nice review, albeit focusing on four-dimensional $\mathcal{N} = 2$ supergravity.)

The general idea of the superconformal approach is to develop an off-shell formulation involving the Weyl multiplet that is locally gauge invariant under the superconformal group. The resulting conformal supergravity may then be broken down to Poincaré supergravity by introducing a conformal compensator in the hypermultiplet sector and introducing expectation values for some of its fields. One advantage of this method is that the off-shell formulation admits a superconformal tensor calculus which enables one to construct supersymmetric invariants of arbitrary order in curvature. This is in fact the approach taken in [107], which worked out the supersymmetric completion of the $A \wedge \mathrm{Tr}\, R \wedge R$ term in $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets.

The basic construction of [107] involves conformal supergravity (*i.e.* the Weyl multiplet) coupled to a set of $n_V + 1$ conformal vector multiplets and a single compensator hypermultiplet. The resulting Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = \mathcal{L}_0^{(V)} + \mathcal{L}_0^{(H)} + \mathcal{L}_1, \tag{III.3}$$

where $\mathcal{L}_0$ corresponds to the two-derivative terms and $\mathcal{L}_1$ the four-derivative terms. We have further broken up $\mathcal{L}_0$ into contributions $\mathcal{L}_0^{(V)}$ from the vector multiplets and $\mathcal{L}_0^{(H)}$ from the hypermultiplet.

As formulated in [107], the full Lagrangian $\mathcal{L}$ contains a set of auxiliary fields which we wish to eliminate in order to make direct comparison to the on-shell Lagrangian (III.2).

To do so, we simply integrate out the auxiliary fields using their equations of motion, and the remainder of this section is devoted to this process. As an important shortcut, we note that when working to linear order in the correction terms in $\mathcal{L}_1$, we only need to substitute in the lowest order expressions for the auxiliary fields [7]. For this reason, we first examine the two-derivative Lagrangian before turning to the four-derivative terms contained in $\mathcal{L}_1$.

3.2.1 The leading two-derivative action

We begin with the vector multiplet contribution to the two-derivative Lagrangian [107]

$$\begin{aligned} e^{-1}\mathcal{L}_0^{(V)} &= \mathcal{N}(\tfrac{1}{2}D - \tfrac{1}{4}R + 3v^2) + 2\mathcal{N}_I v^{\mu\nu} F^I_{\mu\nu} + \mathcal{N}_{IJ}\tfrac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} + \tfrac{1}{24}c_{IJK}\epsilon^{\mu\nu\rho\lambda\sigma}A^I_\mu F^J_{\nu\rho}F^K_{\lambda\sigma} \\ &\quad -\mathcal{N}_{IJ}\big(\tfrac{1}{2}\mathcal{D}^\mu M^I \mathcal{D}_\mu M^J + Y^I_{ij}Y^{J\,ij}\big)\,, \end{aligned} \tag{III.4}$$

where M^I, A^I_μ and Y^I_{ij} $(I, J = 1, 2, \ldots, n_v + 1)$ denote, respectively, the scalar fields, the gauge fields and the $SU(2)$-triplet auxiliary fields in the $n_v + 1$ vector multiplets. In addition, the scalar D and the two-form $v_{\mu\nu}$ are auxiliary fields coming from the Weyl multiplet. The prepotential $\mathcal{N}$ and its functional derivatives are given by the standard expressions

$$\mathcal{N} = \tfrac{1}{6}c_{IJK}M^I M^J M^K, \qquad \mathcal{N}_I = \tfrac{1}{2}c_{IJK}M^J M^K, \qquad \mathcal{N}_{IJ} = c_{IJK}M^K. \tag{III.5}$$

For future reference, we also note the useful relations

$$\mathcal{N}_I M^I = 3\mathcal{N}, \qquad \mathcal{N}_{IJ}M^J = 2\mathcal{N}_I. \tag{III.6}$$

Turning next to the hypermultiplet Lagrangian, we have [107]

$$e^{-1}\mathcal{L}_0^{(H)} = 2\big[\mathcal{D}^\mu \mathcal{A}_i^{\bar\alpha}\mathcal{D}_\mu \mathcal{A}^i_\alpha + \mathcal{A}_i^{\bar\alpha}\,(g\,M)^2\,\mathcal{A}^i_\alpha + 2gY^{ij}_{\alpha\beta}\mathcal{A}_i^{\bar\alpha}\mathcal{A}_j^\beta\big] + \mathcal{A}^2(\tfrac{1}{4}D + \tfrac{3}{8}R - \tfrac{1}{2}v^2)\,. \tag{III.7}$$

In general, $\mathcal{A}^i_\alpha$ are a set of $4 \times n_H$ hypermatter scalars carrying both the SU(2) index i and the index $\alpha = 1, 2, \ldots, 2n_H$ of USp$(2n_H)$. (We use the SU(2) index raising convention

$A^i = \epsilon^{ij} A_j$ and $A_i = A^j \epsilon_{ji}$ with $\epsilon_{12} = \epsilon^{12} = 1$). Note that we have gauged a subgroup G of USp$(2n_H)$, so that the covariant derivative appearing above is given by

$$\mathcal{D}_\mu \mathcal{A}_i^\alpha = \partial_\mu \mathcal{A}_i^\alpha - g A_\mu^I t_I \mathcal{A}_i^\alpha + \mathcal{A}_j^\alpha V_\mu{}^j{}_i \,, \tag{III.8}$$

where t_I are the generators of the gauge symmetry and where V_μ^{ij} is an additional auxiliary field belonging to the Weyl multiplet. Finally, we have defined $M \equiv M^I t_I$, where M^I are the vector multiplet scalars.

For simplicity, we focus on a single compensator and choose the conventional gauging of the diagonal U(1) in the SU(2) R-symmetry. In this case, the action of M on the hyperscalars is given by

$$M \mathcal{A}_i^\alpha = M^I t_I \mathcal{A}_i^\alpha = M^I P_I (i\sigma^3)^\alpha_\beta \mathcal{A}_i^\beta \,, \tag{III.9}$$

while the covariant derivative becomes

$$\mathcal{D}_\mu \mathcal{A}_i^\alpha = \partial_\mu \mathcal{A}_i^\alpha - g A_\mu^I P_I (i\sigma^3)^\alpha_\beta \mathcal{A}_i^\beta + \mathcal{A}_j^\alpha V_\mu{}^j{}_i \,. \tag{III.10}$$

Here P_I denote the charges associated with the gauging. Furthermore, $\mathcal{A}^2 \equiv \mathcal{A}_i^{\bar\alpha} \mathcal{A}_\alpha^i = \mathcal{A}_i^\beta d_\beta{}^\alpha \mathcal{A}_\alpha^i$, where the metric $d_\beta{}^\alpha$ is arranged to be a delta function as appropriate for a compensator [107].

Combining (III.4) with (III.7), the complete two-derivative action is given by

$$\begin{aligned} e^{-1}\mathcal{L}_0 &= \tfrac{1}{4} D(2\mathcal{N} + \mathcal{A}^2) + R\left(\tfrac{3}{8}\mathcal{A}^2 - \tfrac{1}{4}\mathcal{N}\right) + v^2(3\mathcal{N} - \tfrac{1}{2}\mathcal{A}^2) \\ &\quad + 2\mathcal{N}_I v^{\mu\nu} F_{\mu\nu}^I + \mathcal{N}_{IJ}(\tfrac{1}{4} F_{\mu\nu}^I F^{J\,\mu\nu} - \tfrac{1}{2}\mathcal{D}^\mu M^I \mathcal{D}_\mu M^J) + \tfrac{1}{24} c_{IJK} \epsilon^{\mu\nu\rho\lambda\sigma} A_\mu^I F_{\nu\rho}^J F_{\lambda\sigma}^K \\ &\quad - \mathcal{N}_{IJ} Y_{ij}^I Y^{J\,ij} + 2\big[\mathcal{D}^\mu \mathcal{A}_i^{\bar\alpha} \mathcal{D}_\mu \mathcal{A}_\alpha^i + \mathcal{A}_i^{\bar\alpha} (g\,M)^2 \mathcal{A}_\alpha^i + 2g Y_{\alpha\beta}^{ij} \mathcal{A}_i^{\bar\alpha} \mathcal{A}_j^\beta\big] \,. \end{aligned} \tag{III.11}$$

At the two-derivative level, the auxiliary field D plays the role of a Lagrange multiplier, yielding the constraint

$$2\mathcal{N} + \mathcal{A}^2 = 0 \,. \tag{III.12}$$

Thus we can recover the standard very special geometry constraint $\mathcal{N} = 1$ by setting $\mathcal{A}^2 = -2$. (This fixing of the dilatational gauge transformation is in fact the purpose of the conformal compensator). This then brings the Lagrangian to the following form:

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{1}{2}D(\mathcal{N}-1) - \tfrac{1}{4}R(\mathcal{N}+3) + v^2(3\mathcal{N}+1) + 2\mathcal{N}_I v^{\mu\nu} F^I_{\mu\nu} \\ &\quad + \mathcal{N}_{IJ}\left(\tfrac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} - \tfrac{1}{2}\mathcal{D}^\mu M^I \mathcal{D}_\mu M^J\right) + \tfrac{1}{24}\, c_{IJK}\epsilon^{\mu\nu\rho\lambda\sigma} A^I_\mu F^J_{\nu\rho} F^K_{\lambda\sigma} \\ &\quad - \mathcal{N}_{IJ} Y^I_{ij} Y^{J\,ij} + 2\left[\mathcal{D}^\mu \mathcal{A}^{\bar\alpha}_i \mathcal{D}_\mu \mathcal{A}^i_\alpha + \mathcal{A}^{\bar\alpha}_i\,(g\,M)^2\,\mathcal{A}^i_\alpha + 2g Y^{ij}_{\alpha\beta}\mathcal{A}^{\bar\alpha}_i \mathcal{A}^\beta_j\right]. \end{aligned} \tag{III.13}$$

Integrating out the auxiliary fields

The action (III.13) can be written in a more familiar on-shell form by integrating out the auxiliary fields. We will do this in two steps by first eliminating the fields $\mathcal{A}^\alpha_i$, V^{ij}_μ and Y^I_{ij} and then eliminating D and $v_{\mu\nu}$.

We start by fixing the SU(2) symmetry by taking $\mathcal{A}^\alpha_i = \delta^\alpha_i$, which identifies the indices in the hypermultiplet scalar. The equation of motion for V^{ij}_μ is then given by

$$V^{ij}_\mu = g P_I (i\sigma^3)^{ij} A^I_\mu\,, \tag{III.14}$$

which also results in $\mathcal{D}_\mu \mathcal{A}^\alpha_i = 0$. Turning next to Y^I_{ij}, we first note that

$$Y^{ij}_{\alpha\beta}\,\mathcal{A}^{\bar\alpha}_i \mathcal{A}^\beta_j = Y^{I\,ij} P_I\,(i\sigma^3)_{ij}\,. \tag{III.15}$$

Varying (III.13) with respect to Y^I_{ij} then gives us the equation of motion

$$Y^I_{ij} = 2(\mathcal{N}^{-1})^{IJ} P_J (i\sigma^3)_{ij}\,. \tag{III.16}$$

Using the above to eliminate $\mathcal{A}^\alpha_i$, V^{ij}_μ and Y^I_{ij} from the two-derivative action (III.13), we end up with

$$\begin{aligned} e^{-1}\mathcal{L}_0 &= \tfrac{1}{2}D(\mathcal{N}-1) - \tfrac{1}{4}R(\mathcal{N}+3) + v^2(3\mathcal{N}+1) + 2\mathcal{N}_I v^{\mu\nu} F^I_{\mu\nu} \\ &+ \mathcal{N}_{IJ}\left(\tfrac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} - \tfrac{1}{2}\partial^\mu M^I \partial_\mu M^J\right) + \tfrac{1}{24} c_{IJK}\epsilon^{\mu\nu\rho\lambda\sigma} A^I_\mu F^J_{\nu\rho} F^K_{\lambda\sigma} \\ &+ 8g^2(\mathcal{N}^{-1})^{IJ} P_I P_J + 4g^2 (P_I M^I)^2\,, \end{aligned} \tag{III.17}$$

where the last line corresponds to the gauged supergravity potential

$$V = -4g^2[2(\mathcal{N}^{-1})^{IJ}P_I P_J + (P_I M^I)^2]\,. \tag{III.18}$$

Note that, with abelian gauging, the covariant derivative acts trivially on the vector multiplet scalars, $\mathcal{D}_\mu M^I = \partial_\mu M^I$.

To remove the remaining auxiliary fields D and $v_{\mu\nu}$ from (III.17) we must turn to the equations of motion for this system. Varying the action with respect to D, $v_{\mu\nu}$, M^I and A^I_μ yields, respectively,

$$0 = \tfrac{1}{2}(\mathcal{N}-1)\,, \tag{III.19}$$

$$0 = 2(3\mathcal{N}+1)v_{\mu\nu} + 2\mathcal{N}_I F^I_{\mu\nu}\,, \tag{III.20}$$

$$\begin{aligned} 0 = {} & \tfrac{1}{2}\mathcal{N}_I(D - \tfrac{1}{2}R + 6v_{\mu\nu}v^{\mu\nu}) + 2\,\mathcal{N}_{IJ}F^J_{\mu\nu}v^{\mu\nu} + \tfrac{1}{4}\,c_{IJK}\,F^J_{\mu\nu}F^{K\,\mu\nu} + \mathcal{N}_{IJ}\Box M^J \\ & + \tfrac{1}{2}\,c_{IJK}\,\partial_\mu M^J\partial^\mu M^K - g^2\frac{\delta V}{\delta M^I}\,, \end{aligned} \tag{III.21}$$

$$0 = -\nabla^\nu[4\mathcal{N}_I v_{\nu\mu} + \mathcal{N}_{IJ}F^J_{\nu\mu}] + \tfrac{1}{8}C_{IJK}\epsilon_\mu{}^{\nu\rho\lambda\sigma}F^J_{\nu\rho}F^K_{\lambda\sigma}\,. \tag{III.22}$$

In addition, the Einstein equation is given by:

$$\begin{aligned} 0 \;=\; & \tfrac{1}{4}(\mathcal{N}+3)(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) + \tfrac{1}{4}(\mathcal{N}-1)Dg_{\mu\nu} - \tfrac{1}{4}(\nabla_\mu\nabla_\nu\mathcal{N} - g_{\mu\nu}\Box\mathcal{N}) \\ & + \tfrac{1}{2}\mathcal{N}_{IJ}(\partial_\mu M^I\partial_\nu M^J - \tfrac{1}{2}g_{\mu\nu}\partial_\lambda M^I\partial^\lambda M^J) - 2(3\mathcal{N}+1)(v_{\mu\lambda}v_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}v_{\lambda\sigma}v^{\lambda\sigma}) \\ & - 4\mathcal{N}_I(F^I_{(\mu}{}^\lambda v_{\nu)\lambda} - \tfrac{1}{4}g_{\mu\nu}F^I_{\lambda\sigma}v^{\lambda\sigma}) - \tfrac{1}{2}\mathcal{N}_{IJ}(F^I_{\mu\lambda}F^{J\lambda}_\nu - \tfrac{1}{4}g_{\mu\nu}F^I_{\lambda\sigma}F^{J\,\lambda\sigma}) - \tfrac{1}{2}g_{\mu\nu}V\,. \end{aligned} \tag{III.23}$$

We are now in a position to start solving for the auxiliary fields D and $v_{\mu\nu}$. Inserting the very special geometry constraint $\mathcal{N} = 1$ (enforced by the equation of motion for D) into (III.20) yields

$$v_{\mu\nu} = -\tfrac{1}{4}\mathcal{N}_I F^I_{\mu\nu}\,. \tag{III.24}$$

We may now eliminate $\mathcal{N}$ and $v_{\mu\nu}$ from the lowest order Maxwell and Einstein equations to obtain

$$
\begin{aligned}
\nabla^\nu[(\mathcal{N}_I\mathcal{N}_J-\mathcal{N}_{IJ})F^J_{\nu\mu}] &= -\tfrac{1}{8}C_{IJK}\epsilon_\mu{}^{\nu\rho\lambda\sigma}F^J_{\nu\rho}F^K_{\lambda\sigma},\\
R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R &= -\tfrac{1}{2}\mathcal{N}_{IJ}(\partial_\mu M^I\partial_\nu M^J-\tfrac{1}{2}g_{\mu\nu}\partial_\lambda M^I\partial^\lambda M^J)\\
&\quad -\tfrac{1}{2}(\mathcal{N}_I\mathcal{N}_J-\mathcal{N}_{IJ})(F^I_{\mu\lambda}F^{J\lambda}_\nu-\tfrac{1}{4}g_{\mu\nu}F^I_{\lambda\sigma}F^{J\,\lambda\sigma})+\tfrac{1}{2}g_{\mu\nu}V\,.
\end{aligned}
\tag{III.25}
$$

Turning next to the scalar equations of motion, we note that the n_v+1 equations may be decomposed into n_v equations for the constrained scalars M^I, along with one equation for the Lagrange multiplier D. To solve for D, we multiply the scalar equation by M^I and obtain:

$$
\begin{aligned}
D-\tfrac{1}{2}R+6v_{\mu\nu}v^{\mu\nu} &= -\tfrac{8}{3}\mathcal{N}_I F^I_{\mu\nu}v^{\mu\nu}-\tfrac{1}{6}\mathcal{N}_{IJ}F^I_{\mu\nu}F^{J\,\mu\nu}-\tfrac{1}{3}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J\\
&\quad -\tfrac{4}{3}\mathcal{N}_I\Box M^I+\tfrac{2}{3}M^I\frac{\delta V}{\delta M^I}.
\end{aligned}
\tag{III.26}
$$

Substituting in R and $v_{\mu\nu}$ then allows us to express the auxiliary field D entirely in terms of physical fields:

$$
\begin{aligned}
D &= -\tfrac{7}{12}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J-\tfrac{4}{3}\mathcal{N}_I\Box M^I+\tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J-\tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu}-\tfrac{5}{6}V+\tfrac{2}{3}M^I\frac{\delta V}{\delta M^I}\\
&= -\tfrac{7}{12}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J-\tfrac{4}{3}\mathcal{N}_I\Box M^I+\tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J-\tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu}\\
&\quad +2g^2[6P_IP_J(\mathcal{N}^{-1})^{IJ}-P_IP_JM^IM^J]\,.
\end{aligned}
\tag{III.27}
$$

By using (III.26), the equation of motion for the constrained scalars (III.21) can be rewritten in the following form:

$$
\left(\delta^J_I-\frac{\mathcal{N}_IM^J}{3}\right)\Big[c_{JKL}(\partial_\mu M^K\partial^\mu M^L+2M^K\Box M^L)
-(\mathcal{N}_{JK}\mathcal{N}_L-\tfrac{1}{2}c_{JKL})F^KF^L-\frac{\delta V}{\delta M^J}\Big]=0.
\tag{III.28}
$$

We now have all the ingredients we need to write down the on-shell two-derivative Lagrangian:

$$
\begin{aligned}
e^{-1}\mathcal{L} &= -R-\tfrac{1}{2}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J-\tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J-\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu}\\
&\quad +\tfrac{1}{24}\,c_{IJK}\,\epsilon^{\mu\nu\rho\lambda\sigma}A^I_\mu F^J_{\nu\rho}F^K_{\lambda\sigma}+4g^2[2(\mathcal{N}^{-1})^{IJ}P_IP_J+(P_IM^I)^2]\,,
\end{aligned}
\tag{III.29}
$$

where now the M^I are a set of constrained scalars satisfying the very special geometry condition $\mathcal{N} = 1$. The Lagrangian perfectly matches the bosonic sector of the standard two-derivative $\mathcal{N} = 2$ supergravity action coupled to n_v vector multiplets. The resulting equations of motion are given by (III.25) and (III.28).

Here, we are mainly concerned with the truncation of (III.29) to the case of pure supergravity. This is accomplished by setting the scalars to constants and by defining a single graviphoton A_μ according to[2]

$$M^I = \bar{M}^I, \qquad A^I_\mu = \bar{M}^I A_\mu. \tag{III.30}$$

While the constants $\bar{M}^I$ are arbitrary moduli in the ungauged case, in the gauged cause they must lie at a critical point of the potential (III.18) given by solving

$$\left(\delta^J_I - \frac{\mathcal{N}_I M^J}{3}\right)\frac{\delta V}{\delta M^J} = 0. \tag{III.31}$$

By demanding that the critical point is supersymmetric, we find that the constant scalars satisfy[3]:

$$P_I \bar{M}^I = \frac{3}{2}\,, \qquad (\bar{\mathcal{N}}^{-1})^{IJ} P_I P_J = \frac{3}{8}\,. \tag{III.32}$$

in which case the potential becomes $\bar{V} = -12g^2$. The resulting Lagrangian for the bosonic fields of the supergravity multiplet $(g_{\mu\nu}, A_\mu)$ then reads

$$e^{-1}\mathcal{L} = -R - \tfrac{3}{4}F^2_{\mu\nu} + \tfrac{1}{4}\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2\,, \tag{III.33}$$

which reproduces the conventional on-shell supergravity Lagrangian (III.2) once the graviphoton is rescaled according to $A_\mu \to A_\mu/\sqrt{3}$.

While this completes the analysis relevant to the leading, two-derivative action, we note that the expression for D simplifies further in the case of constant scalars. Substituting

[2] Note that our definition differs by a factor of 1/3 from the conventional one where $A_\mu = A^I_\mu \mathcal{N}_I$.

[3] These expressions can be obtained by making use of the hyperino and gauging SUSY variations, as well as the equation of motion for the auxiliary field Y^I_{ij}. We refer the reader to [107] for more details.

(III.30) and (III.32) into the expression (III.27) for D yields the simple result

$$D = \tfrac{1}{4}(\bar{\mathcal{N}}_I\bar{\mathcal{N}}_J - \tfrac{1}{2}\bar{\mathcal{N}}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu} = \tfrac{3}{2}F^2_{\mu\nu}\,. \tag{III.34}$$

By taking $\mathcal{N}=1$, we see that this explicit form of D does not play a role in the leading expression for the two-derivative Lagrangian. However, it will become relevant in the discussion of higher derivative corrections, which we turn to next.

3.2.2 Higher-derivative corrections in gauged SUGRA

We now turn to the four-derivative corrections to the action (III.3), which we parameterize by $\mathcal{L}_1$. For convenience, we separate the contributions to $\mathcal{L}_1$ present in the ungauged theory from those coming strictly from the gauging, $\mathcal{L}_1 = \mathcal{L}_1^{\text{ungauged}} + \mathcal{L}_1^{\text{gauged}}$. The two are given by:

$$\begin{aligned}
e^{-1}\mathcal{L}_1^{\text{ungauged}} \;=\;& \tfrac{1}{24}c_{2I}\Big[\tfrac{1}{16}\epsilon_{\mu\nu\rho\lambda\sigma}A^{I\,\mu}R^{\nu\rho\alpha\beta}R^{\lambda\sigma}{}_{\alpha\beta} + \tfrac{1}{8}M^I C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tfrac{1}{12}M^I D^2 + \tfrac{1}{6}F^I_{\mu\nu}v^{\mu\nu}D \\
& -\tfrac{1}{3}M^I C_{\mu\nu\rho\sigma}v^{\mu\nu}v^{\rho\sigma} - \tfrac{1}{2}F^{I\,\mu\nu}C_{\mu\nu\rho\sigma}v^{\rho\sigma} + \tfrac{8}{3}M^I v_{\mu\nu}\nabla^\nu\nabla_\rho v^{\mu\rho} \\
& -\tfrac{16}{9}M^I v^{\mu\rho}v_{\rho\nu}R^\nu_\mu - \tfrac{2}{9}M^I v^2 R + \tfrac{4}{3}M^I\nabla^\mu v^{\nu\rho}\nabla_\mu v_{\nu\rho} + \tfrac{4}{3}M^I\nabla^\mu v^{\nu\rho}\nabla_\nu v_{\rho\mu} \\
& -\tfrac{2}{3}M^I\epsilon_{\mu\nu\rho\lambda\sigma}v^{\mu\nu}v^{\rho\lambda}\nabla_\delta v^{\sigma\delta} + \tfrac{2}{3}F^{I\,\mu\nu}\epsilon_{\mu\nu\rho\lambda\sigma}v^{\rho\delta}\nabla_\delta v^{\lambda\sigma} + F^{I\,\mu\nu}\epsilon_{\mu\nu\rho\lambda\sigma}v^\rho{}_\delta\nabla^\lambda v^{\sigma\delta} \\
& -\tfrac{4}{3}F^{I\,\mu\nu}v_{\mu\rho}v^{\rho\lambda}v_{\lambda\nu} - \tfrac{1}{3}F^{I\,\mu\nu}v_{\mu\nu}v^2 + 4M^I v_{\mu\nu}v^{\nu\rho}v_{\rho\lambda}v^{\lambda\mu} - M^I(v^2)^2\Big]\,,
\end{aligned} \tag{III.35}$$

$$\begin{aligned}
e^{-1}\mathcal{L}_1^{\text{gauged}} \;=\;& \tfrac{1}{24}c_{2I}\Big[-\tfrac{1}{12}\epsilon_{\mu\nu\rho\lambda\sigma}\,A^{I\,\mu}R^{\nu\rho\,ij}(U)R^{\lambda\sigma}_{ij}(U) \\
& -\tfrac{1}{3}\,M^I R^{\mu\nu\,ij}(U)R_{\mu\nu\,ij}(U) - \tfrac{4}{3}\,Y^I_{ij}v_{\mu\nu}R^{\mu\nu\,ij}(U)\Big]\,,
\end{aligned} \tag{III.36}$$

where

$$R_{\mu\nu}{}^{ij}(U) = \partial_\mu V_\nu^{ij} - V^i_{\mu k}V^{kj}_\nu - (\mu\leftrightarrow\nu)\,. \tag{III.37}$$

As we can see, the constants c_{2I} parameterize the magnitude of these contributions. Notice that the scalar D no longer acts as a Lagrange multiplier, since it now appears quadratically in $\mathcal{L}_1$. In fact, by varying the full action $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with respect to D, with $\mathcal{L}_0$ as in

(III.17), we obtain the modified very special geometry constraint

$$\mathcal{N} = 1 - \frac{c_{2I}}{72}(DM^I + F^{I\,\mu\nu}v_{\mu\nu})\,, \tag{III.38}$$

which encodes information about how the scalars M^I are affected by higher-derivative corrections.

Integrating out the auxiliary fields

As in the two-derivative case, in order to obtain a Lagrangian written solely in terms of the physical fields of the theory we need to eliminate the auxiliary fields D, $v_{\mu\nu}$, $V^i_{\mu\nu}$ and Y^I_{ij} from $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. In Sec. 3.2.1 we solved for the auxiliary fields by neglecting higher order corrections, and then integrated them out of the two-derivative action. It turns out that the lowest order expressions for the auxiliary fields are sufficient when working to linear order in the c_{2I} [7]. This allows us to reuse the results of the previous section for the auxiliary fields, which we summarize here:

$$V^{ij}_\mu = gP_I(i\sigma^3)^{ij}A^I_\mu\,, \tag{III.39}$$

$$Y^I_{ij} = 2(\mathcal{N}^{-1})^{IJ}P_J(i\sigma^3)_{ij}\,, \tag{III.40}$$

$$v_{\mu\nu} = -\tfrac{1}{4}\mathcal{N}_I F^I_{\mu\nu}\,, \tag{III.41}$$

$$D = \tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J - \tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu}\,. \tag{III.42}$$

While it is valid to use these lowest order expressions, it is important to realize that the scalar fields are modified because of (III.38). This modification leads to additional contributions to the two-derivative on-shell action (III.29), which combines with $\mathcal{L}_1$ to yield the complete action at linear order in c_{2I}.

In principle, we may work with the full system of supergravity coupled to n_V vector multiplets. However, here we focus on the truncation to pure supergravity, where the scalars M^I are taken to be non-dynamical. Even so, they are not entirely trivial. While at the two-derivative level, we may simply set them to constants according to (III.30), here we must allow for the modification (III.38) by defining

$$M^I = \bar{M}^I + c_2 \hat{M}^I, \qquad A^I_\mu = \bar{M}^I A_\mu, \qquad c_2 \equiv c_{2I} \bar{M}^I\,, \tag{III.43}$$

where $\hat{M}^I$ are possible scalar fluctuations that enter at $\mathcal{O}(c_2)$. Substituting this into the expressions (III.41) and (III.42) for the auxiliary fields then yields

$$v_{\mu\nu} = -\tfrac{3}{4} F_{\mu\nu} + \mathcal{O}(c_2), \qquad D = \tfrac{3}{2} F^2 + \mathcal{O}(c_2)\,, \tag{III.44}$$

which match the lowest order expressions for constant scalars. The modified very special geometry constraint (III.38) can now be simplified further, and becomes

$$\mathcal{N} = 1 - \frac{c_2}{96} F^2 + \mathcal{O}(c_2^2). \tag{III.45}$$

In general, a solution to the fluctuating scalars $\hat{M}^I$ ought to come from the equations of motion. However, as a shortcut, we make the ansatz that $\hat{M}^I$ is proportional to $\bar{M}^I$. The modified constraint (III.45) is then enough to fix the correction to the scalars to be

$$M^I = \bar{M}^I \left[1 - \frac{c_2}{288} F^2 + \mathcal{O}(c_2^2)\right]\,. \tag{III.46}$$

Consistency with the equations of motion will presumably demand an appropriate relation between the various c_{2I} coefficients. However, since the vectors will be truncated out, we only care about the combination c_2 given in (III.43), and will not work out this relation explicitly.

We are now ready to integrate out both the scalars M^I and the auxiliary fields from the two-derivative action $\mathcal{L}_0$ given in (III.13). By making use of the corrections[4] to the leading order scalar expressions (III.32)

$$P_I M^I = \frac{3}{2}\left[1 - \frac{c_2}{288} F^2\right]\,, \qquad (\mathcal{N}^{-1})^{IJ} P_I P_J = \frac{3}{8}\left[1 + \frac{c_2}{288} F^2\right]\,, \tag{III.47}$$

[4]These can be easily verified using $P_I = \frac{1}{4} \bar{\mathcal{N}}_{IJ} \bar{M}^J$.

we find that the contribution coming from $\mathcal{L}_0$ yields the following terms:

(III.48)
$$e^{-1}\mathcal{L}_0 = -R - \frac{3}{4}F^2 + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 + \frac{c_2}{24}\left[\frac{1}{16}RF^2 + \frac{1}{64}(F^2)^2 - \frac{5}{4}g^2F^2\right] .$$

Turning next to the four-derivative contributions, we note that, since such terms are already linear in c_2, we may simply use the leading order solution for the scalars. The gauging contribution (III.36) is then particularly simple

$$e^{-1}\mathcal{L}_1^{\text{gauged}} = -\frac{c_2}{64}g^2\,\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu F^{\nu\rho}F^{\lambda\sigma} . \qquad \text{(III.49)}$$

On the other hand, the contribution to $\mathcal{L}_1^{\text{ungauged}}$ is given by:

$$\begin{aligned} e^{-1}\mathcal{L}_1^{\text{ungauged}} \quad = \quad & \frac{c_2}{24}\Big[\frac{1}{16}\,\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu R^{\nu\rho\delta\gamma}R^{\lambda\sigma}{}_{\delta\gamma} + \frac{1}{8}\,C^2_{\mu\nu\rho\sigma} + \frac{3}{16}C_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} - F^{\mu\rho}F_{\rho\nu}R^\nu_\mu \\ & -\frac{1}{8}RF^2 + \frac{3}{2}\,F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho} + \frac{3}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho} + \frac{3}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} \\ & +\frac{1}{8}\,\epsilon_{\mu\nu\rho\lambda\sigma}F^{\mu\nu}(3F^{\rho\lambda}\nabla_\delta F^{\sigma\delta} + 4F^{\rho\delta}\nabla_\delta F^{\lambda\sigma} + 6F^\rho{}_\delta\nabla^\lambda F^{\sigma\delta}) \\ & +\frac{45}{64}F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu} - \frac{45}{256}(F^2)^2\Big] . \end{aligned} \qquad \text{(III.50)}$$

The full on-shell Lagrangian is thus given by

$$\begin{aligned} e^{-1}\mathcal{L} \quad = \quad & -R - \frac{3}{4}F^2\Big(1 + \frac{5}{72}c_2g^2\Big) + \frac{1}{4}\Big(1 - \frac{1}{16}c_2g^2\Big)\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 \\ & +\frac{c_2}{24}\Big[\frac{1}{16}RF^2 + \frac{1}{64}(F^2)^2\Big] + \mathcal{L}_1^{\text{ungauged}} . \end{aligned} \qquad \text{(III.51)}$$

Finally, we may redefine A_μ to write the kinetic term in canonical form:

$$A_\mu^{\text{final}} = \sqrt{3}\Big(1 + \frac{5}{144}c_2g^2\Big)\,A_\mu^{\text{old}} . \qquad \text{(III.52)}$$

The Lagrangian then becomes:

$$\begin{aligned} \mathcal{L} \quad = \quad & -R - \frac{1}{4}F^2 + \frac{1}{12\sqrt{3}}\Big(1 - \frac{1}{6}c_2g^2\Big)\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 \\ & +\frac{c_2}{24}\Big[\frac{1}{48}RF^2 + \frac{1}{576}(F^2)^2\Big] + \mathcal{L}_1^{\text{ungauged}} , \end{aligned} \qquad \text{(III.53)}$$

with

$$
\begin{aligned}
e^{-1}\mathcal{L}_1^{\text{ungauged}} \quad = \quad & \frac{c_2}{24}\Big[\frac{1}{16\sqrt{3}}\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu R^{\nu\rho\delta\gamma}R^{\lambda\sigma}{}_{\delta\gamma}+\frac{1}{8}C^2_{\mu\nu\rho\sigma}+\frac{1}{16}C_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}-\frac{1}{3}F^{\mu\rho}F_{\rho\nu}R^\nu_\mu \\
& -\frac{1}{24}RF^2+\frac{1}{2}F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho}+\frac{1}{4}\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho}+\frac{1}{4}\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} \\
& +\frac{1}{32\sqrt{3}}\epsilon_{\mu\nu\rho\lambda\sigma}F^{\mu\nu}(3F^{\rho\lambda}\nabla_\delta F^{\sigma\delta}+4F^{\rho\delta}\nabla_\delta F^{\lambda\sigma}+6F^\rho{}_\delta\nabla^\lambda F^{\sigma\delta}) \\
& +\frac{5}{64}F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu}-\frac{5}{256}(F^2)^2\Big]\,. \qquad \text{(III.54)}
\end{aligned}
$$

3.3 Anomaly matching and AdS/CFT

In the above section, we have written out the on-shell five-dimensional $\mathcal{N}=2$ gauged supergravity Lagrangian up to four-derivative order. Restoring Newton's constant, this takes the form

$$
e^{-1}\mathcal{L}=\frac{1}{16\pi G_5}\left[-R-\frac{1}{4}F^2+\frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma}+12g^2+\frac{c_2}{192}C^2_{\mu\nu\rho\sigma}+\cdots\right], \qquad \text{(III.55)}
$$

where we have only written out a few noteworthy terms. Given this Lagrangian, it is natural to make the appropriate AdS/CFT connection to $\mathcal{N}=1$ super-Yang Mills theory. Before we do so, however, we present a brief review of the AdS/CFT dictionary in the case of $\mathcal{N}=4$ super-Yang Mills.

The standard AdS/CFT setup relates IIB string theory on $\mathrm{AdS}_5\times S^5$ to $\mathcal{N}=4$ super-Yang Mills with gauge group SU(N) and 't Hooft coupling $\lambda=g_{YM}^2N$. The standard AdS/CFT dictionary then reads

$$
\frac{L^4}{\alpha'^2}=4\pi g_sN=g_{YM}^2N, \qquad \text{(III.56)}
$$

where L is the 'radius' of AdS_5. This duality may be approached more directly by reducing IIB supergravity on S^5, yielding $\mathcal{N}=8$ gauged supergravity in five dimensions. Just as in the $\mathcal{N}=2$ case of (III.55), this theory is determined in terms of two gravity-side parameters, G_5 (Newton's constant) and g (the gauged supergravity coupling constant).

These are related to the parameters of the AdS/CFT dictionary (III.56) according to

$$g = \frac{1}{L}, \qquad N^2 = \frac{\pi L^3}{2G_5}. \tag{III.57}$$

Since the range of $\mathcal{N}=1$ gauge theories is much richer than that of $\mathcal{N}=4$ SYM, it is worth rewriting the above AdS/CFT relations in terms of more general invariants of the gauge theory. This may be elegantly done through anomaly matching, and in particular by making a connection through the holographic Weyl anomaly [112]. Note that a discussion of the $\mathcal{N}=1$ SCFT description of the higher derivative theory was already given in [107], where special emphasis was placed on the technique of a-maximization. Here we wish to provide a more complete discussion of the relation between the gravity parameters G_5, g and c_2 and the gauge theory data.

3.3.1 The Weyl anomaly

For a four-dimensional field theory in a curved background, the Weyl anomaly may be parameterized by two coefficients, commonly denoted a and c (or equivalently b and b')

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} C - \frac{a}{16\pi^2} E, \tag{III.58}$$

where

$$C = C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \tfrac{1}{3}R^2 \tag{III.59}$$

is the square of the four-dimensional Weyl tensor, and

$$E = \widetilde{R}^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \tag{III.60}$$

is the four-dimensional Euler invariant. At the two-derivative level, the holographic computation of the $\mathcal{N}=4$ SYM Weyl anomaly gives $a = c = N^2/4$ [112]. Combining this with (III.57) then allows us to write

$$a = c = \frac{\pi L^3}{8G_5}, \tag{III.61}$$

which has the advantage of being completely general, independent of the particular gauge theory dual.

The prescription for obtaining the holographic Weyl anomaly for higher derivative gravity was worked out in [20, 151], and later extended in [77] for general curvature squared terms. The result is that, for an action of the form

$$e^{-1}\mathcal{L} = \frac{1}{2\kappa^2}\left(-R + 12g^2 + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\sigma}^2 + \cdots\right) , \tag{III.62}$$

the holographic Weyl anomaly may be written as [77]

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = \frac{2L}{16\pi G_5}\left[\Big(-\frac{L}{24} + \frac{5\alpha}{3} + \frac{\beta}{3} + \frac{\gamma}{3}\Big)R^2 + \Big(\frac{L}{8} - 5\alpha - \beta - \frac{3\gamma}{2}\Big)R_{\mu\nu}^2 + \frac{\gamma}{2}R_{\mu\nu\rho\sigma}^2\right], \tag{III.63}$$

where L is related to g (to linear order) by

$$g = \frac{1}{L}\left[1 - \frac{1}{6L^2}(20\alpha + 4\beta + 2\gamma)\right]. \tag{III.64}$$

Comparison of (III.58) with (III.63) then gives the curvature-squared correction to (III.61)

$$\begin{aligned} a &= \frac{\pi L^3}{8G_5}\left[1 - \frac{4}{L^2}(10\alpha + 2\beta + \gamma)\right] \\ c &= \frac{\pi L^3}{8G_5}\left[1 - \frac{4}{L^2}(10\alpha + 2\beta - \gamma)\right]. \end{aligned} \tag{III.65}$$

Turning now to the $\mathcal{N} = 2$ gauged supergravity Lagrangian of (III.55), we see that the curvature-squared corrections are proportional to the square of the five-dimensional Weyl tensor. This gives

$$(\alpha, \beta, \gamma) = \frac{c_2}{192}\left(\frac{1}{6}, -\frac{4}{3}, 1\right), \tag{III.66}$$

so that

$$a = \frac{\pi L^3}{8G_5}, \qquad c = \frac{\pi L^3}{8G_5}\left(1 + \frac{c_2}{24L^2}\right), \qquad g = \frac{1}{L}. \tag{III.67}$$

Note that the AdS radius is unshifted from that of the lowest order theory. This is because AdS is conformally flat, so that the Weyl-squared correction in (III.55) has no effect on the background. Finally, we may solve for c_2 to obtain

$$\frac{c_2}{24} = \frac{c-a}{ag^2}. \tag{III.68}$$

This is the key relation connecting the four-derivative terms in the gauged supergravity Lagrangian to the $\mathcal{N}=1$ gauge theory data.

3.3.2 The R-current anomaly

A consistency check on the form of c_2 comes from the gravitational contribution to the anomalous divergence of the $\mathrm{U}(1)_R$ current $\langle \partial_\mu(\sqrt{g}\,\mathcal{R}^\mu)\rangle$, since the latter is related by supersymmetry to the conformal anomaly $\langle T^\mu_\mu\rangle$.

The CFT $U(1)$ anomaly is given by

$$\delta_I(\Lambda) Z_{CFT} = \int \Lambda^I \left[\frac{\mathrm{tr}(G_I G_J G_K)}{24\,\pi^2} F^J \wedge F^K + \frac{\mathrm{tr}\, G_I}{192\,\pi^2} R_{ab} \wedge R^{ab}\right], \tag{III.69}$$

where G_I is a global $U(1)_I$ generator, and the trace is taken to be a sum over all the fermion loops. The AdS/CFT relation $Z_{CFT} = \exp(-I_{bulk})$ then connects this field theory anomaly to the coefficients of the Chern-Simons terms in the bulk supergravity:

$$I_{bulk} = \cdots + \int \left[\frac{\mathrm{tr}(G_I G_J G_K)}{24\,\pi^2} A^I \wedge F^J \wedge F^K + \frac{\mathrm{tr}\, G_I}{192\,\pi^2} A^I \wedge R_{ab} \wedge R^{ab}\right], \tag{III.70}$$

where the ellipses denote the gauge invariant part of the action. Comparison to the $A\wedge R\wedge R$ term of (III.35) gives

$$\mathrm{tr}\, G_I = -\frac{\pi c_{2I}}{8 G_5}. \tag{III.71}$$

To relate $c_2 \equiv c_{2I}\bar{M}^I$ to the central charges, we can use the relation

$$a = \frac{3}{32}(3\mathrm{tr}R^3 - \mathrm{tr}R), \quad c = \frac{1}{32}(9\mathrm{tr}R^3 - 5\mathrm{tr}R)\,, \tag{III.72}$$

provided we can relate G_I appropriately to the U(1) charges R. A few comments are needed to explain how to identify the R-charge correctly. First of all, the R-charge is a particular linear combination of the G_I, proportional to $\bar{M}^I G_I$. Also, the supercharge Q_α should have R-charge one. The U(1) charges of Q_α can be read off from the coupling between the gauge fields and the graviphoton in the gravity side, and the algebra is given by $[G_I, Q_\alpha] = P_I Q_\alpha$. This uniquely determines the R-charge as

$$R = \frac{\bar{M}^I G_I L}{P_I \bar{M}^I} \quad \to \quad \mathrm{tr} R = -\frac{1}{P_I \bar{M}^I} \frac{\pi c_2 L}{8G_5} . \tag{III.73}$$

Recall that the combination $P_I \bar{M}^I = 3/2$ can be determined from the vacuum solution, (III.32). By plugging this equation into (III.72), we obtain

$$\frac{c_2}{24} = \frac{8G_5}{\pi L}(c-a) . \tag{III.74}$$

In addition, the gravitational constant also can be determined from the U(1) anomaly. Eq. (III.70) implies

$$\mathrm{tr}(G_I G_J G_K) = \frac{\pi}{8G_5}\left(12 c_{IJK} - \frac{g^2}{3} c_{(I} P_J P_{K)}\right) . \tag{III.75}$$

By multiplying $\bar{M}^I \bar{M}^J \bar{M}^K$ on both sides, we obtain

$$\frac{27}{8L^3} \mathrm{tr} R^3 = \frac{\pi}{8G_5}\left(12 - \frac{3c_2}{4L^2}\right) . \tag{III.76}$$

The formula for the central charges (III.72) and (III.74) then gives

$$\frac{1}{16\pi G_5} = \frac{a}{2\pi^2 L^3} . \tag{III.77}$$

Using this relation, (III.74) can be rewritten as

$$\frac{c_2}{24L^2} = \frac{c-a}{a} . \tag{III.78}$$

These results agree with those found through the holographic Weyl anomaly calculations, as expected for consistency.

Extracting the R-current anomaly from the $\mathcal{N}=2$ case

Since the U(1) normalization may be somewhat obscure, we may perform an additional check by making contact with the $\mathcal{N}=2$ SCFT literature. In fact, one can extract the c_2 result (III.68) from the analysis of [4], which studied R-symmetry anomalies in the $\mathcal{N}=2$ SCFT dual to $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_2$. Of course, the appropriate supersymmetric CFT that is dual to our bulk $\mathcal{N}=2$ AdS_5 theory has $\mathcal{N}=1$ supersymmetry. Nevertheless, one can still use the analysis of [4], after carefully rewriting it in the language of $\mathcal{N}=1$ anomalies. Before doing so, we will need to make a few general comments on the connection between the CFT R-current anomalies and the dual supergravity description.

The four-dimensional CFT R-current anomaly is sensitive to the amount of supersymmetry, and is given by [6]:

$$\partial_\mu(\sqrt{g}\,\mathcal{R}^\mu)_{\mathcal{N}=1} = \frac{c-a}{12\pi^2}\,\tilde{R}R + \frac{5a-3c}{9\pi^2}\,\tilde{F}F\,, \tag{III.79}$$

$$\partial_\mu(\sqrt{g}\,\mathcal{R}^\mu)_{\mathcal{N}=2} = \frac{c-a}{4\pi^2}\,\tilde{R}R + \frac{3(c-a)}{\pi^2}\,\tilde{F}F\,, \tag{III.80}$$

where F is the flux associated with the R-symmetry. The R-symmetry of $\mathcal{N}=2$ SCFTs is $\mathrm{U}(1)_R \times \mathrm{SU}(2)_R$. The $\mathrm{U}(1)_R$ symmetry of its $\mathcal{N}=1$ subalgebra is

$$\mathcal{R}_{\mathcal{N}=1} = \frac{1}{3}\mathcal{R}_{\mathcal{N}=2} + \frac{4}{3}I_3, \tag{III.81}$$

where I_1, I_2, I_3 are $SU(2)_R$ generators. The factor of $1/3$ in the relation above can also be seen in the gravitational contributions to $\partial_\mu(\sqrt{g}\,\mathcal{R}^\mu)$ in (III.79) and (III.80). Recall that the mixed U(1)-gravity-gravity anomaly $\partial_\mu(\sqrt{g}\,\mathcal{R}^\mu) \propto \tilde{R}R$ is represented in the bulk by the mixed gauge-gravity Chern-Simons interaction $\propto \int_{\mathrm{AdS}_5} A \wedge \mathrm{tr}(R \wedge R)$. Thus, the bulk CS term associated to the $\mathcal{N}=1$ SCFT will be $1/3$ of that corresponding to $\mathcal{N}=2$.

Furthermore, when using the results of [4], we will have to be careful with how the U(1) gauge field is normalized. In the AdS/CFT dictionary, the normalization of the gauge field kinetic term

$$S_{AdS_5} = \int d^4x\, dz\, \sqrt{-g}\, \frac{F_{\mu\nu}F^{\mu\nu}}{4\, g_{SG}^2} \tag{III.82}$$

can be extracted by looking at the two-point function of the dual CFT currents sourced by the gauge field $A_\mu(\vec{x}) = A_\mu(\vec{x}, z)|_{\text{boundary}}$. For a four-dimensional CFT, the general form of the two point function of such currents is given by [73]:

$$\langle J_i(x) J_j(y) \rangle = \frac{B}{(2\pi)^4} \big(\Box \delta_{ij} - \partial_i \partial_j \big) \frac{1}{(x-y)^4}\,, \tag{III.83}$$

where B is a numerical coefficient which is related to the normalization of the gauge kinetic term:

$$B \propto \frac{1}{g_{SG}^2}\,. \tag{III.84}$$

For the $\mathcal{N}=2$ computation of [4] one finds $B=8$, while for the case of $\mathcal{N}=1$ supersymmetry [5] we read off $B = 8/3$. Notice that the two results are again off by a factor of 3. We now have all the ingredients we need to apply the ($\mathcal{N}=2$ SCFT) analysis of [4] to our case ($\mathcal{N}=1$ SCFT). We have seen that both the gauge kinetic term normalization and the coefficient of the mixed gauge-gravity CS term will have to be adjusted.

The five-dimensional supergravity action of [4] takes the form

$$\begin{aligned} S &= \frac{N^2}{\pi^2 L^3} \int \sqrt{-g}\, \frac{F_{\mathcal{R}}^2}{4} + \frac{N}{16\pi^2 L} \int A^{\mathcal{R}} \wedge \mathrm{tr}(R \wedge R) \\ &= \frac{N^2}{4\pi^2 L^3} \int \Big[\sqrt{-g}\, F_{\mathcal{R}}^2 - \frac{L^2}{16\, N}\, \epsilon_{\mu\nu\rho\lambda\sigma} A^\mu R^{\nu\rho\delta\gamma} R^{\lambda\sigma}{}_{\delta\gamma} \Big]\,, \end{aligned} \tag{III.85}$$

where $A^{\mathcal{R}}$ is the gauge field that couples canonically to the R-current. This was the effective supergravity Lagrangian which was appropriate for comparison to the $\mathcal{N}=2$ SCFT. Since we are interested in comparing to a CFT with $\mathcal{N}=1$ SUSY, we will need to rescale both terms by appropriate factors of $1/3$:

$$S \to \frac{N^2}{4\pi^2 L^3} \int \Big[\sqrt{-g}\, \frac{1}{3}\, F_{\mathcal{R}}^2 - \frac{L^2}{3 \cdot 16\, N}\, \epsilon_{\mu\nu\rho\lambda\sigma} A^\mu R^{\nu\rho\delta\gamma} R^{\lambda\sigma}{}_{\delta\gamma} \Big]\,. \tag{III.86}$$

Finally, we rescale the graviphoton, $A^{\mathcal{R}} = (\sqrt{3}/2)A$, to obtain a canonical gauge kinetic term:

$$S \to \frac{N^2}{4\pi^2 L^3} \int \Big[\sqrt{-g}\ \frac{F^2}{4} - \frac{L^2}{32\sqrt{3}\,N}\,\epsilon_{\mu\nu\rho\lambda\sigma} A^\mu R^{\nu\rho\delta\gamma} R^{\lambda\sigma}{}_{\delta\gamma}\Big]\,. \tag{III.87}$$

This is the action which should be compared to ours:

$$S_{us} = \frac{N^2}{4\pi^2 L^3} \int \sqrt{g}\Big[-R - \frac{F^2}{4} + \frac{c_2}{24\cdot 16\sqrt{3}}\,\epsilon_{\mu\nu\rho\lambda\sigma}\, A^\mu R^{\nu\rho\delta\gamma} R^{\lambda\sigma}{}_{\delta\gamma} + \ldots\Big]\,, \tag{III.88}$$

finally giving us

$$c_2 = \frac{12L^2}{N} = 24L^2\frac{c-a}{a}\,, \tag{III.89}$$

in agreement with (III.68) and (III.78).

3.4 R-Charged Black Hole Solutions

The embedding of the lowest order five-dimensional $\mathcal{N}=2$ gauged $U(1)^3$ supergravity into IIB supergravity was done in [57]. If the three $U(1)$ charges are taken to be equal, we end up with the minimal supergravity system that we have considered above, (III.2). The static stationary non-extremal solutions are well known, and were found in [13]. For the truncation to minimal supergravity, they take the form

$$\begin{aligned} ds^2 &= H^{-2} f dt^2 - H\Big(f^{-1}dr^2 + r^2 d\Omega_{3,k}^2\Big),\\ A &= \sqrt{\frac{3(kQ+\mu)}{Q}}\Big(1-\frac{1}{H}\Big)dt, \end{aligned} \tag{III.90}$$

where the metric functions H and f are:

$$\begin{aligned} H(r) &= 1+\frac{Q}{r^2},\\ f(r) &= k - \frac{\mu}{r^2} + g^2 r^2 H^3\,. \end{aligned} \tag{III.91}$$

Here μ is a non-extremality parameter and $d\Omega_{3,k}^2$ for $k = 1$, 0, or -1 corresponds to the unit metric of a spherical, flat, or hyperbolic 3-dimensional geometry, respectively.

3.4.1 Higher order corrected R-charged Black Hole Solutions

We wish to find corrections to the R-charged solutions (III.90) given the higher derivative Lagrangian (III.53). To this end, as in [132] we treat c_2 as a small parameter and expand the metric and gauge field as follows:

$$
\begin{aligned}
H(r) &= 1+\frac{Q}{r^2}+c_2 h_1(r)\,, \\
f(r) &= k-\frac{\mu}{r^2}+g^2r^2H^3+c_2 f_1(r)\,, \\
A &= \sqrt{\frac{3(kQ+\mu)}{Q}}\Big(1-\frac{1+c_2a_1(r)}{H}\Big)dt\,,
\end{aligned}
\tag{III.92}
$$

where h_1, f_1, and a_1 parameterize the corrections to the background geometry. Solving the equations of motion for the theory, we arrive at:

$$
\begin{aligned}
h_1 &= -\frac{Q(kQ+\mu)}{72r^6H_0^2}\,, \\
f_1 &= \frac{-5g^2Q(kQ+\mu)}{72r^4}+\frac{\mu^2}{96r^6H_0}\,, \\
a_1 &= \frac{Q}{144r^6H_0^3}\big[4(kQ+\mu)-3\mu-\frac{3Q\mu}{r^2}\big]\,.
\end{aligned}
\tag{III.93}
$$

The new corrected geometry is therefore given by

$$
\begin{aligned}
H(r) &= H_0(r)+\frac{c_2}{24}\left[\frac{-Q(kQ+\mu)}{3r^6H_0^2}\right], \\
f(r) &= f_0(r)+\frac{c_2}{24}\left[-\frac{8g^2Q(kQ+\mu)}{3r^4}+\frac{\mu^2}{4r^6H_0}\right], \\
A_t(r) &= A_{t\,0}(r)-\frac{c_2}{24}\frac{\sqrt{3Q(kQ+\mu)}}{2r^8H_0^4}\left[2(kQ+\mu)r^2-\mu r^2H_0\right],
\end{aligned}
\tag{III.94}
$$

where H_0, f_0, and A_0 refer to the background solutions (III.90) and (III.91). Finally, we should note that in the literature Q and μ are sometimes written in terms of a parameter β, defined by $\sinh^2\beta = kQ/\mu^2$.

We will state the $k=0$ and $k=1$ solutions explicitly, since they have several interesting applications: the former to studies of the hydrodynamic regime of the theory, and the latter to the issue of horizon formation for small black holes. For $k=0$, the solution is given by

$$\begin{aligned} H(r) &= H_0(r) + \frac{c_2}{24}\left[\frac{-Q\mu}{3\,r^6 H_0^2}\right], \\ f(r) &= f_0(r) + \frac{c_2}{24}\left[-\frac{8g^2\mu Q}{3r^4} + \frac{\mu^2}{4\,r^6 H_0}\right], \\ A_t(r) &= A_{t\,0}(r) - \frac{c_2}{24}\left[\frac{\sqrt{3Q\mu}}{2r^8 H_0^4}(\mu r^2 - Q\mu)\right]. \end{aligned} \tag{III.95}$$

while for $k=1$ it is given by

$$\begin{aligned} H(r) &= H_0(r) - \frac{c_2}{24}\left[\frac{Q(Q+\mu)}{3r^2(r^2+Q)^2}\right], \\ f(r) &= f_0(r) + \frac{c_2}{24}\left[-\frac{8g^2 Q(Q+\mu)}{3r^4} + \frac{\mu^2}{4r^6 H_0}\right], \\ A_t(r) &= A_{t\,0}(r) - \frac{c_2}{24}\left[\frac{\sqrt{3Q(Q+\mu)}}{2r^8 H_0^4}\Big((2Q+\mu)r^2 - Q\mu\Big)\right]. \end{aligned} \tag{III.96}$$

3.4.2 Conditions for Horizon Formation

We would like to conclude this section with some comments on the structure of the horizon for the solutions that we have found. In particular, we are interested in whether higher derivative corrections will facilitate or hinder the formation of a horizon. In the standard two-derivative theory, the BPS-saturated limit ($\mu=0$) of the $k=1$ solution (III.90)-(III.91) describes a geometry with a naked singularity, the so-called superstar [147]. Furthermore, even if the non-extremality parameter is turned on, one finds that a horizon develops only given a certain critical amount, $\mu \geq \mu_c$ [13]. It is therefore natural to ask what happens to such geometries once we start incorporating curvature corrections. For the superstar, we would like to see hints of horizon formation. In the non-extremal case, on the other hand, it would be nice to determine whether the inclusion of higher-derivative corrections leads to a smaller (larger) critical value μ_c, increasing (decreasing) the parameter space for the appearance of a horizon. However, one should keep in mind that our arguments are only suggestive, since our analysis is perturbative, while the formation of a horizon is a non-perturbative process. Moreover, given that even in the non-extremal case turning on μ does not guarantee the presence of a horizon, it is not clear at all whether higher derivative corrections can be enough to push the superstar to develop a horizon. A more proper analysis would involve looking directly at the SUSY conditions, and asking whether they are compatible with having a superstar solution with a finite horizon. In fact, there are already studies which seem to indicate [142] that this may not be possible.

The spherically symmetric solutions presented in (III.96) are of the form:

$$ds^2 = F_1(r)\, dt^2 - F_2(r)\, dr^2 - F_3(r)\, d\Omega_3^2\,. \tag{III.97}$$

Horizons appear at zeroes of the function $F_1(r)$. One can make arguments about their existence without having to solve explicitly for their exact location. Notice that $F_1(r)$ is a positive function for large r. Thus, a sufficient condition for having at least one horizon is

$$F_1(r_{min}) \leq 0\,, \tag{III.98}$$

where r_{min} is a (positive) minimum of $F_1(r)$. This was the reasoning used in [13] to study the properties of the horizon of the non-extremal solution.

For the corrected superstar solution we have, expanding in c_2:

$$F_1 \equiv \frac{f}{H^2} = \frac{f_0 + c_2(f_1 - 2f_0 h_1 H_0^{-1})}{H_0^2} + \mathcal{O}(c_2^2)\,. \tag{III.99}$$

It is easy to see that, to leading order, the numerator does not vanish. With the inclusion of higher-derivative terms, however, it picks up a negative contribution, hinting at the possibility of a horizon. Furthermore, the minimum of the function $F \equiv f_0 + c_2(f_1 - 2f_0 h_1 H_0^{-1})$ will shift. Let's see precisely how that happens. To lowest order, its minimum is given by $x^{(0)}_{min} = 2Q$, which in turn gives us $F(x^{(0)}_{min}) = 1 + 27g^2Q/4$. Including higher order corrections, we find

$$x_{min} = x^{(0)}_{min} + c_2 x^{(1)}_{min} = 2Q - c_2\frac{81g^2Q - 4}{4374Qg^2}\,. \tag{III.100}$$

Now we have

$$F(x_{min}) = 1 + 27g^2Q/4 + c_2(\frac{1}{972Q} - \frac{g^2}{48}),$$

which tells us that the minimum of the function will be slightly closer to zero as long as $g^2Q > 4/81$.

The analysis of the conditions for the existence of a horizon in the non-extremal case ($\mu \neq 0$) is significantly more involved. The expression for the corrected horizon radius in terms of the original, two-derivative horizon radius r_0 is:

$$r_H = r_0\left(1 + \frac{c_2}{24}\left\{\frac{g^4H_0^4(3Q^2 - 26Qr_0^2 + 3r_0^4) - 2g^2H_0^2(13Q - 3r_0^2) + 3}{24H_0r_0[g^2H_0^2(Q - 2r_0^2) - 1]}\right\}\right). \tag{III.101}$$

Notice that we traded μ in favor or r_0 in the expression above by making use of $f_0(r_0) = 0$, *i.e.* the relation $\mu/r_0^2 = 1 + g^2r_0^2H_0^3$. As we mentioned above, in the two-derivative case one finds a critical value μ_{crit} above which a horizon will form. It would certainly be interesting to explore for which parameter values r_H decreases or increases, and more importantly, how the (corrected) critical value of μ is affected by the curvature corrections. We leave this to future studies.

3.5 Thermodynamics

We may now study some of the basic thermodynamic properties of the non-extremal solutions constructed above. With an eye towards AdS/CFT in the Poincaré patch, we will focus on the $k = 0$ solution (III.95), although the analysis may easily be carried out for the other cases as well. We begin with the entropy, which for Einstein gravity is characterized by the area of the event horizon. In the presence of higher derivative terms, however, this relation is modified, and the entropy is no longer given by the area law. Instead, we may turn to the Noether charge method developed in [171] (see also [120, 119]).

The original Noether charge method is only applicable to a theory with general covariance, but has been extended to a theory with gravitational Chern-Simons terms in [166].

Our action includes a mixed Chern-Simons term of the form $A \wedge R \wedge R$. But as long as we keep this term as it is, with a bare gauge potential, the general covariance is unbroken and we can still use the original formulation. In the absence of covariant derivatives of the Riemann tensor, the entropy formula is given by [171]

$$S = -2\pi \int_{\Sigma} d^{D-2}x\sqrt{-h}\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\,\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{III.102}$$

where Σ denotes the horizon cross section, h is the induced metric on the it and $\epsilon_{\mu\nu}$ is the binormal to the horizon cross section.

For the metric ansatz (III.90) the only non-vanishing component of the binormal $\epsilon_{\mu\nu}$ is

$$\epsilon_{tr} = -\epsilon_{rt} = H^{-1/2}\,. \tag{III.103}$$

Applying the prescription (III.102) to the action (**??**), we obtain, to linear order in c_2,

(III.104)

$$\begin{aligned} S &= \frac{A}{8G_5}\left[-g^{\mu\rho}g^{\nu\sigma} + \frac{c_2}{24}\left(-\tfrac{1}{4}C^{\mu\nu\rho\sigma} - \tfrac{1}{32}g^{\mu\rho}g^{\nu\sigma}F^2 + \tfrac{5}{12}g^{\nu\sigma}F^{\mu\lambda}F^{\rho}{}_{\lambda} - \tfrac{1}{16}F^{\mu\nu}F^{\rho\sigma}\right)\right]\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{r=r_+} \\ &= \frac{A}{4G_5}\left[1 + c_2\,\frac{\mu(Q+3r_0^2)}{48(r_0^2+Q)^3}\right], \end{aligned}$$

where $A = \int \sqrt{-h}\; d\Omega_{3,0}$ is the area of the horizon for the solution to the higher derivative theory. Also, r_+ denotes the radius of the event horizon for the corrected black brane solution, while r_0 is the horizon location for the original, two-derivative solution (III.91). The former can be found by requiring that the $g_{tt} = f(r)/H(r)^2$ component of the corrected metric vanishes[5]. Similarly, r_0 satisfies $f_0(r_0) = 0$. Notice that the non-extremality parameter μ can be expressed entirely in terms of r_0 and Q:

$$f_0(r_0) = 0 \;\;\Rightarrow\;\; \mu = \frac{g^2(r_0^2+Q)^3}{r_0^2}\,. \tag{III.105}$$

We can therefore eliminate μ from (III.105), and write the entropy in the following form:

$$S = \frac{A}{4G_5}\left[1 + c_2 g^2\frac{Q+3r_0^2}{48\,r_0^2}\right]. \tag{III.106}$$

[5]To linear order in the expansion parameter c_2, this coincides with demanding that $f(r)$ vanishes.

The first term above is simply the contribution coming from the area, while the remaining $\mathcal{O}(c_2)$ term is the expected deviation from the area law.

In order to arrive at the entropy density, we need one more ingredient, which is the relation between the corrected and uncorrected horizon radii r_+ and r_0:

$$r_+ = r_0\left(1+\frac{c_2}{24}\frac{g^2(r_0^2+Q)(3Q^2-26Qr_0^2+3r_0^4)}{24r_0^4(Q-2r_0^2)}\right). \tag{III.107}$$

This is because the area A appearing in (III.106) is computed using r_+. This expression allows us to write the entropy per unit three-brane spatial volume entirely in terms of r_0 as well as the physical parameters of the theory

$$\begin{aligned} s &= \frac{(r_0^2+Q)^{3/2}}{4G_5L^3}\left(1+\frac{c_2}{24}\frac{g^2(3Q^2-14Qr_0^2-21r_0^4)}{8r_0^2(Q-2r_0^2)}\right) \\ &= \frac{2(r_0^2+Q)^{3/2}}{\pi L^6}\left(a+(c-a)\frac{3Q^2-14Qr_0^2-21r_0^4}{8r_0^2(Q-2r_0^2)}\right). \end{aligned} \tag{III.108}$$

In the second line we have used the relations (III.67) to replace the gravitational quantities G_5 and c_2 by the central charges of the dual CFT. Notice that the lowest order term above matches the two-derivative entropy computation of [165].

While r_0 is the coordinate location of the horizon in the lowest order computation, it is not in itself a physically relevant parameter. Instead, it may be viewed as a proxy for the Hawking temperature associated with the non-extremal solution. A simple way of computing this temperature is to identify it with the inverse of the periodicity of Euclidean time τ. The relevant components of the metric are given by

$$ds^2 = H^{-2}fd\tau^2 + Hf^{-1}dr^2 + \cdots, \tag{III.109}$$

and the horizon is located at $f(r_+)=0$. Expanding near the horizon and identifying the proper period of τ to remove the conical singularity yields the temperature

$$T_H = \frac{(r_0^2+Q)^{1/2}}{2\pi L^2}\left[\frac{(2r_0^2-Q)}{r_0^2}+\frac{c_2}{24L^2}\frac{(3Q^3+4Q^2r_0^2+59Qr_0^4-10r_0^6)}{8r_0^4(2r_0^2-Q)}\right]. \tag{III.110}$$

In principle, we may invert this expression to obtain r_0 as a function of temperature T_H and charge Q. This then allows us to rewrite the entropy density as a function of charge and temperature, $s = s(T_H, Q)$. In practice, however, non-trivial R-charge introduces a new scale, so that the entropy density/temperature relation no longer takes the simple form $s \sim T^3$ resulting from simple dimensional analysis.

Our interest in studying higher order corrections to R-charged AdS_5 black holes is also motivated by our desire to investigate corrections to the hydrodynamic regime of the dual theory. It is natural to apply the results of this work to the calculation of η/s, the shear viscosity to entropy ratio, which has recently received a great deal of attention. In particular, our present construction of higher-derivative corrected R-charged black holes allows for a generalization of the finite coupling shear viscosity calculation to the case of finite (R-charge) chemical potential. We present this calculation in the following sections.

3.6 Overview of η/s in AdS/CFT

Over the past decade the development of the AdS/CFT correspondence [138, 96, 175] has led to a new way of thinking about strongly coupled gauge theories. Although the original and best studied example of the AdS/CFT duality connects $\mathcal{N} = 4$ supersymmetric Yang-Mills to type IIB string theory on $\text{AdS}_5 \times S^5$, the duality has been extended to a variety of cases, and can describe confining gauge theories with features that are qualitatively similar to QCD. In recent years the AdS/CFT correspondence has proven to be a valuable tool for better understanding thermal and hydrodynamic properties of field theories at strong coupling. In particular, it has been applied to the realm of heavy ion collisions, with the aim of providing a more realistic description of the strongly coupled quark-gluon plasma (QGP).

In the context of RHIC physics, a quantity that has played a special role is the ratio of shear viscosity to entropy density, η/s (see *e.g.* [98] and references therein). Weak coupling calculations in thermal field theory predict $\eta/s \gg 1$, while elliptic flow measurements at RHIC seem to indicate a very small ratio $0 \lesssim \eta/s \lesssim 0.3$, showing that the QGP behaves like a nearly ideal fluid, and is in the strong coupling regime. Motivated by such observations, there has been a large effort to apply AdS/CFT methods to the calculation of various transport coefficients. The AdS/CFT "program" is particularly valuable given that lattice methods (which work well for equilibrium, or thermodynamic, quantities) fail for non-equilibrium processes.

Furthermore, developments resulting from the AdS/CFT correspondence prompted Kovtun, Son and Starinets (KSS) to postulate a bound [126] for η/s, according to which all fluids would obey

$$\frac{\eta}{s} \geq \frac{1}{4\pi}\,. \tag{III.111}$$

The bound, which seems to be satisfied by all substances in nature, was later shown [33] to be saturated in all gauge theories with a dual supergravity description in the large N and $\lambda = g_{YM}^2 N$ limit. Moreover, the universal value $\eta/s = 1/4\pi \sim 0.08$ falls into the experimental range observed at RHIC. Finite λ corrections to the leading supergravity result were explored in [35], which considered curvature terms of the form $\sim \alpha'^3 R^4$ in Type IIB supergravity on $AdS_5 \times S^5$. The result was that the leading finite λ corrections increase the ratio in the direction consistent with the bound:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 + 15\,\zeta(3)\lambda^{-3/2}\right]. \tag{III.112}$$

However, η/s bound violations were subsequently observed in the presence of curvature squared terms [122, 27, 26, 40]. In the context of the AdS/CFT correspondence, such terms correspond to finite N corrections and lead to [37]

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{c-a}{a}\right), \tag{III.113}$$

where a and c are the central charges of the dual CFT. Thus, violation will occur provided $c - a > 0$. The central charges are known to be equal in the large N limit [112], with $a = c = \mathcal{O}(N^2)$, but differ for finite N. For the supergravity examples studied so far, the leading $1/N$ corrections on the CFT side lead to $c - a \geq 0$, implying violation of the bound by finite N effects [37]. (It is an interesting question on its own to ask whether one can have string theory constructions whose dual description allows for $c - a < 0$.)

In this paper we investigate what happens to the η/s ratio in the presence of non-zero chemical potential. In particular, we focus on the chemical potential corresponding to turning on a $\mathrm{U}(1)_R$ background of the $\mathcal{N} = 2$ system. To leading order in the supergravity approximation, the R-charge chemical potential does not affect the calculation of η/s, as was shown in [139, 165, 136]. However, it is interesting to examine whether this is still the case once higher derivative corrections are included. Furthermore, if η/s is affected by R-charge, it would be useful to see whether the KSS bound violation gets larger or smaller as a function of chemical potential[6].

We work in the framework of $D = 5$, $\mathcal{N} = 2$ gauged supergravity, which is dual to $\mathcal{N} = 1$ super-Yang Mills theory. In particular, we are interested in supersymmetric higher derivative terms, which have a highly constrained structure[7]. The four-derivative corrections to the leading order supergravity include a mixed gauge-gravitational Chern Simons term $A \wedge \mathrm{Tr}\,(R \wedge R)$. The supersymmetric completion of this term was done in [107], where an off-shell action was obtained for $D = 5$, $\mathcal{N} = 2$ gauged supergravity at the four-derivative level. In [54] we derived the corresponding on-shell Lagrangian, found corrected R-charged black hole solutions, and studied their thermodynamic properties. We will use many of the results of [54] to compute the shear viscosity. Our main result is that turning on R-charge

[6] Ideally, one could imagine tuning the chemical potential to match observations. However, it should be noted that the R-charge chemical potential we are investigating is not the same as the more physically relevant chemical potential related to non-zero baryon number density.

[7] Four derivative corrections in the presence of a chemical potential have been partially discussed in [89, 41], where R^2 and F^4 corrections were considered, respectively. The supersymmetric Lagrangian, however, has RF^2 and $\nabla F \nabla F$-type terms as well which were not previously considered.

not only leads to violation of the bound, but enhances the effect, pushing η/s further below $1/4\pi$. Furthermore, while the dependence of η and s individually on the R-charge is quite complicated, the ratio η/s is remarkably simple.

The general picture that emerges from such studies is that if we are interested in describing properties of the QGP (or other strongly coupled systems), we can try tuning the parameters available to us (whether N, λ or the chemical potential), as long as we remain within the regime of validity of the supergravity approximation. Moreover, it is an interesting fundamental question whether violations of the bound can be related to any constraints on the dual gravitational side or consistency requirement of the underlying string theory. For instance, one may be able to relate the violation of the η/s bound to the weak gravity conjecture of [8], according to which there should be some states whose M/Q ratio is below the BPS bound. While this is an interesting avenue to explore[8], the solutions that we have considered do not admit a nice extremal BPS black hole limit (since the extremal solution is the superstar geometry, with a naked singularity), and therefore do not lend themselves easily to such an analysis.

3.7 Computation of the shear viscosity

Before presenting the result for η/s let us briefly recall the higher derivative lagrangain and the R-charged black hole geometries discussed in the previous sections. Our starting point is five-dimensional $\mathcal{N} = 2$ gauged supergravity. The physical fields in this theory are the metric $g_{\mu\nu}$, graviphoton A_μ and gravitino ψ_μ. The supersymmetric four-derivative corrections were obtained in [107] using the superconformal tensor calculus methods worked out in [127, 17, 76, 18]. By integrating out the auxiliary fields, the Lagrangian may be put

[8]See [121] for investigating the effect of higher derivatives on the weak gravity conjecture.

into the form [54] [9]

$$
\begin{aligned}
16\pi G_5 e^{-1}\mathcal{L} &= -R - \frac{1}{4}F^2 + \frac{1}{12\sqrt{3}}\Big(1-4\bar{c}_2\Big)\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 \\
&+\frac{\bar{c}_2}{g^2}\Big[\frac{1}{16\sqrt{3}}\,\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu R^{\nu\rho\delta\gamma}R^{\lambda\sigma}{}_{\delta\gamma} + \frac{1}{8}\,C^2_{\mu\nu\rho\sigma} + \frac{1}{16}C_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} - \frac{1}{3}F^{\mu\rho}F_{\rho\nu}R^\nu_\mu \\
&\quad -\frac{1}{48}RF^2 + \frac{1}{2}\,F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho} + \frac{1}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho} + \frac{1}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} \\
&\quad +\frac{1}{32\sqrt{3}}\,\epsilon_{\mu\nu\rho\lambda\sigma}F^{\mu\nu}(3F^{\rho\lambda}\nabla_\delta F^{\sigma\delta} + 4F^{\rho\delta}\nabla_\delta F^{\lambda\sigma} + 6F^\rho{}_\delta\nabla^\lambda F^{\sigma\delta}) \\
&\quad +\frac{5}{64}F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu} - \frac{41}{2304}(F^2)^2\Big]. \qquad \text{(III.114)}
\end{aligned}
$$

The four-derivative corrections are determined in terms of a single new dimensionless parameter $\bar{c}_2$ (corresponding to $c_2 g^2/24$ in the notation of [54]). Holographic computation of the Weyl anomaly [112, 20, 151, 77] allows G_5 and $\bar{c}_2$ to be expressed in terms of the anomaly coefficients a and c of the dual $\mathcal{N}=1$ gauge theory. This was worked out in [37, 54], with the result

$$
g^3 G_5 = \frac{\pi}{8a}, \qquad \bar{c}_2 = \frac{c-a}{a}. \qquad \text{(III.115)}
$$

Nonextremal R-charged black hole solutions to the lowest order $\mathcal{N}=2$ gauged supergravity were found in [13], and the corrections linear in $\bar{c}_2$ were worked out in [54]. Using a parameterization convenient for the shear viscosity calculation, the flat-horizon black holes are given by the metric

$$
ds^2 = \frac{g^2 r_0^2}{u}\Big[\frac{f(u)}{H(u)^2}dt^2 - H(u)d\vec{x}^{\,2}\Big] - \frac{H(u)}{4g^2u^2f(u)}du^2, \qquad \text{(III.116)}
$$

and the gauge field

$$
A_t = gr_0\sqrt{\frac{3(1+q)^3}{q}}\left[1 - \frac{1}{1+qu} - \frac{\bar{c}_2}{2}q(1+q)^3\frac{u^3(1-qu)}{(1+qu)^4}\right]. \qquad \text{(III.117)}
$$

The metric functions $f(u)$ and $H(u)$ are given by

$$
\begin{aligned}
f &= (1+qu)^3 - (1+q)^3u^2 + \bar{c}_2\left[-\frac{8}{3}q(1+q)^3u^3 + \frac{1}{4}(1+q)^6\frac{u^4}{1+qu}\right], \\
H &= 1 + qu - \frac{\bar{c}_2}{3}q(1+q)^3\frac{u^3}{(1+qu)^2}. \qquad \text{(III.118)}
\end{aligned}
$$

[9] We follow the conventions of [107] and take $[\nabla_\mu, \nabla_\nu]v^\sigma = R_{\mu\nu\rho}{}^\sigma\, v^\rho$ and $R_{ab} = R_{ac}{}^c{}_b$.

The above solution is fixed in terms of two parameters, r_0 (related to non-extremality) and dimensionless q (related to the R-charge). At the two-derivative level, the horizon is located at $u = 1$, while the boundary of AdS_5 is at $u = 0$. At linear order in $\bar{c}_2$, however, the horizon location gets shifted to

$$u_+ = 1 + \frac{\bar{c}_2}{12}\frac{(1+q)(3-26q+3q^2)}{2-q}. \tag{III.119}$$

The temperature and entropy density were obtained in [54]

$$\begin{aligned} T &= \frac{g^2 r_0(2-q)(1+q)^{1/2}}{2\pi}\left[1-\frac{\bar{c}_2}{8}\frac{10-59q-4q^2-3q^3}{(2-q)^2}\right], \\ s &= \frac{(gr_0)^3(1+q)^{3/2}}{4G_5}\left[1+\frac{\bar{c}_2}{8}\frac{21+14q-3q^2}{2-q}\right]. \end{aligned} \tag{III.120}$$

Note that, for $q = 0$, we may write the entropy density in terms of the temperature as

$$s = 2\pi^2 a\left[1+\frac{9}{4}\frac{c-a}{a}\right]T^3, \tag{III.121}$$

where we used the holographic relations (III.115). This reduces to the familiar $s = \pi^2 N^2 T^3/2$ [95] for $\mathcal{N} = 4$ SYM, where $a = c = N^2/4$.

We compute the shear viscosity using the Kubo formula which relates the shear viscosity to the two point function of the stress-tensor on the boundary. Holographically, since the stress tensor is dual to the metric, this is computed by performing a metric perturbation. Following the methods developed in [35, 122] we introduce a scalar channel perturbation to the metric

$$g_{xy} \to g_{xy} + h_{xy}, \tag{III.122}$$

where, for convenience, we define $h^x{}_y = \phi(t,u,\vec{x})$. Expanding the Lagrangian (III.114) to second order in the perturbation yields

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int\frac{d^4k}{(2\pi)^4}\int_0^1 du\Big[A\phi_k''\phi_{-k}+B\phi_k'\phi_{-k}'+C\phi_k'\phi_{-k}+D\phi_k\phi_{-k} \\ &\qquad +E\phi_k''\phi_{-k}''+F\phi_k''\phi_{-k}'\Big], \end{aligned} \tag{III.123}$$

where the fourier components of ϕ are defined by

$$\phi(t,u,\vec{x}\,) = \int d^3x dt\, \phi_k(u) e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{III.124}$$

We note that this parameterization of the action with coefficients $A,\ldots,F$ was originally used in [35] to handle the R^4 correction of IIB supergravity. However, it is general enough to accommodate the present case. The coefficients are even functions of the momentum, and are given explicitly in the appendix.

Varying this action with respect to ϕ yields a fourth order differential equation. However, since the higher derivative terms are multiplied by $\bar{c}_2$, we may reduce the order of the equation by working perturbatively in $\bar{c}_2$. To see this, we first consider the lowest order equation of motion

$$\phi'' + \left(\frac{f_0'}{f_0} - \frac{1}{u}\right)\phi' + \frac{\bar{\omega}^2 H_0^3}{u f_0^2}\phi \;\; = \;\; 0\,, \tag{III.125}$$

where we have defined the dimensionless frequency

$$\bar{\omega}^2 = \frac{\omega^2}{4g^4 r_0^2}\,. \tag{III.126}$$

The lowest order metric functions

$$f_0 = (1+qu)^3 - (1+q)^3 u^2, \qquad H_0 = 1+qu, \tag{III.127}$$

are obtained by setting $\bar{c}_2 = 0$ in (III.118). Taking additional derivatives of (III.125) allows us to eliminate ϕ''' and ϕ'''' terms in the full equation of motion. The result is rather simple:

$$\phi'' + \left(\frac{f'}{f} - \frac{1}{u} - \bar{c}_2\frac{(1+q)^3 u}{(1+qu)^3}\right)\phi' + \frac{\bar{\omega}^2 H^3}{u f^2}\,\phi = 0. \tag{III.128}$$

Notice that the form of this equation is almost identical to that of (III.125), the lowest order equation of motion, modified only by the presence of the corrected metric functions f and H as well as one new term, which is explicitly $\mathcal{O}(\bar{c}_2)$.

Since the function $f(u)$ vanishes linearly at the horizon u_+, the point $u = u_+$ is a regular singular point of the equation of motion (III.128). This suggests that we write

$$\phi(u) = f(u)^\nu F(u), \tag{III.129}$$

where $F(u)$ is assumed to be regular at the horizon. The exponent ν is then obtained by solving the indicial equation. In the hydrodynamic limit, the lowest order solution is known [139, 165] and is given by:

(III.130)

$$\phi_0 = f_0(u)^{\nu_0}\left\{1-\frac{\nu_0}{2}\left[\Delta \ln\frac{(\Xi-\alpha_1-1+2\alpha_3 u)(\Xi+\alpha_1+1)}{(\Xi+\alpha_1+1-2\alpha_3 u)(\Xi-\alpha_1-1)}+3\ln\left(1+(\alpha_1+1)u-\alpha_3 u^2\right)\right]\right\},$$

where

(III.131)

$$\alpha_1 \equiv 3q, \qquad \alpha_2 \equiv 3q^2, \qquad \alpha_3 \equiv q^3, \qquad \Xi \equiv (1+q)(1+4q)^{1/2}, \qquad \Delta \equiv -3\frac{q+1}{\Xi}.$$

The exponent ν_0 is given by

$$\nu_0 = -\frac{i\bar{\omega}}{(2-q)(1+q)^{1/2}}, \tag{III.132}$$

and may be re-expressed as $\nu_0 = -i\omega/4\pi T_0$, where T_0 is the lowest order temperature given in (III.120). Note that we have chosen incoming wave boundary conditions at the horizon as appropriate to the shear viscosity calculation.

Adding higher derivative terms will have two effects on this solution, one being a correction to the function $F(u)$ and the other a modification of the exponent ν defined above. For the exponent, solving the indicial equation gives

$$\nu = -\frac{i\bar{\omega}}{(2-q)(1+q)^{1/2}}\left(1+\frac{\bar{c}_2}{8}\frac{10-59q-4q^2-3q^3}{(q-2)^2}\right) = -\frac{i\omega}{4\pi T}, \tag{III.133}$$

where the relation to the temperature (III.120) is valid to linear order in $\bar{c}_2$. We may now substitute $\phi(u) = f(u)^\nu F(u)$ into the equation of motion (III.128) and linearize in

$\bar{c}_2$ to obtain an equation for $F(u)$. While this is difficult to solve exactly, since we only need a solution in the hydrodynamic regime, it is sufficient to work to first order in ω (or equivalently ν). The solution for $F(u)$ is quite complicated and can be found in the appendix.

Given this solution, it remains to evaluate the on-shell value of the action. As explained in [35], the bulk action (III.123) must be paired with an appropriate generalization of the Gibbons-Hawking term. In general, the fourth order equation of motion yields a boundary value problem for the two-point function where additional data must be specified (*e.g.* fields and their first derivatives at the endpoints). However, when working perturbatively in $\bar{c}_2$, the equation of motion reduces to a second order one, given by (III.128). This allows us to use a generalized Gibbons-Hawking term of the form

$$\mathcal{K} = -A\phi_k\phi'_{-k} - \frac{F}{2}\phi'_k\phi'_{-k} + E(p_1\phi'_k + 2p_0\phi_k)\phi'_{-k}, \tag{III.134}$$

where

$$p_1 = \frac{f'_0}{f_0} - \frac{1}{u}, \qquad p_2 = \frac{\bar{\omega}^2 H_0^3}{u f_0^2} \tag{III.135}$$

are the coefficients in the lowest order equation of motion (III.125).

Evaluating the on-shell action then amounts to evaluating a boundary term

$$S = \int \frac{d^4k}{(2\pi)^4}\mathcal{F}_k\Big|_0^1, \tag{III.136}$$

where

$$\begin{aligned}\mathcal{F}_k \;&=\; \frac{1}{16\pi G_5}\Big[\Big(B - A - \frac{F'}{2}\Big)\phi'_k\phi_{-k} + \frac{1}{2}(C - A')\phi_k\phi_{-k} - E'\phi''_k\phi_{-k} \\ &\qquad + E\phi''_k\phi'_{-k} - E\phi'''_k\phi_{-k} - E\Big(\frac{f'_0}{f_0} - \frac{1}{u}\Big)\phi'_k\phi'_{-k} + 2E\frac{\bar{\omega}^2 H_0^3}{u f_0^2}\phi'_k\phi_{-k}\Big].\end{aligned} \tag{III.137}$$

In order to compute the shear viscosity we need only the limit of the above action as u approaches the AdS boundary (*i.e.* $u \to 0$). It turns out that only the first and third terms contribute. This yields a value for the shear viscosity via the Kubo relation

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\lim_{u\to 0}(2\,\mathrm{Im}\,\mathcal{F}_k) = \frac{(gr_0)^3}{16\pi G_5}(q+1)^{3/2}\Big(1+\frac{\bar{c}_2}{8}\frac{5+6q+5q^2}{2-q}\Big). \tag{III.138}$$

Finally, dividing this by the entropy density (III.120) gives a value for the shear viscosity to entropy density ratio of

$$\frac{\eta}{s} = \frac{1}{4\pi}\Big[1-\bar{c}_2(1+q)\Big] = \frac{1}{4\pi}\Big[1-\frac{c-a}{a}(1+q)\Big], \tag{III.139}$$

where we have rewritten $\bar{c}_2$ in terms of the anomaly coefficients c and a using (III.115).

3.8 Discussion

The expression for η/s, given in (III.139), is surprisingly simple, given that both η and s are individually rather more complicated functions of the parameter q. This is presumably related to some form of universality, which holds even in an R-charged background[10]. It is instructive to examine the contribution of the various terms in the four-derivative action to the result (III.139). We find that only four terms in (III.114) are important. Writing

$$\begin{aligned} 16\pi G_5 e^{-1}\mathcal{L} &= -R-\frac{1}{4}F^2+\cdots+\frac{\bar{c}_2}{g^2}\Big[\alpha_1 C^2_{\mu\nu\rho\sigma}+\alpha_2 C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \\ &\qquad +\alpha_3\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho}+\alpha_4\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu}+\cdots\Big], \end{aligned} \tag{III.140}$$

we may arrive at the result

$$\frac{\eta}{s} = \frac{1}{4\pi}\Big[1-4\bar{c}_2\big(2\alpha_1-q(\alpha_1+6\alpha_2-6\alpha_3+3\alpha_4)\big)\Big]. \tag{III.141}$$

Note that setting α_i to their actual values in (III.114) reproduces (III.139).

The shear viscosity to entropy density ratio was independently derived in [144], where it was found to depend only on terms explicitly involving the Riemann tensor [*i.e.* the α_1 and α_2 terms in (III.140)]. This appears to differ from the result found above. However,

[10] Of course, the simplest result possible would have been to obtain η/s independent of q. But this is clearly not the case here.

by the use of Bianchi identities and integration by parts we can cast the gradient terms into the form

$$\begin{aligned}\alpha_3\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho}+\alpha_4\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} &= \\ (2\alpha_3-\alpha_4)\Big[-F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho}+F^{\mu\rho}F_{\rho\nu}R^\nu_\mu-\tfrac{1}{2}R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\Big].\end{aligned} \tag{III.142}$$

The first two terms do not contribute to the η/s ratio, while the last term will add to the original α_2 term to give an effective $\tilde{\alpha}_2=\alpha_2-\alpha_3+\alpha_4/2$, so that (III.141) may be rewritten as

$$\frac{\eta}{s}=\frac{1}{4\pi}\Big[1-4\bar{c}_2\big(2\alpha_1-q(\alpha_1+6\tilde{\alpha}_2)\big)\Big]. \tag{III.143}$$

This agrees with the result of [144] provided the difference in signature conventions is taken into account.

Finally, we return to the $\mathcal{N}=1$ SYM shear viscosity result of (III.139). In order to express this in terms of physical quantities, we wish to relate the parameter q to the R-charge chemical potential and temperature. Since q only enters into (III.139) at the next-leading order, we can use the leading order expressions in pinning down q. The chemical potential for R-charge Φ is identified as the difference of A_t between horizon and boundary [47, 58]. At lowest order, (III.117) yields

$$\Phi=gr_0\sqrt{3q(1+q)}. \tag{III.144}$$

Comparing this to the temperature

$$T_0=\frac{g^2r_0}{2\pi}(2-q)(1+q)^{1/2}, \tag{III.145}$$

allows us to write

$$q=\frac{3}{2\bar{\Phi}^2}\left(1+\frac{4}{3}\bar{\Phi}^2-\sqrt{1+\frac{8}{3}\bar{\Phi}^2}\right), \tag{III.146}$$

where $\bar{\Phi} = g\Phi/2\pi T$ is the dimensionless chemical potential. Note that q is an increasing function with respect to $\bar{\Phi}$, with $q = 0$ when $\bar{\Phi} = 0$. The possible value of q ranges as

$$0 \leq q \leq 2. \tag{III.147}$$

Substituting (III.146) into (III.139) then gives

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{c-a}{a}\left(1 + \frac{3}{2\bar{\Phi}^2}\left(1 + \frac{4}{3}\bar{\Phi}^2 - \sqrt{1 + \frac{8}{3}\bar{\Phi}^2}\right)\right)\right]. \tag{III.148}$$

Since q is non-negative, this demonstrates that turning on an R-charge chemical potential only increases violation of the η/s bound, provided $c - a > 0$. Taking the range (III.147) into account, we see that adjusting the R-charge yields a range of values

$$\frac{1}{4\pi}\left(1 - 3\frac{c-a}{a}\right) \leq \frac{\eta}{s} \leq \frac{1}{4\pi}\left(1 - \frac{c-a}{a}\right), \tag{III.149}$$

where we have again assumed $c - a > 0$.

In conclusion, we have explored the effect of a background R-charge on the shear viscosity to entropy density ratio η/s. While the leading order ratio $\eta/s = 1/4\pi$ is universal, R-charge corrections do turn up at the $1/N$ order. For known theories with a holographic dual, where $c - a > 0$, the conjectured $1/4\pi$ bound is generally violated for arbitrary chemical potential. We caution, however, that this is a parametrically small violation appearing at $\mathcal{O}(1/N)$ in the large N limit. In principle, it would be desirable to obtain a more robust result. However, this is hindered by difficulties in obtaining exact solutions to the full equations of motion (*i.e.* beyond the linearized limit). While this can be done in certain cases such as Gauss-Bonnet gravity, the natural supersymmetric organization of the higher derivative Lagrangian (III.114) is not of this form. It would be interesting to see if a modified universality relation for η/s can be obtained for arbitrary forms of the higher derivative gravity theory.

CHAPTER IV

Consistent truncations of IIB supergravity on squashed Sasaki-Einstein manifolds

In this chapter we present a consistent truncation of IIB supergravity on Sasaki-Einstein manifolds. A detailed analysis of the bosonic reduction of IIB is presented, followed by the reduction of the fermionic sector. This chapter is based on work published in [133, 134] in collaboration with Jim Liu and Zhichen Zhao.

4.1 Motivations for Studying Massive Truncations of String/M-theory

Recent developments in AdS/CFT have expanded the scope of applications from the realm of strongly coupled relativistic gauge theories to various condensed matter systems whose dynamics are expected to be described by a strongly coupled theory. These include systems with behavior governed by a quantum critical point [114, 110], as well as cold atoms and similar systems exhibiting non-relativistic conformal symmetry [164, 11]. Much current attention is also directed towards holographic descriptions of superfluids and superconductors [97, 108, 113, 109].

The main feature used in the construction of a dual model of superconductivity is the existence of a charged scalar field in the dual AdS background [108, 109]. Turning on temperature and non-zero chemical potential corresponds to working with a charged black hole in AdS. Then, as the temperature is lowered, the charged scalar develops an instability and condenses, so that the black hole develops scalar hair[1]. This condensate breaks the U(1) symmetry, and is a sign of superconductivity (in the case where the U(1) is "weakly gauged" on the boundary).

The basic model dual to a 2+1 dimensional superconductor is simply that of a charged scalar coupled to a Maxwell field and gravity, and may be described by a Lagrangian of the form

$$\mathcal{L}_4 = R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}^2 - |\partial_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2. \tag{IV.1}$$

The properties of the system may then be studied for various values of mass m and charge q. While this is a perfectly acceptable framework, a more complete understanding demands that this somewhat phenomenological Lagrangian be embedded in a more complete theory such as string theory, or at least its supergravity limit. For AdS_4 duals of 2+1 dimensional superconductors, this was examined at the linearized level in [63], and embedded into $D = 11$ supergravity at the full non-linear level in [82, 84, 85] for the case $m^2L^2 = -2$ and $q = 2$. Similarly, a IIB supergravity model for an AdS_5 dual to 3+1 dimensional superconductors was constructed in [99] with $m^2L^2 = -3$ and $q = 2$.

The AdS_4 model of [82, 84, 85] and the AdS_5 model of [99] are based on Kaluza-Klein truncations on squashed Sasaki-Einstein manifolds. They both have the unusual feature where the $q = 2$ charged scalar arises from the massive level of the Kaluza-Klein truncation. This appears to go against the standard lore of consistent truncations, where it was thought that truncations keeping only a finite number of massive modes would necessarily be inconsistent. A heuristic argument is that states in the Kaluza-Klein tower carry charges under the internal symmetry, and hence would couple at the non-linear level to source higher and higher states, all the way up the Kaluza-Klein tower. This hints that one way to obtain a consistent truncation is simply to truncate to singlets of the

[1] Recent models have generalized this construction to encompass both p-wave [101, 158] and d-wave [50] condensates.

internal symmetry group, and indeed such a construction is consistent. An example of this is a standard torus reduction, where only zero modes on the torus are kept. On the other hand, sphere reductions to maximal gauged supergravities in $D = 4$, 5 and 7 do not follow this rule, as they are expected to be consistent, even though some of the lower-dimensional fields (such as the non-abelian graviphotons) are charged under the R-symmetry. In fact, the issue of Kaluza-Klein consistency is not yet fully resolved, and often must be treated on a case by base basis. This has led us to explore the squashed Sasaki-Einstein compactifications to see if additional consistent massive truncations may be found.

In addition to embedding holographic models of superconductivity into string theory, several groups have demonstrated the embedding of dual non-relativistic CFT backgrounds into string theory [115, 137, 2]. These geometries where originally constructed from a toy model of a massive vector field coupled to gravity with a negative cosmological constant [164, 11] of the form (given here for a deformation of AdS_5):

$$\mathcal{L}_5 = R + \frac{12}{L^2} - \frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2, \tag{IV.2}$$

with mass related to the scaling exponent z according to $m^2L^2 = z(z+2)$. The $z = 2$ and $z = 4$ models ($m^2L^2 = 8$ and $m^2L^2 = 24$, respectively) were subsequently realized within IIB supergravity in terms of consistent truncations retaining a massive vector (along with possibly other fields as well) [115, 137, 2]. These results have further opened up the possibility of obtaining large classes of consistent truncations retaining massive modes of various spin.

4.1.1 Consistent massive truncations of IIB supergravity

For the most part, the massive consistent truncations used in the study of AdS/condensed matter systems have not been supersymmetric[2]. Nevertheless this has motivated us to

[2] The massive truncation given in [82] is supersymmetric, although the connection to a holographic superconductor was done through the non-supersymmetric skew-whiffed case.

investigate the possibility of obtaining new supersymmetric massive truncations of IIB supergravity. In particular, we are mainly interested in reducing IIB supergravity on a Sasaki-Einstein manifold to obtain gauged supergravity in $D = 5$ coupled to possibly massive supermultiplets.

Following the construction of $D = 11$ supergravity [53] and the realization that it admits an $\mathrm{AdS}_4 \times S^7$ vacuum solution [75], it was soon postulated that the Kaluza-Klein reduction on the sphere would give rise to gauged $\mathcal{N} = 8$ supergravity at the "massless" Kaluza-Klein level [69, 67, 68]. This notion was reinforced by a linearized Kaluza-Klein mode analysis demonstrating that the full spectrum of Kaluza-Klein excitations falls into supermultiplets of the $D = 4$, $\mathcal{N} = 8$ superalgebra $\mathrm{OSp}(4|8)$ [19, 160, 42]. However, demonstrating full consistency of the non-linear reduction to gauged $\mathcal{N} = 8$ supergravity has remained elusive. Nevertheless, all indications are that the reduction is consistent [62], and this has in fact been demonstrated for the related case of reducing to $D = 7$ on S^4 [148, 149].

The story is similar for the case of IIB supergravity reduced on S^5. A linearized Kaluza-Klein mode analysis demonstrates that the spectrum of Kaluza-Klein excitations falls into complete supermultiplets of the $D = 5$, $\mathcal{N} = 8$ superalgebra $\mathrm{SU}(2, 2|4)$, with the lowest one corresponding to the ordinary $\mathcal{N} = 8$ supergravity multiplet [104, 124]. In this case, only partial results are known about the full non-linear reduction to gauged supergravity, but there is strong evidence for its consistency [123, 57, 135, 59].

More generally, it was conjectured in [157, 66] and [86], that, for any supergravity reduction, it is always possible to consistently truncate to the supermultiplet containing the massless graviton. This is a non-trivial statement, as the truncation must satisfy rather restrictive consistency conditions related to the gauge symmetries generated by the isometries of the internal manifold [65, 118]. This conjecture has recently been shown to be true for Sasaki-Einstein reductions of IIB supergravity on SE_5 [34] and $D = 11$

supergravity on SE_7 [86], yielding minimal $D = 5$, $\mathcal{N} = 2$ and $D = 4$, $\mathcal{N} = 2$ gauged supergravity, respectively (see also [83, 87]).

While states in the same supermultiplet do not necessarily have the same mass in gauged supergravity, the minimal supergravity multiplets, which contain the graviton, gravitino and a graviphoton, are in fact massless. Thus one may suspect that truncations to massless supermultiplets are necessarily consistent. However, it turns out that this is not the case. This was explicitly demonstrated in [118], where, for example, it was shown to be inconsistent to retain the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ vector multiplets that naturally arise in the compactification of IIB supergravity on $T^{1,1}$.

For many of the above reasons, it has often been a challenge to explore consistent supersymmetric truncations, even at the massless Kaluza-Klein level. However, bosonic truncations retaining massive breathing and squashing modes [25] have been known to be consistent for some time. In this case, consistency is guaranteed by retaining only singlets under the internal symmetry group $\mathrm{SU}(4) \times \mathrm{U}(1)$ for the squashed S^7 or $\mathrm{SU}(3) \times \mathrm{U}(1)$ for the squashed S^5. The supersymmetry of background solutions involving the breathing and squashing modes was explored in [131], where it was further conjectured that a supersymmetric consistent truncation could be found that retains the full breathing/squashing supermultiplet.

Although this massive consistent truncation conjecture was made for squashed sphere compactifications, it naturally generalizes to compactification on more general internal spaces, such as Sasaki-Einstein spaces. For $D = 11$ supergravity compactified on a squashed S^7, written as U(1) bundled over CP^3, truncation of the $\mathcal{N} = 8$ Kaluza-Klein spectrum to SU(4) singlets under the decomposition $\mathrm{SO}(8) \supset \mathrm{SU}(4) \times \mathrm{U}(1)$ yields the $\mathcal{N} = 2$ supergravity multiplet[3]

$$\text{(IV.3)} \qquad n=0: \qquad \mathcal{D}(2,1)_0 = D(3,2)_0 + D(\tfrac{5}{2},\tfrac{3}{2})_{-1} + D(\tfrac{5}{2},\tfrac{3}{2})_1 + D(2,1)_0,$$

at the massless ($n = 0$) Kaluza-Klein level. No SU(4) singlets survive at the first ($n = 1$) massive Kaluza-Klein level, and the breathing and squashing modes finally make their appearance at the second ($n = 2$) Kaluza-Klein level in a massive vector multiplet [131]

$$\begin{aligned} n=2: \qquad \mathcal{D}(4,0)_0 \quad &= \quad D(5,1)_0 + D(\tfrac{9}{2},\tfrac{1}{2})_{-1} + D(\tfrac{9}{2},\tfrac{1}{2})_1 + D(\tfrac{11}{2},\tfrac{1}{2})_{-1} + D(\tfrac{11}{2},\tfrac{1}{2})_1 \\ \text{(IV.4)} \qquad\qquad &\qquad +D(4,0)_0 + D(5,0)_0 + D(5,0)_{-2} + D(5,0)_2 + D(6,0)_0. \end{aligned}$$

Replacing S^7 by SE_7 amounts to replacing CP^3 by an appropriate Kahler-Einstein base B. In this case, the internal isometry is generically reduced from SU(4) $\times$ U(1). Nevertheless, the notion of truncating to SU(4) singlets may simply be replaced by the prescription of truncating to zero modes on the base B. This procedure was in fact done in [82], which constructed the non-linear Kaluza-Klein reduction for all the bosonic fields contained in the above supermultiplets (IV.3) and (IV.4) and furthermore verified the $\mathcal{N} = 2$ supersymmetry.

For the case of IIB supergravity compactified on SE_5, it is straightforward to generalize the squashed S^5 conjecture of [131]. In this case, however, the Kaluza-Klein spectrum is more involved, and is given in Table 4.1. A curious feature shows up here in that an additional LH+RH chiral matter multiplet shows up at the 'massless' Kaluza-Klein level. The $E_0 = 4$ scalar in this multiplet corresponds to the IIB axi-dilaton, while the additional $E_0 = 3$ charged scalar is precisely the charged scalar constructed in the holographic model of [99]. At the higher Kaluza-Klein levels, the breathing and squashing mode scalars correspond to the $E_0 = 8$ and $E_0 = 6$ scalars in the massive vector multiplet. In addition, consistent truncations involving the $E_0 = 5$ ($m^2L^2 = 8$) doublet of vectors in the semi-

[3] The OSp(4|2) super-representations $\mathcal{D}(E_0, s)_q$ and SO(2,3) representations $D(E_0, s)_q$ are labeled by energy E_0, spin s and U(1) charge q under OSp(4|2) $\supset$ SO(2,3) $\times$ U(1) $\supset$ SO(2) $\times$ SO(3) $\times$ U(1).

n	Multiplet	$SU(2,2\|1)$	$SO(2,4)\times U(1)$
0	supergraviton	$\mathcal{D}(3,\frac{1}{2},\frac{1}{2})_0$	$D(4,1,1)_0+D(3\frac{1}{2},1,\frac{1}{2})_{-1}+D(3\frac{1}{2},\frac{1}{2},1)_1+D(3,\frac{1}{2},\frac{1}{2})_0$
0	LH chiral	$\mathcal{D}(3,0,0)_2$	$D(3\frac{1}{2},\frac{1}{2},0)_1+D(3,0,0)_2+D(4,0,0)_0$
0	RH chiral	$\mathcal{D}(3,0,0)_{-2}$	$D(3\frac{1}{2},0,\frac{1}{2})_{-1}+D(3,0,0)_{-2}+D(4,0,0)_0$
1	LH massive gravitino	$\mathcal{D}(4\frac{1}{2},0,\frac{1}{2})_1$	$D(5\frac{1}{2},\frac{1}{2},1)_1+D(5,\frac{1}{2},\frac{1}{2})_0+D(5,0,1)_2$ $+D(6,0,1)_0+D(4\frac{1}{2},0,\frac{1}{2})_1+D(5\frac{1}{2},0,\frac{1}{2})_{-1}$
1	RH massive gravitino	$\mathcal{D}(4\frac{1}{2},\frac{1}{2},0)_{-1}$	$D(5\frac{1}{2},1,\frac{1}{2})_{-1}+D(5,\frac{1}{2},\frac{1}{2})_0+D(5,1,0)_{-2}$ $+D(6,1,0)_0+D(4\frac{1}{2},\frac{1}{2},0)_{-1}+D(5\frac{1}{2},\frac{1}{2},0)_1$
2	massive vector	$\mathcal{D}(6,0,0)_0$	$D(7,\frac{1}{2},\frac{1}{2})_0+D(6\frac{1}{2},\frac{1}{2},0)_{-1}+D(6\frac{1}{2},0,\frac{1}{2})_1$ $+D(7\frac{1}{2},0,\frac{1}{2})_{-1}+D(7\frac{1}{2},\frac{1}{2},0)_1+D(6,0,0)_0$ $+D(7,0,0)_{-2}+D(7,0,0)_2+D(8,0,0)_0$

Table 4.1: The truncated Kaluza-Klein spectrum of IIB supergravity on squashed S^5 [131], or equivalently on SE_5. Here n denotes the Kaluza-Klein level. The consistent truncation is expected to terminate at level $n=2$ with the breathing mode supermultiplet.

long LH+RH massive gravitino multiplet and the $E_0 = 7$ ($m^2L^2 = 24$) vector in the massive vector multiplet were constructed in [115, 137, 2] in the context of investigating non-relativistic conformal backgrounds in string theory.

What we have seen so far is that massive consistent truncations of IIB supergravity have been obtained keeping various subsets of the bosonic fields identified in Table 4.1. The goal of this paper is to construct a complete non-linear Kaluza-Klein reduction of IIB supergravity on SE_5 retaining all the bosonic fields in the multiplets up to the $n=2$ level. This complements the massive Kaluza-Klein truncation of $D=11$ supergravity [82], and provides another example of a consistent truncation retaining the breathing mode supermultiplet. We proceed in Section 4.2 with the Sasaki-Einstein reduction of IIB supergravity. Then in Section 4.3 we connect the full non-linear reduction with the linearized Kaluza-Klein analysis of [104, 124] and show how the bosonic fields in Table 4.1 are related to the original IIB fields. In Section 4.4 we relate the complete non-linear reduction to previous results by performing additional truncations to a subset of active fields. Finally, we conclude in Section 4.5 with some further speculation on massive consistent truncations of supergravity.

For a discussion of the $\mathcal{N}=4$ nature of the general reduction on Sasaki-Einstein manifolds see [43, 88] which independently worked out the massive consistent truncation of IIB supergravity on SE_5. Also [163] reports related results for a particular truncation of these theories.

4.2 Sasaki-Einstein reduction of IIB supergravity

The bosonic field content of IIB supergravity consists of the NSNS fields (g_{MN}, B_{MN}, ϕ) and the RR potentials (C_0, C_2, C_4). Because of the self-dual field strength $F_5^+ = dC_4$, it is not possible to write down a covariant action. However, we may take a bosonic Lagrangian of the form

$$\mathcal{L}_{\rm IIB} = R * 1 - \frac{1}{2\tau_2^2} d\tau \wedge * d\bar{\tau} - \frac{1}{2}\mathcal{M}_{ij} F_3^i \wedge * F_3^j - \frac{1}{4}\widetilde{F}_5 \wedge * \widetilde{F}_5 - \frac{1}{4}\epsilon_{ij} C_4 \wedge F_3^i \wedge F_3^j, \tag{IV.5}$$

where self-duality $\widetilde{F}_5 = *\widetilde{F}_5$ is to be imposed by hand after deriving the equations of motion.

We have given the Lagrangian in an SL(2,$\mathcal{R}$) invariant form where

$$\tau = C_0 + ie^{-\phi}, \qquad \mathcal{M} = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}, \tag{IV.6}$$

and where

$$F_3^i = dB_2^i, \qquad B_2^i = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \qquad \widetilde{F}_5 = dC_4 + \tfrac{1}{2}\epsilon_{ij} B_2^i \wedge dB_2^j. \tag{IV.7}$$

The equations of motion following from (IV.61) and the self-duality of $\widetilde{F}_5$ are

$$\begin{aligned} d\widetilde{F}_5 &= \tfrac{1}{2}\epsilon_{ij} F_3^i \wedge F_3^j, \qquad \widetilde{F}_5 = *\widetilde{F}_5, \\ d(\mathcal{M}_{ij} * F_3^j) &= -\epsilon_{ij} \widetilde{F}_5 \wedge F_3^j, \\ \frac{d * d\tau}{\tau_2} + i\frac{d\tau \wedge * d\tau}{\tau_2^2} &= -\frac{i}{2\tau_2} G_3 \wedge * G_3, \end{aligned} \tag{IV.8}$$

and the Einstein equation (in Ricci form)

$$\begin{aligned} R_{MN} &= \frac{1}{2\tau_2^2}\partial_{(M}\tau\partial_{N)}\bar{\tau} + \frac{1}{4}\mathcal{M}_{ij}\left(F^i_{MPQ}F_N^{j\,PQ} - \frac{1}{12}g_{MN}F^i_{PQR}F^{j\,PQR}\right) \\ &\quad + \frac{1}{4\cdot 4!}\widetilde{F}_{MPQRS}\widetilde{F}_N{}^{PQRS}. \end{aligned} \tag{IV.9}$$

In the above we have introduced the complex three-form $G_3 = F_3^2 - \tau F_3^1$. If desired, this allows us to rewrite the three-form equation of motion as

$$d*G = -i\frac{d\tau}{2\tau_2}\wedge *(G_3+\bar{G}_3) + i\widetilde{F}_5\wedge G_3. \tag{IV.10}$$

4.2.1 The reduction ansatz

Before writing out the reduction ansatz, we note a few key features of Sasaki-Einstein manifolds. A Sasaki-Einstein manifold has a preferred U(1) isometry related to the Reeb vector. This allows us to write the metric as a U(1) fibration over a Kahler-Einstein base B

$$ds^2(SE_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \tag{IV.11}$$

where $d\mathcal{A} = 2J$ with J the Kahler form on B. Moreover, B admits an SU(2) structure defined by the (1,1) and (2,0) forms J and Ω satisfying

$$J\wedge\Omega = 0, \qquad \Omega\wedge\bar{\Omega} = 2J\wedge J = 4*_4 1, \qquad *_4 J = J, \qquad *_4\Omega = \Omega, \tag{IV.12}$$

as well as

$$dJ = 0, \qquad d\Omega = 3i(d\psi + \mathcal{A})\wedge\Omega. \tag{IV.13}$$

Note that we are taking the 'unit radius' Einstein condition $R_{ij} = 4g_{ij}$ on the Sasaki-Einstein manifold, which corresponds to $R_{ab} = 6g_{ab}$ on the Kahler-Einstein base.

For the reduction, we write down the most general decomposition of the bosonic IIB fields consistent with the isometries of B. For the metric, we take

$$ds_{10}^2 = e^{2A}ds_5^2 + e^{2B}ds^2(B) + e^{2C}(\eta + A_1)^2, \tag{IV.14}$$

where $\eta = d\psi + \mathcal{A}$. Since A_1 gauges the U(1) isometry, it will be related to the $D = 5$ graviphoton. Note, however, that the graviphoton receives additional contributions from the five-form.

The three-form and five-form field strengths can be expanded in a basis of invariant tensors on B. For the three-forms, we work with the potentials

$$B_2^i = b_2^i + b_1^i \wedge (\eta + A_1) + b_0^i \Omega + \bar{b}_0^i \bar{\Omega}. \tag{IV.15}$$

The scalars b_0^i are complex, while the remaining fields are real. Note that we do not include a term of the form $\widetilde{b}_0^i J$ in the ansatz, as this field will act simply as a St$\ddot{u}$ckelburg field in the five-dimensional theory. In particular, it does not give rise to any new dynamics in the equations of motion as it can be repackaged as a total derivative plus terms which would simply shift b_2^i and b_1^i,

$$2\widetilde{b}_0^i J = d(\widetilde{b}_0^i \wedge (\eta + A_1)) - d\widetilde{b}_0^i \wedge (\eta + A_1) - \widetilde{b}_0^i F_2. \tag{IV.16}$$

Taking $F_3^i = dB_2^i$ gives

$$\begin{aligned} F_3^i &= (db_2^i - b_1^i \wedge F) + db_1^i \wedge (\eta + A_1) - 2b_1^i \wedge J + Db_0^i \wedge \Omega + D\bar{b}_0^i \wedge \bar{\Omega} \\ &\quad +3ib_0^i \Omega \wedge (\eta + A_1) - 3i\bar{b}_0^i \bar{\Omega} \wedge (\eta + A_1), \end{aligned} \tag{IV.17}$$

where D is the U(1) gauge covariant derivative

$$Db_0^i = db_0^i - 3iA_1 b_0^i. \tag{IV.18}$$

For convenience, we write this as

$$F_3^i = g_3^i + g_2^i \wedge (\eta + A_1) + g_1^i \wedge J + f_1^i \wedge \Omega + \bar{f}_1^i \wedge \bar{\Omega} + f_0^i \wedge \Omega \wedge (\eta + A_1) + \bar{f}_0^i \wedge \bar{\Omega} \wedge (\eta + A_1), \tag{IV.19}$$

where our notation is such that the g^i's are real and the f^i's are complex.

For the self-dual five-form, we take

$$\widetilde{F}_5 = (1+*)[(4+\phi_0) *_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1) + q_2 \wedge \Omega \wedge (\eta + A_1) + \bar{q}_2 \wedge \bar{\Omega} \wedge (\eta + A_1)], \tag{IV.20}$$

where $*_4 1$ denotes the volume form on the Kahler-Einstein base B. Note that we have pulled out a constant background component

$$\widetilde{F}_5 = 4(1+*)\mathrm{vol}(SE_5), \tag{IV.21}$$

which sets up the Freund-Rubin compactification[4]. The two-forms q_2 are complex, while the other fields are real. For later convenience, we take the explicit 10-dimensional dual in the metric (IV.64) to obtain

$$\begin{aligned}\widetilde{F}_5 \;=\;& (4+\phi_0) *_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1) + q_2 \wedge \Omega \wedge (\eta + A_1)\\ & + \bar{q}_2 \wedge \bar{\Omega} \wedge (\eta + A_1) + e^{5A-4B-C}(4+\phi_0) * 1 - e^{3A-4B+C} * \mathbb{A}_1 \wedge (\eta + A_1)\\ & + e^{A-C} * p_2 \wedge J + e^{A-C} * q_2 \wedge \Omega + e^{A-C} * \bar{q}_2 \wedge \bar{\Omega},\end{aligned} \tag{IV.22}$$

where $*$ now denotes the Hodge dual in the $D=5$ spacetime.

4.2.2 Reduction of the equations of motion

In order to obtain the reduction, it is now simply a matter of inserting the above decompositions into the IIB equations of motion. The $\widetilde{F}_5$ equation yields

$$\begin{aligned} d(e^{A-C} * p_2) &= 2e^{3A-4B+C} * \mathbb{A}_1 - p_2 \wedge F_2 + \epsilon_{ij} g_1^i \wedge g_3^j,\\ Dq_2 &= 3ie^{A-C} * q_2 + \epsilon_{ij}(f_1^i \wedge g_2^j - f_0^i g_3^j),\end{aligned} \tag{IV.23}$$

along with the constraints

$$\begin{aligned} \phi_0 &= -\tfrac{2i}{3}\epsilon_{ij}(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j),\\ p_2 &= \tfrac{1}{4}\epsilon_{ij} g_1^i \wedge g_1^j - d[A_1 + \tfrac{1}{4}\mathbb{A}_1 + \tfrac{i}{6}\epsilon_{ij}(f_0^i \bar{f}_1^j - \bar{f}_0^i f_1^j)].\end{aligned} \tag{IV.24}$$

The implication of this is that $\widetilde{F}_5$ gives rise to two physical $D=5$ fields, namely a massive vector $\mathbb{A}_1$ and a complex antisymmetric tensor q_2 satisfying an odd-dimensional self-duality equation and with $m^2 = 9$. The mass of $\mathbb{A}_1$ is not directly apparent from (IV.23) as it

[4] For simplicity, we have assumed a unit radius ($L=1$) compactification.

mixes with A_1 from the metric to yield the massless graviphoton as well as a $m^2 = 24$ massive vector.

The F_3^i equation yields

$$\begin{aligned}
D(e^{3A+C}\mathcal{M}_{ij} * f_1^j) &= -3ie^{5A-C}\mathcal{M}_{ij}f_0^j * 1 + \epsilon_{ij}[(4+\phi_0)e^{5A-4B-C}f_0^j * 1 - q_2 \wedge g_3^j \\
&\quad + e^{A-C} * q_2 \wedge g_2^j + e^{3A-4B+C} * \mathbb{A}_1 \wedge f_1^j], \\
d(e^{A+4B-C}\mathcal{M}_{ij} * g_2^j) &= \mathcal{M}_{ij}[e^{-A+4B+C} * g_3^j \wedge F + 4e^{3A+C} * g_1^j] \\
&\quad + \epsilon_{ij}[-2e^{A-C} * p_2 \wedge g_1^j - \mathbb{A}_1 \wedge g_3^j - 4e^{A-C}(*q_2 \wedge \bar{f}_1^j + *\bar{q}_2 \wedge f_1^j)], \\
d(e^{-A+4B+C}\mathcal{M}_{ij} * g_3^j) &= \epsilon_{ij}[-(4+\phi_0)g_3^j + \mathbb{A}_1 \wedge g_2^j - 2p_2 \wedge g_1^j - 4(q_2 \wedge \bar{f}_1^j + \bar{q}_2 \wedge f_1^j) \\
&\quad + 4e^{A-C}(\bar{f}_0^j * q_2 + f_0^j * \bar{q}_2)].
\end{aligned} \tag{IV.25}$$

These correspond to a pair of charged scalars f_0^i, a pair of $m^2 = 8$ massive vectors g_1^i and a pair of massive antisymmetric tensors b_2^i.

The ten-dimensional Einstein equation (IV.9) reduces to a five-dimensional Einstein equation, as well as the equations of motion for the breathing and squashing modes B and C and the graviphoton A_1. In particular, in the natural vielbein basis, the frame components of the ten-dimensional Ricci tensor corresponding to the reduction (IV.64) are given by

$$\begin{aligned}
{}^{10}R_{\alpha\beta} &= e^{-2A}[R_{\alpha\beta} - \nabla_\alpha\nabla_\beta(3A+4B+C) - \eta_{\alpha\beta}\partial_\gamma A\partial^\gamma(3A+4B+C) - \eta_{\alpha\beta}\Box A \\
&\quad + 3\partial_\alpha A\partial_\beta A - 4\partial_\alpha B\partial_\beta B - \partial_\alpha C\partial_\beta C + 4(\partial_\alpha A\partial_\beta B + \partial_\alpha B\partial_\beta A) \\
&\quad + (\partial_\alpha A\partial_\beta C + \partial_\alpha C\partial_\beta A)] - \tfrac{1}{2}e^{2C-4A}F_{\alpha\gamma}F_\beta{}^\gamma, \\
{}^{10}R_{ab} &= \delta_{ab}[6e^{-2B} - 2e^{2C-4B} - e^{-2A}(\Box B + \partial_\gamma B\partial^\gamma(3A+4B+C))], \\
{}^{10}R_{99} &= 4e^{2C-4B} + \tfrac{1}{4}e^{2C-4A}F_{\gamma\delta}F^{\gamma\delta} - e^{-2A}(\Box C + \partial_\gamma C\partial^\gamma(3A+4B+C)), \\
{}^{10}R_{\alpha 9} &= \tfrac{1}{2}e^{C-3A}[\nabla^\gamma F_{\alpha\gamma} + F_{\alpha\gamma}\partial^\gamma(A+4B+3C)].
\end{aligned} \tag{IV.26}$$

The α and β indices correspond to the $D = 5$ spacetime, while a and b correspond to the Kahler-Einstein base B and 9 corresponds to the U(1) fiber direction. The covariant derivatives and frame indices on the right hand side of these quantities are with respect to the $D=5$ metric. In order to reduce to the $D=5$ Einstein frame metric, we now choose $3A+4B+C=0$, or

$$A=-\tfrac{4}{3}B-\tfrac{1}{3}C. \tag{IV.27}$$

For convenience, we will retain A in the expressions below. However, it is not independent, and should always be thought of as a shorthand for (IV.27).

Equating the ten-dimensional Ricci tensor (IV.26) to the stress tensor formed out of F_3^i and $\widetilde{F}_5$ of (IV.67) and (IV.22), we obtain the $D=5$ Einstein equation

$$\begin{aligned} R_{\alpha\beta} \;=\;& \tfrac{1}{3}\eta_{\alpha\beta}(-24e^{2A-2B}+4e^{5A+3C}+\tfrac{1}{2}e^{8A}(4+\phi_0)^2)+\tfrac{28}{3}\partial_\alpha B\partial_\beta B+\tfrac{8}{3}\partial_{(\alpha}B\partial_{\beta)}C \\ & +\tfrac{4}{3}\partial_\alpha C\partial_\beta C+\tfrac{1}{2\tau_2^2}\partial_{(\alpha}\tau\partial_{\beta)}\bar{\tau}+\tfrac{1}{2}e^{2C-2A}(F_{\alpha\gamma}F_\beta{}^\gamma-\tfrac{1}{6}\eta_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta})+\tfrac{1}{2}e^{-8B}\mathbb{A}_\alpha\mathbb{A}_\beta \\ & +e^{A-C}[(p_{\alpha\gamma}p_\beta{}^\gamma-\tfrac{1}{6}\eta_{\alpha\beta}p_{\gamma\delta}p^{\gamma\delta})+4(q_{(\alpha}{}^\gamma\bar{q}_{\beta)\gamma}-\tfrac{1}{6}\eta_{\alpha\beta}q_{\gamma\delta}\bar{q}^{\gamma\delta})] \\ & +\mathcal{M}_{ij}[\tfrac{2}{3}e^{5A-C}\eta_{\alpha\beta}(f_0^i\bar{f}_0^j+\bar{f}_0^if_0^j)+\tfrac{1}{2}e^{-2A-2C}(g^i_{\alpha\gamma}g^{j\,\gamma}_\beta-\tfrac{1}{6}\eta_{\alpha\beta}g^i_{\gamma\delta}g^{j\,\gamma\delta}) \\ & +\tfrac{1}{4}e^{-4A}(g^i_{\alpha\gamma\delta}g^{j\,\gamma\delta}_\beta-\tfrac{2}{9}\eta_{\alpha\beta}g^i_{\gamma\delta\epsilon}g^{j\,\gamma\delta\epsilon})+e^{-4B}(g^i_\alpha g^j_\beta+2(f^i_\alpha\bar{f}^j_\beta+\bar{f}^i_\alpha f^j_\beta))], \end{aligned} \tag{IV.28}$$

as well as the B, C and A_1 equations of motion

$$\begin{aligned}
d*dB &= [6e^{2A-2B} - 2e^{5A+3C} - \tfrac{1}{4}e^{8A}(4+\phi_0)^2]*1 - \tfrac{1}{4}e^{-8B}\mathbb{A}_1\wedge *\mathbb{A}_1\\
&\quad +\mathcal{M}_{ij}[\tfrac{1}{8}e^{-2A-2C}g_2^i\wedge *g_2^j + \tfrac{1}{8}e^{-4A}g_3^i\wedge *g_3^j - \tfrac{1}{2}e^{5A-C}(f_0^i\bar{f}_0^j + \bar{f}_0^i f_0^j)*1\\
&\quad -\tfrac{1}{4}e^{-4B}(g_1^i\wedge *g_1^j + 2(f_1^i\wedge *\bar{f}_1^j + \bar{f}_1^i\wedge *f_1^j))],\\
d*dC &= [4e^{5A+3C} - \tfrac{1}{4}e^{8A}(4+\phi_0)^2]*1 + \tfrac{1}{2}e^{2C-2A}F_2\wedge *F_2 + \tfrac{1}{4}e^{-8B}\mathbb{A}_1\wedge *\mathbb{A}_1\\
&\quad -\tfrac{1}{2}e^{A-C}(p_2\wedge *p_2 + 4q_2\wedge *\bar{q}_2) + \mathcal{M}_{ij}[-\tfrac{3}{8}e^{-2A-2C}g_2^i\wedge *g_2^j\\
&\quad +\tfrac{1}{8}e^{-4A}g_3^i\wedge *g_3^j - \tfrac{3}{2}e^{5A-C}(f_0^i\bar{f}_0^j + \bar{f}_0^i f_0^j)*1\\
&\quad +\tfrac{1}{4}e^{-4B}(g_1^i\wedge *g_1^j + 2(f_1^i\wedge *\bar{f}_1^j + \bar{f}_1^i\wedge *f_1^j))],\\
d(e^{2C-2A}*F_2) &= (4+\phi_0)e^{-8B}*\mathbb{A}_1 - p_2\wedge p_2 - 4q_2\wedge\bar{q}_2\\
&\quad +\mathcal{M}_{ij}[4e^{-4B}*(f_0^i\bar{f}_1^j + \bar{f}_0^i f_1^j) + e^{-4A}*g_3^i\wedge g_2^j]. \qquad \text{(IV.29)}
\end{aligned}$$

Note that, in order to obtain the $D = 5$ Einstein equation, we had to shift the reduction of ${}^{10}R_{\alpha\beta}$ an appropriate combination of ${}^{10}R_{ab}$ and ${}^{10}R_{99}$ in order to remove the $\eta_{\alpha\beta}\Box A$ component in the first line of (IV.26).

The IIB equations of motion thus reduce to (IV.23), (IV.25), (IV.28) and (IV.29) as well as the axi-dilaton equation, which we have not written down explicitly, but which will be shown to be consistent below.

4.2.3 The effective five-dimensional Lagrangian

We now wish to construct an effective $D = 5$ Lagrangian which reproduces the above equations of motion. This may be done by noting that the $D = 5$ Einstein equation (IV.28) arises naturally from a Lagrangian of the form

$$\begin{aligned}
\mathcal{L} &= R*1+(24e^{2A-2B}-4e^{5A+3C}-\tfrac{1}{2}e^{8A}(4+\phi_0)^2)*1-\tfrac{28}{3}dB\wedge *dB-\tfrac{8}{3}dB\wedge *dC\\
&\quad -\tfrac{4}{3}dC\wedge *dC-\tfrac{1}{2\tau_2^2}d\tau\wedge *d\bar\tau-\tfrac{1}{2}e^{2C-2A}F_2\wedge *F_2-\tfrac{1}{2}e^{-8B}\mathbb{A}_1\wedge *\mathbb{A}_1\\
&\quad -e^{A-C}(p_2\wedge *p_2+4q_2\wedge *\bar q_2)+\mathcal{M}_{ij}[-2e^{5A-C}(f_0^i\bar f_0^j+\bar f_0^i f_0^j)*1\\
&\quad -\tfrac{1}{2}e^{-2A-2C}g_2^i\wedge *g_2^j-\tfrac{1}{2}e^{-4A}g_3^i\wedge *g_3^j-e^{-4B}(g_1^i\wedge *g_1^j+2(f_1^i\wedge *\bar f_1^j+\bar f_1^i\wedge *f_1^j))]\\
&\quad +\mathcal{L}_{CS}.
\end{aligned}\tag{IV.30}$$

We have included a Chern-Simons piece $\mathcal{L}_{CS}$ which cannot be determined from the Einstein equation.

It is now possible to verify that (IV.30) reproduces all the terms in the equations of motion (IV.23), (IV.25) and (IV.29) involving the metric (*ie* the Hodge *). The remaining terms may be obtained from the addition of the topological piece

$$\begin{aligned}
\mathcal{L}_{CS} &= \tfrac{2i}{3}(q_2\wedge d\bar q_2-\bar q_2\wedge dq_2)-4A_1\wedge q_2\wedge\bar q_2+2\epsilon_{ij}b_2^i\wedge db_2^j\\
&\quad +\tfrac{4i}{3}[(\bar q_2-\tfrac{i}{6}\epsilon_{ij}\bar f_0^i g_2^j)\wedge\epsilon_{kl}(f_1^k\wedge g_2^l-f_0^k g_3^l)-(q_2+\tfrac{i}{6}\epsilon_{ij}f_0^i g_2^j)\wedge\epsilon_{kl}(\bar f_1^k\wedge g_2^l-\bar f_0^k g_3^l)]\\
&\quad -A_1\wedge(p_2-\tfrac{1}{4}\epsilon_{ij}g_1^i\wedge g_1^j)\wedge(p_2-\tfrac{1}{4}\epsilon_{kl}g_1^k\wedge g_1^l)\\
&\quad -2[\tfrac{1}{4}\mathbb{A}_1+\tfrac{i}{6}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j)]\wedge\epsilon_{kl}(g_1^k\wedge g_3^l-\tfrac{1}{4}g_1^k\wedge g_1^l\wedge F_2).
\end{aligned}\tag{IV.31}$$

Here we recall the definitions

$$f_0^i=3ib_0^i,\qquad f_1^i=Db_0^i,\qquad g_1^i=-2b_1^i,\qquad g_2^i=db_1^i,\qquad g_3^i=db_2^i-b_1^i\wedge F_2,\tag{IV.32}$$

implicit in (IV.17) and (IV.67). Furthermore, ϕ_0 and p_2 are given by (IV.24). Note that, while $\mathbb{A}_1$ is massive, and does not have a gauge invariance associated with it, it is natural to make the shift

$$\mathbb{A}_1\to\mathbb{A}_1'-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j),\tag{IV.33}$$

so that

$$p_2=\tfrac{1}{4}\epsilon_{ij}g_1^i\wedge g_1^j-F_2-\tfrac{1}{4}\mathbb{F}_2',\tag{IV.34}$$

where $\mathbb{F}_2' = d\mathbb{A}_1'$.

We now turn to the axi-dilaton equation obtained from (IV.30). Since τ only shows up in the kinetic term and in $\mathcal{M}_{ij}$, we see that the τ equation of motion obtained from the $D = 5$ Lagrangian reproduces that obtained from the original IIB Lagrangian. This is because the quantity in the square brackets multiplying $\mathcal{M}_{ij}$ in (IV.30) is the straightforward reduction of $-\frac{1}{2}F_3^i \wedge *F_3^j$ in the original IIB Lagrangian (IV.61).

4.3 Matching the linearized Kaluza-Klein analysis

The complete $D = 5$ Lagrangian, as given by (IV.30) and (IV.31), is somewhat opaque. Thus in this section, we demonstrate that it in fact contains the fields corresponding to the Kaluza-Klein mass spectrum noted in Table 4.1. To do this, it is sufficient to look at the linearized level. We first note that the effective $D = 5$ fields are the complex scalars (τ, b_0^i), real scalars (B, C), one-form potentials $(A_1, b_1^i, \mathbb{A}_1)$, pair of real two-forms (b_2^i), the complex two-form (q_2), and of course the metric $(g_{\mu\nu})$. The $D = 5$ equations of motion (IV.23), (IV.25) and (IV.29) may be linearized on the matter fields to obtain the set

$$\begin{aligned}
d * db_0^i &= (9\delta_j^i + 12i\mathcal{N}^i{}_j)b_0^j * 1,\\
d * db_1^i &= -8 * b_1^i,\\
d * db_2^i &= -4\mathcal{N}^i{}_j db_2^j,\\
dq_2 &= 3i * q_2,\\
d * F_2 &= 4 * \mathbb{A}_1, \qquad\qquad d * F_2 + \tfrac{1}{4} d * \mathbb{F}_2 = -2 * \mathbb{A}_1,\\
d * dB &= 4(7B + C) * 1, \qquad d * dC = 16(B + C) * 1.
\end{aligned} \tag{IV.35}$$

Here we have introduced

$$\mathcal{N} = \mathcal{M}^{-1}\epsilon = \frac{1}{\tau_2}\begin{pmatrix} -\tau_1 & 1 \\ -|\tau|^2 & \tau_1 \end{pmatrix}, \tag{IV.36}$$

with eigenvalues $+i$ and $-i$, corresponding to eigenvectors $\begin{pmatrix} 1 & \tau \end{pmatrix}^T$ and $\begin{pmatrix} 1 & \bar{\tau} \end{pmatrix}^T$, respectively.

The first equation in (IV.35) then decomposes into a pair of equations for the complex scalars $b_0^{m^2=-3}$ and $b_0^{m^2=21}$ with masses $m^2 = -3$ and $m^2 = 21$ according to

$$b_0^i = \begin{pmatrix} 1 \\ \tau \end{pmatrix} b_0^{m^2=-3} + \begin{pmatrix} 1 \\ \bar{\tau} \end{pmatrix} b_0^{m^2=21}. \tag{IV.37}$$

The second equation is that of an SL(2,$\mathbb{R}$) doublet of real vectors b_1^i with mass $m^2 = 8$. The third equation can be converted to an odd-dimensional self-duality equation [168] $db_2^i = 4\mathcal{N}^i{}_j * b_2^j$, for a doublet of antisymmetric tensors b_2^i with mass $m^2 = 16$. The fourth equation is already in odd-dimensional self-duality form, and shows that the complex antisymmetric tensor q_2 has mass $m^2 = 9$.

The vector equations can be diagonalized

$$d * (F_2 + \tfrac{1}{6}\mathbb{F}_2) = 0, \qquad d * \mathbb{F}_2 = -24 * \mathbb{A}_1, \tag{IV.38}$$

to identify the massless graviphoton $A_1 + \frac{1}{6}\mathbb{A}_1$ and the massive $m^2 = 24$ vector $\mathbb{A}_1$. Finally the B and C equations may be diagonalized to identify the $m^2 = 32$ breathing and $m^2 = 12$ squashing modes

$$d * d\rho = 32\rho * 1, \qquad d * d\sigma = 12\sigma * 1, \tag{IV.39}$$

where

$$B = \rho + \tfrac{1}{2}\sigma, \qquad C = \rho - 2\sigma. \tag{IV.40}$$

It is now possible to see how the above linearized modes are organized into $\mathcal{N} = 2$ supermultiplets. As shown in Table 4.1, at the zeroth Kaluza-Klein level, we have the graviton supermultiplet

$$\mathcal{D}(3, \tfrac{1}{2}, \tfrac{1}{2})_0 = D(4, 1, 1)_0 + D(3\tfrac{1}{2}, 1, \tfrac{1}{2})_{-1} + D(3\tfrac{1}{2}, \tfrac{1}{2}, 1)_1 + D(3, \tfrac{1}{2}, \tfrac{1}{2})_0, \tag{IV.41}$$

with bosonic fields being the graviton $g_{\mu\nu}$ and the massless graviphoton $A_1 + \frac{1}{6}\mathbb{A}_1$. Still at the zeroth level, there is also a LH+RH chiral multiplet

$$\begin{aligned}\mathcal{D}(3,0,0)_2 &= D(3\tfrac{1}{2},\tfrac{1}{2},0)_1 + D(3,0,0)_2 + D(4,0,0)_0,\\ \mathcal{D}(3,0,0)_{-2} &= D(3\tfrac{1}{2},0,\tfrac{1}{2})_{-1} + D(3,0,0)_{-2} + D(4,0,0)_0.\end{aligned} \tag{IV.42}$$

The charged $E_0 = 3$ scalar corresponds to the $m^2 = -3$ scalar $b_0^{m^2=-3}$, while the complex $E_0 = 4$ scalar is the axi-dilaton τ.

At the first Kaluza-Klein level, we have a semi-long LH+RH massive gravitino multiplet

$$\begin{aligned}\mathcal{D}(4\tfrac{1}{2},0,\tfrac{1}{2})_1 &= D(5\tfrac{1}{2},\tfrac{1}{2},1)_1 + D(5,\tfrac{1}{2},\tfrac{1}{2})_0 + D(5,0,1)_2 + D(6,0,1)_0\\ &\quad + D(4\tfrac{1}{2},0,\tfrac{1}{2})_1 + D(5\tfrac{1}{2},0,\tfrac{1}{2})_{-1},\\ \mathcal{D}(4\tfrac{1}{2},\tfrac{1}{2},0)_{-1} &= D(5\tfrac{1}{2},1,\tfrac{1}{2})_{-1} + D(5,\tfrac{1}{2},\tfrac{1}{2})_0 + D(5,1,0)_{-2} + D(6,1,0)_0\\ &\quad + D(4\tfrac{1}{2},\tfrac{1}{2},0)_{-1} + D(5\tfrac{1}{2},\tfrac{1}{2},0)_1.\end{aligned} \tag{IV.43}$$

The bosonic field content is an SL(2,$\mathbb{R}$) doublet of $m^2 = 8$ ($E_0 = 5$) vectors b_1^i, a charged $m^2 = 9$ ($E_0 = 5$) anti-symmetric tensor q_2, and a doublet of $m^2 = 16$ ($E_0 = 6$) anti-symmetric tensors b_2^i.

At the second Kaluza-Klein level, we have a massive vector multiplet

$$\begin{aligned}\mathcal{D}(6,0,0)_0 &= D(7,\tfrac{1}{2},\tfrac{1}{2})_0 + D(6\tfrac{1}{2},\tfrac{1}{2},0)_{-1} + D(6\tfrac{1}{2},0,\tfrac{1}{2})_1 + D(7\tfrac{1}{2},0,\tfrac{1}{2})_{-1} + D(7\tfrac{1}{2},\tfrac{1}{2},0)_1\\ &\quad + D(6,0,0)_0 + D(7,0,0)_{-2} + D(7,0,0)_2 + D(8,0,0)_0.\end{aligned} \tag{IV.44}$$

The massive $E_0 = 7$ vector is the $m^2 = 24$ mode $\mathbb{A}_1$. The real $E_0 = 6$ and $E_0 = 8$ scalars are the $m^2 = 12$ squashing and $m^2 = 32$ breathing modes, σ and ρ, respectively. The charged $E_0 = 7$ scalar is $b_0^{m^2=21}$ with $m^2 = 21$. This identification of the linearized fields with the Kaluza-Klein modes is shown in Table 4.3.

For the case of IIB supergravity on S^5, is interesting to note that these fields lie at the lowest level of the massive trajectories in the Kaluza-Klein mode decomposition of the

n	Multiplet	State	Field
0	supergraviton	$D(4,1,1)_0$	$g_{\mu\nu}$
		$D(3,\frac{1}{2},\frac{1}{2})_0$	$A_1+\frac{1}{6}\mathbb{A}_1$
0	LH+RH chiral	$D(3,0,0)_{\pm 2}$	$b_0^{m^2=-3}$
		$D(4,0,0)_0+D(4,0,0)_0$	τ
1	LH+RH massive gravitino	$D(5,\frac{1}{2},\frac{1}{2})_0+D(5,\frac{1}{2},\frac{1}{2})_0$	b_1^i
		$D(5,0,1)_2+D(5,1,0)_{-2}$	q_2
		$D(6,0,1)_0+D(6,1,0)_0$	b_2^i
2	massive vector	$D(7,\frac{1}{2},\frac{1}{2})_0$	$\mathbb{A}_1$
		$D(6,0,0)_0$	σ
		$D(7,0,0)_{\pm 2}$	$b_0^{m^2=21}$
		$D(8,0,0)_0$	ρ

Table 4.2: Identification of the bosonic states in the Kaluza-Klein spectrum with the linearized modes in the reduction.

$D=10$ fields [104, 124]. We note that the massive Kaluza-Klein tower is built out of scalar, vector and tensor harmonics on S^5, and the lowest harmonics generally have simple behavior on the internal sphere coordinates. For example, the lowest scalar harmonic is the constant mode on the sphere, while the lowest vector harmonics generate the Killing vectors on the sphere. It is presumably the simplicity of the lowest harmonics that allows the truncation to be consistent, even at the non-linear level.

Although the harmonics on SE_5 are more involved (see *e.g.* [46] for the case of $T^{1,1}$), it is clear that the decomposition (IV.15) and (IV.22) of the $D=10$ fields in terms of invariant tensors on SE_5 is equivalent to the truncation to the lowest harmonics on the sphere. This appears to be an essential feature guaranteeing the consistency of the massive truncation, and hence we do not expect to be able to keep any additional multiplets in the Kaluza-Klein tower beyond the $n=2$ level.

4.4 Further truncations

In order to make a connection with previous results on massive consistent truncations of IIB supergravity, we note that the semi-long LH+RH massive gravitino multiplet at the first Kaluza-Klein level may be truncated out by setting

$$b_1^i=0,\qquad b_2^i=0,\qquad q_2=0. \tag{IV.45}$$

It is easy to see that this truncation is consistent, since the respective equations of motion for q_2 in (IV.23) and g_2^i and g_3^i in (IV.25) are trivially satisfied in this case. The resulting $D=5$ Lagrangian takes the form

$$\begin{aligned}\mathcal{L} \;=\; & R*1+(24e^{2A-2B}-4e^{5A+3C}-\tfrac{1}{2}e^{8A}(4+\phi_0)^2)*1-\tfrac{28}{3}dB\wedge *dB-\tfrac{8}{3}dB\wedge *dC\\ & -\tfrac{4}{3}dC\wedge *dC-\tfrac{1}{2\tau_2^2}d\tau\wedge *d\bar\tau-\tfrac{1}{2}e^{2C-2A}F_2\wedge *F_2-e^{A-C}(F_2+\tfrac{1}{4}\mathbb{F}_2')\wedge *(F_2+\tfrac{1}{4}\mathbb{F}_2')\\ & -\tfrac{1}{2}e^{-8B}[\mathbb{A}_1'-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j)]\wedge *[\mathbb{A}_1'-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j)]\\ & -2\mathcal{M}_{ij}[e^{5A-C}(f_0^i\bar f_0^j+\bar f_0^i f_0^j)*1+e^{-4B}(f_1^i\wedge *\bar f_1^j+\bar f_1^i\wedge *f_1^j)]\\ & -A_1\wedge(F_2+\tfrac{1}{4}\mathbb{F}_2')\wedge(F_2+\tfrac{1}{4}\mathbb{F}_2'), \qquad \text{(IV.46)}\end{aligned}$$

where

$$f_0^i=3ib_0^i,\qquad f_1^i=Db_0^i,\qquad \phi_0=-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar f_0^j-\bar f_0^i f_0^j). \qquad \text{(IV.47)}$$

A further truncation to the massless $\mathcal{N}=2$ supergravity sector may be obtained by setting

$$b_0^i=0,\qquad B=0,\qquad C=0,\qquad \mathbb{A}_1=0, \qquad \text{(IV.48)}$$

along with taking a constant background for the axi-dilaton, $\tau=\tau_0$. This leaves only $g_{\mu\nu}$ and A_1, and yields the standard Lagrangian for the bosonic fields of minimal gauged supergravity

$$\mathcal{L}=R*1+12g^2*1-\tfrac{1}{2}F_2\wedge *F_2-\tfrac{1}{3\sqrt{3}}A_1\wedge F_2\wedge F_2, \qquad \text{(IV.49)}$$

where we have rescaled the graviphoton, $A_1\to\frac{1}{\sqrt{3}}A_1$, so that it has a canonical kinetic term, and where we have restored the dimensionful gauged supergravity coupling g.

4.4.1 Truncation to the zeroth Kaluza-Klein level

Beyond the truncation to minimal supergravity discussed above, the first nontrivial truncation involves keeping only the lowest Kaluza-Klein level fields $\{\tau, b_0^{m^2=-3}\}$ dynamical.

In what follows we will denote $b_0^{m^2=-3}$ simply as b so that $(b_0^1, b_0^2) = (b, \tau b)$. This truncation is not as simple as setting all other fields to zero, as the equations of motion demand certain constraints to be satisfied. For this case we start with the Lagrangian (C.9), obtained by setting $b_2^i = b_1^i = q_2 = 0$. We then impose the constraints

$$b_0^{m^2=21} = 0, \qquad e^{4B} = e^{-4C} = 1 - 4\tau_2|b|^2, \qquad \mathbb{A}_1 = -4i\tau_2(bD\bar{b} - \bar{b}Db) + 4|b|^2 d\tau_1. \tag{IV.50}$$

These in turn imply that

$$\phi_0 = -24\tau_2|b|^2, \qquad p_2 = -dA_1. \tag{IV.51}$$

To guarantee consistency, we have to check four constraints from the equations of motion (the B, C, f_0^i, and combined Maxwell Equation). They are all verified to hold identically, and hence the truncation to the supergravity plus the LH+RH chiral multiplet is consistent. The Lagrangian is given by

$$\begin{aligned}
\mathcal{L} \;=\; & R * 1 + \left[24(1 - 3\tau_2|b|^2)e^{-4B} - 4e^{-8B} - \tfrac{1}{2}e^{-8B}(4 + \phi_0)^2\right] * 1 - 8dB \wedge *dB \\
& -\tfrac{3}{2}F_2 \wedge *F_2 - \tfrac{1}{2}e^{-8B}\mathbb{A}_1 \wedge *\mathbb{A}_1 - 8e^{-4B}\tau_2 Db \wedge *D\bar{b} - 2ie^{-4B}(\bar{b}Db \wedge *d\bar{\tau} - bD\bar{b} \wedge *d\tau) \\
& -\tfrac{1}{2\tau_2^2}(1 + 8e^{-4B}\tau_2|b|^2)d\tau \wedge *d\bar{\tau} - A_1 \wedge F_2 \wedge F_2.
\end{aligned} \tag{IV.52}$$

This expression can be simplified by defining $\lambda \equiv 4\tau_2|b|^2$, giving

$$\begin{aligned}
\mathcal{L} \;=\; & R * 1 + \frac{6(2 - 3\lambda)}{(1 - \lambda)^2} * 1 - \frac{d\lambda \wedge *d\lambda}{2(1 - \lambda)^2} - \frac{(1 + \lambda)d\tau \wedge *d\bar{\tau}}{2(1 - \lambda)\tau_2^2} - \frac{3}{2}F_2 \wedge *F_2 - \frac{\mathbb{A}_1 \wedge *\mathbb{A}_1}{2(1 - \lambda)^2} \\
& - \frac{8\tau_2 Db \wedge *D\bar{b}}{1 - \lambda} - \frac{2i}{1 - \lambda}(\bar{b}Db \wedge *d\bar{\tau} - bD\bar{b} \wedge *d\tau) - A_1 \wedge F_2 \wedge F_2.
\end{aligned} \tag{IV.53}$$

If we further truncate the model by setting $\tau = ie^{-\phi_0} = ig_s^{-1}$, which is consistent with the equation of motion for τ given in (IV.8), this reproduces the model used in [99] to describe a holographic superconductor using a $m^2 = -3$ and $q = 2$ charged scalar. If we denote $b = \sqrt{g_s} f e^{i\theta}$, the truncated Lagrangian reads

$$\begin{aligned}
\mathcal{L} \;=\; & R * 1 - \tfrac{3}{2}F_2 \wedge *F_2 - A_1 \wedge F_2 \wedge F_2 \\
& + 12\frac{(1 - 6f^2)}{(1 - 4f^2)^2} * 1 - 8\frac{df \wedge *df}{(1 - 4f^2)^2} - 8f^2\frac{(d\theta - 3A_1) \wedge *(d\theta - 3A_1)}{(1 - 4f^2)^2}.
\end{aligned} \tag{IV.54}$$

A further redefinition $f = \frac{1}{2}\tanh\frac{\eta}{2}$ then reproduces the Lagrangian given in [99].

4.4.2 Truncation to the second Kaluza-Klein level

Starting with the Lagrangian (C.9) with $b_2^i = b_1^i = q_2 = 0$, it is possible to retain the $b_0^{m^2=21}$ scalar by setting $b_0^{m^2=-3} = 0$. In this case, we first let $b_0^2 = \bar{\tau} b_0^1$ and define $b_0^1 = \sqrt{g_s} h e^{i\xi}$, so that (h, ξ) describe the $m^2 = 21$ scalar. Again, the scalar equations of motion lead to constraints, and in particular the first equation in (IV.25) yields the equation of motion for h and ξ as well as

$$d(e^{3A+C} * d\tau) + ie^{3A+C}\frac{1}{\tau_2} d\tau \wedge *d\tau = 0. \tag{IV.55}$$

This is simply the τ equation of motion without any sources, and the simplest thing to do is to set τ to be constant, $\tau = ie^{-\phi_0} = ig_s^{-1}$. The remaining field content is then $\{g_{\mu\nu}, A_1, \rho, \sigma, b_0^{m^2=21}, \mathbb{A}_1\}$, corresponding to the supergravity multiplet coupled to the massive vector multiplet. It is now straightforward to complete the truncation, and the Lagrangian is given by

$$\begin{aligned}
\mathcal{L} \;=\;& R * 1 + \big(24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho}(1+6h^2)^2\big) * 1 - \tfrac{40}{3} d\rho \wedge *d\rho - 5 d\sigma \wedge *d\sigma \\
& -\tfrac{1}{2} e^{\frac{16}{3}\rho-4\sigma} F_2 \wedge *F_2 - e^{-\frac{8}{3}\rho+2\sigma}(F_2 + \tfrac{1}{4}\mathbb{F}_2') \wedge *(F_2 + \tfrac{1}{4}\mathbb{F}_2') \\
& -\tfrac{1}{2} e^{-8\rho-4\sigma}(\mathbb{A}_1' + 8h^2\Gamma) \wedge *(\mathbb{A}_1' + 8h^2\Gamma) - A_1 \wedge (F_2 + \tfrac{1}{4}\mathbb{F}_2') \wedge (F_2 + \tfrac{1}{4}\mathbb{F}_2') \\
& -8\big(e^{-4\rho-2\sigma} dh \wedge *dh + e^{-4\rho-2\sigma} h^2 \Gamma \wedge *\Gamma + e^{-\frac{28}{3}\rho+2\sigma} h^2 * 1\big),
\end{aligned} \tag{IV.56}$$

where we have defined $\Gamma = d\xi - 3A_1$.

We can further truncate this by removing the $m^2 = 21$ scalar (*i.e.* by setting $h = \xi = 0$), giving the Lagrangian

$$\begin{aligned}
\mathcal{L} \;=\;& R * 1 + (24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho}) * 1 - \tfrac{40}{3} d\rho \wedge *d\rho - 5 d\sigma \wedge *d\sigma \\
& -\tfrac{1}{2} e^{\frac{16}{3}\rho-4\sigma} F_2 \wedge *F_2 - e^{-\frac{8}{3}\rho+2\sigma}(F_2 + \tfrac{1}{4}\mathbb{F}_2') \wedge *(F_2 + \tfrac{1}{4}\mathbb{F}_2') - \tfrac{1}{2} e^{-8\rho-4\sigma} \mathbb{A}_1' \wedge *\mathbb{A}_1' \\
& -A_1 \wedge (F_2 + \tfrac{1}{4}\mathbb{F}_2') \wedge (F_2 + \tfrac{1}{4}\mathbb{F}_2'),
\end{aligned} \tag{IV.57}$$

which corresponds to the $m^2 = 24$ massive vector field truncation of [137].

4.4.3 Non-supersymmetric truncations

All the truncations we have listed so far have field content which fills the bosonic sector of AdS_5 supermultiplets and so are presumably supersymmetric truncations. It is also useful to consider truncations which contain dynamical fields belonging to different supermultiplets, without keeping the entire multiplet. In this sense these truncations are not supersymmetric, although they are perfectly consistent truncations and solutions of the ten-dimensional equations of motion. For these truncations, we start with the complete Lagrangian given in (IV.30) and (IV.31).

Massive vector field

The first non-supersymmetric truncation we will discuss involves keeping the $m^2 = 8$ vector field, b_1^i, and has already been noted in [137]. The field content in this truncation consists of one component of b_1^i (denoted b_1), τ_2, ρ, σ and $g_{\mu\nu}$. Note that the graviphoton is turned off here so that even at the lowest level this cannot be supersymmetric. Furthermore, by keeping only one component of b_1^i, the τ equation of motion demands that we must set $\tau_1 = 0$. With this field content, the $D = 10$ constraints (IV.24) are trivially satisfied with $\phi_0 = 0$ and $p_2 = 0$, and the Lagrangian (IV.30) becomes [137]

$$\begin{aligned} \mathcal{L} &= R * 1 + (24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho}) * 1 - \tfrac{40}{3} d\rho \wedge * d\rho - 5 d\sigma \wedge * d\sigma \\ &\quad - \tfrac{1}{2\tau_2^2} d\tau_2 \wedge * d\tau_2 - \tfrac{1}{2}\tau_2 e^{\frac{4}{3}\rho + 4\sigma} db_1 \wedge * db_1 - 4\tau_2 e^{-4\rho-2\sigma} b_1 \wedge * b_1. \end{aligned} \tag{IV.58}$$

Complex massive anti-symmetric tensor

We can also truncate to theories containing the $m^2 = 9$ complex anti-symmetric tensor field q_2. The field content here is given by, q_2, $\mathbb{A}_1$, B, C, τ, $g_{\mu\nu}$ and A_1. The $D = 10$ constraints become $\phi_0 = 0$ and $p_2 = -dA_1 - \frac{1}{4} d\mathbb{A}_1$. All the other equations of motion are either satisfied by setting the rest of the fields to zero or can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L} &= R*1+(24e^{-\frac{16}{3}\rho-\sigma}-4e^{-\frac{16}{3}\rho-6\sigma}-8e^{-\frac{40}{3}\rho})*1-\tfrac{40}{3}d\rho\wedge*d\rho-5d\sigma\wedge*d\sigma\\ &\quad -\tfrac{1}{2}e^{\frac{16}{3}\rho-4\sigma}F_2\wedge*F_2-e^{-\frac{8}{3}\rho+2\sigma}(p_2\wedge*p_2+4q_2\wedge*\bar{q}_2)-\tfrac{1}{2\tau_2^2}d\tau\wedge*d\bar{\tau}\\ &\quad -\tfrac{1}{2}e^{-8\rho-4\sigma}\mathbb{A}_1\wedge*\mathbb{A}_1+\tfrac{2i}{3}(q_2\wedge d\bar{q}_2-\bar{q}_2\wedge dq_2)-A_1\wedge p_2\wedge p_2-4A_1\wedge q_2\wedge\bar{q}_2.\end{aligned}\tag{IV.59}$$

Note that it is consistent to further truncate to a constant axi-dilaton $\tau=\tau_0$.

Real massive anti-symmetric tensor

Along similar lines to the case above for a massive vector field, we can set $A_1=0$ and make a truncation including the $m^2=16$ real anti-symmetric tensor doublet b_2^i by keeping only the graviton coupled to b_2^i, τ, ρ and σ. Again, the equations of motion for the other fields are trivially satisfied, and the constraints are also trivial $\phi_0=0$ and $p_2=0$. This leaves the Lagrangian

$$\begin{aligned}\mathcal{L} &= R*1+(24e^{-\frac{16}{3}\rho-\sigma}-4e^{-\frac{16}{3}\rho-6\sigma}-8e^{-\frac{40}{3}\rho})*1-\tfrac{40}{3}d\rho\wedge*d\rho-5d\sigma\wedge*d\sigma\\ &\quad -\tfrac{1}{2\tau_2^2}d\tau\wedge*d\bar{\tau}-\tfrac{1}{2}e^{\frac{20}{3}\rho}\mathcal{M}_{ij}db_2^i\wedge*db_2^j+2\epsilon_{ij}b_2^i\wedge*db_2^j.\end{aligned}\tag{IV.60}$$

As in the previous truncation, it is consistent to further truncate to $\tau=\tau_0$.

4.5 Discussion

In the above, we have examined massive reductions of 10-dimensional IIB supergravity on Sasaki-Einstein manifolds. By utilizing the structure of SE_5, we have given a general decomposition of the IIB fields based on the invariant tensors associated with the internal manifold. The field content obtained in five-dimensions completes the bosonic sector of various AdS_5 supermultiplets, and in particular they fill out the lowest Kaluza-Klein tower up to the breathing mode supermultiplet. This proves, at least at the level of the bosonic fields, the conjecture raised in [131] that a consistent massive truncation may be obtained by truncating to the singlet sector on the Kahler-Einstein base B (which is CP^2 for the squashed S^5) and further restricting to the level of the breathing mode multiplet and below.

As suggested at the end of Section 4.3, it is this truncation to constant modes on the base B that ensures the consistency of the reduction. In a sense, this is a generalization of the old consistency criterion of restricting to singlets of the internal isometry group, except that here restricting to singlets of an appropriate subgroup turned out to be sufficient. For this reason, we believe it is not that the breathing mode is special in itself which allows for a consistent truncation retaining its supermultiplet, but rather that in the examples given here and in [82], the breathing mode superpartners so happen to be the lowest harmonics in their respective Kaluza-Klein towers. It is an unusual feature of Kaluza-Klein compactifications on curved internal spaces that states originating from different levels of the harmonic expansion may combine into a single supermultiplet. Thus, while the breathing mode is always the lowest state in its tower (being a constant mode on the internal space), its superpartners may carry excitations on the internal space. This does not occur for the $\mathcal{N} = 2$ compactification of IIB supergravity on SE_5 (nor does it for $D = 11$ supergravity on SE_7). However, in extended theories, such as IIB supergravity on the round S^5, the superpartners will involve non-trivial harmonics. In particular, the $\mathcal{N} = 8$ superpartners to the breathing mode include a massive spin-2 excitation of the graviton involving the second harmonic (d-waves) on the sphere. Thus we believe it to be unlikely that an $\mathcal{N} = 8$ massive truncation with the breathing mode multiplet will be consistent.

Consistent truncations of the type discussed here have recently been of particular interest in the growing literature on AdS/CFT applications to condensed matter systems. Until recently a strictly phenomenological approach has been taken in this area. In these systems the inclusion of a scalar condensate is required in the gravity theory to source an operator whose expectation value acts as an order parameter describing superconductor/superfluid phase transitions in the strongly coupled system. In the phenomenological approach, the origin of this scalar and its properties have not been of immediate interest; rather the general behavior was determined and many interesting similarities to real condensed matter systems have been noted. However, this approach lacks strong theoretical control in that systems are described by a set of free parameters which can be tuned to provide the property of interest. Recently there has been some work to embed these models in UV complete theories, where the parameters are no longer free but are determined by the underlying features of the theory, such as an origin in string theory. The discussion here has put these reductions into a more general framework and gives further examples of UV complete systems whose duals may have useful applications in the AdS/CMT correspondence.

Given that the fields in these truncations fall into specific supermultiplets it is an obvious and relevant question to discuss their fermionic partners. This would involve reducing the supersymmetry variations and fermion equations in ten-dimensions down to five-dimensions and determining the complete supersymmetric action of these truncations. This is also of interest in terms of AdS/CMT where there has been much interest in describing fermion behavior in condensed matter systems such as the Fermi-liquid theory using the holographic correspondence. In particular, the full supersymmetric action could give us examples of specific interactions studied in these systems coupling scalar condensates to the fermionic excitations [51, 72, 102]. This is the topic of the remaining sections in this chapter.

4.6 The bosonic reduction of IIB supergravity on SE_5

The reduction of the bosonic sector of IIB supergravity on a squashed Sasaki-Einstein manifold carried out in the previous sections was first done in [43, 133, 88, 163]. From an $\mathcal{N} = 2$ point of view, the resulting theory has on-shell fields corresponding to that of five-dimensional gauged supergravity coupled to a massive hypermultiplet, massive gravitino multiplet and massive vector multiplet [131, 133].

Before turning to the fermions, we review the reduction of the bosonic sector, following the notations and conventions of [133]. In particular, here we highlight the truncation to the $\mathcal{N}=2$ sector presented in section 4.4.

Although IIB supergravity does not admit a covariant action, we may take a bosonic Lagrangian of the form

$$\mathcal{L}_{\text{IIB}} = R*1 - \frac{1}{2\tau_2^2}d\tau\wedge *d\bar{\tau} - \frac{1}{2}\mathcal{M}_{ij}F_3^i\wedge *F_3^j - \frac{1}{4}\widetilde{F}_5\wedge *\widetilde{F}_5 - \frac{1}{4}\epsilon_{ij}C_4\wedge F_3^i\wedge F_3^j, \tag{IV.61}$$

where self-duality $\widetilde{F}_5 = *\widetilde{F}_5$ is to be imposed by hand after deriving the equations of motion. Here we have chosen to write the Lagrangian in an SL(2,$\mathbb{R}$) invariant form using

$$\tau = C_0 + ie^{-\phi}, \qquad \mathcal{V} = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix} -\tau_1 & 1 \\ \tau_2 & 0 \end{pmatrix}, \qquad \mathcal{M} = \mathcal{V}^T\mathcal{V} = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}. \tag{IV.62}$$

For convenience when coupling to fermions, we also introduce the complexified vielbein $v_i = \mathcal{V}^1{}_i - i\mathcal{V}^2{}_i$, so that

$$v_iF_3^i = \tau_2^{-1/2}(F_3^2 - \tau F_3^1) = \tau_2^{-1/2}G_3, \tag{IV.63}$$

where $G_3 = F_3^2 - \tau F_3^1$.

The reduction ansatz follows by taking a metric of the squashed Sasaki-Einstein form

$$ds_{10}^2 = e^{2A}ds_5^2 + e^{2B}ds^2(B) + e^{2C}(\eta + A_1)^2, \tag{IV.64}$$

where $d\eta = 2J$ and where we set $3A+4B+C=0$ to remain in the Einstein frame. The key to the reduction is to expand the remaining bosonic fields in terms of the invariant forms J and Ω based on the SU(2) structure of the base B and satisfying

$$J\wedge\Omega = 0, \qquad \Omega\wedge\bar{\Omega} = 2J\wedge J = 4*_4 1, \qquad *_4J = J, \qquad *_4\Omega = \Omega, \tag{IV.65}$$

as well as

$$dJ = 0, \qquad d\Omega = 3i(d\psi + \mathcal{A})\wedge\Omega. \tag{IV.66}$$

The bosonic reduction follows by expanding the three-form and five-form field strengths in a basis of invariant tensors on B. Since we will truncate out the massive gravitino multiplet, we set the corresponding bosonic fields to zero. (The complete reduction is given in [133].) In this case, the three-form gives rise to two complex scalars b^i, and is given by

$$F_3^i = f_1^i \wedge \Omega + \bar{f}_1^i \wedge \bar{\Omega} + f_0^i \wedge \Omega \wedge (\eta + A_1) + \bar{f}_0^i \wedge \bar{\Omega} \wedge (\eta + A_1), \tag{IV.67}$$

where

$$f_1^i = Db^i, \qquad f_0^i = 3ib^i, \tag{IV.68}$$

with D the U(1) gauge covariant derivative

$$Db^i = db^i - 3iA_1 b^i. \tag{IV.69}$$

Furthermore, introducing

$$b^i = \begin{pmatrix} 1 \\ \tau \end{pmatrix} b^{m^2=-3} + \begin{pmatrix} 1 \\ \bar{\tau} \end{pmatrix} b^{m^2=21}, \tag{IV.70}$$

it is easy to see that

$$v_i f_0^i = 6\sqrt{\tau_2}\, b^{m^2=21}, \qquad \bar{v}_i f_0^i = -6\sqrt{\tau_2}\, b^{m^2=-3}, \tag{IV.71}$$

while

$$\begin{aligned} v_i f_1^i &= -2i\sqrt{\tau_2}[Db^{m^2=21} + \tfrac{i}{2\tau_2}(b^{m^2=-3}d\tau + b^{m^2=21}d\bar{\tau})], \\ \bar{v}_i f_1^i &= 2i\sqrt{\tau_2}[Db^{m^2=-3} - \tfrac{i}{2\tau_2}(b^{m^2=-3}d\tau + b^{m^2=21}d\bar{\tau})]. \end{aligned} \tag{IV.72}$$

These expressions will show up extensively in the fermion reduction below.

For the self-dual five-form, we have

$$\widetilde{F}_5 = (1 + *)[(4 + \phi_0) *_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1)], \tag{IV.73}$$

where $*_4 1$ denotes the volume form on the Kahler-Einstein base B. The fields ϕ_0 and p_2 are constrained by

$$\begin{aligned} \phi_0 &= -\tfrac{2i}{3}\epsilon_{ij}(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j), \\ p_2 &= -d[A_1 + \tfrac{1}{4}\mathbb{A}_1 + \tfrac{i}{6}\epsilon_{ij}(f_0^i \bar{f}_1^j - \bar{f}_0^i f_1^j)]. \end{aligned} \tag{IV.74}$$

Hence the only additional field arising from the five-form is the vector $\mathbb{A}_1$.

Finally, we note that the bosonic field content of this massive truncation is that of gauged supergravity coupled to a hypermultiplet with fields $(\tau, b^{m^2=-3})$ and a massive vector multiplet with fields $(B, C, b^{m^2=21}, \mathbb{A}_1)$. This massive multiplet is actually a vector combined with a hypermultiplet. However, since we are working on shell, one of the scalars has been absorbed into the massive vector. If desired, this scalar may be restored by an appropriate Stueckelberg shift of $\mathbb{A}_1$.

4.7 Reduction of the IIB fermions

We are now prepared to examine the fermionic sector of IIB supergravity [159]. For simplicity in working out the reduction, we follow a Dirac convention throughout. In this case, the fermions consist of a spin-$\frac{3}{2}$ gravitino Ψ_M and a spin-$\frac{1}{2}$ dilatino λ, with opposite chiralities

$$\Gamma_{11}\Psi_M = \Psi_M, \qquad \Gamma_{11}\lambda = -\lambda. \tag{IV.75}$$

Our Dirac conventions are detailed in Appendix C.1. In particular, as opposed to [159], we are using a mostly plus metric signature.

In the following we always work to lowest order in the fermions. In this case, the IIB supersymmetry variations on the fermions are given by [159]

$$\begin{aligned}
\delta\lambda &= \frac{i}{2\tau_2}\Gamma^A\partial_A\tau\epsilon^c - \frac{i}{24}\Gamma^{ABC}v_iF^i_{ABC}\epsilon, \\
\delta\Psi_M &= \mathcal{D}_M\epsilon \equiv \left(\nabla_M + \frac{i}{4\tau_2}\partial_M\tau_1 + \frac{i}{16\cdot 5!}\Gamma^{ABCDE}\widetilde{F}_{ABCDE}\Gamma_M\right)\epsilon \\
&\quad + \frac{i}{96}\left(\Gamma_M{}^{ABC} - 9\delta^A_M\Gamma^{BC}\right)v_iF^i_{ABC}\epsilon^c.
\end{aligned} \tag{IV.76}$$

The supersymmetry parameter ϵ is chiral with $\Gamma_{11}\epsilon = \epsilon$, and the complexified $SL(2,\mathbb{R})$ vielbein, v_i, was defined above in (IV.63). In addition the fermion equations of motion are [159]

$$\begin{aligned}
0 &= \Gamma^M\mathcal{D}_M\lambda - \frac{i}{8\cdot 5!}\Gamma^{MNPQR}F_{MNPQR}\lambda, \\
0 &= \Gamma^{MNP}\mathcal{D}_N\Psi_P + \frac{i}{48}\Gamma^{NPQ}\Gamma^M v_i^* F^{i*}_{NPQ}\lambda - \frac{i}{4\tau_2}\Gamma^N\Gamma^M\partial_N\tau\lambda^c,
\end{aligned} \tag{IV.77}$$

where the supercovariant derivative acting on the gravitino is defined in the gravitino variation (IV.76). On the other hand, the supercovariant derivative acting on the dilatino takes the form

$$\mathcal{D}_M\lambda = \left(\nabla_M + \frac{3i}{4\tau_2}\partial_M\tau_1\right)\lambda - \frac{i}{2\tau_2}\Gamma^N\partial_N\tau\Psi^c_M + \frac{i}{24}\Gamma^{NPQ}v_iF^i_{NPQ}\Psi_M, \tag{IV.78}$$

and is defined so that $\nabla_M\epsilon$ terms drop out of the variation $\mathcal{D}_M\delta\lambda$, as appropriate to supercovariantization.

4.7.1 Killing spinors on SE_5

The starting point of the fermion reduction is the construction of Killing spinors on SE_5. Starting with the undeformed Sasaki-Einstein metric

$$ds^2(SE_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \tag{IV.79}$$

the Killing spinor equations then follow from the internal components of the gravitino variation in (IV.76) with a constant five-form flux

$$\tilde{F}_5 = 4 *_5 1 + 4 *_4 1 \wedge (d\psi + \mathcal{A}) \tag{IV.80}$$

and take the form

$$0 = \delta\Psi_a = \hat{\mathcal{D}}_a\eta \equiv [\hat{\nabla}_a - \mathcal{A}_a\partial_\psi + \tfrac{1}{2}J_{ab}\tau^b\tau^9 + \tfrac{i}{2}\tau_a]\eta,$$

$$0 = \delta\Psi_9 = [\partial_\psi - \tfrac{1}{4}J_{ab}\tau^{ab} + \tfrac{i}{2}\tau_9]\eta. \tag{IV.81}$$

We proceed by assigning a U(1) charge q to the Killing spinor η, so that $\partial_\psi\eta = iq\eta$. Furthermore, since $(J_{ab}\tau^{ab})^2 = -8(1-\tau^9)$, we see that $J_{ab}\tau^{ab}$ has eigenvalues $(4i, -4i, 0, 0)$ with corresponding τ^9 eigenvalues $(-1, -1, 1, 1)$. The variation $\delta\Psi_9$ then vanishes for the charges $q = (\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. The $\mathcal{N} = 2$ Killing spinor is thus obtained by taking $q = \frac{3}{2}$ and $J_{ab}\tau^{ab}\eta = 4i\eta$.

Having exhausted the content of the $\delta\Psi_9$ equation, we now turn to integrability of $\delta\Psi_a$, which gives the requirement

$$0 = \tau^b[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b]\eta = \tau^b[\delta_{ab}(\tau^9 - 1) - iJ_{ab}(\tau^9 + 2q)]\eta. \tag{IV.82}$$

For $q = \frac{3}{2}$ and $\tau^9\eta = -\eta$, this gives the condition $J_{ab}\tau^b\eta = i\tau_a\eta$, which is easily seen to be consistent with the above requirement that $J_{ab}\tau^{ab}\eta = 4i\eta$. After defining $\eta = e^{3i\psi/2}\tilde{\eta}$, we are finally left with the condition

$$[\hat{\nabla}_a - \tfrac{3i}{2}\mathcal{A}_a]\tilde{\eta} = 0, \tag{IV.83}$$

which is solved by taking $\tilde{\eta}$ to be a gauge covariantly constant spinor on the Kahler-Einstein base [90].

To summarize the above, the system (IV.81) may be solved to yield a single complex Killing spinor η satisfying

$$\partial_\psi\eta = \tfrac{3i}{2}\eta, \qquad \tau^9\eta = -\eta, \qquad \tau^b J_{ab}\eta = i\tau_a\eta, \qquad \tau^b\Omega_{ab}\eta = 0. \tag{IV.84}$$

The final condition may be obtained by multiplying the penultimate one by Ω_{ca} on both sides and making use of the identity $\Omega_{ca}J_{ab} = -i\Omega_{cb}$, which follows from the relations [88]

$$\Omega_{ac}\Omega^{bc} = 0, \qquad \Omega_{ac}\bar{\Omega}^{bc} = 2\delta_a{}^b - 2iJ_a{}^b. \tag{IV.85}$$

The Killing spinor η and its conjugate η^c provide a natural basis of invariant spinors in which to expand the fermions. Furthermore, as discussed in [10], these represent singlets of the $SU(2)$ structure group, thus ensuring consistency of the reduction. Note that η and η^c are related by

$$\tau^b\bar{\Omega}_{ab}\eta = 2\tau_a\eta^c, \tag{IV.86}$$

and η^c satisfies the conjugated relations

$$\partial_\psi\eta^c = -\tfrac{3i}{2}\eta^c, \qquad \tau^9\eta^c = -\eta^c, \qquad \tau^b J_{ab}\eta^c = -i\tau_a\eta^c, \qquad \tau^b\bar{\Omega}_{ab}\eta^c = 0. \tag{IV.87}$$

4.7.2 IIB spinor decomposition

We are now in a position to present the fermion decomposition ansatz by expanding the ten-dimensional fermions in terms of η and η^c. Although we will ultimately truncate away the massive gravitino multiplet, we find it instructive to start with the complete ansatz. This allows us to identify which fermions belong in which multiplets, and hence will guide the truncation.

Starting with the IIB dilatino, since it has negative chirality, it may be decomposed as[5]

$$\lambda = e^{-A/2}\lambda\otimes\eta\otimes\begin{bmatrix}0\\1\end{bmatrix} + e^{-A/2}\lambda'\otimes\eta^c\otimes\begin{bmatrix}0\\1\end{bmatrix}. \tag{IV.88}$$

The IIB transformation parameter ϵ and gravitino Ψ_A each have positive chirality. Thus we expand the gravitino in ten dimensional flat indices as

$$\begin{aligned}
\Psi_\alpha &= e^{-A/2}\psi_\alpha\otimes\eta\otimes\begin{bmatrix}1\\0\end{bmatrix} + e^{-A/2}\psi'_\alpha\otimes\eta^c\otimes\begin{bmatrix}1\\0\end{bmatrix},\\
\Psi_a &= e^{-A/2}\psi\otimes\tau_a\eta\otimes\begin{bmatrix}1\\0\end{bmatrix} + e^{-A/2}\psi'\otimes\tau_a\eta^c\otimes\begin{bmatrix}1\\0\end{bmatrix},\\
\Psi_9 &= e^{-A/2}\psi_9\otimes\tau_9\eta\otimes\begin{bmatrix}1\\0\end{bmatrix} + e^{-A/2}\psi'_9\otimes\tau_9\eta^c\otimes\begin{bmatrix}1\\0\end{bmatrix},
\end{aligned} \tag{IV.89}$$

and the transformation parameter as

$$\epsilon = e^{A/2}\varepsilon\otimes\eta\otimes\begin{bmatrix}1\\0\end{bmatrix}. \tag{IV.90}$$

[5]Note that this is a slight abuse of notation, in that λ shows up as both ten-dimensional and five-dimensional fields. The correct interpretation will be obvious from the context.

Note that in all the above we have included relevant warp factors to account for the breathing and squashing modes.

While we have started with a theory with 32 real supercharges, only a quarter of these are preserved in the $\mathrm{AdS}_5 \times SE_5$ background. By focusing on supersymmetries generated by (IV.90), we are thus restricting our study to five-dimensional supersymmetry parameterized by a single Dirac spinor. This corresponds to an $\mathcal{N} = 2$ theory, and provides a motivation for us to remove the massive gravitino from subsequent consideration. (If desired, the full spontaneously broken $\mathcal{N} = 4$ symmetry may be obtained by introducing an $\varepsilon \otimes \eta^c$ component in (IV.90). However, we will not pursue this here.)

4.7.3 Linearized analysis and the $\mathcal{N} = 2$ supermultiplet structure

Before presenting the fermionic reduction, it is instructive to analyze the linearized equations of motion. Doing so allows us to group the effective five-dimensional fermions into the relevant $\mathcal{N} = 2$ supermultiplets as highlighted in [133]. We start by noting that the five-dimensional fermions consist of the two gravitini ψ_α and ψ'_α, two dilatini λ and λ' and four additional spin-1/2 fields ψ, ψ', ψ_9 and ψ'_9 arising from the internal components of the ten-dimensional gravitino.

In the linearized theory, the equations are greatly simplified and the fermions satisfy free massive Dirac and Rarita-Schwinger equations. The λ and λ' equations are naturally diagonal and the gravitino equations are diagonalized by the following modes,

$$\hat{\psi}_\alpha = \psi_\alpha + \tfrac{i}{3}\gamma_\alpha\left(4\psi + \psi_9\right), \qquad \psi^{m=11/2} = 4\psi + \psi_9, \qquad \psi^{m=-9/2} = \psi - \psi_9,$$

$$\hat{\psi}'_\alpha = \psi'_\alpha + \tfrac{i}{10}\left(\gamma_\alpha + 2\nabla_\alpha\right)\left(4\psi' + \psi'_9\right), \qquad \psi'^{m=5/2} = \psi' - \psi'_9. \tag{IV.91}$$

n	Multiplet	State	Field
0	supergraviton	$D(4,1,1)_0$	$g_{\mu\nu}$
		$D(3\frac{1}{2},1,\frac{1}{2})_{-1}+D(3\frac{1}{2},\frac{1}{2},1)_1$	$\hat{\psi}_\mu$
		$D(3,\frac{1}{2},\frac{1}{2})_0$	$A_1+\frac{1}{6}\mathbb{A}_1$
0	LH+RH chiral	$D(3,0,0)_{\pm 2}$	$b^{m^2=-3}$
		$D(3\frac{1}{2},\frac{1}{2},0)_1+D(3\frac{1}{2},0,\frac{1}{2})_{-1}$	λ'
		$D(4,0,0)_0+D(4,0,0)_0$	τ
1	LH+RH massive gravitino	$D(5\frac{1}{2},\frac{1}{2},1)_1+D(5\frac{1}{2},1,\frac{1}{2})_{-1}$	$\hat{\psi}'_\mu$
		$D(5,\frac{1}{2},\frac{1}{2})_0+D(5,\frac{1}{2},\frac{1}{2})_0$	b_1^i
		$D(5,0,1)_2+D(5,1,0)_{-2}$	q_2
		$D(6,0,1)_0+D(6,1,0)_0$	b_2^i
		$D(4\frac{1}{2},0,\frac{1}{2})_1+D(4\frac{1}{2},\frac{1}{2},0)_{-1}$	$\psi'^{m=5/2}$
		$D(5\frac{1}{2},0,\frac{1}{2})_{-1}+D(5\frac{1}{2},\frac{1}{2},0)_1$	λ
2	massive vector	$D(7,\frac{1}{2},\frac{1}{2})_0$	$\mathbb{A}_1$
		$D(6\frac{1}{2},\frac{1}{2},0)_{-1}+D(6\frac{1}{2},0,\frac{1}{2})_1$	$\psi^{m=-9/2}$
		$D(7\frac{1}{2},0,\frac{1}{2})_{-1}+D(7\frac{1}{2},\frac{1}{2},0)_1$	$\psi^{m=11/2}$
		$D(6,0,0)_0$	σ
		$D(7,0,0)_{\pm 2}$	$b^{m^2=21}$
		$D(8,0,0)_0$	ρ

Table 4.3: Identification of the bosonic and fermionic states in the Kaluza-Klein spectrum with the linearized modes in the reduction.

In all, the linearized modes satisfy,

$$\begin{aligned}
&\gamma^{\mu\alpha\beta}\nabla_\alpha\hat{\psi}_\beta=\tfrac{3}{2}\gamma^{\mu\alpha}\hat{\psi}_\alpha, &&\gamma^{\mu\alpha\beta}\nabla_\alpha\hat{\psi}'_\beta=-\tfrac{7}{2}\gamma^{\mu\alpha}\hat{\psi}'_\alpha,\\
&\gamma^\alpha\nabla_\alpha\lambda=\tfrac{7}{2}\lambda, &&\gamma^\alpha\nabla_\alpha\lambda'=-\tfrac{3}{2}\lambda',\\
&\gamma^\alpha\nabla_\alpha\psi^{m=11/2}=\tfrac{11}{2}\psi^{m=11/2}, &&\gamma^\alpha\nabla_\alpha\psi^{m=-9/2}=-\tfrac{9}{2}\psi^{m=-9/2},\\
&\gamma^\alpha\nabla_\alpha\psi'^{m=5/2}=\tfrac{5}{2}\psi'^{m=5/2}.
\end{aligned}\tag{IV.92}$$

Note that the massive gravitino obtains its mass by absorbing the spin-1/2 combination $4\psi'+\psi'_9$.

As with the fields in the bosonic truncation, we have arrived at a field content which, in the case of the round five-sphere, saturates the lowest harmonic in each of the respective Kaluza-Klein towers as determined in [104, 124]. Noting that, in five dimensions, the relation between the conformal weight Δ and mass m of the fermions is $|m|=\Delta-2$, we can map the fermion fields into $\mathcal{N}=2$ AdS multiplets. First, it is straightforward to see that $\hat{\psi}_\mu$ has $m=3/2$, corresponding to a massless spin-3/2 field in AdS$_5$. Hence it should be identified with the massless gravitino sitting in the supergraviton multiplet. Also at the zeroth Kaluza-Klein level, the LH+RH chiral multiplet contains an $m = 3/2$ fermion which may be identified as λ'. At level $n = 1$, the massive gravitino multiplet has three fermions; one spin-3/2 particle with $m = -7/2$ corresponding to the massive gravitino $\hat{\psi}'_\mu$ and two spin-1/2 particles with $m = 5/2$ corresponding to $\psi'^{m=5/2}$ and $m = 7/2$ corresponding to λ. Finally, at the $n = 2$ Kaluza-Klein level, the massive vector multiplet contains two spin-1/2 particles, $\psi^{m=-9/2}$ and $\psi^{m=11/2}$. These identifications will be further justified by examining the supersymmetry transformations. The complete field content of the supermultiplets is shown in Table 4.3, where the bosonic fields are fully defined in [133].

4.8 The Five-dimensional Theory and $\mathcal{N} = 2$ Supergravity

The linearized analysis above demonstrates that the fields ψ'_α, ψ', ψ'_9 and λ belong to the massive gravitino multiplet. We thus proceed with the $\mathcal{N} = 2$ truncation by setting these to zero

$$\psi'_\alpha = 0, \qquad \psi' = 0, \qquad \psi'_9 = 0, \qquad \lambda = 0. \tag{IV.93}$$

It is straightforward to show that this is a consistent truncation, provided the bosonic fields in the massive graviton multiplet are set to zero[6]. Moreover, other than just simplifying the resulting equations, this truncation is natural when explicitly discussing $\mathcal{N} = 2$ supersymmetry as the massive gravitino should really be thought of as descending from a spontaneously broken $\mathcal{N} = 4$ theory.

4.8.1 Supersymmetry Variations

We start with the reduction of the IIB supersymmetry variations given in (IV.76). Inserting the fermion ansätze (IV.88), (IV.89) and (IV.90) into the IIB variations, we

[6] The consistency of this truncation in the bosonic sector has been previously shown in [133, 88, 43].

arrive at the following five-dimensional variations[7]

$$\delta\hat{\psi}_\alpha \equiv \mathcal{D}_\alpha\varepsilon = \Big[D_\alpha + \tfrac{i}{24}e^{C-A}\left(\gamma_\alpha{}^{\nu\rho} - 4\delta_\alpha{}^\nu\gamma^\rho\right)\left(F_{\nu\rho} - 2e^{-2B-2C}p_{\nu\rho}\right) + \tfrac{1}{12}\gamma_\alpha\left(4e^{A-2B+C} + 6e^{A-C} - (4+\phi_0)e^{A-4B-C}\right)\Big]\varepsilon - e^{-2B}\left(v_i f^i_\alpha - \tfrac{i}{3}e^{A-C}v_i f^i_0\gamma_\alpha\right)\varepsilon^c, \tag{IV.94}$$

$$\delta\psi^{m=11/2} = \Big[-\tfrac{i}{2}\gamma^\mu\partial_\mu(4B+C) - \tfrac{3}{8}e^{-4B}\gamma^\mu\mathbb{A}_\mu + \tfrac{1}{8}e^{C-A}\gamma^{\mu\nu}\left(F_{\mu\nu} + e^{-2B-2C}p_{\mu\nu}\right) - ie^{A-2B+C} - \tfrac{3i}{2}e^{A-C} + \tfrac{5i}{8}(4+\phi_0)e^{A-4B-C}\Big]\varepsilon + e^{-2B}\left(\tfrac{3i}{4}\gamma^\mu v_i f^i_\mu + \tfrac{7}{4}e^{A-C}v_i f^i_0\right)\varepsilon^c, \tag{IV.95}$$

$$\delta\psi^{m=-9/2} = \Big[-\tfrac{i}{2}\gamma^\mu\partial_\mu(B-C) - \tfrac{1}{4}e^{-4B}\gamma^\mu\mathbb{A}_\mu - \tfrac{1}{8}e^{C-A}\gamma^{\mu\nu}\left(F_{\mu\nu} + e^{-2B-2C}p_{\mu\nu}\right) - \tfrac{3i}{2}e^{A-2B+C} + \tfrac{3i}{2}e^{A-C}\Big]\varepsilon + e^{-2B}\left(\tfrac{i}{2}\gamma^\mu v_i f^i_\mu - \tfrac{1}{2}e^{A-C}v_i f^i_0\right)\varepsilon^c, \tag{IV.96}$$

$$\delta\lambda' = -\tfrac{1}{2\tau_2}\gamma^\mu\partial_\mu\tau\varepsilon^c - ie^{-2B}\left(\gamma^\mu v_i\bar{f}^i_\mu - ie^{A-C}v_i\bar{f}^i_0\right)\varepsilon. \tag{IV.97}$$

The gauge covariant derivative D_α acting on ε is given by $D_\alpha \equiv \nabla_\alpha - \frac{3i}{2}(A_\alpha + \frac{1}{6}e^{-4B}\mathbb{A}_\alpha) + \frac{i}{4\tau_2}\partial_\alpha\tau_1$, where the latter term descends from the traditional charge with respect to the U(1) compensator field, Q_M, in the ten dimensional IIB theory [159]. Furthermore, we have defined the five-dimensional supercovariant derivative $\mathcal{D}_\alpha$ through the gravitino variation in (IV.94).

There are several facts worth noting about these expressions. Firstly, we see that these variations fit nicely into the multiplet structure as presented in Table 4.3. In particular, the dilatino variation is built out of τ and $\bar{v}_i f^i$, both of which belong to the LH+RH chiral multiplet, since the latter corresponds to $b^{m^2=-3}$ according to (IV.71). On the other hand, $\delta\psi^{m=11/2}$ and $\delta\psi^{m=-9/2}$ contain only terms involving fields from the graviton and massive vector multiplets. [Note that the combination $F_2 + e^{-2B-2C}p_2$ appearing in (IV.96) and (IV.97) essentially selects the field strength of the massive vector $\mathbb{A}_1$, as can be seen from the definition of p_2 given in (IV.74)]. These observations give further justification for the

[7] Note that with the Dirac matrix conventions described in the appendix we have $\epsilon^c = i\varepsilon^c \otimes \eta^c \otimes \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

multiplet structure presented in section 4.7.3.

Furthermore, since the breathing mode is $\rho \sim 4B + C$, and the squashing mode is $\sigma \sim B - C$, we can identify $\psi^{m=11/2}$ with the fermionic partner of the breathing mode and $\psi^{m=-9/2}$ as the fermionic partner of the squashing mode as first demonstrated in [131]. Finally, from the gauge covariant derivative, it is evident that the combination $A_\mu + \frac{1}{6}e^{-4B}\mathbb{A}_\mu$ may be identified with the graviphoton, which is consistent with the linearized analysis in [133]. (The combination $F_2 - 2e^{-2B-2C}p_2$ appearing in the gravitino variation is similarly the effective graviphoton field strength.)

The gravitino variation (IV.94) is particularly interesting, as we may attempt to read off an $\mathcal{N} = 2$ superpotential from the term proportional to $\gamma_\alpha \varepsilon$

$$W = 2e^{A-2B+C} + 3e^{A-C} - \frac{1}{2}(4 + \phi_0)e^{A-4B-C}. \tag{IV.98}$$

Recalling the relations $3A + 4B + C = 0$ and $\phi_0 = -\frac{2i}{3}\epsilon_{ij}\left(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j\right)$, we see that the scalar potential can be written as

$$V = 2(\mathcal{G}^{-1})^{ij}\partial_i W \partial_j W - \frac{4}{3}W^2, \tag{IV.99}$$

where $(\mathcal{G}^{-1})^{ij}$ is the inverse scalar metric which can be read off from the scalar kinetic terms in the Lagrangian and $\{i, j\}$ run over all scalars in the theory.

To verify (IV.99), we made use of the fact that the scalar metric given in [133] is composed of three independent components, pertaining to the independent sets of scalars $\{B, C\}$, $\{b_0^1, b_0^2\}$ and τ, with explicit components

$$(\mathcal{G}^{-1}_{\{B,C\}})^{ij} = \frac{1}{16}\begin{pmatrix} 1 & -1 \\ -1 & 7 \end{pmatrix}, \qquad (\mathcal{G}^{-1}_{\{b_0^1, b_0^2\}})^{ij} = \frac{e^{4B}}{4\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \qquad \mathcal{G}^{-1}_\tau = \tau_2^2. \tag{IV.100}$$

Inserting these expressions into (IV.99) then exactly reproduces the scalar potential appearing in the bosonic Lagrangian. This is, however, a somewhat surprising relation as the actual gravitino variation (IV.94) contains not only the term proportional to the superpotential written above, but another term involving $v_i f_0^i \varepsilon^c$ where $v_i f_0^i$ is proportional to $b_0^{m^2=21}$, as indicated in (IV.71). Based on general $\mathcal{N}=2$ gauged supergravity arguments, this should conceivably also contribute to the scalar potential, but is not taken into account by (IV.99).

4.8.2 Equations of Motion

Turning to the equations of motion, the reduction of the dilatino equation is the most straightforward. After a bit of manipulation, we obtain

(IV.101)

$$\begin{aligned} 0 &= \left[\gamma^\mu \mathcal{D}_\mu + \tfrac{i}{8}\gamma^{\mu\nu}\left(e^{C-A}F_{\mu\nu} - 2e^{-A-2B-C}p_{\mu\nu}\right) - \tfrac{1}{4}(4+\phi_0)e^{A-4B-C} + e^{A-2B+C} + \tfrac{3}{2}e^{A-C}\right]\lambda' \\ &\quad -e^{-2B}v_i\left[\tfrac{4}{5}\gamma^\mu \bar{f}_\mu^i + \tfrac{28i}{15}\bar{f}_0^i\right]\psi^{m=11/2} - e^{-2B}v_i\left[\tfrac{4}{5}\gamma^\mu \bar{f}_\mu^i - \tfrac{4i}{5}\bar{f}_0^i e^{A-C}\right]\psi^{m=-9/2}, \end{aligned}$$

where the supercovariant derivative acting on the dilatino is defined by

$$\text{(IV.102)}\quad \mathcal{D}_\mu\lambda' \equiv D_\mu\lambda' - K(\lambda')\hat{\psi}_\mu = \left[\nabla_\mu + \tfrac{3i}{4\tau_2}\partial_\mu\tau_1 + \tfrac{3i}{2}\left(A_\mu + \tfrac{1}{6}e^{-4B}\mathbb{A}_\mu\right)\right]\lambda' - K(\lambda')\hat{\psi}_\mu.$$

The supercovariantization term $K(\lambda')$ acting on $\hat{\psi}_\mu$ is given by the right hand side of the dilatino variation (IV.97) with ε replaced by $\hat{\psi}_\mu$ (and similarly ε^c replaced by $\hat{\psi}_\mu^c$).

Starting with the IIB gravitino, we arrive at three equations, corresponding to the α, a, and 9 components. After a fair bit of manipulations, and the appropriate redefinitions given in the first line of (IV.91), we obtain the $\psi^{m=11/2}$ and $\psi^{m=-9/2}$ equations

(IV.103)

$$\begin{aligned}
0 \quad = \quad & \Big[\gamma^\mu \mathcal{D}_\mu + \tfrac{3i}{5} e^{-4B} \gamma^\mu \mathbb{A}_\mu - \tfrac{i}{120} e^{C-A} \gamma^{\mu\nu} F_{\mu\nu} - \tfrac{11i}{60} e^{-A-2B-C} \gamma^{\mu\nu} p_{\mu\nu} \\
& + e^A \left(-\tfrac{17}{12}(4+\phi_0) e^{-4B-C} + \tfrac{1}{15} e^{-2B+C} - \tfrac{1}{10} e^{-C}\right)\Big] \psi^{m=11/2} \\
& + \Big[\tfrac{3i}{5} e^{-4B} \gamma^\mu \mathbb{A}_\mu + \tfrac{i}{5} e^{C-A} \gamma^{\mu\nu} F_{\mu\nu} - \tfrac{i}{10} e^{-A-2B-C} \gamma^{\mu\nu} p_{\mu\nu} + e^A \left(\tfrac{12}{5} e^{-2B+C} - \tfrac{12}{5} e^{-C}\right)\Big] \psi^{m=-9/2} \\
& + v_i e^{-2B} \Big[\left(-\tfrac{2}{5} \gamma^\mu f^i_\mu + \tfrac{34i}{15} e^{A-C} f^i_0\right) \psi^{c\, m=11/2} + \left(\tfrac{3}{5} \gamma^\mu f^i_\mu - \tfrac{7i}{5} e^{A-C} f^i_0\right) \psi^{c\, m=-9/2}\Big] \\
& + \bar{v}_i e^{-2B} \left(\tfrac{3}{4} \gamma^\mu f^i_\mu + \tfrac{7i}{4} e^{A-C} f^i_0\right) \lambda',
\end{aligned}$$

(IV.104)

$$\begin{aligned}
0 \quad = \quad & \Big[\gamma^\mu \mathcal{D}_\mu + \tfrac{2i}{5} e^{-4B} \gamma^\mu \mathbb{A}_\mu - \tfrac{3i}{40} e^{C-A} \gamma^{\mu\nu} F_{\mu\nu} - \tfrac{3i}{20} e^{-A-2B-C} \gamma^{\mu\nu} p_{\mu\nu} \\
& + e^A \left(\tfrac{1}{4}(4+\phi_0) e^{-4B-C} + \tfrac{13}{5} e^{-2B+C} + \tfrac{9}{20} e^{-C}\right)\Big] \psi^{m=-9/2} \\
& + \Big[\tfrac{2i}{5} e^{-4B} \gamma^\mu \mathbb{A}_\mu + \tfrac{2i}{15} e^{C-A} \gamma^{\mu\nu} F_{\mu\nu} - \tfrac{i}{15} e^{-A-2B-C} \gamma^{\mu\nu} p_{\mu\nu} + e^A \left(\tfrac{8}{5} e^{-2B+C} - \tfrac{8}{5} e^{-C}\right)\Big] \psi^{m=11/2} \\
& + v_i e^{-2B} \Big[\left(\tfrac{2}{5} \gamma^\mu f^i_\mu - \tfrac{14i}{5} e^{A-C} f^i_0\right) \psi^{c\, m=11/2} + \left(-\tfrac{3}{5} \gamma^\mu f^i_\mu - \tfrac{3i}{5} e^{A-C} f^i_0\right) \psi^{c\, m=-9/2}\Big] \\
& + \bar{v}_i e^{-2B} \left(\tfrac{1}{2} \gamma^\mu f^i_\mu - \tfrac{i}{2} e^{A-C} f^i_0\right) \lambda'.
\end{aligned}$$

As in the dilatino case, we have defined the supercovariant derivatives

(IV.105)

$$\begin{aligned}
\mathcal{D}_\mu \psi^{m=11/2} &= \Big[\nabla_\mu + \tfrac{i}{4\tau_2} \partial_\mu \tau_1 - \tfrac{3i}{2}(A_\mu + \tfrac{1}{6} e^{-4B} \mathbb{A}_\mu)\Big] \psi^{m=11/2} - K(\psi^{m=11/2}) \hat{\psi}_\mu, \\
\mathcal{D}_\mu \psi^{m=-9/2} &= \Big[\nabla_\mu + \tfrac{i}{4\tau_2} \partial_\mu \tau_1 - \tfrac{3i}{2}(A_\mu + \tfrac{1}{6} e^{-4B} \mathbb{A}_\mu)\Big] \psi^{m=-9/2} - K(\psi^{m=-9/2}) \hat{\psi}_\mu,
\end{aligned}$$

with $K(\psi^{m=11/2})$ and $K(\psi^{m=-9/2})$ similarly obtained from the variations (IV.96) and (IV.97), respectively.

Finally, the gravitino equation takes the form

(IV.106)

$$0 = \gamma^{\mu\nu\rho} \mathcal{D}_\nu \hat{\psi}_\rho - \tfrac{8}{15} \tilde{K}(\psi^{m=11/2}) \gamma^\mu \psi^{m=11/2} - \tfrac{4}{5} \tilde{K}(\psi^{m=-9/2}) \gamma^\mu \psi^{m=-9/2} - \tfrac{1}{2} \tilde{K}(\lambda') \gamma^\mu \lambda',$$

where the supercovariant derivative acting on the gravitino is given by the right hand side of the gravitino variation (IV.94), and where the $\tilde{K}$ terms are essentially the Dirac conjugates of K. The above equations have the appropriate structure to be obtained from an effective $\mathcal{N}=2$ Lagrangian of the form[8]

(IV.107)

$$\begin{aligned} e^{-1}\mathcal{L} &= \bar{\hat{\psi}}_\mu \gamma^{\mu\nu\rho}\mathcal{D}_\nu\hat{\psi}_\rho + \tfrac{8}{15}\bar{\psi}^{m=11/2}\gamma^\mu D_\mu\psi^{m=11/2} + \tfrac{4}{5}\bar{\psi}^{m=-9/2}\gamma^\mu D_\mu\psi^{m=-9/2} + \tfrac{1}{2}\bar{\lambda}'\gamma^\mu D_\mu\lambda' \\ &\quad + \Big[\bar{\hat{\psi}}_\mu\Big(-\tfrac{8}{15}\tilde{K}(\psi^{m=11/2})\gamma^\mu\psi^{m=11/2} - \tfrac{4}{5}\tilde{K}(\psi^{m=-9/2})\gamma^\mu\psi^{m=-9/2} - \tfrac{1}{2}\tilde{K}(\lambda')\gamma^\mu\lambda'\Big) + \text{h.c.}\Big] \\ &\quad + \cdots . \end{aligned}$$

The full fermionic Lagrangian (to quadratic order in the fermions) is given in Appendix C.2.

Although we have worked only to quadratic order in the fermions, it is clear from the nature of the invariant spinors η and η^c that higher spinor harmonics would not be excited by this subset of states. Thus, if desired, the consistent truncation may be extended to the four-fermi terms as well. However, we expect this to be quite tedious and not particularly worth pursuing.

4.9 A supersymmetric holographic superconductor

In this final section we demonstrate the consistency of a particularly interesting truncation to the lowest Kaluza-Klein level, namely the supersymmetric completion of the bosonic truncation first demonstrated in [99]. As we demonstrate, this is a fully consistent truncation, so long as we keep all fields in the graviton and LH+RH chiral multiplets. However, it is a nontrivial truncation, in that it is not consistent to naively set the other fields in the above reduction to zero. Instead, the "backreaction" on the truncated fields must be taken into account, effectively setting these modes equal to something depending on the

[8] Note that some care must be taken when considering the conjugate spinor terms. Nevertheless, the various conjugate terms do assemble themselves properly into a consistent effective fermionic Lagrangian. This is one place where a more conventional symplectic-Majorana approach would allow the manipulations to be more transparent.

dynamical fields. Due to this backreaction on the non dynamical fields, the resulting Lagrangian is nonlinear and so describes a non-trivial coupling of $\mathcal{N}=2$ supergravity with a single hypermultiplet.

In the bosonic sector the truncation amounts to keeping only $\{\tau, b^{m^2=-3}\}$ and the graviton and graviphoton dynamical. In what follows, we will denote $b^{m^2=-3}$ simply as b so that $(b_0^1, b_0^2) = (b, \tau b)$. This requires the following constraints on the other terms in the reduction [43, 133]

$$b^{m^2=21} = 0, \qquad e^{4B} = e^{-4C} = 1 - 4\tau_2|b|^2, \qquad \mathbb{A}_1 = -4i\tau_2(bD\bar{b} - \bar{b}Db) + 4|b|^2 d\tau_1, \tag{IV.108}$$

and

$$\phi_0 = -24\tau_2|b|^2, \qquad p_2 = -dA_1. \tag{IV.109}$$

For the fermions, by analyzing the supersymmetry transformations of the spin-$\frac{1}{2}$ fields in this truncation, it is evident that if we set

$$\psi = -\psi_9 = -\frac{i}{2} b\tau_2^{1/2} e^{-2B}\lambda', \tag{IV.110}$$

the resulting system will be consistent with the supersymmetry transformations. It turns out that under this identification the fermion equations of motion also degenerate into a single expression, resulting in a theory containing only λ' and $\hat{\psi}_\mu$ in the fermionic sector.

Moving directly to the Lagrangian, we write this as a sum of bosonic and fermionic contributions $\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f$, where

$$\begin{aligned}\mathcal{L}_b \;=\;& R * 1 + \frac{6(2-3\chi)}{(1-\chi)^2} - \frac{d\chi \wedge * d\chi}{2(1-\chi)^2} * 1 - \frac{(1+\chi) d\tau \wedge * d\bar{\tau}}{2(1-\chi)\tau_2^2} - \frac{3}{2} F_2 \wedge * F_2 - \frac{\mathbb{A}_1 \wedge * \mathbb{A}_1}{2(1-\chi)^2} \\ & - \frac{8\tau_2 Db \wedge * D\bar{b}}{1-\chi} - \frac{2i}{1-\chi}(\bar{b}Db \wedge * d\bar{\tau} - bD\bar{b} \wedge * d\tau) - A_1 \wedge F_2 \wedge F_2,\end{aligned} \tag{IV.111}$$

and

$$
\begin{aligned}
e^{-1}\mathcal{L}_f \quad = \quad & \bar{\hat{\psi}}_\alpha \gamma^{\alpha\beta\sigma} D_\beta \hat{\psi}_\sigma + \tfrac{3i}{8}\bar{\hat{\psi}}_\alpha \left(\gamma^{\alpha\beta\rho\sigma} + 2g^{\alpha\beta}g^{\rho\sigma}\right) F_{\beta\rho}\hat{\psi}_\sigma + \tfrac{1}{2}\bar{\tilde{\lambda}}\gamma^\alpha D_\alpha \tilde{\lambda} + \tfrac{3i}{16}\bar{\tilde{\lambda}}\gamma^{\mu\nu}F_{\mu\nu}\tilde{\lambda} \\
& + \tfrac{1}{2}e^{-4B}\left(3\tau_2(b\bar{D_\mu b} - \bar{b}D_\mu b)\bar{\tilde{\lambda}}\gamma^\mu\tilde{\lambda} + \tfrac{3}{2}(1+8\tau_2|b|^2)\bar{\tilde{\lambda}}\tilde{\lambda}\right) \\
& + e^{-4B}\left(-\tfrac{3}{2}\bar{\hat{\psi}}_\alpha\gamma^{\alpha\sigma}\hat{\psi}_\sigma + \tau_2(\bar{b}D_\beta b - b\bar{D_\beta b})\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\hat{\psi}_\sigma\right) \\
& + \tau_2^{1/2}e^{-4B}\left(D_\mu b\bar{\hat{\psi}}_\alpha\gamma^\mu\gamma^\alpha\tilde{\lambda} + 3b\bar{\hat{\psi}}_\alpha\gamma^\alpha\tilde{\lambda} + h.c.\right) \\
& + \frac{e^{-2B}}{\tau_2^{1/2}}\left(-b\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\partial_\beta\tau\hat{\psi}^c_\sigma + \tau_2^{1/2}\bar{\hat{\psi}}_\alpha\gamma^\mu\partial_\mu\tau\gamma^\alpha\tilde{\lambda}^c + h.c.\right),
\end{aligned}
\tag{IV.112}
$$

where we have defined $\tilde{\lambda} \equiv e^{-2B}\lambda'$, $\chi = \tau_2|b|^2$ and we have redefined the gauge covariant derivative acting on b as $D_\mu b = \left(\partial_\mu - 3iA_\mu - \frac{i}{2\tau_2}\partial_\mu\tau_1\right)b$, and similarly for $\tilde{\lambda}$ and $\hat{\psi}_\alpha$.

This truncation is of interest for many of the condensed matter applications of the AdS/CFT correspondence involving the coupling of a charged scalar and fermion. In particular the original motivation for the bosonic truncation was in describing a superconducting phase transition using holographic methods within a controlled system, *i.e,* one which is derived directly from a UV complete theory. This truncation hence completes the story by demonstrating the embedding into a fully supersymmetric theory. It would be interesting to consider the dynamics of this theory, and whether there is a supersymmetric superconducting phase transition. Note however that this analysis would be complicated by the presence of the gravitino. After all, it is not consistent to simply set the gravitino field defined here to be zero. Since the gravitino couples to the supercurrent, this suggests that the holographic superconductor model of [99] in fact has an underlying (although spontaneously broken) supersymmetry.

While the truncation first presented in [99] did not include the axi-dilaton, as in the bosonic case, it is consistent to fix τ as well. This simplifies the Lagrangian to be

$$\begin{aligned}
e^{-1}\mathcal{L} \;=\;& R - \tfrac{3}{4}F_{\mu\nu}F^{\mu\nu} - e^{-1}A_1 \wedge F_2 \wedge F_2 \\
&+12\frac{(1-6f^2)}{(1-4f^2)^2} - 8\frac{\partial_\mu f \partial^\mu f}{(1-4f^2)^2} - 8f^2\frac{(\partial_\mu\theta - 3A_\mu)(\partial^\mu\theta - 3A^\mu)}{(1-4f^2)^2} \\
&+\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}D_\beta\hat{\psi}_\sigma + \tfrac{1}{2}\bar{\tilde{\lambda}}\gamma^\alpha D_\alpha\tilde{\lambda} + \tfrac{3i}{8}\bar{\hat{\psi}}_\alpha\left(\gamma^{\alpha\beta\rho\sigma} + 2g^{\alpha\beta}g^{\rho\sigma}\right)F_{\beta\rho}\hat{\psi}_\sigma + \tfrac{3i}{16}\bar{\tilde{\lambda}}\gamma^{\mu\nu}F_{\mu\nu}\tilde{\lambda} \\
&\frac{1}{1-4f^2}\left(\tfrac{3}{4}(1+8f^2)\bar{\tilde{\lambda}}\tilde{\lambda} - \tfrac{3}{2}\bar{\hat{\psi}}_\alpha\gamma^{\alpha\sigma}\hat{\psi}_\sigma - if^2(\partial_\mu\theta - 3A_\mu)\left(3\bar{\tilde{\lambda}}\gamma^\mu\tilde{\lambda} + 2\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\hat{\psi}_\sigma\right)\right) \\
&+\left(\frac{e^{i\theta}}{1-4f^2}\left((\partial_\mu f + if(\partial_\mu\theta - 3A_\mu))\,\bar{\hat{\psi}}_\alpha\gamma^\mu\gamma^\alpha\tilde{\lambda} + 3f\bar{\hat{\psi}}_\alpha\gamma^\alpha\tilde{\lambda}\right) + h.c.\right), \qquad \text{(IV.113)}
\end{aligned}$$

where we have defined $b = \sqrt{g_s}fe^{i\theta}$ and $\tau = ig_s^{-1}$.

Finally, it is worth noting that although this theory involves a charged scalar coupled to the fermion $\tilde{\lambda}$, it lacks the Majorana coupling $\phi\lambda\lambda$ that has been of recent interest in studies involving gapped fermions in the bosonic condensate [51, 72, 102]. While this coupling is allowed by charge conservation, the explicit reduction shows that it is not present. More generally, examination of Table 4.3 demonstrates that the $b^{m^2=21}$ scalar in the massive vector multiplet may have such a coupling, and in fact the equations of motion (IV.103) and (IV.104) show that it is exists for both $\psi^{m=11/2}$ and $\psi^{m=-9/2}$. It would be curious to see if this $b^{m^2=21}$ scalar may play a role in novel models of holographic superconductors.

CHAPTER V

Final remarks

We have presented various excursions away from the standard AdS/CFT paradigm. First, concentrating on bulk physics, we discussed the effects of higher derivative interactions to the effective five-dimensional Lagrangian. From the boundary perspective, these terms describe deviations from the large-N or large λ limit, depending on their ten-dimensional origin. We presented a discussion of the effects on R-charged black holes, highlighting some technical details concerning field redefinitions and Gibbons-Hawking surface terms. Additionally we constructed the supersymmetric four-derivative completion of minimal gauged supergravity and examined the effects on the shear viscosity to entropy density ratio.

In the final chapter we presented results of consistent truncations of IIB supegravity on squashed Sasaki-Einstein manifolds. The novel feature of these truncations is that they include a finite number of massive supergravity multiplets. The hope is that these and similar truncations can be utilized to construct new solutions of IIB/M-theory which describe dual gauge theories where conformal invariance or in some cases even Lorentz invariance is broken. Additionally these types of constructions may provide insights into interesting regimes of both the quark/gluon plasma and various condensed matter systems.

There has been somewhat of a paradigm shift in the string theory community over the past fifteen years. While a large portion of the community continues to pursue the goal of realizing a UV complete theory of the standard model, many practitioners have shifted attention to the utility of string theory in describing strongly coupled gauge theories. As discussed, the relevant gauge theories are similar to QCD, which describes the quark/gluon plasma being studied experimentally in large particle colliders, although a precise string theory dual to QCD itself has not been discovered. In addition these gauge theories can potentially be applied to effective field theory descriptions of various condensed matter systems studied in table-top experiments.

The possibility that string theory may provide the mathematical machinery to describe novel states of matter actually seen in a laboratory is an exciting development. However, the application of string theory to these areas should not be considered a test of string theory as a fundamental theory of gravity or in relation to its prospects as a completion to the standard model of particle physics. The distinction as to whether string theory is a UV complete description of physics is still an unanswered question.

This thesis presented two avenues of exploration towards more realistic models within the developing paradigm of gauge/string duality. By studying the inclusion of higher derivative terms in the action we can probe small deviations from the regime of large number of degrees of freedom (colors) and large coupling in the gauge theory. Second, the discussion on consistent truncations makes progress in understanding the duals to theories with lower supersymmetry. Much work is to be done in this area to make solid predictions for any real laboratory system. The material presented here represents a small subset of the possible approaches towards a description of these systems in this context. It is hoped that the progress made herein will provide some additional insight into more realistic settings.

APPENDICES

APPENDIX A

Shear Viscosity

A.1 Explicit Form of Quadratic Action and Solution to Metric Fluctuation

The quadratic action for the scalar channel perturbation ϕ is given in (III.123) in terms of six coefficients $A, \ldots, F$. Here we present their explicit forms:

(A.1)

$$
\begin{aligned}
A(u) &= \frac{4}{u} f_0 + \bar{c}_2 \Bigg[-\frac{\omega^2}{g^2}\frac{H_0^2}{3} + \frac{2uf_0(1+q)^3(5qu-1)}{H_0^3} - \frac{32g^2qu^2(1+q)^3}{3} + \frac{g^2u^3(1+q)^6}{H_0} \Bigg], \\
B(u) &= \frac{3f_0}{u} + \bar{c}_2 \Bigg[-\frac{\omega^2}{g^2} H_0^2 + \frac{g^2(4qu+1)^2H_0^3}{3u} - \frac{g^2u(1+q)^3(56q^2u^2+7qu+11)}{6} \\
&\qquad + \frac{g^2u^3(1+q)^6(26q^2u^2-17qu+17)}{6H_0^3} - 8g^2(1+q)^3qu^2 + \frac{3g^2u^3(1+q)^6}{4H_0} \Bigg], \\
C(u) &= \frac{2g^2(4qu-3)H_0^2}{u^2} - \frac{2g^2(1+q)^3(2qu+1)}{H_0} \\
&\quad + \bar{c}_2 \Bigg[-\frac{\omega^2}{6uf_0}\Big((4qu+1)H_0^4 - (1+q)^3(-11qu^3+13u^2)H_0\Big) \\
&\qquad - \frac{g^2(1+q)^3(4q^2u^2+45qu+3)}{3H_0} + \frac{g^2u^2(1+q)^6(4q^3u^3-7q^2u^2-32qu+15)}{2H_0^4} \Bigg], \\
D(u) &= \frac{2g^2H_0^3 - g^2qu^3(1+q)^3}{u^3H_0^2} + \omega^2 \frac{H_0^3}{4u^2f_0} \\
&\quad + \bar{c}_2 \Bigg[\frac{\omega^4}{g^2}\frac{H_0^5}{12uf_0^2} + \frac{\omega^2g^2(1+q)^3}{48f_0^2}\Big(2(31qu-9)H_0^3 - 3u^2(1+q)^3(5q^2u^2-4qu+11)\Big) \\
&\qquad - \frac{19g^2q(1+q)^3}{3H_0^2} - \frac{3g^2u(1+q)^6(6q^2u^2-17qu+1)}{2H_0^5} \Bigg], \\
E(u) &= \bar{c}_2 \frac{4uf_0^2}{3g^2H_0}, \\
F(u) &= \bar{c}_2\, f_0\, \frac{2(2(4qu+1)H_0^3 - u^2(1+q)^3(7qu+4))}{3H_0^2}.
\end{aligned}
$$

Here we also present the $\mathcal{O}(\bar{c}_2)$ solution for ϕ. Writing $\phi(u) = f(u)^\nu F(u)$, we may expand $F(u)$ to first order in both $\bar{c}_2$ and ω

(A.2)
$$F(u) = F_0(u,\omega) + \bar{c}_2(F_{10}(u) + \omega F_{11}(u)).$$

Since $F(u)$ satisfies a second order equation (after linearizing in $\bar{c}_2$ and using the lowest order equation of motion), it is consistent to choose the boundary conditions such that $F(u)$ is normalized at the boundary ($F(0) = 1$) and is regular at the horizon.

The function $F_0(u,\omega)$ is given by the expression in the curly brackets in (III.130), while the remaining functions are

$$
\begin{aligned}
F_{10}(u) &= 0, \\
F_{11}(u) &= \frac{(1+q)^{3/2}(11q^5+4q^4+179q^3-10q^2-8q-16)}{32q^2(1+q)^2(q-2)^3}\Big[i\ln(q^3u^2-3qu-u-1)+\pi\Big] \\
&+\frac{i(q+1)^{3/2}(60q^6+99q^5+648q^4-69q^3-154q^2-104q-16)}{16(4q+1)^{3/2}(q+1)^2(q-2)^3}\times \\
&\qquad\left[\tanh^{-1}\frac{-(1+3q)}{(4q+1)^{1/2}(q+1)}-\tanh^{-1}\frac{2q^3u-(1+3q)}{(4q+1)^{1/2}(q+1)}\right] \\
&-\frac{i\ln(1+qu)(1+q)^{3/2}}{8q^2}-\frac{i(q+1)^{3/2}(-4q^5+21q^4+143q^3-21q^2-39q-6)}{8q^4(4q+1)(q-2)^2} \\
&-\frac{i(q+1)^{3/2}(4q^7-27q^6+64q^5+511q^4+137q^3-128q^2-57q-6)qu^2}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2} \\
&+\frac{i(-12q^6+102q^5+605q^4+63q^3-177q^2-63q-6)u}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2} \\
&-\frac{i(4q^5+21q^4+143q^3-21q^2-39q-6)}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2}. \qquad \text{(A.3)}
\end{aligned}
$$

APPENDIX B

Charge to Mass Ratio

B.1 The mass to charge ratio in AdS

For the case of asymptotically AdS solutions with a flat boundary, i.e. $k = 0$, $g \neq 0$, we find that the mass to charge ratio is:

$$
\begin{aligned}
\frac{m}{q} &= \left(\frac{m}{q}\right)_0 \left(1 - \tilde{\alpha}_1 g^2 \frac{(d-1)(7d^3 - 27d^2 + 8d + 32)}{2(d-2)(3d-7)}\right. \\
&\qquad\qquad \left. -\tilde{\alpha}_2 g^2 \frac{(d-1)(2d^3 - 3d^2 - 19d + 32)}{2(d-2)(3d-7)} - \alpha_3 g^2 \frac{(d-3)(d-4)}{2}\right) \\
&= \left(\frac{m}{q}\right)_0 \left(1 - \alpha_1 g^2 \frac{(d-1)(7d^3 - 27d^2 + 8d + 32)}{2(d-2)(3d-7)}\right. \\
&\qquad\qquad -\alpha_2 g^2 \frac{(d-1)(2d^3 - 3d^2 - 19d + 32)}{2(d-2)(3d-7)} \\
&\qquad\qquad \left. -\alpha_3 g^2 \frac{(2d^4 - 10d^3 + 21d^2 - 37d + 36)}{(d-2)(3d-7)}\right),
\end{aligned} \tag{B.1}
$$

where

$$
\left(\frac{m}{q}\right)_0 = g r_+ \sqrt{\frac{2(d-2)^3}{(d-1)(d-3)^2}}\,. \tag{B.2}
$$

We note that if the redshift factor λ had not been taken into account, the correction to the mass/charge ratio for the $k = 0$ Gauss-Bonnet term ($\tilde{\alpha}_1 = 0, \tilde{\alpha}_2 = 0$) would have vanished. It is precisely the addition of the redshift factor which is responsible for generating the correction.

For the case of asymptotically AdS solutions with a spherical horizon, i.e. $k=1$, $g\neq 0$, the expressions are rather more complicated. For $d=4$, we find

$$\begin{aligned}\left(\frac{m}{q}\right)_{d=4} &= \left(\frac{m}{q}\right)_0\left(1-12\tilde{\alpha}_1 g^2-\tilde{\alpha}_2\frac{(54\beta^2+21\beta+1)}{5r_+^2(1+2\beta)}\right)\\ &= \left(\frac{m}{q}\right)_0\left(1-12\alpha_1 g^2-\alpha_2\frac{(54\beta^2+21\beta+1)}{5r_+^2(1+2\beta)}-4\alpha_3\frac{(24\beta^2+6\beta+1)}{5r_+^2(1+2\beta)}\right),\end{aligned} \tag{B.3}$$

where $\beta=g^2r_+^2$, and

$$\left(\frac{m}{q}\right)_{0,d=4}=\frac{2(1+2\beta)}{\sqrt{1+3\beta}}\,. \tag{B.4}$$

For $d=5$, we have

$$\begin{aligned}\left(\frac{m}{q}\right)_{d=5} = &\left(\frac{m}{q}\right)_0\left(1-\tilde{\alpha}_1\frac{(816\beta^3+1024\beta^2+300\beta+1)}{6r_+^2(1+2\beta)(2+3\beta)}\right.\\ &\left.-\tilde{\alpha}_2\frac{(336\beta^3+392\beta^2+132\beta+11)}{6r_+^2(1+2\beta)(2+3\beta)}-\alpha_3\frac{(3\beta^2+2\beta-2)}{r_+^2(2+3\beta)}\right)\\ = &\left(\frac{m}{q}\right)_0\left(1-\alpha_1\frac{(816\beta^3+1024\beta^2+300\beta+1)}{6r_+^2(1+2\beta)(2+3\beta)}\right.\\ &\left.-\alpha_2\frac{(336\beta^3+392\beta^2+132\beta+11)}{6r_+^2(1+2\beta)(2+3\beta)}-\alpha_3\frac{(564\beta^3+586\beta^2+216\beta+31)}{6r_+^2(1+2\beta)(2+3\beta)}\right),\end{aligned} \tag{B.5}$$

where

$$\left(\frac{m}{q}\right)_{0,d=5}=\frac{\sqrt{3}(2+3\beta)}{2\sqrt{1+2\beta}}. \tag{B.6}$$

This result corresponds to (II.116) given in Section 2.2.5.

Similarly, for $d=6$:

$$\begin{aligned}\left(\frac{m}{q}\right)_{d=6} = &\left(\frac{m}{q}\right)_0\left(1-1\tilde{\alpha}_1\frac{(15500\beta^3+23445\beta^2+8325\beta+81)}{22r_+^2(3+4\beta)(3+5\beta)}\right.\\ &\left.-\tilde{\alpha}_2\frac{(275\beta^2+195\beta+27)}{4r_+^2(3+5\beta)}-3\alpha_3\frac{(20\beta^3+27\beta^2-7\beta-9)}{r_+^2(3+4\beta)(3+5\beta)}\right)\\ = &\left(\frac{m}{q}\right)_0\left(1-\alpha_1\frac{(15500\beta^3+23445\beta^2+8325\beta+81)}{22r_+^2(3+4\beta)(3+5\beta)}\right.\\ &\left.-\alpha_2\frac{(275\beta^2+195\beta+27)}{4r_+^2(3+5\beta)}-3\alpha_3\frac{(3340\beta^3+4549\beta^2+2153\beta+369)}{22r_+^2(3+4\beta)(3+5\beta)}\right),\end{aligned} \tag{B.7}$$

where

$$\left(\frac{m}{q}\right)_{0,d=6}=\frac{2\sqrt{2}(3+4\beta)}{3\sqrt{3+5\beta}}\,. \tag{B.8}$$

A general d-dimensional expression may be obtained in principle, although it is not expected to be particularly illuminating.

APPENDIX C

Fermion Conventions and Reduced Action

C.1 Dirac Matrix Conventions

We work with a mostly plus metric signature, and take the conventional Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. Note, in particular, that Γ^0 is anti-hermitian, so that $(\Gamma^0)^\dagger = -\Gamma^0$ and $(\Gamma^i)^\dagger = \Gamma^i$. The ten-dimensional Chirality matrix is given by

$$\Gamma^{11} \equiv \frac{1}{10!}\epsilon_{A_1\cdots A_{10}}\Gamma^{A_1}\cdots\Gamma^{A_{10}} = \Gamma^0\cdots\Gamma^9, \tag{C.1}$$

and squares to the identity.

Corresponding to the metric reduction (IV.64), we decompose the ten-dimensional Dirac matrices according to

$$\begin{aligned} \Gamma^\alpha &\equiv \gamma^\alpha \otimes 1_4 \otimes \sigma_1, \\ \Gamma^a &\equiv 1_4 \otimes \tau^a \otimes \sigma_2, \\ \Gamma^9 &\equiv 1_4 \otimes \tau^9 \otimes \sigma_2, \end{aligned} \tag{C.2}$$

where γ^α are Dirac matrices in the $5d$ spacetime with $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and τ^a are Dirac matrices in the $5d$ internal space with $\tau^9 \equiv \tau^5\tau^6\tau^7\tau^8$. The Chirality matrix Γ^{11} is then given by

$$\Gamma^{11} = \Gamma^0\cdots\Gamma^9 = 1_4 \otimes 1_4 \otimes \sigma_3. \tag{C.3}$$

We furthermore take the following conventions for the A, C and D intertwiners which map between different representations of the Dirac matrices

$$A_{10}\Gamma_M A_{10}^{-1} = \Gamma_M^\dagger, \qquad C_{10}^{-1}\Gamma_M C_{10} = -\Gamma_M^T, \qquad D_{10}^{-1}\Gamma_M D_{10} = -\Gamma_M^*. \tag{C.4}$$

Here C_{10} denotes the charge conjugation matrix. These may be decomposed as

$$A_{10} = A_{4,1} \otimes A_5 \otimes \sigma_1, \qquad C_{10} = C_{4,1} \otimes C_5 \otimes \sigma_2, \qquad D_{10} = iD_{4,1} \otimes D_5 \otimes \sigma_3, \tag{C.5}$$

where the five-dimensional intertwiners are defined as

$$\begin{aligned} A_{4,1}\gamma_\mu A_{4,1}^{-1} &= -\gamma_\mu^\dagger, \qquad & C_{4,1}^{-1}\gamma_\mu C_{4,1} &= \gamma_\mu^T, \qquad & D_{4,1}^{-1}\gamma_\mu D_{4,1} &= -\gamma_\mu^* \\ A_5\tau_a A_5^{-1} &= \tau_a^\dagger, \qquad & C_5^{-1}\tau_a C_5 &= \tau_a^T, \qquad & D_5^{-1}\tau_a D_5 &= \tau_a^*. \end{aligned} \tag{C.6}$$

It turns out the following is a consistent decomposition:

$$A_{10} = \Gamma_0 = \gamma_0 \otimes 1 \otimes \sigma_1, \qquad C_{10} = C_{4,1} \otimes C_5 \otimes \sigma_2, \qquad D_{10} = i\gamma_0 C_{4,1} \otimes C_5 \otimes \sigma_3. \tag{C.7}$$

The five dimensional charge conjugation matrices on both spacetime and the internal manifold satisfy

$$C_5 = -C_5^T = C_5^* = -C_5^{-1}. \tag{C.8}$$

Finally, we define the charge conjugate of a spinor in any dimension to be $\psi^c = CA^T\psi^*$, which is equivalent to $\psi^c = -\Gamma_0 C_{10}\psi^*$. Therefore, letting χ and η be spinors on M and SE_5, respectively, the charge conjugates are given by $\chi^c = -\gamma_0 C_{4,1}\chi^*$ and $\eta^c = C_5\eta^*$.

C.2 The Reduced Lagrangian

The bosonic Lagrangian with the massive gravitino multiplet removed was presented in [133], and takes the form

$$\begin{aligned}
\mathcal{L}_b \;=\;& R*1+(24e^{2A-2B}-4e^{5A+3C}-\tfrac{1}{2}e^{8A}(4+\phi_0)^2)*1-\tfrac{28}{3}dB\wedge *dB-\tfrac{8}{3}dB\wedge *dC\\
&-\tfrac{4}{3}dC\wedge *dC-\tfrac{1}{2\tau_2^2}d\tau\wedge *d\bar{\tau}-\tfrac{1}{2}e^{2C-2A}F_2\wedge *F_2-e^{A-C}(F_2+\tfrac{1}{4}\mathbb{F}_2')\wedge *(F_2+\tfrac{1}{4}\mathbb{F}_2')\\
&-\tfrac{1}{2}e^{-8B}[\mathbb{A}_1'-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)]\wedge *[\mathbb{A}_1'-\tfrac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)]\\
&-2\mathcal{M}_{ij}[e^{5A-C}(f_0^i\bar{f}_0^j+\bar{f}_0^i f_0^j)*1+e^{-4B}(f_1^i\wedge *\bar{f}_1^j+\bar{f}_1^i\wedge *f_1^j)]\\
&-A_1\wedge(F_2+\tfrac{1}{4}\mathbb{F}_2')\wedge(F_2+\tfrac{1}{4}\mathbb{F}_2'), \qquad \text{(C.9)}
\end{aligned}$$

where $\mathbb{A}_1'=\mathbb{A}_1+\frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)$, and where $\mathbb{F}_2'=d\mathbb{A}_1'$.

The corresponding fermionic Lagrangian may be obtained from the equations of motion presented in Section 4.8.2. At quadratic order in the fermions, we have

$$
\begin{aligned}
e^{-1}\mathcal{L}_f \;=\;& \bar{\hat{\psi}}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \hat{\psi}_\rho \\
&+\Big[-\tfrac{8}{15}\bar{\psi}^{m=11/2}\gamma^\mu K(\psi^{m=11/2})\hat{\psi}_\mu - \tfrac{4}{5}\bar{\psi}^{m=-9/2}\gamma^\mu K(\psi^{m=-9/2})\hat{\psi}_\mu \\
&\qquad\qquad -\tfrac{1}{2}\bar{\lambda}'\gamma^\mu K(\lambda')\hat{\psi}_\mu + h.c.\Big] \\
&+\tfrac{8}{15}\bar{\psi}^{m=11/2}\Big[\gamma^\mu D_\mu + \tfrac{3i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu - \tfrac{i}{120}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \tfrac{11i}{60}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad\qquad +e^A\left(-\tfrac{17}{12}(4+\phi_0)e^{-4B-C} + \tfrac{1}{15}e^{-2B+C} - \tfrac{1}{10}e^{-C}\right)\Big]\psi^{m=11/2} \\
&+\tfrac{4}{5}\psi^{m=-9/2}\Big[\gamma^\mu D_\mu + \tfrac{2i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu - \tfrac{3i}{40}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \tfrac{3i}{20}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad\qquad +e^A\left(\tfrac{1}{4}(4+\phi_0)e^{-4B-C} + \tfrac{13}{5}e^{-2B+C} + \tfrac{9}{20}e^{-C}\right)\Big]\psi^{m=-9/2} \\
&+\tfrac{1}{2}\bar{\lambda}'\Big[\gamma^\mu D_\mu + \tfrac{i}{8}\gamma^{\mu\nu}\left(e^{C-A}F_{\mu\nu} - 2e^{-A-2B-C}p_{\mu\nu}\right) \\
&\qquad\qquad -\tfrac{1}{4}(4+\phi_0)e^{A-4B-C} + e^{A-2B+C} + \tfrac{3}{2}e^{A-C}\Big]\lambda' \\
&+\tfrac{8}{15}\Big[\bar{\psi}^{m=11/2}\Big(\tfrac{3i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu + \tfrac{i}{5}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \tfrac{i}{10}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad\qquad +e^A\left(\tfrac{12}{5}e^{-2B+C} - \tfrac{12}{5}e^{-C}\right)\Big)\psi^{m=-9/2} + h.c.\Big] \\
&+\tfrac{8}{15}\Big[v_i e^{-2B}\bar{\psi}^{m=11/2}\left(-\tfrac{2}{5}\gamma^\mu f^i_\mu + \tfrac{34i}{15}e^{A-C}f^i_0\right)\psi^{c\,m=11/2} + h.c.\Big] \\
&+\tfrac{8}{15}\Big[v_i e^{-2B}\bar{\psi}^{m=11/2}\left(\tfrac{3}{5}\gamma^\mu f^i_\mu - \tfrac{7i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2} + h.c.\Big] \\
&+\tfrac{4}{5}\Big[v_i e^{-2B}\bar{\psi}^{m=-9/2}\left(-\tfrac{3}{5}\gamma^\mu f^i_\mu - \tfrac{3i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2} + h.c.\Big] \\
&+\tfrac{8}{15}\Big[\bar{v}_i e^{-2B}\bar{\psi}^{m=11/2}\left(\tfrac{3}{4}\gamma^\mu f^i_\mu + \tfrac{7i}{4}e^{A-C}f^i_0\right)\lambda' + h.c.\Big] \\
&+\tfrac{4}{5}\Big[\bar{v}_i e^{-2B}\bar{\psi}^{m=-9/2}\left(\tfrac{1}{2}\gamma^\mu f^i_\mu - \tfrac{i}{2}e^{A-C}f^i_0\right)\lambda' + h.c.\Big],
\end{aligned} \tag{C.10}
$$

and the full Lagrangian up to quadratic order in the fermions is given by

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f. \tag{C.11}$$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Allan Adams, Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, and Riccardo Rattazzi. Causality, analyticity and an IR obstruction to UV completion. *JHEP*, 10:014, 2006.

[2] Allan Adams, Koushik Balasubramanian, and John McGreevy. Hot Spacetimes for Cold Atoms. *JHEP*, 11:059, 2008.

[3] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys.Rept.*, 323:183–386, 2000.

[4] Ofer Aharony, Jacek Pawelczyk, Stefan Theisen, and Shimon Yankielowicz. A note on anomalies in the AdS/CFT correspondence. *Phys. Rev.*, D60:066001, 1999.

[5] D. Anselmi, D. Z. Freedman, Marcus T. Grisaru, and A. A. Johansen. Universality of the operator product expansions of SCFT(4). *Phys. Lett.*, B394:329–336, 1997.

[6] D. Anselmi and A. Kehagias. Subleading corrections and central charges in the AdS/CFT correspondence. *Phys. Lett.*, B455:155–163, 1999.

[7] Philip C. Argyres, Adel M. Awad, Gregory A. Braun, and F. Paul Esposito. Higher-derivative terms in N = 2 supersymmetric effective actions. *JHEP*, 07:060, 2003.

[8] Nima Arkani-Hamed, Lubos Motl, Alberto Nicolis, and Cumrun Vafa. The string landscape, black holes and gravity as the weakest force. *JHEP*, 06:060, 2007.

[9] Dumitru Astefanesei, Nabamita Banerjee, and Suvankar Dutta. (Un)attractor black holes in higher derivative AdS gravity. *JHEP*, 11:070, 2008.

[10] Ibrahima Bah, Alberto Faraggi, Juan I. Jottar, Robert G. Leigh, and Leopoldo A. Pando Zayas. Fermions and D=11 Supergravity On Squashed Sasaki-Einstein Manifolds. 2010.

[11] Koushik Balasubramanian and John McGreevy. Gravity duals for non-relativistic CFTs. *Phys. Rev. Lett.*, 101:061601, 2008.

[12] K. Behrndt, Ali H. Chamseddine, and W. A. Sabra. BPS black holes in N = 2 five dimensional AdS supergravity. *Phys. Lett.*, B442:97–101, 1998.

[13] K. Behrndt, Mirjam Cvetic, and W. A. Sabra. Non-extreme black holes of five dimensional N = 2 AdS supergravity. *Nucl. Phys.*, B553:317–332, 1999.

[14] Paolo Benincasa and Alex Buchel. Transport properties of N = 4 supersymmetric Yang-Mills theory at finite coupling. *JHEP*, 01:103, 2006.

[15] Paolo Benincasa, Alex Buchel, and Roman Naryshkin. The shear viscosity of gauge theory plasma with chemical potentials. *Phys. Lett.*, B645:309–313, 2007.

[16] E. A. Bergshoeff and M. de Roo. THE QUARTIC EFFECTIVE ACTION OF THE HETEROTIC STRING AND SUPERSYMMETRY. *Nucl. Phys.*, B328:439, 1989.

[17] Eric Bergshoeff et al. Weyl multiplets of N = 2 conformal supergravity in five dimensions. *JHEP*, 06:051, 2001.

[18] Eric Bergshoeff et al. N = 2 supergravity in five dimensions revisited. *Class. Quant. Grav.*, 21:3015–3042, 2004.

[19] B. Biran, A. Casher, F. Englert, M. Rooman, and P. Spindel. THE FLUCTUATING SEVEN SPHERE IN ELEVEN-DIMENSIONAL SUPERGRAVITY. *Phys. Lett.*, B134:179, 1984.

[20] Matthias Blau, K. S. Narain, and Edi Gava. On subleading contributions to the AdS/CFT trace anomaly. *JHEP*, 09:018, 1999.

[21] Nikolay Bobev, Nick Halmagyi, Krzysztof Pilch, and Nicholas P. Warner. Supergravity Instabilities of Non-Supersymmetric Quantum Critical Points. *Class.Quant.Grav.*, 27:235013, 2010.

[22] David G. Boulware and Stanley Deser. String Generated Gravity Models. *Phys. Rev. Lett.*, 55:2656, 1985.

[23] Peter Breitenlohner and Daniel Z. Freedman. Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. *Phys.Lett.*, B115:197, 1982.

[24] Peter Breitenlohner and Daniel Z. Freedman. Stability in Gauged Extended Supergravity. *Annals Phys.*, 144:249, 1982.

[25] M. S. Bremer, M. J. Duff, Hong Lu, C. N. Pope, and K. S. Stelle. Instanton cosmology and domain walls from M-theory and string theory. *Nucl. Phys.*, B543:321–364, 1999.

[26] Mauro Brigante, Hong Liu, Robert C. Myers, Stephen Shenker, and Sho Yaida. The Viscosity Bound and Causality Violation. *Phys. Rev. Lett.*, 100:191601, 2008.

[27] Mauro Brigante, Hong Liu, Robert C. Myers, Stephen Shenker, and Sho Yaida. Viscosity Bound Violation in Higher Derivative Gravity. *Phys. Rev.*, D77:126006, 2008.

[28] Yves Brihaye and Eugen Radu. Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method. *JHEP*, 09:006, 2008.

[29] Yves Brihaye and Eugen Radu. Five-dimensional rotating black holes in Einstein-Gauss-Bonnet theory. *Phys. Lett.*, B661:167–174, 2008.

[30] Alex Buchel. Resolving disagreement for eta/s in a CFT plasma at finite coupling. *Nucl. Phys.*, B803:166–170, 2008.

[31] Alex Buchel. Shear viscosity of boost invariant plasma at finite coupling. *Nucl. Phys.*, B802:281–306, 2008.

[32] Alex Buchel. Shear viscosity of CFT plasma at finite coupling. *Phys. Lett.*, B665:298–304, 2008.

[33] Alex Buchel and James T. Liu. Universality of the shear viscosity in supergravity. *Phys. Rev. Lett.*, 93:090602, 2004.

[34] Alex Buchel and James T. Liu. Gauged supergravity from type IIB string theory on Y(p,q) manifolds. *Nucl. Phys.*, B771:93–112, 2007.

[35] Alex Buchel, James T. Liu, and Andrei O. Starinets. Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory. *Nucl. Phys.*, B707:56–68, 2005.

[36] Alex Buchel and Robert C. Myers. Causality of Holographic Hydrodynamics. *JHEP*, 08:016, 2009.

[37] Alex Buchel, Robert C. Myers, and Aninda Sinha. Beyond eta/s = 1/4pi. *JHEP*, 03:084, 2009.

[38] Alex Buchel and Leopoldo A. Pando Zayas. Hagedorn vs. Hawking-Page transition in string theory. *Phys. Rev.*, D68:066012, 2003.

[39] Rong-Gen Cai. Gauss-Bonnet black holes in AdS spaces. *Phys. Rev.*, D65:084014, 2002.

[40] Rong-Gen Cai, Zhang-Yu Nie, Nobuyoshi Ohta, and Ya-Wen Sun. Shear Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling. *Phys. Rev.*, D79:066004, 2009.

[41] Rong-Gen Cai, Zhang-Yu Nie, and Ya-Wen Sun. Shear Viscosity from Effective Couplings of Gravitons. *Phys. Rev.*, D78:126007, 2008.

[42] A. Casher, F. Englert, H. Nicolai, and M. Rooman. THE MASS SPECTRUM OF SUPERGRAVITY ON THE ROUND SEVEN SPHERE. *Nucl. Phys.*, B243:173, 1984.

[43] Davide Cassani, Gianguido Dall'Agata, and Anton F. Faedo. Type IIB supergravity on squashed Sasaki-Einstein manifolds. *JHEP*, 05:094, 2010.

[44] Alejandra Castro, Joshua L. Davis, Per Kraus, and Finn Larsen. 5D Black Holes and Strings with Higher Derivatives. *JHEP*, 06:007, 2007.

[45] Alejandra Castro, Joshua L. Davis, Per Kraus, and Finn Larsen. String Theory Effects on Five-Dimensional Black Hole Physics. *Int. J. Mod. Phys.*, A23:613–691, 2008.

[46] Anna Ceresole, Gianguido Dall'Agata, Riccardo D'Auria, and Sergio Ferrara. Spectrum of type IIB supergravity on AdS(5) x T(11): Predictions on N = 1 SCFT's. *Phys. Rev.*, D61:066001, 2000.

[47] Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, and Robert C. Myers. Charged AdS black holes and catastrophic holography. *Phys. Rev.*, D60:064018, 1999.

[48] Christos Charmousis. Higher order gravity theories and their black hole solutions. *Lect. Notes Phys.*, 769:299–346, 2009.

[49] W. A. Chemissany, Mees de Roo, and Sudhakar Panda. alpha'-Corrections to Heterotic Superstring Effective Action Revisited. *JHEP*, 08:037, 2007.

[50] Jiunn-Wei Chen, Ying-Jer Kao, Debaprasad Maity, Wen-Yu Wen, and Chen-Pin Yeh. Towards A Holographic Model of D-Wave Superconductors. *Phys. Rev.*, D81:106008, 2010.

[51] Jiunn-Wei Chen, Ying-Jer Kao, and Wen-Yu Wen. Peak-Dip-Hump from Holographic Superconductivity. *Phys. Rev.*, D82:026007, 2010.

[52] Z. W. Chong, Mirjam Cvetic, H. Lu, and C. N. Pope. General non-extremal rotating black holes in minimal five- dimensional gauged supergravity. *Phys. Rev. Lett.*, 95:161301, 2005.

[53] E. Cremmer, B. Julia, and Joel Scherk. Supergravity theory in 11 dimensions. *Phys. Lett.*, B76:409–412, 1978.

[54] Sera Cremonini, Kentaro Hanaki, James T. Liu, and Phillip Szepietowski. Black holes in five-dimensional gauged supergravity with higher derivatives. *JHEP*, 12:045, 2009.

[55] Sera Cremonini, Kentaro Hanaki, James T. Liu, and Phillip Szepietowski. Higher derivative effects on eta/s at finite chemical potential. *Phys. Rev.*, D80:025002, 2009.

[56] Sera Cremonini, James T. Liu, and Phillip Szepietowski. Higher Derivative Corrections to R-charged Black Holes: Boundary Counterterms and the Mass-Charge Relation. *JHEP*, 1003:042, 2010.

[57] Mirjam Cvetic et al. Embedding AdS black holes in ten and eleven dimensions. *Nucl. Phys.*, B558:96–126, 1999.

[58] Mirjam Cvetic and Steven S. Gubser. Phases of R-charged black holes, spinning branes and strongly coupled gauge theories. *JHEP*, 04:024, 1999.

[59] Mirjam Cvetic, Hong Lu, C. N. Pope, A. Sadrzadeh, and Tuan A. Tran. Consistent SO(6) reduction of type IIB supergravity on S(5). *Nucl. Phys.*, B586:275–286, 2000.

[60] Mirjam Cvetic, Shin'ichi Nojiri, and Sergei D. Odintsov. Black hole thermodynamics and negative entropy in deSitter and anti-deSitter Einstein-Gauss-Bonnet gravity. *Nucl. Phys.*, B628:295–330, 2002.

[61] Maro Cvitan, Predrag Dominis Prester, Silvio Pallua, and Ivica Smolic. Extremal black holes in D=5: SUSY vs. Gauss-Bonnet corrections. *JHEP*, 11:043, 2007.

[62] B. de Wit and H. Nicolai. The Consistency of the S**7 Truncation in D=11 Supergravity. *Nucl.Phys.*, B281:211, 1987.

[63] Frederik Denef and Sean A. Hartnoll. Landscape of superconducting membranes. *Phys. Rev.*, D79:126008, 2009.

[64] Eric D'Hoker and Daniel Z. Freedman. Supersymmetric gauge theories and the AdS / CFT correspondence. pages 3–158, 2002.

[65] M. J. Duff, B. E. W. Nilsson, C. N. Pope, and N. P. Warner. ON THE CONSISTENCY OF THE KALUZA-KLEIN ANSATZ. *Phys. Lett.*, B149:90, 1984.

[66] M. J. Duff and C. N. Pope. CONSISTENT TRUNCATIONS IN KALUZA-KLEIN THEORIES. *Nucl. Phys.*, B255:355–364, 1985.

[67] M.J. Duff. ULTRAVIOLET DIVERGENCES IN EXTENDED SUPERGRAVITY. 1982.

[68] M.J. Duff and C.N. Pope. KALUZA-KLEIN SUPERGRAVITY AND THE SEVEN SPHERE. 1983.

[69] M.J. Duff and D.J. Toms. KALUZA-KLEIN KOUNTERTERMS. 1982.

[70] Ethan Dyer and Kurt Hinterbichler. Boundary Terms, Variational Principles and Higher Derivative Modified Gravity. *Phys. Rev.*, D79:024028, 2009.

[71] Kazem Bitaghsir Fadafan. R^2 curvature-squared corrections on drag force. *JHEP*, 12:051, 2008.

[72] Thomas Faulkner, Gary T. Horowitz, John McGreevy, Matthew M. Roberts, and David Vegh. Photoemission 'experiments' on holographic superconductors. *JHEP*, 03:121, 2010.

[73] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Correlation functions in the CFT(d)/AdS($d+1$) correspondence. *Nucl. Phys.*, B546:96–118, 1999.

[74] M. D. Freeman, C. N. Pope, M. F. Sohnius, and K. S. Stelle. HIGHER ORDER sigma MODEL COUNTERTERMS AND THE EFFECTIVE ACTION FOR SUPERSTRINGS. *Phys. Lett.*, B178:199, 1986.

[75] Peter G. O. Freund and Mark A. Rubin. Dynamics of Dimensional Reduction. *Phys. Lett.*, B97:233–235, 1980.

[76] Tomoyuki Fujita and Keisuke Ohashi. Superconformal tensor calculus in five dimensions. *Prog. Theor. Phys.*, 106:221–247, 2001.

[77] Masafumi Fukuma, So Matsuura, and Tadakatsu Sakai. Higher-derivative gravity and the AdS/CFT correspondence. *Prog. Theor. Phys.*, 105:1017–1044, 2001.

[78] Maxime Gabella, Jerome P. Gauntlett, Eran Palti, James Sparks, and Daniel Waldram. The central charge of supersymmetric AdS_5 solutions of type IIB supergravity. *Phys. Rev. Lett.*, 103:051601, 2009.

[79] Maxime Gabella, Jerome P. Gauntlett, Eran Palti, James Sparks, and Daniel Waldram. AdS_5 Solutions of Type IIB Supergravity and Generalized Complex Geometry. *Commun. Math. Phys.*, 299:365–408, 2010.

[80] Davide Gaiotto. N=2 dualities. 2009.

[81] Davide Gaiotto and Juan Maldacena. The gravity duals of N=2 superconformal field theories. 2009.

[82] Jerome P. Gauntlett, Seok Kim, Oscar Varela, and Daniel Waldram. Consistent supersymmetric Kaluza–Klein truncations with massive modes. *JHEP*, 04:102, 2009.

[83] Jerome P. Gauntlett, Eoin O Colgain, and Oscar Varela. Properties of some conformal field theories with M-theory duals. *JHEP*, 02:049, 2007.

[84] Jerome P. Gauntlett, Julian Sonner, and Toby Wiseman. Holographic superconductivity in M-Theory. *Phys. Rev. Lett.*, 103:151601, 2009.

[85] Jerome P. Gauntlett, Julian Sonner, and Toby Wiseman. Quantum Criticality and Holographic Superconductors in M- theory. *JHEP*, 02:060, 2010.

[86] Jerome P. Gauntlett and Oscar Varela. Consistent Kaluza-Klein Reductions for General Supersymmetric AdS Solutions. *Phys. Rev.*, D76:126007, 2007.

[87] Jerome P. Gauntlett and Oscar Varela. D=5 SU(2)xU(1) Gauged Supergravity from D=11 Supergravity. *JHEP*, 02:083, 2008.

[88] Jerome P. Gauntlett and Oscar Varela. Universal Kaluza-Klein reductions of type IIB to N=4 supergravity in five dimensions. *JHEP*, 06:081, 2010.

[89] Xian-Hui Ge, Yoshinori Matsuo, Fu-Wen Shu, Sang-Jin Sin, and Takuya Tsukioka. Viscosity Bound, Causality Violation and Instability with Stringy Correction and Charge. *JHEP*, 10:009, 2008.

[90] G. W. Gibbons, Sean A. Hartnoll, and C. N. Pope. Bohm and Einstein-Sasaki metrics, black holes and cosmological event horizons. *Phys. Rev.*, D67:084024, 2003.

[91] Amit Giveon, Dan Gorbonos, and Merav Stern. Fundamental Strings and Higher Derivative Corrections to d-Dimensional Black Holes. *JHEP*, 02:012, 2010.

[92] Marcus T. Grisaru and D. Zanon. SIGMA MODEL SUPERSTRING CORRECTIONS TO THE EINSTEIN- HILBERT ACTION. *Phys. Lett.*, B177:347, 1986.

[93] David J. Gross and John H. Sloan. The Quartic Effective Action for the Heterotic String. *Nucl. Phys.*, B291:41, 1987.

[94] David J. Gross and Edward Witten. Superstring Modifications of Einstein's Equations. *Nucl. Phys.*, B277:1, 1986.

[95] S. S. Gubser, Igor R. Klebanov, and A. W. Peet. Entropy and Temperature of Black 3-Branes. *Phys. Rev.*, D54:3915–3919, 1996.

[96] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from non-critical string theory. *Phys. Lett.*, B428:105–114, 1998.

[97] Steven S. Gubser. Breaking an Abelian gauge symmetry near a black hole horizon. *Phys. Rev.*, D78:065034, 2008.

[98] Steven S. Gubser. Heavy ion collisions and black hole dynamics. *Int. J. Mod. Phys.*, D17:673–678, 2008.

[99] Steven S. Gubser, Christopher P. Herzog, Silviu S. Pufu, and Tiberiu Tesileanu. Superconductors from Superstrings. *Phys. Rev. Lett.*, 103:141601, 2009.

[100] Steven S. Gubser, Igor R. Klebanov, and Arkady A. Tseytlin. Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory. *Nucl. Phys.*, B534:202–222, 1998.

[101] Steven S. Gubser and Silviu S. Pufu. The gravity dual of a p-wave superconductor. *JHEP*, 11:033, 2008.

[102] Steven S. Gubser, Fabio D. Rocha, and Pedro Talavera. Normalizable fermion modes in a holographic superconductor. *JHEP*, 10:087, 2010.

[103] Monica Guica, Lisa Huang, Wei Li, and Andrew Strominger. R**2 corrections for 5D black holes and rings. *JHEP*, 10:036, 2006.

[104] M. Gunaydin and N. Marcus. The Spectrum of the s**5 Compactification of the Chiral N=2, D=10 Supergravity and the Unitary Supermultiplets of U(2, 2/4). *Class. Quant. Grav.*, 2:L11, 1985.

[105] Jan B. Gutowski and Harvey S. Reall. General supersymmetric AdS(5) black holes. *JHEP*, 04:048, 2004.

[106] Jan B. Gutowski and Harvey S. Reall. Supersymmetric AdS(5) black holes. *JHEP*, 02:006, 2004.

[107] Kentaro Hanaki, Keisuke Ohashi, and Yuji Tachikawa. Supersymmetric Completion of an R^2 Term in Five- Dimensional Supergravity. *Prog. Theor. Phys.*, 117:533, 2007.

[108] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Building a Holographic Superconductor. *Phys. Rev. Lett.*, 101:031601, 2008.

[109] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Holographic Superconductors. *JHEP*, 12:015, 2008.

[110] Sean A. Hartnoll, Pavel K. Kovtun, Markus Muller, and Subir Sachdev. Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes. *Phys. Rev.*, B76:144502, 2007.

[111] S. W. Hawking and Simon F. Ross. Duality between Electric and Magnetic Black Holes. *Phys. Rev.*, D52:5865–5876, 1995.

[112] M. Henningson and K. Skenderis. The holographic Weyl anomaly. *JHEP*, 07:023, 1998.

[113] C. P. Herzog, P. K. Kovtun, and D. T. Son. Holographic model of superfluidity. *Phys. Rev.*, D79:066002, 2009.

[114] Christopher P. Herzog, Pavel Kovtun, Subir Sachdev, and Dam Thanh Son. Quantum critical transport, duality, and M-theory. *Phys. Rev.*, D75:085020, 2007.

[115] Christopher P. Herzog, Mukund Rangamani, and Simon F. Ross. Heating up Galilean holography. *JHEP*, 11:080, 2008.

[116] Diego M. Hofman. Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT. *Nucl. Phys.*, B823:174–194, 2009.

[117] Diego M. Hofman and Juan Maldacena. Conformal collider physics: Energy and charge correlations. *JHEP*, 05:012, 2008.

[118] P. Hoxha, R. R. Martinez-Acosta, and C. N. Pope. Kaluza-Klein consistency, Killing vectors, and Kaehler spaces. *Class. Quant. Grav.*, 17:4207–4240, 2000.

[119] Vivek Iyer and Robert M. Wald. Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev.*, D50:846–864, 1994.

[120] Ted Jacobson, Gungwon Kang, and Robert C. Myers. On black hole entropy. *Phys. Rev.*, D49:6587–6598, 1994.

[121] Yevgeny Kats, Lubos Motl, and Megha Padi. Higher-order corrections to mass-charge relation of extremal black holes. *JHEP*, 12:068, 2007.

[122] Yevgeny Kats and Pavel Petrov. Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory. *JHEP*, 01:044, 2009.

[123] Alexei Khavaev, Krzysztof Pilch, and Nicholas P. Warner. New vacua of gauged N = 8 supergravity in five dimensions. *Phys. Lett.*, B487:14–21, 2000.

[124] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen. The Mass Spectrum of Chiral N=2 D=10 Supergravity on S**5. *Phys. Rev.*, D32:389, 1985.

[125] P. Kovtun, D. T. Son, and A. O. Starinets. Viscosity in strongly interacting quantum field theories from black hole physics. *Phys. Rev. Lett.*, 94:111601, 2005.

[126] Pavel Kovtun, Dam T. Son, and Andrei O. Starinets. Holography and hydrodynamics: Diffusion on stretched horizons. *JHEP*, 10:064, 2003.

[127] Taichiro Kugo and Keisuke Ohashi. Supergravity tensor calculus in 5D from 6D. *Prog. Theor. Phys.*, 104:835–865, 2000.

[128] Hari K. Kunduri, James Lucietti, and Harvey S. Reall. Supersymmetric multi-charge AdS(5) black holes. *JHEP*, 04:036, 2006.

[129] James T. Liu and W. A. Sabra. Mass in anti-de Sitter spaces. *Phys. Rev.*, D72:064021, 2005.

[130] James T. Liu and Wafic A. Sabra. Hamilton-Jacobi Counterterms for Einstein-Gauss-Bonnet Gravity. *Class. Quant. Grav.*, 27:175014, 2010.

[131] James T. Liu and H. Sati. Breathing mode compactifications and supersymmetry of the brane-world. *Nucl. Phys.*, B605:116–140, 2001.

[132] James T. Liu and Phillip Szepietowski. Higher derivative corrections to R-charged AdS_5 black holes and field redefinitions. *Phys. Rev.*, D79:084042, 2009.

[133] James T. Liu, Phillip Szepietowski, and Zhichen Zhao. Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds. *Phys. Rev.*, D81:124028, 2010.

[134] James T. Liu, Phillip Szepietowski, and Zhichen Zhao. Supersymmetric massive truncations of IIb supergravity on Sasaki-Einstein manifolds. *Phys.Rev.*, D82:124022, 2010.

[135] Hong Lu, C. N. Pope, and Tuan A. Tran. Five-dimensional N = 4, SU(2) x U(1) gauged supergravity from type IIB. *Phys. Lett.*, B475:261–268, 2000.

[136] Kengo Maeda, Makoto Natsuume, and Takashi Okamura. Viscosity of gauge theory plasma with a chemical potential from AdS/CFT. *Phys. Rev.*, D73:066013, 2006.

[137] Juan Maldacena, Dario Martelli, and Yuji Tachikawa. Comments on string theory backgrounds with non- relativistic conformal symmetry. *JHEP*, 10:072, 2008.

[138] Juan Martin Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[139] J. Mas. Shear viscosity from R-charged AdS black holes. *JHEP*, 03:016, 2006.

[140] R. R. Metsaev and Arkady A. Tseytlin. Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor. *Nucl. Phys.*, B293:385, 1987.

[141] Thomas Mohaupt. Black hole entropy, special geometry and strings. *Fortsch. Phys.*, 49:3–161, 2001.

[142] Wolfgang Mueck and Alessio Celi. Private communications.

[143] Robert C. Myers. HIGHER DERIVATIVE GRAVITY, SURFACE TERMS AND STRING THEORY. *Phys. Rev.*, D36:392, 1987.

[144] Robert C. Myers, Miguel F. Paulos, and Aninda Sinha. Holographic Hydrodynamics with a Chemical Potential. *JHEP*, 06:006, 2009.

[145] Robert C. Myers, Miguel F. Paulos, and Aninda Sinha. Quantum corrections to eta/s. *Phys. Rev.*, D79:041901, 2009.

[146] Robert C. Myers and Jonathan Z. Simon. Black Hole Thermodynamics in Lovelock Gravity. *Phys. Rev.*, D38:2434–2444, 1988.

[147] Robert C. Myers and Oyvind Tafjord. Superstars and giant gravitons. *JHEP*, 11:009, 2001.

[148] Horatiu Nastase, Diana Vaman, and Peter van Nieuwenhuizen. Consistent nonlinear K K reduction of 11d supergravity on AdS(7) x S(4) and self-duality in odd dimensions. *Phys. Lett.*, B469:96–102, 1999.

[149] Horatiu Nastase, Diana Vaman, and Peter van Nieuwenhuizen. Consistency of the AdS(7) x S(4) reduction and the origin of self-duality in odd dimensions. *Nucl. Phys.*, B581:179–239, 2000.

[150] Shin'ichi Nojiri and Sergei D. Odintsov. Brane-world cosmology in higher derivative gravity or warped compactification in the next-to-leading order of AdS/CFT correspondence. *JHEP*, 07:049, 2000.

[151] Shin'ichi Nojiri and Sergei D. Odintsov. On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence. *Int. J. Mod. Phys.*, A15:413–428, 2000.

[152] Shin'ichi Nojiri and Sergei D. Odintsov. Anti-de Sitter black hole thermodynamics in higher derivative gravity and new confining-deconfining phases in dual CFT. *Phys. Lett.*, B521:87–95, 2001.

[153] Shin'ichi Nojiri, Sergei D. Odintsov, and Sachiko Ogushi. Cosmological and black hole brane world universes in higher derivative gravity. *Phys. Rev.*, D65:023521, 2002.

[154] Miguel F. Paulos. Higher derivative terms including the Ramond-Ramond five- form. *JHEP*, 10:047, 2008.

[155] Jacek Pawelczyk and Stefan Theisen. AdS(5) x S(5) black hole metric at O(alpha'**3). *JHEP*, 09:010, 1998.

[156] G. Policastro, D. T. Son, and A. O. Starinets. The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma. *Phys. Rev. Lett.*, 87:081601, 2001.

[157] C.N. Pope. CONSISTENCY OF TRUNCATIONS IN KALUZA-KLEIN. 1985.

[158] Matthew M. Roberts and Sean A. Hartnoll. Pseudogap and time reversal breaking in a holographic superconductor. *JHEP*, 08:035, 2008.

[159] John H. Schwarz. Covariant Field Equations of Chiral N=2 D=10 Supergravity. *Nucl. Phys.*, B226:269, 1983.

[160] E. Sezgin. THE SPECTRUM OF THE ELEVEN-DIMENSIONAL SUPERGRAVITY COMPACTIFIED ON THE ROUND SEVEN SPHERE. *Phys. Lett.*, B138:57–62, 1984.

[161] Alfred D. Shapere and Yuji Tachikawa. Central charges of N=2 superconformal field theories in four dimensions. *JHEP*, 09:109, 2008.

[162] Jonathan Z. Simon. HIGHER DERIVATIVE LAGRANGIANS, NONLOCALITY, PROBLEMS AND SOLUTIONS. *Phys. Rev.*, D41:3720, 1990.

[163] Kostas Skenderis, Marika Taylor, and Dimitrios Tsimpis. A consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure. *JHEP*, 06:025, 2010.

[164] D. T. Son. Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry. *Phys. Rev.*, D78:046003, 2008.

[165] Dam T. Son and Andrei O. Starinets. Hydrodynamics of R-charged black holes. *JHEP*, 03:052, 2006.

[166] Yuji Tachikawa. Black hole entropy in the presence of Chern-Simons terms. *Class. Quant. Grav.*, 24:737–744, 2007.

[167] C. Teitelboim and J. Zanelli. Dimensionally continued topological gravitation theory in Hamiltonian form. *Class. Quant. Grav.*, 4:L125, 1987.

[168] P.K. Townsend, K. Pilch, and P. van Nieuwenhuizen. Selfduality in Odd Dimensions. *Phys.Lett.*, 136B:38, 1984.

[169] Cumrun Vafa. The string landscape and the swampland. 2005.

[170] Justin F. Vazquez-Poritz. Drag force at finite 't Hooft coupling from AdS/CFT. 2008.

[171] Robert M. Wald. Black hole entropy is the Noether charge. *Phys. Rev.*, D48:3427–3431, 1993.

[172] James T. Wheeler. Symmetric Solutions to the Gauss-Bonnet Extended Einstein Equations. *Nucl. Phys.*, B268:737, 1986.

[173] D. L. Wiltshire. SPHERICALLY SYMMETRIC SOLUTIONS OF EINSTEIN-MAXWELL THEORY WITH A GAUSS-BONNET TERM. *Phys. Lett.*, B169:36, 1986.

[174] David L. Wiltshire. BLACK HOLES IN STRING GENERATED GRAVITY MODELS. *Phys. Rev.*, D38:2445, 1988.

[175] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[176] Barton Zwiebach. Curvature Squared Terms and String Theories. *Phys. Lett.*, B156:315, 1985.