Recent higher order QCD results : the β-function at 4-loops and the first moment of g_1 in $O(\alpha_S^3)$

S.A. Larin*, T. van Ritbergen† and J.A.M. Vermaseren‡

*Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312, Russia
† Randall Laboratory, University of Michigan, Ann Arbor, MI 48109, USA
‡ NIKHEF, P.O. Box 41882, 1009 DB, Amsterdam, The Netherlands

Abstract. We discuss the analytical calculation of the renormalization group β-function at the 4-loop order of perturbative Quantum Chromodynamics. In addition we have obtained the order α_s^3 contribution to the Ellis-Jaffe sum rule for the structure function g_1 of polarized deep inelastic lepton-nucleon scattering.

THE FOUR LOOP β-FUNCTION IN QCD

The renormalization group β-function in Quantum Chromodynamics (QCD) has a history of more than 20 years. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [1]. The two-loop QCD β-function was calculated in [2]. The three-loop QCD β-function was calculated in Ref. [3] within the minimal subtraction (MS) scheme [4]. The MS-scheme belongs to the class of massless schemes where the β-function does not depend on masses of the theory and (only) the first two coefficients of the β-function are scheme-independent. In spite of its scheme dependence at higher orders the β-function is an important object since it governs (within a given scheme) the scale dependence of the strong coupling constant which is the basic expansion parameter in perturbative calculations.

In this section we discuss the recent analytical four-loop calculation [5] of the QCD β-function in the $\overline{\text{MS}}$-scheme. The definition of the 4-dimensional β-function is

$$\begin{aligned}\frac{\partial a_s}{\partial \ln \mu^2} &= \beta(a_s)\\ &= -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6) \qquad (1)\end{aligned}$$

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in which $a_s = \alpha_s/4\pi = g^2/16\pi^2$, $g = g(\mu^2)$ is the renormalized strong coupling constant of the standard QCD Lagrangian. (We should note at this point that various other normalizations of the beta function coefficients β_i are often used.) μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme.

To calculate the β-function we need to calculate the renormalization constant Z_{a_s} of the coupling constant

$$a_B = Z_{a_s} a_s \tag{2}$$

where a_B is the bare (unrenormalized) charge. We obtain this renormalization constant in the 4-loop order by calculating the following three renormalization constants of the Lagrangian: Z_{hhg} for the ghost-ghost-gluon vertex, Z_h for the inverted ghost propagator and Z_g for the inverted gluon propagator. Then by virtue of the Ward identities one has $Z_{a_s} = Z_{hhg}^2/(Z_h^2 Z_g)$. This is from a calculational point of view one of the simplest ways to obtain Z_{a_s} at higher orders (but several other choices are possible as well)

The actual calculation of the renormalization constants Z_{hhg}, Z_h and Z_g in the 4-loop order is done using a technique based on the direct calculation of 4-loop massive vacuum (bubble) integrals (i.e. massive integrals with no external momenta). This technique which is described in more detail in Ref. [5] involves the introduction of an auxiliary mass parameter and provides a procedure that is well suited for the automatic evaluation of huge numbers of Feynman diagrams. This is of vital importance since there are approximately 50000 4-loop diagams contributing to the ghost-ghost-gluon vertex, ghost propagator and gluon propagator combined. The obtained $\overline{\mathrm{MS}}$ β-function for QCD reads

$$\begin{aligned}
\beta_0 &= 11 - \frac{2}{3} n_f \\
\beta_1 &= 102 - \frac{38}{3} n_f \\
\beta_2 &= \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \\
\beta_3 &= \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) n_f \\
&\quad + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) n_f^2 + \frac{1093}{729} n_f^3
\end{aligned} \tag{3}$$

where n_f is the number of (active) quark flavours and ζ is the Riemann zeta-function ($\zeta_3 = 1.2020569\cdots$) . In Ref. [5] the beta-function was obtained for an arbitrary compact semi-simple Lie group, but we quoted here only the result for QCD (i.e. the group SU(3)).

Another prominent renormalization group quantity, the quark mass anomalous dimension, has recently been calculated at 4-loops for an arbitrary compact semi-simple Lie group [6].

THE α_S^3 APPROXIMATION OF QCD TO THE ELLIS-JAFFE SUM RULE

Polarized deep inelastic electron-nucleon scattering is described by the hadronic tensor

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4 z e^{iqz}\langle p,s|J_\mu(z)J_\nu(0)|p,s\rangle = W_{\mu\nu}^{\text{spin average}}(x,Q^2) + \\ +i\epsilon_{\mu\nu\rho\sigma}q_\rho\left(\frac{s_\sigma}{p\cdot q}g_1(x,Q^2)+\frac{s_\sigma p\cdot q - p_\sigma q\cdot s}{(p\cdot q)^2}g_2(x,Q^2)\right) \quad (4)$$

Here J_μ is the electromagnetic quark current $x = Q^2/(2p\cdot q)$ is the Bjorken scaling variable and $Q^2 = -q^2$ is the square of the transferred momentum. $|p,s\rangle$ is the nucleon state. The polarization vector of the nucleon is expressed as $s_\sigma = \overline{U}(p,s)\gamma_\sigma\gamma_5 U(p,s)$ where $U(p,s)$ is the nucleon spinor.

In the present section we will focus on the first Mellin moment of the structure function g_1, the Ellis-Jaffe sum-rule. Moments of deep inelastic structure functions can be expressed [7] in terms of quantities that appear in the operator product expansion (OPE) of the two currents J_μ. In particular the Ellis-Jaffe sum-rule is expressed as

$$\int_0^1 dx g_1^{p(n)}(x,Q^2) = C^{\rm ns}(1,a_s(Q^2))(\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8) \\ +C^{\rm s}(1,a_s(Q^2))\exp\left(\int_{a_s(\mu^2)}^{a_s(Q^2)} da_s'\frac{\gamma^s(a_s')}{\beta(a_s')}\right)\frac{1}{9}a_0(\mu^2) \quad (5)$$

where the plus (minus) sign before $|g_A|$ corresponds to the proton (neutron) target. $C^{\rm s}$ and $C^{\rm ns}$ are the flavour singlet and non-singlet coefficient functions that appear in the relevant Operator Product Expansion. $\gamma^s(a_s)$ is the anomalous dimension of the axial singlet current (see further below). $\alpha_s = 4\pi a_s$ is the strong coupling constant. The proton matrix elements of the axial currents are defined as

$$\begin{array}{lclcl} |g_A|s_\sigma & = & 2\langle p,s|J_\sigma^{5,3}|p,s\rangle & = & (\Delta u - \Delta d)s_\sigma, \\ a_8 s_\sigma & = & 2\sqrt{3}\langle p,s|J_\sigma^{5,8}|p,s\rangle & = & (\Delta u + \Delta d - 2\Delta s)s_\sigma, \\ a_0(\mu^2)s_\sigma & = & \langle p,s|J_\sigma^5|p,s\rangle & = & (\Delta u + \Delta d + \Delta s)s_\sigma = \Delta\Sigma(\mu^2)s_\sigma. \end{array} \quad (6)$$

Here $|g_A|$ is the absolute value of the constant of the neutron beta-decay , $g_A/g_V = -1.2601 \pm 0.0025$ [8]. $a_8 = 0.579 \pm 0.025$ [8,9] is the constant of hyperon decays. We use the notation $\Delta q(\mu^2)s_\sigma = \langle p,s|\overline{q}\gamma_\sigma\gamma_5 q|p,s\rangle$, $q = u,d,s$, for the polarized quark distributions. We omit the contributions of the nucleon matrix elements for quarks heavier than the s-quark but it is straightforward to include them. The matrix element of the singlet axial current $a_0(\mu^2)$ can be redefined in a proper invariant way as a constant $\hat{a}_0$

$$\hat{a}_0 = \exp\left(-\int^{a_s(\mu^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)}\right) a_0(\mu^2) \equiv \Delta\Sigma_{\rm inv} \tag{7}$$

The singlet anomalous dimension $\gamma^s(a_s)$ determines the renormalization scale dependence of the axial singlet current i.e. $d[J^5_\sigma]_R/(d\ln\mu^2) = \gamma^s[J^5_\sigma]_R$ where subscript R means that a current is renormalized. Since $\hat{a}_0$ is renormalization group invariant it should be considered as a physical constant on the same ground as the constants g_A and a_8.

The flavour non-singlet contribution to the Ellis-Jaffe sum rule is known in the order a_s^3 from [10] where the polarized Bjorken sum rule $\int_0^1 dx(g_1^p - g_1^n)$ was calculated in this order. To obtain the singlet contribution to the Ellis-Jaffe sum rule in the a_s^3 order one needs to calculate C^s in the order a_s^3 and $\gamma^s(a_s)$ in the order a_s^4. The most difficult part of this calculation is to obtain $\gamma^s(a_s)$ in the a_s^4 order (since it is a 4-loop calculation) and this can be done with the same method that was used to obtain the β-function in the 4-loop order.

Further details on the calculations can be found in Ref. [11] where we obtained the following result for the Ellis-Jaffe sum rule

$$\int_0^1 dx g_1^{p(n)}(x,Q^2) = \left[1 + \left(\frac{\alpha_s}{\pi}\right) d_1^{\rm ns} + \left(\frac{\alpha_s}{\pi}\right)^2 d_2^{\rm ns} + \left(\frac{\alpha_s}{\pi}\right)^3 d_3^{\rm ns}\right] (\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8)$$
$$+ \left[1 + \left(\frac{\alpha_s}{\pi}\right) d_1^{\rm s} + \left(\frac{\alpha_s}{\pi}\right)^2 d_2^{\rm s} + \left(\frac{\alpha_s}{\pi}\right)^3 d_3^{\rm s}\right] \frac{1}{9}\hat{a}_0$$

$$\begin{aligned}
d_1^{\rm ns} &= -1 \\
d_2^{\rm ns} &= -\tfrac{55}{12} + \tfrac{1}{3}n_f \\
d_3^{\rm ns} &= -\tfrac{13841}{216} - \tfrac{44}{9}\zeta_3 + \tfrac{55}{2}\zeta_5 + n_f(\tfrac{10339}{1296} + \tfrac{61}{54}\zeta_3 - \tfrac{5}{3}\zeta_5) + n_f^2(-\tfrac{115}{648}) \\
d_1^{\rm s} &= (1/\beta_0)\left[-11 + n_f(\tfrac{8}{3})\right] \\
d_2^{\rm s} &= (1/\beta_0)^2\left[-\tfrac{6655}{12} + n_f(\tfrac{235}{2} + \tfrac{242}{3}\zeta_3) + n_f^2(-\tfrac{85}{18} - \tfrac{88}{9}\zeta_3) + n_f^3(\tfrac{16}{81} + \tfrac{8}{27}\zeta_3)\right] \\
d_3^{\rm s} &= (1/\beta_0)^3\left[-\tfrac{18422371}{216} - \tfrac{58564}{9}\zeta_3 + \tfrac{73205}{2}\zeta_5 + n_f(\tfrac{46351373}{1296} + \tfrac{312785}{54}\zeta_3 - \tfrac{113135}{9}\zeta_5)\right. \\
&\quad + n_f^2(-\tfrac{2353243}{432} - \tfrac{30976}{27}\zeta_3 + \tfrac{13310}{9}\zeta_5) + n_f^3(\tfrac{4647815}{11664} + \tfrac{22594}{243}\zeta_3 - \tfrac{220}{3}\zeta_5) \\
&\quad \left. + n_f^4(-\tfrac{235867}{17496} - \tfrac{2440}{729}\zeta_3 + \tfrac{320}{243}\zeta_5) + n_f^5(\tfrac{386}{2187} + \tfrac{32}{729}\zeta_3)\right]
\end{aligned} \tag{8}$$

where $\alpha_s = \alpha_s(Q^2)$, $\beta_0 = 11 - 2/3n_f$ is the 1-loop coefficient of the beta function and $\hat{a}_0 = \Delta\Sigma_{\rm inv}$ is the invariant matrix element of the singlet axial current defined in Eq. (7). n_f is the number of (active) quark flavours and ζ is the Riemann zeta-function. In particular, for $n_f = 3$ we find

$$\int_0^1 dx g_1^{p(n)} = \left[1 - \left(\frac{\alpha_s}{\pi}\right) - 3.583\left(\frac{\alpha_s}{\pi}\right)^2 - 20.215\left(\frac{\alpha_s}{\pi}\right)^3\right] (\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8)$$
$$+ \left[1 - 0.3333\left(\frac{\alpha_s}{\pi}\right) - 0.5496\left(\frac{\alpha_s}{\pi}\right)^2 - 4.4473\left(\frac{\alpha_s}{\pi}\right)^3\right] \frac{1}{9}\hat{a}_0 \tag{9}$$

TABLE 1. Second and third-order coefficients for the Ellis-Jaffe sum rule.

n_f	non-singlet		singlet	
	$(\alpha_s/\pi)^2$	$(\alpha_s/\pi)^3$	$(\alpha_s/\pi)^2$	$(\alpha_s/\pi)^3$
3	-3.58333	-20.21527	-0.54959	-4.44725
4	-3.25000	-13.85026	1.08153	4.87423
5	-2.91667	-7.84019	2.97845	13.07103
6	-2.58333	-2.18506	5.27932	20.73034

In table 1 we have listed the numerical values of the second and third-order coefficients for the Ellis-Jaffe sum rule for $n_f = 3, 4, 5, 6$. One can observe the sign-constant character of perturbative QCD series both for non-singlet and singlet contributions. The series tends to preserve its sign-constant character even when perturbative coefficients of the singlet contribution change their signs around the value $n_f = 4$.

One can see that the obtained perturbative coefficients of the Ellis-Jaffe sum rule grow rather moderately. If we assume that the error of the truncated asymptotic series is determined by the last calculated term, then the obtained α_s^3 approximation for this sum rule provides a good theoretical framework for extraction of the fundamental constant $\hat{a}_0 = \Delta\Sigma_{\text{inv}}$, the invariant axial proton charge, from experiment.

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