Azimuthal clumping instabilities in a *Z*-pinch wire array

Trevor Strickler, Y. Y. Lau, R. M. Gilgenbach, M. E. Cuneo,[a] and T. A. Mehlhorn[a]
Department of Nuclear Engineering and Radiological Sciences, University of Michigan, Ann Arbor, Michigan 48109-2104

(Received 8 November 2004; accepted 15 February 2005; published online 7 April 2005)

A simple model is constructed to evaluate the temporal evolution of azimuthal clumping instabilities in a cylindrical array of current-carrying wires. An analytic scaling law is derived, which shows that randomly seeded perturbations evolve at the rate of the fastest unstable mode, almost from the start. This instability is entirely analogous to the Jeans instability in a self-gravitating disk, where the mutual attraction of gravity is replaced by the mutual attraction among the current-carrying wires. [DOI: 10.1063/1.1886828]

I. INTRODUCTION

The most intense x-ray pulses in the world, with x-ray yield in the megajoule range and x-ray powers in the hundreds of terawatts have been generated by wire *Z* pinches and the Sandia *Z* machine.[1–5] The energy conversion efficiency, from wall plug to x rays, exceeds 10%. In virtually all *Z*-pinch experiments, ranging from low to high currents, and from low wire-number to high wire-number arrays, a host of hydromagnetic activities have been observed. These experiments, and their simulations and models,[6–8] focused mostly on the radial and axial perturbations of the wires, and these perturbations are in the form of radial jets and axial striations,[7–11] or of some peculiar caterpillar structures on the boundary of the metallic plumes.[12] In this paper, we focus on the azimuthal clumping instability that is unique to a high wire-number array. This instability is entirely analogous to the Jeans instability of self-gravitating systems,[13] with the gravitational attraction between matter being replaced by the mutual attraction of neighboring wires that carry currents in the same direction. This instability was revealed in our (unpublished) simulations of a 300-wire array,[14] and was previously studied by Samokhin[15] and by Hammer and Ryutov.[16] The analytic theory in Refs. 15 and 16 showed that the most unstable azimuthal clumping mode was one in which two neighboring wires paired up.

In this paper, we go one step further. We use a simple model to evaluate the temporal evolution of randomly seeded perturbations which are composed of all azimuthal clumping modes in a high wire-number array. We show that after about one e-fold, these azimuthal perturbations grow at a rate essentially determined by the fastest growing mode—one that corresponds to pairing of two neighboring wires, i.e., the π mode which is described below. A simple analytic scaling law for this temporal evolution is constructed and the theory is compared with our simulation. Toward the end of this paper, we shall comment on the various aspects of this instability.

II. MODEL

Consider an array of N wires, each carrying a current I_w in the z direction, arranged in a circle of radius R [Fig. 1(a)]. We assume that the wire radius r_w is much smaller than the wire separation $d=2\pi R/N$ and that the mass per unit length of the wire is m_L. In addition to $r_w \ll d \ll R$, we further assume that the backposts of the return current are sufficiently far away so that they have negligible effects on the dynamical evolution. In a continuum description, this array carries a z-directed surface current $K=I_w/d$ with a surface mass density $\sigma=m_L/d=Nm_L/2\pi R$. Without any perturbation, this cylindrical array undergoes a radially inward acceleration g ($g>0$) as a result of the self-magnetic field,

$$g=\frac{\mu_0 K^2}{2\sigma}=\frac{\mu_0 d K^2}{2m_L}=\frac{\mu_0 I_w^2}{2dm_L}=\frac{\mu_0 I_w K}{2m_L}=\frac{\mu_0 K^2 \pi R}{Nm_L}, \qquad (1)$$

where μ_0 is the free space permeability. In the Cartesian model, this array lies on the y axis [Fig. 1(b)]. There is a static equilibrium in such an infinite array. In the continuum limit, σg becomes the pressure on the current sheet as a result of the self-magnetic field $H_0=K/2$ [Fig. 1(b)].

In the cylindrical array of N wires, the displacement from equilibrium of the nth wire, ξ_n is related to that of the $(n+1)$th wire by $\xi_{n+1}=\xi_n \exp(-jm2\pi/N)$ according to the Floquet theorem on the eigenmodes in a periodic structure. The azimuthal mode number m is restricted in the solution of ξ_n,

$$\xi_n=\xi_0 e^{-jn(m2\pi/N)}, \quad m=0,1,2,\ldots,N/2, \qquad (2)$$

where ξ_0 is an arbitrary constant. We may use the magnetron terminology to designate the $m=0$ mode as the 2π mode and the $m=N/2$ as the π mode on account of the phase shift of the perturbation in neighboring wires.[17] In Eq. (2) and hereafter, $N/2$ is to be replaced by $(N-1)/2$ if N is an odd integer. In the Cartesian analog treated below [Fig. 1(b)], the phase shift between neighboring wires is $k_y d$, where $k_y=m/R=2\pi m/(Nd)$ is wave number in the y direction and m takes on the integer values specified by Eq. (2). We assume that there is no axial variation nor axial displacement of the wires.

To calculate the natural mode of oscillation in the Cartesian array [Fig. 1(b)], let (x_n, y_n) be the small signal dis-

[a]Sandia National Laboratories, Albuquerque, NM 87185.

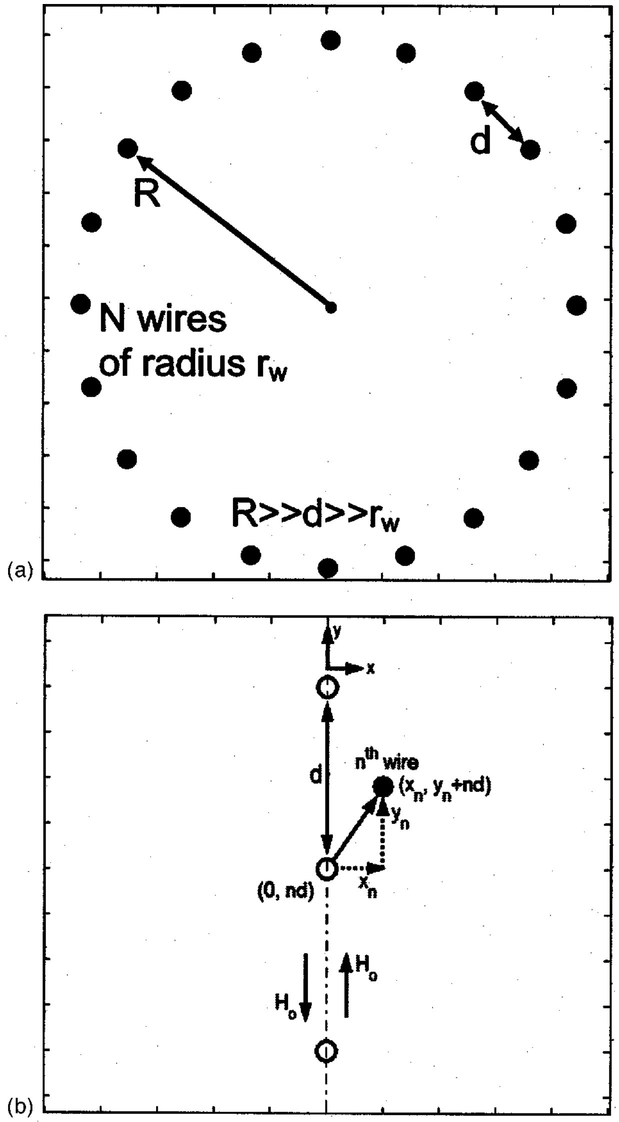

FIG. 1. (a) A circular array of N metallic wires, each carrying a current in the z direction, out of the plane of paper. (b) The Cartesian analog. Also shown is the displacement, x_n and y_n, of the nth wire from its unperturbed position $(x,y)=(0,nd)$.

placement of the nth wire from its equilibrium coordinates $(0,nd)$, where n takes on all positive and negative integers, and zero. It suffices to focus on the equation of motion for just one wire, say the $n=0$ wire, since ξ_n in Eq. (2) stands for both x_n and y_n. Let $\mathbf{S}_n=[x_n-x_0, nd+y_n-y_0]$ be the instantaneous vector from the zeroth wire to the nth wire [Fig. 2]. The force per unit length on the zeroth wire, by the parallel current on the nth wire, is easily shown to be $\mathbf{F}_n=(\mathbf{S}_n/S_n^2)\mu_0 I_w^2/(2\pi)$ by the Biot–Savart law, where S_n is the magnitude of $\mathbf{S}_n$. Since x_n (y_n) is related to x_0 (y_0) in the form of Eq. (2), $\mathbf{S}_n$, and therefore $\mathbf{F}_n$, are both functions of x_0 and y_0. Linearize $\mathbf{F}_n$ to first order in x_0 and y_0, sum $\mathbf{F}_n$ over all integer values of n (positive and negative) to obtain the total force per unit length on the zeroth wire, apply the following formulas to this infinite sum:

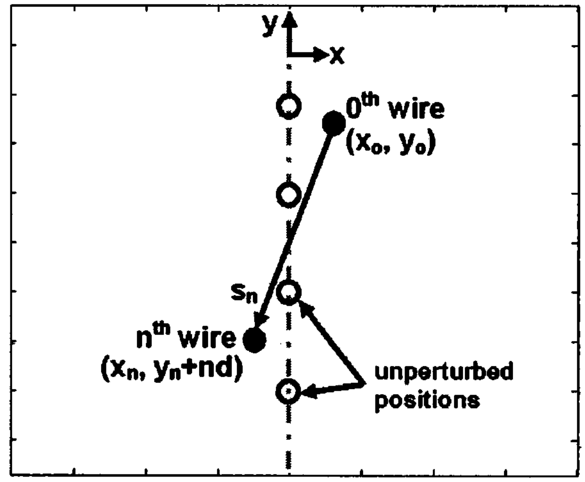

FIG. 2. The vector $\mathbf{S}_n$, from the instantaneous coordinates of the zeroth wire (x_0,y_0) to the instantaneous coordinates of the nth wire $(x_n, nd+y_n)$.

$$\sum_{n=1}^{\infty}\frac{\cos(zn)}{n^2}=\frac{\pi^2}{6}-\frac{\pi|z|}{2}+\frac{z^2}{4}, \quad -\pi\leqslant z\leqslant\pi$$

and obtain the linearized force law for the zeroth wire in component form,

$$\ddot{x}_0=-4\gamma_p^2\left[\frac{m}{N}\left(1-\frac{m}{N}\right)\right]x_0, \quad m=0,1,2,\ldots,N/2, \tag{3}$$

$$\ddot{y}_0=4\gamma_p^2\left[\frac{m}{N}\left(1-\frac{m}{N}\right)\right]y_0, \quad m=0,1,2,\ldots,N/2, \tag{4}$$

$$\gamma_p=\sqrt{\frac{\pi g}{2d}}, \tag{5}$$

where the dot denotes a time derivative. Note that γ_p is the growth rate of the π mode $(m=N/2)$ according to Eq. (4). In the above infinite sum, the nth term decays like $1/n^2$ and it represents the combined force on the zeroth wire by the nth and the $-n$th wire. Thus, a circular array of N wires may be adequately represented by a Cartesian array of infinite number of wires if $(N/4)^2 \gg 1$, as far as the azimuthal clumping instability is concerned, since in this case the wire at the top of the circular array, say, will at most experience the $N/4$ wires to its left, as well as the $N/4$ wires to its right.

From Eqs. (3) and (4), we note that the x and y components of motions are decoupled. This is obvious from the direction of $\mathbf{F}_n$, which, being parallel to $\mathbf{S}_n$ [Fig. 2], has a nonzero component in x (y) only if x_n (y_n) is nonzero. Note from Eq. (3) that in this planar model, the x-directed (radial) motion is always stable. This is also obvious from Fig. 1(b), which shows that there is no distinction if there were an acceleration either in the $+x$ direction or $-x$ direction. Thus, the sign of this acceleration should not matter as far as x-directed motions are concerned. The lateral motion, i.e., the y-directed or azimuthal motion, is unstable according to Eq. (4). This is the clumping instability, simply due to the fact that current filaments flowing in the same direction have a tendency to attract each other. The mechanism is entirely analogous to the Jeans instability in self-gravitating disks.[13]

Including the exponential time dependence $\exp(\gamma_m t)$, in Eq. (2), Eq. (4) then gives the growth rate of the lateral (clumping) instability of the mth mode as

$$\gamma_m = 2\gamma_p \sqrt{\frac{m}{N}\left(1-\frac{m}{N}\right)}, \quad m=0,1,2,\ldots,N/2. \tag{6}$$

Equation (4) also admits damping modes whose amplitudes vary as $\exp(-\gamma_m t)$. The 2π mode ($m=0$) is stable, it corresponds to a constant, static, azimuthal displacement on each wire and is thus ignored. From Eq. (6), we see that all other modes with $m>0$ have the growth rate increasing monotonically with m, with the π mode ($m=N/2$) having the highest growth rate γ_p. This π mode leads to the merging of two neighboring wires. Its growth rate is identical to Eq. 66 of Hammer and Ryutov,[16] its being the maximum value, which also agrees with the earlier work of Samokhin.[15] For small values of m, Eq. (6) may be approximated by Ref. 16, $\gamma_m = 2\gamma_p (m/N)^{1/2} = (k_y g)^{1/2}$, which is the same expression of the Jeans instability for a self-gravitating disk in the long wavelength limit.[13] Note that this Jeans instability in the wire Z pinch is robust, because the (large) value of k_y is *fixed* by the wire separation d as shown in Eq. (5).

Note that in a high wire-number array, the unstable modes are heavily crowded. For example, for $N=300$, all 50 modes with $m=100-150$ have their growth rates within 6% of the most unstable mode, the π mode ($m=150$), according to Eq. (6). Therefore, in the following section, we study the temporal evolution of initial perturbations, which we assume to be randomly and uniformly distributed among all modes, i.e., the spectral equivalent of white noise.

III. TEMPORAL EVOLUTION OF INITIAL RANDOM PERTURBATIONS

The equations of motion for the lateral displacements in the N-wire array may be described by a $2N$-tuple vector, $\mathbf{Y} = (y_1, y_2, \ldots, y_N, Dy_1, Dy_2, \ldots, Dy_N)$, where y_n is the lateral displacement and Dy_n is the lateral velocity of the nth wire at time t. This state vector $\mathbf{Y}$ may be represented as a linear superposition of the eigenvectors $\mathbf{e}_m$ and $\mathbf{f}_m$, respectively, of the growing modes and decaying modes,

$$\mathbf{Y}(t) = \sum_{m=1}^{N/2} a_m \mathbf{e}_m e^{\gamma_m t} + \sum_{m=1}^{N/2} b_m \mathbf{f}_m e^{-\gamma_m t}, \tag{7}$$

where the coefficients a_m and b_m depend on the initial condition at $t=0$, and all eigenvectors in the combined sets of $\{\mathbf{e}_m\}$ and $\{\mathbf{f}_m\}$ are independent and mutually orthonormal. The nth component of the eigenvectors $\mathbf{e}_m$ and $\mathbf{f}_m$ is in the form of Eq. (2).

We now assume that at time $t=0$, the wires are at rest but they have some initial random perturbations in their azimuthal positions. The condition of zero initial velocity on each wire implies that $|a_m| = |b_m|$ for all m in Eq. (7). If we further assume that the initial random azimuthal displacements are uniformly distributed among all modes, then all the a_m's and b_m's in Eq. (7) will have the same magnitude; that is, $|a_i| = |b_j| = |a_1|$ for all $i,j = 1,2,3,\ldots,N/2$. We next take the inner product of Eq. (7) to obtain,

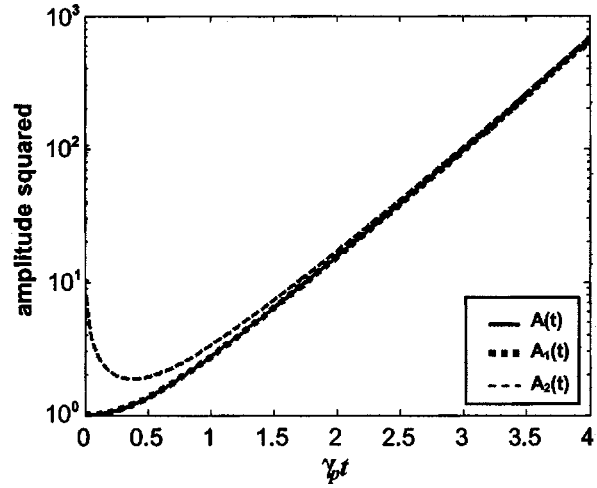

FIG. 3. Energy gain of randomly distributed initial perturbations in a $N=300$ array according to $A(t)$, $A_1(t)$, and $A_2(t)$. Note that the π mode growth rate is observed almost from the start.

$$A(t) \equiv \frac{|\mathbf{Y}(t)|^2}{|\mathbf{Y}(0)|^2} = \frac{2}{N}\sum_{m=1}^{N/2} \cosh(2\gamma_m t)$$
$$= \frac{2}{N}\sum_{m=1}^{N/2} \cosh\left[4\gamma_p t \sqrt{\frac{m}{N}\left(1-\frac{m}{N}\right)}\right], \tag{8}$$

upon using the mutual orthonormality of $\{\mathbf{e}_m\}$ and $\{\mathbf{f}_m\}$ and Eq. (6). Equation (8) gives the energy gain at time t from its initial value at time $t=0$; its square root gives the amplitude gain in the lateral (azimuthal) perturbations in the same time interval, as a result of the clumping (Jeans) instability.

Useful estimates may be obtained when there is a large number of wires. For $N \gg 1$, the finite sum in Eq. (8) may be approximated by an integral. Let $x=m/N$. Equation (8) may then be approximated by $A_1(t)$, where

$$A_1(t) = 2\int_0^{1/2} dx \cosh[4\gamma_p t\sqrt{x(1-x)}]. \tag{9}$$

A saddle point calculation of Eq. (9) yields a further approximation $A_2(t)$ to this integral,

$$A_2(t) = \frac{\sqrt{\pi}}{4}\frac{e^{2\gamma_p t}}{\sqrt{\gamma_p t}}. \tag{10}$$

Shown in Fig. 3 are the plots of $A(t)$, its approximation $A_1(t)$, and its further approximation $A_2(t)$, as a function of $\gamma_p t$ for an $N=300$ array. The excellent agreement between $A(t)$ and $A_1(t)$ is apparent, and for $\gamma_p t > 1$, even the simple asymptotic form of $A_2(t)$ appears adequate. Figure 3 shows that the randomly distributed initial perturbations evolve at the fastest growth rate γ_p of the π mode almost from the start, and that the simple scaling law, Eq. (10), is valid after one e-fold.

We may use Eq. (10) to estimate the amplitude gain for an inwardly accelerating array, for which the geometrical factors such as R and d [Fig. 1(a)], and therefore the "grav-

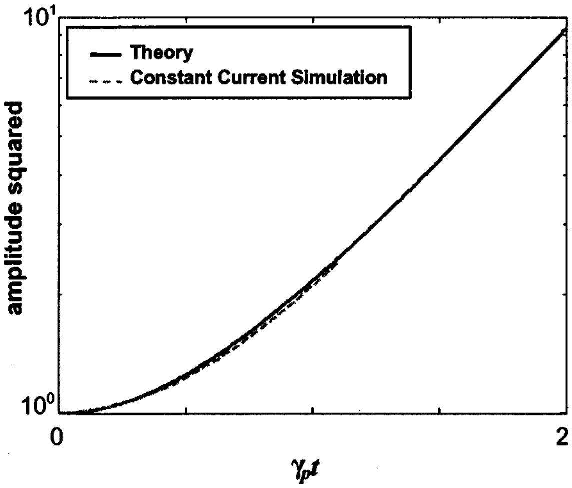

FIG. 4. Comparison of simulation of 300 wires with the analytic formula, Eq. (8).

ity" g are all functions of time. Taking the square root of Eq. (10), the expected amplitude gain over time t reads,

$$a(t) \equiv \sqrt{A(t)} \approx \frac{\pi^{1/4}}{2} \frac{e^{\Gamma}}{\Gamma^{1/4}}, \tag{11}$$

where

$$\Gamma = \int_0^t \gamma_p(t)dt = \int_0^t \sqrt{\frac{\pi g(t)}{2d(t)}}dt \tag{12}$$

is the total number of e-folds associated with the amplitude gain in the π mode. Once more, we expect that Eq. (11) provides a good approximation whenever $\Gamma > 1$ [Fig. 3].

Figure 4 shows a comparison between the analytic theory, Eq. (8), and a numerical simulation of a cylindrical array of $N=300$ wires. The simulations were based on the discrete wire inductance equations developed in Ref. 14, with a minor reformulation so that the total wire array current could be externally imposed as a driving term, and with the addition of terms to account for the change in wire geometry as the wires move. These time-dependent ordinary differential equations were simultaneously solved to model the motion of the wires under the force of the combined magnetic fields. The geometry was chosen to mimic typical experimental configuration used on Sandia's Z machine (a 1 cm long array 2 cm diameter with three hundred 11.5 μm tungsten wires,[2,4] all surrounded by a 2.4–3.0 cm diameter coaxial return conductor). Rather than using a fast current pulse as the driving current term, we used a constant total current 18 MA (about 60 kA per wire) for the simulation. We verified that each wire carried about the same current to within 5% maximum deviation over the course of our simulation run. Also, we seed the clumping instability with an initial randomly distributed azimuthal wire displacement with a standard deviation of 11.1 μm (typical perturbations from Ref. 2 are $\pm 21 \pm 4$ μm). To find the squared amplitude of the azimuthal clumping instability as a function of time, we calculate at each time step the sum of the squares of each interwire gap displacement from equilibrium. To account for the fact that the wire array implodes, which changes the ideal equilibrium interwire gap over the course of the simulation, we adjust at each time step the square displacement calculations accordingly. In Fig. 4, we show the square amplitude of the gap displacements as calculated by the simulation plotted along side the analytic theory we developed. As the figure illustrates, the theory very closely matches the discrete wire simulation, even though the theory is based on a linear wire array and the simulation was a cylindrical model. Note that the plotted simulation data covers 56 ns of simulation time, and that the transient growth is accurately predicted by the theory.

IV. REMARKS

In the discrete wire-array model studied in this paper, the azimuthal wave number of the fastest growing mode in the clumping (Jeans) instability is *predetermined* by the wire separation, as shown in Eq. (5). This is a unique feature, not shared by other types of instabilities whose axial wave number of the most virulent mode is often unknown. The most unstable clumping mode, designated in this paper as the π mode, is heavily crowded in a high wire-number array. We have constructed a time-domain solution which accounts for mode competition, establishes the transient growth, and provides the scaling law, all of which are confirmed by numerical simulations.

While our paper has focused only on the non-kink-type modes with $k=0$, where k is the axial wave number on the wire, Hammer and Ryutov[16] have shown that the π mode is the most potent among all instabilities in a wire array, as long as $kd < 1$, in which case the π-mode growth rate is also insensitive to the internal current distributions on each wire. This does not mean that three-dimensional effects are unimportant, however.

The driving mechanism of the Jeans instability is solely due to the axial current. In the wire Z-pinch experiments, this axial current, at some stage, may be shared or even predominantly carried by the plasma corona. When this happens, the metallic wires may be detached from the plasma corona, as far as the Jeans instability is concerned. Attention should then be shifted to the plasma corona, and the Jeans instability may acquire rather different characteristics from what is described in this paper. For example, the substantial thickness of the plasma corona (compared with the original wire size) is expected to have a stabilizing influence according to the concept of "plasma reduction factor" in the dynamics of electron sheets[18] and of self-gravitating disks.[19] Without an explicit specification of the current profile in the plasma corona, one can no longer identify the azimuthal mode number (and possibly the radial mode number) of the most unstable mode. The radial Rayleigh–Taylor instability, which is absent in the *metallic wire* array [cf. Eq. (3)] but might occur in the radially imploding *plasma* corona, may also mask or couple to the azimuthal instability that is studied in this paper.

ACKNOWLEDGMENTS

This work was supported by DOE through a subcontract to the University of Michigan from the Sandia National Laboratories. The research by one of the authors (T.S.) was performed under appointment to the Fusion Energy Sciences Fellowship program administered by the Oak Ridge Institute for Science and Education under a contract between the U.S. Department of Energy and the Oak Ridge Associated Universities.

[1] C. A. Coverdale, C. Deeney, and M. R. Douglas, Phys. Rev. Lett. **88**, 065001 (2002); Laser Part. Beams **19**, 497 (2001).
[2] M. E. Cuneo, R. A. Vesey, J. L. Porter *et al.*, Phys. Plasmas **8**, 2257 (2001).
[3] L. I. Rudakov, A. L. Velikovich, J. Davis *et al.*, Phys. Rev. Lett. **84**, 3326 (2000).
[4] M. E. Cuneo, R. A. Vesey, J. L. Porter, Jr. *et al.*, Phys. Rev. Lett. **88**, 215004 (2002).
[5] D. B. Sinars, M. E. Cuneo, E. P. Yu *et al.*, Phys. Rev. Lett. **93**, 145002–1 (2004).
[6] J. H. Hammer, M. Tabak, S. C. Wilks *et al.*, Phys. Plasmas **6**, 2129 (1999).
[7] J. P. Chittenden, S. V. Lebedev, S. N. Bland, J. Ruiz-Camacho, F. N. Beg, and M. G. Haines, Laser Part. Beams **19**, 323 (2001); Phys. Plasmas **8**, 2305 (2001); Phys. Rev. E **61**, 4370 (2000).
[8] M. P. Desjarlais and B. M. Marder, Phys. Plasmas **6**, 2057 (1999).
[9] K. M. Chandler, D. A. Hammer, D. B. Sinars, S. A. Pikus, and T. A. Shelkovenko, IEEE Trans. Plasma Sci. **30**, 577 (2002).
[10] S. V. Lebedev, F. N. Beg, S. N. Bland *et al.*, Phys. Rev. Lett. **85**, 98 (2000).
[11] S. A. Pikuz, T. A. Shelkovenko, A. R. Mingaleev, D. A. Hammer, and H. P. Neves, Phys. Plasmas **6**, 4272 (1999).
[12] M. D. Johnston, Y. Y. Lau, R. M. Gilgenbach, T. S. Strickler, M. C. Jones, M. E. Cuneo, and T. A. Mehlhorn, Appl. Phys. Lett. **83**, 4915 (2003).
[13] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, New York, 1960).
[14] T. S. Strickler, R. M. Gilgenbach, M. D. Johnston, and Y. Y. Lau, IEEE Trans. Plasma Sci. **31**, 1384 (2003).
[15] A. A. Samokhin, J. Appl. Mech. Tech. Phys. **29**, 243 (1988).
[16] J. H. Hammer and D. D. Ryutov, Phys. Plasmas **6**, 3302 (1999).
[17] N. M. Kroll and W. E. Lamb, J. Appl. Phys. **19**, 166 (1948).
[18] M. Chodorow and C. Susskind, *Fundamentals of Microwave Electronics* (McGraw-Hill, New York, 1964), p. 99.
[19] F. H. Shu, "The dynamics and large scale structures of spiral galaxies," Ph.D. thesis, Harvard University, 1968.