

Averaging of the Nonlinear Dynamics of Flapping Wing Micro Air Vehicles for Symmetrical Flapping

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First order equations of motion for a flapping wing micro-air vehicle are presented. The first order, longitudinal equations of motion are obtained from the second-order, strongly coupled, multi-body equations of motion using an approximate inverse. The nonlinear dynamics of the longitudinal equations of motion are averaged using two different methods: local averaging over a fully flapping cycle and local averaging over quarter-flapping cycles. Open loop simulations are presented, near a hover condition, for both averaged systems. The results for the locally averaged system are not consistent with the solution to the full system - suggesting that the neglecting of individual contributions to the dynamics, due to averaging over the entirety of a flapping cycle, is not a valid approach. Better results are obtained when the equations of motion are averaged over a quarter of a flapping cycle. The quarter-cycle results show a significant improvement in the accuracy of the position and orientation of the simulations versus the cycle averaged simulations.

Nomenclature

b_w	wing semi-span, m
c_w	wing chord, m
C_N	normal force coefficient for force and moment calculations
C_T	tangential force coefficient for force and moment calculations
f	flapping frequency of the wings, Hz
\bar{F}_{aero}	aerodynamic forces expressed in the body, B , frame, N
\mathbf{I}_i	mass moment of inertia matrix for the i th rigid body, kg-m ²
k_N	normal force coefficient for averaged forces and moments
k_T	tangential force coefficient for averaged forces and moments
m_i	mass of the i th rigid body
m_{sys}	total mass of the system
\bar{M}_{aero}	aerodynamic moments expressed in the B frame, N-m
\bar{M}_g	gravity moments due to the wings, expressed in the B frame, N-m
M_R, M_L	aerodynamic moments of the right and left wings in the B frame, N-m
p, q, r	angular velocity components of the body, rad/s
p_{RW}, q_{RW}, r_{RW}	angular velocity components of the right wing with respect to the right stroke plane, rad/s
p_{LW}, q_{LW}, r_{LW}	angular velocity components of the left wing with respect to the left stroke plane, rad/s
Q_{RW}, Q_{LW}	control moment to achieved desired acceleration/motion of the wings, kg-m ² /s ²
\mathbf{R}_B	rotation matrix from the inertial frame to body frame
$\mathbf{R}_R, \mathbf{R}_L$	rotation matrix from the right (left) stroke plane frame to the right (left) wing frame
\mathbf{R}_β	rotation matrix from the body frame to stroke plane frame
u, v, w	translational velocity of the central body expressed in the body frame with respect to the inertial frame, m/s
\bar{v}_i	velocity of the i th rigid body, m/s
X, Y, Z	inertial position of the central body, m
α_m	amplitude of the pitch angle, rad

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α_R, α_L	pitch angle of the right and left wings, rad
$\dot{\alpha}_R, \dot{\alpha}_L$	time derivative of the pitch angle, rad/s
β_R, β_L	stroke plane angle of the right and left wings, rad
δ_R, δ_L	deviation (elevation) angle of the right and left wings, rad
ζ_m	amplitude of the flapping angle motion, rad
ζ_R, ζ_L	flapping angle of the right and left wings, rad
$\dot{\zeta}_R, \dot{\zeta}_L$	time derivative of the flapping angle, rad/s
θ	pitch angle of the central body, rad
$\bar{\rho}_{ci}$	reference vector from the wing joint to wing center of mass in body frame, m
$\bar{\rho}_{ac,i}$	reference vector from the wing joint to wing aerodynamic center of pressure in wing frame, m
$\rho_{ci,x}, \rho_{ci,y}, \rho_{ci,z}$	components of the reference vectors $\bar{\rho}_{ci}$, m
$\rho_{ac,x}, \rho_{ac,y}, \rho_{ac,z}$	components of the reference vectors $\bar{\rho}_{ac,i}$, m
ω	angular frequency of the wings, $2\pi f$, rad/s
$\bar{\omega}_i$	angular velocity of the i th rigid body, rad/s
$\tilde{\omega}_{i,sp}$	skew (cross) matrix of the angular velocity of the wings with respect to the stroke plane
$\Omega_{RW,d}, \Omega_{LW,d}$	desired angular acceleration of the right and left wings, rad/s ²
<i>Subscript</i>	
i	rigid body number
j	generalized coordinate/quasi-velocity number
R	right wing
L	left wing

I. Introduction

The multibody dynamics of flapping wing micro-air vehicles (FWMAVs) are strongly coupled and only numerical solutions exist. The commonly reported approach to the dynamics of FWMAV is to ignore the mass of the wings, since the mass of the wings typically accounts for less than 5% of the total body mass¹ and the wings generally beat/flap fast enough to not excite the rigid body modes of the central body.² In Ref. 3, we showed that neglecting the mass of the wings may be problematic in dynamics and control studies - different dynamic behavior is exhibited by flight dynamics models that include the mass of the wings versus those that do not. Sun and Wang conclude in Ref. 4 that the rigid body approximation may not be suitable for dynamics models based on larger insects such as the hawkmoth and crane fly. Furthermore, in Ref. 5, Bolender states that for the proper development of control laws for FWMAVs, the mass of the wings needs to be included in the calculation.

The purpose of this study is to approximate the effects of the wings on the central body using averaging theory. The classic applications of averaging theory is to dynamical systems with ‘small perturbations.’ The multibody equations of motion, through the use of an approximate inverse, can be placed into first order form similar to the classic form presented in Refs. 6 and 7. The calculation of the first order equations results in the standard aircraft equations of motion plus perturbations of order ε and ε^2 , where the perturbations contain the inertial and mass effects of the wings on the central body. With proper simplifications based on an assumption of symmetrical flapping of the wings, with respect to the central body, and limits on the initial conditions for the system, the equations of motion can be reduced to longitudinal equations of motion for the FWMAV central body. The aerodynamic forces and moments generated by the motion of the wings are included in the standard aircraft equations of motion. The classic applications of averaging theory to dynamical systems is in the context of studying vibrations. Averaging has recently been applied in the context of flapping wing micro-air vehicles to the aerodynamic forces and moments in Refs. 8–18. The use of averaging theory in Refs. 8–18 is applied to the aerodynamic forces and moments to obtain ‘cycle’-averaged forces and moments for control studies. Averaging theory has been applied to the dynamics of biomimetic nonlinear systems in Refs. 19,20.

Averaging theory is applied to the entire flight dynamics equations of motion. Due to the assumption of symmetrical flapping, the equations of motion reduce to a set of longitudinal equations of motion. The initial averaging solution is based on local averaging, detailed in Ref. 21, where it is also referred to as ‘crude’ averaging. The local averaging is carried over the entirety of the flapping cycle, where the effects during the flapping cycle are summed. Averaging over the entirety of the flapping cycle tends to hide the nonlinear

effects of the system and results in poor accuracy of solutions, when compared to the numerical solution of the full dynamics. The derivation of $1/4$ -cycle averaged values and formulation of simpler functions for these values shows a significant improvement versus crude averaging for a whole flapping cycle. The accuracy of the numerical solutions improves in both position and orientation.

The paper will be detailed in the following manner. Section II will present a brief overview of the multibody flight dynamics model. Section III will detail the simplifications that can be made to the model in the case of symmetrical flapping. Section IV will present the averaged results for a hover condition using ‘crude’ averaging. Section V will present the averaged system in the case of a hover condition, but with the numerical solution calculated on the dynamics model averaged over $1/4$ -cycles. Section VI will present the conclusion of the paper and future work.

II. Multibody Flight Dynamics Model

The model used to obtain the equations of motion is derived using D’Alembert’s Principle for Multiple Rigid Bodies, a treatment of the method is available in Refs. 22–24. We used the method previously to derive the equations of motion for a two wing FWMAV in Ref. 25 and compared to a ‘wingless’ model in Ref. 3, a two wing FWMAV with a tail and control mass in Ref. 26, and a four wing FWMAV in Ref. 27. In brevity, the FWMAV is modeled as a system of three rigid bodies, two wings and a central body, where each wing is afforded three separate degrees of freedom. The degrees of freedom for the wings are the flapping angle, the pitch angle (relative to the stroke plane), and the deviation (elevation) angle. The total degrees of freedom for the system are twelve. The twelve generalized coordinates, \mathbf{q}_j , associated with the twelve degrees of freedom are

$$\mathbf{q}_j = \left[X \quad Y \quad Z \quad \psi \quad \theta \quad \phi \quad \delta_R \quad \alpha_R \quad \zeta_R \quad \delta_L \quad \alpha_L \quad \zeta_L \right]. \quad (1)$$

The associated quasi-velocities of the system are

$$\mathbf{u}_j = \left[u \quad v \quad w \quad p \quad q \quad r \quad p_{RW} \quad q_{RW} \quad r_{RW} \quad p_{LW} \quad q_{LW} \quad r_{LW} \right]. \quad (2)$$

The derivation of the equations of motion results in a strongly coupled system of second order equations. To perform any approximation analysis of the system, the equations need to be transformed into first order form. The equations of motion can be written in the following form

$$\mathbf{M}\dot{\mathbf{x}} = \begin{bmatrix} \bar{F}_{aero} + \bar{F}_g - \sum_{i=1}^3 (\dot{\mathbf{v}}_{i,red} + \ddot{\rho}_{ci,red}) \\ \bar{M}_{aero} + \bar{M}_g - \sum_{i=1}^3 (\mathbf{I}_i \bar{\omega}_1 \times \bar{\omega}_i + \bar{\omega}_i \times \mathbf{I}_i \bar{\omega}_1 + m_i \bar{\rho}_{ci} \times \dot{\mathbf{v}}_{i,red}) \\ Q_{RW} - (\mathbf{I}_2 \bar{\omega}_1 \times \bar{\omega}_2 + \bar{\omega}_2 \times \mathbf{I}_2 \bar{\omega}_2 + m_2 \bar{\rho}_{c2} \times \dot{\mathbf{v}}_{2,red}) \\ Q_{LW} - (\mathbf{I}_3 \bar{\omega}_1 \times \bar{\omega}_3 + \bar{\omega}_3 \times \mathbf{I}_3 \bar{\omega}_3 + m_3 \bar{\rho}_{c3} \times \dot{\mathbf{v}}_{3,red}) \end{bmatrix}, \quad (3)$$

where $\mathbf{M} \in \mathbb{R}^{12 \times 12}$ and $\dot{\mathbf{x}} \in \mathbb{R}^{12 \times 1}$. The components of $\dot{\mathbf{x}}$ are the time derivatives of the quasi-velocities as defined in Equation 2. The inverse of \mathbf{M} is approximated according to

$$\mathbf{M}^{-1} = (A + \varepsilon E)^{-1} = A^{-1} + \varepsilon A^{-1} E A^{-1}, \quad (4)$$

where ε is the mass of the wings and A is block diagonal. The use of the approximate matrix inversion results in the equations of motion being transformed to the following form

$$\dot{\mathbf{x}} = f^{(0)}(\mathbf{x}, t) + \varepsilon f^{(1)}(\mathbf{x}, t) + \varepsilon^2 f^{(2)}(\mathbf{x}, t), \quad (5)$$

where ε is the mass of the wings, as previously defined, and we use the notation used by Bogoliubov and Mitropolsky.⁷ The superscript in the notation of Equation 5 denotes the order of the term in reference to ε .

III. Reduced Model - Symmetrical Flapping

If the wings’ motion is symmetrical with respect to the central body, the equations can be significantly reduced. Furthermore, if the assumption is made that initial body roll angle, ϕ , and body angular velocities, p and r , are zero, then the equations of motion reduce to longitudinal equations of motion. The assumption

of symmetrical flapping requires several specifications. For one, the stroke plane angles of the wings are equal with respect to the body frame, such that $\beta_R = \beta_L$. Two, the wings angles are equal, such that $\delta_R = \delta_L$, $\alpha_R = \alpha_L$, and $\zeta_R = \zeta_L$. The resultant aerodynamic force in the body frame in the lateral direction, along the \hat{b}_y -axis, is identically zero. The resultant aerodynamic moments causing yaw and rolling motion are both identically zero. The detailed derivation of the cancellation of the aerodynamic forces and moments in the B frame is available in Ref. 3. Identical wing angles, and associated time derivatives of the wing angles, produces angular velocities where $p_{RW} = -p_{LW}$, $q_{RW} = q_{LW}$, and $-r_{RW} = r_{LW}$. Furthermore, with symmetrical flapping and wings with the same length and mass parameters, the inertia terms for $I_{xy,w}$ and $I_{yz,w}$ are equal in magnitude and opposite sign (where the appropriate wing is substituted for w). The remaining inertia tensor terms ($I_{xx,w}$, $I_{xz,w}$, $I_{yy,w}$, and $I_{zz,w}$) are equal in both sign and magnitude.

The right hand side of the equation in Equation 3 contains terms that need to be further defined. The terms $\dot{\mathbf{v}}_{i,red}$ and $\ddot{\rho}_{ci,red}$ are the reduced form of the acceleration of the i th rigid body ($\dot{\mathbf{v}}_i$) and the acceleration of the reference vector for the i th rigid body ($\ddot{\rho}_{ci}$). The reduced terms contain only x components and not any \dot{x} components; the \dot{x} components are absorbed into \mathbf{M} . The reduced form of the acceleration of the rigid bodies is calculated according to

$$\dot{\mathbf{v}}_{i,red} = m_i (\bar{\omega}_1 \times \bar{\mathbf{v}}_i). \quad (6)$$

The reduced form of the acceleration of the reference vector is calculated according to

$$\ddot{\rho}_{c2,red} = \mathbf{R}_{\beta_R}^T \mathbf{R}_R^T \tilde{\omega}_{2,sp} \tilde{\omega}_{2,sp} \bar{\rho}_{c2,w} + 2 \bar{\omega}_2 \times \left(\mathbf{R}_{\beta_R}^T \mathbf{R}_R^T \tilde{\omega}_{2,sp} \bar{\rho}_{c2,w} \right) + \bar{\omega}_2 \times (\bar{\omega}_2 \times \bar{\rho}_{c2}), \quad (7)$$

where the formulation for the right wing is used as an example. The terms Q_{RW} and Q_{LW} represent the control moments for the right and left wings. Using the approximate matrix inversion, we can calculate the control moments, to ε precision, required to achieve the desired value for the time derivative of the angular velocities of the wings. The control moments from the right wing are calculated according to

$$Q_{RW} = \mathbf{R}_{\beta_R}^T \mathbf{I}_2 \mathbf{R}_{\beta_R}^T \begin{bmatrix} \dot{p}_{RW,d} \\ \dot{q}_{RW,d} - \frac{M_R + M_L}{I_{yy,sys}} \\ \dot{r}_{RW,d} \end{bmatrix} \quad (8)$$

and for the left wing are calculated according to

$$Q_{LW} = \mathbf{R}_{\beta_L}^T \mathbf{I}_3 \mathbf{R}_{\beta_L}^T \begin{bmatrix} \dot{p}_{LW,d} \\ \dot{q}_{LW,d} - \frac{M_R + M_L}{I_{yy,sys}} \\ \dot{r}_{LW,d} \end{bmatrix}. \quad (9)$$

For future reference, we define the following

$$\Omega_{RW,d} = \begin{bmatrix} \dot{p}_{RW,d} \\ \dot{q}_{RW,d} - \frac{1}{I_{yy,sys}} (M_R + M_L) \\ \dot{r}_{RW,d} \end{bmatrix} \quad \text{and} \quad \Omega_{LW,d} = \begin{bmatrix} \dot{p}_{LW,d} \\ \dot{q}_{LW,d} - \frac{1}{I_{yy,sys}} (M_R + M_L) \\ \dot{r}_{LW,d} \end{bmatrix}, \quad (10)$$

where d in the subscript denotes the desired values of the time derivative of the wing angular velocities.

A. \dot{u} - longitudinal velocity of central body

The derivative of the longitudinal velocity of the central body, for symmetrical flapping, is calculated according to the following two equations. The effects of the translation and rotation of the central body on the derivative of the longitudinal velocity is

$$\dot{u}_1 = \frac{1}{m_{sys}} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{\rho_{c2,z} + \rho_{c3,z}}{I_{yy,sys}} & 0 \end{bmatrix} \begin{bmatrix} F_x - m_{sys} g \sin \theta - \left(\sum_{i=1}^3 (\dot{\mathbf{v}}_{i,red} + \ddot{\rho}_{ci,red}) \right) \cdot \hat{b}_x \\ 0 \\ F_z + m_{sys} g \cos \theta - \left(\sum_{i=1}^3 (\dot{\mathbf{v}}_{i,red} + \ddot{\rho}_{ci,red}) \right) \cdot \hat{b}_z \\ 0 \\ \left(\bar{M}_{aero} + \bar{M}_g \right) \\ - \sum_{i=1}^3 \left(\mathbf{I}_i \bar{\omega}_1 \times \bar{\omega}_i + \bar{\omega}_i \times \mathbf{I}_i \bar{\omega}_1 + m_i \bar{\rho}_{ci} \times \dot{\mathbf{v}}_{i,red} \right) \cdot \hat{b}_y \\ 0 \end{bmatrix}. \quad (11)$$

The effects of the wings are

$$\dot{u}_2 = \left(\left[\left(\tilde{\rho}_{c2}^T \mathbf{R}_{\beta_R}^T + \mathbf{R}_{\beta_R}^T \Theta_{RW} \right) \mathbf{R}_{\beta_R} \mathbf{I}_2^{-1} \mathbf{R}_{\beta_R} \quad \left(\tilde{\rho}_{c3}^T \mathbf{R}_{\beta_L}^T + \mathbf{R}_{\beta_L}^T \Theta_{LW} \right) \mathbf{R}_{\beta_L} \mathbf{I}_3^{-1} \mathbf{R}_{\beta_L} \right] \begin{bmatrix} Q_{RW} \\ Q_{LW} \end{bmatrix} \right) \cdot \hat{b}_x. \quad (12)$$

Putting the \dot{u} equation in the form of Equation 5, we can write

$$\dot{x}_1 = f_1^{(0)}(x, t) + \varepsilon f_1^{(1)}(x, t) + \varepsilon^2 f_1^{(2)}(x, t). \quad (13)$$

The first term, $f_1^{(0)}$ represents the standard equations of motion for an aircraft in the absence of yawing and rolling motion:

$$f_1^{(0)} = \frac{1}{m_{sys}} F_x - g \sin \theta - qw. \quad (14)$$

The second term, $f_1^{(1)}$, represents the first order effects of the wings on the motion of the central body:

$$\begin{aligned} f_1^{(1)} &= \frac{1}{m_{sys}} \left(\left(\tilde{\rho}_{c2}^T \mathbf{R}_{\beta_R}^T + \mathbf{R}_{\beta_R}^T \Theta_{RW} \right) \Omega_{RW,d} + \left(\tilde{\rho}_{c3}^T \mathbf{R}_{\beta_L}^T + \mathbf{R}_{\beta_L}^T \Theta_{LW} \right) \Omega_{LW,d} \right) \cdot \hat{b}_x \\ &\quad + \left((\rho_{c2,z} + \rho_{c3,z}) \frac{(M_R + M_L)}{m_{sys} I_{yy,sys}} \right) - \frac{1}{m_{sys}} (\ddot{\rho}_{c2,red} + \ddot{\rho}_{c3,red}) \cdot \hat{b}_x. \end{aligned} \quad (15)$$

The third term, $f_1^{(2)}$ represents the second order effects of the wings on the motion of the central body. The second order effects are

$$f_1^{(2)} = \frac{(\rho_{c2,z} + \rho_{c3,z})}{m_{sys} I_{yy,sys}} \left[\left(\bar{M}_g - \sum_{i=2}^3 (\mathbf{I}_i \bar{\omega}_1 \times \bar{\omega}_i + \bar{\omega}_i \times \mathbf{I}_i \bar{\omega}_i + m_i \bar{\rho}_{ci} \times \dot{\bar{\mathbf{v}}}_{i,red}) \right) \cdot \hat{b}_y \right]. \quad (16)$$

B. \dot{w} - vertical velocity of central body

The equations of motion for the vertical velocity of the central body as in Section A. As with the equation for \dot{u} , we desire to write \dot{w} in the following form:

$$\dot{x}_2 = f_2^{(0)}(x, t) + \varepsilon f_2^{(1)}(x, t) + \varepsilon^2 f_2^{(2)}(x, t). \quad (17)$$

The first term of \dot{x}_2 has the same form as the standard aircraft equations of motion for vertical velocity:

$$f_2^{(0)} = \frac{1}{m_{sys}} F_z + g \cos \theta + qw. \quad (18)$$

The first order effects of the wings on the translation on the vertical velocity of the central body are

$$\begin{aligned} f_2^{(1)} &= \frac{1}{m_{sys}} \left(\left(\tilde{\rho}_{c2}^T \mathbf{R}_{\beta_R}^T + \mathbf{R}_{\beta_R}^T \Theta_{RW} \right) \Omega_{RW,d} + \left(\tilde{\rho}_{c3}^T \mathbf{R}_{\beta_L}^T + \mathbf{R}_{\beta_L}^T \Theta_{LW} \right) \Omega_{LW,d} \right) \cdot \hat{b}_z \\ &\quad - \left((\rho_{c2,x} + \rho_{c3,x}) \frac{(M_R + M_L)}{m_{sys} I_{yy,sys}} \right) - \frac{1}{m_{sys}} (\ddot{\rho}_{c2,red} + \ddot{\rho}_{c3,red}) \cdot \hat{b}_z \end{aligned} \quad (19)$$

The second order effects of the wings on the vertical translation of the central body are

$$f_2^{(2)} = \frac{(\rho_{c2,x} + \rho_{c3,x})}{m_{sys} I_{yy,sys}} \left[\left(-\bar{M}_g + \sum_{i=2}^3 (\mathbf{I}_i \bar{\omega}_1 \times \bar{\omega}_i + \bar{\omega}_i \times \mathbf{I}_i \bar{\omega}_i + m_i \bar{\rho}_{ci} \times \dot{\bar{\mathbf{v}}}_{i,red}) \right) \cdot \hat{b}_y \right]. \quad (20)$$

C. \dot{q} - pitch velocity of central body

The equations of motion for \dot{q} can be placed in the same form as \dot{u} and \dot{w} , such that

$$\dot{x}_3 = f_3^{(0)}(x, t) + \varepsilon f_3^{(1)}(x, t) + \varepsilon^2 f_3^{(2)}(x, t). \quad (21)$$

The first term, $f_3^{(0)}$, is simply -

$$f_3^{(0)} = \frac{M_R + M_L}{I_{yy,sys}}. \quad (22)$$

The first order effects of the mass of the wings on the pitch velocity of the central body are

$$\begin{aligned} f_3^{(1)} &= \frac{1}{m_{sys}I_{yy,sys}} ((\rho_{c2,z} + \rho_{c3,z})(F_x - m_{sys}(g \sin \theta + qw)) - (\rho_{c2,x} + \rho_{c3,x})(F_z + m_{sys}(g \cos \theta + qu))) \\ &+ \frac{1}{I_{yy,sys}} \left(\bar{M}_g - \sum_{i=2}^3 (\mathbf{I}_i \bar{\omega}_1 \times \bar{\omega}_i + \bar{\omega}_i \times \mathbf{I}_i \bar{\omega}_i + m_i \bar{\rho}_{ci} \times \dot{\bar{\mathbf{v}}}_{i,red}) \right) \cdot \hat{\mathbf{b}}_y \\ &+ \frac{1}{I_{yy,sys}} \left(\mathbf{R}_{\beta_R}^T \mathbf{I}_2 \mathbf{R}_{\beta_R}^T \Omega_{RW,d} + \mathbf{R}_{\beta_L}^T \mathbf{I}_3 \mathbf{R}_{\beta_L}^T \Omega_{LW,d} \right) \cdot \hat{\mathbf{b}}_y. \end{aligned} \quad (23)$$

The second order effects of the mass of the wings on the pitch velocity of the central body are

$$\begin{aligned} f_3^{(2)} &= \frac{1}{m_{sys}I_{yy,sys}} \left((\rho_{c2,x} + \rho_{c3,x}) \left((\ddot{\rho}_{c2,red} + \ddot{\rho}_{c3,red}) \cdot \hat{\mathbf{b}}_z \right) \right. \\ &\quad \left. - (\rho_{c2,z} + \rho_{c3,z}) \left((\ddot{\rho}_{c2,red} + \ddot{\rho}_{c3,red}) \cdot \hat{\mathbf{b}}_x \right) \right). \end{aligned} \quad (24)$$

D. Periodic System

As detailed in the previous subsections, the approximate inverse when coupled with symmetrical flapping produces longitudinal equations of motion for the central body. The equations of motion are the standard aircraft equations of motion plus ε and ε^2 perturbations containing the inertial and mass effects of the wings on the system. The system of equations is periodic, such that

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}(t + T), \quad (25)$$

as long as the wing angles (α, ζ) , are themselves T -periodic. For example, the resultant aerodynamic force on the central body in the $\hat{\mathbf{b}}_x$ direction (the thrust force) with $\delta_R = \delta_L = 0$, is

$$\begin{aligned} F_x &= F_{x,R} + F_{x,L} \\ &= (\cos \beta_R \cos \alpha_R \cos \zeta_R - \sin \beta_R \sin \alpha_R) F_{T,R} + (\cos \beta_R \sin \alpha_R \cos \zeta_R + \sin \beta_R \cos \alpha_R) F_{N,R} \\ &\quad + (\cos \beta_L \cos \alpha_L \cos \zeta_L - \sin \beta_L \sin \alpha_L) F_{T,L} + (\cos \beta_L \sin \alpha_L \cos \zeta_L + \sin \beta_L \cos \alpha_L) F_{N,L}. \end{aligned} \quad (26)$$

which is T -periodic as long as the wing angles are T -periodic. The same periodic result can be shown for the resultant force in the $\hat{\mathbf{b}}_z$ direction, the aerodynamic moment affecting the pitch angle of the central body, and all of the $f^{(1)}$ and $f^{(2)}$ terms in the equations of motion.

IV. Symmetrical Hovering Using ‘Crude’ Averaging

The local averaging of the dynamic equations is obtained from the following equation

$$\dot{\hat{\mathbf{x}}}(t) = \frac{1}{T} \int_0^T f(\hat{\mathbf{x}}, \tau) d\tau, \quad (27)$$

where the implicit functions of time are assumed to remain constant and explicit functions of time are integrated over a specified period, T . For a hover condition, we make the assumption that only two degrees of freedom per wing are needed, such that the deviation angle from the stroke plane will remain zero, e.g. $\delta_R = \delta_L \equiv 0$. The assumption reduces the complexity of several terms in the equations of motion, to include the resultant aerodynamic forces and moments, the inertia tensors of the wings when expressed in the body frame, and the orientation of the centers of mass of the wings with respect to the origin of the body frame.

A. Aerodynamic Model

In order to average the equations of motion, an aerodynamic model is required. The aerodynamic model chosen is based on the model developed and presented by Deng, Schenato, et al. in Refs. 13 and 14. The

model is a quasi-steady/blade-element model that includes translational and rotational wing effects. We assume the flapping angle, ζ , and the pitch angle of the wing, α , have the following form

$$\zeta(t) = \zeta_m \sin(2\pi ft) \quad \text{and} \quad \alpha(t) = \alpha_m \text{sign}(\dot{\zeta}). \quad (28)$$

The angle of attack is constant during each half-stroke. The choice is made to simplify the calculation of the various integrals used to obtain the $1/4$ -cycle equations to be presented. It is important to note that a pitch angle of the form $\alpha(t) = \alpha_m \cos(2\pi ft)$ will produce qualitatively similar results. The normal and tangential forces on the wing generated by the motion of the wings are calculated according to

$$F_T = \frac{1}{2} \rho_{sl} A_w C_T (\hat{r}_2 b_w \omega \zeta_m)^2 \cos^2(\omega t) \quad (29)$$

and

$$F_N = \frac{1}{2} \rho_{sl} A_w C_N (\hat{r}_2 b_w \omega \zeta_m)^2 \cos^2(\omega t). \quad (30)$$

In Equations 29 and 30, $\hat{r}_2 b_w$ denotes the aerodynamic center of pressure of the wing for velocity calculations and $\omega = 2\pi f$, the flapping frequency. The aerodynamic center of pressure is based off of the geometry of the wing and is detailed in Refs. 13 and 14. The coefficients in Equations 29 and 30 are calculated according to

$$C_T = -0.4 \text{sign}(\dot{\zeta}) \cos^2(2\alpha) \quad \text{and} \quad C_N = -3.4 \text{sign}(\dot{\zeta}) \sin(\alpha). \quad (31)$$

In Equation 31, the signum function is used to ensure proper orientation of the forces on the wings. The flapping velocity of the wings is assumed to be much greater than the velocity of the freestream air, since the FWMAV is considered to be at (or near) a hovering condition.

B. Averaged Forces

Rotating from the wing frame to the stroke plane frame will not alter the time average of the aerodynamic forces, just the orientation of the forces in the body frame. We can examine, individually, the thrust and lift forces in the body frame for a parallel stroke plane frame without loss of generality. The resultant thrust and lift forces are

$$F_x = (\cos \alpha \cos \zeta) F_T + (\sin \alpha \cos \zeta) F_N \quad (32)$$

and

$$F_z = -(\sin \alpha) F_T + (\cos \alpha) F_N \quad (33)$$

The average of the forces is obtained according to

$$\bar{F}_x = \frac{1}{2\pi} \int_0^{2\pi} F_x d\tau \quad \text{and} \quad \bar{F}_z = \frac{1}{2\pi} \int_0^{2\pi} F_z d\tau. \quad (34)$$

The thrust and lift forces are not continuous over the interval $[0, 2\pi]$, but are continuous over each quarter-stroke. The average of the the lift and thrust forces is calculated by a summation of four integrals, each integral over a quarter-stroke: $[0, \frac{1}{2}\pi]$, $[\frac{1}{2}\pi, \pi]$, $[\pi, \frac{3}{2}\pi]$, and $[\frac{3}{2}\pi, 2\pi]$. The average of the thrust force in the stroke plane is identically zero. For simplicity in presentation, the following constants are defined (based on the averaged values):

$$k_T = 0.2 \rho_{sl} A_w \cos^2(2\alpha_m) (\hat{r}_2 b_w \omega \zeta_m)^2 \quad \text{and} \quad k_N = 1.7 \rho_{sl} A_w \sin(\alpha_m) (\hat{r}_2 b_w \omega \zeta_m)^2. \quad (35)$$

The average of the lift force in the stroke plane is calculated according to

$$\bar{F}_{z,sp} = -\frac{1}{2} k_T \sin(\alpha_m) - \frac{1}{2} k_N \cos(\alpha_m). \quad (36)$$

The integrals of individual terms are calculated using assistance from Refs. 28–30. Figure 1 shows the instantaneous lift force and the averaged lift force for a flapping frequency of 21 Hz, a flapping amplitude of 60 degrees, and a maximum angle of attack of 34.4212 degrees. The wings are considered to be thin, flat plates with constant chord. The dimensions of the wings are set to 51.9 mm for the semi-span and 18.9 mm for the chord. The average of the aerodynamic pitching moment is zero, over one flapping cycle, consistent with Refs. 9 and 11.

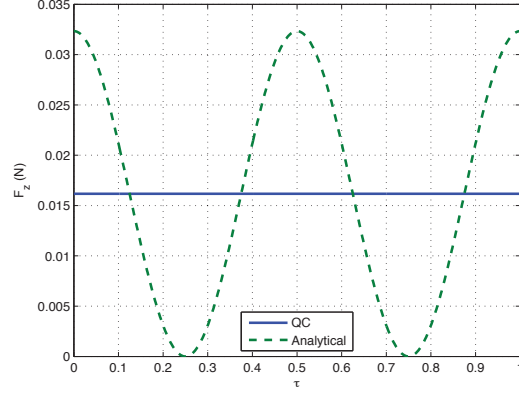


Figure 1. Instantaneous Lift Force and Average Lift Force

C. Averaged Equations and Simulation Results

To achieve the averaged equations of motion, we assume that the control moments produce the desired equations of motion of the wings. Therefore, we can define the angular velocity of the wings according to

$$\begin{bmatrix} p_{RW} \\ q_{RW} \\ r_{RW} \end{bmatrix} = \begin{bmatrix} \sin \alpha_R \dot{\zeta}_R \\ \dot{\alpha}_R \\ -\cos \alpha_R \dot{\zeta}_R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_{LW} \\ q_{LW} \\ r_{LW} \end{bmatrix} = \begin{bmatrix} -\sin \alpha_L \dot{\zeta}_L \\ \dot{\alpha}_L \\ \cos \alpha_L \dot{\zeta}_L \end{bmatrix}. \quad (37)$$

With symmetrical flapping, $p_{RW} = -p_{LW}$ and $-r_{RW} = r_{LW}$. The period for the calculation of the averaged equations is set to be $\frac{1}{f}$, where f denotes the flapping frequency. The local averaging produces the following equations of motion for the translation of the central body, where the averaged variables are denoted by a ‘bar,’

$$\dot{\bar{x}}_1 = \frac{\bar{F}_x}{m_{sys}} - g \sin \bar{\theta} - \bar{q}\bar{w} + \frac{2}{m_{sys}} m_w \rho_w \omega \zeta_m \sin(\alpha_m) \cos \beta \left(\frac{2 \sin(\zeta_m)}{\pi \zeta_m} \right) \bar{q} \quad (38)$$

$$+ \frac{1}{m_{sys} I_{yy,sys}} m_w \rho_w \hat{r}_2 b_w \sin \beta (1 - \mathbf{J}_0(2\zeta_m) - \mathbf{J}_2(2\zeta_m)) (k_T \sin(\alpha_m) - k_N \cos(\alpha_m)) \quad (39)$$

and

$$\dot{\bar{x}}_2 = \frac{\bar{F}_z}{m_{sys}} + g \cos \bar{\theta} + \bar{q}\bar{u} - \frac{2}{m_{sys}} m_w \rho_w \omega \zeta_m \sin(\alpha_m) \sin \beta \left(\frac{2 \sin(\zeta_m)}{\pi \zeta_m} \right) \bar{q} \quad (40)$$

$$- \frac{1}{m_{sys} I_{yy,sys}} m_w \rho_w \hat{r}_2 b_w \cos \beta (1 - \mathbf{J}_0(2\zeta_m) - \mathbf{J}_2(2\zeta_m)) (k_T \sin(\alpha_m) - k_N \cos(\alpha_m)) \quad (41)$$

where the terms ρ_w and m_w denote the distance to the center of mass of the wing from the wing joint and the mass of the wing, respectively. Through the calculation of numerous individual integrals, over the period T , the majority of the effects of the wings on the central body are zero, which is consistent with the method and results in Ref. 4. In order to calculate the average of the variable \dot{x}_3 , the term $I_{yy,sys}$ is averaged separately in order to obtain some form of an analytical averaging solution. The time-varying form of $I_{yy,sys}$, for symmetrical flapping, is

$$I_{yy,sys} = I_{yy,1} + 2 (I_{xx,w} \cos^2 \alpha + I_{yy,w} \cos^2 \zeta + I_{zz,w} \sin^2 \alpha \sin^2 \zeta). \quad (42)$$

The averaged result is

$$\bar{I}_{yy,sys} = I_{yy,1} + 2 I_{xx,w} \cos^2(\alpha_m) + I_{yy,w} (1 + \mathbf{J}_0(2\zeta_m)) + I_{zz,w} (\sin^2(\alpha_m) (1 - \mathbf{J}_0(2\zeta_m))). \quad (43)$$

As stated previously, the average of the aerodynamic pitching moment, over one flapping-cycle, is identically zero. Likewise, the average over one flapping cycle of all of the terms in $f_3^{(1)}$ is also zero. Averaging over one flapping cycle predicts no change in the pitch angle of the FWMAV. Therefore, the averaged equation for the pitch velocity of the FWMAV is

$$\dot{\bar{x}}_3 = 0. \quad (44)$$

The numerical solution for the perturbed system, to precision of ε , and the averaged system are presented in Figure 2. The morphological parameters of the FWMAV are based off a hawkmoth and the parameters presented in Ref. 31, specifically specimen F1. The FWMAV is started with an initial height of 5 m, initial pitch angle of 15° , stroke plane angle of -15° , and zero translational/rotational velocity. The flapping amplitude of the wings, ζ_m , is set to 60° . The amplitude of the pitch angle, α_m , is 34.4212° . The numerical

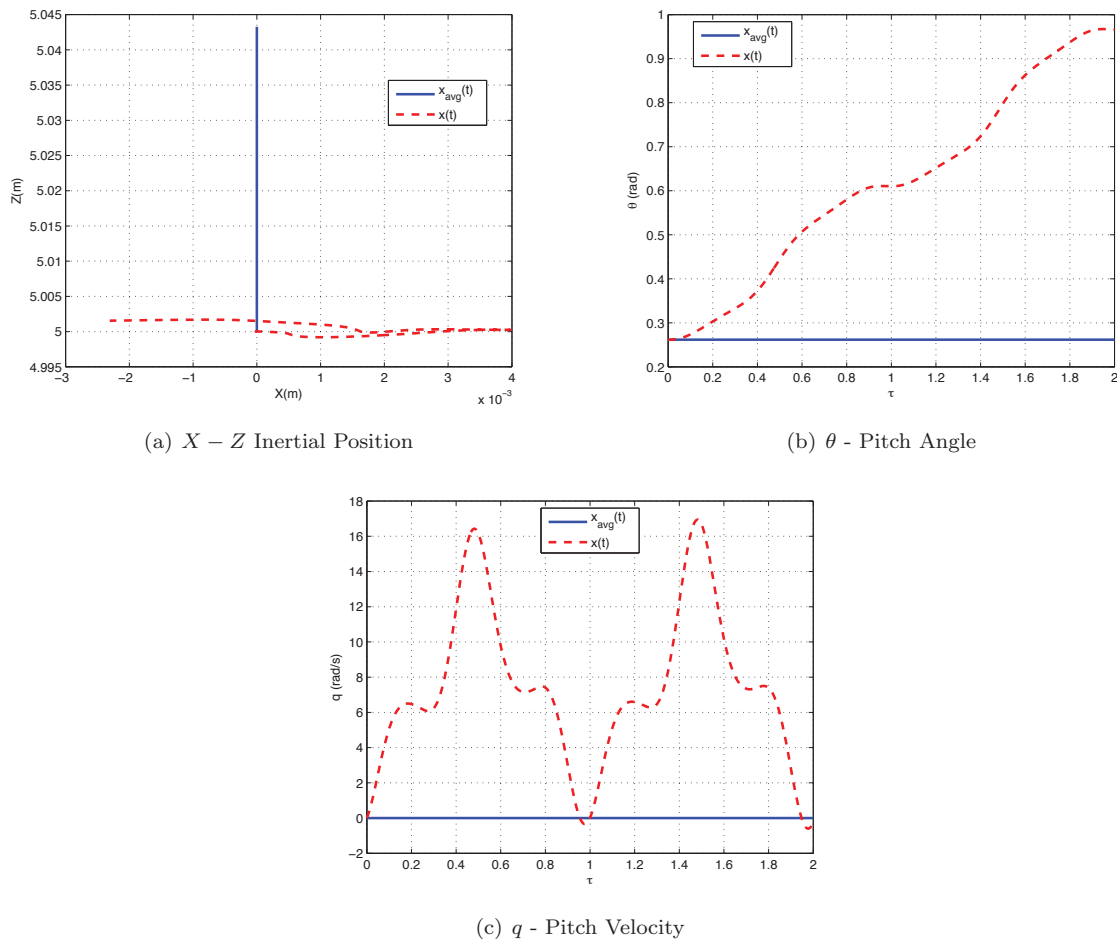


Figure 2. Simulation Results for Local Averaging, $\beta = -15^\circ$, $\theta_o = 15^\circ$

results in Fig. 2 are poor. The averaged equations do not match the first order equations of motion. Of special concern is the pitch angle of the FWMAV. Averaging, based on the first order equations of motion, predicts the change in the pitch attitude of the FWMAV to be zero for symmetrical flapping. The averaged equations predict a constant pitch angle. Without control, the pitch angle continues to increase. The pitch attitude change is definitely non-zero. Furthermore, the averaged equations predict an increase in altitude and little change in the longitudinal position.

V. Symmetrical Hovering Solution Using Averaging in $1/4$ Flapping Cycles

The standard form of differential equations for averaging is^{7,21}

$$\dot{x}(t) = \varepsilon f(x, t). \quad (45)$$

According to Ref. 21, crude averaging is not encouraged due to inaccurate results. In order to place an equation of the form in Equation 5 in the standard form, the analytical solution of the unperturbed system is required.²¹ The equation of the form in Equation 5, to ε precision, is solved for $\varepsilon = 0$ and a given initial condition. The explicit solution, denoted by $x = h(y, t)$ is then composed into the ε portion of the equation according to the method of variation of constants. The formula is

$$\frac{\partial h(t, y)}{\partial t} + \left[\frac{\partial h}{\partial y}(y, t) \right] \frac{dh}{dt} = f^{(0)}(h(y, t), t) + \varepsilon f^{(1)}(h(y, t), t, \varepsilon), \quad (46)$$

where $\left[\frac{\partial h}{\partial y} \right]$ denotes the Jacobian of the solution $x = h(y, t)$. The first terms on the left and right sides of the equation cancel and the resultant perturbation problem in the standard form is²¹

$$\dot{h} = \varepsilon \left[\frac{\partial h}{\partial y}(y, t) \right]^{-1} f^{(1)}(h(y, t), t, \varepsilon). \quad (47)$$

The issue with the perturbation problem in the standard form is the explicit solution of the unperturbed system. The analytical solution of the longitudinal aircraft equations of motion does not exist. An additional issue is that the aerodynamic forces and moments are generated by piecewise continuous explicit functions of time. The approach, presented here, is to utilize the knowledge of dynamic changes during a flapping cycle and use that knowledge to construct piecewise continuous equations that will accurately approximate the dynamics of the flapping micro-air vehicle. For example, in Section IV-C, the thrust force, in the stroke plane, and the aerodynamic pitching moment are identically zero when averaged over a flapping cycle. The thrust force may be written as

$$\bar{F}_{x,sp} = \frac{1}{2} \text{sign}(\dot{\zeta}) (k_T \cos(\alpha_m) + k_N \sin(\alpha_m)) (\mathbf{J}_0(\zeta_m) + \mathbf{J}_2(\zeta_m)), \quad (48)$$

where \mathbf{J}_n denotes a Bessel function of the first kind, order n . The aerodynamic pitching moment may be written as

$$M = (\sin \alpha \sin \zeta) \hat{r}_2 b_w F_T - (\cos \alpha \sin \zeta) \hat{r}_2 b_w F_N - \text{sign}(\dot{\zeta}) \frac{c_w}{4} (\cos \zeta) F_N. \quad (49)$$

When averaged over the four $1/4$ cycles, the result is

$$\begin{aligned} \bar{M}_{aero} &= \text{sign}(\zeta) \hat{r}_2 b_w \frac{1}{\zeta_m} \mathbf{H}_1(\zeta_m) (\cos(\alpha_m) k_N - \sin(\alpha_m) k_T) \\ &+ \frac{1}{8} \text{sign}(\dot{\zeta}) c_w k_N (\mathbf{J}_0(\zeta_m) + \mathbf{J}_2(\zeta_m)). \end{aligned} \quad (50)$$

\mathbf{H}_1 denotes the first order Struve function of the first kind. In the stroke plane, the averaged lift and $1/4$ -cycle averaged lift are identical. A comparison of the instantaneous thrust, lift, and aerodynamic pitching moment and the respective $1/4$ -cycle results are in Figures 3 and 4. The lift and thrust forces are resolved into the B frame of the FWMAV. As a result, the thrust force will not have a time average of zero, nor will the lift force be constant over a flapping cycle. If the stroke plane were identically zero, or parallel with the longitudinal axis of the body, then the lift force would be constant and the thrust would average to zero over the course of one flapping cycle.

The first order equations of motion are averaged, in their entirety, to produce piecewise continuous functions that will approximate the equations derived in Section III. The $1/4$ -cycle equations of motions are simulated versus the ε^1 equations presented in Section II. Two sets of simulation results are presented. The first set of results, in Figure 5, demonstrate the numerical solution from an initial height of 5 m, initial pitch angle of 0 rad, and zero initial translational and rotational velocities. The solid line is the $1/4$ -cycle solution, denoted by QC in the legend, and the dashed line is the ε^1 solution. The simulations are presented for three flapping cycles, with $\zeta_m = 60^\circ$, $\alpha_m = 34.4212^\circ$, and $f = 21Hz$.

Figure 5(a) shows the inertial position of the FWMAV. The positions are much closer in magnitude than the results presented in Section IV-C. Furthermore, a qualitative consistency is demonstrated between the two numerical solutions that is not present in the comparison between the cycle-averaged solution and the ε^1 solution. Figure 5(b) shows the pitch angle comparison and Fig. 5(c) shows the relationship between the pitch velocity. The pitch angles match closely for three flapping cycles and also demonstrate a qualitative consistency. After three flapping cycles, the error between the two predictions of the pitch angles is approximately 11.55%.

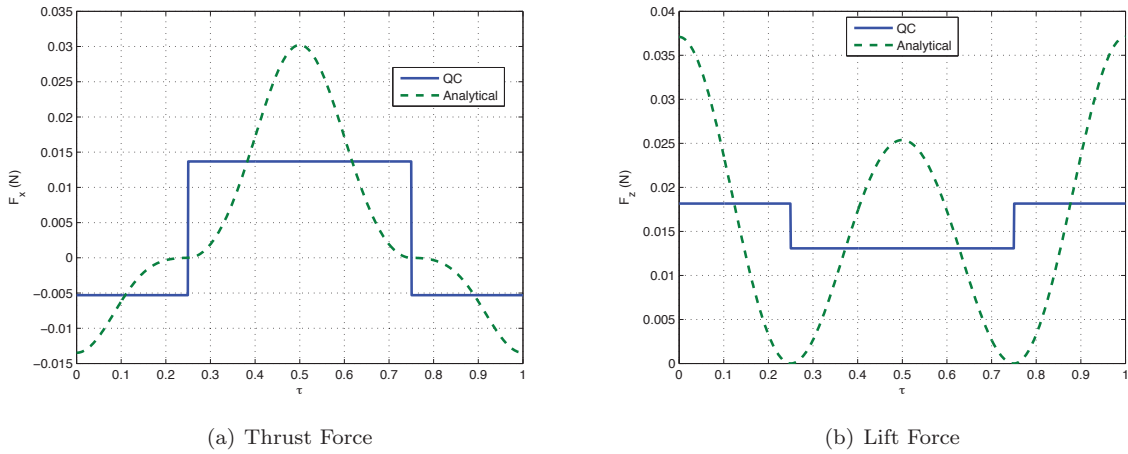


Figure 3. Comparison of Instantaneous and $1/4$ -Cycle Averaged Thrust and Lift

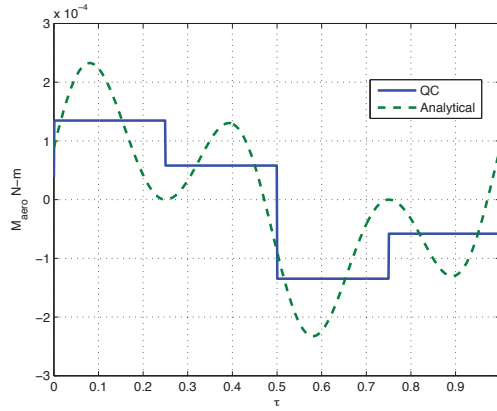


Figure 4. Comparison of Instantaneous and $1/4$ -Cycle Averaged Pitching Moment

Figure 6 shows the results for a stroke plane angle, β , of -25° , and initial pitch angle of 25° . The simulations are qualitatively consistent and the magnitude is an improvement over the fully cycle averaged solution. For a range of values of $\beta = 0 : -1 : -25$ and $\theta_o = 0 : 1 : 25$, the average error in the pitch angle at the end of three flapping cycles is 11.21° . The piecewise continuous equations obtained from the $1/4$ -cycle averaging methods produces an estimate of the longitudinal dynamics of a FWMAV that are superior to the results obtained from averaging the equations of motion over the entirety of a flapping cycle. The piecewise continuous equations may provide a more efficient basis for the computation of control algorithms and stability analyses of flapping wing micro air vehicles.

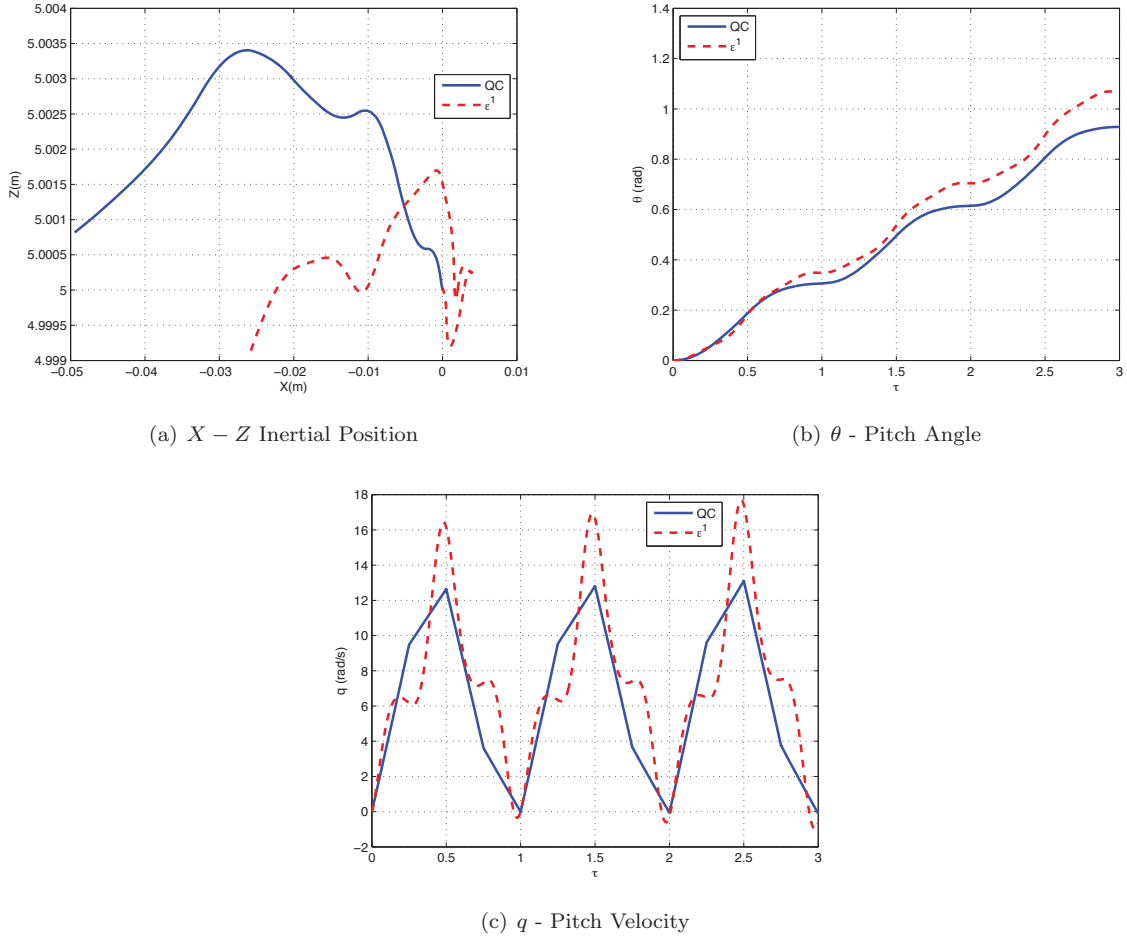


Figure 5. Simulation results for $1/4$ -Cycle Averaging, $\beta = 0$, $\theta_o = 0$

VI. Conclusion

The paper presented the averaging of the longitudinal equations of motion of a flapping wing micro-air vehicle. Averaging results were presented for two different averaging methods: local averaging over one flapping cycle and local averaging over $1/4$ -cycles. Combining the local averaging over $1/4$ -cycles results in a piecewise continuous description of the dynamics of the FWMAV. The accuracy of the results obtained by local averaging are not good, nor should they closely match the behavior. Without averaging in the standard form, no guarantees can be made about the accuracy of the approximation. Local averaging results in a time-averaged aerodynamic pitching moment equal to zero. The results lead to the conclusion that averaging of individual portions of the equations of motion, over the entirety of the flapping cycle, do not produce a good result when compared with the numerical solution of the full equations. The $1/4$ -cycle averaged results show an improvement over the full cycle averaged results. The piecewise continuous dynamics will enable further studies into controlling and stabilizing the dynamics of a flapping wing micro-air vehicle that include the all important inertial and mass effects of the wings.

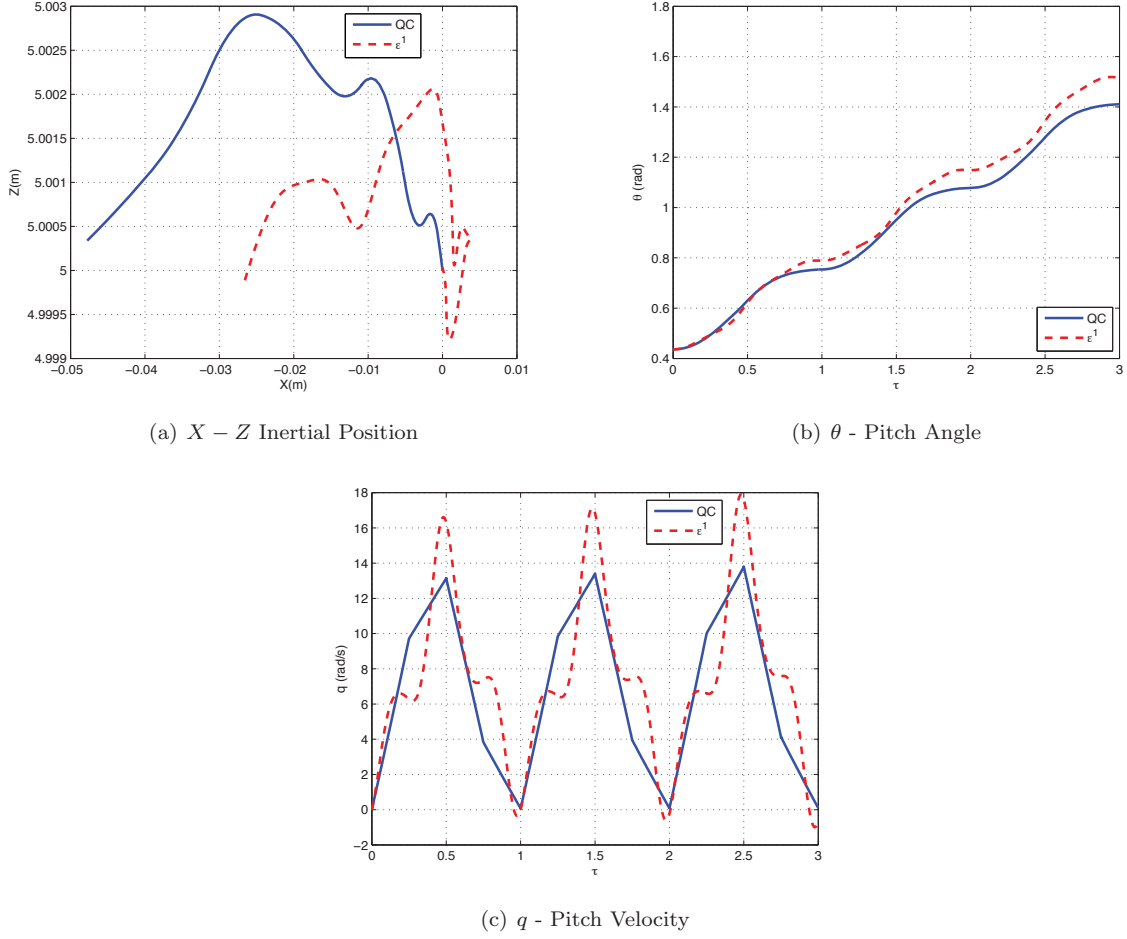


Figure 6. Simulation results for $1/4$ -Cycle Averaging, $\beta = -25^\circ$, $\theta_o = 25^\circ$

Appendix

The following terms, previously listed in the equations of motion defined throughout the paper, are defined for clarity. The reference vector from the wing joint to the center of mass of the wings are $\bar{\rho}_{c2}$ for the right wing and $\bar{\rho}_{c3}$ for the left wing. The distance from the wing joint to the center of mass of the wing in the wing frame is defined by ρ_i , where $i = 2$ for the right wing and $i = 3$ for the left wing. The skew-matrix representation of $\bar{\rho}_{ci}$ is denoted by $\tilde{\rho}_{ci}$. The definition of the skew-matrix and individual reference vectors are defined for the right wing

$$\tilde{\rho}_{c2} = \begin{bmatrix} 0 & -\rho_{c2,z} & \rho_{c2,y} \\ \rho_{c2,z} & 0 & -\rho_{c2,x} \\ -\rho_{c2,y} & \rho_{c2,x} & 0 \end{bmatrix}, \text{ where } \bar{\rho}_{c2} = \mathbf{R}_{\beta_R}^T \mathbf{R}_R^T \begin{bmatrix} 0 \\ \rho_2 \\ 0 \end{bmatrix}, \quad (51)$$

and

$$\tilde{\rho}_{c3} = \begin{bmatrix} 0 & -\rho_{c3,z} & \rho_{c3,y} \\ \rho_{c3,z} & 0 & -\rho_{c3,x} \\ -\rho_{c3,y} & \rho_{c3,x} & 0 \end{bmatrix} \text{ where } \bar{\rho}_{c3} = \mathbf{R}_{\beta_L}^T \mathbf{R}_L^T \begin{bmatrix} 0 \\ -\rho_3 \\ 0 \end{bmatrix}. \quad (52)$$

The rotation matrix from the right (left) stroke plane to the right (left) wing frame is defined by \mathbf{R}_R (\mathbf{R}_L). The rotation matrix from the body frame to the right (left) wing frame is defined by \mathbf{R}_{β_R} (\mathbf{R}_{β_L}). The fixed vectors defining the center the wing joint positions in the B frame are \bar{r}_R and \bar{r}_L . The associated

skew-matrices for each of the reference vectors are

$$\tilde{r}_R = \begin{bmatrix} 0 & -R_z & R_y \\ R_z & 0 & -R_x \\ -R_y & R_x & 0 \end{bmatrix} \quad \text{and} \quad \tilde{r}_L = \begin{bmatrix} 0 & -L_z & L_y \\ L_z & 0 & -L_x \\ -L_y & L_x & 0 \end{bmatrix}. \quad (53)$$

For the analysis presented, the wing joints are assumed to be mounted such that only R_y and L_y are non-zero, with the relation that $L_y = -R_y$ in the B frame. The following terms result from the calculation of the acceleration of the wing center of mass, with respect to an inertial fixed frame, in the B frame. The full calculation for the right wing reference vector, $\ddot{\bar{\rho}}_{c2}$ is

$$\ddot{\bar{\rho}}_{c2} = \frac{\partial}{\partial t} \dot{\bar{\rho}}_{c2} + \dot{\bar{\omega}}_2 \times \bar{\rho}_{c2} + \bar{\omega}_2 \times (\bar{\omega}_2 \times \bar{\rho}_{c2}). \quad (54)$$

In order to construct the matrix M defined in Section II, the following matrices are defined relating the acceleration of the wing angles ($\ddot{\delta}, \ddot{\zeta}, \ddot{\alpha}$) to the translational and rotational acceleration of the body of the FWMAV. The matrix Θ_{RW} denotes the contribution of the right wing and Θ_{LW} denotes the contribution of the left wing.

$$\Theta_{RW} = m_2 \rho_2 \begin{bmatrix} \mathbf{R}_R^T(1,3) & 0 & -\mathbf{R}_R^T(1,1) \\ \mathbf{R}_R^T(2,3) & 0 & -\mathbf{R}_R^T(2,1) \\ \mathbf{R}_R^T(3,3) & 0 & -\mathbf{R}_R^T(3,1) \end{bmatrix} \quad \text{and} \quad \Theta_{LW} = m_3 \rho_3 \begin{bmatrix} -\mathbf{R}_L^T(1,3) & 0 & \mathbf{R}_L^T(1,1) \\ -\mathbf{R}_L^T(2,3) & 0 & \mathbf{R}_L^T(2,1) \\ -\mathbf{R}_L^T(3,3) & 0 & \mathbf{R}_L^T(3,1) \end{bmatrix}. \quad (55)$$

The acceleration of the wing angles is contained in the terms $\frac{\partial}{\partial t} \dot{\bar{\rho}}_{c2}$ and $\dot{\bar{\omega}}_2 \times \bar{\rho}_{c2}$.

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