Decay of Solutions to the Wave Equation in Static Spherically Symmetric Spacetimes

by

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Gloria Patri, et Filio, et Spiritui Sancto.
Sicut erat in principio, et nunc, et semper, et in saecula saeculorum.
Amen.

To Audrey, with love and gratitude

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PREFACE

We study the decay of solutions to the wave equation in two general classes of static spherically symmetric spacetimes: black hole geometries and particle-like (non-singular, asymptotically flat) geometries. We prove decay in L^∞_{loc} as $t \to \infty$ for a broad class of black hole spacetimes and obtain analogous decay for a broad class of particle-like spacetimes. We also obtain a t^{-1} pointwise decay rate in the case of spherically symmetric initial data. Our results apply to particle-like and black hole solutions of the Einstein/Yang-Mills equations with gauge group $SU(2)$, thus yielding the first analysis of the long-time asymptotics of the wave equation in these spacetimes.

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CHAPTER I

Introduction

The decay of linear waves on various geometric backgrounds from general relativity (GR) is an area of much active research. Indeed, one of the great open questions in GR is the global nonlinear stability of a Kerr black hole.[1] Thus far, this problem has been considered too difficult, and energy has therefore been restricted to studying the linear stability of the Kerr metric. Investigating the linear stability of the Kerr metric requires studying the evolution of scalar waves, electromagnetic waves, gravitational waves, and Dirac particles on a Kerr background. These problems are also difficult and only partial results are known (more details can be found in the next section). However, two other simplifications may be made: one can restrict to the scalar wave case and one can consider the Schwarzschild metric (which is the special case of a spherically symmetric Kerr metric). The decay of scalar waves on a Schwarzschild background is the first of the problems mentioned above to have been completely analyzed.

Our goal in this research was to understand the stability of metrics appearing as

[1] We recall that, physically, a black hole is a region in spacetime from which even light rays cannot escape. The *event horizon* is the boundary of this region. One can state this mathematically by defining a black hole $\mathscr{B}$ within an asymptotically flat spacetime manifold $\mathscr{M}$ as $\mathscr{B} = \mathscr{M} \setminus J^-(\mathscr{I}^+)$, where $\mathscr{I}^+$ is future null infinity and $J^-(\mathscr{I}^+)$ is the causal past of future null infinity. A Kerr black hole is an axially symmetric black hole, and a Schwarzschild black hole is a spherically symmetric black hole. Since we will be primarily concerned with spherically symmetric spacetimes in this document, we informally think of a black hole as a manifold with a metric of the form $ds^2 = -T^{-2}(r)dt^2 + K^2(r)dr^2 + r^2 d\Omega^2$ with $K(r_0) = \infty$ and $K(r) > 0$ for $r > r_0$. We will define a class of spherically symmetric black holes precisely in Chapter III by introducing additional assumptions on the coefficients T, K.

solutions of the Einstein/Yang-Mills (EYM) equations. Taking note of the hierarchy of difficulty above and regarding these metrics as a particular example of a more general spherically symmetric metric, we endeavored to study the decay of the scalar wave equation on a spherically symmetric black hole (SSBH) background and on a spherically symmetric particle-like (SSPL) background (these backgrounds are defined precisely in later sections of this document). We were able to prove the decay of solutions to the scalar wave equation on a large class of such static spherically symmetric backgrounds.

We obtained our results by formulating this wave equation as a Hamiltonian evolution equation and deriving an integral spectral representation for the solution using the functional calculus. We then analyze this representation formula and use the Riemann-Lebesgue lemma to obtain decay. Our basic approach is not novel – it is in fact one of the standard approaches in obtaining decay results for linear evolution equations. The novelty in our work lies in the judicious use of coordinate changes so that this approach may be applied, as well as a determination of sufficient conditions to require of a spherically symmetric background to obtain decay. We acknowledge that it is desirable to also have a rate of decay with these results, but the generality of the metric backgrounds we allow precludes, for now, a more refined analysis which would yield more detailed decay information. We do, however, obtain decay rates for spherically symmetric initial data.

The rest of this document is organized as follows. In chapter II we give more detailed background and motivation, in chapter III we study the wave equation on an SSBH background, and in chapter IV we study the wave equation on an SSPL background.

CHAPTER II

Motivation & Background

2.1 Motivation

The classical (homogeneous) scalar wave equation in three spatial dimensions reads

$$\phi_{tt} - \Delta\phi = 0$$

for $x \in \mathbb{R}^3, t > 0$ and with initial data $\phi(x,0) = g(x)$ and $\phi_t(x,0) = h(x)$ for $x \in \mathbb{R}^3$. For regular enough g, h the solution ϕ of this equation is given by the so-called *Kirchhoff formula*, which reads

$$\phi(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y) \cdot (y - x)\right) dS(y)$$

for $x \in \mathbb{R}^3, t > 0$ (c.f. [18]).[1] In the case where g, h are smooth with compact support, one easily deduces the result $|\phi(x,t)| \leq c/t$ for each $x \in \mathbb{R}^3, t > 0$, where c is a (*uniform*) constant depending only on the data g, h. This implies therefore that for any fixed $x \in \mathbb{R}^3$, $\phi(x,t) \to 0$ as $t \to \infty$ with a rate t^{-1}.

One can also consider the wave equation on any other geometric background, but we will be primarily concerned with those coming from GR. Our motivation for this study is three-fold. First, whether the spacetimes coming from GR are stable or not comprise at least one of the great open questions in GR (c.f. [16]).

[1] Note: $B(x,t) := \{y \in \mathbb{R}^3 : |x - y| < t\}$.

Indeed, in the special cases of the Minkowski, Schwarzschild, and Kerr metrics (to be discussed more later), much effort has been devoted to answering this question. However, except in the simplest case of the Minkowski spacetime, determining the global nonlinear stability currently seems to be too difficult. Attention has therefore been focused on determining the linear stability of these metrics. For simplicity we consider scalar wave perturbations. Since linear stability of a spacetime under certain wave perturbations is equivalent to decay of solutions to the scalar wave equation on that spacetime, we investigate this linear stability by studying solutions of the wave equation.[2] Second, it is our hope that the results and methods contained herein might aid in studying the nonlinear stability of various spacetimes. Third, we believe that a more complete understanding of the wave equation in any physical situation is itself desirable.

2.2 Geometric Preliminaries

The mathematical description of the spacetimes coming from GR relies heavily on Riemannian geometry, and we therefore give an overview of the relevant topics from Riemannian geometry. We refer to [17] for a more comprehensive discussion of these ideas. Consider then a smooth four-dimensional manifold (spacetime) $\mathcal{M}$ endowed with a smooth metric g. In a given coordinate system, g has components g_{ij}, $0 \leq i, j \leq 3$. There is a unique, symmetric connection that is compatible with the metric g (i.e. covariantly constant). Written in coordinates, this connection (the Christoffel symbols) is given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

[2] We again reference [16] for a insightful discussion outlining the link between linear stability and wave equations.

(Note that the Einstein summation convention is to be applied above and throughout this document. This convention stipulates that repeated indices are summed over; e.g. $A_i B_i := \sum_j A_j B_j$, where the sum takes places over all valid indices j.) With the connection Γ, one can then define a coordinate invariant (covariant) notion of differentiation on the manifold $\mathcal{M}$. In particular, given a vector field T with coordinates T^i, we define the ij component of the covariant derivative of T to be

$$\nabla_j T^i = \frac{\partial T^i}{\partial x^j} + \Gamma^i_{kj} T^k.$$

Similarly for a covector S with coordinates S_i, we define the ij component of the covariant derivative of S to be

$$\nabla_j S_i = \frac{\partial S_i}{\partial x^j} - \Gamma^k_{ij} S_k.$$

Given a scalar field ϕ, we define the ith component of the covariant derivative of ϕ to be

$$\nabla_i \phi = \frac{\partial \phi}{\partial x^i},$$

which of course coincides with the usual notion of partial differentiation. One can then extend this notion of differentiation to the more general tensors of type (p, q), but we ignore this here since it is irrelevant to our study. The main utility of the definitions above is that these covariant derivatives transform like tensors. This is important because Einstein postulates that the equations of physics should be tensor equations (i.e., physics does not depend on the frame of reference), and a tensor equation is an equation that holds in all frames (more precisely, if a tensor is zero in one frame, it is zero in any other frame).

A covariant notion of differentiation allows us to define a covariant definition of the divergence operator. Given a vector field T with components T^i, one defines the divergence of T to be

$$\operatorname{div}(T) = \nabla_i T^i.$$

It can be shown that

$$\operatorname{div}(T) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} T^i \right),$$

where[3] $|g| := |\det g_{ij}|$. With a covariant definition of the divergence, we can then formulate a covariant notion of the Laplacian by computing the divergence of the covariant derivative of a scalar. Note however, that in a Lorentzian manifold (i.e. the manifold has signature $(1, -1, -1, -1)$), this might more appropriately be called a covariant generalization of the d'Alembert (wave) operator, since the resulting operator reduces to the wave operator on flat metrics. Since we are concerned with Lorentzian manifolds, we write $\Box$ rather than Δ for this operator. More precisely, given a scalar field ϕ, we define $\Box_g \phi$ by

$$\Box_g \phi = \nabla_i \left(g^{ij} \nabla_j \phi \right) = g^{ij} \nabla_i \nabla_j \phi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial \phi}{\partial x^j} \right), \tag{2.1}$$

where $g_{ij} g^{jk} = \delta_i^k$ (i.e. g^{ij} are the components of the inverse of the matrix (g_{ij})). We note that $\nabla_j \phi$ is a covector; since we wish to take the divergence of this object, we raise the index (transform it to a vector) by contracting with the inverse metric and obtain $g^{ij} \nabla_j \phi$. Of course, this doesn't fundamentally change anything since the metric is covariantly constant.

It is easy to see that if the metric g is the Minkowski metric (i.e., in the coordinates (t, x, y, z), $g = diag(1, -1, -1, -1)$) then $\Box_g = \Box$, the classical wave operator. So $\Box_g \phi = 0$ is indeed a generalization of the classical wave equation.

Now given a symmetric connection Γ on $\mathcal{M}$, we define the Riemann curvature

[3] Here we view the g_{ij} as components of a 4×4 matrix.

tensor by

$$R^i_{qkl} = -\left(\frac{\partial \Gamma^i_{ql}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^l} + \Gamma^i_{pk}\Gamma^p_{ql} - \Gamma^i_{pl}\Gamma^p_{qk}\right).$$

With the Riemann curvature tensor one then defines the Ricci tensor by $R_{ql} = R^i_{qil}$ and the scalar curvature by $R = g^{lq}R_{ql}$. Einstein's equations (without cosmological constant) then read

$$R_{ab} - \frac{1}{2}Rg_{ab} = \lambda T_{ab}, \tag{2.2}$$

where the left-hand side is typically denoted by G_{ab} and is referred to as the Einstein tensor, $\lambda = \frac{8\pi G}{c^4}$ is a universal constant, and T_{ab} is the stress-energy tensor (the stress-energy tensor is where one encapsulates the energy content of the space). It is a remarkable fact that the Einstein tensor is divergence free (i.e., $\nabla_i G^i_j = 0$), and this in turn guarantees *automatically* the conservation of energy and momentum, since $\nabla_i G^i_j = 0$ implies $\nabla_i T^i_j = 0$, and these equations imply the conservation of momentum and energy (c.f. [1]). Note that the Einstein equations are thus equations for the metric g (albeit complicated and highly nonlinear equations).

Of particular importance are the Einstein equations *in vacuo* (i.e. $T_{ab} \equiv 0$). There are two famous (non-flat) solutions to the Einstein equations in empty space: the Schwarzschild solution and the Kerr solution. The Schwarzschild solution describes a spherically symmetric black hole spacetime. In spatially spherical coordinates (t, r, θ, φ) the line element of the Schwarzschild metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2.3}$$

where M is related to the mass m of the black hole by $M = \frac{Gm}{c^2}$ (note that using the assumed symmetry of the metric (i.e. $g_{ij} = g_{ji}$), one can obtain the metric coefficients from the line element by observing that $ds^2 := \langle v, v\rangle_g$ for $v = (dt, dr, d\theta, d\varphi)^t$). Now

(2.3) describes a static, spherically symmetric geometry with a singularity at $r = 0$ and $r = 2M$, but the singularity at $r = 2M$, the event horizon, is an artifact of the coordinates and can be transformed away. This can be done, for example, by employing the so-called Kruskal coordinates (see [1]). The singularity at $r = 0$ cannot, however, be transformed away. To see this, we refer to [4], where it is shown that $R_{abcd}R^{abcd} = \frac{48m^2}{r^6}$ (though a simple calculation in a computer algebra system yields the same result). Since this number must be coordinate invariant, the Schwarzschild metric is singular at $r = 0$.

The Kerr solution describes an axially symmetric black hole spacetime. Its representation in coordinates is therefore more complex than the Schwarzschild metric. The line element is given by

$$(2.4) \quad ds^2 = \frac{d}{U}\left(dt - a\sin^2\theta d\varphi\right)^2 - U\left(\frac{dr^2}{d} + d\theta^2\right) - \frac{\sin^2\theta}{U}\left(adt - (r^2 + a^2)d\varphi\right)^2,$$

where $U = r^2 + a^2\cos^2\theta$, $d = r^2 - 2Mr + a^2$, $M^2 > a^2$, and the larger root of d ($r_1 := M + \sqrt{M^2 + a^2}$) is the radius of the black hole, with M and a describing the mass and angular momentum per unit mass of the black hole. Clearly, the case $a = 0$ corresponds to the Schwarzschild metric, and therefore the Schwarzschild metric is a special case of the Kerr metric. In fact, since having *precisely* zero angular momentum is a non-generic property in nature, a Schwarzschild black hole is a non-generic physical phenomenon, and the more plausible object in nature would be a Kerr black hole with $|a| \ll M$. We refer to the survey [19] by Finster, Kamran, Smoller, and Yau for a detailed discussion of these metrics, and we refer to [1] for an interesting discussion of precisely what is meant when one says the Kerr metric describes a *rotating* black hole.

It is an active area of research to rigorously establish the stability of these metrics, and the decay of solutions to the equation $\Box_g\phi = 0$ can be thought of as a "first step" in understanding the full nonlinear stability of the metrics (c.f. [16] and [11]).

2.3 Relevant Results

2.3.1 Minkowski

The problem of decay for linear scalar waves in Minkowski space is classical and is easily solved using the Kirchhoff formula. However, more can be said: Christodoulou and Klainerman actually proved the global *nonlinear* stability of the Minkowski metric in their book [7].

2.3.2 Schwarzschild

The problem of decay for solutions to $\Box_g \phi = 0$, where g is the Schwarzschild metric (2.3) and the initial data is smooth and compactly supported outside the event horizon is well understood (in fact some of the results listed below consider data that does not necessarily vanish at the horizon). First let us write out (2.1) in the Schwarzschild metric:

$$\left[\frac{\partial^2}{\partial t^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 - 2Mr\right)\frac{\partial}{\partial r} + \Delta_{S^2}\right)\right]\phi = 0,$$

where Δ_{S^2} is the Laplacian on the unit two-sphere; i.e.,

$$\Delta_{S^2} = \frac{\partial}{\partial \cos\theta}\sin^2\theta\frac{\partial}{\partial \cos\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

In [22], Kronthaler showed that the unique global solution to this problem decays pointwise as $t \to \infty$. Then in [23] he obtains a t^{-3} decay rate for spherically symmetric initial data (and improves this to t^{-4} for initially static spherically symmetric initial data). In [15] Donninger, Schlag, and Soffer obtain t^{-3}, t^{-4} decay rates for *full* solutions with no symmetry requirement (the higher rate for momentarily static data). The same authors also proved t^{-2-2l}, t^{-3-2l} decay rates for each angular mode of the solution in [16].[4] Tataru also obtains a t^{-3} decay rate in [31]; his results apply as well to a slowly rotating Kerr background. (Indeed all results for a slowly rotating Kerr background apply to Schwarzschild, but we mention this one in particular.) In [24] Luk obtains a uniform $t^{-\frac{3}{2}+\varepsilon}$ decay rate on any compact region, including inside the horizon. We also reference [5], [10], and [11].

No result has been published yet in regards to the *nonlinear* stability of the Schwarzschild metric.

2.3.3 Kerr

The problem of decay for solutions of $\Box_g \phi = 0$, where g is the Kerr metric (2.4) (and we again assume that the initial data is smooth and compactly supported outside the event horizon) is much more difficult than the corresponding problem for the Schwarzschild metric. In this case, $\Box_g \phi = 0$ reads

$$\frac{1}{U}\left[-\frac{\partial}{\partial r} d \frac{\partial}{\partial r} + \frac{1}{d}\left(\left(r^2+a^2\right)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \varphi}\right)^2 \right.$$
$$\left. - \frac{\partial}{\partial \cos\theta}\sin^2\theta \frac{\partial}{\partial \cos\theta} - \frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi}\right)^2\right]\phi = 0,$$

The difficulty in this case arises from the presence of the cross-terms $dt, d\varphi$ in the metric. These terms admit the existence of the *ergosphere*, a region outside of the event horizon in which $\frac{\partial}{\partial t}$ is *spacelike*, and therefore, the energy density of the associated field will fail to be non-negative in the ergosphere (c.f. [19]). The difficulty is then that without a positive semi-definite scalar product that can be defined in the ergosphere, energy methods are not directly applicable. Finster, Kamran, Smoller, and Yau were able to overcome this and show pointwise decay for solutions with finite angular momentum (i.e. finite angular momentum of the wave perturbation).

[4] Price's law conjectures that the solution of the wave equation on a Schwarzschild background decays pointwise as t^{-3-2l}, t^{-4-2l}. Donninger, Schlag, and Soffer obtain *uniform* decay rates and this may explain the loss of one order of decay.

The lack of a positive definite scalar product is precisely why this result is restricted to finite angular momentum (in particular, an argument like that in [22] does not apply). This seems to be the most general result for the decay problem in the Kerr geometry in the sense that there is no restriction on the angular momentum of the black hole.

The case of a slowly rotating Kerr background (i.e. $a \ll M$) is well studied at this point. In this case, Dafermos and Rodnianski were able to demonstrate the uniform boundedness of solutions to the wave equation in [12]. Andersson and Blue obtain decay rates in [2], and Luk improves upon these in [26] to again obtain uniform $t^{-\frac{3}{2}}$ decay rates. Tataru obtains a pointwise t^{-3} decay rate in [31] .

Finally, Luk demonstrated in [25], using his decay results, that a global in time solution of the semilinear equation $\Box_g \phi = F(D\phi)$ exists and decays with a quantitative rate to the trivial solution, given that F satisfies a null condition made precise in the paper. We mention this since it intimates the connection between the decay of linear wave equations and nonlinear wave equations.

2.4 Relevant Techniques

Throughout the literature, there are essentially two methods used to study the decay problem in the Schwarzschild geometry. One is the vector field method; the other involves obtaining integral spectral representations for solutions and analyzing these formulae. We note here that the latter is our method of choice, so we give only a rough sketch of the vector field method.

2.4.1 The Vector Field Method

We shall illustrate the vector field method for the case of Minkowski spacetime and follow the discussion outlined in [24]. We refer to Dafermos and Rodnianski's lecture notes [11] for a detailed discussion of the vector field method in the Schwarzschild geometry and the paper [2] by Andersson and Blue for a discussion of the vector field method in the Kerr geometry.

In spatially Cartesian coordinates (t, x, y, z), the Minkowski metric η is given by

$$\eta = \mathrm{diag}(1, -1, -1, -1).$$

Then consider a scalar solution ϕ of the linear wave equation $\Box_\eta \phi = 0$ and define the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha \phi \partial_\alpha \phi.$$

(Note again that we always use the metric to raise and lower indices, i.e. $A^i = g^{ij}A_j$.) Using that $\Box_\eta \phi = 0$, it is straightforward to show that $\nabla^\mu T_{\mu\nu} = 0$, i.e. the energy-momentum tensor is divergence free. Then, given a vector field V with coordinates V^μ, one defines the associated current

$$J^V_\mu = V^\nu T_{\mu\nu}, \text{ and } K^V = \frac{1}{2}T_{\mu\nu}\left(\nabla^\mu V^\nu + \nabla^\nu V^\mu\right).$$

Using the divergence free condition for the energy-momentum tensor and the symmetry of the metric, we see that

$$\nabla^\mu J^V_\mu = K^V.$$

Now recall that a vector field X is a *Killing field* if $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$; equivalently, by raising indices, this becomes $\nabla^\mu X^\nu + \nabla^\nu X^\mu = 0$. Therefore, if V is a Killing vector, then $K^V = 0$ and this then implies that J^V_μ is divergence free. In particular, one can consider the Killing vector ∂_t and then apply the divergence theorem to integration over the region $[t_0, t_1] \times \mathbb{R}^3$ to obtain conservation of energy for compactly supported ϕ:

$$\int_{\mathbb{R}^3} (\partial_t \phi)^2 + \sum_{i=1}^{3} (\partial_{x_i}\phi)^2 dx\bigg|_{t=t_1} = \int_{\mathbb{R}^3} (\partial_t \phi)^2 + \sum_{i=1}^{3} (\partial_{x_i}\phi)^2 dx\bigg|_{t=t_0}.$$

If instead of global estimates we are interested in local estimates, we can obtain a similar estimate in a bounded region of space Ω and ϕ compactly supported in Ω:

$$\int_\Omega (\partial_t\phi)^2 + \sum_{i=1}^3 (\partial_{x_i}\phi)^2 dx \bigg|_{t=t_1} = \int_\Omega (\partial_t\phi)^2 + \sum_{i=1}^3 (\partial_{x_i}\phi)^2 dx \bigg|_{t=t_0}.$$

Now, each of the vector fields $\partial_t, \partial_{x_i}$, $i = 1, 2, 3$ commutes with $\Box_\eta$ and therefore, the conservation law above can be applied to the derivatives of ϕ as well (for if ϕ solves $\Box_\eta\phi = 0$, then $\Box_\eta(\partial\phi) = 0$ for $\partial = \partial_t, \partial_{x_i}, i = 1, 2, 3$). Thus we can control the L^2 norms of the derivatives of ϕ in terms of the data. We can then use Sobolev inequalities to control ϕ. For example, using Morrey's inequality one has

$$\|\phi\|_\infty \leq C\|\phi\|_{W^{1,6}(\mathbb{R}^3)}$$

and using the Gagliardo-Nirenberg-Sobolev inequality one has

$$\|\phi\|_{L^6(\mathbb{R}^3)} \leq C\|D\phi\|_{L^2(\mathbb{R}^3)} \text{ and } \|D\phi\|_{L^6(\mathbb{R}^3)} \leq C\|D^2\phi\|_{L^2(\mathbb{R}^3)},$$ [5]

which holds for compactly supported functions (c.f. [18]).[6] Putting these together we obtain a universal constant $C > 0$ so that

$$\|\phi(t,\cdot)\|_{L^\infty(R^3)} \leq C\left(\|D\phi(0,\cdot)\|_{L^2(\mathbb{R}^3)} + \|D^2\phi(0,\cdot)\|_{L^2(\mathbb{R}^3)}\right).$$

Thus we obtain uniform boundedness of ϕ.

Notice that different vector fields V can produce different integral identities (for V Killing, these are conservation laws). For V not Killing, these identities may still be useful. For example, for an arbitrary vector field V and a region of spacetime $[t_0, t_1] \times \Omega$, we can again employ the divergence theorem to find

$$\int_{[t_0,t_1]\times\Omega} K^V dxdt = \int_\Omega J_0^V dx_{t_1} - \int_\Omega J_0^V dx_{t_0}.$$

[5] By D we mean the usual gradient and by D^2 we mean the Hessian.

[6] Thus we make use of the compact support of the initial data and the finite speed of propagation inherent in a hyperbolic equation.

Although this not a conservation law unless the left-hand integral is zero, it may contain terms that we are satisfied to use to estimate ϕ (this will depend on the choice of V). The choice of such vector fields and the following estimations is the essence of the vector field method.

2.4.2 Integral Spectral Representations

We will present here a method for obtaining an integral representation formula for our solution. We specialize to Schwarzschild for this discussion and cite [22] for what follows. We first recall the wave equation in the Schwarzschild geometry:

$$\left[\frac{\partial^2}{\partial t^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 - 2Mr\right)\frac{\partial}{\partial r} + \Delta_{S^2}\right)\right]\phi = 0,$$

and we shall suppose we are interested in the Cauchy problem with smooth, compactly supported initial data in $(2M, \infty) \times S^2$. The first step in the reduction is to introduce the Regge-Wheeler (or tortoise) coordinate, which is defined by

$$u(r) = r + 2M \log\left(\frac{r}{2M} - 1\right).$$

This transforms the radial coordinate constrained to the exterior of the black hole $(r > 2M)$ to the coordinate u which varies over $(-\infty, \infty)$. Indeed, u is a smooth, increasing function of r, $u \to -\infty$ as $r \searrow 2M$, and $u \to \infty$ as $r \to \infty$. Moreover, upon making the additional substitution $\psi = r\phi$, the equation for ψ in the coordinates (t, u, θ, φ) reads

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)\right]\psi = 0,$$

and the initial data for this problem is also smooth and compactly supported.

This equation can be reformulated as the Euler-Lagrange equation corresponding to the action

$$S(\psi) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\int_{0}^{2\pi} \mathcal{L}(\psi, \nabla\psi)\, d\varphi\, d(\cos\theta)\, du\, dt,$$

where $\nabla = (\partial_t, \partial_u, \partial_{\cos\theta}, \partial_\varphi)^T$ and the Lagrangian $\mathcal{L}$ is given by

$$2\mathcal{L} = |\psi_t|^2 - |\psi_u|^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{1}{r^2\sin^2\theta}|\psi_\varphi|^2 + \frac{\sin^2\theta}{r^2}|\psi_{(\cos\theta)}|^2\right).$$

As the Lagrangian is invariant under time translations, Noether's theorem guarantees the existence of a conserved quantity $E = E(\psi)$ (the energy), where

$$E(\psi) = \int_{-\infty}^{\infty}\int_{-1}^{1}\int_0^{2\pi} \mathcal{E}\frac{d\varphi}{\pi} d(\cos\theta) du,$$

and

$$\begin{aligned}2\mathcal{E} &= 2\left(\frac{\partial\mathcal{L}}{\partial\psi_t}\psi_t - \mathcal{L}\right)\\ &= |\psi_t|^2 + |\psi_u|^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{1}{r^2\sin^2\theta}|\psi_\varphi|^2 + \frac{\sin^2\theta}{r^2}|\psi_{(\cos\theta)}|^2\right).\end{aligned}$$

Note that $\mathcal{E}$ is positive in the exterior region (which, as we have remarked, would not be the case in the ergosphere in a Kerr background). We note also that it is an easy calculation to verify directly that E is conserved for all smooth, compactly supported solutions ψ.

The next step is to reformulate the problem as a Hamiltonian equation. Thus we define $\Psi = (\psi, i\psi_t)^T$, which allows us to write our problem in the Hamiltonian form

$$i\partial_t\Psi = H\Psi,\ \Psi|_{t=0} = \Psi_0,$$

where H is the Hamiltonian given by

$$H = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

and A is the differential operator $A = -\partial_u^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)$. Next, E induces an inner product $\langle\cdot,\cdot\rangle$ on $C_0^\infty(\mathbb{R}\times S^2)^2$ via polarization and from the energy conservation it follows that H is a symmetric operator on $C_0^\infty(\mathbb{R}\times S^2)^2$ with respect to the induced inner product.

The next observation is that the angular dependence of the wave equation is manifest only in the Δ_{S^2} term. Using that the spherical harmonics $Y_{lm}, l \in \mathbb{N}_0, |m| \leq l$ are smooth eigenfunctions of Δ_{S^2} (with corresponding eigenvalues $-l(l+1)$) and that the Y_{lm} form an orthonormal basis of $L^2(S^2)$, we can decompose Ψ as

$$\Psi(t, u, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \Psi^{lm}(t, u) Y_{lm}(\theta, \varphi),$$

this sum converging, for fixed (t, u), in $L^2(S^2)$ (c.f. [9]). It's also easy to see that $\Psi^{lm}(u) \in C_0^{\infty}(\mathbb{R})$. The action of the Hamiltonian is therefore expressible as

$$H\Psi = \sum_{l=0}^{\infty} \sum_{|m| \leq l} H_l \Psi^{lm}(u) Y_{lm}(\theta, \varphi),$$

where

$$H_l = \begin{pmatrix} 0 & 1 \\ -\partial_u^2 + V_l & 0 \end{pmatrix},$$

and $V_l = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right)$. Therefore, the problem has been reduced to

$$i\partial_t \Psi^{lm} = H_l \Psi^{lm}, \; \Psi^{lm}\big|_{t=0} = \Psi_0^{lm},$$

where H_l is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_l$, which is the inner product $\langle \cdot, \cdot \rangle$ decomposed onto the angular mode l. The symmetry of H_l is inherited from the symmetry of H. Moreover, an application of Stone's theorem yields that H_l is essentially self-adjoint on a domain containing $C_0^{\infty}(\mathbb{R})^2$(we are not concerned with the specifics of the domain of definition here). We then use Stone's formula along with an explicit construction of the resolvent operator in terms of the Jost solutions to obtain an explicit integral representation of Ψ^{lm}. We then analyze the spectral representation for our solution (which turns out to be the Fourier transform of some function) and use the Riemann-Lebesgue lemma to obtain the (modal) decay of our solution. Finally, using Sobolev inequalities, we can show that the decay of the full solution follows from the decay of the modal solutions.

We have described this method as it was demonstrated in [22], but the method also applies to more general spacetime backgrounds. In particular, the method applies to general static spherically symmetric black hole and particle-like geometries (which will be defined precisely later).

2.5 The Motivating Application

2.5.1 The Einstein/Yang-Mills Equations

We refer to the papers [28], [29], and [30] for what follows. The Einstein/Yang-Mills (EYM) equations with gauge group G can be written as

$$R_{ij} - \frac{1}{2}Rg_{ij} = \sigma T_{ij}, \qquad d^* F_{ij} = 0,$$

where T_{ij} is the stress-energy tensor associated to the electromagnetic field $F_{ij}dx^i \wedge dx_j$ and F_{ij} are the coefficients of the curvature 2-form of the gauge field B_α, which can be written as

$$F = \sum_{\mu<\nu} \left[\frac{\partial B_\nu}{\partial x^\mu} - \frac{\partial B_\mu}{\partial x^\nu} + [B_\mu, B_\nu] \right] dx^\mu \wedge dx^\nu,$$

i.e., $F_{ij} = \frac{\partial B_\nu}{\partial x^\mu} - \frac{\partial B_\mu}{\partial x^\nu} + [B_\mu, B_\nu]$, and $R_{ij} - \frac{1}{2}Rg_{ij}$ is comprised solely of g_{ij} and its derivatives (see also [17]).

2.5.2 Solutions with Gauge Group $SU(2)$

If we specialize now to the case $G = SU(2)$ and look for static spherically symmetric solutions (i.e. solutions depending only on r), then Bartnik and McKinnon showed in [3] that the metric (line element) can be written in the coordinates (t, r, θ, φ) as

$$ds^2 = -T(r)^{-2}dt^2 + A(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2.5}$$

and the curvature 2-form as

$$F = w'\tau_1 dr \wedge d\theta + w'\tau_2 dr \wedge (\sin\theta d\varphi) - (1 - w^2)\tau_3 d\theta \wedge (\sin\theta d\varphi),$$

where $' = \frac{d}{dr}$ and τ_1, τ_2, τ_3 form a basis for the Lie algebra $su(2)$. The Einstein/Yang-Mills equations are then expressible as a system of three coupled ordinary differential equations:

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1-w^2)^2}{r^2}, \tag{2.6}$$

$$r^2 A w'' + \left[r(1-A) - \frac{(1-w^2)^2}{r} \right] w' + w(1-w^2) = 0, \tag{2.7}$$

$$2rA\frac{T'}{T} = \frac{(1-w^2)^2}{r^2} + (1 - 2w'^2)A - 1. \tag{2.8}$$

We note that (2.6), (2.7) decouple from T and the problem reduces to analysis of (2.6) and (2.7).

Bartnik and McKinnon in [3] first analyzed these equations and found numerical solutions which are nonsingular and asymptotically flat. These are referred to as *particle-like* solutions.[7] The existence of such solutions is in stark contrast to the case of the vacuum Einstein equations or the pure Yang-Mills equations, since neither of these admits nontrivial, nonsingular, spherically symmetric, asymptotically flat solutions. Indeed, the only nontrivial, spherically symmetric, asymptotically flat solution of the vacuum Einstein equations is the Schwarzschild solution (c.f. [1]). The corresponding result for the pure Yang-Mills equations can be found in [8] and [14]. The discovery of such solutions for the Einstein/Yang-Mills equations is remarkable, since it implies that the gravitational attractive force can balance the repulsive weak nuclear force (c.f. [3]). Smoller and Wasserman then proved rigorously in [28] the

[7] A heuristic explanation for this name is to recall the Reissner-Nordström metric, which in appropriate units units (i.e. $G = 1$) is given by $ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$, where $q^2 = Q^2 + M^2$, Q being the total the electric charge and M being the total magnetic charge. Let us forget for a moment the singularity in the metric at the origin $r = 0$. The zeros of the dt coefficient (equivalently, the singularities in the dr coefficient) are given by $r = m \pm \sqrt{m^2 - q^2}$, which are real if and only if $m \geq |q|$. Thus for a massive object (i.e. a black hole with small charge) there are singularities in the metric. But when $|q| > m$ (i.e. for a electron or proton) the metric is nonsingular. Thus one could think of a particle-like metric as a "nonsingular generalization of the metric around a proton or electron."

existence of countably many such solutions. They also show that each solution has a finite (ADM[8]) mass and that the corresponding metrics decay to the Minkowski metric at infinity.

Smoller and Wasserman also showed in [29] that there are black hole solutions to the EYM equations; i.e., solutions in which $A(r_0) = 0$ for some $r_0 > 0$ and $A(r) > 0$ for each $r > r_0$.

The particlelike and black hole solutions of the EYM equations will thus serve as a model for our study. We will study more general classes of metrics, but it will be useful to keep the physical examples in mind.

[8]For Arnold, Deser, and Misner.

CHAPTER III

The Wave Equation in Black Hole Geometries

Our goal in this chapter is to generalize decay results in the Schwarzschild metric to a more general class of spherically symmetric black hole geometries. We consider a metric given by

$$ds^2 = g_{ij}dx^i dx^j = -T^{-2}(r)dt^2 + K^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where $r > 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$. This clearly generalizes the Schwarzschild geometry, as can be seen by making the identifications

$$T(r) = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} = K(r).$$

We must impose natural (i.e. physical) conditions on the coefficients T, K. To that end, we assume there is a singularity in K at $r = r_0 > 0$ and near the singularity we have the following asymptotics:

$$T(r) = c_1(r - r_0)^{-\frac{1}{2}} + O(1) \text{ and } K(r) = c_2(r - r_0)^{-\frac{1}{2}} + O(1) \tag{3.2}$$

for some constants $c_1, c_2 > 0$, as well as

$$T'(r) = c_3(r - r_0)^{-\frac{3}{2}} + O(r - r_0)^{-\frac{1}{2}} \text{ and } K'(r) = c_4(r - r_0)^{-\frac{3}{2}} + O(r - r_0)^{-\frac{1}{2}} \tag{3.3}$$

for some nonzero constants c_3, c_4. We assume smoothness away from the horizon: $T, K \in C^\infty(r_0, \infty)$; and we assume that in the far-field, the metric asymptotically approaches Minkowski flat-space:

$$T(r) = 1 + O\left(\frac{1}{r}\right) \text{ and } K(r) = 1 + O\left(\frac{1}{r}\right) \tag{3.4}$$

as $r \to \infty$. We assume that for each $r \in (r_0, \infty)$, $T(r) \neq 0$ and $K(r) \neq 0$; and finally, we impose restrictions on the far-field decay:

$$\frac{T'(r)}{T(r)} + \frac{K'(r)}{K(r)} = O\left(\frac{1}{r^2}\right) \tag{3.5}$$

as $r \to \infty$. We note that the Schwarzschild metric, the Reissner-Nordström metric with $m^2 > q^2$, and the metrics given by black hole solutions of the EYM equations (c.f. [29]) satisfy these conditions. It is easy to see the first two cases, and we will show in Section 3.8 that black hole solutions to the EYM equations satisfy these conditions. For the purposes of this paper, a geometry (3.1) satisfying the above conditions will be referred to as a *spherically symmetric black hole* (SSBH). We note also that the work [22] served as a model for solving this problem, and we rely on results therein in a few places.

3.1 Preliminary Notions

We begin by calculating that, according to (2.1), the wave equation in the geometry (3.1) takes the form

$$\Box\zeta = \left(-T^2\partial_t^2 + \frac{1}{r^2}\partial_r\left(\frac{r^2\partial_r}{K^2}\right) + \frac{T}{K^3}\partial_r\left(\frac{K}{T}\right)\partial_r + \frac{1}{r^2}\Delta_{S^2}\right)\zeta = 0, \tag{3.6}$$

where $T = T(r)$ and $K = K(r)$. We introduce the coordinate $u = u(r)$ by

$$u(r) = -\int_r^\infty \frac{K(\alpha)T(\alpha)}{\alpha^2} d\alpha, \tag{3.7}$$

which maps the interval (r_0, ∞) to the interval $(-\infty, 0)$.[1] This is a simple consequence of the asymptotics (3.2) and (3.4). We note also that since T, K are ev-

[1] We note that this is *not* a generalization of the Regge-Wheeler coordinate. We do not use the Regge-Wheeler coordinate here since the wave equation obtained from the standard substitution $\psi = r\zeta$ does not have an everywhere positive potential. If the potential is not everywhere non-negative, then we cannot readily apply Hilbert space methods since the resulting inner product is not necessarily positive semi-definite.

erywhere positive, u is indeed a valid coordinate change and the inverse mapping $r = r(u)$ is well-defined. Then the wave equation (3.6) on $\mathbb{R} \times (r_0, \infty) \times S^2$ is equivalent to

$$\left(-r^4\partial_t^2 + \partial_u^2 + \frac{r^2}{T^2}\Delta_{S^2}\right)\psi = 0 \tag{3.8}$$

on $\mathbb{R} \times (-\infty, 0) \times S^2$, where $T = T(r)$, $r = r(u)$, and $\psi(u) = \zeta(r(u))$.[2] The Cauchy problem for the wave equation in the coordinates (t, u, θ, ϕ) then reads[3]

$$\begin{cases} \left(-r^4(u)\partial_t^2 + \partial_u^2 + \frac{r^2(u)}{T^2(r(u))}\Delta_{S^2}\right)\psi(t,u,\theta,\phi) = 0 \text{ on } \mathbb{R} \times (-\infty,0) \times S^2, \\ (\psi, i\psi_t)(0,u,\theta,\phi) = \Psi_0(u,\theta,\phi) \in C_0^\infty\left((-\infty,0)\times S^2\right)^2. \end{cases} \tag{3.9}$$

Let us settle first the question of existence and uniqueness for the problem (3.9).

Theorem III.1. *The Cauchy problem* (3.9) *in the geometry of an SSBH has a unique, smooth solution that exists for all times t. Furthermore, this solution is compactly supported in (u, θ, ϕ) for each time t.*

Proof. To prove the theorem, we wish to apply the theory of symmetric hyperbolic systems in section 5.3 of [20] to the auxiliary PDE

$$\left(\partial_t^2 - \partial_s^2 - \frac{\Delta_{S^2}}{r^2T^2} - \frac{1}{T^2K^2r}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\xi(t,s,\theta,\phi) = 0, \tag{3.10}$$

where

$$s(u) = \int_{u(2r_0)}^{u} r^2(\alpha)d\alpha, \tag{3.11}$$

$r = r(u(s))$, and the arguments of T, T', K, K' are also $r(u(s))$. Note that we may

[2] We will frequently suppress the arguments of the functions in our formula for notational convenience. The argument should always be clear from the context.

[3] We use the compact form $(\psi, i\psi_t)$ for the data in what follows, since this is most convenient when we reformulate this as a Hamiltonian problem later.

consider s as a function of r by considering[4]

$$
\begin{aligned}
s(u(r)) &= \int_{u(2r_0)}^{u(r)} r^2(\alpha)d\alpha \\
&= \int_{2r_0}^{r} T(\alpha)K(\alpha)d\alpha. \qquad (3.12)
\end{aligned}
$$

The PDE (3.10) is equivalent to (3.8) upon making the change of coordinate $s = s(u)$ and letting $\xi = r\psi$. We consider (3.10) because in this coordinate we will be able to prove that the solution of (3.9) is compactly supported for each t. This is not obvious when working in the u variable. Let us also note that $s(u)$ maps the interval $(-\infty, 0)$ monotonically onto $\mathbb{R}$. We will prove the theorem first for the Cauchy problem

$$
\begin{cases}
\left(\partial_t^2 - \partial_s^2 - \frac{\Delta_{S^2}}{r^2T^2} - \frac{1}{T^2K^2r}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\xi = 0 \text{ on } \mathbb{R}\times\mathbb{R}\times S^2, \\
(\xi, i\xi_t)(0, s, \theta, \phi) = \Xi_0(s, \theta, \phi) \in C_0^\infty(\mathbb{R}\times S^2)^2,
\end{cases} \qquad (3.13)
$$

and then use this to obtain results about the Cauchy problem (3.9).[5] To this end, we must work in local coordinates on S^2. So let us consider the chart $(U, (\theta, \phi))$ where U is an open, relatively compact subset of S^2 and (θ, ϕ) are well-defined on $\bar{U}$. Then, letting $\Gamma = (\xi_t, \xi_u, \partial_{(\cos\theta)}\xi, \partial_\phi\xi, \xi)^T$, we can cast the PDE in (3.13) as a first-order system:

$$
A_0\partial_t\Gamma + A_1\partial_u\Gamma + A_2\partial_{(\cos\theta)}\Gamma + A_3\partial_\phi\Gamma + B\Gamma = 0, \qquad (3.14)
$$

where the matrices $A_0, \dots, A_3, B$ are defined as follows:

$$
A_0 := \operatorname{diag}\left(1, 1, \frac{\sin^2\theta}{r^2T^2}, \frac{1}{r^2T^2}\frac{1}{\sin^2\theta}, 1\right), (A_1)_{12} = -1 = (A_1)_{21},
$$

$$
(A_2)_{13} = -\frac{\sin^2\theta}{r^2T^2} = (A_2)_{31}, (A_3)_{14} = -\frac{1}{r^2T^2}\frac{1}{\sin^2\theta} = (A_3)_{41},
$$

$$
B_{13} = \frac{2\cos\theta}{r^2T^2}, B_{15} = -\frac{1}{T^2K^2r}\left(\frac{T'}{T} + \frac{K'}{K}\right), B_{51} = -1,
$$

[4]Note that this *is* a generalization of the Regge-Wheeler coordinate.

[5]We note that the following argument is based on a similar argument in [22], but we present it for completeness.

and all other matrix entries are zero. Upon multiplying this system by T^2, we obtain a symmetric hyperbolic system on $\mathbb{R} \times \mathbb{R} \times U$ (c.f. [20]), since each A_i is symmetric and A_0 is uniformly positive definite on this region. Further, since the initial data Ξ_0 has compact support, we can restrict the system to $\mathbb{R} \times V \times U$, where V is open, relatively compact, and $\operatorname{supp} \Xi_0(u, \theta, \phi) \subset V \subset \mathbb{R}$ for each $(\theta, \phi) \in S^2$. Since we can cover S^2 by finitely many such charts, the theory of symmetric hyperbolic systems guarantees the existence and uniqueness of a smooth solution ξ of (3.13) defined for all $t < \varepsilon_1$ for some $\varepsilon_1 > 0$ (note that since our matrices are smooth and the data is smooth and compactly supported, ε_1 is independent of the data). Moreover, this solution propagates with finite speed and thus there exists an $0 < \varepsilon \leq \varepsilon_1$ so that ξ has compact support in $V \times S^2$ for all times $t \leq \varepsilon$. Therefore, we can repeat this argument for the Cauchy problem with data $(\xi(\varepsilon, u, \theta, \phi), i\xi_t(\varepsilon, u, \theta, \phi))^T$ and obtain a unique, smooth solution ξ defined for $t \leq 2\varepsilon$ which has compact support in a possibly larger, though still open and relatively compact set for all times $t \leq 2\varepsilon$. Repeating the argument yields a global solution ξ of the Cauchy problem (3.13) which is smooth, unique, and compactly supported for all times t.

Since we have already observed that a solution ψ of (3.9) yields a solution of (3.13) under the coordinate change $s = s(u)$ and the identification $\xi = r\psi$ and vice versa, the theorem follows. $\square$

We now observe that the Cauchy problem admits a conserved energy:

Proposition III.2. *A solution of the Cauchy problem* (3.9) *admits a conserved energy $E(\psi)$ given by*

$$
\begin{aligned}
E(\psi) = \int_0^{2\pi}\int_{-1}^{1}\int_{-\infty}^{0} & r^4(u)\,(\psi_t)^2 + (\psi_u)^2 \\
& + \frac{r^2(u)}{T^2(r(u))}\left(\frac{1}{\sin^2\theta}\left(\partial_\phi \psi\right)^2 + \sin^2\theta\left(\partial_{(\cos\theta)}\psi\right)^2\right) du\, d(\cos\theta)\, d\phi; \qquad (3.15)
\end{aligned}
$$

i.e. $\frac{d}{dt}E(\psi) = 0$.

Proof. We know that (3.9) admits a globally defined, smooth, unique solution which is compactly supported for all times t. Thus, the energy $E(\psi)$ is well-defined. Moreover, an easy calculation shows that $\frac{d}{dt}E(\psi) = 0$, since ψ solves (3.9). $\square$

Next, we wish to cast the Cauchy problem (3.9) as a first-order Hamiltonian system. To this end we define $\Psi := (\psi, i\psi_t)^T$; then $i\partial_t \Psi = H\Psi$, where

$$H = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

and $A = -\frac{1}{r^4}\partial_u^2 - \frac{\Delta_{S^2}}{r^2 T^2}$. Therefore the Cauchy problem (3.9) is equivalent to the problem

$$\begin{cases} i\partial_t \Psi = H\Psi \text{ on } \mathbb{R} \times (-\infty, 0) \times S^2, \\ \Psi(0, u, \theta, \phi) = \Phi_0(u, \theta, \phi) \in C_0^\infty\left((-\infty, 0) \times S^2\right)^2. \end{cases} \tag{3.16}$$

Theorem III.1 then implies that the problem (3.16) has a unique, smooth solution Ψ that is defined for all times t and compactly supported for each t.

Let us next observe that the energy in (3.15) defines an inner product on the space $C_0^\infty\left((-\infty, 0) \times S^2\right)^2$: for $\Psi, \Gamma \in C_0^\infty\left((-\infty, 0) \times S^2\right)^2$ with $\Psi = (\psi_1, \psi_2)^T$ and $\Gamma = (\gamma_1, \gamma_2)^T$, we can define the scalar product $\langle \Psi, \Gamma \rangle$ by

$$\begin{aligned} &\int_0^{2\pi}\int_{-1}^{1}\int_{-\infty}^{0} r^4 \psi_2 \overline{\gamma_2} + (\partial_u \psi_1)\overline{(\partial_u \gamma_1)} \\ &\quad + \frac{r^2}{T^2}\left(\frac{1}{\sin^2\theta}(\partial_\phi \psi_1)\overline{(\partial_\phi \gamma_1)} + \sin^2\theta (\partial_{(\cos\theta)}\psi_1)\overline{(\partial_{(\cos\theta)}\gamma_1)}\right) du d(\cos\theta) d\phi. \end{aligned} \tag{3.17}$$

We next show that with respect to this inner product, H is symmetric on the domain $C_0^\infty\left((-\infty, 0) \times S^2\right)^2$.

Proposition III.3. *The operator H is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$ on the domain $C_0^\infty\left((-\infty, 0) \times S^2\right)^2$.*

Proof. Consider a solution Ψ of (3.16). Upon making the identification $\Psi = (\psi, i\psi_t)^T$, we know that ψ solves (3.9). We have that $\langle \Psi, \Psi \rangle = E(\psi)$ and therefore that $\frac{d}{dt}\langle \Psi, \Psi \rangle = 0$ for solutions of (3.16). On the other hand,

$$\begin{aligned} \frac{d}{dt}\langle \Psi, \Psi \rangle &= \langle \partial_t \Psi, \Psi \rangle + \langle \Psi, \partial_t \Psi \rangle \\ &= -i\langle H\Psi, \Psi \rangle + i\langle \Psi, H\Psi \rangle, \end{aligned}$$

which shows that $\langle H\Psi, \Psi \rangle = \langle \Psi, H\Psi \rangle$ for any Ψ solving (3.16). Note that this expression holds for each t, and in particular at $t = 0$. Thus, $\langle H\Psi_0, \Psi_0 \rangle = \langle \Psi_0, H\Psi_0 \rangle$. But the initial data Ψ_0 can be chosen arbitrarily in $C_0^\infty\left((-\infty, 0) \times S^2\right)^2$, which shows that, after a simple polarization argument, H is symmetric on the space $C_0^\infty\left((-\infty, 0) \times S^2\right)^2$ with respect to the inner product $\langle \cdot, \cdot \rangle$. $\square$

We next observe that the only manifestation of the angular variables (θ, ϕ) in the problem (3.9) is in the spherical Laplacian. Since any smooth function on S^2 can be expanded into an absolutely and uniformly convergent series in terms of spherical harmonics (c.f. [9]), we may therefore write

$$\Psi(t, u, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{|m| \le l} \Psi^{lm}(t, u) Y_{lm}(\theta, \phi), \tag{3.18}$$

where the $Y_{lm}(\theta, \phi)$ are the spherical harmonics (i.e. $\Delta_{S^2} Y_{lm} = -l(l+1)Y_{lm}$) and this series converges uniformly and absolutely for each fixed $(t, u) \in R \times (-\infty, 0)$. Furthermore, we know that $\Psi^{lm} = (\psi_1^{lm}, \psi_2^{lm})^T$, where $\psi_i^{lm} = \langle \psi_i, Y_{lm} \rangle_{L^2(S^2)}$. It is clear therefore that $\Psi^{lm}(t, u)$ is smooth and for each t, $\Psi^{lm}(t, u) \in C_0^\infty\left(-\infty, 0\right)^2$. Thus, for any $\Psi, \Gamma \in C_0^\infty\left((-\infty, 0) \times S^2\right)^2$, we can decompose the scalar product

$\langle \Psi, \Gamma \rangle$ according to

$$\begin{aligned} \langle \Psi, \Gamma \rangle &= \sum_{l=0}^{\infty} \sum_{|m| \leq l} \langle \Psi^{lm}, \Gamma^{lm} \rangle_l \\ &= \sum_{l=0}^{\infty} \sum_{|m| \leq l} \int_{-\infty}^{0} r^4 \psi_2^{lm} \overline{\gamma_2^{lm}} + (\partial_u \psi_1^{lm}) \overline{(\partial_u \gamma_1^{lm})} + \frac{r^2}{T^2} l(l+1) \psi_1^{lm} \overline{\gamma_1^{lm}} du, \end{aligned} \tag{3.19}$$

which follows from integrating by parts.

The action of the Hamiltonian also simplifies under the modal decomposition:

$$H\Psi(t, u, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} H_l \Psi^{lm}(t, u) Y_{lm}(\theta, \phi),$$

where

$$H_l = \begin{pmatrix} 0 & 1 \\ -\frac{1}{r^4} \partial_u^2 + \frac{l(l+1)}{r^2 T^2} & 0 \end{pmatrix}. \tag{3.20}$$

Therefore, the components Ψ^{lm} in the spherical harmonic decomposition of Ψ solve a reduced equation:

Proposition III.4. *Consider the solution* Ψ *of* (3.16). *The component functions* Ψ^{lm} *in the spherical harmonic decomposition of* Ψ (3.18) *solve the problem*

$$\begin{cases} i\partial_t \Psi^{lm} = H_l \Psi^{lm} \text{ on } \mathbb{R} \times (-\infty, 0), \\ \Psi^{lm}(0, u) = \Psi_0^{lm} \in C_0^\infty(-\infty, 0)^2. \end{cases} \tag{3.21}$$

Proof. The proposition follows from the discussion above and the uniqueness of the spherical harmonic decomposition. $\square$

Our strategy therefore is to solve problem (3.21) and then sum up according to (3.18) to obtain a solution of (3.16). We note as well that H_l is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_l$ on the domain $C_0^\infty(-\infty, 0)^2$, since for

$\Psi^{lm}, \Gamma^{lm} \in C_0^\infty(-\infty, 0)^2$ we have

$$\begin{aligned}\langle H_l \Psi^{lm}, \Gamma^{lm} \rangle_l &= \langle H(\Psi^{lm} Y_{lm}), \Gamma^{lm} Y_{lm} \rangle \\ &= \langle \Psi^{lm} Y_{lm}, H(\Gamma^{lm} Y_{lm}) \rangle \\ &= \langle \Psi^{lm}, H_l \Gamma^{lm} \rangle_l.\end{aligned}$$

This also implies that the energy $E_l(\Psi^{lm}) := \langle \Psi^{lm}, \Psi^{lm} \rangle_l$ is conserved for smooth, compactly supported solutions of (3.21), since we have

$$\begin{aligned}\frac{d}{dt}\langle \Psi^{lm}, \Psi^{lm} \rangle_l &= \langle \partial_t \Psi^{lm}, \Psi^{lm} \rangle_l + \langle \Psi^{lm}, \partial_t \Psi^{lm} \rangle_l \\ &= -i \langle H_l \Psi^{lm}, \Psi^{lm} \rangle_l + i \langle \Psi^{lm}, H \Psi^{lm} \rangle_l \\ &= 0,\end{aligned}$$

by the symmetry of H_l.

3.2 The Hamiltonian

Let us rewrite H_l as

$$H_l = \begin{pmatrix} 0 & 1 \\ -\frac{1}{r^4}\partial_u^2 + V_l(u) & 0 \end{pmatrix}, \tag{3.22}$$

where $V_l(u) = \frac{l(l+1)}{r^2 T^2}$. (Recall the arguments are $r = r(u)$ and $T = T(r(u))$.) We wish to construct a self-adjoint extension of H_l, and we therefore need to find a Hilbert space on which H_l is densely defined. To this end, let us define $\mathscr{H}^1_{V_l,0}$ as the completion of $C_0^\infty(-\infty, 0)$ within the Hilbert space

$$\mathscr{H}^1_{V_l}(-\infty, 0) := \left\{ \psi : \psi_u \in L^2(-\infty, 0) \text{ and } r^2 V_l^{\frac{1}{2}} \psi \in L^2(-\infty, 0) \right\}.$$

Let us also define $\mathscr{H}_{r^2,0}$ as the completion of $C_0^\infty(-\infty, 0)$ within the Hilbert space

$$\mathscr{H}_{r^2}(-\infty, 0) := \left\{ \psi : r^2 \psi \in L^2(-\infty, 0) \right\}.$$

Finally, we define Hilbert space

$$\mathscr{H} := \mathscr{H}^1_{V_l,0} \otimes \mathscr{H}_{r^2,0}$$

endowed with the inner product $\langle \cdot, \cdot \rangle_l$ to be the underlying Hilbert space on which H_l is densely defined.

As in [22] we next construct a self-adjoint extension of H_l:

Proposition III.5. *The operator H_l with domain $\mathscr{D}(H_l) = C_0^\infty(-\infty, 0)^2$ is essentially self-adjoint in the Hilbert space $\mathscr{H}$.*

Proof. To prove this, we use the following version of Stone's theorem (c.f. [27], Sec. VIII.4):

Theorem III.6 (Stone's Theorem). *Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathscr{H}$. Then there is a self-adjoint operator A on $\mathscr{H}$ so that $U(t) = e^{itA}$. Furthermore, let $\mathscr{D}$ be a dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly differentiable. Then i^{-1} times the strong derivative of $U(t)$ is essentially self-adjoint on $\mathscr{D}$ and its closure is A.*

Now consider the Cauchy problem (3.21). By the theory of symmetric hyperbolic systems, the problem (3.21) has a unique, smooth, global solution Ψ^{lm} that is compactly supported for all times t (we prove this similarly to Theorem III.1). Thus, for $t \in \mathbb{R}$ we define the solution operators

$$U(t) : C_0^\infty(-\infty, 0)^2 \mapsto C_0^\infty(-\infty, 0)^2 \text{ by}$$

$$U(t)\Psi_0^{lm} = \Psi^{lm}(t) = \left(\psi^{lm}(t), i\partial_t \psi^{lm}(t)\right)^T .$$

Note that $U(t)$ leaves the dense subspace $C_0^\infty(-\infty, 0)^2$ invariant for all times t and also, by the energy conservation, the $U(t)$ are unitary with respect to the energy inner product. Therefore the $U(t)$ extend to unitary operators on $\mathscr{H}$. The uniqueness of Ψ^{lm} guarantees that $U(0) = I$ and $U(t)U(s) = U(t+s)$ for all $s, t \in \mathbb{R}$. Thus, the $U(t)$ form a one-parameter unitary group. The fact that the solutions are smooth in t and u guarantees that this group is strongly continuous on $\mathscr{H}$ and strongly differentiable on $C_0^\infty(-\infty, 0)^2$. Then, for $\gamma_1, \gamma_2 \in C_0^\infty(-\infty, 0)$,

$$i^{-1} \lim_{h \to 0} \frac{1}{h} \left(U(h)(\gamma_1, \gamma_2)^T - (\gamma_1, \gamma_2)^T \right) = -H_l(\gamma_1, \gamma_2)^T.$$

Thus, by Stone's theorem, H_l is essentially self-adjoint on $\mathscr{H}$ with self-adjoint closure $\bar{H}_l$ and $U(t) = e^{-it\bar{H}_l}$. □

To obtain a representation of the solution Ψ^{lm} of (3.21), we will use Stone's formula which relates the spectral projections of a self-adjoint operator to the resolvent. Recall that the spectral projections P_Ω onto a measurable set Ω of such an operator are defined by $P_\Omega = \chi_\Omega(A)$, where $\chi_\Omega(x)$ is the usual indicator function of Ω and we use the functional calculus to define $\chi_\Omega(A)$. We recall Stone's formula in the following theorem:

Theorem III.7 (Stone's Formula). *For a self-adjoint operator A, the following holds*

$$\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} \right] d\lambda, \tag{3.23}$$

where the limit is taken in the strong operator topology.

We refer to [27], chapter VII for a proof. According to Stone's formula, to understand the spectral projections of $\bar{H}_l$, we must investigate the resolvent operator $(\bar{H}_l - \omega)^{-1} : \mathscr{H} \mapsto \mathscr{H}$. Since $\bar{H}_l$ is self-adjoint, it follows immediately that $(\bar{H}_l - \omega)^{-1}$ exists and is bounded for each $\omega \in \mathbb{C} \setminus \mathbb{R}$. Now fix $\omega \in \mathbb{C} \setminus \mathbb{R}$ and consider the eigenvalue equation

$$\bar{H}_l \Phi = \omega \Phi. \tag{3.24}$$

Note that since $\omega \notin \sigma(\bar{H}_l)$, this equation does not have solutions in $\mathscr{H}$. Nonetheless, we will be able to construct the resolvent out of special solutions of this equation (the Jost solutions). To this end, let us observe that (3.24) is equivalent to the differential equation

$$-\zeta''(u) - \omega^2 r^4 \zeta + \frac{r^2}{T^2} l(l+1)\zeta = 0 \tag{3.25}$$

on the interval $(-\infty, 0)$ where $\zeta = \phi_1$ or ϕ_2. This ODE is difficult to solve explicitly, so let us use the coordinate $s = s(u)$ defined in (3.11) and define

$$\eta(s) = r(u(s))\zeta(u(s)). \tag{3.26}$$

Inserting these into (3.25), we obtain the equivalent ODE

$$-\eta''(s) - \omega^2 \eta(s) + \left(\frac{l(l+1)}{r^2 T^2} - \frac{1}{r T^2 K^2} \left(\frac{T'}{T} + \frac{K'}{K} \right) \right) \eta(s) = 0. \tag{3.27}$$

To investigate this ODE, we need to look at the potential

$$W_l(s) := \left(\frac{l(l+1)}{r^2 T^2} - \frac{1}{r T^2 K^2} \left(\frac{T'}{T} + \frac{K'}{K} \right) \right). \tag{3.28}$$

Observe that for large r, we have $s \sim r + O(\log r)$, and for $r \searrow r_0$ we have $s \sim c_1 c_2 \log(r - r_0) + O(1)$, which follows from (3.2), (3.3), and (3.12). Then invoking the asymptotics (3.4) and (3.5), we see that $|W_l(s)| \sim O\left(\frac{l(l+1)}{s^2}\right)$ for $l \neq 0$ and $|W_0(s)| \sim O\left(\frac{1}{s^3}\right)$ as $s \to \infty$. Noting also that $W_l(s) \sim O(r - r_0)$ near the horizon, we find that $|W_l(s)| \leq \alpha_1 e^{\alpha_2 s}$ as $s \to -\infty$ for some constants $\alpha_1, \alpha_2 > 0$.

We now return to equation (3.27) and prescribe asymptotic boundary conditions to determine a pair of fundamental solutions. In the case Im $(\omega) > 0$, we require

$$\lim_{s \to -\infty} e^{i\omega s} \eta_{1,\omega}(s) = 1, \text{ and } \lim_{s \to -\infty} \left(e^{i\omega s} \eta_{1,\omega}(s) \right)' = 0 \tag{3.29}$$

and

$$\lim_{s\to\infty} e^{-i\omega s}\eta_{2,\omega}(s) = 1, \text{ and } \lim_{s\to\infty} \left(e^{-i\omega s}\eta_{2,\omega}(s)\right)' = 0, \tag{3.30}$$

whereas in the case Im $(\omega) < 0$, we require

$$\lim_{s\to-\infty} e^{-i\omega s}\eta_{1,\omega}(s) = 1, \text{ and } \lim_{s\to-\infty} \left(e^{-i\omega s}\eta_{1,\omega}(s)\right)' = 0 \tag{3.31}$$

and

$$\lim_{s\to\infty} e^{i\omega s}\eta_{2,\omega}(s) = 1, \text{ and } \lim_{s\to\infty} \left(e^{i\omega s}\eta_{2,\omega}(s)\right)' = 0. \tag{3.32}$$

Now these two solutions $\eta_{1,\omega}, \eta_{2,\omega}$ must be linearly independent, for if they were not, the exponential decay at $s = \pm\infty$ would yield a nonzero vector in the kernel of $(\bar{H}_l - \omega)^{-1}$. But since $\bar{H}_l$ is essentially self-adjoint, the spectrum is contained on the real line, and thus, for $\omega \in \mathbb{C} \setminus \mathbb{R}$, the kernel of $(\bar{H}_l - \omega)^{-1}$ is trivial. Thus, $\eta_{1,\omega}$ and $\eta_{2,\omega}$ and linearly independent and form a fundamental set of solutions of (3.27), and thus the Wronskian $w(\eta_{1,\omega}, \eta_{2,\omega}) := \eta_{1,\omega}(s)\eta'_{2,\omega}(s) - \eta'_{1,\omega}(s)\eta_{2,\omega}(s)$ is non-vanishing.[6] By Abel's theorem, $w(\eta_{1,\omega}, \eta_{2,\omega})$ is independent of s.

3.3 Constructing the Jost Solutions

Let us now construct the solutions $\eta_{1,\omega}, \eta_{2,\omega}$. In this section we will write

$$\eta_{1,\omega} = \eta^1(\lambda, \omega, s) \text{ and } \eta_{2,\omega} = \eta^2(\lambda, \omega, s), \tag{3.33}$$

where $\lambda = l + \frac{1}{2}$. The λ dependence in these functions is important in a more general setting, so we make the λ dependence explicit for generality. We cite [13] for the idea of this construction. We focus first on the solution with boundary conditions at

[6] In general linear independence of two functions does not guarantee that their Wronskian does not vanish. However, when the two functions solve the same homogeneous second-order linear ODE, this is sufficient.

$s = \infty$ and restrict ourselves for the moment to Im $\omega \leq 0$, $\omega \neq 0$. We first write the ODE (3.27) as

$$\eta''(s)+\left(\omega^2 - \frac{\lambda^2 - \frac{1}{4}}{s^2}\right)\eta(s) = \left(\left(\lambda^2 - \frac{1}{4}\right)\left[\frac{1}{r^2T^2} - \frac{1}{s^2}\right] - \frac{1}{rT^2K^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\eta(s),$$

and we find the Green's function for the operator on the left-hand side with zero boundary conditions at $s = \infty$ is given by

$$B(\lambda, \omega, s, y) = \Theta(y-s)\frac{i}{2\omega}\left(\eta_0^2(\lambda, \omega, y)\eta_0^2(\lambda, -\omega, s) - \eta_0^2(\lambda, \omega, s)\eta_0^2(\lambda, -\omega, y)\right), \tag{3.34}$$

where

$$\eta_0^2(\lambda, \omega, s) = \left(\frac{1}{2}\pi\omega s\right)^{\frac{1}{2}} e^{-\frac{i\pi}{2}\left(\lambda+\frac{1}{2}\right)} H_\lambda^{(2)}(\omega s), \tag{3.35}$$

Θ is the usual Heaviside function, and $H_\lambda^{(2)}$ is the Hankel function of the second kind (we reference [33] and [34] for information about the Hankel functions). Note that $\lim_{s\to\infty} \eta_0^2(\lambda, \omega, s)e^{i\omega s} = 1$. Thus, if we require

$$\lim_{s\to\infty} \eta^2(\lambda, \omega, s)e^{i\omega s} = 1, \tag{3.36}$$

then the equivalent integral equation for η^2 is

$$\eta^2(\lambda, \omega, s) = \eta_0^2(\lambda, \omega, s) + \int_s^\infty B(\lambda, \omega, s, y)W_l(y)\eta^2(\lambda, \omega, y)dy. \tag{3.37}$$

This is "almost" a Volterra integral equation of the second kind, excepting the infinite interval; and the existence/smoothness properties of the solution are easy to establish (c.f. Appendix B of [16]). Moreover, the smoothness then guarantees that these solutions actually solve the ODE (3.27). However, these results are insufficient for our methods since we need to have asymptotics for these solutions. To this end we shall construct the solution of this equation as a Neumann integral series (at least for $s > 0$, but by uniqueness it extends to a solution on the whole line due to the smoothness of the potential $W_l(s)$). So we write

$$\eta^2(\lambda,\omega,s) = \sum_{n=0}^{\infty} \eta_n^2(\lambda,\omega,s), \tag{3.38}$$

where

$$\eta_{n+1}^2 = \int_s^{\infty} B(\lambda,\omega,s,y) W_l(y) \eta_n^2(\lambda,\omega,y) dy. \tag{3.39}$$

To address convergence, we appeal to the following facts proved in appendix A of [13]: we have

$$|\eta_0^2(\lambda,\omega,s)| \le C \left(\frac{|\omega| s}{1+|\omega| s}\right)^{-\lambda+\frac{1}{2}} e^{(\mathrm{Im}\ \omega)s} \tag{3.40}$$

and for $0 < s < y$ we have

$$|B(\lambda,\omega,s,y)| \le C e^{|\mathrm{Im}\ \omega| y + (\mathrm{Im}\ \omega)s} \left(\frac{y}{1+|\omega| y}\right)^{\lambda+\frac{1}{2}} \left(\frac{s}{1+|\omega| s}\right)^{-\lambda+\frac{1}{2}} \tag{3.41}$$

where C depends on λ. With these facts it is easy to show by induction that

$$|\eta_n^2(\lambda,\omega,s)| \le C \frac{(CQ(s))^n}{n!} \left(\frac{|\omega| s}{1+|\omega| s}\right)^{-\lambda+\frac{1}{2}} e^{(\mathrm{Im}\ \omega)s}, \tag{3.42}$$

where

$$Q(s) = \int_s^{\infty} \frac{y|W_l(y)|}{1+|\omega| y} e^{(|\mathrm{Im}\ \omega| + \mathrm{Im}\ \omega)y} dy. \tag{3.43}$$

Note that for $\mathrm{Im}\ \omega \le 0$, Q is finite for all $s \in [0,\infty)$ and $\|Q\|_{L^1([0,\infty))} < \infty$, owing to the integrability of W and our requirement that $\mathrm{Im}\ \omega \le 0$. Thus η^2 exists (for $\mathrm{Im}\ \omega \le 0$ and $\omega \ne 0$), and the following bounds are obvious

$$|\eta^2(\lambda,\omega,s)| \le C e^{(\mathrm{Im}\ \omega)s} \left(\frac{|\omega| s}{1+|\omega| s}\right)^{-\lambda+\frac{1}{2}} e^{CQ(s)}, \tag{3.44}$$

and

$$|\eta^2(\lambda,\omega,s) - \eta_0^2(\lambda,\omega,s)| \le C e^{(\mathrm{Im}\ \omega)s} \left(\frac{|\omega| s}{1+|\omega| s}\right)^{-\lambda+\frac{1}{2}} (e^{CQ(s)} - 1). \tag{3.45}$$

It is also straightforward to show that η^2 is analytic in ω for fixed s (for Im $\omega < 0$). Furthermore, we easily obtain the following estimates:

$$\left|\frac{d}{ds}\eta_0^2(\lambda,\omega,s)\right| \leq C|\omega|e^{(\mathrm{Im}\ \omega)s}\left(\frac{|\omega|s}{1+|\omega|s}\right)^{-\lambda-\frac{1}{2}} \tag{3.46}$$

and

$$\begin{aligned}&\left|\frac{d}{ds}\eta^2(\lambda,\omega,s) - \frac{d}{ds}\eta_0^2(\lambda,\omega,s)\right| \\ &\quad\leq C\left(\frac{|\omega|s}{1+|\omega|s}\right)^{-\lambda-\frac{1}{2}} e^{(\mathrm{Im}\ \omega)s}\int_s^\infty \left(\frac{|\omega|y}{1+|\omega|y}\right)^{-\lambda+\frac{1}{2}} e^{CQ(y)}|W_l(y)|dy.\end{aligned} \tag{3.47}$$

From (3.44) we see a possible singularity in η^2 at $\omega = 0$, but this singularity is removable. Indeed, repeating the above arguments with the initial function $\eta_0^{2,0}(\lambda,\omega,s) = \omega^{\lambda-\frac{1}{2}}\left(\frac{1}{2}\pi\omega s\right)^{\frac{1}{2}} e^{-\frac{i\pi}{2}\left(\lambda+\frac{1}{2}\right)}H_\lambda^{(2)}(\omega s)$ yields a solution $\eta^{2,0}$ of the integral equation

$$\eta^{2,0}(\lambda,\omega,s) = \eta_0^{2,0}(\lambda,\omega,s) + \int_s^\infty B(\lambda,\omega,s,y)W_l(y)\eta^{2,0}(\lambda,\omega,y)dy$$

in the region $|\omega| \leq 1, \mathrm{Im}\ \omega \leq 0$. By uniqueness, one has $\omega^{\lambda-\frac{1}{2}}\eta^2(\lambda,\omega,s) = \eta^{2,0}(\lambda,\omega,s)$, and $\eta^{2,0}$ is defined up to $\omega = 0$ with

$$\lim_{\omega\to 0,\mathrm{Im}\ \omega\leq 0} \omega^{\lambda-\frac{1}{2}}\eta^2(\lambda,\omega,s) = \eta^{2,0}(\lambda,0,s) \tag{3.48}$$

pointwise in s, and $\eta^{2,0}$ has the following asymptotics:

$$\lim_{s\to\infty} \eta^{2,0}(\lambda,0,s)s^l = (-i)^l(2l-1)!!. \tag{3.49}$$

So we have solved the ODE (3.27) with boundary conditions at $s = \infty$ for Im $\omega \leq 0$. For Im $\omega > 0$, we obtain a solution of $\eta^2(\lambda,\omega,s)$ of this BVP by defining $\eta^2(\lambda,\omega,s) = \overline{\eta^2(\lambda,\bar{\omega},s)}$. The uniqueness guarantees that this is indeed a solution and it obviously has properties similar to those discussed above, and moreover, η^2 is continuous in ω as $\omega \to 0$ for fixed s.

A similar construction produces a solution $\eta^1(\lambda, \omega, s)$ of (3.27) with boundary conditions at $s = -\infty$ with properties analogous to those of η^2. In fact, the exponential decay of the potential as $s \to -\infty$ makes the construction easier and we obtain that η^1 is analytic in ω for Im $\omega < \beta$ for some $\beta > 0$. We obtain a solution η^1 on all of $\mathbb{C}$ via the conjugation process used above.

3.4 The Resolvent

We now use the Jost solutions to construct the resolvent. In this section we will consider l (and therefore λ) fixed, and we may therefore write $\eta_{1,\omega}(s) = \eta^1(\lambda, \omega, s)$ and $\eta_{2,\omega}(s) = \eta^2(\lambda, \omega, s)$. We then use the the definition (3.26) to obtain two solutions $\zeta_{1,\omega}, \zeta_{2,\omega}$ of (3.25) from $\eta_{1,\omega}, \eta_{2,\omega}$, and in the case $\omega = 0$, we again use (3.26) to obtain a solution $\zeta_{2,0}$ from $\eta_{2,0}$. It's easy to check that $w(\eta_{1,\omega}, \eta_{2,\omega}) = w(\zeta_{1,\omega}, \zeta_{2,\omega})$, and it therefore follows that for Im $\omega \neq 0$, $\{\zeta_{1,\omega}, \zeta_{2,\omega}\}$ forms a fundamental set of solutions for the ODE (3.25) with non-vanishing Wronskian. Thus we may define the following function for Im $\omega \neq 0$

$$h_\omega(u, v) := -\frac{1}{w(\zeta_{1,\omega}, \zeta_{2,\omega})} \begin{cases} \zeta_{1,\omega}(u)\zeta_{2,\omega}(v), & u \leq v \\ \zeta_{1,\omega}(v)\zeta_{2,\omega}(u), & u > v. \end{cases} \tag{3.50}$$

An easy calculation shows that $h_\omega(u, v)$ satisfies the distributional equations

$$\begin{gathered} \left(-d_u^2 - r^4\omega^2 + \frac{r^2}{T^2} l(l+1)\right) h_\omega(u, v) = \delta(u - v) \\ \left(-d_v^2 - r^4\omega^2 + \frac{r^2}{T^2} l(l+1)\right) h_\omega(u, v) = \delta(u - v) \end{gathered} \tag{3.51}$$

where the arguments on the left are $r = r(v)$ and $r = r(u)$ on the right. We next use the function $h_\omega(u, v)$ to construct the resolvent $(\bar{H}_l - \omega)^{-1}$. The argument below follows one presented in [22] but again we give it for completeness.

Proposition III.8. *For any $\omega \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(\bar{H}_l - \omega)^{-1}$ can be represented as an integral operator with kernel*

$$k_\omega(u,v) = \delta(u-v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + r^4(v) h_\omega(u,v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix}. \tag{3.52}$$

Proof. Consider the integral operator K_ω with kernel given by $k_\omega(u,v)$ acting on the domain $\mathscr{D}(K_\omega) := \left\{(\bar{H}_l - \omega)\Psi : \Psi \in C_0^\infty(-\infty,0)^2\right\}$. We claim first that $\mathscr{D}(K_\omega)$ is a dense subset of $\mathscr{H}$. To this end, let $\xi \in \mathscr{H}$ be arbitrary. Because the resolvent exists, $(\bar{H}_l - \omega) : \mathscr{D}(\bar{H}_l) \mapsto \mathscr{H}$ is onto and there exists $\gamma \in \mathscr{D}(\bar{H}_l)$ so that $(\bar{H}_l - \omega)\gamma = \xi$. Since $\bar{H}_l$ is the closure of H_l, there is a sequence $\{\gamma_n\} \subset C_0^\infty(-\infty, 0)$ so that $\gamma_n \to \gamma$ and $\bar{H}_l \gamma_n \to \bar{H}_l \gamma$ as $n \to \infty$. Thus, $\{(\bar{H}_l - \omega)\gamma_n\}$ converges to $(\bar{H}_l - \omega)\gamma = \xi$, and therefore $\mathscr{D}(K_\omega)$ is dense in $\mathscr{H}$.

Now, for an arbitrary $\Gamma = (\gamma_1, \gamma_2)^T \in C_0^\infty(-\infty,0)^2$, we have

$$\begin{aligned}
(K_\omega(\bar{H}_l - \omega)\Gamma)(u) &:= \int_{-\infty}^0 k_\omega(u,v)(\bar{H}_l - \omega)\Gamma(v)dv \\
&= (0, -\omega\gamma_1 + \gamma_2)^T + \int_{-\infty}^0 h_\omega(u,v) \begin{pmatrix} \left(-d_u^2 - r^4\omega^2 + \frac{r^2}{T^2} l(l+1)\right)\gamma_1 \\ \left(-d_u^2 - r^4\omega^2 + \frac{r^2}{T^2} l(l+1)\right)\gamma_1 \omega \end{pmatrix} dv \\
&= (\gamma_1, \gamma_2)^T \\
&= \Gamma,
\end{aligned}$$

where we have used (3.51). Thus, $K_\omega(\bar{H}_l - \omega) = I$ on $C_0^\infty(-\infty,0)^2$ and hence $K_\omega = (\bar{H}_l - \omega)^{-1}$ on $\mathscr{D}(K_\omega)$. Since $(\bar{H}_l - \omega)^{-1}$ is bounded and $\mathscr{D}(K_\omega)$ is dense, the claim follows. $\square$

We can now apply Stone's formula to $\bar{H}_l$ to state that for each $\Psi \in \mathscr{H}$ we have

$$\begin{aligned}
\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right)\Psi(u) &= \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left[(\bar{H}_l - (\omega + i\varepsilon))^{-1} - (\bar{H}_l - (\omega - i\varepsilon))^{-1}\right] \Psi(u) d\omega \\
&= \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left(\int_{-\infty}^0 \left(k_{\omega + i\varepsilon}(u,v) - k_{\omega - i\varepsilon}(u,v)\right) \Psi(v) dv \right) d\omega, \qquad (3.53)
\end{aligned}$$

where the limit is taken in $\mathscr{H}$.

3.5 A Representation Formula

In this section we obtain an integral representation formula for the solution of the Cauchy problem (3.21) via (3.53). We begin first with a proposition:

Proposition III.9. *The Wronskian $w(\zeta_{1,\omega}, \zeta_{2,\omega})$ does not vanish for $\omega \in \mathbb{R}$ (considering $\zeta_{2,0}$ when $\omega = 0$).*

Proof. We first note again that $w(\eta_{1,\omega}, \eta_{2,\omega}) = w(\zeta_{1,\omega}, \zeta_{2,\omega})$ and it therefore suffices to prove the proposition for the η solutions. For the $\omega = 0$ case, we observe that $\zeta_{1,0}, \zeta_{2,0}$ solve the ODE

$$\zeta''(u) = \frac{r^2}{T^2} l(l+1)\zeta(u), \tag{3.54}$$

subject to the asymptotic boundary conditions

$$\lim_{u \nearrow 0} s(u)^l \zeta_{2,0} r(u) = (-i)^l (2l-1)!!$$

$$\lim_{u \to -\infty} \zeta_{1,0}(u) r(u) = 1.$$

Thus, equation (3.54) with the asymptotic boundary conditions above implies that the solution $\zeta_{1,0}$ is convex. Similarly, since the solution $\zeta_{2,0}$ must be either real or purely imaginary depending on whether l is odd or even, (3.54) implies that either Re $(\zeta_{2,0})$ or Im $(\zeta_{2,0})$ is strictly convex or concave (again depending on l). In any case, this observation coupled with the asymptotic boundary conditions imply that $\zeta_{1,0}$ and $\zeta_{2,0}$ are linearly independent and thus that $w(\zeta_{1,0}, \zeta_{2,0}) \neq 0$.

In the case $\omega \in \mathbb{R} \setminus \{0\}$ we argue as in [22]. It is easy to show, using the asymptotic boundary conditions (3.29) – (3.32), that $w(\text{Re }(\eta_{j,\omega}), \text{Im }(\eta_{j,\omega})) \neq 0$ for $j = 1, 2$. Next, for $j \in \{1, 2\}$, consider $y_j := \frac{\eta'_{j,\omega}}{\eta_{j\omega}}$. An easy calculation shows that

$$\text{Im }(y_j) = \frac{w(\text{Re }(\eta_{j,\omega}), \text{Im }(\eta_{j,\omega}))}{|\eta_{j,\omega}|^2}.$$

Note that y_j is well-defined since $w(\text{Re }(\eta_{j,\omega}), \text{Im }(\eta_{j,\omega})) \neq 0$. Thus, Im $(y_j) \neq 0$ and either Im $(y_j) > 0$ or Im $(y_j) < 0$ by continuity for all $s \in (-\infty, \infty)$. Moreover, using the boundary conditions again, it's easy to show that Im (y_1) and Im (y_2) have different signs. Therefore

$$w(\eta_{1,\omega}, \eta_{2,\omega}) = \eta_{1,\omega}\eta_{2,\omega}(y_2 - y_1) \neq 0,$$

and hence, $w(\zeta_{1,\omega}, \zeta_{2,\omega}) \neq 0$. $\square$

As a consequence, we have

Corollary III.10. *The function* $h_\omega(u, v)$ *defined in* (3.50) *is continuous in* (ω, u, v) *for* $\omega \in \{Im\ (\omega) \leq 0\}$ *and* $(u, v) \in (-\infty, 0)^2$.

Proof. Note that the analytic dependence on ω of the ODE (3.25) guarantees that the ζ solutions depend at least continuously on ω for Im $(\omega) \leq 0, \omega \neq 0$. Moreover, since $h_\omega(u, v)$ is invariant under the substitution $\omega^l \zeta_{2,\omega}$ for $\zeta_{2,\omega}$, the previous proposition yields the claim. $\square$

Next observe that $\overline{h_\omega(u, v)} = h_{\bar{\omega}}(u, v)$ due to the definitions of η^1, η^2 for Im $\omega > 0$, and hence, $\overline{k_\omega(u, v)} = k_{\bar{\omega}}(u, v)$. We can then simplify (3.53) to read

$$\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right)\Psi(u) = \lim_{\varepsilon \searrow 0} -\frac{1}{\pi}\int_a^b \left(\int_{-\infty}^0 \text{Im }(k_{\omega - i\varepsilon}(u, v))\Psi(v)dv\right)d\omega, \tag{3.55}$$

where this converges in $\mathscr{H}$-norm. Since $\Psi \in C_0^\infty(-\infty, 0)^2$ and the integrand is continuous, for any bounded interval $[a, b]$ we are integrating a continuous integrand over a compact region, and if we consider this limit as a pointwise limit in u, then for any fixed u we may exchange the limit and the integration (by Lebesgue's Dominated

Convergence Theorem).[7] This observation coupled with the norm convergence yields

$$\frac{1}{2}\left(P_{[a,b]}+P_{(a,b)}\right)\Psi(u)=-\frac{1}{\pi}\int_a^b\left(\int_{\mathrm{supp}\Psi}\mathrm{Im}\ (k_\omega(u,v))\Psi(v)dv\right)d\omega. \tag{3.56}$$

Note that this yields that $P_{\{a\}}=0$ for any $a\in\mathbb{R}$, and thus that $P_{[a,b]}=P_{(a,b)}$. This in turn implies that the spectrum $\sigma(\bar{H}_l)$ is absolutely continuous. In particular, this yields

$$P_{(a,b)}\Psi(u)=-\frac{1}{\pi}\int_a^b\left(\int_{\mathrm{supp}\Psi}\mathrm{Im}\ (k_\omega(u,v))\Psi(v)dv\right)d\omega \tag{3.57}$$

for any $\Psi\in C_0^\infty(-\infty,0)^2$ and any bounded interval (a,b).

We would next like to rewrite the integrand in (3.57) in a more useful form. To this end, let us observe that for $\omega\in\mathbb{R}\setminus\{0\}$, the pair $\{\zeta_{1,\omega},\overline{\zeta_{1,\omega}}\}$ forms a fundamental system for the ODE (3.25).[8] Therefore, there exist constants (constant in u,v) $\lambda(\omega),\mu(\omega)$[9] so that

$$\zeta_{2,\omega}(u)=\lambda(\omega)\zeta_{1,\omega}(u)+\mu(\omega)\overline{\zeta_{1,\omega}}(u) \tag{3.58}$$

for $\omega\in\mathbb{R}\setminus\{0\}$. From the boundary conditions (3.29) – (3.32) (and the fact that $w(\eta_{1,\omega},\eta_{2,\omega})=w(\zeta_{1,\omega},\zeta_{2,\omega})$), it's easy to see that $w(\zeta_{1,\omega},\zeta_{2,\omega})=-2i\omega\mu(\omega)$ and thus $\mu(\omega)\neq 0$. Now, let us make the following definitions

$$\gamma_{1,\omega}(u)=\mathrm{Re}\ (\zeta_{1,\omega}(u)),\ \text{ and } \gamma_{2,\omega}(u)=\mathrm{Im}\ (\zeta_{1,\omega}(u)), \tag{3.59}$$

as well as

$$\Gamma^a_\omega(u)=(\gamma_{a,\omega}(u),\omega\gamma_{a,\omega}(u))^T. \tag{3.60}$$

Then for $\omega\neq 0$, we find that

$$\mathrm{Im}\ (h_\omega(u,v))=-\frac{1}{2\omega}\sum_{a,b=1}^{2}\alpha_{ab}(\omega)\gamma_{a,\omega}(u)\gamma_{b,\omega}(v), \tag{3.61}$$

[7] We can do this because our data is compactly supported, and we therefore don't need to distinguish between high and low energies here; c.f. [16]. However, this separation of high and low energies is essential in a more refined analysis.

[8] This is easy to check.

[9] These constants are sometimes called transmission or reflection coefficients in the literature.

where the coefficients are given by

$$\alpha_{11}(w) = 1 + \text{Re}\ \left(\frac{\lambda(\omega)}{\mu(\omega)}\right),$$
$$\alpha_{22}(\omega) = 1 - \text{Re}\ \left(\frac{\lambda(\omega)}{\mu(\omega)}\right),$$
$$\alpha_{12}(\omega) = -\text{Im}\ \left(\frac{\lambda(\omega)}{\mu(\omega)}\right) = \alpha_{21}(\omega). \tag{3.62}$$

This expression extends to $\omega = 0$ by continuity of h, and we may therefore write

$$\int\limits_{\text{supp}\Psi} \text{Im}\ (k_\omega(u,v))\Psi(v)dv = -\frac{1}{2\omega} \int\limits_{\text{supp}\Psi} r^4 \sum_{a,b=1}^{2} \alpha_{ab}(\omega)\gamma_{a,\omega}(u)\gamma_{b,\omega}(v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi(v)dv$$
$$= -\frac{1}{2\omega^2} \sum_{a,b=1}^{2} \alpha_{ab}(\omega)\Gamma^a_\omega(u) \int\limits_{\text{supp}\Psi} (\omega^2\gamma_{2,\omega}(v)\psi_1(v) + \omega\gamma_{2,\omega}(v)\psi_2(v))r^4 dv$$
$$= -\frac{1}{2\omega^2} \sum_{a,b=1}^{2} \alpha_{ab}(\omega)\Gamma^a_\omega(u)\langle\Gamma^b_\omega, \Psi\rangle_l, \tag{3.63}$$

where we have used the fact that $\omega^2 r^4\gamma_{2,\omega}(v) = (-d_v^2 + r^4 V_l(v))\gamma_{2,\omega}(v)$ and we integrated by parts. We also note that the inner product in the last line of (3.63) is well-defined because $\Psi \in C_0^\infty(-\infty, 0)^2$.

We next use (3.63) to obtain a representation formula for the solution Ψ^{lm} of the Cauchy problem (3.21):

Proposition III.11. *The solution Ψ^{lm} of the Cauchy problem* (3.21) *can be represented as*

$$\Psi^{lm}(t,u) = e^{-it\bar{H}_l}\Psi_0^{lm}(u)$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\frac{1}{\omega^2} \sum_{a,b=1}^{2} \alpha_{ab}(\omega)\Gamma^a_\omega(u)\langle\Gamma^b_\omega, \Psi_0\rangle_l d\omega, \tag{3.64}$$

where the integral converges in norm in $\mathscr{H}$.

Proof. Using (3.63) in (3.57) and applying the spectral theorem, for any $n \in \mathbb{N}$ we have

$$e^{-it\bar{H}_l}P_{(-n,n)}\Psi_0(u) = \frac{1}{2\pi}\int_{-n}^{n} e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2}\alpha_{ab}(\omega)\Gamma^a_\omega(u)\langle\Gamma^b_\omega, \Psi_0^{lm}\rangle_l d\omega.$$

Furthermore, since $e^{-it\bar{H}_l}$ is unitary, we have

$$e^{-it\bar{H}_l}P_{(-n,n)}\Psi_0 \to e^{-it\bar{H}_l}\Psi_0$$

in $\mathscr{H}$ as $n \to \infty$ which yields the claim. □

3.6 Decay

We now obtain decay results from the representation formula (3.64). To this end, we state a proposition.

Proposition III.12. *For fixed $u \in (-\infty, 0)$, the integrand in the representation* (3.64) *is in $L^1(\mathbb{R}, \mathbb{C}^2)$ as a function of ω. In particular, the representation* (3.64) *holds pointwise for $u \in (-\infty, 0)$.*

Proof. Since the integrand is continuous in ω, we are only concerned with $|\omega| \gg 1$. We must therefore investigate the asymptotic behavior of $\zeta_{1,\omega}(u)$ for $|\omega| \gg 1$, but according to the definition (3.26), we shall first analyze the asymptotic behavior of $\eta_{1,\omega}$. To do this we first construct $\eta_{1,\omega}$ in a form more conducive to analyzing large $|\omega|$ asymptotics. We consider $|\omega| \geq 1$ and first solve

$$\eta''_{1,\omega}(s) + \omega^2\eta_{1,\omega}(s) = W_l(s)\eta_{1,\omega}(s) \tag{3.65}$$

by converting this into the integral equation

$$\eta_{1,\omega}(s) = e^{i\omega s} - \int_{-\infty}^{s}\frac{\sin(\omega(s-\tilde{s}))}{\omega}W_l(\tilde{s})\eta_{1,\omega}(\tilde{s})d\tilde{s}. \tag{3.66}$$

The uniqueness of the solution to (3.66) (which is easy to show) and the uniqueness of the solution to (3.37)[10] guarantee that $\eta_{1,\omega}$ is the unique solution to both of

[10] To be more precise, the solution of the analog to (3.37) with boundary conditions at $s = -\infty$.

these integral equations (since these integral equations are derived from the same differential equation). We then write

$$\eta_{1,\omega}(s) = \sum_{k=0}^{\infty} \eta_{1,\omega}^{(k)}(s), \tag{3.67}$$

where $\eta_{1,\omega}^{(0)}(s) = e^{i\omega s}$ and

$$\eta_{1,\omega}^{(k+1)}(s) = -\int_{-\infty}^{s} \frac{1}{\omega} \sin(\omega(s-\tilde{s}))W_l(\tilde{s})\eta_{1,\omega}^{(k)}(\tilde{s})d\tilde{s},$$

which yields the following estimate

$$\left|\eta_{1,\omega}^{(k+1)}(s)\right| \leq \int_{-\infty}^{s} \frac{1}{|\omega|}|W_l(\tilde{s})| \cdot |\eta_{1,\omega}^{(k)}(\tilde{s})|d\tilde{s},$$

and if we assume, by way of induction, that

$$\left|\eta_{1,\omega}^{(k)}(s)\right| \leq \frac{1}{k!}\left(\int_{-\infty}^{s} \frac{1}{|\omega|}|W_l(\tilde{s})|d\tilde{s}\right)^k, \tag{3.68}$$

we then have

$$\begin{aligned}
\left|\eta_{1,\omega}^{(k+1)}(s)\right| &\leq \int_{-\infty}^{s} \frac{1}{|\omega|}|W_l(\tilde{s})|\frac{1}{k!}\left(\int_{-\infty}^{\tilde{s}} \frac{1}{|\omega|}|W_l(\hat{s})|d\hat{s}\right)^k \\
&= \int_{-\infty}^{s} \frac{d}{d\tilde{s}}\left[\frac{1}{(k+1)!}\int_{-\infty}^{\tilde{s}} \frac{1}{|\omega|}|W_l(\hat{s})|d\hat{s}\right]^{k+1} d\tilde{s} \\
&= \frac{1}{(k+1)!}\left(\int_{-\infty}^{s} \frac{1}{|\omega|}|W_l(\tilde{s})|d\tilde{s}\right)^{k+1}.
\end{aligned}$$

Since the induction hypothesis (3.68) obviously holds for $k = 0$, (3.68) holds for each $k \in \mathbb{N}$.

Therefore, we have the following estimate on $\eta_{1,\omega}$ from (3.67):

$$\begin{aligned}
|\eta_{1,\omega}(s)| &\leq \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{-\infty}^{s} \frac{1}{|\omega|}|W_l(\tilde{s})|d\tilde{s}\right)^k \\
&\leq e^{\frac{1}{|\omega|}\|W_l\|_{L^1}} \\
&\leq 1 + O\left(\frac{1}{|\omega|}\right),
\end{aligned}$$

for $|\omega| \gg 1$, since we know $\|W_l\|_{L^1} < \infty$. We can then use this to obtain an estimate on $\zeta_{1,\omega}(u)$ via (3.26). In particular, for fixed u we find

$$|\zeta_{1,\omega}(u)| \leq C\left(1 + \frac{1}{|\omega|}\right). \tag{3.69}$$

Next we analyze $\langle \Psi_0^{lm}, \Gamma_\omega^b \rangle_l$. We have

$$\langle \Psi_0^{lm}, \Gamma_\omega^b \rangle_l = \int\limits_{\mathrm{supp}\Psi_0^{lm}} r^4 (\psi_0^{lm})_2 \omega \gamma_{2,\omega}(u) + (\psi_0^{lm})_1' \gamma_{2,\omega}'(u) + \frac{r^2}{T^2} l(l+1)(\psi_0^{lm})_1 \gamma_{2,\omega} du,$$

but since $\gamma_{2,\omega}$ solves the ODE (3.25), we rearrange terms to find

$$\gamma_{2,\omega} = \frac{1}{\omega^2 r^4}\left(-\gamma_{2,\omega}'' + \frac{r^2}{T^2} l(l+1)\gamma_{2,\omega}\right).$$

Substituting this in the expression above and integrating by parts twice yields

$$\begin{aligned}\langle \Psi_0^{lm}, \Gamma_\omega^b \rangle_l = \frac{1}{\omega^2}\int_{\mathrm{supp}\Psi_0^{lm}} & -\gamma_{2,\omega}(u)\left(\omega(\psi_0^{lm})_2 - \frac{1}{r^4}(\psi_0^{lm})_1'' + \frac{l(l+1)}{r^2T^2}(\psi_0^{lm})_1\right)'' \\ & + \gamma_{2,\omega}(u)\frac{r^2}{T^2} l(l+1)\left(\omega(\psi_0^{lm})_2 - \frac{(\psi_0^{lm})_1''}{r^4} + \frac{l(l+1)}{r^2T^2}\right) du\end{aligned}$$

Since r, T are smooth and $\Psi_0^{lm} \in C_0^\infty(-\infty, 0)^2$, we can iterate this argument as many times as we like to obtain arbitrary polynomial decay in ω.

It remains to analyze the coefficients $\alpha_{ab}(\omega)$. To this end, consider $\lambda(\omega), \mu(\omega)$; these satisfy

$$w(\eta_{2,\omega}, \eta_{1,\omega}) = 2i\omega\mu(\omega) \text{ and } w(\eta_{2,\omega}, \overline{\eta_{1,\omega}}) = 2i\omega\lambda(\omega).$$

One proceeds exactly as in [22] (and uses the fact $w(\zeta_{2,\omega}, \zeta_{1,\omega}) = w(\eta_{2,\omega}, \eta_{1,\omega})$) to find $w(\eta_{2,\omega}, \eta_{1,\omega}) = 2i\omega + O(1)$ and $w(\eta_{2,\omega}, \overline{\eta_{1,\omega}}) = O(1)$, which implies

$$\mu(\omega) = 1 + O\left(\frac{1}{\omega}\right) \text{ and } \lambda(\omega) = O(1)$$

for $|\omega|$ large. Thus according to (3.62) the coefficients α_{ab} are bounded.

Putting all of this together, we have shown that the integrand in the representation (3.64) is in $L^1(\mathbb{R}, \mathbb{C}^2)$. Furthermore, this implies that the integral converges pointwise, and thus that the representation (3.64) holds for each $u \in (-\infty, 0)$. $\square$

As a simple corollary we now obtain decay:

Corollary III.13. *The solution Ψ^{lm} of the reduced Cauchy problem* (3.21) *tends to zero as $t \to \infty$ for fixed $u \in (-\infty, 0)$.*

Proof. According to the representation formula (3.64) and the above theorem, Ψ^{lm} is the Fourier transform of an absolutely integrable function. Thus by the Riemann-Lebesgue lemma, $\Psi^{lm}(t, u) \to 0$ for fixed u as $t \to \infty$. $\square$

Our next goal is to show decay of the solution Ψ of the problem (3.16). By the uniqueness and convergence of the spherical harmonic decomposition, one obtains a solution of (3.16) from solutions of (3.21) via (3.18) and vice versa. In particular, this implies that the solution Ψ of the problem (3.16) has the representation

$$\Psi(t, u, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} e^{-it\bar{H}_l} \Psi_0^{lm}(t, u) Y^{lm}(\theta, \phi). \tag{3.70}$$

We then have the following theorem:

Theorem III.14. *Consider the Cauchy problem for the wave equation on an SSBH background where the data is smooth and compactly supported in $(r_0, \infty) \times S^2$. The solution of this problem decays to zero in $L^\infty_{loc}((r_0, \infty) \times S^2)$ as $t \to \infty$.*

Proof. Proving that the decay of each angular mode of Ψ implies the decay of Ψ can be done exactly as in [22]. Since we have demonstrated the modal decay above, the theorem is proved. $\square$

Remark III.15. *An outline of the proof given in [22] is as follows: Define*

$$\Psi^N = \sum_{l=N}^{\infty} \sum_{|m|\leq l} \Psi^{lm} Y_{lm}.$$

Due to the modal energy conservation, there is an N_0 *so that* $\|\Psi^N\| < \varepsilon$ *uniformly in* t *for* $N \geq N_0$. *Then since* $H\Psi^N$ *solves* (3.16) *with data* $H\Psi_0^N$ *we can find* N_1 *so that* $\|H\Psi^N\| < \varepsilon$ *for uniformly in* t *for* $N \geq N_1$*; similarly there is an* N_2 *so that* $\|H^2\Psi^N\| < \varepsilon$ *uniformly in* t *for* $N \geq N_2$. *For any compact set* $K \subset\subset (-\infty, 0) \times S^2$*, the positivity of the energy density yields a constant* $C(K)$ *so that*

$$\|\Gamma\|_{H^1(K)\times L^2(K)} \leq C(K)\|\Gamma\|$$

for any $\Gamma \in C_0^\infty((-\infty) \times S^2)^2$. *Since* $H\Psi^N = (\psi_2^N, A\psi_1^N)$ *where* A *is an elliptic operator, ones uses the estimate*

$$\|f\|_{H^2(\Omega_1)} \leq C\left(\|Af\|_{H^0(\Omega_2)} + \|f\|_{H^1(\Omega_2)}\right)$$

for Ω_1, Ω_2 *compact and* $\Omega_1 \subset\subset \Omega_2$ *(c.f. [32]). Putting these facts together, one finds a constant* $C(K)$ *and a large number* N *so that*

$$\begin{aligned}\|\Psi^N\|_{H^2(K)\times H^2(K)} &\leq C(K)\left(\|\Psi^N\| + \|H\Psi^N\| + \|H^2\Psi^N\|\right) \\ &< \varepsilon.\end{aligned}$$

Then by Sobolev embedding we find $\|\Psi^N\|_{L^\infty_{loc}(K)} < \varepsilon$ *for a possibly larger* N. *Therefore*

$$|\Psi(t, u, \theta, \phi)| \leq \sum_{l=0}^{N-1} \sum_{|m|\leq l} \left|\Psi^{lm}(t, u) Y_{lm}(\theta, \phi)\right| + \varepsilon$$

uniformly in t. *Since* $\Psi^{lm}(t, u, \theta, \phi) \to 0$ *for fixed* (u, θ, ϕ) *as* $t \to \infty$*, it then follows that* $\Psi(t, u, \theta, \phi) \to 0$ *as* $t \to \infty$.

Remark III.16. *We note again that on the Kerr background, the lack of a positive definite inner product means that one cannot necessarily bound* $\|\cdot\|_{H^1(K)\times L^2}$ *in terms of* $\|\cdot\|$. *Thus the above argument fails on a Kerr background.*

3.7 Decay Rates for Spherically Symmetric Data

In this section we consider the Cauchy problem with spherically symmetric initial data. In this case, the solution is given exactly by the $l = 0$ modal solution derived above. So we need only to refine our previous results in the case $l = 0$. We recall from the previous section that we have

$$\Psi(u,t) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\left(\int_{\text{supp}\Psi_0} \text{Im } k_\omega(u,v)\Psi_0(v)dv\right)d\omega. \tag{3.71}$$

From this we obtain

$$\Psi(u(s),t) = \int_{\mathbb{R}} e^{-i\omega t} f(s,\omega)d\omega \tag{3.72}$$

where

(3.73)
$$f(s,\omega) = \int_U \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \frac{\Theta(\tilde{s}-s)\eta_{1,\omega}(s)\eta_{2,\omega}(\tilde{s}) + \Theta(s-\tilde{s})\eta_{1,\omega}(\tilde{s})\eta_{2,\omega}(s)}{w(\eta_{1,\omega},\eta_{2,\omega})}\Psi_0(u(\tilde{s}))d\tilde{s},$$

and $U := s(\text{supp}\Psi_0)$; this follows by changing variables and recalling the definition of k_ω. Thus if we show that $f(s,\omega)$ is C^1 with respect to ω and that $f_\omega(s,\omega)$ decays sufficiently fast at infinity, then we may integrate by parts in the ω integral and obtain a t^{-1} rate of decay for Ψ. To this end, we must estimate $\eta_{1,\omega}, \eta_{2,\omega}, \partial_s\eta_{1,\omega}, \partial_s\eta_{2,\omega}$ as well as the ω derivatives of these functions in the $l = 0$ case. We do this explicitly for $\eta_{2,\omega}$ terms; the $\eta_{1,\omega}$ terms can be handled similarly.

Recall that $\eta_{2,\omega}$ solves the integral equation (3.37), and note that in (3.37) with $\lambda = 1/2$, the function $B(1/2,\omega,s,y)$ simplifies to

$$B(1/2,\omega,s,y) = \frac{\sin(\omega(s-y))}{\omega} \tag{3.74}$$

and the initial function simplifies to $e^{-i\omega s}$. Thus, $\eta_{2,\omega}$ solves the integral equation

$$\eta_{2,\omega}(s) = e^{-i\omega s} + \int_s^\infty \frac{\sin(\omega(s-y))}{\omega}W_0(y)\eta_{2,\omega}(y)dy. \tag{3.75}$$

Observe next that for large s we have $W_0(s) \sim O(s^{-3})$. This fact guarantees that the kernel in the above integral equation has an ω derivative that is integrable.[11] The smoothness of the initial function as well as the smoothness and integrability of the kernel then allow us to conclude that $\eta_{2,\omega}$ exists and is smooth in s (of course we already knew this) and is C^1 with respect to ω (c.f. [16]). However, this is not enough for our purposes; we need estimates on the behavior of $\eta_{2,\omega}$ and its derivatives for large ω. Thus we consider $|\omega| \geq 1$ and we construct $\eta_{2,\omega}$ as a series. In particular we write

$$\eta_{2,\omega}(s) = \sum_{n=0}^{\infty} \eta_{2,\omega}^{n}(s), \tag{3.76}$$

where $\eta_{2,\omega}^{0}(s) = e^{-i\omega s}$ and

$$\eta_{2,\omega}^{n+1}(s) = \int_s^\infty \frac{\sin(\omega(s-y))}{\omega} W_0(y)\eta_{2,\omega}^{n}(y)dy. \tag{3.77}$$

We demonstrate convergence as in the previous section by noting the estimate

$$\left|\eta_{2,\omega}^{n}(s)\right| \leq \frac{1}{n!}\left(\int_s^\infty \frac{|W_0(y)|}{|\omega|}dy\right)^n \tag{3.78}$$

which is easily proved by induction. This bounds $\eta_{2,\omega}$ according to

$$|\eta_{2,\omega}(s)| \leq \exp\left(\int_s^\infty \frac{|W_0(y)|}{|\omega|}dy\right). \tag{3.79}$$

Next, we invoke the estimates

$$\left|\frac{\sin(\omega(s-y))}{\omega}\right| \leq \frac{1}{|\omega|},$$

$$\left|\partial_\omega\left(\frac{\sin(\omega(s-y))}{\omega}\right)\right| \leq \frac{1+|s|+|y|}{|\omega|}$$

as well as (3.78) to estimate $\left|\partial_\omega \eta_{2,\omega}^{n}\right|$. In particular, we find immediately that

$$\left|\partial_\omega \eta_{2,\omega}^{0}(s)\right| \leq |s|.$$

[11] However, second derivative of the kernel is not integrable, which restricts us to a t^{-1} decay rate.

We then suppose by way of induction that

$$\left|\partial_\omega \eta^n_{2,\omega}(s)\right| \le (1+|s|)\frac{n+1}{n!}\left(\int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|+|s|)dy\right)^n. \tag{3.80}$$

Due to the smoothness of the kernel and the integrability of the ω derivative of the kernel and the bounds (3.78), (3.80), we may compute $\partial_\omega \eta^{n+1}_{2,\omega}$ by differentiating under the integral in (3.77). We find

$$\begin{aligned}\left|\partial_\omega \eta^{n+1}_{2,\omega}(s)\right| \le & \int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|+|s|)\frac{1}{n!}\left(\int_y^\infty \frac{|W_0(\alpha)|}{|\omega|}d\alpha\right)^n dy \\ & + \int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|)\frac{n+1}{n!}\left(\int_y^\infty \frac{|W_0(\alpha)|}{|\omega|}(1+|\alpha|+|y|)d\alpha\right)^n dy \\ & \le (1+|s|)\frac{n+2}{(n+1)!}\left(\int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|+|s|)dy\right)^{n+1}.\end{aligned}$$

Therefore by induction, (3.80) holds for all n. Thus, the series

$$\sum_{n=0}^\infty \partial_\omega \eta^n_{2,\omega}(s)$$

converges uniformly in s on compact sets and since the series (3.76) also converges uniformly, we can differentiate the series (3.76) term-by-term with respect to ω. This yields an estimate on $\partial_\omega \eta_{2,\omega}$:

$$\begin{aligned}&|\partial_\omega \eta_{2,\omega}(s)| \\ &\le (1+|s|)\left(1+\int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|+|s|)dy\right)\exp\left(\int_s^\infty \frac{|W_0(y)|}{|\omega|}(1+|y|+|s|)dy\right).\end{aligned} \tag{3.81}$$

In particular this shows that for fixed s, $\partial_\omega \eta_{2,\omega}$ is bounded by a constant for all $|\omega| \ge 1$. This holds similarly for $\eta_{2,\omega}$. However, this is still not sufficient since the representation formula in (3.72) also contains s derivatives of $\eta_{2,\omega}$ in the Wronskian term. Therefore we must estimate $\partial_s \eta_{2,\omega}$ and $\partial_\omega \partial_s \eta_{2,\omega}$ for large ω. We consider now

$$\partial_s \eta^{n+1}_{2,\omega}(s) = \int_s^\infty \cos(\omega(s-y))W_0(y)\eta^n_{2,\omega}(y)dy, \tag{3.82}$$

the differentiation under the integral being justified by the smoothness and integrability of the kernel and the estimate (3.78). Moreover, one easily finds that $\left|\partial_s \eta^0_{2,\omega}(s)\right| \leq |\omega|$. An easy induction then yields

$$\left|\partial_s \eta^n_{2,\omega}(s)\right| \leq \frac{|\omega|}{n!}\left(\int_s^\infty |W_0(y)|dy\right)^n . \tag{3.83}$$

Therefore since the series

$$\sum_{n=0}^{\infty} \partial_s \eta^n_{2,\omega}(s) \tag{3.84}$$

and the series (3.76) converge uniformly in s, we can differentiate (3.76) term-by-term in s and we obtain the estimate

$$|\partial_s \eta_{2,\omega}(s)| \leq |\omega| \exp\left(\int_s^\infty |W_0(y)|dy\right). \tag{3.85}$$

Next note that $\left|\partial_\omega \partial_s \eta^0_{2,\omega}(s)\right| \leq 1 + |s\omega|$, and then assume by way of induction that

$$\left|\partial_\omega \partial_s \eta^n_{2,\omega}(s)\right| \leq \frac{n+1}{n!}(1 + |s\omega| + |\omega|)\left(\int_s^\infty (1+|y|)|W_0(y)|dy\right)^n . \tag{3.86}$$

Then by statements analogous to those above, we may compute $\partial_\omega \partial_s \eta^{n+1}_{2,\omega}$ by differentiating under the integral in (3.82). This yields

$$\begin{aligned}
\left|\partial_\omega \partial_s \eta^{n+1}_{2,\omega}(s)\right| \leq &\int_s^\infty (|s| + |y|)|W_0(y)|\frac{1}{n!}\left(\int_y^\infty |W_0(\alpha)|d\alpha\right)^n dy \\
&+ \int_s^\infty |W_0(y)|(1 + |y\omega| + |\omega|)\frac{n+1}{n!}\left(\int_y^\infty (1+|\alpha|)|W_0(\alpha)d\alpha)\right)^n dy \\
&\leq (1 + |s\omega| + |\omega|)\frac{n+2}{(n+1)!}\left(\int_s^\infty (1+|y|)|W_0(y)|dy\right)^{n+1} .
\end{aligned}$$

Thus by induction, (3.86) holds for all n. The series

$$\sum_{n=0}^{\infty} \partial_\omega \partial_s \eta^n_{2,\omega}(s)$$

therefore converges uniformly in s and we may therefore compute $\partial_\omega \partial_s \eta_{2,\omega}(s)$ by differentiating the series (3.84) term-by-term. Thus we obtain the following estimate on $\partial_\omega \partial_s \eta_{2,\omega}$:

$$|\partial_\omega \partial_s \eta_{2,\omega}(s)|$$

(3.87)
$$\leq (1+|s\omega|+|\omega|)\left((1+\int_s^\infty (1+|y|)|W_0(y)|dy\right)\exp\left(\int_s^\infty (1+|y|)|W_0(y)|dy\right).$$

Thus for fixed s and large $|\omega|$, $\partial_s\eta_{2,\omega}$ $\partial_\omega\partial_s\eta_{2,\omega}$ are $O(\omega)$. One can repeat the arguments above to obtain the same results for $\eta_{1,\omega}$. We next recall that since the $\eta_{j,\omega}$ solve (3.27) we may write

$$\eta_{j,\omega}(s) = \frac{1}{\omega^2}\left(-\eta''_{j,\omega} + W_0(s)\eta_{j,\omega}\right)$$

as well as

$$\partial_\omega \eta_{j,\omega}(s) = -\frac{2}{\omega^3}\left(-\eta''_{j,\omega}(s) + W_0(s)\eta_{j,\omega}(s)\right) + \frac{1}{\omega^2}\left(-\partial_\omega\eta_{j,\omega}(s) + W_0(s)\eta_{j,\omega}(s)\right).$$

Thus if we were to differentiate with respect to ω under the $\tilde{s}$ integral in (3.72), use the estimates we derived above, and substitute the formulas above repeatedly (using integration by parts), we would find that this quantity has arbitrary polynomial decay in ω. Thus the differentiation under the integral is permitted[12] and that $\partial_\omega f(s,\omega)$ has arbitrary polynomial decay in ω. An integration by parts in ω in (3.72) is justified[13] and from the Riemann-Lebesgue lemma we obtain a $o\left(t^{-1}\right)$ decay rate. We state this in the following theorem:

Theorem III.17. *Consider the wave equation on an SSBH background and suppose the data is spherically symmetric and compactly supported in $(r_0,\infty)\times S^2$. Then the solution ψ obeys*

$$|\psi(t,r,\theta,\phi)| \leq \frac{c}{1+t},$$

where $c>0$ depends only on r.

[12] We can check this carefully using Lebesgue's Dominated Convergence Theorem.
[13] The boundary terms vanish due to the ω decay results proved in the previous section.

3.8 Application to the EYM Equations

It was shown by Smoller, Wasserman, and Yau in [29] that there exists infinitely many black hole solutions of the SU(2) EYM equations. These solutions correspond to a metric of the form

$$ds^2 = -T^{-2}(r)dt^2 + A^{-1}(r)dr^2 + r^2 d\Omega^2,$$

which has a singularity at some horizon radius $r = r_0 > 0$ (i.e. $A(r_0) = 0$) and are smooth in the region (r_0, ∞). Moreover, the metric coefficients decay to unity at a rate $O(r^{-1})$ (see section 4 of [29]) and are bounded away from zero away from the singularity. It remains then to analyze the asymptotic behavior near the singularity and the asymptotic decay of the derivatives. To that end, let us state explicitly the differential equations satisfied by T, A:

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1 - w^2)^2}{r^2}, \tag{3.88}$$

$$2rA\left(\frac{T'}{T}\right) = \frac{(1 - w^2)^2}{r^2} + (1 - 2w'^2)A - 1. \tag{3.89}$$

There is a third equation allowing one to solve for w, but since we only wish to deduce asymptotics, we omit the equation and instead recall the relevant facts about w.

Proposition III.18. *The function w satisfies the following*

$$\lim_{r\to\infty} w^2(r) = 1, \tag{3.90}$$

$$\lim_{r\to\infty} rw'(r) = 0, \tag{3.91}$$

$$\lim_{r\searrow r_0} w^2(r) < 1, \tag{3.92}$$

$$\lim_{r\searrow r_0} |w'(r)| < \infty. \tag{3.93}$$

Moreover, the following inequality also holds,

$$\left(r_0 - \frac{(1-w^2(r_0))^2}{r_0}\right) \neq 0. \tag{3.94}$$

For proof, we refer to [29]. Now, since $A(r)$ is smooth on $[r_0, \infty)$ and $A(r_0) = 0$, a Taylor expansion yields

$$A(r) = A'(r_0) \cdot (r - r_0) + O(r - r_0)^2, \tag{3.95}$$

where, from (3.88), we have

$$A'(r_0) = \frac{1}{r_0^2}\left(r_0 - \frac{\left(1 - w(r_0)^2\right)^2}{r_0}\right) \neq 0,$$

according to (3.94). Also, (3.89) gives

$$\begin{aligned} \left(\frac{T'(r)}{T(r)}\right) &= \frac{\left(1 - w(r_0)^2\right)^2}{2r_0^3 A'(r_0)(r - r_0)} - \frac{1}{2rA'(r_0)(r - r_0)} + O(1) \\ &= \frac{-1}{2(r - r_0)} + O(1). \end{aligned} \tag{3.96}$$

Thus,

$$\frac{d}{dr}\log(T) = \frac{d}{dr}\log\left[(r - r_0)^{-\frac{1}{2}}\right] + O(1),$$

which implies that

$$\frac{d}{dr}\log\left[T \cdot (r - r_0)^{\frac{1}{2}}\right] = O(1).$$

Integrating this from r_0 to r, we obtain

$$T(r) \cdot (r - r_0)^{\frac{1}{2}} = c_1 + O(r - r_0)$$

for some constant c_1, and thus that

$$T(r) = c_1(r - r_0)^{-\frac{1}{2}} + O(r - r_0)^{\frac{1}{2}}$$

as $r \searrow r_0$. Moreover, using this in (3.96) yields

$$T'(r) = c_2(r - r_0)^{-\frac{3}{2}} + O(r - r_0)^{-\frac{1}{2}}$$

for some constant c_2. Note also that, in applying the results of this paper to the EYM equations, we make the identification $K^2 = A^{-1}$, and therefore, $K(r) = A^{-\frac{1}{2}}(r)$. Thus from (3.95) we find

$$K(r) = c_3(r - r_0)^{-\frac{1}{2}} + O(1)$$

for r near r_0. Finally, we have

$$K'(r) = -\frac{A'(r)}{2A^{\frac{3}{2}}(r)},$$

so we can write

$$\begin{aligned} K'(r) &= -\frac{1}{2}\frac{(A'(r_0) + O(r - r_0))}{(A'(r_0)(r - r_0) + O(r - r_0)^2)^{\frac{3}{2}}} \\ &= c_4(r - r_0)^{-\frac{3}{2}} + O(r - r_0)^{-\frac{1}{2}} \end{aligned}$$

for some constant c_4.

Now for the far-field decay condition, observe that from (3.88), we have

$$A'(r) = -\frac{2(w')^2}{r} + O\left(\frac{1}{r^2}\right),$$

since $A = 1 + O(r^{-1})$. From the relationship between A and K, this implies that

$$\left(\frac{K'}{K}\right) = \frac{(w')^2}{r} + O\left(\frac{1}{r^2}\right).$$

Similarly, from (3.89), we have

$$\left(\frac{T'}{T}\right) = -\frac{(w')^2}{r} + O\left(\frac{1}{r^2}\right).$$

Putting these two observations together yields

$$\frac{T'}{T} + \frac{K'}{K} = O\left(\frac{1}{r^2}\right)$$

for r tending to infinity.[14]

[14] The decay of w' actually yields that $\frac{T'}{T}$ and $\frac{K'}{K}$ are *each* bounded by $\frac{c}{r^2}$ for large r.

Thus, black hole solutions of the EYM equations do indeed satisfy the conditions of a generalized Schwarzschild black hole and we conclude that solutions of the Cauchy problem for the wave equation in these geometries must decay according to Theorem III.14. Furthermore, by Theorem III.17, this solution decays at least as fast as t^{-1} when the initial data is spherically symmetric.

CHAPTER IV

The Wave Equation in Particle-like Geometries

4.1 Introduction

Our goal in this chapter is to study the wave equation on a non-singular asymptotically flat geometric background. In particular, we wish to show that the solution decays as $t \to \infty$ and obtain a rate of decay when the initial data is spherically symmetric.

To this end consider a 4-dimensional Riemannian manifold $\mathscr{M}$ endowed with a metric g where the metric g is given by

$$ds^2 = g_{ij}dx^i dx^j = -T^{-2}(r)dt^2 + K^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

and where $r \geq 0$, $0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. We assume that the metric coefficients are globally smooth: $T, K \in C^\infty[0,\infty)$; we also assume that the metric is not degenerate: $T, K > 0$. [1] We further assume

$$K(0) = 1, \tag{4.2}$$

$$T'(0) = K'(0) = 0, \tag{4.3}$$

$$T(r) \sim 1 + O\left(\frac{1}{r}\right) \text{ and } K(r) \sim 1 + O\left(\frac{1}{r}\right) \text{ as } r \to \infty, \tag{4.4}$$

[1] Note that since we shall assume $T, K \to 1$ as $r \to \infty$, this implies T and K are bounded away from zero.

and finally

$$\frac{T'(r)}{T(r)} + \frac{K'(r)}{K(r)} \sim O\left(\frac{1}{r^2}\right) \text{ as } r \to \infty. \tag{4.5}$$

In other words, we are assuming that the $t =$ const. hyperplanes are similar to the Euclidean space $\mathbb{R}^3$ up to order r^2 near the origin, and in the far-field limit $\mathscr{M}$ is the Minkowski space $\mathbb{R}^{1+3}$ up to order r^{-1}.[2] The assumption (4.5) is equivalent to assuming that $\frac{d}{dr}\log(TK) = O\left(\frac{1}{r^2}\right)$ for large r, so we assume control on the rate at which the log of TK tends to 0, which is related to the rate at which $TK \to 1$. We call a metric satisfying these conditions a *spherically symmetric particle-like geometry* (SSPLG). These assumptions are satisfied for the important examples of particle-like geometries.[3]

We propose to study the Cauchy problem for the wave equation in this geometry. Since there is a boundary at $r = 0$, we must impose a boundary condition there. When considering black hole solutions in Chapter III, we required that the data be compactly supported away from the horizon and showed that the solution never reaches the boundary. Thus the natural boundary conditions were that the solution is zero at the horizon and at infinity. In the particle-like case we must take a different approach since there is no a priori reason why the solution of the wave equation in a particle-like geometry should be always supported away from the origin. Therefore we must determine the proper (i.e. physical) boundary condition at the origin. Changing to coordinates (t, x, y, z) in which the spatial coordinates are standard Euclidean coordinates, the metric g becomes

$$ds^2 = g_{ij}dx^i dx^j, \tag{4.6}$$

[2]The far-field conditions are identical to those imposed on an SSBH.
[3]For example, Minkowski and particle-like solutions of Einstein/Yang-Mills (EYM) with gauge group SU(2). We demonstrate that particle-like solutions of the EYM equations define an SSPLG in Section 4.6.

where the nonzero metric coefficients are given by

$$
\begin{aligned}
g_{11} &= -T^{-2}(r) \\
g_{22} &= \frac{x^2K^2(r) + y^2 + z^2}{r^2} \\
g_{24} = g_{42} &= \frac{xz(K^2(r) - 1)}{r^2} \\
g_{23} = g_{32} &= \frac{xy(K^2(r) - 1)}{r^2} \\
g_{33} &= \frac{x^2 + y^2K^2(r) + z^2}{r^2} \\
g_{34} = g_{43} &= \frac{yz(K^2(r) - 1)}{r^2} \\
g_{44} &= \frac{x^2 + y^2 + z^2K^2(r)}{r^2},
\end{aligned}
\tag{4.7}
$$

and $r = \sqrt{x^2 + y^2 + z^2}$. Note that these coefficients are globally smooth (the conditions $K(0) = 1, K'(0) = 0$ guaranteeing smoothness at the origin). This is obvious for each term except, perhaps, for the diagonal terms g_{ii}, since $K^2(r) - 1 = O(r^2)$ near the origin. Consider, for example, g_{22}. We can write

$$
g_{22} = \frac{1}{r^2}\left(\frac{x^2}{K^2} + y^2 + z^2\right) = 1 + \frac{x^2}{r^2K^2}\left(1 - K^2\right),
$$

from which we see that g_{22} is globally smooth. Similar arguments demonstrate smoothness of the other diagonal terms. The wave equation in this geometry is given by

$$
0 = g^{ij}\nabla_i\nabla_j\zeta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right)\zeta =: \Box\zeta.
\tag{4.8}
$$

Since this is a Lorentzian metric, the Laplacian will be a hyperbolic operator and we therefore have finite speed of propagation. This coupled with compactly supported initial data suggests the asymptotic boundary condition $\zeta(t, x, y, z) \to 0$ as $r \to \infty$.

We therefore study the Cauchy problem[4]

$$(4.9)\qquad \begin{cases} \Box\zeta(t,x,y,z) = 0, (x,y,z) \in \mathbb{R}^3, t > 0 \\ (\zeta, i\zeta_t)(0,x,y,z) = Z_0(x,y,z) \in C_0^\infty(\mathbb{R}^3)^2, (x,y,z) \in \mathbb{R}^3. \end{cases}$$

(We omit the asymptotic boundary condition at infinity since we will show that it is necessarily satisfied by the solution of (4.9).) Next we write out explicitly $\Box\zeta = 0$ in Cartesian coordinates:

$$(4.10)\qquad \zeta_{tt} = \sum_{i,j=1}^{3} a_{ik}\zeta_{x_i x_j} + \sum_{i=1}^{3} b_i \zeta_{x_i},$$

where the coefficients are given by

$$(4.11)\qquad \begin{aligned} a_{11} &= \frac{1}{r^2T^2}\left(\frac{x^2}{K^2} + y^2 + z^2\right), \\ a_{22} &= \frac{1}{r^2T^2}\left(x^2 + \frac{y^2}{K^2} + z^2\right), \\ a_{33} &= \frac{1}{r^2T^2}\left(x^2 + y^2 + \frac{z^2}{K^2}\right), \\ a_{12} = a_{21} &= \frac{(1-K^2)xy}{r^2T^2K^2}, \\ a_{13} = a_{31} &= \frac{(1-K^2)xz}{r^2T^2K^2}, \\ a_{23} = a_{32} &= \frac{(1-K^2)yz}{r^2T^2K^2}, \\ b_i &= \left[\frac{2}{r^2K^2}(1-K^2) - \frac{1}{rK^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right]. \end{aligned}$$

We note that we will frequently suppress the arguments of functions to ease notation. We can show as before that these coefficients are globally smooth. If we now let $v = (\zeta_x, \zeta_y, \zeta_z, \zeta_t)^T$, then we can write equation (4.10) as

$$(4.12)\qquad A\partial_t v - A_1\partial_x v - A_2\partial_y v - A_3\partial_z v - Bv = 0,$$

[4] We again use the compact form $(\psi, i\psi_t)$ for the data in what follows, since this is most convenient when we reformulate this as a Hamiltonian problem later.

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ A_i = \begin{pmatrix} 0 & 0 & 0 & a_{i1} \\ 0 & 0 & 0 & a_{i2} \\ 0 & 0 & 0 & a_{i3} \\ a_{i1} & a_{i2} & a_{i3} & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}.$$

Then, since the eigenvalues of A are $1, T^{-2}, T^{-2}$, and K^{-2} and these are all bounded away from zero, A is uniformly positive definite and the system in (4.12) is therefore a symmetric hyperbolic system (in the sense of section 5.3 in [20]). Accordingly, there exists a unique, global, smooth solution that propagates with finite speed. Coupling this with the initial data yields a solution ζ of (4.9) that is unique, smooth, globally defined, and compactly supported for each t.

We now wish to use this solution to understand the wave equation in the coordinates (t, r, θ, ϕ). In particular, it is necessary to obtain a natural boundary condition to impose at $r = 0$. Since ζ is smooth in spatially Cartesian coordinates, in spatially spherical coordinates we must have

$$\left.\frac{\partial \zeta}{\partial r}\right|_{r=0} = 0. \tag{4.13}$$

Now the wave equation in the coordinates (t, r, θ, ϕ) reads (we abuse notation slightly and consider $\zeta = \zeta(t, r, \theta, \phi)$)

$$-T^2 \zeta_{tt} + \frac{1}{K^2}\zeta_{rr} + \left(\frac{2}{K^2 r} - \frac{1}{K^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\zeta_r + \frac{\Delta_{S^2}}{r^2}\zeta = 0. \tag{4.14}$$

We are therefore interested in solving the Cauchy problem

(4.15)
$$\begin{cases} -T^2\zeta_{tt} + \frac{1}{K^2}\zeta_{rr} + \left(\frac{2}{K^2 r} - \frac{1}{K^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\zeta_r + \frac{\Delta_{S^2}}{r^2}\zeta = 0 \text{ on } \mathbb{R}\times(0,\infty)\times S^2 \\ \frac{\partial\zeta}{\partial r}\Big|_{r=0} = 0 \\ (\zeta, i\zeta_t)(0,r,\theta,\phi) = Z_0(r,\theta,\phi) \in \mathscr{A}^2, \end{cases}$$

where

$$\mathscr{A} := \big\{\psi \in C^\infty([0,\infty)\times S^2) : \psi_r\big|_{r=0} = 0 \text{ and there exists } R > 0$$
$$\text{so that } \psi(r,\theta,\phi) \equiv 0 \text{ for } r > R\big\}\,. \tag{4.16}$$

Proposition IV.1. *The Cauchy problem* (4.15) *has a globally smooth, unique solution ζ. Moreover, $\zeta \in \mathscr{A}$ for each time t.*

Proof. This follows at once from the previous discussion when we change to spherical coordinates and consider $\zeta = \zeta(t,r,\theta,\phi)$. □

Let us now define the coordinate $u = u(r)$ by

$$u(r) = -\int_r^\infty \frac{K(r')T(r')}{(r')^2}dr', \tag{4.17}$$

which maps the interval $(0,\infty)$ to $(-\infty,0)$.[5] We record some asymptotics of u which will be useful later. For large r, we have $u \nearrow 0$ according to

$$u(r) = -\frac{1}{r} + O\left(\frac{1}{r^2}\right) \text{ and } \frac{1}{u} = -r + O(1). \tag{4.18}$$

For r small we have $u \to -\infty$ according to

$$u(r) = -\frac{1}{r} + O(1) \text{ and } \frac{1}{u} = -r + O(r^2). \tag{4.19}$$

[5] We remark again that we use this coordinate precisely because the wave equation in this variable has a positive definite energy. We will exploit this as in Chapter III.

If we let $\psi(t,u,\theta,\phi) = \zeta(t,r(u),\theta,\phi)$, then ψ satisfies

$$\left(-r^4\partial_t^2 + \partial_u^2 + \frac{r^2}{T^2}\Delta_{S^2}\right)\psi = 0 \text{ on } \mathbb{R}\times(-\infty,0)\times S^2. \tag{4.20}$$

Furthermore, ψ is the unique, global, smooth solution of the Cauchy problem[6]

$$\begin{cases} \left(-r^4\partial_t^2 + \partial_u^2 + \frac{r^2}{T^2}\Delta_{S^2}\right)\psi = 0 \text{ on } \mathbb{R}\times(-\infty,0)\times S^2 \\ \psi_u = O\left(\frac{1}{u^3}\right) \text{ as } u\to-\infty \\ (\psi, i\psi_t)(0,r,\theta,\phi) = \Psi_0(r,\theta,\phi) \in \mathscr{B}^2, \end{cases} \tag{4.21}$$

where $\psi \in \mathscr{B}$ if and only if $\psi \in C^\infty((-\infty,0)\times S^2)$ and

(i) there exists $u_0 < 0$ so that $\psi(u,\theta,\phi) \equiv 0$ for all $u > u_0$;

(ii) $\psi_u = O\left(\frac{1}{u^3}\right)$ as $u \to -\infty$; and

(iii) ψ and all its derivatives have finite limits as $u \to -\infty$.

Observe that $\mathscr{B}$ and $\mathscr{A}$ are related to each other, since for any $\zeta(r,\theta,\phi) \in \mathscr{A}$, $\psi(u,\theta,\phi) := \zeta(r(u),\theta,\phi) \in \mathscr{B}$. To see this, note that $\zeta_r = O(r)$ for small r, and thus we have

$$\psi_u = \zeta_r\frac{dr}{du} = \zeta_r\frac{r^2}{KT} = O(r^3)$$

for small r. Owing to the asymptotics in (4.19) for small r, it follows that $\psi_u = O\left(\frac{1}{u^3}\right)$ as $u \to -\infty$. Recalling also that ζ is smooth up to the origin, it follows that ψ and all the derivatives of ψ have finite limits as $u \to -\infty$. In fact, one easily finds that $\partial_u^m\psi = O\left(\frac{1}{u^{m+2}}\right)$ for $m \geq 1$ as $u \to -\infty$.

We also note that $\psi \in \mathscr{B}$ for all times t. This follows from the above observations and the fact that $\zeta \in \mathscr{A}$ for all times t. Thus the energy

(4.22)

$$E(\psi) := \int_0^{2\pi}\int_{-1}^{1}\int_{-\infty}^{0} r^4(\psi_t)^2 + (\psi_u)^2 + \frac{r^2}{T^2}\left(\frac{1}{\sin^2\theta}(\partial_\phi\psi)^2 + \sin^2\theta(\partial_{\cos\theta}\psi)^2\right) du\, d(\cos\theta)\, d\phi$$

[6]The condition on ψ as $u \to -\infty$ is precisely the condition $\zeta_r|_{r=0} = 0$ after changing variables and employing the asymptotics (4.18) and (4.19).

is well-defined. Moreover, the summability guarantees that we may compute $\frac{d}{dt}E(\psi)$ by differentiating under the integral.[7] Integrating by parts, using the asymptotics (4.19) to account for the boundary terms, and using the equation (4.20) yields that $\frac{d}{dt}E(\psi) = 0$; i.e. the energy is conserved.

We next let $\Psi = (\psi, i\psi_t)^T$ and recast (4.21) as a Hamiltonian system; i.e. Ψ is the unique global solution in $\mathscr{B}^2$ for all times t of the Cauchy problem

$$\begin{cases} i\partial_t \Psi = H\Psi \text{ on } \mathbb{R}\times(-\infty,0)\times S^2 \\ \Psi(0,u,\theta,\phi) = \Psi_0(u,\theta,\phi) \in \mathscr{B}^2, \end{cases} \tag{4.23}$$

where the Hamiltonian H is given by

$$H = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \text{ and } A = -\frac{1}{r^4}\partial_u^2 - \frac{\Delta_{S^2}}{r^2T^2}. \tag{4.24}$$

We can also see that the energy functional induces an inner product on $\mathscr{B}^2$. Indeed, for $\Psi, \Gamma \in \mathscr{B}^2$, the inner product $\langle \Psi, \Gamma\rangle$ is given by

$$\begin{aligned} \langle \Psi, \Gamma\rangle = \int_0^{2\pi}\int_{-1}^{1}\int_{-\infty}^{0} & r^4\psi_2\overline{\gamma_2} + (\partial_u\psi_1)\overline{(\partial_u\gamma_1)} \\ & + \frac{r^2}{T^2}\left(\frac{1}{\sin^2\theta}(\partial_\phi\psi_1)\overline{(\partial_\phi\gamma_1)} + \sin^2\theta(\partial_{\cos\theta}\psi_1)\overline{(\partial_{\cos\theta}\gamma_1)}du d(\cos\theta)\right)d\phi, \end{aligned} \tag{4.25}$$

where $\Gamma = (\gamma_1, \gamma_2)^t$ and $\Psi = (\psi_1, \psi_2)^t$.

As in Chapter III, the Hamiltonian is symmetric with respect to this inner product.

Proposition IV.2. *H is symmetric with respect to $\langle\cdot,\cdot\rangle$ on $\mathscr{B}^2$.*

Proof. The proof is identical to that of Proposition III.3. □

[7] We can do this since according to the asymptotics (4.18) and (4.19) the coefficients on the first and third terms in the integrand decay at least as fast as $\frac{1}{u^2}$ as $u \to -\infty$, and the ψ_u term decays as $\frac{1}{u^3}$. Then, using that all the derivatives of ψ have finite limits as $u \to -\infty$, we can apply Lebesgue's dominated convergence theorem to differentiate under the integral.

We can use the spherically symmetry to reduce this from a three-dimensional problem to a one-dimensional problem by projecting our solution onto the spherical harmonics:

$$\Psi(t,u,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{|m|\leq l}\Psi^{lm}(t,u)Y_{lm}(\theta,\phi), \tag{4.26}$$

where, again, the Y_{lm} are the spherical harmonics. The inner product $\langle\cdot,\cdot\rangle$ decomposes as

$$\begin{aligned}\langle\Psi,\Gamma\rangle&=\sum_{l=0}^{\infty}\sum_{|m|\leq l}\langle\Psi^{lm},\Gamma^{lm}\rangle_l\\&=\sum_{l=0}^{\infty}\sum_{|m|\leq l}\int_{-\infty}^{0}r^4\psi_2^{lm}\overline{\gamma_2^{lm}}+(\partial_u\psi_1^{lm})\overline{(\partial_u\gamma_1^{lm})}+\frac{r^2}{T^2}l(l+1)\psi_1^{lm}\overline{\gamma_1^{lm}}du,\end{aligned} \tag{4.27}$$

and the action of the Hamiltonian decomposes as

$$H\Psi=\sum_{l=0}^{\infty}\sum_{|m|\leq l}H_l\Psi^{lm}Y_{lm}, \tag{4.28}$$

where

$$H_l=\begin{pmatrix}0&1\\A_l&0\end{pmatrix}\text{ and }A_l=-\frac{1}{r^4}\partial_u^2+\frac{l(l+1)}{r^2T^2}. \tag{4.29}$$

We also note that[8] H_l is symmetric on $\mathscr{C}_l^2$, where $\psi\in\mathscr{C}_l$ if and only if $\psi\in C^\infty(-\infty,0)$ and

(i) there exists $u_0<0$ so that $\psi(u,\theta,\phi)\equiv 0$ for all $u>u_0$;

(ii) $\psi_u=O\left(\frac{1}{u^3}\right)$ as $u\to-\infty$;

(iii) ψ and all its derivatives have finite limits as $u\to-\infty$;

(iv) if $l\neq 0$, $\psi=O\left(\frac{1}{u^2}\right)$ as $u\to-\infty$.

[8]This is different from the black hole case, since we must now account for the fact that our solution of the original problem (4.15) need not be supported away from the origin.

The symmetry statement follows since, for $\Psi^{lm}, \Gamma^{lm} \in \mathscr{C}_l^2$, we have

$$\begin{aligned}\langle H_l\Psi^{lm}, \Gamma^{lm}\rangle_l &= \langle H(\Psi^{lm}Y_{lm}), \Gamma^{lm}Y_{lm}\rangle \\ &= \langle \Psi^{lm}Y_{lm}, H(\Gamma^{lm}Y_{lm})\rangle \\ &= \langle \Psi^{lm}, H_l\Gamma^{lm}\rangle_l.\end{aligned}$$

The component functions Ψ^{lm} are global, smooth solutions of the Cauchy problem

$$\begin{cases} i\partial_t\Psi^{lm} = H_l\Psi^{lm} \text{ on } \mathbb{R}\times(-\infty,0) \\ \Psi^{lm}(0,u) = \Psi_0^{lm}(u) \in \mathscr{C}_l^2 \end{cases} \tag{4.30}$$

and $\Psi^{lm} \in \mathscr{C}_l^2$ for each time t. That the Ψ^{lm} satisfy conditions (i) - (iii) has been demonstrated; we must still verify condition (iv). This follows from the fact that if $\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{|m|\leq l}\psi^{lm}(r)Y_{lm}(\theta,\phi)$ and ψ is well-defined at the origin, then $\psi^{lm}(0) = 0$ for $l \neq 0$. Then since the solution ζ of (4.15) was smooth up to the origin with $\partial_r\zeta(t,0,\theta,\phi) = 0$, it follows that $\zeta^{lm} = O(r^2)$ near the origin for $l \neq 0$ (ζ^{lm} being the component functions in the spherical harmonic expansion of ζ). Translating this in terms of the u variable implies that, indeed, for $l \neq 0$, $\Psi^{lm} = O\left(\frac{1}{u^2}\right)$ as $u \to -\infty$. The symmetry of the Hamiltonian then implies as before that the energy $E_l(\Psi^{lm}) := \langle\Psi^{lm}, \Psi^{lm}\rangle_l$ is conserved for solutions of (4.30), and energy conservation implies that Ψ^{lm} are the unique solutions of (4.30) in $\mathscr{C}_l^2$.

4.2 Spectral Analysis & The Hamiltonian

Our goal now is to derive a representiation formula for Ψ^{lm} from which we might deduce decay. To that end, we wish to apply Stone's formula (Theorem III.7) to H_l, which expresses the spectral projections of H_l in terms of the resolvent. However, Stone's formula applies to self-adjoint operators, so we must first find a self-adjoint extension of H_l and therefore we must find a Hilbert space on which H_l is densely defined. Let us first note that we can write

$$\langle \Psi, \Gamma \rangle_l = \langle \psi_1, \gamma_1 \rangle_{l_1} + \langle \psi_2, \gamma_2 \rangle_{l_2},$$

where, of course, $\langle \cdot, \cdot \rangle_{l_1}, \langle \cdot, \cdot \rangle_{l_2}$ correspond to the terms in the integral in (4.27) acting on the first and second components of the input functions, respectively. Then we let

$$\mathscr{H}_{r^2} := \left(\left\{ \psi : r^2 \psi \in L^2(-\infty, 0) \right\}, \langle \cdot, \cdot \rangle_{l_1} \right)$$

and

$$\mathscr{H}^1_{V_l} := \left(\left\{ \psi : \psi_u \in L^2(-\infty, 0) \text{ and } r^2 V_l^{\frac{1}{2}} \psi \in L^2(-\infty, 0) \right\}, \langle \cdot, \cdot \rangle_{l_2} \right),$$

where $V_l = \frac{l(l+1)}{r^2 T^2}$. Then we take $\mathscr{H}_{r^2,0}, \mathscr{H}^1_{V_l,0}$ to be the completion of $\mathscr{C}_l$ within $\mathscr{H}_{r^2}, \mathscr{H}^1_{V_l}$, respectively. Finally, we take $\mathscr{H} = \mathscr{H}_{r^2,0} \oplus \mathscr{H}^1_{V_l,0}$.

Proposition IV.3. *The operator H_l with domain $\mathscr{D}(H_l) = \mathscr{C}_l^2$ is essentially self-adjoint in the Hilbert space $\mathscr{H}$.*

Proof. To prove this, we again use Stone's Theorem (Theorem III.6). Now consider the Cauchy problem (4.30). We have already demonstrated that there is a unique solution Ψ^{lm} to this problem in $\mathscr{C}_l^2$ for each time t. Therefore we may define the operators

$$U(t) : \mathscr{C}_l^2 \mapsto \mathscr{C}_l^2 \text{ by}$$

$$U(t)\Psi_0^{lm} = \Psi^{lm}(t).$$

The energy conservation guarantees that the $U(t)$ are unitary on $\mathscr{C}_l^2$ with respect to $\langle \cdot, \cdot \rangle_l$ and they therefore extend to unitary operators on $\mathscr{H}$. The uniqueness of the solution to (4.30) guarantees that $U(0) = Id.$ and $U(t)U(s) = U(t+s)$ for all $t, s \in \mathbb{R}$, and thus the $U(t)$ form a one-parameter unitary group on $\mathscr{H}$.

We next wish to show that the $U(t)$ are strongly continuous on $\mathscr{H}$. Thus let $\Psi \in \mathscr{H}$. Then there exists $(\Psi_n) \subset \mathscr{C}_l^2$ such that $\Psi_n \to \Psi$ and we have

$$\|U(t)\Psi - \Psi\| \leq \|U(t)\Psi - U(t)\Psi_n\| + \|U(t)\Psi_n - \Psi_n\| + \|\Psi_n - \Psi\|.$$

Thus since the $U(t)$ are unitary and since $U(t)$ is obviously strongly continuous on $\mathscr{C}_l^2$, it follows that the $U(t)$ are strongly continuous on $\mathscr{H}$. Moreover, the smoothness of the solution guarantees that the $U(t)$ are strongly differentiable on $\mathscr{C}_l^2$. A simple calculation shows that for $(\psi_1, \psi_2)^T \in \mathscr{C}_l^2$,

$$\lim_{h \searrow 0} \frac{1}{h} \left(U(h)(\psi_1, \psi_2)^T - (\psi_1, \psi_2)^t \right) = (-i\psi_2, -iA_l\psi_1)^T = -iH_l(\psi_1, \psi_2)^T,$$

and thus that i^{-1} times the strong derivative of $U(t)$ is $-H_l$.

Therefore, since $\mathscr{C}_l^2$ is invariant under $U(t)$, H_l is essentially self-adjoint on $\mathscr{C}_l^2$. $\square$

From this we conclude that H_l has a unique self-adjoint extension $\bar{H}_l$ defined on a domain dense in $\mathscr{H}$ containing $\mathscr{C}_l^2$.

4.3 The Jost Solutions

As in Chapter III, to utilize Stone's formula we must study the resolvent of $\bar{H}_l$. To this end, we consider the eigenvalue equation

$$\bar{H}_l \Gamma = \omega \Gamma. \tag{4.31}$$

Since $\bar{H}_l$ is self-adjoint on a domain in $\mathscr{H}$, it follows that $\sigma(\bar{H}_l) \subset \mathbb{R}$ and that the resolvent $(\bar{H}_l - \omega)^{-1} : \mathscr{H} \mapsto \mathscr{H}$ exists for all $\omega \in \mathbb{C} \setminus \mathbb{R}$. Thus, the eigenvalue equation (4.31) has no solutions in $\mathscr{H}$ for $\operatorname{Im}\omega \neq 0$. However, (4.31) is equivalent to the ODE

$$-\gamma''(u) - \omega^2 r^4 \gamma + \frac{r^2}{T^2} l(l+1)\gamma = 0 \text{ on } (-\infty, 0), \tag{4.32}$$

where the arguments are $r = r(u)$ and $T = T(r(u))$. We will construct the resolvent out of solutions to this ODE.

To solve this ODE, let us first note that if we consider the coordinate $s(u)$ given by

$$s(u) = \int_{-\infty}^{u} r^2(u')du' \tag{4.33}$$

and let

$$\eta(s) = r(u(s))\gamma(u(s)), \tag{4.34}$$

then η solves the ODE

$$-\eta''(s) - \omega^2\eta(s) + \left(\frac{l(l+1)}{r^2T^2} - \frac{1}{rT^2K^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\eta = 0 \text{ on } (0,\infty). \tag{4.35}$$

Let us note that we may regard s as a function of r by considering $s(u(r))$, which yields

$$s(u(r)) = \int_0^r K(r')T(r')dr'. \tag{4.36}$$

We now look to construct two linearly independent solutions of the ODE (4.35), one satisfying boundary conditions at $s = 0$ and the other satisfying asymptotic boundary conditions at $s = \infty$. In what follows, we will let $\lambda = l + \frac{1}{2}$, so that $l(l+1) = \lambda^2 - \frac{1}{4}$. Again we cite [13] for the following construction.[9]

4.3.1 The Solution with Boundary Conditions at $s = 0$

We first consider the solution of (**??**) satisfying boundary conditions at $s = 0$: call this solution $\eta^1(\lambda, \omega, s)$. We shall require

$$\lim_{s \searrow 0} \eta^1(\lambda, \omega, s) s^{-\lambda - \frac{1}{2}} = 1. \tag{4.37}$$

[9] One should note carefully that we are deriving solutions with boundary conditions at $s = \infty$ (as in Chapter III) and $s = 0$. We did *not* require a solution with asymptotics at $s = 0$ in the previous chapter. However, as in the case of a black hole, the solution with boundary conditions at $s = 0$ will be analytic in ω and thus the solution with boundary conditions at $s = \infty$ will determine the properties of the solution. This fact suggests to us that the wave equation on a particle-like background should have decay properties similar to those on an SSBH background.

We will construct $\eta^1(\lambda, \omega, s)$ as a series, so let us define

$$\eta_0^1(\lambda,\omega,s) = \left(\frac{2}{\omega}\right)^{\lambda} \Gamma(\lambda+1)\sqrt{s}J_\lambda(\omega s) \text{ for } \omega \neq 0, \tag{4.38}$$

where $\Gamma(\cdot)$ is the gamma function and $J_\lambda(\cdot)$ is the Bessel function of the first kind (c.f. [33] on Bessel and Hankel functions and [34] on Bessel functions). Then we rewrite the ODE (4.35) as

(4.39)
$$\eta''(s)+\left(\omega^2 - \frac{\lambda^2-\frac{1}{4}}{s^2}\right)\eta(s) = \left(\left(\lambda^2 - \frac{1}{4}\right)\left[\frac{1}{r^2T^2} - \frac{1}{s^2}\right] - \frac{1}{rT^2K^2}\left(\frac{T'}{T} + \frac{K'}{K}\right)\right)\eta(s).$$

The Green's function for the operator on the left-hand side of the above equation (satisfying zero boundary conditions at $s = 0$) is

$$G(\lambda,\omega,s,y) = \Theta(s-y)\frac{1}{2\lambda}\left(\eta_0^1(\lambda,\omega,s)\eta_0^1(-\lambda,\omega,y) - \eta_0^1(-\lambda,\omega,s)\eta_0^1(\lambda,\omega,y)\right), \tag{4.40}$$

where $\Theta(\cdot)$ is the usual Heaviside function.

One then obtains the integral equation

$$\eta^1(\lambda,\omega,s) = \eta_0^1(\lambda,\omega,s) + \int_0^s G(\lambda,\omega,s,y)W(y)\eta^1(\lambda,\omega,y)dy, \tag{4.41}$$

where

$$W(y) = \left(\lambda^2 - \frac{1}{4}\right)\left(\frac{1}{r^2T^2} - \frac{1}{y^2}\right) + V(y), V(y) = -\frac{1}{rT^2K^2}\left(\frac{K'}{K} + \frac{T'}{T}\right). \tag{4.42}$$

Note that since W is integrable, a smooth solution of this integral equation will indeed be a solution of the ODE (4.35) with boundary conditions (4.37).

We next show that $W(y)$ is integrable. To this end, note that $V(s) = O(1)$ as $s \to 0$ (due to $K'(0) = 0 = T'(0)$) and that $V(s) = O\left(\frac{1}{s^3}\right)$ as $s \to \infty$, since $s \sim r$ for large s and the condition (4.5). To study the other term in W, first note that $s \sim T(0)r + O(r^3)$ for small s, and this implies that

$$\left|\frac{1}{r^2T^2} - \frac{1}{s^2}\right| = O(1)$$

for small s. Finally, for large s, we use the conditions (4.4) which imply that

$$\left|\frac{1}{r^2T^2} - \frac{1}{s^2}\right| = O\left(\frac{\log s}{s^3}\right).$$

These statements together yield that W is $O(1)$ near the origin and decays like $\frac{\log s}{s^3}$ as $s \to \infty$, and so $\|W\|_{L^1(0,\infty)} < \infty$.

It is shown in Appendix A of [13] that for $0 < y < s$ we have

$$|G(\lambda,\omega,s,y)| \leq Ce^{|\mathrm{Im}\ \omega|(s-y)} \left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}} \left(\frac{y}{1+|\omega|y}\right)^{-\lambda+\frac{1}{2}}, \tag{4.43}$$

for some $C > 0$ depending on λ. We then write

$$\eta^1(\lambda,\omega,s) = \sum_{n=0}^{\infty} \eta^1_n(\lambda,\omega,s) \tag{4.44}$$

for

$$\eta^1_n(\lambda,\omega,s) = \int_0^s G(\lambda,\omega,s,y)W(y)\eta^1_{n-1}(\lambda,\omega,y)dy. \tag{4.45}$$

In the same appendix, it is shown that

$$|\eta^1_0(\lambda,\omega,s)| \leq Ce^{|\mathrm{Im}\ \omega|s} \left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}}. \tag{4.46}$$

Using (4.43) and (4.46) it is easy to show by induction that

$$|\eta^1_n(\lambda,\omega,s)| \leq Ce^{|\mathrm{Im}\ \omega|s} \frac{(C\cdot P(s))^n}{n!} \left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}}, \tag{4.47}$$

where

$$P(s) = \int_0^s \frac{yW(y)}{1+|\omega|y}dy. \tag{4.48}$$

Thus the series (4.44) is bounded term-by-term by an exponential series, and converges uniformly since $0 < P(s) < M$ for all s and some large M. The following bounds are then obvious:

$$|\eta^1(\lambda,\omega,s)| \leq Ce^{|\mathrm{Im}\ \omega|s} e^{CP(s)} \left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}}, \tag{4.49}$$

as well as

$$(4.50) \qquad |\eta^1(\lambda,\omega,s)-\eta_0^1(\lambda,\omega,s)| \le Ce^{|\mathrm{Im}\,\omega|s}\left(e^{CP(s)}-1\right)\left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}}.$$

We now check the smoothness of η_0^1. First we study $\frac{\partial G}{\partial s}$. To this end, we observe that a short calculation gives

$$(4.51) \qquad \frac{d}{ds}\eta_0^1(\lambda,\omega,s) = \frac{1}{2s}\eta_0^1(\lambda,\omega,s)+\lambda\eta_0^1(\lambda-1,\omega,s)-\frac{1}{\lambda+1}\left(\frac{\omega}{2}\right)^2\eta_0^1(\lambda,\omega,s).$$

Using the bound (4.46) we can bound $\frac{d}{ds}\eta_0^1(\lambda,\omega,s)$:

$$(4.52) \qquad \left|\frac{d}{ds}\eta_0^1(\lambda,\omega,s)\right| \le Ce^{|\mathrm{Im}\,\omega|s}\left(\frac{s}{1+|\omega|s}\right)^{\lambda-\frac{1}{2}},$$

and C is some constant depending on λ. We can then use this to bound $\frac{\partial G}{\partial s}$:

$$(4.53) \qquad \left|\frac{\partial G}{\partial s}(\lambda,\omega,s,y)\right| \le Ce^{|\mathrm{Im}\,\omega|(s+y)}\left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}}\left(\frac{y}{1+|\omega|y}\right)^{-\lambda-\frac{1}{2}}.$$

Thus (4.53) and (4.49) enable us to compute $\frac{d}{ds}\eta^1(\lambda,\omega,s)$ as

$$(4.54) \qquad \frac{d}{ds}\eta^1(\lambda,\omega,s) = \frac{d}{ds}\eta_0^1(\lambda,\omega,s)+\int_0^s\frac{\partial G}{\partial s}(\lambda,\omega,s,y)W(y)\eta^1(\lambda,\omega,y)dy.$$

This yields

$$(4.55) \qquad \left|\frac{d}{ds}\eta^1(\lambda,\omega,s)-\frac{d}{ds}\eta_0^1(\lambda,\omega,s)\right| \le Ce^{3|\mathrm{Im}\,\omega|s}\left(\frac{s}{1+|\omega|s}\right)^{\lambda+\frac{1}{2}},$$

since P is bounded as $s\to\infty$ and W is integrable. We can carry out a similar procedure to bound higher derivatives of G and conclude that η^1 is smooth in s for any fixed $\omega\neq 0$. Thus on can show that η^1 solves the ODE (4.35) and (obviously) satisfies the boundary conditions (4.37).

We also claim that η^1 is analytic in ω in the region $\omega\neq 0$; we will show this using Morera's theorem. First note that $\eta_0^1(\lambda,\omega,s)$, for fixed $s\in(0,\infty)$, is analytic for $\omega\neq 0$. Assume that the same holds for $\eta_n^1(\lambda,\omega,s)$, and recall the definition

(4.45). It's easy to show the continuity of η^1_{n+1} in ω using the dominated convergence theorem, the analyticity of G in ω, and the induction hypothesis. Then let C be a closed contour in $\mathbb{C} \setminus \{0\}$ and consider

$$\int_C \eta^1_{n+1}(\lambda, \omega, s) d\omega = \int_C \int_0^s G(\lambda, \omega, s, y) W(y) \eta^1_n(\lambda, \omega, y) dy d\omega.$$

Using the integrability of W and the bounds (4.43), (4.47), we may interchange the order of integration, and the analyticity of G and η^1_n then yields that the integral is zero. Morera's theorem then guarantees that η^1_{n+1} is analytic in ω, and by induction, this holds for each $n \in \mathbb{N}$. Furthermore, the uniform convergence of the series (4.44) then yields that $\eta^1(\lambda, \omega, s)$ is analytic in ω in the region $\mathbb{C} \setminus \{0\}$ for fixed $s \in (0, \infty)$.[10]

We note also that the only restriction on ω is that $\omega \neq 0$, but we claim that $\eta^1(\lambda, \omega, s)$ can be extended analytically to $\omega = 0$. To this end, we note that (4.41) has a unique solution. Since any solution of (4.35) with the specified asymptotics at $s = 0$ must solve the integral equation (4.41), there is only one solution of (4.35) with these asymptotics at $s = 0$. Now, to show that η^1 may be extended analytically to $\omega = 0$, we first rewrite the ODE (4.35) as

(4.56)
$$\eta''(s) - \frac{\lambda^2 - \frac{1}{4}}{s^2} \eta(s) = \left(\left(\lambda^2 - \frac{1}{4} \right) \left[\frac{1}{r^2 T^2} - \frac{1}{s^2} \right] - \frac{1}{r T^2 K^2} \left(\frac{T'}{T} + \frac{K'}{K} \right) - \omega^2 \right) \eta(s).$$

The operator on the left-hand side here (with zero boundary conditions at $s = 0$) has the Green's function $\Theta(s - y) \left(s^{\lambda + \frac{1}{2}} y^{-\lambda + \frac{1}{2}} - y^{\lambda + \frac{1}{2}} s^{-\lambda + \frac{1}{2}} \right)$. We thus obtain the integral equation

$$\eta^{1,0}(\lambda, \omega, s) = s^{\lambda + \frac{1}{2}} + \int_0^s \left(W(y) - \omega^2 \right) \sqrt{sy} \left[\left(\frac{s}{y} \right)^{\lambda} - \left(\frac{y}{s} \right)^{\lambda} \right] \eta^{1,0}(\lambda, \omega, y) dy. \tag{4.57}$$

We again solve this with a series:

$$\eta^{1,0}(\lambda, \omega, s) = \sum_{n=0}^{\infty} \eta^{1,0}_n(\lambda, \omega, s), \tag{4.58}$$

[10] This argument also applies to the analogous fact in the previous chapter.

where $\eta^{1,0}(\lambda,\omega,s)=s^{\lambda+\frac{1}{2}}$ and

$$\eta^{1,0}_{n+1}(\lambda,\omega,s)=\int_0^s\left(W(y)-\omega^2\right)\sqrt{sy}\left[\left(\frac{s}{y}\right)^{\lambda}-\left(\frac{y}{s}\right)^{\lambda}\right]\eta^{1,0}_n(\lambda,\omega,y)dy.$$

For $0<y<s$ we use the obvious bound

$$\left[\left(\frac{s}{y}\right)^{\lambda}-\left(\frac{y}{s}\right)^{\lambda}\right]\leq 2\left(\frac{s}{y}\right)^{\lambda}$$

and we can easily show by induction that

$$|\eta^{1,0}_n(\lambda,\omega,s)|\leq\frac{s^{\lambda+\frac{1}{2}}}{n!\lambda^n}\left(\tilde{P}(s)\right)^n, \tag{4.59}$$

where

$$\tilde{P}(s)=\int_0^s\left(|W(y)|+|\omega|^2\right)dy.$$

We thus obtain the bound

$$\left|\eta^{1,0}(\lambda,\omega,s)\right|\leq s^{\lambda+\frac{1}{2}}\exp\left(\frac{\tilde{P}(s)}{\lambda}\right). \tag{4.60}$$

Thus $\eta^{1,0}$ exists and (4.58) converges uniformly on compact sets. Now a simple induction shows that each term in the series (4.58) is analytic, and thus $\eta^{1,0}$ is analytic in ω, say, for $|\omega|\leq 1$. We can follow a procedure similar to the one above to verify that $\eta^{1,0}$ is smooth in s for $s>0$ and fixed ω. Thus, $\eta^{1,0}$ is a solution of the ODE (4.35) and, due to the boundary conditions, it also solves the integral equation (4.41). Using the uniqueness above, it follows that $\eta^1=\eta^{1,0}$.[11] Thus, η^1 may be extended analytically to $\omega=0$. We remark that the uniqueness also guarantees that $\eta^1(\lambda,\bar{\omega},s)=\overline{\eta^1(\lambda,\omega,s)}$, and we note that, as can be seen from the construction above, $\eta^{1,0}$ is real-valued for $\omega\in\mathbb{R}$, and hence, η^1 is real valued for $\omega\in\mathbb{R}$.

[11]One might justifiably ask why we bother at all with η^1. The reason is that the asymptotics for large ω are imperative to obtain a decay result, but it is difficult to analyze $\eta^{1,0}$ for large ω.

4.3.2 The Solution with Boundary Conditions at $s = \infty$

We wish now to construct a solution of (4.35) satisfying asymptotic boundary conditions as $s \to \infty$. This construction can be done as in Chapter III, and we are thus justified in stating the existence of a function $\eta^2(\lambda, \omega, s)$ satisfying

$$\lim_{s\to\infty} \eta^2(\lambda, \omega, s) e^{i\omega s} = 1 \tag{4.61}$$

and obeying the bounds

$$|\eta^2(\lambda, \omega, s)| \leq C e^{(\mathrm{Im}\ \omega)s} \left(\frac{|\omega| s}{1 + |\omega| s} \right)^{-\lambda + \frac{1}{2}} e^{CQ(s)}, \tag{4.62}$$

where

$$Q(s) = \int_s^\infty \frac{y|W(y)|}{1 + |\omega| y} e^{(|\mathrm{Im}\ \omega| + \mathrm{Im}\ \omega)y} dy.; \tag{4.63}$$

$$|\eta^2(\lambda, \omega, s) - \eta_0^2(\lambda, \omega, s)| \leq C e^{(\mathrm{Im}\ \omega)s} \left(\frac{|\omega| s}{1 + |\omega| s} \right)^{-\lambda + \frac{1}{2}} (e^{CQ(s)} - 1); \tag{4.64}$$

$$\left| \frac{d}{ds} \eta_0^2(\lambda, \omega, s) \right| \leq C|\omega| e^{(\mathrm{Im}\ \omega)s} \left(\frac{|\omega| s}{1 + |\omega| s} \right)^{-\lambda - \frac{1}{2}}; \tag{4.65}$$

and

$$\begin{aligned} &\left| \frac{d}{ds} \eta^2(\lambda, \omega, s) - \frac{d}{ds} \eta_0^2(\lambda, \omega, s) \right| \\ &\quad \leq C \left(\frac{|\omega| s}{1 + |\omega| s} \right)^{-\lambda - \frac{1}{2}} e^{(\mathrm{Im}\ \omega)s} \int_s^\infty \left(\frac{|\omega| y}{1 + |\omega| y} \right)^{-\lambda + \frac{1}{2}} e^{CQ(y)} |W(y)| dy \end{aligned} \tag{4.66}$$

for $\mathrm{Im}\ \omega \leq 0$, $\omega \neq 0$. Moreover, we can extend $\eta^2(\lambda, \omega, s)$ to $\omega = 0$ and obtain a solution for $\mathrm{Im}\ \omega > 0$ via $\eta^2(\lambda, \omega, s) = \overline{\eta^2(\lambda, \bar{\omega}, s)}$ as before.

4.3.3 Constructing the Resolvent

For $\mathrm{Im}\ \omega < 0$, the behavior of η^1 at the origin and the exponential decay of η^2 as $s \to \infty$ imply that η^1, η^2 are linearly independent, since their being linearly dependent would force the existence of a nontrivial vector in the kernel of $(\bar{H}_l-\omega)^{-1}$. However, since $\bar{H}_l$ is self-adjoint, the spectrum is real and thus the kernel of $(\bar{H}_l-\omega)^{-1}$ is trivial. Thus, as in Chapter III, $w(\eta^1,\eta^2)=0$.

For $\omega\in\mathbb{R}$, η^1 is real and thus has constant phase. However, for $\omega\neq 0$ the boundary conditions (3.36) imply that η^2 is of variable phase. This implies that η^1,η^2 are linearly independent for real $\omega\neq 0$ and thus that the Wronskian is nonzero for real $\omega\neq 0$.

For $\omega=0$ we consider the extensions of η^1,η^2 to $\omega=0$. We recall that, according to the definition (4.34), there exist solutions $\gamma^1(\lambda,\omega,u),\gamma^2(\lambda,\omega,u)$ of (4.32) corresponding to $\eta^1(\lambda,\omega,s),\eta^2(\lambda,\omega,s)$, respectively. Using the asymptotics of u in (4.19) and the definition of s, we find[12]

$$s\sim-\frac{T(0)}{u}\left(1+O\left(\frac{1}{u}\right)\right)$$

for $u\to-\infty$. Now let us investigate the asymptotic behavior of γ^1,γ^2. For γ^2 we have

$$\begin{aligned}1&=\lim_{s\to\infty}\eta^2(\lambda,\omega,s)\\&=\lim_{u\nearrow 0}r(u)\gamma^2(\lambda,\omega,u)\\&=\lim_{u\nearrow 0}\left(-\frac{1}{u}+O(1)\right)\gamma^2(\lambda,\omega,u).\end{aligned}$$

This implies that γ^2 decays as $u\nearrow 0$. For γ^1 we have

$$\begin{aligned}1&=\lim_{s\to 0}s^{-\lambda-\frac{1}{2}}\eta^1(\lambda,\omega,s)\\&=\lim_{u\to-\infty}\left(\frac{-u}{T(0)}\right)^{\lambda+\frac{1}{2}}\left(1+O\left(\frac{1}{u}\right)\right)r(u)\gamma^1(\lambda,\omega,u)\\&=\lim_{u\to-\infty}\left(\frac{-u}{T(0)}\right)^{\lambda+\frac{1}{2}}\left(1+O\left(\frac{1}{u}\right)\right)\left(-\frac{1}{u}+O\left(\frac{1}{u^2}\right)\right)\gamma^1(\lambda,\omega,u).\end{aligned}$$

[12] Since the definition of s obviously implies $s\sim T(0)r+O(r^3)$ for small r.

This implies that γ^1 either decays to zero or to a constant as $u \to -\infty$ (depending on λ). From (4.32) with $\omega = 0$, we see that γ^1 and γ^2 are either strictly concave or convex.[13] Thus, γ^1 and γ^2 must be linearly independent. In particular, since they solve the same ODE, the Wronskian $w(\gamma^1, \gamma^2)$ is nonzero. Furthermore, an easy calculation shows that $w(\eta^1, \eta^2) = w(\gamma^1, \gamma^2)$. We have thus shown that $w(\gamma^1(\lambda, \omega, u), \gamma^2(\lambda, \omega, u))$ is never zero.

Thus, the function $h(\omega, u, v)$ defined by

$$h(\omega, u, v) = -\frac{1}{w(\gamma^1(\lambda, \omega, u), \gamma^2(\lambda, \omega, u))} \begin{cases} \gamma^1(\lambda, \omega, u)\gamma^2(\lambda, \omega, v), & u \leq v \\ \gamma^1(\lambda, \omega, v)\gamma^2(\lambda, \omega, u), & v < u \end{cases} \tag{4.67}$$

is well-defined.[14] It is clear that h is continuous in u, v for fixed $\omega \in \mathbb{C}$, but moreover, h is also continuous in ω over all of $\mathbb{C}$ for fixed u, v. The only possible difficulty comes near $\omega = 0$; however since h is unchanged if we consider $\omega^{\lambda - \frac{1}{2}}\gamma^2$ instead of γ^2, the continuity follows. We next claim that

Proposition IV.4. *The function $h(\omega, u, v)$ defined in (4.67) satisfies*

$$\int_{-\infty}^{0} h(\omega, u, v) \left(-\partial_v^2 - r^4(v)\omega^2 + \frac{r^2}{T^2} l(l+1) \right) \gamma(v) dv = \gamma(u) \tag{4.68}$$

for any $\gamma \in C_0^\infty(-\infty, 0)$. A similar result holds if we integrate over u instead of v.

Proof. This follows from a simple calculation where we split the integral into $\int_{-\infty}^{u}$ and $\int_{u}^{0}$ and integrate by parts in these regions (which is justified since h is smooth in v in these regions). A similar calculation proves the second statement. $\square$

Let us now compute the resolvent. To this end, we define an integral operator S_ω

[13] This is obvious for γ^1; for γ^2 we refer to the corresponding discussion in Chapter III where we invoke the specific asymptotics of $\eta^{2,0}$ as $\omega \to 0$.

[14] Since we are considering a fixed mode, we omit the functional dependence of λ in h.

acting on the domain $\mathscr{D}(S_\omega) = \left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in \mathscr{C}_l^2\right\}$, with $S_\omega\Phi$ being given by

(4.69)
$$(S_\omega\Phi)(u) = \int_{-\infty}^{0} \left[\delta(u-v)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + r^4(v)h(\omega,u,v)\begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix}\right]\Phi(v)dv.$$

We next claim that $S_\omega = (\bar{H}_l - \omega)^{-1}$ on $\mathscr{H}$. To see this, first note that $\mathscr{D}(S_\omega)$ is dense in $\mathscr{H}$. This can be argued easily, and indeed, the argument is identical to the one presented in [22]. However, we can go further; we claim that, in fact, the set $\left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in C_0^\infty(-\infty,0)^2\right\}$ is dense in $\mathscr{D}(S_\omega)$ and thus dense in $\mathscr{H}$. To prove it, we must show that for each $\Phi \in \mathscr{C}^2$ there exists a sequence $\Phi_n \in C_0^\infty(-\infty,0)^2$ such that $(H_l - \omega)(\Phi_n - \Phi) \to 0$ as $n \to \infty$ (in the norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle_l$). Recalling the specific form of H_l in (4.29), we compute for $\Phi = (\phi_1,\phi_2)^T \in \mathscr{C}^2$,

$$\begin{aligned}
&\|\left(\bar{H}_l - \omega\right)\Phi\|^2 \\
&= \|(H_l - \omega)\Phi\|^2 \\
&= \int_{-\infty}^{0} r^4|A_l\phi_1 - \omega\phi_2|^2 + |\partial_u\phi_2 - \omega\partial_u\phi_1|^2 + \frac{r^2}{T^2}l(l+1)|\phi_2 - \omega\phi_1|^2 du \\
&\leq 2\int_{-\infty}^{0} r^4|A_l\phi_1|^2 + r^4|\omega|^2|\phi_2|^2 + |\partial_u\phi_2|^2 + |\omega|^2|\partial_u\phi_1|^2 \\
&\qquad + \frac{r^2}{T^2}l(l+1)|\phi_2|^2 + \frac{r^2}{T^2}l(l+1)|\omega|^2|\phi_1|^2 du.
\end{aligned}$$

Now, if we define $\psi_i(r) = \phi_i(u(r))$, then we get $\partial_r\psi_i(r)\frac{r^2}{KT} = \partial_u\phi_i(u(r))$, owing to the definition of $u(r)$ in (4.17). We also find $\partial_u^2\phi_i(u(r)) = \frac{r^2}{KT}\partial_r\left(\partial_r\psi_i(r)\frac{r^2}{KT}\right)$. Now plugging in the specific form of A_l and changing from the u variable to the r variable, we find

$$\begin{aligned}
\|(\bar{H}_l - \omega)\|^2 \leq 2\int_0^\infty &\frac{2}{KTr^2}\left|\partial_r\left(\partial_r\psi_1\frac{r^2}{KT}\right)\right|^2 + \frac{2K(l(l+1))^2}{T^3r^2}|\psi_1|^2 + KTr^2|\omega|^2|\psi_2|^2 \\
&+ \frac{r^2}{KT}|\partial_r\psi_2|^2 + |\omega|^2\frac{r^2}{KT}|\partial_r\psi_1|^2 + \frac{Kl(l+1)}{T}|\psi_2|^2 + \frac{K}{T}l(l+1)|\omega|^2|\psi_1|^2 dr
\end{aligned}$$

The first two terms in the above integrand appear troublesome. First we study the second term when $l \neq 0$. We recall that $\phi_1 \in \mathscr{C}_l$ implies that ψ_1 vanishes outside of a large ball (say of radius R). Thus, for some $r_0 > 0$, we have

$$\begin{aligned}
\int_0^\infty \frac{|\psi_1|^2}{r^2} dr &= \int_0^R \frac{|\psi_1|^2}{r^2} dr \\
&= \int_0^{r_0} \frac{|\psi_1|^2}{r^2} dr + \int_{r_0}^R \frac{|\psi_1|^2}{r^2} dr \\
&\leq \frac{1}{r_0^2} \int_0^\infty |\psi_1|^2 dr + C \int_0^{r_0} |\psi_1| dr, \text{ since } \psi_1 = O(r^2) \text{ near } r = 0 \\
&\leq \frac{1}{r_0^2} \int_0^\infty |\psi_1|^2 dr + 2Cr_0^2 \int_0^{r_0} |\psi_1|^2 dr, \text{ using Hölder} \\
&\leq C \int_0^\infty |\psi_1|^2 dr.
\end{aligned}$$

Thus, this term is actually bounded. To study the first term, we simply observe that, due to the smoothness and specific behavior of T, K, and the fact $\partial_r \psi_1 = O(r)$ near the origin, the absolute value will have at least one power of r inside it, and this will control the r^2 in the denominator. Using that T, K are bounded above and bounded away from zero, as well as the fact that ψ_1, ψ_2 vanish outside of a large ball, we can bound the above by the H^2 norms of ψ_1, ψ_2. More precisely,

$$\|(H_l - \omega)\Phi\|^2 \leq C(1 + |\omega|^2)\|\Psi\|^2_{H^2(0,\infty)^2},$$

where C depends on l and the support of Ψ. Now, since $C_0^\infty(0,\infty)^2$ is dense in $H^2(0,\infty)^2$, this shows that for $(\Psi_n) \subset C_0^\infty(0,\infty)^2$ with $\|\Psi - \Psi_n\| \to 0$ as $n \to \infty$, we can find a sequence $(\Phi_n) = (\Psi(r(u))_n)$ so that $(\Phi_n) \subset C_0^\infty(-\infty, 0)^2$ and $\|(H_l - \omega)(\Phi - \Phi_n)\| \to 0$ as $n \to \infty$. Therefore, given $\Phi \in \mathscr{C}_l^2$, we consider $\Psi(r) := \Phi(u(r))$. Then Ψ is surely in $H^2(0,\infty)^2$, and we may therefore find a sequence $(\Psi_n) \subset C_0^\infty(0,\infty)^2$ so that $\Psi_n \to \Psi$ as $n \to \infty$. Then define a sequence $(\Phi_n) \subset C_0^\infty(-\infty, 0)$ by $\Phi_n(u) = \Psi_n(r(u))$. By the above we know that

$(H_l - \omega)(\Phi - \Phi_n) \to 0$ as $n \to \infty$. Thus $\left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in C_0^\infty(-\infty, 0)^2\right\}$ is dense in the set $\left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in \mathscr{C}_l^2\right\}$, and our claim is proved.

Now using equation (4.68) it is easy to check that for $\Psi \in C_0^\infty(-\infty, 0)$, we have $(S_\omega(\bar{H}_l - \omega)\Psi)(u) = \Psi(u)$. In other words, $S_\omega = (\bar{H}_l - \omega)^{-1}$ on

$$\left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in C_0^\infty(-\infty, 0)^2\right\}.$$

Then since the resolvent is a bounded operator and $\left\{(\bar{H}_l - \omega)\Gamma : \Gamma \in C_0^\infty(-\infty, 0)^2\right\}$ is dense in $\mathscr{D}(S_\omega)$, which is dense in $\mathscr{H}$, it follows that $S_\omega = (\bar{H}_l - \omega)^{-1}$ on $\mathscr{H}$.

Now, according to Stone's formula (Theorem III.7), if we let $k(\omega, u, v)$ denote the kernel of the operator S_ω, then for any $\Psi \in \mathscr{H}$ we have

$$\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right)\Psi(u) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \int_{-\infty}^0 (k(\omega + i\varepsilon, u, v) - k(\omega - i\varepsilon, u, v))\Psi(v)dvd\omega.$$

Recalling that $\eta^i(\lambda, \bar{\omega}, s) = \overline{\eta^i(\lambda, \omega, s)}$ and noting that the same must therefore hold for the γ^i, this implies that $h(\omega + i\varepsilon, u, v) = \overline{h(\omega - i\varepsilon, u, v)}$ for $\omega \in \mathbb{R}$, and thus $k(\omega + i\varepsilon, u, v) = \overline{k(\omega - i\varepsilon, u, v)}$. This yields

$$\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right)\Psi(u) = \lim_{\varepsilon \searrow} -\frac{1}{\pi} \int_a^b \int_{-\infty}^0 \operatorname{Im}(k(\omega - i\varepsilon, u, v))\Psi(v)dv \tag{4.70}$$

and we note again that this converges in $\mathscr{H}$. In particular, we would like to derive a spectral representation for the data Ψ_0^{lm}. We would like to consider the representation in (4.70) and interchange the limit and the integral, so we must analyze $\operatorname{Im}(k(\omega - i\varepsilon, u, v))$. Indeed, we know that by properties of γ^1, $h(\omega - i\varepsilon, u, v)$ tends to a constant or decays as $u = -\infty$. There is however a factor of r^4 in $\operatorname{Im}(k)$ to enforce decay. Indeed, as $u \to -\infty$, $r^4 \sim O\left(\frac{1}{u^4}\right)$, and since Ψ_0^{lm} tends to a constant as $u \to -\infty$, we see that we are justified in switching the order of the limit and the integration (also using the continuity of Im k), for fixed u. From the norm convergence implied in Stone's formula, we thus obtain the spectral representation of Ψ_0^{lm}:

$$\frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right)\Psi_0^{lm}(u) = -\frac{1}{\pi}\int_a^b \int_{-\infty}^0 \mathrm{Im}(k(\omega, u, v))\Psi_0^{lm}(v)dvd\omega. \tag{4.71}$$

This yields that $P_{\{a\}} = 0$ for any $a \in \mathbb{R}$ and that the spectrum of $\bar{H}_l$ is absolutely continuous. Thus we have

$$P_{(a,b)}\Psi_0^{lm}(u) = -\frac{1}{\pi}\int_a^b \int_{-\infty}^0 \mathrm{Im}(k(\omega, u, v))\Psi_0^{lm}(v)dvd\omega. \tag{4.72}$$

Finally, using the spectral theorem and the fact that $e^{-it\bar{H}_l}$ is unitary, we obtain as before a representation for $\Psi^{lm}(t, u)$:

$$\Psi^{lm}(t, u) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t} \int_{-\infty}^0 \mathrm{Im}(k(\omega, u, v))\Psi_0^{lm}(v)dvd\omega. \tag{4.73}$$

4.4 Decay

To show that the solution Ψ^{lm} decays, we would like to use the Riemann-Lebesgue lemma and the representation formula (4.73). In particular, if we show that the integrand within the ω-integral is in $L^1(\mathbb{R}, \mathbb{C}^2)$, then the Riemann-Lebesgue lemma guarantees that for fixed $u \in (-\infty, u)$, $\Psi^{lm}(t, u) \to 0$ as $t \to \infty$. To this end, we first derive a more useful form of the integrand. First we claim that the pair $\{\eta^2, \overline{\eta^2}\}$ forms a fundamental set for the ODE (4.35) for $\omega \in \mathbb{R} \setminus \{0\}$. To verify this, we first compute the Wronskian $w(\eta_0^2, \overline{\eta_0^2})$. An easy calculation shows that $w(\eta_0^2, \overline{\eta_0^2}) = 2i \neq 0$. Now, (4.64) and (4.66) imply that

$$w(\eta^2, \overline{\eta^2}) = w(\eta_0^2, \overline{\eta_0^2}) = 2i \neq 0, \tag{4.74}$$

since $w(\eta^2, \overline{\eta^2})$ is constant in s. Thus, the pair $\{\eta^2, \overline{\eta^2}\}$ forms a fundamental set for $\omega \in \mathbb{R} \setminus \{0\}$. This implies that $\{\gamma^2, \overline{\gamma^2}\}$ forms a fundamental set for (4.32) while $\omega \in \mathbb{R} \setminus \{0\}$. Thus there exist numbers $c(\omega), d(\omega)$ depending only on ω such that

$$\gamma^1(\lambda, \omega, u) = c(\omega)\gamma^2(\lambda, \omega, u) + d(\omega)\overline{\gamma^2(\lambda, \omega, u)} \tag{4.75}$$

where $d(\omega) \neq 0$ for all ω. Note then that (4.74) implies that $w(\gamma^1, \gamma^2) = -2id(\omega)$ and $w(\gamma^1, \overline{\gamma^2}) = 2ic(\omega)$.

Next, we let $\phi^1_\omega = \text{Re } \gamma^2, \phi^2_\omega = \text{Im } \gamma^2$, and $\Phi^a_\omega = (\phi^a_\omega, \omega\phi^a_\omega)^T$ (note that we are dropping the λ argument, since it is superfluous for these purposes, and we denote the ω dependence by a subscript). A short calculation then shows

$$\text{Im } h_\omega(u,v) = \frac{1}{2\omega}\sum_{a,b=1}^{2} \alpha_{ab}\phi^a_\omega(u)\phi^b_\omega(v) \tag{4.76}$$

where

$$\alpha_{11} = 1 + \text{Re}\left(\frac{c}{d}\right), \alpha_{22} = 1 - \text{Re}\left(\frac{c}{d}\right), \alpha_{12} = \alpha_{21} = -\text{Im}\left(\frac{c}{d}\right). \tag{4.77}$$

Now if we write $\Psi^{lm}_0 = (\psi^{lm}_{0,1}, \psi^{lm}_{0,2})^T$, we have

$$\int_{-\infty}^{0} \text{Im}(k_\omega(u,v))\Psi^{lm}(v)dv = \frac{1}{2\omega^2}\sum_{a,b=1}^{2}\alpha_{ab}\Phi^a_\omega(u)\int_{-\infty}^{0} r^4\phi^b_\omega(v)(\omega^2\psi^{lm}_{0,1} + \omega\psi^{lm}_{2,0})dv.$$

Now let us use the fact that $\phi^b(v)$ satisfies $-\partial_v^2\phi^b_\omega + \frac{r^2}{T^2}l(l+1)\phi^b_\omega = \omega^2 r^4\phi^b_\omega$. We substitute this into the above integral to obtain

$$\begin{aligned}\int_{-\infty}^{0} &\text{Im}(k_\omega(u,v))\Psi^{lm}_0(v)dv \\ &= \frac{1}{2\omega^2}\sum_{a,b=1}^{2}\alpha_{ab}\Phi^a_\omega(u)\int_{-\infty}^{0}\psi^{lm}_{0,1}\left(-\partial_v^2 + \frac{r^2}{T^2}l(l+1)\right)\phi^b_\omega + \omega r^4\psi^{lm}_{0,2}\phi^b_\omega dv.\end{aligned}$$

We now introduce the additional assumption that $\Psi^{lm}_0 \in C^\infty_0(-\infty,0)^2$.[15] Owing to this assumption, we may integrate by parts in the above integral and obtain

$$\begin{aligned}\int_{-\infty}^{0} &\text{Im}(h_\omega(u,v))\Psi^{lm}_0(v)dv \\ &= \frac{1}{2\omega^2}\sum_{a,b=1}^{2}\Phi^a_\omega(u)\int_{-\infty}^{0}(\partial_v\psi^{lm}_{0,1})(\partial_v\phi^b_\omega) + \frac{r^2}{T^2}l(l+1)\phi^b_\omega\psi^{lm}_{0,1} + \omega r^4\psi^{lm}_{0,2}\phi^b_\omega dv \\ &= \frac{1}{2\omega^2}\sum_{a,b=1}^{2}\alpha_{ab}\Phi^a_\omega(u)\langle\Psi^{lm}_0, \Phi^b_\omega\rangle_l, \end{aligned} \tag{4.78}$$

[15] This corresponds to assuming that the data in problem (4.15) is supported away from the origin. But note that, to work in the u coordinate as we have done, which maps the interval $(0,\infty)$ to $(-\infty,0)$, with $r=0$ corresponding to $u=-\infty$, this does not seem like an unreasonable requirement.

and so

$$\Psi^{lm}(t,u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2}\alpha_{ab}(\omega)\Phi_\omega^a(u)\langle\Psi_0^{lm},\Phi_\omega^b\rangle_l d\omega. \tag{4.79}$$

We have already demonstrated that the integrand above is continuous in ω, so to show the integrand is in $L^1(\mathbb{R},\mathbb{C}^2)$, we need only to analyze it for $|\omega| \gg 1$. First we recall the formulas for $c(\omega), d(\omega)$: $w(\gamma^1,\gamma^2) = -2id(\omega), w(\gamma^1,\overline{\gamma^2}) = 2ic(\omega)$. Let us fix $s = s_0 \in (0,\infty)$ and we will compute $w(\eta^1,\eta^2)(s)$. Indeed, recalling the bounds $(4.66), (4.64), (4.55), (4.50)$ and considering $\omega \in \mathbb{R}$, we have

$$\left|\eta^1(\lambda,\omega,s) - \eta_0^1(\lambda,\omega,s)\right| = O\left(\frac{1}{|\omega|^{\lambda+\frac{3}{2}}}\right), \tag{4.80}$$

$$\left|\frac{d}{ds}\eta^1(\lambda,\omega,s) - \frac{d}{ds}\eta_0^1(\lambda,\omega,s)\right| = O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right), \tag{4.81}$$

$$\left|\eta^2\lambda,\omega,s) - \eta_0^2(\lambda,\omega,s)\right| = O\left(\frac{1}{|\omega|}\right), \tag{4.82}$$

and

$$\left|\frac{d}{ds}\eta^2(\lambda,\omega,s) - \frac{d}{ds}\eta_0^2(\lambda,\omega,s)\right| = O(1). \tag{4.83}$$

Thus we have $w(\eta^1,\eta^2) = w(\eta_0^1,\eta_0^2) + O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right)$. Then an easy calculation shows that $w(\eta_0^1,\eta_0^2) = O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right)$, which implies that $w(\eta^1,\eta^2) = O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right)$. We can show similarly that $w(\eta^1,\overline{\eta^2}) = O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right)$. Thus, we have that $c, d = O\left(\frac{1}{|\omega|^{\lambda+\frac{1}{2}}}\right)$. This implies that

$$|\alpha_{ab}| \leq 1 + O\left(|\omega|^{\lambda+\frac{1}{2}}\right). \tag{4.84}$$

Next we note that $|\phi_\omega^a| \leq |\gamma^2|$. From the bound (4.62) with $\omega \in \mathbb{R}$, we have that (considering $s = s(u)$) $\gamma^2(\lambda,\omega,u) = O(1)$ for large ω. Finally we look at the term

$\langle \Psi_0^{lm}, \Phi_\omega^b \rangle_l$. We have

$$\begin{aligned}\langle \Psi_0^{lm}, \Phi_\omega^b \rangle_l &= \int_{-\infty}^{0} (\partial_v \psi_{0,1}^{lm})(\partial_v \phi_\omega^b) + \frac{r^2}{T^2} l(l+1)\gamma_\omega^b \psi_{0,1}^{lm} + \omega r^4 \psi_{0,2}^{lm} \phi_\omega^b dv \\ &= \int_{-\infty}^{0} (-\partial_v^2 \psi_{1,0}^{lm} + \omega r^4 \psi_{lm}^2 + \frac{r^2}{T^2} l(l+1)\psi_{1,0}^{lm})\phi_\omega^b dv\end{aligned}$$

$$= \frac{1}{\omega^2}\int_{-\infty}^{0}(-\partial_v^2 \psi_{1,0}^{lm} + \omega r^4 \psi_{lm}^2 + \frac{r^2}{T^2} l(l+1)\psi_{1,0}^{lm}) \left(-\partial_v^2 \phi_\omega^b + \frac{r^2}{T^2} + \frac{r^2}{T^2} l(l+1)\phi_\omega^b\right) \frac{1}{r^4} dv,$$

where we used that $-\partial_v^2 \phi_\omega^b + \frac{r^2}{T^2} l(l+1)\phi_\omega^b = \omega^2 r^4 \phi_\omega^b$. Integrating by parts and iterating this argument ad infinitum, we find that this term has arbitrary polynomial decay in ω. This polynomial decay is enough to then guarantee that the integrand in (4.79) is in $L^1(\mathbb{R}, \mathbb{C}^2)$, and then by the Riemann-Lebesgue lemma, we are assured that $\Psi^{lm}(t,u) \to 0$ for fixed u as $t \to \infty$. That the modal decay implies decay of the full solution Ψ follows almost exactly as in Chapter III. Translating this back into the r-coordinate, this implies that for fixed $r \in [0,\infty)$, the solution ζ of (4.15), under the additional requirement that $Z_0 \in C_0^\infty(R^3 \setminus \{0\})$ decays as $t \to \infty$. Thus we have the following theorem:

Theorem IV.5. *Consider problem* (4.15) *in an SSPLG. If the initial data is smooth and compactly supported in* $(0,\infty)\times S^2$*, then the solution decays to zero in* $L^\infty_{loc}([0,\infty)\times S^2)$ *as* $t \to \infty$*.*

4.5 Decay Rates for Spherically Symmetric Data

Using arguments almost identical to those presented in Section 3.7, we can prove the following theorem:

Theorem IV.6. *Consider the wave equation on an SSPLG and suppose the initial data is spherically symmetric and compactly supported in* $(0,\infty) \times S^2$*. Then the* *solution ψ obeys*

$$|\psi(t,r,\theta,\phi)| \leq \frac{c}{1+t},$$

where $c > 0$ depends only on r.

4.6 Application to Particle-like Solutions of EYM

Finally, we note that particle-like solutions of the SU(2) EYM equations satisfy the conditions (4.2) – (4.5), c.f. [28]. The behavior at the origin follows by simple Taylor expansions and the far-field behavior follows exactly as in Section 3.8. Therefore, according to Theorem IV.5, solutions of the wave equation in an SU(2) EYM particle-like geometry with data that is smooth and compactly supported in $(0,\infty)\times S^2$ decay in L^∞_{loc} as $t \to \infty$. Moreover, if the initial data is also assumed to be spherically symmetric, then by Theorem IV.6 the solution decays pointwise at least as fast as t^{-1}.

CHAPTER V

Concluding Remarks

In this chapter we will recapitulate the contents of this document and reflect on possible future studies related to this work.

5.1 Summary

In Chapter II we presented a comprehensive overview of the study of linear scalar waves on static spherically symmetric spacetimes. We introduced the necessary notions from differential geometry to facilitate the study of the wave equation on curved geometric backgrounds. We also presented Einstein's equations and considered some special solutions of those equations. We then summarized previous work in this area and gave an overview of the two methods generally used in obtaining these results. The last topic in Chapter II was an introduction to the Einstein-Yang/Mills equations, with special attention given to static solutions with $SU(2)$ gauge group.

In Chapter III we defined a generalization of a Schwarzschild black hole (an SSBH) and discussed its relevance. We then considered the wave equation on this spacetime (the initial data being smooth and compactly supported away from the horizon) and worked to derive an integral spectral representation of the solution. We did this by formulating the wave equation (in a proper coordinate system) as a Hamiltonian evolution equation and using the functional calculus to derive an integral representation for the action of the propagator e^{-itH_l} on each angular mode of the solution. This representation turned out to be the Fourier transform of an absolutely integrable function, and we therefore appealed to the Riemann-Lebesgue lemma to conclude decay of each angular mode of the solution. We then used the energy norm to bootstrap from the modal decay to decay of the full solution. We then considered the case of spherically symmetric initial data and derived a t^{-1} pointwise decay rate for the solution. Finally, we showed that the results in Chapter III applied to black hole solutions of the EYM equations.

In Chapter IV we defined a spherically symmetric particle-like geometry (SSPLG) and discussed its properties and some relevant examples. We then proceeded as in Chapter III. There are many similarities between Chapter IV and Chapter III, but the differences are not immaterial. In particular, we had to consider the boundary at $r = 0$; this required considerable effort to properly formulate the wave equation, and we also had to take special care in defining the Hilbert space on which the Hamiltonian was defined. The Jost solutions were also markedly different in this case, due to the fact that we were working over the domain $(0, \infty)$ instead of $(-\infty, \infty)$. In particular, the construction of the solution with asymptotics at $s = 0$ was analogous to the construction in Chapter III, but required different special functions; we also had to take a different approach in verifying that these solutions were linearly independent. Computing the resolvent also required much more effort in this case, due to the more complicated Hilbert space on which the Hamiltonian was defined. Regardless of these differences, we were able to obtain results similar to those in Chapter III. In particular, we obtained decay of each angular mode of the solution and decay of the full solution; we obtained a t^{-1} pointwise decay rate when the initial data is spherically symmetric; and we were able to apply these results to particle-like solutions of the EYM equations.

5.2 Physical Implications

Here we would like to briefly identify some physical implications of these results. We recall that the Einstein equations constitute a nonlinear, coupled, hyperbolic system of partial differential equations. Coupling the Einstein equations with the Einstein/Yang-Mills equations[1] results in a system of hyperbolic partial differential equations coupled with the equations for the Yang-Mills field. Since we therefore have an evolution equation for the metric, a natural question to ask is whether the metric component of a solution to the EYM equations is stable. In particular, if we perturb the metric from a solution to the EYM equations, should we expect the perturbed metric to remain close to the original metric?[2] This is a question of global, nonlinear stability and it is too difficult for us to answer. However, the decay of the wave equation on the metrics from EYM solutions does imply the linear stability of these metrics under a certain class of perturbations (typically called *axial* perturbations).[3] Of course, linear stability does not imply global, nonlinear stability, but linear instability would preclude nonlinear stability. Therefore, one is at least justified in investigating the nonlinear stability of these spacetimes.

5.3 Possible Directions

This study has raised a number of questions that are worth studying. Chapter III showed decay and obtained a rate of decay for the first angular mode of the solution of the wave equation on an SSBH background, but it would be desirable to have a decay rate for each angular mode of the solution and a decay rate for the full

[1] We consider in particular the EYM equations with gauge group $SU(2)$.
[2] We are essentially asking if these spacetimes are observable or not.
[3] We reference [6] for a detailed discussion of this idea in the case of a Schwarzschild black hole.

solution. We focused much effort on this problem, but the generality of the class of spherically symmetric black holes made it difficult to make progress on this. However, for a well-chosen subset of this class, one should be able to employ the methods in [16] and [15], in combination with our results, to obtain these decay rates. In particular, this should be done first and foremost for black hole solutions of the EYM equations. The difficulty in obtaining decay rates for each angular mode of the solution lies in ruling out the presence of a "zero energy resonance" (see [16]). If there is no zero energy resonance, one could obtain results analogous to those obtained by Donninger, Schlag, and Soffer – namely, complete understanding of the solution's long-time asymptotics.

Another direction in the black hole case is to consider only axially symmetric black holes. These are analogous to the Kerr metric, though one would like to generalize this to include, for example, EYM black holes (c.f. [21]). One would then like to answer the same questions addressed above, though this will be much more difficult since the results for even the special case of the Kerr metric are unsatisfactory. It would also be interesting to study different types of linear perturbations on the aforementioned backgrounds; in particular, it would be interesting to see analogous results for electromagnetic waves, gravitational waves, and Dirac particles. Finally, with these (and stronger) decay results, one hopes to investigate nonlinear wave equations on these backgrounds.

In the particle-like case as well it would be desirable to have decay rates for each angular mode of the solution as well as a decay rate for the full solution. The generality of a spherically symmetric particle-like geometry again makes it difficult to make progress in this direction. However, one may be able to use our results in conjunction with an analysis similar to that in [16] to obtain decay rates for each angular mode (at least for a subset of particle-like geometries). If this could be done, it may also be possible to obtain decay rates on the full solution. Finally, as in the black hole case, it would be interesting to study axially symmetric non-singular asymptotically flat backgrounds, as well as different types of perturbations.

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