LOCAL VOLUMES

by

Aurel Mihai Fulger

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Doctoral Committee:

Professor Robert K. Lazarsfeld, Chair
Professor Mircea Immanuel Mustaţă
Professor William Fulton
Professor Mattias Jonsson
Professor James P. Tappenden

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Introduction

In this dissertation we study a notion of local volume for Cartier divisors on arbitrary blow–ups of normal complex algebraic varieties of dimension greater than one, with a distinguished point. Although not directly related, this theory bears many similarities to the well–studied case of volumes of Cartier divisors on projective varieties. We use this notion to define and study local invariants of normal isolated singularities, generalizing work on surfaces done by Wahl in [Wah90]. We also compare this volume of isolated singularities to a different generalization by Boucksom, de Fernex, and Favre from [BdFF11].

Plurigenera of smooth complex projective varieties have been the subject of much research ([Iit77], [KM98], [Siu98], etc.). The rate of growth of the plurigenera leads to the the notion of the volume of a variety, which has played and important role in birational geometry in recent decades [1] (cf. [KM98], [BCHM07], [Tsu04], [HM05], [HMX10], etc.). One would hope for local analogues that could be used in the study of singularities. Indeed, local plurigenera have been studied in [Wat80], [Yau77], [Ish90], and in [Mor87] as invariants of isolated singularities appearing on normal complex algebraic varieties.

The *geometric genus* of a normal complex algebraic isolated singularity (X, x) of dimension n at least two is defined as

$$p_g(X,x) =_{\text{def}} \dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_{\widetilde{X}})_x$$

[1] For instance, the fact that the volume need not be an integer gave the first proof that varieties do not in general have smooth minimal models

for $\pi : \widetilde{X} \to X$ an arbitrary resolution of singularities. Work of S.S.T. Yau in [Yau77] shows that this invariant of the singularity can be computed analytically on X as

$$p_g(X,x) = \dim \frac{H^0(U \setminus \{x\}, \mathcal{O}_X^{an}(K_X))}{L^2(U \setminus \{x\})},$$

where U is a sufficiently small Stein neighborhood of x in X, and $L^2(U \setminus \{x\})$ is the set of all square integrable forms on $U \setminus \{x\}$. Motivated by this alternate description, Kimio Watanabe introduced the *plurigenera* of (X,x) in [Wat80] as:

$$\delta_m(X,x) =_{\text{def}} \dim \frac{H^0(U \setminus \{x\}, \mathcal{O}_X^{an}(mK_X))}{L^{2/m}(U \setminus \{x\})},$$

with $L^{2/m}(U \setminus \{x\})$ denoting the set of holomorphic $m-$canonical forms ω on the sufficiently small $U \setminus \{x\}$ that satisfy $\int_{U \setminus \{x\}} (\omega \wedge \bar{\omega})^{1/m} < \infty$. In the case of surfaces, these invariants can be used to classify some log–canonical singularities (see [Ish90], [Oku98], [Oku00], [Wad03], [Wat80]). For example, by [Oku98], a normal surface singularity (X,x) is a quotient singularity if, and only if, $\delta_4(X,x) = \delta_6(X,x) = 0$.

The proofs of [Sak77, Thm.1.1, Thm.2.1], and remarks in [Ish90] provide an algebro–geometric approach to plurigenera at the expense of working again on resolutions. Let $\pi : \widetilde{X} \to X$ be a log–resolution of (X,x), with E the reduced fiber over x, let U be an arbitrary affine neighborhood of x, and let $\widetilde{U}$ be the preimage of U in $\widetilde{X}$ via π. In the algebraic category,

$$\delta_m(X,x) = \dim \frac{H^0(\widetilde{U} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{U}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E))} = \dim \frac{\mathcal{O}_X(mK_X)}{\pi_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E)},$$

with the last equality holding because U is affine, for choices of Weil canonical divisors on X and $\widetilde{X}$ such that $\pi_* K_{\widetilde{X}} = K_{\widetilde{X}}$.

The growth rate of $\delta_m(X,x)$, as m varies, is studied in [Ish90] and [Wat80]. It is shown that $\delta_m(X,x)$ grows at most like m^n. Generalizing work in [Wah90] for the case of surfaces, we define the *volume of the normal isolated singularity* (X,x) of dimension n by

$$\mathrm{vol}(X,x) =_{\mathrm{def}} \limsup_{m\to\infty} \frac{\delta_m(X,x)}{m^n/n!}.$$

A different notion of plurigenera, which we find more convenient to work with, was introduced by Morales ([Mor87]) as:

$$\lambda_m(X,x) =_{\mathrm{def}} \dim \frac{\mathcal{O}_X(mK_X)}{\pi_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}+mE)},$$

where $\pi : (\widetilde{X}, E) \to (X,x)$ is an arbitrary log–resolution. One sees that $\lambda_m(X,x)$ vanishes if x is a smooth point of X or, more generally, if X is $\mathbb{Q}-$Gorenstein with log–canonical singularities. The Morales plurigenera are independent of the log–resolution, and are local invariants around x. Results of Ishii in [Ish90] show that the Morales plurigenera and the Watanabe plurigenera have the same asymptotic behavior. In particular,

$$\mathrm{vol}(X,x) = \limsup_{m\to\infty} \frac{\lambda_m(X,x)}{m^n/n!}.$$

For surfaces, $\mathrm{vol}(X,x)$ has been studied by Wahl in [Wah90], and shown to be a characteristic number of the link of the singularity. In particular, its behavior under pull–backs by ramified maps was analyzed. The vanishing of $\mathrm{vol}(X,x)$ in the two dimensional case is also well understood. We review Wahl's work in Chapter II, where we also study our generalizations of his results to higher dimension.

Apart from normal surface singularities, which are automatically isolated, another important class of examples is that of cone singularities. Let (V,H) be a polarized complex projective manifold of dimension n, with H sufficiently positive, and let X be the cone $\mathrm{Spec} \bigoplus_{m\geq 0} H^0(V, \mathcal{O}(mH))$ whose vertex 0 is an isolated singularity. By explicit computation, or by [Wat80, Thm.1.7],

$$\lambda_m(X,0) = \sum_{k\geq 1} \dim H^0(V, \mathcal{O}(mK_V - kH)).$$

We will see that this leads to the formula:

$$\mathrm{vol}(X,0) = (n+1)\cdot \int_0^\infty \mathrm{vol}_V(K_V - tH)dt.$$

The volume under the integral is the volume of line bundles on projective varieties in the sense of [Laz04, Ch.2.2.C]. All isolated surface singularities have rational volume, but cone singularities provide examples of isolated singularities with irrational volume $\mathrm{vol}(X,x)$ already in dimension three.

In Chapter I, we introduce a local invariant that includes the volume of isolated singularities as a special case. Let X be a normal complex algebraic variety of dimension $n \geq 2$, and let x be a point on X. Fix a projective birational morphism $\pi : X' \to X$. For an arbitrary Cartier divisor D on X', define the *local volume of D at x* by

$$\mathrm{vol}_x(D) =_{\mathrm{def}} \limsup_{m\to\infty} \frac{h^1_x(mD)}{m^n/n!},$$

where

$$h^1_x(D) =_{\mathrm{def}} \dim H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(D)).$$

We do not assume that X' is also normal. We show that $\mathrm{vol}_x(D)$ is finite. When $\pi : (\widetilde{X}, E) \to (X,x)$ is a log–resolution of a normal complex isolated singularity of dimension n, we will see that

$$\mathrm{vol}(X,x) = \mathrm{vol}_x(K_{\widetilde{X}} + E).$$

The inspiration for the asymptotic construction of $\mathrm{vol}_x(D)$ comes from its global counterpart. Given D a Cartier divisor on a complex projective variety X of dimension n, write $h^0(mD) =_{\mathrm{def}} \dim H^0(X, \mathcal{O}_X(mD))$. The Riemann–Roch problem motivates the construction of the finite asymptotic invariant

$$\mathrm{vol}(D) =_{\mathrm{def}} \limsup_{m\to\infty} \frac{h^0(mD)}{m^n/n!}.$$

For example, when D is ample, $\operatorname{vol}(D) = (D^n)$. Volumes of Cartier divisors have been well studied for their importance in birational geometry (see [Laz04, Ch.2.2.C]). Analogues for higher cohomologies, the asymptotic cohomology functions, have been studied more recently by Küronya in [Kur06]. These functions and the local volumes have many similar properties. In the local setting we prove:

Theorem. *With notation as above,* vol_x *is well defined, $n-$homogeneous and continuous on* $N^1(X'/X)_{\mathbb{R}}$.

Here, $N^1(X'/X)_{\mathbb{R}}$ denotes the additive group of $\mathbb{R}-$Cartier divisors on X' modulo numerical equivalence on the fibers of π. A difference between vol_x and the volume of divisors on projective varieties is that whereas the latter increases in all effective directions, vol_x decreases in effective directions that contract to x and increases in effective directions without components contracting to x. This behavior proves quite useful. Following ideas in [LM09], we present a convex–geometric approach to local volumes that allows us to prove the following:

Proposition. *For any Cartier divisor D on X', we can replace* $\limsup$ *in the definition of* $\operatorname{vol}_x(D)$ *by* $\lim$*:*

$$\operatorname{vol}_x(D) = \lim_{m\to\infty} \frac{h^1_x(mD)}{m^n/n!}.$$

In the style of [LM09, Thm.3.8], we obtain a Fujita–type approximation result. If $\mathcal{I}$ is a fractional ideal sheaf on X, following [CHST05] or [Cut10], we define its local multiplicity, also known as $\epsilon-$multiplicity, at x by the formula:

$$\widehat{h}^1_x(\mathcal{I}) =_{\text{def}} \limsup_{m\to\infty} \frac{\dim H^1_{\{x\}}(\mathcal{I}^m)}{m^n/n!}.$$

This generalizes the concept of Hilbert–Samuel multiplicity, as we see in Chapter I.

Theorem. *Let D be a Cartier divisor on X', such that $\mathfrak{a}_p =_{\text{def}} \pi_*\mathcal{O}_{X'}(pD)$ coincides with $\mathfrak{b}^p$ on $X \setminus \{x\}$, for some coherent fractional ideal sheaf $\mathfrak{b}$ on $X \setminus \{x\}$, and for* *all* $p \geq 1$. *Then*

$$\operatorname{vol}_x(D) = \lim_{p\to\infty} \frac{\widehat{h}^1_x(\pi_*\mathcal{O}_{X'}(pD))}{p^n}.$$

Two other problems that are well understood in the projective case are the vanishing and log–concavity for volumes of Cartier divisors (see [Laz04, Ch.2.2.C]). We know that volumes vanish outside the big cone, and that $\operatorname{vol}^{1/n}$ is a concave function on the same big cone, i.e.,

$$\operatorname{vol}(D+D')^{1/n} \geq \operatorname{vol}(D)^{1/n} + \operatorname{vol}(D')^{1/n},$$

for all big Cartier divisors D and D' on X. In the local setting we find analogous results when working with divisors supported on the fiber over x. Denote by $\operatorname{Exc}_x(\pi)$ the real vector space spanned by all such divisors.

Proposition. *On* X'*, let* D *be a Cartier divisor supported on the fiber over* x*. Then* $\operatorname{vol}_x(D) = 0$ *if, and only if,* D *is an effective divisor. When* D *is an arbitrary Cartier divisor, then* $\operatorname{vol}_x(D) = 0$ *if, and only if,* $h^1_x(m\widetilde{D}) = 0$ *for all* $m \geq 0$*, where* $\widetilde{D}$ *is the pullback of* D *to the normalization of* X'.

Proposition. *The function* $\operatorname{vol}_x^{1/n}$ *is convex on* $\operatorname{Exc}_x(\pi)$*, i.e.,*

$$\operatorname{vol}_x(D+D')^{1/n} \leq \operatorname{vol}_x(D)^{1/n} + \operatorname{vol}_x(D')^{1/n},$$

for all big $\mathbb{R}-$*Cartier divisors* D *and* D' *on* X' *that are supported over* x*, but it may fail to be so on* $N^1(X'/X)_{\mathbb{R}}$.

Returning to the setting of normal complex isolated singularities, we generalize to higher dimension some of the properties established by Wahl in [Wah90] for local volumes of isolated surface singularities.

Proposition. *Let* $f : (X,x) \to (Y,y)$ *be a finite map of complex normal isolated singularities of dimension* n*, with* $f^{-1}\{y\} = x$*. Then*

$$\operatorname{vol}(X,x) \geq (\deg f) \cdot \operatorname{vol}(Y,y).$$

Equality holds if f is unramified outside y.

Corollary.

(i) If $f:(X,x)\to(Y,y)$ is a finite map of normal isolated singularities as above, and $\mathrm{vol}(X,x)$ *vanishes, then* $\mathrm{vol}(Y,y)=0$.

(ii) If (X,x) admits an endomorphism of degree at least two, then $\mathrm{vol}(X,x)=0$.

Unlike the two dimensional case, we show in Example II.29 that $\mathrm{vol}(X,x)$ is not a topological invariant of the link of the singularity in dimension at least three. For surfaces, the vanishing of $\mathrm{vol}(X,x)$ is equivalent to (X,x) being log–canonical in the sense of [Wah90, Rem.2.4]. In arbitrary dimension, as a corollary to [Ish90, Thm.4.2], we show:

Proposition. *If (X,x) is a normal isolated singularity of dimension n, then* $\mathrm{vol}(X,x)$ *vanishes if, and only if, $\lambda_m(X,x)=0$ for all $m\geq 0$.*

In the $\mathbb{Q}-$Gorenstein case, the conclusion of the previous result is the same as saying that (X,x) has log–canonical singularities, but by [BdFF11] this is not the case in general. We also construct another notion of volume that is useful for the study of canonical singularities in the sense of [dFH09]:

$$\mathrm{vol}_\gamma(X,x) =_{\mathrm{def}} \mathrm{vol}_x(K_{\widetilde{X}}),$$

where $\pi:\widetilde{X}\to X$ is a resolution of a normal isolated singularity (X,x). We will see that $\mathrm{vol}_\gamma(X,x)$ is also independent of the resolution.

Proposition. *The volume* $\mathrm{vol}_\gamma(X,x)$ *vanishes if, and only if, (X,x) has canonical singularities in the sense of [dFH09].*

On surfaces, by [Wah90], the volume $\mathrm{vol}(X,x)$ can be computed as $-P\cdot P$, where P is the nef part of the relative Zariski decomposition of $K_{\widetilde{X}}+E$, for any good resolution

$$\pi : (\widetilde{X}, E) \to (X, x).$$

Building on the theory of $b-$divisors, this definition is generalized by Boucksom, de Fernex, and Favre to higher dimension in [BdFF11] to produce another notion of volume for a normal isolated singularity, denoted $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$. This new volume is studied in Chapter III. We are able to show that

$$\mathrm{vol}_{\mathrm{BdFF}}(X, x) \geq \mathrm{vol}(X, x).$$

By the same [BdFF11], the two notions of volume differ in general, but coincide in the $\mathbb{Q}-$Gorenstein case, and we are able to extend this to the numerically Gorenstein case (cf. [BdFF11]). All normal surface singularities are numerically Gorenstein. The volume $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ enjoys similar properties to those of $\mathrm{vol}(X, x)$ concerning the behavior with respect to finite covers, and is better suited for the study of log–canonical singularities. On the other hand, $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ is usually hard to compute because all birational models of X may influence it, as opposed to $\mathrm{vol}(X, x)$, which is computed on any log-resolution of (X, x). As we will see, combining techniques in [BdFF11] with results in our study of vol_x, the volume $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ can also be computed for some cone singularities. It can also achieve irrational values.

The thesis is organized as follows. Chapter I develops the theory of local volumes, motivated by generalizing Hilbert–Samuel multiplicities. We compute several examples, before presenting a convex–geometric approach to local volumes and proving our version of the Fujita approximation theorem. We next investigate the vanishing and the convexity for $\mathrm{vol}_x^{1/n}$. Chapter II is dedicated to the volume of isolated singularities associated to the plurigenera in the sense of Watanabe or Morales, and to $\mathrm{vol}_\gamma(X, x)$, an asymptotic invariant associated to Knöller's plurigenera. We generalize to higher dimension results for surfaces in [Wah90] that we briefly review in the first section, translate to volumes some of the results of Ishii ([Ish90]), and give examples. In Chapter III, we compare our notion of volume with the one appearing in [BdFF11]. By studying the impact that the theory of vol_x has on $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$, we are able to give a nontrivial computation for $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ that yields an irrational result.

This dissertation expands on results of the author in [F1]. In previous work, in [F2], we have studied numerical pseudoeffective cycles on projective bundles over curves, describing the pseudoeffective cones in terms of the numerical data of the Harder–Narasimhan filtration of a defining locally free sheaf.

Background and conventions

Unless otherwise stated, we work over the field of complex numbers $\mathbb{C}$, and we use the notation of [Laz04].

Asymptotic cohomology

For a Cartier divisor D on a projective variety X of dimension n, we consider the asymptotic cohomology functions studied by Küronya in [Kur06]:

$$\widehat{h}^i(D) =_{\text{def}} \limsup_{m\to\infty} \frac{h^i(X, \mathcal{O}(mD))}{m^n/n!}.$$

When $i = 0$, we recover the volume function $\mathrm{vol}(D)$ from [Laz04, Ch.2.2.C]. The results of [Kur06] that we will call upon are summarized in the following:

Proposition. *For any i, the functions $\widehat{h}^i$ depend only on the numerical equivalence classes of divisors on X, and they are $n-$homogeneous:*

$$\widehat{h}^i(a \cdot D) = a^n \cdot \widehat{h}^i(D),$$

for any $a \geq 0$. Thus they descend to $N^1(X)_{\mathbb{Q}}$ where they are continuous and satisfy a Lipschitz–type estimate that allows us to extend them continuously to functions

$$\widehat{h}^i : N^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geq 0}.$$

The relative setting

Let $\pi : Y \to X$ be a projective (or just proper) morphism of algebraic varieties. A Cartier divisor D on Y is called:

- $\pi-$*trivial* if $D = \pi^* L$ for some Cartier divisor L on X. Two Cartier divisors D and D' are $\pi-$*linearly equivalent* if D is linearly equivalent to $D' + \pi^* L$ for some Cartier divisor L on X.

- $\pi-$*numerically trivial* if its restriction to any fiber of π is numerically trivial, i.e., if $D \cdot C = 0$ for any curve C such that $\pi(C)$ is a point. The set of $\pi-$numerical equivalence classes is an abelian group of finite rank denoted $N^1(Y/X)$.

- $\pi-$*ample (nef)* if the restriction to each fiber of π is ample (nef).

- $\pi-$*movable* if its class in $N^1(Y/X)_{\mathbb{R}}$ lies in the closed convex cone generated by divisors whose $\pi-$base locus have codimension at least two in Y. The $\pi-$base locus of D is the vanishing locus of the ideal sheaf on Y arising as the image of the canonical evaluation morphism:

$$\pi^* \pi_* \mathcal{O}_Y(D) \otimes \mathcal{O}_Y(-D) \to \mathcal{O}_Y.$$

Cohomology with supports

We point to [Gro62] for a detailed study of cohomology with supports, or [Har77, Exer.III.2.3] for a quick introduction that is sufficient for our purposes. We will mostly use the following three properties when Y is a closed point.

Proposition (Long exact sequence)**.** *If $Y \subset X$ is a closed subset, and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of sheaves of abelian groups on X, then we have a long exact sequence

$$0 \to H^0_Y(X, \mathcal{F}) \to H^0_Y(X, \mathcal{G}) \to H^0_Y(X, \mathcal{H}) \to H^1_Y(X, \mathcal{F}) \to \dots$$

Proposition (Excision)**.** *If $Y \subset X$ is a closed subset and $U \subset X$ is an open subset such that $Y \subset U$, then*

$$H^i_Y(X, \mathcal{F}) \simeq H^i_Y(U, \mathcal{F}|_U)$$

for any sheaf $\mathcal{F}$ of abelian groups on X.

Proposition (Restriction sequence)**.** *With notation as above, there is a long exact sequence*

$$0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X \setminus Y, \mathcal{F}) \to H^1_Y(X, \mathcal{F}) \to \dots$$

Resolutions of singularities

Let X be a normal complex variety, and let Y be a subscheme. A *log–resolution* $\pi : (\widetilde{X}, E) \to (X, Y)$ is a birational morphism from a nonsingular variety $\widetilde{X}$ to X, with $E = \pi^{-1}\{Y\}$ a reduced divisor supporting the vanishing locus of the invertible ideal sheaf $\mathcal{I}(Y) \cdot \mathcal{O}_{\widetilde{X}}$, such that $E \cup \mathrm{Exc}(\pi)$ is a simple normal crossings divisor on $\widetilde{X}$. Note that a log–resolution of (X, Y) factors through $Bl_Y X$. When $Y = \{x\}$ is an isolated singularity, we say that the log–resolution π is a *good resolution* if it is an isomorphism outside x. The existence of such resolutions follows in the complex setting from Hironaka's ([Hir64]) celebrated results.

Coherent fractional ideal sheaves

A coherent subsheaf $\mathcal{I}$ of the constant fraction field sheaf of a quasiprojective variety X is called a *coherent fractional ideal sheaf.* Typical examples are constructed by pushing forward invertible sheaves via projective birational morphisms. For $\mathcal{I}$ a coherent fractional ideal sheaf, there exists a (sufficiently negative) Cartier divisor D on X such that

$$\mathcal{J} := \mathcal{I} \cdot \mathcal{O}_X(D)$$

is an actual ideal sheaf. Using this, the blow–up $Bl_{\mathcal{I}}X$ of X along $\mathcal{I}$ is defined as $Bl_{\mathcal{J}}X$. If π denotes the blow–down map $Bl_{\mathcal{J}}X \to X$ with its relative Serre invertible sheaf $\mathcal{O}_{\mathcal{J}}(1)$ (see [Har77, II.7]), then $Bl_{\mathcal{I}}X$ is naturally endowed, via [Har77, Lem.II.7.9], with the relative Serre invertible sheaf

$$\mathcal{O}_{\mathcal{I}}(1) := \mathcal{O}_{\mathcal{J}}(1) \otimes \pi^* \mathcal{O}_X(-D).$$

CHAPTER I

Local volumes

This chapter is devoted to building the theory of local volumes for Cartier divisors on a relatively projective birational modification of a normal complex quasi-projective variety of dimension at least two with a distinguished point. We compare many properties of these volumes to their counterparts in the theory of volumes of Cartier divisors on projective varieties as presented in [Laz04, Ch.2.2.C]. In the first section we describe the algebraic motivation behind this theory, coming from (generalizations of) the Hilbert–Samuel multiplicity of $\mathfrak{m}-$primary ideals in normal noetherian domains. Next, we define the local volumes, study them variationally, discus their behavior under finite maps and give examples. In the third section we adapt some of the methods of [LM09] to present a convex–geometric approach to local volumes. We obtain a Fujita–type approximation result. We also discuss convexity and vanishing properties for local volumes.

I.1 Multiplicities

Although what motivated the author in constructing local volumes was the attempt to extend constructions of Wahl on surfaces to higher dimension in order to study normal isolated singularities, in this expository section we will see that one can develop this theory naturally starting from the Hilbert–Samuel multiplicity.

I.1.1 The Hilbert–Samuel multiplicity

We review the Hilbert–Samuel multiplicity for $\mathfrak{m}-$primary ideal sheaves.

Definition I.1. Let $(R, \mathfrak{m}, \mathbb{C})$ be a local noetherian ring of dimension n, and let $\mathfrak{a} \subset R$ be an $\mathfrak{m}-$primary ideal. Recall that the Hilbert–Samuel multiplicity of $\mathfrak{a}$ is defined as the limit:

$$e(\mathfrak{a}) =_{\text{def}} \lim_{r\to\infty} \frac{\text{length}(R/\mathfrak{a}^r)}{r^n/n!}.$$

That the limit exists and it is finite is a consequence of the polynomial behavior of the lengths. The local nature of the definition allows us to define the Hilbert–Samuel multiplicity when $\mathfrak{m}$ is the maximal ideal sheaf corresponding to a closed point x on some scheme X, and when $\mathfrak{a}$ is some $\mathfrak{m}-$primary, coherent (fractional) ideal sheaf. In this setting, one obtains an intersection theoretic interpretation of multiplicity.

Remark I.2. With notation as above, assume that X is integral, and let

$$\pi : Bl_{\mathfrak{a}}X \to X$$

be the blow–up with relative $\mathcal{O}(1) = \mathcal{O}(-E)$ for some Cartier divisor E that is supported over x. Then

$$e(\mathfrak{a}) = -(-E)^n.$$

We assume henceforth that X is normal and of dimension $n \geq 2$. Then the long exact sequence for cohomology with supports contains

$$H^0_{\{x\}}(X, \mathcal{O}_X) \to H^0_{\{x\}}(X, \mathcal{O}_X/\mathfrak{a}) \to H^1_{\{x\}}(X, \mathfrak{a}) \to H^1_{\{x\}}(X, \mathcal{O}_X). \tag{I.1.1}$$

The first term is zero because X is integral of positive dimension, hence a function vanishing outside x also vanishes at x. Because $\mathfrak{a}$ is $\mathfrak{m}-$primary, it is then co–supported at x, therefore

$$H^0_{\{x\}}(X, \mathcal{O}_X/\mathfrak{a}) = H^0(X, \mathcal{O}_X/\mathfrak{a}),$$

and its dimension over k is length$(\mathcal{O}_X/\mathfrak{a})$. For the last term, by excision one can replace X by some suitable affine neighborhood of x and then consider the restriction sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X \setminus \{x\}, \mathcal{O}_X) \to H^1_{\{x\}}(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to \dots$$

The first map is an isomorphism because X is normal of dimension $n \geq 2$. The last term is zero because we have assumed that X is affine. It follows that the last term in (I.1.1) is zero, hence

$$H^1_{\{x\}}(X, \mathfrak{a}) = H^0_{\{x\}}(X, \mathcal{O}_X/\mathfrak{a}) = H^0(X, \mathcal{O}_X/\mathfrak{a}),$$

and their dimension is the colength of $\mathfrak{a}$. Denote

$$h^1_x(\mathfrak{a}) = \dim_k H^1_{\{x\}}(X, \mathfrak{a}).$$

The associated asymptotic cohomology function

$$\widehat{h}^1_x(\mathfrak{a}) =_{\text{def}} \limsup_{r \to \infty} \frac{h^1_x(\mathfrak{a}^r)}{r^n/n!}$$

clearly equals $e(\mathfrak{a})$. We know that the lim sup is a true limit in this case.

I.1.2 The $\epsilon-$multiplicity

In this subsection we present a generalization of the notion of Hilbert–Samuel multiplicity for coherent fractional ideal sheaves that are not necessarily $\mathfrak{m}-$primary. When working with such ideal sheaves, the notion of colength does not readily make sense. Fortunately, $h^1_x(\mathfrak{a})$ still does. Under the assumption that X is normal of dimension n at least two, choosing a point x on X, and reasoning as in the previous subsection, we find that

$$H^1_{\{x\}}(X, \mathfrak{a}) = \frac{\imath_* \imath^* \mathfrak{a}}{\mathfrak{a}},$$

where $\imath : X \setminus \{x\} \to X$ is the canonical embedding. For convenience, we write

$$\widetilde{\mathfrak{a}} =_{\mathrm{def}} \imath_* \imath^* \mathfrak{a}.$$

Algebraically, $\widetilde{\mathfrak{a}}$ is obtained from $\mathfrak{a}$ by removing all the $\mathfrak{m}-$primary components.

$$\widetilde{\mathfrak{a}} = (\mathfrak{a} : \mathfrak{m}^\infty) =_{\mathrm{def}} \bigcup_{p \geq 0} (\mathfrak{a} : \mathfrak{m}^p).$$

Here, $\mathfrak{m}$ denotes the maximal ideal sheaf on X corresponding to x. Note that $h^1_x(\mathfrak{a}) = \dim H^1(X, \mathfrak{a})$ is a finite number. When $\mathfrak{a}$ is an ideal sheaf, this follows because $\mathfrak{a}$ and $\widetilde{\mathfrak{a}}$ coincide outside x, and because $\mathfrak{a}$ is quasi–coherent ([Har77, Prop.II.5.8]), and an ideal sheaf itself by the conditions on X, therefore it is coherent. By the local nature of our invariants, we can assume that X is projective. For H sufficiently ample, $\mathfrak{a} \otimes \mathcal{O}_X(-H)$ is an ideal sheaf, and

$$\mathfrak{a} \otimes \widetilde{\mathcal{O}(-H)} = \widetilde{\mathfrak{a}} \otimes \mathcal{O}_X(-H)$$

by the projection formula. Thus we can reduce to the case when $\mathfrak{a}$ is an ideal sheaf. The asymptotic invariant

$$\widehat{h}^1_x(\mathfrak{a}) = \limsup_{r \to \infty} \frac{h^1_x(\mathfrak{a}^r)}{r^n/n!},$$

also known as the $\epsilon-$multiplicity of $\mathfrak{a}$, has been previously studied in [CHST05] and [Cut10]. In [CHST05], it is shown that the numbers $h^1_x(\mathfrak{a}^r)$ no longer necessarily have polynomial behavior, and one can construct ideals with irrational $\epsilon-$multiplicity. It is therefore no longer clear that the lim sup in the definition is a true limit, but the result holds true non–trivially by [Cut10]. We will also obtain this as a consequence of our Fujita–type approximation result, Theorem I.36. The finiteness of $\epsilon(\mathfrak{a})$ is also proved in [CHST05]. We will see this as a special case of finiteness for local volumes.

Example I.3 (Monomial ideals)**.** Let I be a monomial ideal in $\mathbb{C}[X_1, \ldots, X_n]$, and let $\mathfrak{m} = (X_1, \ldots, X_n)$ be the irrelevant ideal corresponding to the origin 0 of $\mathbb{C}^n$.

Then

$$\widetilde{I^k} = (I^k : \mathfrak{m}^\infty) = \bigcap_{i=1}^{n} (I^k : X_i^\infty).$$

For an arbitrary monomial ideal J, the ideal $(J : X_i^\infty)$ can be computed as $\varphi_i^{-1}\varphi_i(J)$, where

$$\varphi_i : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, \hat{X}_i, \ldots, X_n]$$

is the evaluation map determined by $\varphi_i(X_j) = X_j$ for $j \neq i$ and $\varphi_i(X_i) = 1$. Geometrically, J is determined by the set $A(J)$ of $n-$tuples of nonnegative numbers $(a_1, \ldots, a_n)$ such that $X_1^{a_1} \cdot \ldots \cdot X_n^{a_n}$ belongs to J. Then $A(J : X_i^\infty)$ is obtained by taking the integer coordinate points in the preimage of the image of $A(J)$ via the projection onto the coordinate hyperplane that does not contain the $i-$th coordinate axis. Subsequently, $A(J : \mathfrak{m}^\infty) = \bigcap_{i=1}^n A(J : X_i^\infty)$ and

$$\dim H^1_{\{0\}}(J) = \#(A(J : \mathfrak{m}^\infty) \setminus A(J)).$$

Let $P(J)$ denote the convex span of $A(J)$ in $\mathbb{R}^n$, and let $\widetilde{P}(J)$ be the polyhedron obtained by intersecting the preimages of the images of the projection of $P(J)$ onto each of the coordinate hyperplanes. Then one checks that

$$\widehat{h}^1_{\{0\}}(I) = n! \cdot \mathrm{vol}(\widetilde{P}(I) \setminus P(I)),$$

where the volume used in the right–hand side is the Euclidean one.

In Figure I.1 we see the example of the ideal $I = (X^3,\ XY^3) \subset \mathbb{C}[X, Y]$. The polyhedron $P(I)$ is the convex span of the set of all lattice points in the set

$$\{(a, b)\ :\ a \geq 1,\ b \geq 3\} \cup \{(a, b)\ :\ a \geq 3,\ b \geq 0\},\ \text{i.e.,}$$

$$P(I) = \{(x, y) \in \mathbb{R}^2\ :\ x \geq 1,\ y \geq 0,\ 3x + 2y \geq 9\}.$$

The projection of $P(I)$ onto the x axis is the halfline $x \geq 1$, and the projection onto the y axis is $y \geq 0$. These give

$$\widetilde{P}(I) = \{(x, y) \in \mathbb{R}^2\ :\ x \geq 1,\ y \geq 0\}.$$

The $\epsilon-$multiplicity of I is then $\widehat{h}^1_{\{0\}}(I) = 2! \cdot \frac{3 \cdot 2}{2} = 6$.

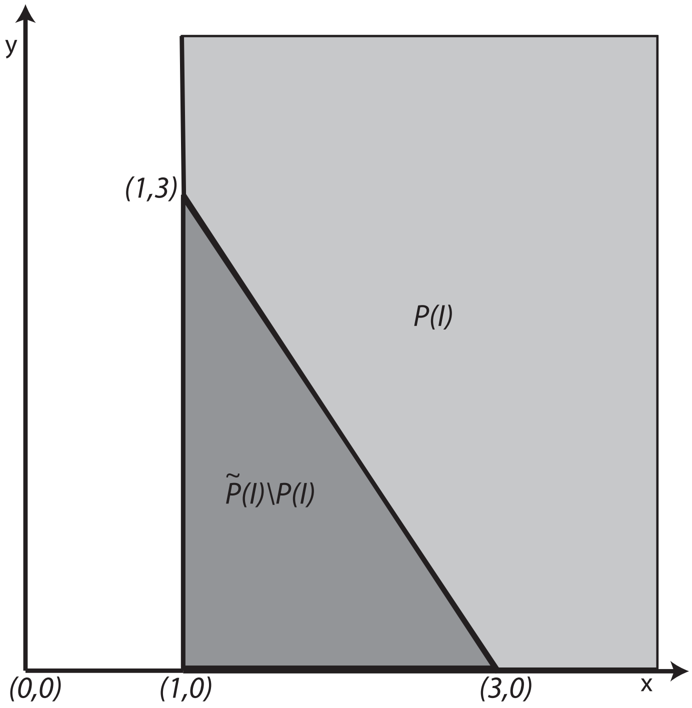

Figure I.1: $I = (X^3, XY^3) \subset \mathbb{C}[X, Y]$

We obtain a translation of the classical picture for the computation of the Hilbert–Samuel multiplicity of the ideal (X^2, Y^3), i.e., $\widetilde{P}(I)$ is a translation of the first quadrant. This happens for any ideal I of $\mathbb{C}[X, Y]$, because I can be written as $u \cdot \mathfrak{a}$ for some monomial u and for some $\mathfrak{m}-$primary ideal $\mathfrak{a}$, and then $\widetilde{I} = (u)$ is a principal ideal. In the higher dimensional case, the picture is generally more complicated. $\square$

Example I.4 (Toric ideals)**.** Let I be a monomial ideal inside the toric algebra $\mathbb{C}[S_\sigma]$, where S_σ is the semigroup of integral points of a pointed (it contains no lines through the origin) rational convex cone σ of dimension n. Let $\tau_1, \ldots, \tau_r$ be the minimal ray generators for σ. For a subset V of σ and $0 \leq i \leq r$, let $V_i = \sigma \cap \bigcup_{k \geq 0}(V - k \cdot \tau_i)$. Geometrically, this is the trace left by V inside σ by sliding it in the direction of $-\tau_i$.

If $P(I)$ is the convex hull of the set $A(I)$ defined as in the monomial case, and if

$\widetilde{P}(I)$ is the intersection of all $P(I)_i$, then

$$\widehat{h}^1_{x_\sigma}(I) = n! \cdot \mathrm{vol}(\widetilde{P}(I) \setminus P(I)),$$

where x_σ is the torus fixed point of $\mathrm{Spec}\mathbb{C}[S_\sigma]$. □

I.1.3 Multiplicities for graded sequences of ideal sheaves

In this subsection we extend multiplicities to graded sequences of coherent fractional ideal sheaves.

Definition I.5. Let X be a normal algebraic variety of dimension $n \geq 2$, with a fixed point x. On X, consider a *graded sequence of fractional ideal sheaves* $\mathfrak{a}_\bullet$ (i.e. $\mathfrak{a}_0 = \mathcal{O}_X$ and $\mathfrak{a}_k \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{k+l}$), and define its generalized Hilbert–Samuel multiplicity at x as

$$\widehat{h}^1_x(\mathfrak{a}_\bullet) =_{\mathrm{def}} \limsup_{r\to\infty} \frac{\dim H^1_{\{x\}}(\mathfrak{a}_r)}{r^n/n!}.$$

When $\mathfrak{a}_r = \mathcal{I}^r$ for all r and for some fixed fractional ideal sheaf $\mathcal{I}$, we recover the epsilon multiplicity of $\mathcal{I}$. When $\mathfrak{a}_\bullet$ is a graded sequence of $\mathfrak{m}-$primary ideals in $\mathcal{O}_X$, then $\widehat{h}^1_x(\mathfrak{a}_\bullet)$ coincides with the multiplicity defined in [LM09, Sec.3.2]. If $\mathcal{I}$ is an $\mathfrak{m}-$primary ideal, then $\widehat{h}^1_x(\mathcal{I})$ is the Hilbert–Samuel multiplicity of $\mathcal{I}$ at $\mathfrak{m}$.

The local volumes, as we will see in the next section, are multiplicities in the above sense of graded sequences of fractional ideals arising from a geometric setting as pushforwards of tensor powers of a line bundle.

I.2 The definition and basic properties of local volumes

In this section we define local volumes, compute examples, and prove several properties paralleling the theory of volumes of Cartier divisors on projective varieties.

Assume henceforth that X is a normal complex quasiprojective variety of dimension n at least two, and fix a point $x \in X$. Let $\pi : X' \to X$ be a projective birational morphism, and let D be a Cartier divisor on X'. Using cohomology with supports at x, define

$$h^1_x(D) =_{\text{def}} \dim H^1_{\{x\}}(X, \pi_* \mathcal{O}_{X'}(D)). \tag{I.2.1}$$

We will see in the course of the proof of Proposition I.16 that this is a finite number.

Remark I.6. (i) If U is an open subset of X containing x, let F be the set theoretic fiber of π over x, let V be the preimage of U and denote by $i : U \setminus \{x\} \to U$ and $j : V \setminus F \to V$ the natural open embeddings. By abuse, we denote $\pi|_U^V$ again by π. An inspection of the restriction sequence for cohomology with supports, together with flat base change, reveal

$$h^1_x(D) = \dim \frac{i_* i^* (\pi_* \mathcal{O}_{X'}(D)|_U)}{\pi_* \mathcal{O}_{X'}(D)|_U} = \dim \frac{\pi_* j_* j^* (\mathcal{O}_{X'}(D)|_V)}{\pi_* \mathcal{O}_{X'}(D)|_U}.$$

When U is affine, the last term is equal to $\frac{H^0(V \setminus F, \mathcal{O}_V(D))}{H^0(V, \mathcal{O}_V(D))}$.

(ii) If U is affine, X' is normal and E is the divisorial component of F, then

$$h^1_x(D) = \dim \frac{\bigcup_{k \geq 0} H^0(\pi^{-1}U, \mathcal{O}_{X'}(D + kE))}{H^0(\pi^{-1}U, \mathcal{O}_{X'}(D))}$$

as a study of local sections shows.

I.2.1 The definition of local volumes

Definition I.7. The *local volume* of D at x is the asymptotic limit:

$$\operatorname{vol}_x(D) =_{\text{def}} \limsup_{m \to \infty} \frac{h^1_x(mD)}{m^n / n!}.$$

We will prove that this quantity is finite in Proposition I.16. We will also see in Corollary I.34 that the lim sup in the definition of $\operatorname{vol}_x(D)$ can be replaced by lim.

The excision property of cohomology with supports shows that vol_x is local around x. The term volume is justified by the resemblance of the definition to that of volumes of divisors on projective varieties. We shall see that the two notions share many similar properties.

I.2.2 Examples

Example I.8 (Toric varieties)**.** We use the notation of [Ful97]. Let σ be pointed rational cone of maximal dimension in $N_{\mathbb{R}}$, where N is a lattice isomorphic to $\mathbb{Z}^n$. Denote

$$M = \mathrm{Hom}(N, \mathbb{Z})$$

and let S_σ be the semigroup $\sigma^\vee \cap M$. Let $X(\sigma)$ be the affine toric variety $\mathrm{Spec}\mathbb{C}[S_\sigma]$. The unique torus invariant point of $X(\sigma)$ is denoted x_σ.

Let Σ be a rational fan obtained by refining σ. It determines a proper birational toric modification $\pi : X(\Sigma) \to X(\sigma)$. Let $v_1, \ldots, v_r$ be the first nonzero integer coordinate points on the rays that span σ. Let $v_{r+1}, \ldots, v_{r+s}$ be the first non-zero points of N on the rays in Σ that lie in the relative interior of faces of σ of dimension $2 \leq d \leq n-1$ and denote by $v_{r+s+1}, \ldots, v_{r+s+t}$ the first non-zero points from N on the rays of Σ in the interior of σ. Denote by D_i the Weil divisor on $X(\Sigma)$ associated to the ray containing v_i. A divisor D_i lies over x_σ exactly when its support is a complete variety, which is equivalent to v_i lying in the interior of σ, i.e., when $i > r + s$.

To $D = \sum_{i=1}^{r+s+t} a_i D_i$, a $T-$invariant Cartier divisor on $X(\Sigma)$, we associate the rational convex polyhedra in $M_{\mathbb{R}}$ defined by

$$P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i \text{ for all } i\}.$$

$$P'_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i \text{ for all } i \leq r+s\}.$$

By [Ful97, Lemma on p.66], global sections of $\mathcal{O}_{X(\Sigma)}(mD)$ correspond to points of $(mP_D) \cap M$ and sections defined outside the fiber over x_σ correspond to $(mP'_D) \cap M$. By Remark I.6,

$$h^1_{x_\sigma}(mD) = \#((mP'_D \setminus mP_D) \cap M).$$

Taking asymptotic limits,

$$\mathrm{vol}_{x_\sigma}(D) = n! \cdot \mathrm{vol}(P'_D \setminus P_D).$$

On the right hand side we have the Euclidean volume in $M_{\mathbb{R}}$. Note that this volume is rational and finite, even though P_D and P'_D may be infinite polyhedra. See Example I.49 and Figure I.2 for an explicit computation. $\square$

The surface case, which was studied in [Wah90] and served as the inspiration for our work, gives another set of computable examples.

Example I.9 (Surface case)**.** Let (X, x) be a normal (isolated) surface singularity, and let $\pi : (\widetilde{X}, E) \to (X, x)$ be a good resolution. Any divisor D on $\widetilde{X}$ admits a relative Zariski decomposition $D = P + N$, where P is a relatively nef and exceptional $\mathbb{Q}-$divisor. From [Wah90, Thm.1.6], we have

$$\mathrm{vol}_x(D) = -P \cdot P$$

and this can be computed algorithmically from the data of the intersection numbers of D with the components of E, and from the intersection numbers between components of E (cf. [Wah90, Prop.1.2], see also Proposition II.1). $\square$

As hinted to in the previous section, local volumes generalize the $\epsilon-$multiplicity of ideals.

Example I.10. When $\mathfrak{a}$ is a fractional ideal sheaf on X, and $\mathcal{O}(1)$ denotes the relative Serre bundle on the blow–up $\pi : Bl_{\mathfrak{a}}X \to X$, then using [Laz04, Lemma.5.4.24],

$\pi_*\mathcal{O}(m) = \mathfrak{a}^m$ for sufficiently large m, and consequently

$$\widehat{h}^1_x(\mathfrak{a}) = \mathrm{vol}_x(\mathcal{O}(1)).$$

$\square$

I.2.3 Basic properties of local volumes I

In this subsection we prove that local volumes are finite, $n-$homogeneous, and we study their behavior under pullbacks. Recall that X denotes a normal complex quasiprojective variety of dimension n at least two. We fix a point $x \in X$, and a projective birational morphism $\pi : X' \to X$, where we do not assume that X' is also normal.

Lemma I.11. *With notation as above, let D be a Cartier divisor on X'. Then there exist projective completions $\overline{X}$ and $\overline{X'}$ of X and X' respectively, together with a map $\overline{\pi} : \overline{X'} \to \overline{X}$ extending π and a Cartier divisor $\overline{D}$ on $\overline{X'}$, such that $\overline{D}|_{X'} = D$.*

Proof. Choose arbitrary projective completions $\overline{X}$ and Y of X and X' respectively. The rational map $Y \dashrightarrow \overline{X}$ induced by π can be extended, by resolving its indeterminacies in Y, to $\pi' : Y' \to \overline{X}$, such that $\pi'|_{X'} = \pi$. The Cartier divisor D determines an invertible sheaf $\mathcal{O}_{X'}(D)$, which by [Har77, Ex.II.5.15] extends to a coherent fractional ideal sheaf $\mathcal{I}$ on Y'. We denote by $\overline{X'}$ the blow–up of Y' along $\mathcal{I}$, by $\overline{\pi} : \overline{X'} \to \overline{X}$ the induced map and by $\mathcal{O}_{\overline{Y}}(\overline{D})$ the relative Serre bundle of the blow–up, then $\overline{D}|_{X'} = D$. $\square$

The previous result can be used to reduce questions about the local volume of one divisor D (or of finitely many) to the case when X and X' are projective. We will see that we can reduce the study of the function vol_x to X' normal, or even nonsingular.

Lemma I.12. *With notation as above, let $\mathcal{F}$ be a torsion free coherent sheaf on X' of rank r. Then*

$$\mathrm{vol}_x(D) = \limsup_{m\to\infty} \frac{\dim H^1_{\{x\}}(X, \pi_*(\mathcal{F}(mD)))}{r \cdot m^n/n!}.$$

Proof. By Lemma I.11, since we can extend coherent torsion free sheaves to coherent sheaves with the same property, we can assume that X and X' are projective. Let H be sufficiently ample on X so that there exist short exact sequences

$$0 \to \mathcal{O}^r_{X'}(-\pi^* H) \to \mathcal{F} \to Q \to 0$$

$$0 \to \mathcal{F} \to \mathcal{O}^r_{X'}(\pi^* H) \to R \to 0$$

with torsion quotients Q and R. Such H exists because $\pi^* H$ is a big Cartier divisor.

If Q_m denotes the image of $\pi_*(\mathcal{F}(mD))$ in $\pi_*(Q(mD))$, and R_m is the image of $\pi_*\mathcal{O}^r_{X'}(\pi^* H + mD)$ in $\pi_*(R(mD))$, then

(I.2.2)

$$\dim H^1_{\{x\}}(X, \pi_*(\mathcal{F}(mD))) \leq r \cdot \dim H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(\pi^* H + mD))) + \dim H^0_{\{x\}}(X, R_m),$$

$$r \cdot \dim H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(-\pi^* H + mD))) \leq \dim H^1_{\{x\}}(X, \pi_*(\mathcal{F}(mD))) + \dim H^0_{\{x\}}(X, Q_m).$$

Since the cohomology of twists of torsion sheaves grows submaximally by [Laz04, Ex.1.2.33], from the inequality

$$\dim H^0_{\{x\}}(X, Q_m) \leq \dim H^0(X, Q_m) \leq \dim H^0(X, \pi_*(Q(mD))) = \dim H^0(X', Q(mD))$$

together with the corresponding one for R and (I.2.2), we conclude by the next easy lemma. □

Lemma I.13. *(i) If L is a Cartier divisor on X, then $h^1_x(D + \pi^* L) = h^1_x(D)$.*

(ii) In particular, if D and D' are linearly equivalent on X', then $h^1_x(D) = h^1_x(D')$.

Proof. Cohomology with supports at x is a local invariant by excision. Choosing an affine neighborhood where $\mathcal{O}_X(L)$ is trivial yields the result. $\square$

Corollary I.14. *If $f : Y \to X'$ is projective and birational, then*

$$\mathrm{vol}_x(D) = \mathrm{vol}_x(f^*D).$$

Proof. This is an immediate consequence of applying Lemma I.12 for the torsion–free sheaf of rank one $\mathcal{F} = f_*\mathcal{O}_Y$. $\square$

We also deduce a useful result concerning pullbacks by finite maps.

Proposition I.15. *Let $\pi : X' \to X$ and $\rho : Y' \to Y$ be projective birational morphisms onto normal quasiprojective varieties of dimension n at least two. Let y be a point on Y. Assume $f : X \to Y$ is a finite morphism that has a lift to a generically finite morphism $f' : X' \to Y'$, and let D be a Cartier divisor on Y'. Then*

$$(\deg f) \cdot \mathrm{vol}_y(D) = \sum_{x \in f^{-1}\{y\}} \mathrm{vol}_x(f'^*D).$$

Note that the index family for the sum is understood from the set theoretic perspective, not from the scheme theoretic one.

Proof. Let $i : Y \setminus \{y\} \to Y$ and $j : X \setminus f^{-1}\{y\}$ be the natural open embeddings. As a consequence of Remark I.6,

$$\dim \frac{j_*j^*\pi_*\mathcal{O}_{X'}(f'^*D)}{\pi_*\mathcal{O}_{X'}(f'^*D)} = \sum_{x \in f^{-1}\{y\}} h^1_x(f'^*D).$$

Looking at global sections, and by the finiteness of f,

$$\dim \frac{j_*j^*\pi_*\mathcal{O}_{X'}(f'^*D)}{\pi_*\mathcal{O}_{X'}(f'^*D)} = \dim f_*\left(\frac{j_*j^*\pi_*\mathcal{O}_{X'}(f'^*D)}{\pi_*\mathcal{O}_{X'}(f'^*D)}\right) = \dim \frac{f_*j_*j^*\pi_*\mathcal{O}_{X'}(f'^*D)}{f_*\pi_*\mathcal{O}_{X'}(f'^*D)}.$$

Chasing through the diagram

$$\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dd, "\pi"] & & Y' \arrow[dd, "\rho"] \\
X'' \arrow[ur, hook, "j'"] \arrow[rr, "f''"] \arrow[dd, "\pi'"'] & & Y'' \arrow[ur, hook, "i'"'] \arrow[dd, "\rho'"] & \\
& X \arrow[rr, "f"] & & Y \\
X \setminus f^{-1}\{y\} \arrow[ur, hook, "j"] \arrow[rr, "f'"] & & Y \setminus \{y\} \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

obtained by restricting outside y and its preimages, and applying flat base change ([Har77, Prop. III.9.3]) for the flat open embedding i, one finds that

$$\dim \frac{f_* j_* j^* \pi_* \mathcal{O}_{X'}(f'^* D)}{f_* \pi_* \mathcal{O}_{X'}(f'^* D)} = \dim \frac{i_* i^* \rho_* \mathcal{F}(D)}{\rho_* \mathcal{F}(D)},$$

with $\mathcal{F}$ denoting the torsion–free sheaf $f'_* \mathcal{O}_{X'}$ of rank $\deg(f)$ on Y'. The result is now a consequence of Lemma I.12 and of Corollary I.34. $\square$

We start drawing parallels with the theory of volumes of Cartier divisors on projective varieties.

Proposition I.16 (Finiteness)**.** *With notation as before, let D be a Cartier divisor on X'. Then $\mathrm{vol}_x(D)$ is finite.*

Proof. We can assume that X and X' are projective. Choose H ample on X such that $\pi^* H - D$ is effective. From the restriction sequence for cohomology with supports,

$$h^1_x(mD) \leq h^0(X \setminus \{x\}, \pi_* \mathcal{O}_{X'}(mD)) + h^1(X, \pi_* \mathcal{O}_{X'}(mD)).$$

By the choice of H, we have

$$h^0(X \setminus \{x\}, \pi_* \mathcal{O}_{X'}(mD)) \leq h^0(X \setminus \{x\}, \mathcal{O}_X(mH)) = h^0(X, \mathcal{O}_X(mH)).$$

The last equality holds because X is normal of dimension $n \geq 2$. For any $m \geq 0$, we have a short exact sequence

$$0 \to \mathcal{O}_{X'}(mD) \to \mathcal{O}_{X'}(m \cdot \pi^* H) \to Q_m \to 0$$

that defines Q_m. Pushing forward and taking cohomology, one finds

$$h^1(X, \pi_* \mathcal{O}_{X'}(mD)) \leq h^0(X', Q_m) + h^1(X, \mathcal{O}_X(mH)) \leq$$

$$\leq h^0(X, \mathcal{O}_X(mH)) + h^1(X', \mathcal{O}_{X'}(mD)) + h^1(X, \mathcal{O}_X(mH)).$$

We conclude that

$$\mathrm{vol}_x(D) \leq 2 \cdot \mathrm{vol}(H) + \widehat{h}^1(D) + \widehat{h}^1(H).$$

The right–hand side is finite by [Kur06, Rem.2.2]. $\square$

Remark I.17. Note the when x is a point on a nonsingular curve, even $\dim H^1_{\{x\}}(\mathcal{O}_X)$ is infinite. Therefore the assumption that $\dim X \geq 2$ is crucial.

Proposition I.18 (Homogeneity)**.** *With the same hypotheses as before,*

$$\mathrm{vol}_x(mD) = m^n \cdot \mathrm{vol}_x(D)$$

for any integer $m \geq 0$.

Proof. Following ideas in [Laz04, Lemma.2.2.38] or [Kur06, Prop.2.7], consider

$$a_i =_{\mathrm{def}} \limsup_{k \to \infty} \frac{h^1_x((mk+i)D)}{k^n/n!}.$$

It is easy to see that

$$\mathrm{vol}_x(D) = \max_{i \in \{0, \ldots, m-1\}} \{\frac{a_i}{m^n}\}.$$

On the other hand, Lemma I.12 implies that $a_0 = \ldots = a_{m-1} = \mathrm{vol}_x(mD)$. $\square$

I.2.4 Cones over polarized projective manifolds

Our prototype example, when we can compute local volumes and see an explicit connection to the theory of volumes of divisors on projective varieties, is the case of cones over polarized projective manifolds.

Example I.19 (Cone singularities)**.** Let (V, H) be a nonsingular projective polarized variety of dimension $n-1$. When H is sufficiently positive, the vertex 0 is the isolated singularity of the normal variety

$$X = \operatorname{Spec} \bigoplus_{m \geq 0} H^0(V, \mathcal{O}(mH)).$$

Blowing–up $\{0\}$ yields a resolution of singularities for X that we denote Y. The induced map $\pi : Y \to X$ is isomorphic to the contraction of the zero section E of the geometric vector bundle

$$f : \operatorname{Spec}_{\mathcal{O}_V} \operatorname{Sym}^{\bullet} \mathcal{O}_V(H) \to V.$$

We have $f_* \mathcal{O}_Y = \operatorname{Sym}^{\bullet} \mathcal{O}_V(H)$. Being the zero section, E is isomorphic to V. Concerning divisors on Y, we mention some well known results:

$$\operatorname{Pic}(Y) = f^* \operatorname{Pic}(V),$$

and divisors on Y are determined, up to linear equivalence, by their restriction to E:

$$\mathcal{O}_Y(D) = f^* \mathcal{O}_V(D|_E).$$

The conormal bundle of E in Y is:

$$\mathcal{O}_E(-E) \simeq \mathcal{O}_V(H).$$

Let L be a divisor on V and $D = f^* L$. Since X is affine, Remark I.6 implies

$$\mathrm{h}^1_{\{0\}}(mD) = \dim \frac{\bigcup_{k \geq 0} H^0(Y, \mathcal{O}_Y(mD + kE))}{H^0(Y, \mathcal{O}_Y(mD))} = \sum_{k \geq 1} h^0(\mathcal{O}_V(mL - kH)).$$

We aim to show that

$$\operatorname{vol}_{\{0\}}(D) = n \cdot \int_0^{\infty} \operatorname{vol}(L - tH) dt,$$

the volume on the right hand side being the volume of Cartier divisors on the projective variety V. Note that the integral is actually definite, because H is ample. By homogeneity and a change of variables, we can assume we are computing the integral over the interval $[0,1]$. Since H is ample, the function $t \to \mathrm{vol}(L-tH)$ is decreasing, hence for all $k>0$,

$$\frac{1}{k}\cdot\sum_{i=1}^{k}\mathrm{vol}(L-\frac{i}{k}H)\leq\int_0^1\mathrm{vol}(L-tH)dt\leq\frac{1}{k}\cdot\sum_{i=0}^{k-1}\mathrm{vol}(L-\frac{i}{k}H).$$

For any $\varepsilon>0$, there exists s_0 depending on ε and k such that for $s>s_0$,

$$\frac{n}{k}\cdot\sum_{i=0}^{k-1}\mathrm{vol}(L-\frac{i}{k}H)\leq\frac{n!}{k^n s^{n-1}}\cdot\sum_{i=0}^{k-1}h^0(skL-siH)+\varepsilon=$$

$$\frac{n!}{k^n s^{n-1}}\cdot\sum_{i=1}^{k}h^0(skL-siH)+\varepsilon+\frac{h^0(skL)-h^0(skL-skH)}{(sk)^{n-1}\cdot k/n!}\leq$$

$$\leq\frac{\mathrm{h}^1_{\{0\}}(skD)}{(sk)^n/n!}+\varepsilon+\frac{h^0(skL)-h^0(skL-skH)}{(sk)^{n-1}\cdot k/n!}.$$

Letting s tend to infinity,

$$\frac{n}{k}\cdot\sum_{i=0}^{k-1}\mathrm{vol}(L-\frac{i}{k}H)\leq\frac{\mathrm{vol}_{\{0\}}(kD)}{k^n}+\varepsilon+\frac{\mathrm{vol}(kL)-\mathrm{vol}(kL-kH)}{k^{n-1}\cdot k/n}=$$

$$=\mathrm{vol}_{\{0\}}(D)+\varepsilon+\frac{\mathrm{vol}(L)-\mathrm{vol}(L-H)}{k/n},$$

the equality taking place by the n and $n-1$ homogeneity properties of vol_0 and vol respectively. Taking limits with k and ε, we obtain

$$n\cdot\int_0^\infty\mathrm{vol}(L-tH)dt\leq\mathrm{vol}_{\{0\}}(D).$$

The reverse inequality follows in similar fashion. □

I.2.5 Basic properties of local volumes II

In this subsection we study the variational behavior of local volumes on relative Neron–Severi spaces. As before, X is a normal complex quasiprojective variety of dimension n at least two. We fix a point $x\in X$, and a projective birational morphism $\pi: X'\to X$. Unless otherwise stated, we do not assume that X' is also normal. We say that the Weil divisor D on X' *lies over* x if $D = 0$, or if $\pi(D) = \{x\}$ set theoretically. We know that the volume of Cartier divisors on projective varieties increases in effective directions and variations can be controlled by a result of Siu (see [Laz04, Thm.2.2.15] and [Laz04, Ex.2.2.23]). As we shall soon see, vol_x behaves quite differently depending on whether the effective divisor lies over x or if it has no components with this property. Controlling the variation of volumes in effective directions is our key to proving continuity properties.

Lemma I.20. *On X', let E be an effective Cartier divisor lying over x. Then for any Cartier divisor D on X',*

(i) $h^1_x(D) \geq h^1_x(D+E)$, *and hence* $\mathrm{vol}_x(D) \geq \mathrm{vol}_x(D+E)$.

(ii) $h^1_x(D) - h^1_x(D+E) \leq h^0((D+E)|_E)$.

(iii) If $E = A - B$, with A and B two $\pi-$ample divisors on X', then

$$\mathrm{vol}_x(D) - \mathrm{vol}_x(D+E) \leq n \cdot \mathrm{vol}((D+A)|_E),$$

with the volume in the right–hand side being the volume of divisors on the projective $n-1$ dimensional sub-scheme E of X'.

Proof. Denote by i the natural embedding $X \setminus \{x\} \hookrightarrow X$ and consider the diagram

$$\begin{tikzcd}
\pi_*\mathcal{O}_{X'}(D) \arrow[r, hook] \arrow[d, hook] & \pi_*\mathcal{O}_{X'}(D+E) \arrow[d, hook] \\
i_*i^*\pi_*\mathcal{O}_{X'}(D) \arrow[r, equal] & i_*i^*\pi_*\mathcal{O}_{X'}(D+E)
\end{tikzcd}$$

We get an induced surjection between the cokernels of the vertical maps and part (i) follows by Remark I.6. The same remark, together with the inclusion map

$$\frac{\pi_*\mathcal{O}_{X'}(D+E)}{\pi_*\mathcal{O}_{X'}(D)} \hookrightarrow \pi_*\mathcal{O}_E(D+E)$$

lead to part (ii). A repeated application of (ii) yields

$$h_x^1(mD) - h_x^1(mD + mE) \le \sum_{k=1}^{m} h^0((mD + kE)|_E) \le m \cdot h^0(m(D + A)|_E),$$

with the last inequality following from the assumptions on A and B that imply the effectiveness of $A|_E$ and of $(A - E)|_E$. Part (iii) follows by taking asymptotic limits. □

Quite opposite behavior is observed for effective divisors without components over x. We can control variations in such directions only in the nonsingular case, but we do have the tools to reduce our general questions to this case.

Lemma I.21. *Assume X' is nonsingular, and let F be an effective divisor without components lying over x. There exists a $\pi-$ample divisor $-\Delta_1 - \Delta_2$ with Δ_1 effective lying over x and Δ_2 effective without components over x, such that $-\Delta_1 - \Delta_2 - F$ is $\pi-$very ample. Write $\Delta_1 = M - N$, with M and N two $\pi-$ample divisors. Then for any divisor D,*

(i) $h_x^1(D + F) \ge h_x^1(D)$ *and* $\mathrm{vol}_x(D + F) \ge \mathrm{vol}_x(D)$.

(ii) $h_x^1(D + F) - h_x^1(D) \le h^0(D|_{\Delta_1})$.

(iii) $\mathrm{vol}_x(D + F) - \mathrm{vol}_x(D) \le n \cdot \mathrm{vol}((D + N)|_{\Delta_1})$.

Proof. To justify the existence of Δ_1 and Δ_2, it is enough to show that there exists an antieffective $\pi-$ample divisor. By [Har77, Thm.II.7.17], since π is projective and birational, X' is the blow–up of some ideal sheaf on X. The relative Serre bundle of the blow–up is both negative and $\pi-$ample.

Let i be the natural open embedding $X \setminus \{x\} \hookrightarrow X$. Examining the diagram

$$\begin{tikzcd}
\pi_*\mathcal{O}_{X'}(D) \arrow[r, hook] \arrow[d, hook] & \pi_*\mathcal{O}_{X'}(D + F) \arrow[d, hook] \\
i_*i^*\pi_*\mathcal{O}_{X'}(D) \arrow[r, hook] & i_*i^*\pi_*\mathcal{O}_{X'}(D + F)
\end{tikzcd}$$

we get an induced injective morphism between the cokernels of the vertical maps if we show that

$$\pi_*\mathcal{O}_{X'}(D) = \pi_*\mathcal{O}_{X'}(D+F) \cap i_*i^*\pi_*\mathcal{O}_{X'}(D),$$

the intersection taking place in $i_*i^*\pi_*\mathcal{O}_{X'}(D+F)$. It is enough to show this on the level of sections over open neighborhoods of x. Let U be such an open set on X and let V be its inverse image in X'. Let E be the divisorial support of the set theoretic fiber $\pi^{-1}(x)$. Since X' is in particular normal, we have to show

$$H^0(V, \mathcal{O}_{X'}(D)) = H^0(V, \mathcal{O}_{X'}(D+F)) \cap H^0(V \setminus \{E\}, \mathcal{O}_{X'}(D))$$

inside $H^0(V \setminus \{E\}, \mathcal{O}_{X'}(D+F))$ which is easily checked. Part (i) follows by Remark I.6. Let A be a divisor without components over x that is $\pi-$linearly equivalent to $-\Delta_1 - \Delta_2 - F$. By part (i) and Lemmas I.13 and I.20,

$$h^1_x(D+F) - h^1_x(D) \leq h^1_x(D+F+A+\Delta_2) - h^1_x(D) = h^1_x(D-\Delta_1) - h^1_x(D) \leq h^0(D|_{\Delta_1}).$$

Consider the telescopic sum as in Lemma I.20 and the previous estimate:

$$h^1_x(m(D+F)) - h^1_x(mD) \leq \sum_{k=1}^{m} h^0((mD-(k-1)\Delta_1)|_{\Delta_1}) \leq m \cdot h^0(m(D+N)|_{\Delta_1}),$$

because $(mN+(k-1)\Delta_1)|_{\Delta_1} = ((m-k+1)N+(k-1)M)|_{\Delta_1}$ is effective for any $1 \leq k \leq m$. Part (iii) follows by taking asymptotic limits. □

We aim to prove that $\mathrm{vol}_x(D)$ depends only on the $\pi-$relative numerical class of D in $N^1(X'/X)$.

Lemma I.22. *Let T be a $\pi-$nef divisor on X'. Then for any Cartier divisor D on X', we have*

$$\mathrm{vol}_x(D+T) \geq \mathrm{vol}_x(D).$$

Proof. By Lemma I.12, we can assume that X' is nonsingular. Let then F be a $\pi-$ample divisor on X'. For any $m \geq 1$, there exists $k_m > 0$ such that $k_m(mT+F)$

is $\pi-$linearly equivalent to an effective divisor without components lying over x. By Lemmas I.13 and I.21, and by Proposition I.18, we have

$$\frac{\mathrm{vol}_x(m(D+T)+F)}{m^n} \geq \mathrm{vol}_x(D). \tag{I.2.3}$$

By part *(iii)* of Lemma I.21, with the notation there,

$$\mathrm{vol}_x(m(D+T)+F) - \mathrm{vol}_x(m(D+T)) \leq n \cdot \mathrm{vol}((m(D+T) - \Delta_1 + M)|_{\Delta_1}).$$

Since the support of Δ_1 is of dimension $n-1$, dividing by m^n, and applying Proposition I.18 and the inequality (I.2.3),

$$\mathrm{vol}_x(D+T) = \lim_{m\to\infty} \frac{\mathrm{vol}_x(m(D+T)+F)}{m^n} \geq \mathrm{vol}_x(D).$$

□

Corollary I.23 (Relative numerical invariance)**.** *Let T be a $\pi-$numerically trivial divisor on X'. Then for any Cartier divisor D on X', we have*

$$\mathrm{vol}_x(D+T) = \mathrm{vol}_x(D).$$

Proof. Both T and $-T$ are $\pi-$nef, hence

$$\mathrm{vol}_x(D) \leq \mathrm{vol}_x(D+T) \leq \mathrm{vol}_x((D+T)+(-T)) = \mathrm{vol}_x(D).$$

□

By Corollary I.23, the local volume vol_x is a well defined function on $N^1(X'/X)$. From the homogeneity result in Proposition I.18, it also has a natural extension to $N^1(X'/X)_{\mathbb{Q}}$. By proving a Lipschitz–type estimate on this space, we are able to extend to real coefficients.

Proposition I.24 (Continuity)**.** *With notation as before, the relative numerical real space $N^1(X'/X)_{\mathbb{R}}$ is finite dimensional. Fix a norm $|\cdot|$ on it. Then there exists a positive constant C such that for any A and B in the rational vector space $N^1(X'/X)_{\mathbb{Q}}$*

we have the Lipschitz–type estimate:

$$|\mathrm{vol}_x(B) - \mathrm{vol}_x(A)| \leq C \cdot (\max(|A|, |B|))^{n-1} \cdot |A - B|.$$

Proof. We show that we can assume that X' is nonsingular. Let $f : Y \to X'$ be a resolution of singularities. Then f^* induces an injective morphism

$$N^1(X'/X) \hookrightarrow N^1(Y/X),$$

which does not change the values of vol_x by Corollary I.13. Hence it is sufficient to prove our estimate for X' nonsingular.

We choose $\lambda_1, \ldots, \lambda_k$ a basis for $N^1(X'/X)_{\mathbb{R}}$ composed of integral $\pi-$very ample divisors without components over x. Relative to this basis, we can assume that

$$|(a_1, \ldots, a_k)| = \max_{1 \leq i \leq k} |a_i|.$$

With notation as in Lemma I.21, choose Δ_1 and Δ_2 two effective integral divisors with the first lying over x whereas the second has no components over x such that for all $i \in \{1, \ldots, k\}$, the divisor $-\Delta_1 - \Delta_2 - \lambda_i$ is $\pi-$linearly equivalent to one without components lying over x. Write $\Delta_1 = M - N$ with M and N two $\pi-$ample divisors. Let

$$A = (a_1, \ldots, a_k), \qquad B = (a_1 + b_1, \ldots, a_k + b_k), \qquad N = (\alpha_1, \ldots, \alpha_k)$$

with all entries being rationals. Since our estimate to prove and vol_x are both $n-$homogeneous, we can further assume that all the entries are integers. Note that the α_i are fixed.

If we denote $B_i = (a_1, \ldots, a_i, a_{i+1} + b_{i+1}, \ldots, a_k + b_k)$ and set

$$A_i = \begin{cases} B_{i-1} + b_i N, & \text{if } b_i \geq 0 \\ B_i - b_i N, & \text{if } b_i \leq 0 \end{cases},$$

then

$$|\mathrm{vol}_x(B) - \mathrm{vol}_x(A)| \leq \sum_{i=1}^{k} |\mathrm{vol}_x(B_{i-1}) - \mathrm{vol}_x(B_i)| \leq n \cdot \sum_{i=1}^{k} \mathrm{vol}(A_i|_{|b_i|\Delta_1})$$

by Lemma I.21. Let

$$\alpha = \max_{1\leq i\leq k}(|\alpha_i|).$$

Since $\lambda_i|_{|b_i|\Delta_1}$ is ample for all i, and $\mathrm{vol}(D|_{\Delta_1}) = D^{n-1}\cdot\Delta_1$ if D is $\pi-$ample,

$$n \cdot \sum_{i=1}^{k} \mathrm{vol}(A_i|_{|b_i|\Delta_1}) \leq n(1+\alpha)^{n-1} \cdot \max_{1\leq i\leq k}(|a_i| + |b_i|)^{n-1} \cdot ((\sum_{i=1}^{k}\lambda_i)^{n-1}\cdot\Delta_1) \cdot \sum_{i=1}^{k}|b_i|.$$

Setting

$$C = nk \cdot 2^{n-1}(1+\alpha)^{n-1} \cdot ((\sum_{i=1}^{k}\lambda_i)^{n-1}\cdot\Delta_1)$$

concludes the proof. □

Putting together Propositions I.18 and I.24 with Corollary I.23, we have proved:

Theorem I.25. *Notation being as above,* vol_x *is a well defined, $n-$homogeneous, and locally Lipschitz continuous function on* $N^1(X'/X)_{\mathbb{R}}$.

I.3 Further extensions

We say a few words about extending the results in the previous section to proper generically–finite morphisms, and to algebraically closed fields of arbitrary characteristic.

Remark I.26. By working in an affine neighborhood of $x \in X$, we can remove the assumption that X is quasi-projective.

Remark I.27 (Proper morphisms). Using Chow's lemma ([Har77, Ex.II.4.10]), and adjusting the proof of Lemma I.12, we can extend our results to proper birational morphisms $\pi : X' \to X$.

Remark I.28 (Generically finite morphisms). Let $\pi : X' \to X$ be a generically finite proper morphism with X normal of dimension $n \geq 2$. Let D be a Cartier divisor on X', and let x be a point on X. Denote by $\widetilde{X}$ the normalization of X', by $\widetilde{D}$ the lift of D, and by $\widetilde{Y}$ the normalization of the Stein factorization ([Har77, Cor.III.11.5]) of π. Note that $\widetilde{Y}$ is the Stein factorization of the induced map $\widetilde{X} \to X$ and that the map $\widetilde{X} \to \widetilde{Y}$ is birational. Let $\{y_1, \ldots, y_k\}$ be the set theoretic preimage of x in $\widetilde{Y}$. Then one can define

$$\mathrm{vol}_x(D) =_{\mathrm{def}} \frac{1}{\deg \pi} \sum_{i=1}^{k} \mathrm{vol}_{y_i}(\widetilde{D}).$$

Proposition I.15 and Lemma I.12 make this definition compatible with the birational case, i.e.,

$$\mathrm{vol}_x(D) = \limsup_{m \to \infty} \frac{\dim H^1_{\{x\}}(X, \pi_* \mathcal{O}_{X'}(mD))}{\deg(\pi) \cdot m^n / n!}.$$

Remark I.29 (Positive characteristic). We have used characteristic 0 in studying the variational behavior of local volumes in Lemma I.21 where we reduced to X' being nonsingular, which we could do upon replacing X' by a resolution of singularities. In arbitrary characteristic, over an algebraically closed field, to extend the results of this subsection, one first replaces X' by a regular alteration (see [dJ96]), and applies the discussion above for generically finite proper morphisms to reduce to the case where π is birational and X' is regular. The price to pay is that x is replaced by a finite collection of points, but this is afforded by Proposition I.15 via Corollary I.34, which extends in characteristic p under the assumption that X' is regular.

I.4 A convex–geometric approach to local volumes

Given a projective birational morphism $\pi : X' \to X$ onto the complex normal algebraic variety X of dimension $n \geq 2$, and given $x \in X$, and a Cartier divisor D on X', in this section we realize $\mathrm{vol}_x(D)$ as a volume of a not necessarily convex body arising naturally as the bounded difference of two possibly unbounded convex nested polyhedra. This approach has proven effective in [LM09], in particular for proving that volumes of Cartier divisors are actual limits, and for developing Fujita–type approximation results. It is plausible that one can use this point of view to provide new proofs for the results in the second section of this chapter. By employing techniques similar to [LM09], we extend these results to the local setting.

Assume, unless otherwise stated, that $\pi : (X', E) \to (X, x)$ is a log–resolution of the normal affine (X, x), with x not necessarily an isolated singularity, and let

$$E = E_1 + \ldots + E_k$$

be the irreducible decomposition of the reduced fiber over x. Since X is assumed to be affine, for any divisor D on X', we have by Remark I.6 that

$$H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(D)) = \frac{H^0(X' \setminus E, \mathcal{O}_{X'}(D))}{H^0(X', \mathcal{O}_{X'}(D))}.$$

The dimension of the above vector space is by definition $h^1_x(D)$. Spaces of sections of multiples of line bundles on X' are studied in [LM09] via valuation–like functions defined with respect to a choice of a complete flag. It is important to work with line bundles on X' and not $X' \setminus E$. In this regard, the following lemma helps us handle $H^0(X' \setminus E, \mathcal{O}_{X'}(mD))$ for all $m \geq 0$.

Lemma I.30. *In the above setting, for any divisor D on nonsingular X' there exists $r > 0$ such that for all $m \geq 0$ there is a natural identification*

$$H^0(X' \setminus E, \mathcal{O}_{X'}(mD)) \simeq H^0(X', \mathcal{O}_{X'}(m(D + rE))).$$

Proof. For any divisor L on X', identify

$$H^0(X', \mathcal{O}_{X'}(L)) = \{f \in K(X) \ : \ \mathrm{div}(f) + L \geq 0\}. \tag{I.4.1}$$

With this identification, recall that

$$H^0(X' \setminus E, \mathcal{O}_{X'}(mD)) = \bigcup_{i \geq 0} H^0(X', \mathcal{O}_{X'}(mD + iE)).$$

There exists an inclusion $\mathcal{O}_{X'}(D) \subseteq \pi^*\mathcal{O}_X(H)$ for some effective Cartier (sufficiently ample) divisor H on X. Since X is normal, rational functions defined outside subsets of codimension two or more extend, and so

$$H^0(X' \setminus E, \pi^*\mathcal{O}_X(mH)) = H^0(X \setminus \{x\}, \mathcal{O}_X(mH)) =$$

$$= H^0(X, \mathcal{O}_X(mH)) = H^0(X', \pi^*\mathcal{O}_X(mH)).$$

For all non-negative i and m, the following natural inclusions are then equalities:

$$H^0(X', \mathcal{O}_{X'}(\pi^* mH)) \subseteq H^0(X', \mathcal{O}_{X'}(\pi^* mH + iE)) \subseteq H^0(X' \setminus E, \mathcal{O}_{X'}(\pi^* mH)).$$

Choose r so that the order of $D + rE$ along any irreducible component of E is strictly greater than the order of $\pi^* H$ along the same component. For $s > r$, that $\operatorname{div}(f) + m(D + sE)$ is effective implies that

$$f \in H^0(X', \mathcal{O}_{X'}(m(D+sE))) \subseteq H^0(X', \mathcal{O}_{X'}(m(\pi^* H+sE))) = H^0(X', \mathcal{O}_{X'}(\pi^* mH)),$$

therefore $\operatorname{div}(f)+\pi^* mH$ is also effective. Looking at the orders along the components of E, because of our choice of r, we actually get $f \in H^0(X', \mathcal{O}_{X'}(m(D + rE)))$. $\square$

Consider a complete flag of subvarieties of X', i.e., each is a divisor in the previous subvariety:

$$Y_\bullet : \; X' = Y_0 \supset E_1 = Y_1 \supset \ldots \supset Y_n = \{y\}$$

such that each Y_i is nonsingular at y. Recall that E_1 is a component of E, the reduced fiber of π over x. Following [LM09, 1.1], for any divisor D on X', we construct a valuation like function

$$\nu = \nu_D = (\nu_1, \ldots, \nu_n) : H^0(X', \mathcal{O}_{X'}(D)) \to \mathbb{Z}^n \cup \{\infty\}$$

having the following properties:

(i). $\nu(s) = \infty$ if, and only if, $s = 0$.

(ii). $\nu(s + s') \geq \min\{\nu(s), \nu(s')\}$ for any $s, s' \in H^0(X', \mathcal{O}_{X'}(D))$.

(iii). $\nu_{D_1+D_2}(s_1 \otimes s_2) = \nu_{D_1}(s_1) + \nu_{D_2}(s_2)$ for any divisors D_i on X', and any $s_i \in H^0(\mathcal{O}_{X'}(D_i))$.

Each ν_i is constructed by studying orders of vanishing along the terms of the flag $Y_\bullet$. For $s \in H^0(X', \mathcal{O}_{X'}(D))$, define first $\nu_1(s)$ as the order of vanishing of s along E_1. If f is the rational function corresponding to s via the identification (I.4.1), then $\nu_1(s)$ is the coefficient of E_1 in $\mathrm{div}(f) + D$. A non–unique local equation for Y_1 in Y_0 then determines a section

$$\overline{s} \in H^0(Y_1, \mathcal{O}_{Y_0}(D - \nu_1(s)Y_1)|_{Y_1})$$

having a uniquely defined order of vanishing along Y_2 that we denote $\nu_2(s)$, and the construction continues inductively. More details can be found in [LM09, 1.1]. Note that the ν_i assume only nonnegative values.

For any divisor D on X' and for $m \geq 0$, with r given by Lemma I.30, let

$$I'_m = \nu_{m(D+rE)}(H^0(X', \mathcal{O}_{X'}(m(D + rE)))), \tag{I.4.2}$$

$$I_m = \nu_{m(D+rE)}(H^0(X', \mathcal{O}_{X'}(mD))), \tag{I.4.3}$$

$$B_m = I'_m \setminus I_m. \tag{I.4.4}$$

By construction, $I'_\bullet = \bigcup_{m\geq 0}(I'_m, m)$ and $I_\bullet = \bigcup_{m\geq 0}(I_m, m)$ are semigroups of $\mathbb{N}^{n+1}$. We abuse notation in identifying the sets I_m and (I_m, m). We will soon prove (Lemma I.32) that

$$\#B_m = \dim \frac{H^0(X', \mathcal{O}_{X'}(m(D + rE)))}{H^0(X', \mathcal{O}_{X'}(mD))} = h^1_x(mD).$$

Assuming this result, we aim to show that $\mathrm{vol}_x(D)$ is the normalized volume of the not necessarily convex body B obtained as the difference of two nested polytopes arising as Okounkov bodies of some sub–semigroups of $I'_\bullet$ and $I_\bullet$ respectively, each satisfying the conditions [LM09, (2.3)-(2.5)]. For a semigroup $\Gamma_\bullet \subseteq \mathbb{N}^{n+1}$ with $\Gamma_m = \Gamma_\bullet \cap (\mathbb{N}^n \times \{m\})$, these conditions are as follows:

(Strictness): $\Gamma_0 = \{0\}$.

(Boundedness): $\Gamma_\bullet \subseteq \Theta_\bullet$, for some semigroup $\Theta_\bullet \subseteq \mathbb{N}^{n+1}$ generated by the finite set Θ_1.

(Denseness): $\Gamma_\bullet$ generates $\mathbb{Z}^{n+1}$ as a group.

A semigroup $\Gamma_\bullet$ satisfying the above conditions generates the closed convex cone

$$\Sigma(\Gamma) \subset \mathbb{R}^{n+1}_{\geq 0}.$$

This determines the convex polytope (the associated Okounkov body)

$$\Delta(\Gamma) = \Sigma(\Gamma) \cap (\mathbb{R}^n \times \{1\}).$$

By [LM09, Prop.2.1], with the volume on $\mathbb{R}^n$ normalized so that the volume of the unit cube is one,

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(\Gamma)) = \lim_{m\to\infty} \frac{\#\Gamma_m}{m^n}.$$

Our first challenge is to show that B_m (see I.4.4) is linearly bounded with m. With Lemma I.32 still to prove, we show the following apparently stronger independent result:

Lemma I.31. *For a divisor D on the nonsingular X', with r as in Lemma I.30, there exists $N > 0$ such that for all i and m, with valuation–like functions on*

$$H^0(X', \mathcal{O}_{X'}(m(D + rE)))$$

as above, we have $\nu_i(s) \leq mN$ *for any*

$$s \in H^0(X', \mathcal{O}_{X'}(m(D + rE))) \setminus H^0(X', \mathcal{O}_{X'}(mD)),$$

e.g., $\nu_{m(D+rE)}(s) \in B_m$.

Proof. Let H be a relatively ample integral divisor on X' and assume we have shown that there exists such a linear bound N_1 for ν_1. Since Y_1 is projective, as in [LM09, Lemma.1.10], there exists N_2 such such that for all real number $0 \leq a < N_1$

$$((D + rE - aY_1)|_{Y_1} - N_2 Y_2) \cdot H^{n-2} < 0.$$

This provides the linear bound for ν_2, and one iterates this construction for all $i > 1$. Letting N be the maximum of all N_i completes the proof. We still have to construct N_1. The idea here is to apply a theorem of Izumi that shows that a regular function with a high order of vanishing along E_1 also vanishes to high order along the other E_i. The technical part is to see how to apply this to rational functions giving sections of $\mathcal{O}_{X'}(m(D+rE))$. Since X is assumed to be affine, there exists a rational function g such that

$$G =_{\text{def}} \operatorname{div}(g) - D - rE$$

is effective on X'. With the identification in (I.4.1), for any

$$f \in H^0(X', \mathcal{O}_{X'}(m(D + rE))),$$

the a priori rational function $f \cdot g^m$ is regular on X'. Let

$$\operatorname{div}(f \cdot g^m) = C + \sum_{i=1}^{k} c_i E_i$$

with E_i the components of the reduced fiber E over x, with $c_i \geq 0$ for all i and C an effective divisor without components over x. There exists $R > 1$ such that if $c_1 > 0$, then

$$R > \frac{c_i}{c_j} > \frac{1}{R}$$

for all i, j. This is an analytic result of Izumi ([Izu85]), extended to arbitrary characteristic by Rees ([Ree89]). It follows that even when $c_1 = 0$,

$$\operatorname{div}(f \cdot g^m) = C + \sum_{i=1}^{k} c_i E_i \geq C + \frac{c_1}{R} \cdot E.$$

If s is the regular section associated to f, i.e., its zero locus is

$$Z(s) = \operatorname{div}(f) + m(D + rE),$$

then the above inequality can be rewritten as

$$Z(s) = C - mG + \sum_{i=1}^{k} c_i E_i \geq C - mG + \frac{c_1}{R} \cdot E.$$

If ρ is the maximal coefficient of any E_i in G, and g_1 is the coefficient of E_1, we set $N_1 = R(r + \rho - g_1)$ and see that when $\nu_1(s) = c_1 - mg_1 > mN_1$, then $Z(s) \geq mrE$ showing that

$$s \in H^0(X', \mathcal{O}_{X'}(mD)) \subseteq H^0(X', \mathcal{O}_{X'}(m(D + rE))).$$

□

We now prove that B_m has the expected cardinality.

Lemma I.32. *With notation as above, for all $m \geq 0$, we have $\#B_m = h^1_x(mD)$.*

Proof. Without loss of generality, we can assume that $m = 1$. By Lemma I.31, the set B_1 is a bounded subset of a lattice, therefore it is finite. The idea is to reduce the problem to the projective setting, where we apply [LM09, Lemma.1.3].

Recall that X is assumed to be affine. Let $\overline{\pi} : \overline{X'} \to \overline{X}$ be a compactification of π such that $\overline{X} \setminus X$ is the support of an ample divisor H. By abuse of notation, we use the same symbol for H and its pullback, and we use the same notation for D and its closure in $\overline{X'}$. Note that the pullback of H is big and semiample. For all $m \geq 0$, the natural inclusion

$$H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(m(tH + D + rE))) \subset H^0(X', \mathcal{O}_{X'}(m(D + rE)))$$

is compatible with the valuation like functions $\nu_{m(tH+D+rE)}$ and $\nu_{m(D+rE)}$ that we construct when working over $\overline{X'}$ and X' respectively with the flag $Y_\bullet$ and the obvious compactification that replaces $Y_0 = X'$ by $\overline{X'}$ and leaves the remaining terms unchanged. We have the same compatibility for $\nu_{m(tH+D)}$ and ν_{mD}. Note also that

$$H^0(X', \mathcal{O}_{X'}(D+rE)) = \bigcup_{t\geq 0} H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D+rE))$$

and a similar statement holds for D. When t is sufficiently large so that

$$H^1(\overline{X}, \overline{\pi}_*\mathcal{O}_{\overline{X'}}(D) \otimes \mathcal{O}_{\overline{X}}(tH)) = 0, \tag{I.4.5}$$

excision and the natural cohomology sequence on $\overline{X}$ show that

$$H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(D)) \simeq \frac{H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D+rE))}{H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D))}. \tag{I.4.6}$$

Note that the r provided by Lemma I.30 also works to prove

$$H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D+rE)) = H^0(\overline{X'} \setminus E, \mathcal{O}_{\overline{X'}}(tH+D)).$$

Denote

$$\begin{aligned}
W'_t &= H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D+rE)) \\
W_t &= H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(tH+D)) \\
W' &= \bigcup_{t\geq 0} W'_t = H^0(X', \mathcal{O}_{X'}(D+rE)) \\
W &= \bigcup_{t\geq 0} W_t = H^0(X', \mathcal{O}_{X'}(D)).
\end{aligned}$$

With the intersection taking place in W', note that

$$W_t = W'_t \cap W.$$

Let t be large enough so that the vanishing (I.4.5) takes place, such that $\nu(W'_t)$ contains the set $\mathcal{N}$ of all elements in $\nu(W')$ satisfying the bound in Lemma I.31, and such that $\nu(W_t)$ contains all elements in $\nu(W) \cap \mathcal{N}$. We show that

$$\nu(W'_t) \setminus \nu(W_t) = \nu(W') \setminus \nu(W) = B_1.$$

Since $B_1 \subset \mathcal{N}$ by Lemma I.31, all its elements are in $\nu(W'_t)$ by the choice of t, and are not in $\nu(W_t) \subset \nu(W)$. Therefore $B_1 \subseteq \nu(W'_t) \setminus \nu(W_t)$. Again by the choice of t, any element in $\nu(W'_t) \setminus \nu(W_t)$ that is not in B_1 is also not in $\mathcal{N}$. Let $\sigma \in W'_t$, such that $\nu(\sigma) \in (\nu(W'_t) \setminus \nu(W_t)) \setminus B_1$. Then $\nu(\sigma) \in \nu(W'_t) \setminus \mathcal{N}$, and again by Lemma I.31 we obtain $\sigma \in W$, hence $\sigma \in W \cap W'_t = W_t$, and $\nu(\sigma) \in \nu(W_t)$, which is impossible.

Now $\#B_1 = \#(\nu(W'_t) \setminus \nu(W_t)) = \#\nu(W'_t) - \#\nu(W_t) = h^1_x(D)$ by (I.4.6) and by [LM09, Lemma.1.3], a result that shows $\#\nu(W'_t) = \dim W'_t$, and the analogous result for W_t. $\square$

We next construct subsemigroups $\Gamma'_\bullet \subset I'_\bullet$ and $\Gamma_\bullet \subset I_\bullet$, each satisfying the properties [LM09, (2.3)-(2.5)] mentioned above on page 40, and such that $B_m = \Gamma'_m \setminus \Gamma_m$ for all m. With notation as in the proof of Lemma I.32, and with t sufficiently large so that $tH + D$ is big, let

$$S_m = \nu_{m(tH+D+rE)}(H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(m(tH+D)))).$$

If we pick the flag $Y_\bullet$ so that $Y_n = \{y\}$ is not contained in any E_i for $i > 1$ [1], then

$$S_m = \text{translation of } \nu_{m(tH+D)}(H^0(\overline{X'}, \mathcal{O}_{\overline{X'}}(m(tH+D)))) \text{ by } (mr, 0, 0 \ldots, 0, m) \in \mathbb{N}^{n+1}.$$

It follows from [LM09, Lemma 2.2] that $S_\bullet$ satisfies the conditions [LM09, (2.3)-(2.5)]. By [LM09, Lemma 1.10], there exists a linear bound for $S_\bullet$ in the sense of Lemma I.31. Let N be the greatest of the two linear bounds provided by Lemmas I.31 and [LM09, Lemma 1.10]. If x_i denotes the i-th coordinate on $\mathbb{N}^n$, let

$$\Gamma_m = \{(x_1, \ldots, x_n) \in I_m \;:\; x_i \leq mN \;\text{ for all } 1 \leq i \leq n\}$$

and construct $\Gamma'_\bullet$ similarly. These semigroups satisfy the strictness ([LM09, (2.3)]) and boundedness ([LM09, (2.4)]) conditions. They also each generate $\mathbb{Z}^n$ as a group

[1] We thank Tommaso de Fernex for suggesting this choice

because they contain $S_\bullet$ which does. By Lemma I.31, we have $B_m = \Gamma'_m \backslash \Gamma_m$. Letting $B = \Delta(\Gamma') \setminus \Delta(\Gamma)$, we prove:

Proposition I.33. *With notation as above, we have*

$$\mathrm{vol}_x(D) = n! \cdot \mathrm{vol}_{\mathbb{R}^n}(B),$$

where $\mathrm{vol}_{\mathbb{R}^n}(\bullet)$ *is the usual Euclidean volume on* $\mathbb{R}^n$ *(normalized so that the volume of the unit cube is* 1*).*

Proof.

$$\mathrm{vol}_x(D) = \limsup_{m\to\infty} \frac{h^1_x(mD)}{m^n/n!} = n! \cdot \limsup_{m\to\infty} \frac{\# B_m}{m^n} = n! \cdot \limsup_{m\to\infty} \frac{\#\Gamma'_m - \#\Gamma_m}{m^n}.$$

By [LM09, Prop.2.1], the lim sup is lim, and

$$\lim_{m\to\infty} \frac{\#\Gamma'_m - \#\Gamma_m}{m^n} = \mathrm{vol}_{\mathbb{R}^n}(\Delta(\Gamma') \setminus \Delta(\Gamma)) = \mathrm{vol}_{\mathbb{R}^n}(B).$$

□

Corollary I.34. *Let* $\pi : X' \to X$ *be a projective birational morphism onto the complex normal algebraic* X *of dimension* n *at least two, and let* x *be a point on* X*. Then for any Cartier divisor* D *on* X'*, we have*

$$\mathrm{vol}_x(D) = \lim_{m\to\infty} \frac{h^1_x(mD)}{m^n/n!}.$$

Proof. Let $f : \widetilde{X} \to X'$ be a projective birational morphism such that $\rho = \pi \circ f : \widetilde{X} \to X$ is a log–resolution of (X, x). Since $\mathrm{vol}_x(D)$ is local around x, we can also assume that X is affine. By the proof of Lemma I.12, the sequences $h^1_x(mD)$ and $h^1_x(mf^*D)$ have the same asymptotic behavior. Therefore we have reduced to the setting of Proposition I.33 where we saw that lim replaces lim sup via [LM09, Prop.2.1]. □

Remark I.35. The natural approach to the problem of expressing $\mathrm{vol}_x(D)$ as a volume of a polytope and replacing lim sup by lim is to write $B_m = \Gamma'_m \setminus \Gamma_m$, with $\Gamma'_\bullet$ and

$\Gamma_\bullet$ semigroups constructed on compacfications of π, in the same style as we did for $S_\bullet$, and then apply [LM09, Thm.2.13]. This approach is successful when we have an analogue of [LM09, Lemma.3.9], i.e., when we can show that, at least asymptotically, the groups $H^1(\overline{X}, \overline{\pi}_*\mathcal{O}_{\overline{X'}}(mD) \otimes \mathcal{O}_{\overline{X}}(mH))$ vanish for some ample divisor H on a projective compactification $\overline{\pi}$ of π. We do not know if such a result holds for any Cartier divisor D on X', but we will see it when the graded family $\mathfrak{a}_m = \pi_*\mathcal{O}_{X'}(mD)$ is of the form $\mathfrak{b}^m$ outside x, for all $m \geq 0$, for some coherent fractional ideal sheaf $\mathfrak{b}$ on $X \setminus \{x\}$. This happens for example when D lies over x, or when $D = K_{\widetilde{X}} + aE$ with $a \in \mathbb{Z}$ on a log–resolution $\pi : (\widetilde{X}, E) \to (X, x)$ of a normal isolated singularity.

I.5 A Fujita–type approximation result

The content of the classical Fujita approximation statement is that the volume of a Cartier divisor D on a projective variety X of dimension n can be approximated arbitrarily closely by volumes $\mathrm{vol}(A)$ where A is a nef Cartier $\mathbb{Q}-$divisor on some blow–up $\pi : X' \to X$, such that $\pi^*D - A$ is effective. The Fujita–type approximation result in [LM09, Thm.3.8] states that for any graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}-$primary ideals,

$$\widehat{h}^1_x(\mathfrak{a}_\bullet) = \lim_{p\to\infty} \frac{e(\mathfrak{a}_p)}{p^n}.$$

We remove the $\mathfrak{m}-$primary restriction in a particular case.

Theorem I.36. *Let $\pi : X' \to X$ be a projective birational morphism onto a normal quasiprojective variety X of dimension $n \geq 2$. Fix $x \in X$, and let D be a Cartier divisor on X'. Assume that there exists a coherent fractional ideal sheaf $\mathfrak{b}$ on $X \setminus \{x\}$ such that $\pi_*\mathcal{O}_{X'}(pD)|_{X\setminus\{x\}} = \mathfrak{b}^p$ for all $p \geq 0$. Then*

$$\mathrm{vol}_x(D) = \lim_{p\to\infty} \frac{\widehat{h}^1_x(\pi_*\mathcal{O}_{X'}(pD))}{p^n}.$$

Proof. Since our invariants are local, we can assume that X is projective, and choose an ample divisor A. Up to replacing D by $D - m\pi^* A$ for some large m, we can assume that D is antieffective. Denote

$$\mathfrak{a}_p = \pi_* \mathcal{O}_{X'}(pD).$$

The negativity assumption on D shows that these are actually ideal sheaves. Inspired by [LM09, Lemma.3.9], we claim that there exists an ample divisor H on X such that for every $p, k > 0$,

$$H^1(X, \mathcal{O}_X(pkH) \otimes (\pi_* \mathcal{O}_{X'}(pD))^k) = 0, \tag{I.5.1}$$

and the subspaces $H^0(X, \mathcal{O}_X(pH) \otimes \pi_* \mathcal{O}_{X'}(pD)) \subseteq H^0(X, \mathcal{O}_X(pH))$ determine rational maps

$$\phi_p : X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(pD))$$

that are birational onto their image for all $p > 0$.

Let $\sigma : Y \to X$ be the blow–up of $\mathfrak{a}_1$, and let E be the exceptional divisor. In particular, $\mathcal{O}_Y(-E)$ is $\sigma-$ample. Upon replacing A by a sufficiently high multiple, we can assume that $\sigma^* A - E$ is ample on Y. By [Laz04, Lemma.5.4.24], $\sigma_* \mathcal{O}_Y(-rE) = \mathfrak{a}_1^r$ for $r \gg 0$. From Serre vanishing on Y, from the $\sigma-$ampleness of $-E$, using the Leray spectral sequence, we obtain

$$H^1(X, \mathfrak{a}_1^p \otimes \mathcal{O}_X(pA)) = 0$$

for sufficiently large p. This holds for all $p \geq 1$, if we again replace A by a multiple. The hypothesis on D implies that $\frac{\mathfrak{a}_p^k}{\mathfrak{a}_1^{pk}}$ is supported at x. From the short exact sequences

$$0 \to \mathfrak{a}_1^{pk} \otimes \mathcal{O}_X(pkA) \to \mathfrak{a}_p^k \otimes \mathcal{O}_X(pkA) \to \frac{\mathfrak{a}_p^k}{\mathfrak{a}_1^{pk}} \otimes \mathcal{O}_X(pkA) \to 0,$$

we then deduce (I.5.1) with $H = A$. The birationality of ϕ_p is implied by the case $p = 1$ via the inclusion $H^0(X, \mathfrak{a}_1^p \otimes \mathcal{O}_X(pH)) \hookrightarrow H^0(X, \mathfrak{a}_p \otimes \mathcal{O}_X(pH))$. We can insure that ϕ_1 is birational, if we replace H by a multiple.

We now follow the ideas of the proof of [LM09, Thm.3.8]. Denote

$$W_p = H^0(X, \mathfrak{a}_p \otimes \mathcal{O}_X(pH)),$$

$$W'_p = H^0(X, \widetilde{\mathfrak{a}_p} \otimes \mathcal{O}_X(pH)),$$

where $\widetilde{\mathfrak{a}_p} = \imath_* \imath^* \mathfrak{a}_p$, and $\imath : X \setminus \{x\}$ is the open embedding. Setting $\mathfrak{c}_p = \imath_*(\mathfrak{b}^p)$, note that

$$\widetilde{\mathfrak{a}_p^k} = \widetilde{\mathfrak{a}_{pk}} = \mathfrak{c}_{pk}$$

for all $p, k > 0$. Write $w_p = \dim W_p$, and $w'_p = \dim W'_p$. Following [LM09], define

$$\mathrm{vol}(W_\bullet) = \limsup_{p \to \infty} \frac{w_p}{p^n/n!}.$$

Thanks to [LM09, Rem.2.14], the lim sup can be replaced by lim. Because $\widetilde{\mathfrak{a}_p}/\mathfrak{a}_p$ is supported at x, from the vanishing (I.5.1), one finds

$$h^1_x(\mathfrak{a}_p) = w'_p - w_p$$

for all $p > 0$. Taking limits,

$$\mathrm{vol}_x(D) = \widehat{h}^1_x(\mathfrak{a}_\bullet) = \mathrm{vol}(W'_\bullet) - \mathrm{vol}(W_\bullet). \tag{I.5.2}$$

In particular, with the hypothesis on D, the lim sup in the definition of $\mathrm{vol}_x(D)$ can be replaced by lim. The surjection $\mathfrak{c}_{pk}/\mathfrak{a}_p^k \to \mathfrak{c}_{pk}/\mathfrak{a}_{pk}$ then implies

$$\widehat{h}^1_x(\mathfrak{a}_\bullet) \leq \frac{\widehat{h}^1_x(a_p)}{p^n} \tag{I.5.3}$$

for all $p > 0$. Let

$$V_{p,k} =_{\mathrm{def}} \mathrm{Im}(\mathrm{Sym}^k H^0(X, \mathfrak{a}_p \otimes \mathcal{O}_X(pH)) \to H^0(X, \mathfrak{a}_{pk} \otimes \mathcal{O}_X(pkH))).$$

Note that $V_{p,k} \subset H^0(X, \mathfrak{a}_p^k \otimes \mathcal{O}_X(pkH))$. Then

$$\text{(I.5.4)} \qquad v_{p,k} =_{\text{def}} \dim V_{p,k} =\leq h^0(\mathfrak{a}_p^k \otimes \mathcal{O}_X(pkH)) = w'_{pk} - h^1_x(\mathfrak{a}_p^k),$$

with the equality holding by (I.5.1). By the Fujita approximation theorem for graded linear series ([LM09, Thm.3.5]), for any $\varepsilon > 0$, there exists p_0 such that if $p \geq p_0$, then

$$\lim_{k \to \infty} \frac{v_{k,p}}{p^n k^n / n!} \geq \operatorname{vol}(W_\bullet) - \varepsilon.$$

Together with (I.5.1), and with the remark that every lim sup in our case is lim, we obtain

$$\operatorname{vol}(W_\bullet) - \varepsilon \leq \operatorname{vol}(W'_\bullet) - \frac{\widehat{h}^1_x(\mathfrak{a}_p)}{p^n}$$

for all $p \geq p_0$. From (I.5.2) and (I.5.3), the conclusion follows by taking limits.

□

Corollary I.37. *Let $\mathcal{I}$ be a coherent fractional ideal sheaf on X, and consider the graded family $\mathfrak{a}_p = \mathcal{I}^p$. By the the proof of the theorem, the* lim sup *in the definition of $\widehat{h}^1_x(\mathcal{I})$ can be replaced by* lim, *and this limit is finite by Proposition I.16 (compare [CHST05, Thm.1.3] and [Cut10, Thm.1.3]).*

Remark I.38. Using Lemma I.13 and Lemma I.10, Theorem I.36 implies that $\operatorname{vol}_x(D)$ is the limit of local volumes of $\mathbb{Q}-$Cartier divisors that are nef over X on blow–ups of X', thus realizing the analogy with the global version of the Fujita approximation theorem.

Remark I.39. The highly restrictive condition on D in our Fujita approximation result is automatic when π is an isomorphism outside x, which is the case for good resolutions of normal isolated singularities $\pi : (\widetilde{X}, E) \to (X, x)$. Even when π is only a log–resolution of a normal isolated singularity, the divisor $K_{\widetilde{X}} + E$ satisfies the condition of Theorem I.36 since $X \setminus \{x\}$ is nonsingular. We do not know if Theorem I.36 holds for arbitrary D.

I.6 Vanishing and convexity

Our first objective in this section is to study the vanishing of local volumes. We begin by recalling a few general facts about exceptional Cartier divisors. If $\pi : X' \to X$ is a projective birational morphism of quasiprojective complex varieties with x a point on the normal variety X of dimension n at least two, the relative numerical space $N^1(X'/X)_{\mathbb{R}}$ contains two interesting subspaces. The first and largest of the two is the space of $\pi-$exceptional divisors that we denote $\mathrm{Exc}(\pi)$. Any exceptional divisor is uniquely determined by its relative numerical class (cf. [BdFF11, Lemma.1.9]):

Lemma I.40. *With notation as above, let $\alpha \in N^1(X'/X)_{\mathbb{R}}$. Then there exists at most one exceptional $\mathbb{R}-$Cartier divisor D on X' whose relative numerical class over X is α. In particular, when X' is normal and $\mathbb{Q}-$factorial, the numerical classes of the irreducible $\pi-$exceptional divisors form a basis of* $\mathrm{Exc}(\pi)$.

Proposition I.41. *Assume that X and X' are both normal and $\mathbb{Q}-$factorial. Then*

$$N^1(X'/X)_{\mathbb{R}} = \mathrm{Exc}(\pi).$$

Proof. We observe that any Cartier divisor D on X' is $\pi-$linearly equivalent, over $\mathbb{Q}$, to an exceptional divisor via

$$D = \pi^*(\pi_* D) + (D - \pi^*\pi_* D).$$

The pullback by π is well defined since the Weil divisor $\pi_* D$ is $\mathbb{Q}-$Cartier by assumption, and $D - \pi^*\pi_* D$ is clearly exceptional. □

A subspace of $\mathrm{Exc}(\pi)$ that we have seen is relevant to the study of local volumes is formed by the divisors lying over x. We denote it by $\mathrm{Exc}_x(\pi)$. Studying the behavior of the local volume function on this space will prove important in connecting our work to the study of volumes for some $b-$divisors as developed in [BdFF11]. A particularly useful result, drawing on [KMM87, Lemma.1-3-2], is [dFH09, Lemma.4.5]:

Lemma I.42. *Assume that X' is nonsingular. Let P and N be effective divisors on X' without common components and assume that P is $\pi-$exceptional. Then*

$$\pi_*\mathcal{O}_{X'}(P-N) = \pi_*\mathcal{O}_{X'}(-N).$$

It is natural to ask which divisors in $N^1(X'/X)_{\mathbb{R}}$ have zero local volume over x. The answer to this question is well understood for volumes of Cartier divisors on projective varieties; we know that $\mathrm{vol}(D) > 0$ is equivalent to D being in the interior of the cone of pseudoeffective divisors (see [Laz04, Ch.2.2.C]). In the local setting, we start by looking at the fiber over x.

Proposition I.43. *For $D \in \mathrm{Exc}_x(\pi)$, the vanishing $\mathrm{vol}_x(D) = 0$ is equivalent to D being effective.*

Proof. We can assume that X is projective, that π is a log–resolution, and that D is a divisor with integral coefficients. If D is effective, then $\pi_*\mathcal{O}_{X'}(mD) = \mathcal{O}_X$ for all $m \geq 0$ and so $\mathrm{vol}_x(D) = 0$. Using Lemma I.42, to complete the proof, it is enough to show that if $-D$ is effective, then $\mathrm{vol}_x(D) > 0$.

Let $\mathfrak{m}$ denote the maximal ideal sheaf on X corresponding to x, and let $e(\mathcal{I})$ denote the Hilbert–Samuel multiplicity at x of an $\mathfrak{m}-$primary ideal sheaf $\mathcal{I}$. The idea is to show that there exists $r > 0$ such that for all $m \geq 1$ we have inclusions

$$\pi_*\mathcal{O}_{X'}(mD) \subseteq \mathfrak{m}^{[m/r]},$$

because then $e(\pi_*\mathcal{O}_{X'}(mD)) \geq e(m^{[m/r]})$, leading to $\mathrm{vol}_x(D) \geq e(\mathfrak{m})/r^n > 0$. This is a consequence of a result of Izumi (see [Izu85, Cor.3.5], or the presentation of Rees in [Ree89]). $\square$

For arbitrary Cartier divisors on X' we can also give a precise answer to the question of the vanishing of vol_x, but one that does not provide satisfying geometric intuition.

Proposition I.44. *With the usual notation, if D is a Cartier divisor on X', then* $\mathrm{vol}_x(D) = 0$ *if, and only if,* $h^1_x(m\widetilde{D}) = 0$ *for all* $m \geq 0$*, where* $\widetilde{D}$ *is the pullback of* D *to the normalization of* X'.

Proof. Since $\mathrm{vol}_x(D) = \mathrm{vol}_x(\widetilde{D})$, and since $h^1_x(m\widetilde{D})$ is invariant under pullbacks from the normalization of X' to another birational model of X, we can assume that X' is nonsingular and $\widetilde{D} = D$. One implication is clear. Since vol_x is $n-$homogeneous, we can assume without loss of generality that $h^1_x(D) \neq 0$. This means that D is linearly equivalent to a divisor $F + G$, with F effective (at least in a neighborhood of E) without components over x, and with G a non–effective divisor lying over x. By Lemmas I.13, I.21, and by Proposition I.43, we then have

$$\mathrm{vol}_x(D) = \mathrm{vol}_x(F + G) \geq \mathrm{vol}_x(G) > 0.$$

□

Remark I.45. It is a consequence of Lemmas I.13, I.21, and I.42 that if D is an exceptional divisor (not necessarily effective) without components lying over x on the nonsingular X', then $\mathrm{vol}_x(D) = 0$.

The conclusion of Proposition I.44 is not sufficient for understanding the vanishing of the local volume function on $N^1(X'/X)_{\mathbb{R}}$. We can prove the following partial result:

Proposition I.46. *With the usual notation, let C_x denote the open cone in* $\mathrm{Exc}_x(\pi)$ *spanned by effective classes whose support is the entire divisorial component of the set theoretic fiber* $\pi^{-1}\{x\}$*. Then there exists an open convex cone $\mathcal{C}$ in* $N^1(X'/X)_{\mathbb{R}}$ *such that* $\mathcal{C} \cap \mathrm{Exc}_x(\pi) = C_x$*, and* $\mathrm{vol}_x(D) = 0$ *for any* $D \in \mathcal{C}$.

Proof. We can assume that X' is nonsingular. Fix $E \in C_x$. We first show that for any Cartier divisor D on X' it holds that $\mathrm{vol}_x(D + tE) = 0$ for $t \gg 0$. By the monotonicity properties in Lemmas I.20 and I.21, we can further assume D is effective without components over x. With the notation in Lemma I.21 and by the approximation result there,

$$\mathrm{vol}_x(D + t\Delta_1) = \mathrm{vol}_x(D + t\Delta_1) - \mathrm{vol}_x(t\Delta_1) \leq \mathrm{vol}((t(\Delta_1 + \Delta_2) + N)|_{\Delta_1}) = 0$$

for $t \gg 0$ since $(-\Delta_1 - \Delta_2)|_{\Delta_1}$ is ample and $\Delta_2|_{\Delta_1}$ is effective. There exists positive r such that $rE > \Delta_1$. Then $\mathrm{vol}_x(D + trE) \leq \mathrm{vol}_x(D + t\Delta_1)$ by Lemma I.20 and we conclude that $\mathrm{vol}_x(D + tE) = 0$ for $t \gg 0$.

Working as in the proof of Proposition I.24, the result follows. □

We have seen in Theorem I.25 that vol_x is a continuous and $n-$homogeneous function on $N^1(X'/X)_{\mathbb{R}}$. These properties are shared by volumes of Cartier divisors on projective varieties (see [Laz04, Ch.2.2.C] or [LM09]). In the projective setting, it is known that $\mathrm{vol}^{1/n}$ is concave on the big cone ([LM09, Cor.4.12]), meaning that

$$\mathrm{vol}(\xi + \xi')^{1/n} \geq \mathrm{vol}(\xi)^{1/n} + \mathrm{vol}(\xi')^{1/n}$$

for any classes ξ and ξ' with nonzero volume. In our local setting, it is easy to construct examples of divisors $E - E'$ lying over x such that $\mathrm{vol}_x(E - E')$ and $\mathrm{vol}_x(E' - E)$ are both nonzero. Therefore we cannot expect concavity. Generalizing [BdFF11, Rem.4.17] and [BdFF11, Thm.4.15], results developed in the setting of isolated singularities, we show that $\mathrm{vol}_x^{1/n}$ is convex when we restrict to divisors lying over x.

Proposition I.47. *With notation as above,* $\mathrm{vol}_x^{1/n} : \mathrm{Exc}_x(\pi) \to \mathbb{R}_{\geq 0}$ *is convex.*

Proof. The idea is that by the Fujita approximation result in [LM09, Thm.3.8], when D lies over x, we can understand $\mathrm{vol}_x(D)$ as an asymptotic Hilbert–Samuel multiplicity. Then we apply Teissier's inequality ([Laz04, Ex.1.6.9]). Let $\mathfrak{m}$ denote the maximal ideal corresponding to $x \in X$. For an $\mathfrak{m}-$primary ideal sheaf $\mathcal{I}$ on X, denote by $e(\mathcal{I})$ its Hilbert–Samuel multiplicity.

By the continuity and homogeneity of vol_x, we can reduce to working with Cartier $\mathbb{Z}-$divisors lying over x. Let D and D' be two such, and construct the graded families of $\mathfrak{m}-$primary ideals $\mathfrak{a}_m = \pi_*\mathcal{O}_{X'}(mD)$ and $\mathfrak{a}'_m = \pi_*\mathcal{O}_{X'}(mD')$. By [LM09, Thm.3.8],

$$\mathrm{vol}_x(D) = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m)}{m^n}$$

and a similar equality holds for $\mathrm{vol}_x(D')$. Denoting $\mathfrak{b}_m = \pi_*\mathcal{O}_{X'}(m(D+D'))$, one has

$$\mathfrak{a}_m \cdot \mathfrak{a}'_m \subseteq \mathfrak{b}_m,$$

therefore $e(\mathfrak{b}_m) \leq e(\mathfrak{a}_m \cdot \mathfrak{a}'_m)$. Teissier's inequality in [Laz04, Ex.1.6.9] then implies

$$e(\mathfrak{b}_m)^{1/n} \leq e(\mathfrak{a}_m \cdot \mathfrak{a}'_m)^{1/n} \leq e(\mathfrak{a}_m)^{1/n} + e(\mathfrak{a}'_m)^{1/n}.$$

The conclusion follows again by [LM09, Thm.3.8]. □

Remark I.48. Note that we did not restrict ourselves to working with classes having positive volume as was necessary in the projective setting.

When π is an isomorphism outside x and X is $\mathbb{Q}-$factorial, Propositions I.47 and I.41 show that $\mathrm{vol}_x^{1/n}$ is convex on $N^1(X'/X)_{\mathbb{R}}$. We construct a toric example showing that this does not hold for general π.

Example I.49. Let $\sigma \subset \mathbb{R}^3$ be the cone spanned by the vectors $(0,1,0)$, $(0,0,1)$ and $(1,0,-2)$. Let Σ be a refinement obtained by adding the rays spanned by $(1,1,1)$ and $(1,0,0)$, such that $X(\Sigma)$ is $\mathbb{Q}-$factorial. These determine a proper birational toric morphism $\pi : X(\Sigma) \to X(\sigma)$ that is not an isomorphism outside x_σ. Let $x = x_\sigma$ be the torus fixed point of $X(\sigma)$. On $X(\Sigma)$, let D and E be the torus invariant divisors associated to the rays $(1,0,-2)$ and $(1,1,1)$ respectively. We show that

$$\mathrm{vol}_x(2D - \frac{1}{2}E)^{1/3} + \mathrm{vol}_x(2D - \frac{3}{2}E)^{1/3} < \mathrm{vol}_x(4D-2E)^{1/3} = 2\cdot \mathrm{vol}_x(2D-E)^{1/3}.$$

The idea is to study the function $\mathrm{vol}_x(2D - tE)$. By Example I.8, the volume $\mathrm{vol}_x(2D - tE)$ is computed as the normalized volume of the body

$$B(t) = \{(x, y, z) \in \mathbb{R}^3 \ : \ x \geq 0, \ y \geq 0, \ z \geq 0, \ x - 2z \geq -2, \ x + y + z \leq t\}.$$

Let $S(t)$ be the simplex generated by $(0,0,0)$, $(t,0,0)$, $(0,t,0)$ and $(0,0,t)$. We have $B(t) = S(t)$ for $0 \leq t \leq 1$ and $B(t) \subsetneq S(t)$ for $t > 1$. Figure I.2 shows the polyhedron $B(3/2)$ corresponding to $2D - \frac{3}{2}E$. The desired inequality follows easily from the linearity of $\mathrm{vol}(S(t))^{1/3}$. $\square$

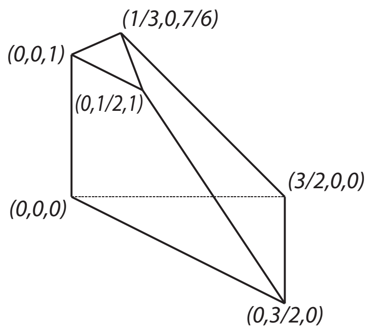

Figure I.2: $B(3/2)$

CHAPTER II

Plurigenera and volumes for normal isolated singularities

In this chapter we introduce a notion of volume for complex normal isolated singularities of dimension at least two. This volume, which we will denote $\mathrm{vol}(X, x)$, is obtained in the second section as an asymptotic invariant associated to the growth rate of the plurigenera in the sense of Morales or Watanabe. We generalize to higher dimension several results of Wahl ([Wah90]) who introduced this volume on surfaces, and translate to our setting several results of Ishii ([Ish90]). The first section is devoted to a brief review of Wahl's work. The third section studies the Knöller plurigenera and the associated volume $\mathrm{vol}_\gamma(X, x)$ that, using results or Ishii ([Ish90]), and of de Fernex and Hacon ([dFH09]), relates to the study of canonical singularities. We end with a series of examples. The results of this chapter are the motivation for our work.

II.1 Wahl's volume for normal surface singularities

In this expository section we review Wahl's work on normal surface singularities. Recall that normal surface singularities are automatically isolated, because normality implies smoothness in codimension one. In [Wah90], whose notation we use throughout this section, Wahl introduced a volume for normal isolated surface singularities as a characteristic number of the link of the singularity. What this means, in our setting, is that this invariant of the singularity is a topological invariant of the link, and it satisfies a certain monotonicity property for finite maps. Before defining this volume, we recall the basics of relative Zariski decompositions for divisors on good resolutions of normal surface singularities.

II.1.1 Relative Zariski decompositions

We review Sakai's work on relative Zariski decompositions. The reference is [Wah90]. Given a good resolution $\pi : (\widetilde{X}, E) \to (X, x)$ of a normal surface singularity[1], denote by $E_1, \ldots, E_s$ the components of E. These meet transversally, no more than two at a point. The intersection form $(E_i \cdot E_j)$ is known to be negative definite. Given a line bundle $\mathcal{L}$ on $\widetilde{X}$, by the nondegeneracy of the intersection form, there exists a unique

$$(a_1, \ldots, a_s) \in \mathbb{Q}^s$$

such that

$$\mathcal{L} \cdot E_i = (\sum_i a_i E_i) \cdot E_j.$$

The intersection on the left makes sense as $\deg(\mathcal{L}|_{E_i})$, because the E_i are all complete smooth curves. This defines an adjoint homomorphism

$$\mathrm{Pic}\widetilde{X} \to \bigoplus_i \mathbb{Q} \cdot E_i =_{\mathrm{def}} \mathbb{E}_{\mathbb{Q}}.$$

We denote the image of $\mathcal{L}$ by L.

Proposition II.1. *[Sakai] Let $L \in \mathbb{E}_{\mathbb{Q}}$ be the image of a $\mathbb{Q}-$Cartier divisor. There exists a unique relative Zariski decomposition $L = P + N$ in $\mathbb{E}_{\mathbb{Q}}$ such that:*

(i) P is $\pi-$nef, i.e., $P \cdot E_i \geq 0$ for all i.

(ii) N is effective, i.e., a nonnegative combination of the E_i.

[1] This means that π is a resolution of singularities restricting to an isomorphism away from x, the scheme theoretic image of x is a Cartier divisor whose reduced support, E, has simple normal crossings

(iii) $P \cdot N = 0$. *This is equivalent to* $P \cdot E_i = 0$ *whenever* E_i *is in the support of* N.

Proof. We only prove the existence of such a decomposition. If L is $\pi-$nef, we set $P = L$ and $N = 0$. Otherwise, let $P_0 = L$, and denote $\mathcal{A}_1 =_{\text{def}} \{j \ : \ P_0 \cdot E_j < 0\}$. Using the negativity of the intersection form, define $N_1 = \sum_{j \in \mathcal{A}_1} b_j E_j$ by $N_1 \cdot E_j = P_0 \cdot E_j < 0$ for all $j \in \mathcal{A}_1$. Let $P_1 = P_0 - N_1$. Note that $P_1 \cdot N_1 = 0$ by construction. Also, N_1 is effective: If $N_1 = A - B$, with A and B nonnegative combinations of the E_j with $j \in \mathcal{A}_1$, sharing no components, then $0 < A \cdot B - B \cdot B = N_1 \cdot B < 0$. If P_1 is $\pi-$nef, we are done. Otherwise, let $\mathcal{A}_2 = \{j \ : \ P_1 \cdot E_j < 0\} \cup \mathcal{A}_1$, and define $N_2 = \sum_{j \in \mathcal{A}_2} b_j E_j$ by $N_2 \cdot E_j = P_1 \cdot E_j < 0$ for $j \in \mathcal{A}_2$. As before, N_2 is effective, and $P_2 =_{\text{def}} P_1 - N_2$ is orthogonal to all curves corresponding to the index set $\mathcal{A}_2$. If it is not $\pi-$nef, then continue this procedure. □

We refer to P as the $\pi-$nef part of L.

Remark II.2. Relative Zariski decompositions are functorial under pullbacks in the following sense: Given $f : (Y, y) \to (X, x)$ a finite map of normal complex isolated surface singularities, i.e., $f^{-1}\{x\} = \{y\}$ as sets, and $\widetilde{f} : (\widetilde{Y}, F) \to (\widetilde{X}, E)$ a generically finite lifting of f between good resolutions, $\widetilde{f}^*$ induces a morphism $\mathbb{E}_{\mathbb{Q}} \to \mathbb{F}_{\mathbb{Q}}$ that preserves the effective and the nef properties for divisors. If $L = P + N$ is the relative Zariski decomposition of L on $\widetilde{X}$, then $\widetilde{f}^*L = \widetilde{f}^*P + \widetilde{f}^*N$ is the relative Zariski decomposition of $\widetilde{f}^*L$ on $\widetilde{Y}$.

II.1.2 The definition and properties of the volume of an isolated normal surface singularity

Definition II.3. Given $\pi : (\widetilde{X}, E) \to (X, x)$ a good resolution of a normal complex isolated surface singularity, define

$$\mathrm{vol}(X, x) =_{\text{def}} -P \cdot P,$$

where $P = P_{\widetilde{X}}$ is the $\pi-$nef part of $K_{\widetilde{X}} + E$.

We collect Wahl's main results (whose generalizations we investigate in the next section) from [Wah90] in the following:

Theorem II.4 (Wahl). *Let (X, x) be a a normal complex isolated surface singularity. Then*

(i) $-P \cdot P$ is independent of the chosen good resolution, i.e., the volume of (X, x) is well defined.

(ii) $-P \cdot P = 0$ if, and only if, (X, x) is log–canonical, which by definition means that $P = 0$.

The next two properties describe what it means for $-P \cdot P$ to be a characteristic number of the link of the singularity.

(iii) $-P \cdot P$ is a topological invariant of the link of (X, x).

(iv) Given $(Y, y) \to (X, x)$ a degree d, finite surjective map of isolated surface singularities,

$$\mathrm{vol}(Y, y) \geq d \cdot \mathrm{vol}(X, x).$$

We have equality when the map is unramified off y.

The next property expresses $\mathrm{vol}(X, x)$ *as a local volume.*

(v) Given $\pi : (\widetilde{X}, E) \to (X, x)$ a good resolution,

$$\dim \frac{H^0(\widetilde{X} \setminus E, \mathcal{O}(n(K_{\widetilde{X}} + E)))}{H^0(\widetilde{X}, \mathcal{O}(n(K_{\widetilde{X}} + E)))} = \frac{n^2}{2}(-P \cdot P) + O(n).$$

Proof. Given $\pi : (\widetilde{X}, E) \to (X, x)$ and $\pi' : (X', F) \to (X, x)$ two good resolutions, we can dominate both by a third, and because rational maps between smooth surfaces can be resolved by a sequence of blow–ups and blow–downs of points, we can assume that $\rho : (X', F) \to (\widetilde{X}, E)$ is the blow–up of a point p that we can choose on E, because good resolutions are isomorphic away from x. The point p is either a smooth point of E, or an intersection of exactly two of its components. Let F_1 be the $\rho-$exceptional component of F over p. Then

$$K_{X'} + F = \rho^*(K_{\widetilde{X}} + E) + \delta \cdot F_1,$$

with $\delta \in \{0, 1\}$ according to the case above that p falls into. By the projection formula,

$$\rho^* P_{\widetilde{X}} \cdot F_1 = 0,$$

and it follows that $P_{X'} = \rho^* P_{\widetilde{X}}$. Independence of the resolution is then a consequence of the projection formula. Part (ii) is clear from the negativity of the intersection form on $\mathbb{E}_{\mathbb{Q}}$. This result does not fully generalize to arbitrary dimension. We do not reproduce the proof of part (iii) here, because it is not a result that generalizes to higher dimension. We prove (iv) in the next section in arbitrary dimension as Theorem II.10. Part (v) says that $\mathrm{vol}(X, x) = \mathrm{vol}_x(K_{\widetilde{X}} + E)$. This is actually the way that we define $\mathrm{vol}(X, x)$ in the next section. The original proof for (v) in the surface case uses a Riemann–Roch–type argument that is very specific to dimension two. A result from [BdFF11], together with work we do in the last chapter, will give a more general proof. □

Corollary II.5.

(i) If $f : (Y, y) \to (X, x)$ is a finite morphism of normal complex isolated surface singularities, such that (Y, y) is log–canonical, then so is (X, x).

(ii) If (X, x) admits an endomorphism of degree at least two, then it is log–canonical.

Remark II.6. The surface case has its computational advantages. [Wah90, Prop. 2.3] gives an algorithm for computing $-P \cdot P$ from the resolution dual graph, which by the conventions of [Wah90] contains the information of the intersection form on $\mathbb{E}_{\mathbb{Q}}$.

II.2 The Morales and the Watanabe plurigenera and $\mathrm{vol}(X, x)$

The *geometric genus* of a normal complex quasiprojective isolated singularity (X, x) of dimension n at least two, is defined as

$$p_g(X, x) =_{\mathrm{def}} \dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_{\widetilde{X}})_x,$$

for $\pi : \widetilde{X} \to X$ an arbitrary resolution of singularities. Work of S.S.T. Yau in [Yau77] shows that this invariant of the singularity can be computed analytically on X as

$$p_g(X, x) = \dim \frac{H^0(U \setminus \{x\}, \mathcal{O}_X^{an}(K_X))}{L^2(U \setminus \{x\})},$$

where U is a sufficiently small Stein neighborhood of x in X, and $L^2(U \setminus \{x\})$ is the set of all square integrable canonical forms on $U \setminus \{x\}$. Motivated by this alternate description, in [Wat80] the *plurigenera* of (X, x) were introduced as

$$\delta_m(X, x) =_{\mathrm{def}} \dim \frac{H^0(U \setminus \{x\}, \mathcal{O}_X^{an}(mK_X))}{L^{2/m}(U \setminus \{x\})},$$

with $L^{2/m}(U \setminus \{x\})$ now denoting the set of holomorphic $m-$canonical forms ω on the sufficiently small $U \setminus \{x\}$ that satisfy $\int_{U \setminus \{x\}}(\omega \wedge \bar{\omega})^{1/m} < \infty$.

The proofs of [Sak77, Thm.2.1], [Sak77, Thm.1.1], and remarks in [Ish90] provide an algebro–geometric approach to plurigenera at the expense of working again on resolutions. Let $\pi : \widetilde{X} \to X$ be a log–resolution of (X, x) with E the reduced fiber over x, let U be an arbitrary affine neighborhood of x and let $\widetilde{U}$ be the preimage of U in $\widetilde{X}$ via π. Then working in the algebraic category,

$$\delta_m(X, x) = \dim \frac{H^0(\widetilde{U} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{U}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E))} = \dim \frac{\mathcal{O}_X(mK_X)}{\pi_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E)},$$

with the last equality holding since U is affine, for choices of Weil canonical divisors on X and $\widetilde{X}$ such that $\pi_*K_{\widetilde{X}} = K_{\widetilde{X}}$ as Weil divisors.

Definition II.7. Generalizing work in [Wah90] for the case of surfaces, the volume of the normal isolated singularity (X, x) of dimension n is defined as

$$\mathrm{vol}(X, x) =_{\mathrm{def}} \limsup_{m\to\infty} \frac{\delta_m(X, x)}{m^n/n!}.$$

We would like to understand this volume as a local volume of some Cartier divisor on a log–resolution of (X, x). For this, it turns out that a more convenient plurigenus is the one introduced by Morales in [Mor87]:

$$\lambda_m(X, x) =_{\mathrm{def}} \dim \frac{H^0(\widetilde{U} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{U}, \mathcal{O}_{\widetilde{X}}(m(K_{\widetilde{X}} + E)))},$$

for $\pi : \widetilde{X} \to X$ a log–resolution with E the reduced fiber over x and $\widetilde{U}$ the inverse image in $\widetilde{X}$ via π of an affine neighborhood of x. By Remark I.6,

$$\lambda_m(X, x) = h^1_x(m(K_{\widetilde{X}} + E)).$$

By [Ish90, Thm.5.2],

$$\mathrm{vol}(X, x) = \limsup_{m\to\infty} \frac{\lambda_m(X, x)}{m^n/n!}$$

and we see that

$$\mathrm{vol}(X, x) = \mathrm{vol}_x(K_{\widetilde{X}} + E)$$

on any log–resolution.

Remark II.8. The classical literature usually requires that we work with good resolutions, i.e., that $\pi : \widetilde{X} \to X$ is a log–resolution that is an isomorphism outside x. To prove that the plurigenera are independent of the log–resolution, one applies the logarithmic ramification formula in [Iit77, Thm.11.5], using that any two log–resolutions can be dominated by a third, and that $X \setminus \{x\}$ is nonsingular.

Remark II.9. If follows from Corollary I.34 that the lim sup in the definition of $\mathrm{vol}(X, x)$ is an actual limit.

Generalizing a result for the volume of surface singularities (see [Wah90, Thm.2.8]), we show that volumes of normal isolated singularities satisfy the following monotonicity property:

Theorem II.10. *Let $f : (X, x) \to (Y, y)$ be a finite morphism of normal isolated singularities, i.e., f is finite and set theoretically $f^{-1}\{y\} = \{x\}$. Then*

$$\operatorname{vol}(X, x) \geq (\deg f) \cdot \operatorname{vol}(Y, y).$$

If f is unramified away from x, then the previous inequality is an equality.

Proof. Let $\rho : (\widetilde{Y}, F) \to (Y, y)$ be a log–resolution of (Y, y). Let Z be the normalization of $\widetilde{Y}$ in the fraction field of X and let $u : (\widetilde{X}, E) \to (X, x)$ be a log–resolution factoring through a log–resolution of Z. We have a diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[dr, "u"] \arrow[drr, "\widetilde{f}"] \arrow[ddr, "\pi"'] & & \\
& Z \arrow[r, "v"] \arrow[d, "\tau"] & \widetilde{Y} \arrow[d, "\rho"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

We can assume that $\widetilde{X}$ has simple normal crossings for both the branching and for the ramification locus of $\widetilde{f}$. We write the reduced branching locus as $F + R$, where R has no components lying over y. Similarly, write the reduced ramification locus as $E + S$ with S having no components lying over x.

A local study of forms with log–poles at the generic points of each component of $E + S$ shows that

$$K_{\widetilde{X}} + E + S = \widetilde{f}^*(K_{\widetilde{Y}} + F + R) + T,$$

where T is an effective divisor that is exceptional for $\widetilde{f}$, hence also exceptional for u. Note that $\widetilde{f}^*R - S$ is effective, and write it as $P + Q$, with P being supported on S

and with Q being $u-$exceptional. Then

$$K_{\widetilde{X}} + E = \widetilde{f}^*(K_{\widetilde{Y}} + F) + P + (Q + T).$$

Since P is supported on S, it has no components over x, so

$$\mathrm{vol}(X, x) = \mathrm{vol}_x(K_{\widetilde{X}} + E) \geq \mathrm{vol}_x(\widetilde{f}^*(K_{\widetilde{Y}} + F) + (Q + T)),$$

by Lemma I.21. Since $Q+T$ is effective and $u-$exceptional and since vol_x is computed by pushing forward to X,

$$\pi_* \mathcal{O}_{\widetilde{X}}(\widetilde{f}^*(K_{\widetilde{Y}} + F) + (Q + T)) = \tau_* v^* \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}} + F),$$

and hence

$$\mathrm{vol}_x(\widetilde{f}^*(K_{\widetilde{Y}} + F) + (Q + T)) = \mathrm{vol}_x(v^*(K_{\widetilde{Y}} + F)).$$

By Proposition I.15,

$$\mathrm{vol}_x(v^*(K_{\widetilde{Y}} + F)) = \deg(f) \cdot \mathrm{vol}_y(K_{\widetilde{Y}} + F) = \deg(f) \cdot \mathrm{vol}(Y, y).$$

When f is unramified outside x, the divisors R, S are zero. Since T is $u-$exceptional, we obtain the required equality. □

Corollary II.11.

(i) If $f : (X, x) \to (Y, y)$ is a finite map of normal isolated singularities and $\mathrm{vol}(X, x)$ vanishes, then $\mathrm{vol}(Y, y) = 0$.

(ii) If (X, x) admits an endomorphism of degree at least two, then $\mathrm{vol}(X, x) = 0$.

In the surface case, [Wah90, Thm.2.8] shows that $\mathrm{vol}(X, x) = 0$ is equivalent to saying that X has log–canonical singularities in the sense of [Wah90, Rem.2.4]. In the $\mathbb{Q}-$Gorenstein case, this coincides with the usual definition of log–canonical. In higher dimension, as an immediate consequence of Proposition I.44, or by [Ish90, Thm.4.2] it follows:

Proposition II.12. *Let (X, x) be a normal complex quasiprojective normal isolated singularity of dimension n at least two. Then* $\mathrm{vol}(X, x) = 0$ *if, and only if, for all (any) log–resolutions $\pi : \widetilde{X} \to X$ with E the reduced fiber over x, one has that*

$$\pi_* \mathcal{O}_{\widetilde{X}}(m(K_{\widetilde{X}} + E)) = \mathcal{O}_X(mK_X),$$

for all non-negative m, i.e., $\lambda_m(X, x) = 0$ for all non–negative m.

In the previous result, we understand $\mathcal{O}_X(mK_X)$ as the sheaf of sections associated to a Weil canonical divisor K_X chosen together with a canonical divisor on $\widetilde{X}$ such that $\pi_* K_{\widetilde{X}} = K_X$ as Weil divisors.

Remark II.13. In the $\mathbb{Q}$-Gorenstein case, the conclusion of Proposition II.12, as in the case of surfaces, is the same as saying that X is log–canonical. This result also appears in [TW90]. In general, following [dFH09], we say X is log–canonical if there exists an effective $\mathbb{Q}-$boundary Δ such that the pair (X, Δ) is log–canonical. With this definition, an inspection of [BdFF11, Ex.4.20] and [BdFF11, Ex.5.4] shows that there exist non $\mathbb{Q}-$Gorenstein isolated singularities (X, x) that are not log–canonical, but $\mathrm{vol}(X, x) = 0$.

Another result of Ishii ([Ish90, Thm.5.6]) that we translate to volumes studies hyperplane sections of normal isolated singularities.

Proposition II.14. *Let (X, x) be an complex normal quasiprojective isolated singularity of dimension n at least three. Let (H, x) be a hyperplane section of (X, x) that is again a normal isolated singularity. If* $\mathrm{vol}(X, x) > 0$*, then* $\mathrm{vol}(H, x) > 0$.

II.3 The Knöller plurigenera

Another notion of plurigenera for a normal isolated singularity (X, x), different from $\delta_m(X, x)$ and $\lambda_m(X, x)$, was introduced by Knöller in [Kno73] and can be defined as

$$\gamma_m(X,x) = \dim \frac{\mathcal{O}_X(mK_X)}{\pi_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})}$$

for $\pi : \widetilde{X} \to X$ an arbitrary resolution of singularities. This is again an invariant of the singularity (X,x), independent of the chosen resolution. The asymptotic behavior of $\gamma_m(X,x) = h^1_x(mK_{\widetilde{X}})$ is studied in [Ish90]. Denoting

$$\mathrm{vol}_\gamma(X,x) =_{\mathrm{def}} \mathrm{vol}_x(K_{\widetilde{X}}),$$

the result in [Ish90, Thm.2.1], or Proposition I.44 can be rephrased as:

Proposition II.15. *For a normal algebraic complex isolated singularity (X,x) of dimension at least two, the following are equivalent:*

(i) $\mathrm{vol}_\gamma(X,x) = 0$

(ii) $\gamma_m(X,x) = 0$ *for all non-negative m.*

The following remark was kindly suggested by T. de Fernex.

Remark II.16. In [dFH09], the authors generalize the notion of canonical singularities to normal varieties that are not necessarily $\mathbb{Q}-$Gorenstein and it is a consequence of [dFH09, Prop.8.2] that a normal variety X has canonical singularities if, and only if, for all sufficiently divisible $m \geq 1$ and all (any) resolution $\pi : \widetilde{X} \to X$, it holds that

$$\pi_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}) = \mathcal{O}_X(mK_X),$$

with K_X and $K_{\widetilde{X}}$ chosen such that $\pi_*K_{\widetilde{X}} = K_X$ as Weil divisors.

When (X,x) is an isolated singularity, since the lim sup in the definition of $\mathrm{vol}_\gamma(X,x)$ is replaceable by lim, by similar arguments as in the case of $\mathrm{vol}(X,x)$, the vanishing $\mathrm{vol}_\gamma(X,x) = 0$ is equivalent to (X,x) being canonical in the sense of [dFH09].

Since in any case $\mathrm{vol}_\gamma(X,x) \geq \mathrm{vol}(X,x)$, we see that $\mathrm{vol}(X,x) = 0$ for canonical singularities.

We show that vol_γ does not exhibit the same monotonicity properties as $\mathrm{vol}(X,x)$ with respect to finite maps of normal isolated singularities by constructing a $\mathbb{Q}$-Gorenstein non–canonical isolated singularity carrying endomorphisms of arbitrarily high degree.

Example II.17. Let (X,x) be the cone over $V=\mathbb{P}^{n-1}$ corresponding to the polarization $H=\mathcal{O}_{\mathbb{P}^{n-1}}(n+1)$. By Examples I.19 and II.27,

$$\mathrm{vol}_\gamma(X,x)=n\cdot\int_0^\infty \mathrm{vol}(K_V+H-tH)dt=n\cdot\int_0^\infty(1-t(n+1))^{n-1}dt=\frac{1}{n+1}>0,$$

therefore (X,x) is not canonical, and the other requirements are met. □

However, we can prove the opposite to the inequality of Theorem II.10 in the unramified case.

Proposition II.18. *Let $f:(X,x)\to(Y,y)$ be a finite morphism of complex normal isolated singularities of dimension n at least two. Assume that f is unramified away from x. Then*

$$\mathrm{vol}_\gamma(X,x)\le(\deg f)\cdot\mathrm{vol}_\gamma(Y,y).$$

Proof. Construct good resolutions $\pi:(\widetilde{X},E)\to(X,x)$ and $\rho:(\widetilde{Y},F)\to(Y,y)$ and a lift $\widetilde{f}:\widetilde{X}\to\widetilde{Y}$ for f. Then the ramification divisor $K_{\widetilde{X}}-\widetilde{f}^*K_{\widetilde{Y}}$ is effective. It is also exceptional for π by assumption. We conclude by Proposition I.15 and Lemma I.20. □

Corollary II.19. *Under the assumptions of the previous proposition, if (Y,y) has canonical singularities, then (X,x) also has canonical singularities.*

Proof. The result is an immediate consequence of the proposition and Remark II.16. □

Remark II.20. In this paper we refer to $\mathrm{vol}(X,x)$ and not to $\mathrm{vol}_\gamma(X,x)$ as the volume of the isolated singularity (X,x). It would be interesting to study all volumes of the form $\mathrm{vol}_x(K_{\widetilde{X}}+aE)$.

II.4 Examples

We begin with a series of examples of normal isolated singularities (X,x) where the volume is zero. We can usually show this by explicit computation of plurigenera, or by exhibiting endomorphisms of degree bigger than one.

Example II.21 ($\mathbb{Q}-$Gorenstein log–canonical case)**.** Let (X,x) be a $\mathbb{Q}-$Gorenstein log–canonical normal isolated singularity of dimension n. It is a consequence of Proposition II.12 that $\mathrm{vol}(X,x)=0$, but we can also compute explicitly that

$$\lambda_m(X,x)=0$$

for all nonnegative, sufficiently divisible m. Pick $\pi:\widetilde{X}\to X$ a log–resolution with E the reduced fiber over x. Since π^*K_X is defined as a $\mathbb{Q}-$divisor, by Lemma I.13,

$$\lambda_m(X,x)=h^1_x(m(K_{\widetilde{X}}+E))=h^1_x(m(K_{\widetilde{X}}+E-\pi^*K_X))$$

for m divisible enough so that mK_X is Cartier. But $K_{\widetilde{X}}+E-\pi^*K_X$ is $\pi-$exceptional and effective by the log-canonical condition, so $h^1_x(m(K_{\widetilde{X}}+E-\pi^*K_X))=0$ for all sufficiently divisible m. By homogeneity, it follows that $\mathrm{vol}(X,x)=0$. □

Example II.22 (Finite quotient isolated singularities)**.** Let G be a finite group acting algebraically on a complex algebraic affine manifold M. Let $X=\mathrm{Spec}(\mathbb{C}[M]^G)$ be the quotient and assume it has a normal isolated singularity x. Then by Proposition I.15 and by the previous example, following ideas in Theorem II.10, we obtain

$$\mathrm{vol}(X,x)=0.$$

□

Example II.23 (Toric isolated singularities)**.** We use the notation in Example I.8. Let σ be an $n-$dimensional pointed rational cone. The condition that $(X(\sigma), x_\sigma)$ be an isolated singularity is the same as saying that all the faces of non-maximal dimension of σ are spanned as cones by a set of elements of N that can be extended to a basis. Affine toric varieties carry Frobenius non–invertible endomorphisms and one checks that they are actually endomorphisms of the singularity $(X(\sigma), x_\sigma)$ i.e. totally ramified at the isolated singularity, so $\mathrm{vol}(X(\sigma), x_\sigma) = 0$ by Corollary II.11.

It can be checked that, for a toric resolution $\pi : (X(\Sigma), E) \to (X(\sigma), x_\sigma)$, the divisor $K_{X(\Sigma)} + E$ is antieffective, without components lying over x_σ. Then

$$\mathrm{vol}(X, x) = 0$$

by Lemma I.21. □

Example II.24 (Cusp singularities)**.** Tsuchihashi's cusp singularities provide yet another example of isolated singularities (X, x) with $\mathrm{vol}(X, x) = 0$. See [BdFF11, 6.3] or [Wat80, Thm.1.16] for explanations and [Tsu83] for more on cusp singularities. □

One of the simplest classes of isolated singularities that may have nonzero volume are quasi–homogeneous singularities.

Example II.25 (Quasi–homogeneous singularities)**.** We follow [Wat80, Def.1.10]. Let $r_0, \dots, r_n$ be positive rational numbers. Call a polynomial $f(x_0, \dots, x_n)$ quasi-homogeneous of type $(r_0, \dots, r_n)$, if it is a linear combination of monomials $x_0^{a_0} \cdot \ldots \cdot x_n^{a_n}$ with $\sum_{i=0}^n a_i r_i = 1$. When such a polynomial is sufficiently general, its vanishing locus in $\mathbb{C}^{n+1}$ has an isolated singularity at the origin. We denote this singularity by $(X(f), 0)$. Let $r(f) = r_0 + \ldots + r_n$. By [Wat80, Exap.1.15],

$$\mathrm{vol}(X(f), 0) = \begin{cases} 0, & \text{if } r(f) \geq 1 \\ \frac{(1-r(f))^n}{r_0 \cdot \ldots \cdot r_n}, & \text{if } r(f) \leq 1 \end{cases}.$$

□

Example II.26 (Surface case)**.** By Example I.9, the volume of a normal isolated surface singularity (X,x) can be computed as

$$\mathrm{vol}(X,x) = -P\cdot P,$$

where $K_{\widetilde{X}} + E = P + N$ is the relative Zariski decomposition on a good resolution $\pi : \widetilde{X} \to X$. In [Wah90, Prop.2.3], an algorithm for computing P is described in terms of the combinatorial data of the dual graph of a good resolution. □

Although the quasi-homogeneous and surface cases provide nonzero examples, they always provide rational values for the volume of the singularity. We will see that cone singularities provide irrational volumes already in dimension three.

Example II.27 (Cone singularities)**.** If $(X,0)$ is a cone singularity constructed as

$$\mathrm{Spec}\bigoplus_{m\geq 0} H^0(V,\mathcal{O}_V(mH))$$

for (V,H) a polarized nonsingular projective variety of dimension $n-1$, then by Example I.19, using that K_Y+E restricts to K_V on E by adjunction,

$$\mathrm{vol}(X,0) = n\cdot\int_0^\infty \mathrm{vol}(K_V - tH)dt.$$

We see right away that $\mathrm{vol}(X,0)>0$ if, and only if, V is of general type. □

In similar flavor to an example of Urbinati in [Urb10], following a suggestion of Lazarsfeld, we show that there exist cone singularities of irrational volume.

Example II.28 (Irrational volume)**.** Choose two general integral classes D and L in the ample cone of $E\times E$, where E is a general elliptic curve. Then, by the Lefschetz Theorem ([Deb05, Thm.6.8]), $2D$ is globally generated and we can construct V, the cyclic double cover (see [Laz04, Prop.4.1.6]) of $E\times E$ over a general section of $2D$.

Let $g : V \to E \times E$ be the cover map. Note that $K_V = g^*D$. The volume of the cone singularity $(X, 0)$ associated to (V, g^*L) is then

$$3 \cdot \int_0^\infty \mathrm{vol}(g^*(D - tL))dt.$$

On abelian varieties, pseudoeffective and nef are equivalent notions for divisors and the volumes of such are computed as self–intersections. Let

$$m =_{\mathrm{def}} \max\{t : \ D - tL \text{ is nef}\}.$$

It is also characterized as the smallest solution to the equation

$$(D - tL)^2 = 0.$$

One can compute,

$$\mathrm{vol}(X, 0) = \frac{4D^2L^2 - 4(DL)^2}{L^2} \cdot m + \frac{2(DL)D^2}{L^2}.$$

The study in [Laz04, Sec.1.5.B] shows that the nef cone of $E \times E$ is a round quadratic cone for general E. Hence general choices for D and L produce a quadratic irrational m. Upon replacing L by a large multiple, we can insure that $(X, 0)$ is normal. $\square$

In [Wah90] it is proved that $\mathrm{vol}(X, x)$ is a topological invariant of the link of the surface singularity (X, x). We give an example showing that this may fail already in dimension three. The idea for the construction comes from [BdFF11, p.36] and [BdFF11, Ex.4.23] where, using the Ehresmann–Feldbau theorem, it is shown that if $f : (V, A) \to T$ is a smooth polarized family of nonsingular projective varieties, then the links of the cone singularities associated to (V_t, A_t) have the same diffeomorphism type as t varies in T. This is used to show that if V is the family of blow–ups of $\mathbb{P}^2$ at ten or more points, and if $(C_t, 0_t)$ denotes the three dimensional cone singularity over (V_t, A_t), for some appropriate polarization A, then the volume $\mathrm{vol}_{\mathrm{BdFF}}(C_t, 0_t)$

(that we discuss in the next section) is positive for very general t, but it does vanish for special values of t. Since the V_t's are all rational surfaces, $\mathrm{vol}(C_t, 0_t) = 0$ for any t, but we can construct an example where $\mathrm{vol}(C_t, 0_t)$ is nonconstant by passing to double covers of the family of blow–ups of $\mathbb{P}^2$ at three distinct points.

Example II.29. Let $g : S \to T$ be the smooth family of blow–ups of $\mathbb{P}^2$ at three distinct points. There are line bundles H and E on S such that for each $t \in T$, the divisor H_t is the pullback of the hyperplane bundle via the blow–down to $\mathbb{P}^2$ and $E_t = E_{t,1} + E_{t,2} + E_{t,3}$ is the exceptional divisor of the blow–up. The geometry of S_t differs according to whether t consists of three collinear or non–collinear points, with the latter being the generic case. In both cases, $3H_t - E_t = -K_{S_t}$ is big and globally generated and $4H_t - E_t$ is ample and globally generated. It follows by [Laz04, Ex.1.8.23] that $4(4H_t - E_t)$ is very ample. By Kodaira vanishing,

$$H^1(S_t, \mathcal{O}(4(4H_t - E_t))) = H^1(S_t, \mathcal{O}(K_{S_t} + (4(4H_t - E_t) - K_{S_t}))) = 0.$$

By Grauert ([Har77, Cor.III.12.9]), $R^1 g_* \mathcal{O}_S(4(4H - E)) = 0$.

Let t_0 be a set of collinear points, and choose a smooth divisor in the linear system $|4(4H_{t_0} - E_{t_0})|$ corresponding to a section s_{t_0}. Because $R^1 g_* \mathcal{O}_S(4(4H - E))$ and $H^1(S_t, \mathcal{O}(4(4H_t - E_t)))$ both vanish, cohomology and base change ([Har77, Thm. III.12.11.(b)]) show that the section s_{t_0} extends in a neighborhood of t_0 to a section s of $4(4H - E)$. By further restricting T, we can assume that s_t vanishes along a smooth divisor for all t (see [Har77, Ex.III.10.2]). Let $h : V \to S$ be the double cover corresponding to s. By [Laz04, Prop.4.1.6], the composition $f : V \to T$ is again a smooth family. We endow it with the fiberwise polarization given by $A = h^*(40H - 3E)$. By above mentioned results on [BdFF11, p.36], the links of the cone singularities $(C_t, 0_t)$ associated to (V_t, A_t) are all diffeomorphic. We compute $\mathrm{vol}(C_t, 0_t)$ and show that we get different answers when the tree points to be blown–

up are collinear than when they are non–collinear. Note that

$$K_t =_{\text{def}} K_{V_t} = h_t^*(K_{S_t} + 2(4H_t - E_t)) = h_t^*(5H_t - E_t).$$

By Example II.27,

$$\operatorname{vol}(C_t, 0_t) = 3 \cdot \int_0^\infty \operatorname{vol}(h^*(5H_t - E_t - s(40H_t - 3E_t)))ds =$$

$$= 6 \cdot \int_0^\infty \operatorname{vol}((5 - 40s)H_t - (1 - 3s)E_t)ds.$$

We are reduced to working with volumes on $\mathbb{P}^2$ blown–up at three distinct points. For this we can use Zariski decompositions (see [Laz04, Thm.2.3.19, Cor.2.3.22]) that can be explicitly computed for $aH_t + bE_t$ with $a, b \in \mathbb{Z}$ to show that $\operatorname{vol}(C_t, 0_t)$ yields different values when t corresponds to collinear points than when it corresponds to non–collinear points.

Assume first that t consists of three collinear points, e.g., $t = t_0$. Using that $|H_t - E_t|$ contains only the -2 curve obtained as the strict transform of the line containing the points on t, one can show that if P_m denotes the nef part of the Zariski decomposition of $mH_t - E_t$, then

$$P_m = \begin{cases} mH_t - E_t & \text{, if } m \geq 3 \\ \frac{m-1}{2}(3H_t - E_t) & \text{, if } 3 \geq m \geq 1 \\ 0 & \text{, if } 1 \geq m \end{cases}$$

The volume of $mH_t - E_t$ is computed as P_m^2 by [Laz04, Cor.2.3.22], therefore

$$\operatorname{vol}(C_t, 0_t) = 6 \cdot \left(\int_0^{2/31} ((5 - 40s)^2 - 3(1 - 3s)^2)ds + \int_{2/31}^{4/37} 6 \cdot \left(\frac{4 - 37s}{2} \right)^2 ds \right).$$

Assume now that t is generic, i.e., it corresponds to non–collinear points. Then $|3H_t - 2E_t|$ contains only a sum of three disjoint -1 curves obtained as the strict transforms of the lines joining any two of the three points in t. In this case,

$$P_m = \begin{cases} mH_t - E_t & \text{, if } m \geq 2 \\ (2m-3)(2H_t - E_t) & \text{, if } 2 \geq m \geq 3/2 \\ 0 & \text{, if } 3/2 \geq m \end{cases}$$

Using these, we compute as before

$$\mathrm{vol}(C_t, 0_t) = 6 \cdot \left(\int_0^{3/34} ((5-40s)^2 - 3(1-3s)^2) ds + \int_{3/34}^{7/71} (7-71s)^2 ds \right).$$

The two integrals produce rational answers that have different classes modulo 11.

To make sure that the isolated cone singularities $(C_t, 0_t)$ are normal, one may replace the polarization A by uA for u sufficiently large. This rescales the $\mathrm{vol}(C_t, 0_t)$ by $\frac{1}{u}$, with no other effect on our computations. $\square$

Remark II.30. We do not know if $\mathrm{vol}(X, x)$ is a topological invariant of the links of $\mathbb{Q}-$Gorenstein normal isolated singularities in arbitrary dimension. By Siu's theorem on the invariance of plurigenera, and because the cone over (V, H) is $\mathbb{Q}-$Gorenstein if, and only if, H is a rational multiple of K_V, no example can arise as above, by coning over a smooth projective polarized family.

CHAPTER III

An alternative notion of volume due to Boucksom, de Fernex, and Favre

In this chapter we prove an inequality between our definition of volume for normal isolated singularities and one other volume, recently introduced by Boucksom, de Fernex, and Favre in [BdFF11], also as a generalization of Wahl's work. We also describe a case when their volume and ours coincide, and compute an example where they do not. The first section is devoted to a brief overview of the constructions and of some results in [BdFF11]. The second section compares the two volumes.

III.1 $b-$divisors and $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$

In this expository section we review the basics of Shokurov's theory of $b-$divisors. We also review the construction and some properties of another notion of volume for normal isolated complex singularities. The reference is [BdFF11].

Let X be a complex normal algebraic variety of dimension n. A Weil canonical divisor K_X on X induces canonical divisors K_{X_π} on all resolutions $\pi : X_\pi \to X$, such that if $f : X_\pi \to X_\delta$ is a proper birational morphism over X between resolutions of X, we have

$$f_* K_{X_\pi} = K_{X_\delta}.$$

The pushforward above is to be understood in the sense of Weil divisors. Via resolutions of singularities and pushforwards, we obtain a Weil divisor K_{X_π} for any proper birational cover $\pi : X_\pi \to X$, with X_π potentially singular. Such an association of an $\mathbb{R}-$Weil divisor to any birational cover of X, an association that is compatible with pushforwards of Weil divisors is called a *$b-$divisor* (over X). Given a $b-$divisor D, and a birational cover π of X, the $\mathbb{R}-$Weil divisor D_π is called the *incarnation* of D in π, or in X_π. Denote the $b-$divisor described above by $K_{\mathfrak{X}}$. Other natural examples can be obtained as follows: Given an $\mathbb{R}-$Cartier divisor D_π on a birational model $\pi : X_\pi \to X$, one constructs the $b-$divisor $\overline{D}$ by pulling D_π back to higher models (which is possible under the Cartier assumption), and pushing forward to lower models. We call such $\overline{D}$ a Cartier $b-$divisor, and we say that it is *determined* on π, or on X_π. A particular case are the Cartier $b-$divisors $Z(\mathfrak{a})$ that are determined, for $\mathfrak{a}$ a coherent fractional ideal sheaf on X, by $\mathcal{O}(1)$ on $Bl_{\mathfrak{a}}X$. We will see that $b-$divisors provide a convenient language for defining pullbacks of Weil divisors and Zariski decompositions in dimension greater than two.

III.1.1 $X-$nef $b-$divisors

A Cartier $b-$divisor is called *$X-$nef (or just nef)* if any of (or all) its determinations are nef over X. A Weil $b-$divisor D is called nef if it is a limit (not componentwise), of nef Cartier $b-$divisors. We prefer the following equivalent definition (see [BdFF11, Lemma.2.9]):

Definition III.1. A $b-$divisor D is $X-$nef if all incarnations D_π with X_π nonsingular (or just $\mathbb{Q}-$factorial) are $\pi-$movable, i.e., the class of D_π in $N^1(X_\pi/X)_{\mathbb{R}}$ is a limit of classes of divisors whose $\pi-$base locus has codimension at least two.

Example III.2. The Cartier $b-$divisors $Z(\mathfrak{a})$ are clearly $X-$nef. More generally, consider $\mathfrak{a}_\bullet$ a graded sequence of coherent fractional ideal sheaves on X having *linearly* *bounded denominators*, i.e., there exists a Weil divisor D on X such that $\mathfrak{a}_m \cdot \mathcal{O}_X(mD)$ is an ideal sheaf for all $m \geq 0$. Then by [BdFF11, Prop.2.1], the sequence $\frac{1}{m}Z(\mathfrak{a}_m)$ converges coefficient–wise to a $b-$divisor $Z(\mathfrak{a}_\bullet)$. This divisor is $X-$nef by [BdFF11, Prop.2.10]. □

III.1.2 Nef envelopes

We will see that nef envelopes play the role of the nef parts of relative Zariski decompositions. They can also be used to define pullbacks of Weil divisors.

Definition III.3. Let D_π be a Weil divisor on a birational cover $X_\pi \to X$. The graded sequence

$$\mathfrak{a}_m =_{\text{def}} \pi_* \mathcal{O}_{X_\pi}(mD_\pi)$$

has denominators linearly bounded by the Weil divisor $-\pi_* D_\pi$, and one defines the nef envelope

$$\mathrm{Env}_\pi(D_\pi) =_{\text{def}} Z(\mathfrak{a}_\bullet).$$

With the conventions of [dFH09], when D is a Weil divisor on X, the incarnation $-\mathrm{Env}_X(-D)_\pi$ plays the role of the pullback of D by π. It coincides with f^*D when the divisor is Cartier. We collect [BdFF11, Prop.2.5, Prop.2.7, Prop.2.11, Cor.2.12] into the following:

Proposition III.4.

(i) $\mathrm{Env}_\pi(D_\pi + D'_\pi) \geq \mathrm{Env}_\pi(D_\pi) + \mathrm{Env}_\pi(D'_\pi)$ *for* D_π *and* D'_π *Weil divisors on* X_π*. We say that* Env_π *is a concave function.*

(ii) $\mathrm{Env}_\pi(t \cdot D_\pi) = t \cdot \mathrm{Env}_\pi(D_\pi)$ *for all* $t > 0$*. This homogeneity does not extend to linearity. It may well happen that* $\mathrm{Env}_\pi(D_\pi) \neq -\mathrm{Env}_\pi(-D_\pi)$*.*

The previous two properties allow us to define envelopes for $\mathbb{R}-$ *Weil divisors.*

(iii) For D an $\mathbb{R}-$Weil divisor on X, the incarnation $\mathrm{Env}_X(D)_X$ *is D.*

(iv) If D is an $X-$nef $b-$divisor, then

$$D \leq \mathrm{Env}_\pi(D_\pi)$$

for all birational covers $\pi : X_\pi \to X$. This result is also called the Negativity Lemma.

(v) $\mathrm{Env}_\pi(D_\pi)$ *is the largest $X-$nef $b-$divisor W such that $W_\pi \leq D_\pi$.*

(vi) $\mathrm{Env}_\pi(D_\pi) = \overline{D}_\pi$*, when D_π is $\mathbb{R}-$Cartier and $\pi-$nef.*

(vii) $\mathrm{Env}_\pi(D_\pi)$ *is $\mathbb{R}-$Cartier if, and only if, one of its incarnations on some resolution of X_π is $X-$nef, in which case, it is determined there.*

Via [BdFF11, Prop.2.14, Cor.2.15], we define nef envelopes for $b-$divisors.

Definition III.5. Given a $b-$divisor D, define the nef envelope of D as the componentwise infimum

$$\mathrm{Env}_{\mathfrak{X}}(D) =_{\mathrm{def}} \inf_\pi \mathrm{Env}_\pi(D_\pi).$$

If this exists, it is also the largest nef $b-$divisor W such that $W \leq D$.

III.1.3 Surfaces

In this subsection we explain how one can use nef envelopes to recover relative Zariski decompositions on surfaces. The following result appears as [BdFF11, Thm.2.20]:

Theorem III.6. *Let X be a normal surface, and let $\pi : X_\pi \to X$ be a log–resolution of $(X, \mathrm{Sing(X)})$.*

(i) If D_π is $\mathbb{R}-$Cartier on X_π, then $\mathrm{Env}_\pi(D_\pi)$ *is $\mathbb{R}-$Cartier, determined on X_π, and the following is a $\pi-$relative Zariski decomposition:*

$$D_\pi = \mathrm{Env}_\pi(D_\pi) + (D - \mathrm{Env}_\pi(D_\pi)).$$

(ii) If D is a Weil divisor on X, then $\mathrm{Env}_X(D)$ *is a Cartier $b-$divisor, determined on any X_π by the Mumford numerical pullback, π^*D.* [1]

Proof. Since $\mathrm{Env}_\pi(D_\pi)$ is a nef $b-$divisor, all incarnations are $\pi-$movable. On smooth surfaces, this is the same as $\pi-$nef. From Proposition III.4.(vii), it follows that $\mathrm{Env}_\pi(D_\pi)$ is $\mathbb{R}-$Cartier. By Proposition III.4.(vi), $\mathrm{Env}_\pi(D_\pi)_\pi$ is the largest $\pi-$nef $\mathbb{R}-$Cartier divisor W on X_π such that $D - W$ is effective. This is one of the characterizations of the $\pi-$nef component of the relative Zariski decomposition of D_π. For part (ii), note that as before, setting $W := \mathrm{Env}_X(D)_\pi$, one has $\mathrm{Env}_X(D) = \overline{W}$. Moreover, $\mathrm{Env}_X(D)_X = D$ by Proposition III.4.(iii), It remains to show that W is $\pi-$numerically trivial. In any case, W is $\pi-$nef. Assume $W \cdot E > 0$ for some component E of the exceptional locus of π. Then $W + \varepsilon \cdot E$ is still $\pi-$nef for sufficiently small $\varepsilon > 0$. Its incarnation in X is D. By Proposition III.4.$(v - vi)$,

$$\overline{W + \varepsilon \cdot E} \leq \mathrm{Env}_X(D).$$

This is impossible at the level of $\pi-$incarnations. □

Remark III.7. Since the numerical pullback is linear, it follows that Env_X is linear on Weil divisors on the normal surface X.

III.1.4 The definition of $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$

We present the definition of [BdFF11] for volumes of normal isolated singularities. Let (X, x) be a normal isolated singularity of dimension n at least two. Throughout, $\mathfrak{m}$ denotes the maximal ideal sheaf corresponding to x.

[1] Recall that π^*D is the unique Weil divisor W on X_π, such that $\pi_*W = D$ and W is $\pi-$numerically trivial.

Definition III.8. We form the $\pi-$exceptional log–discrepancy divisor

$$(\mathcal{A}_{\mathfrak{X}/X})_\pi =_{\text{def}} K_{X_\pi} + \text{Env}_X(-K_X)_\pi + 1_{X_\pi/X},$$

where $1_{X_\pi/X}$ is the reduced divisorial component of the full exceptional locus of π. These glue to form the $b-$divisor $\mathcal{A}_{\mathfrak{X}/X}$.

Intuitively, $-\text{Env}_X(-K_X)_\pi$ computes the pull-back $\pi^* K_X$, which should serve as justification for calling $(\mathcal{A}_{\mathfrak{X}/X})_\pi$ a log–discrepancy divisor. We denote by $\mathcal{A}^0_{\mathfrak{X}/X}$ the divisorial component lying over x of the $b-$divisor $\mathcal{A}_{\mathfrak{X}/X}$. A consequence of the smoothness of $X \setminus \{x\}$ is:

Remark III.9. The $b-$divisor $\mathcal{A}_{\mathfrak{X}/X} - \mathcal{A}^0_{\mathfrak{X}/X}$ is effective and exceptional.

Definition III.10. Let (X, x) be a complex normal quasiprojective isolated singularity of dimension n at least two. The volume of (X, x) in the sense of [BdFF11] is

$$\text{vol}_{\text{BdFF}}(X, x) =_{\text{def}} -(\text{Env}_{\mathfrak{X}}(\mathcal{A}^0_{\mathfrak{X}/X}))^n.$$

Intersections of nef $b-$divisors lying over x are defined in [BdFF11, Def.4.13]. Following [BdFF11], say that D is a Cartier $b-$divisor *over* x, if D admits a determination D_π with π a good resolution of (X, x), such that D_π lies over x. It is important that π is an isomorphism away from x. Given $D_1, \ldots, D_n$ a set of Cartier $b-$divisors over x, not necessarily $X-$nef, let π be a common determination that is a good resolution of (X, x), and define

$$D_1 \cdot \ldots \cdot D_n = (D_1)_\pi \cdot \ldots \cdot (D_n)_\pi.$$

The intersection makes sense because the $(D_i)_\pi$ have compact support, and it does not depend on π. Note that when $D_i = Z(\mathfrak{a}_i)$, with $\mathfrak{a}_i$ an $\mathfrak{m}-$primary ideal sheaf for all $i \in \{1, \ldots, n\}$, we recover the mixed multiplicity ([Laz04, p.91]):

$$-Z(\mathfrak{a}_1) \cdot \ldots \cdot Z(\mathfrak{a}_n) = e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n).$$

Definition III.11. Let $D_1, \ldots, D_n$ be arbitrary nef $\mathbb{R}-$Weil $b-$divisors over x. Then

$$D_1 \cdot \ldots \cdot D_n =_{\text{def}} \inf_{C_i \geq D_i} (C_1 \cdot \ldots \cdot C_n),$$

where the index set are all nef $\mathbb{R}-$Cartier $b-$divisors over x, such that $C_i \geq D_i$ for all i.

This quantity is finite when each D_i is *bounded below*, i.e., $D_i \geq \varepsilon \cdot Z(\mathfrak{m})$ for some $\varepsilon > 0$ and all i. Boundedness from below makes sense for any $\mathbb{R}-$Weil $b-$divisor over x, not necessarily nef. Examples include Cartier $b-$divisors over x (cf. [BdFF11, Lemma 4.7]), and $\mathcal{A}^0_{\mathfrak{X}/X}$ (cf. [BdFF11, Prop.4.6]). Some important properties of these intersection numbers are collected in [BdFF11, Thm.4.14]:

Proposition III.12. *The intersection product $(D_1, \ldots, D_n) \to D_1 \cdot \ldots \cdot D_n$ of nef $\mathbb{R}-$Weil $b-$divisors over x is symmetric, upper semicontinuous, and continuous along monotonic families (for the topology of coefficient–wise convergence). It is also homogeneous, additive, and non–decreasing in each variable. Moreover, $D_1 \cdot \ldots \cdot D_n < 0$ if all D_i are nonzero.*

We will review some of the properties of $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ in the next section.

III.2 $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ vs. $\mathrm{vol}(X, x)$

We compare the two notions of volume for normal isolated singularities in dimension n at least two. We also study a case when they are equal, and compute an example where they are not. A nontrivial result that relates intersections of nef $b-$divisors with multiplicities for graded sequences of $\mathfrak{m}-$primary ideal sheaves is [BdFF11, Rem.4.17]:

Lemma III.13. *For every graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}-$primary ideals, we have*

$$-Z(\mathfrak{a}_\bullet)^n = \widehat{h}^1_x(\mathfrak{a}_\bullet).$$

Theorem III.14. *Let (X,x) be a complex normal quasiprojective isolated singularity of dimension n at least two. Then*

$$\mathrm{vol}_{\mathrm{BdFF}}(X,x) \geq \mathrm{vol}(X,x).$$

Proof. By the definition of the nef envelopes of $b-$divisors, for any resolution $\pi : X_\pi \to X$, we have

$$\mathrm{Env}_{\mathfrak{X}}(\mathcal{A}^0_{\mathfrak{X}/X}) \leq \mathrm{Env}_\pi((\mathcal{A}^0_{\mathfrak{X}/X})_\pi).$$

The monotonicity property of intersection numbers in Proposition III.12 shows

$$\mathrm{vol}_{\mathrm{BdFF}}(X,x) \geq -(\mathrm{Env}_\pi((\mathcal{A}^0_{\mathfrak{X}/X})_\pi))^n.$$

By Lemma III.13, the latter is equal to $\mathrm{vol}_x((\mathcal{A}^0_{\mathfrak{X}/X})_\pi)$, since vol_x and envelopes are both computed from pushforward sheaves. Remark III.9 and Lemma I.42 yield

$$\mathrm{vol}_x((\mathcal{A}^0_{\mathfrak{X}/X})_\pi) = \mathrm{vol}_x((\mathcal{A}_{\mathfrak{X}/X})_\pi) = \mathrm{vol}_x(K_{X_\pi} + \mathrm{Env}_X(-K_X)_\pi + E),$$

where now $\pi : (X_\pi, E) \to (X,x)$ is a log–resolution. Since $\mathrm{vol}(X,x) = \mathrm{vol}_x(K_{X_\pi}+E)$, it suffices to prove that

$$\mathrm{vol}_x((K_{X_\pi} + E) + \mathrm{Env}_X(-K_X)_\pi) \geq \mathrm{vol}_x(K_{X_\pi} + E).$$

Since $\mathrm{Env}_X(-K_X)_\pi$ is $\pi-$movable, there exists a sequence of effective divisors D_m on X_π without components over x, a sequence that converges to $\mathrm{Env}_X(-K_X)_\pi$ in $N^1(X_\pi/X)$. We conclude by the continuity of vol_x and Lemma I.21. □

Remark III.15. When X is $\mathbb{Q}-$Gorenstein, [BdFF11, Prop.5.3] shows that

$$\mathrm{vol}_{\mathrm{BdFF}}(X,x) = \mathrm{vol}(X,x).$$

Aiming to extend this result to the numerically Gorenstein case (see [BdFF11, Def.2.24]), we start with a lemma inspired by the proof of [BdFF11, Prop.5.3] that allows us to compute $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ on a fixed resolution in a particular case:

Lemma III.16. *Let $\pi : (X_\pi, E) \to (X, x)$ be a log–resolution of a normal isolated singularity of dimension n, and assume* $\mathrm{Env}_X(-K_X)_\pi$ *is $\pi-$nef. Then*

$$\mathrm{vol}_{\mathrm{BdFF}}(X, x) = \mathrm{vol}_x(K_{X_\pi} + \mathrm{Env}_X(-K_X)_\pi + E).$$

Proof. Proposition III.4.(vii) proves that $\mathrm{Env}_X(-K_X)$ is Cartier, determined on X_π. Using [BdFF11, Lemma.3.2],

$$\mathcal{A}_{\mathfrak{X}/X} - \overline{(\mathcal{A}_{\mathfrak{X}/X})_\pi}$$

is effective and exceptional over X. The conclusion follows from Lemma I.42, Proposition III.4.$(v) - (vi)$, Remark III.9, and Lemma III.13. □

Proposition III.17. *If X is a numerically Gorenstein, i.e.,*

$$\mathrm{Env}_X(K_X) + \mathrm{Env}_X(-K_X) = 0,$$

then

$$\mathrm{vol}_{\mathrm{BdFF}}(X, x) = \mathrm{vol}(X, x).$$

Proof. The hypothesis implies that $\mathrm{Env}_X(\pm K_X)_\pi$ is $\pi-$numerically trivial on any nonsingular model X_π. We conclude using the numerical invariance of local volumes and Lemma III.16. □

Remark III.18. The result above and Remark III.7 prove that the two volumes are equal on surfaces (see also [BdFF11, Prop.5.1]). However, they may differ in general. [BdFF11, Exap.5.4] provides an example of a cone singularity where $\mathrm{vol}_{\mathrm{BdFF}}(X, x) > \mathrm{vol}(X, x) = 0$.

As [BdFF11, Thm.4.21] proves, the volume $\mathrm{vol}_{\mathrm{BdFF}}(X, x)$ satisfies the same monotonicity property with respect to finite covers that $\mathrm{vol}(X, x)$ does:

Remark III.19. Let $f : (X, x) \to (Y, y)$ be a finite morphism of isolated singularities. Then

$$\mathrm{vol}_{\mathrm{BdFF}}(X, x) \geq (\deg f) \cdot \mathrm{vol}_{\mathrm{BdFF}}(Y, y).$$

Remark III.20. Example II.29 and [BdFF11, Ex.4.23] show that the volumes $\mathrm{vol}(X,x)$ and $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ are not in general topological invariants of the link of the singularity in dimension 3 or higher. We do not know if $\mathrm{vol}(X,x)$ also has this property in the $\mathbb{Q}-$Gorenstein case (see Remark II.30). [BdFF11, Ex.4.23] shows that $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ is not a topological invariant of the link also in the $\mathbb{Q}-$Gorenstein case.

One advantage of $\mathrm{vol}(X,x)$ is that, being determined on any log–resolution, it is usually easy to compute. On the other hand, since every resolution may bring new information to the $b-$divisors that are involved, $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ is usually hard to compute when it is nonzero. Lemma III.16 provides examples when we can realize $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ as a local volume on a fixed birational model. Applying this to cone singularities, we give an example of an irrational $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$.

Lemma III.21. *Let (V,H) be a polarized projective nonsingular variety of dimension $n-1$, let $(X,0)$ be the associated cone singularity, which we assume is normal, and let $\pi:(Y,E)\to(X,0)$ be the contraction of the zero section of* $\mathrm{Spec}_{\mathcal{O}_V}\mathrm{Sym}^{\bullet}\mathcal{O}_V(H)$*. Let $f:Y\to V$ be the vector bundle map. Then*

$$\mathrm{Env}_X(-K_X)_\pi = f^*(-K_V+M\cdot H),$$

with M minimal such that $-K_V+M\cdot H$ is pseudoeffective.

Proof. Note that π is a good resolution, hence

$$\mathcal{O}_X(-mK_X)=\bigcup_{t\geq 0}\pi_*\mathcal{O}_Y(-mK_Y+tE).$$

By coherence, there exists minimal t_m such that

$$\mathcal{O}_X(-mK_X)=\pi_*\mathcal{O}_Y(-mK_Y+t_mE).$$

We get an induced inclusion that is actually an equality outside E:

$$\mathcal{O}_X(-mK_X)\cdot\mathcal{O}_Y\to\mathcal{O}_Y(-mK_Y+t_mE).$$

Using the defining minimality property of t_m, and that E is irreducible, one finds

$$Z(-mK_X)_\pi =_{\text{def}} (\mathcal{O}_X(-mK_X)\cdot\mathcal{O}_Y)^{\vee\vee} = \mathcal{O}_Y(-mK_Y + t_m E).$$

Observe that X is affine, therefore the sheaves $\pi_*\mathcal{O}_Y(-mK_Y+tE)$ are determined by their global sections. But by the relations in Example I.19, and since $K_Y+E = f^*K_V$ by adjunction,

$$H^0(Y, \mathcal{O}_Y(-mK_Y + tE)) = \bigoplus_{k\geq 0} H^0(V, \mathcal{O}_V(-mK_V + (-t-m+k)H))$$

and it follows that t_m is the maximal t such that $\mathcal{O}_V(-mK_V + (-t-m)H)$ has sections. Recall that $\mathrm{Env}_X(-K_X) = \lim_m(Z(-mK_X)/m)$, and set $l = \lim_m(t_m/m)$. One finds that

$$\mathrm{Env}_X(-K_X)_\pi = -K_Y + lE = f^*(-K_V - (l+1)H)$$

with l maximal such that $-(K_V + (l+1)H)$ is pseudoeffective. Manifestly

$$M = -1 - l.$$

□

Corollary III.22. *With the same notation as before, assume that* $\mathrm{Env}_X(-K_X)_\pi$ *is also* $\pi-$*nef. Then*

$$\mathrm{vol}_{\mathrm{BdFF}}(X,0) = \begin{cases} M^n\cdot H^{n-1} & , \text{ if } M\geq 0 \\ 0 & , \text{ if } M<0 \end{cases}.$$

Proof. Since the negative case follows similarly, we assume $M>0$. By Lemma III.16, Example I.19, the preceding result and from the ampleness of H,

$$\mathrm{vol}_{\mathrm{BdFF}}(X,0) = \mathrm{vol}_{\{0\}}(K_Y + E + f^*(-K_V + M\cdot H)) = \mathrm{vol}_{\{0\}}(f^*(M\cdot H)) =$$

$$= n\cdot\int_0^\infty \mathrm{vol}(M\cdot H - tH)dt = M^n\mathrm{vol}(H) = M^n\cdot H^{n-1}.$$

□

Example III.23. As in Example II.27, with notation as in the preceding lemma, let $\mathcal{E}$ be a general elliptic curve. Let D and L be integral ample divisors on $\mathcal{E} \times \mathcal{E}$, let $g : V \to \mathcal{E} \times \mathcal{E}$ be the double cover over a general section of $\mathcal{O}_{\mathcal{E}\times\mathcal{E}}(2D)$, and denote $H = g^*L$. Note that $K_V = g^*D$. Then $\mathrm{Env}_X(-K_X)_\pi$ is $\pi-$nef because its restriction to E, the only positive dimensional fiber, is isomorphic to $-K_V + M \cdot H$, which is pseudoeffective; and on V, nef and pseudoeffective are equivalent notions for pullbacks of divisors from $\mathcal{E} \times \mathcal{E}$, e.g., from [Laz04, Rem.4.1.7]. By the previous corollary, noting that M is positive since $-K_V + M \cdot H = g^*(-D + M \cdot L)$,

$$\mathrm{vol}_{\mathrm{BdFF}}(X, 0) = M^3 H^2.$$

We find that $\mathrm{vol}_{\mathrm{BdFF}}(X, 0)$ can be irrational by producing an example of D and L where M^3 is irrational. The same construction as in Example II.28 works. $\square$

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