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WAVE MAKING: A LOW-SPEED SLENDER-BODY THEORY

by

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ABSTRACT

A slender-body theory for the wave making of a ship in steady translational motion is developed by assuming that the length of the waves generated by the ship is of the same order of magnitude as the beam of the ship. The exact boundary value problem is formulated and then linearized by assuming that the wave length is short.

This linearized problem is solved to two orders of magnitude by the method of matched asymptotic expansions. The first order solution turns out to be a simple variation on the Tuck slender-body theory, while the second order solution turns out to be a diffraction problem accounting for the refraction of the waves away from the body.

Qualitatively, this solution shows some of the same characteristics which Adachi has observed in his experiments on models with long parallel middle bodies. This solution is in many ways similar to the solution which Faltinsen obtained for the diffraction of head seas by a slender ship. Quantitatively, the wave elevations obtained using this theory agree well with experimental measurements obtained from tests on a pointed body of revolution.

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$A(x)$	immersed cross-sectional area of the ship
B	maximum beam of the ship
D	maximum draft of the ship
g	gravitational constant
$h(x,y)$	function defining the surface of the ship
L	length of the ship
$R(x)$	rapidly varying source strength
r	radial distance from the x-axis to a point in the cross plane: $r = \sqrt{y^2+z^2}$
U	ship velocity in units consistent with L , g , etc.
x,y,z	right-handed cartesian coordinates fixed to the ship (See Figure 3)
$\delta(x)$	Dirac delta function
ε	slenderness parameter
$\zeta(x,y)$	free surface elevation
θ	angular location of a point in the cross plane: $\theta = \tan^{-1}(y/z)$
κ	wave number: $\kappa = g/U^2$
λ	fundamental wave length: $\lambda = 2\pi/\kappa$
μ	fictitious viscosity (See page 51)
$\Sigma(x)$	slowly varying component of the rapidly varying source strength: $R(x) = \mathrm{Re}[\Sigma(x)e^{i\kappa x}]$
$\sigma(x)$	slowly varying source strength
$\Phi(x,y,z)$	total velocity potential for the ship fixed in a uniform flow

$\phi(x,y,z)$ normalized perturbation velocity: $\Phi(x,y,z) = Ux + U\phi(x,y,z)$

$\phi_{rw}(x,y,z)$ normalized perturbation velocity for a body and its mirror image in an infinite fluid

$\psi(y,z)$ slowly varying component of the second order near-field velocity potential

$\partial/\partial n$ derivative in the direction of the three-dimensional outward normal vector to the ship

$\partial/\partial N$ derivative in the direction of the two-dimensional outward normal vector in the cross-section of the ship

INTRODUCTION

For hundreds of years, one of the naval architect's goals has been to predict the resistance of a ship using purely theoretical means. William Froude (1868) took the first steps towards this goal when he published his famous paper suggesting that ship resistance could be broken into two components, one due to viscosity and one due to gravity. Through the work of Osborn Reynolds, Froude knew how the viscous resistance could be scaled, and it only remained for him to state how the residuary resistance scaled. Froude developed his similarity law for scaling residuary resistance by studying the wave patterns generated by models and noting at what speeds the wave patterns were similar. He also stated the well known fact that a major portion of the residuary resistance is the wave making resistance.

J. H. Michell (1898) was the first person to come forward with a theory for predicting the wave resistance of a ship which was the forerunner of today's thin-ship theory. Although thin-ship theory has been rigorously studied and elaborated upon by many workers (for example, see the many works by Wrigley, Havelock, and Weinblum), it was not until the early 1960's that a totally new theory was developed. This was the slender-body theory developed independently and essentially simultaneously by H. Maruo, E. O. Tuck, and G. Vosser [see, for example, Tuck (1963)].

As quickly as a slender-body theory for wave making was brought to life, it was dealt a death blow by G. R. G. Lewison (1963). He showed that the wave resistance given by slender-body theory was approximately twice that of thin-ship theory and the latter theory already over estimated the wave resistance compared with experimental results. Thus, slender-body theory as it applied to wave making became dormant for the next decade, at the end of which a number of events occurred which gave new direction to the theory.

First of all, calculations of the wave elevation alongside the ship were completed using the Tuck Slender-Body Theory. These calculations [see Ogilvie (1974)] showed that, although the wave amplitude was too great and the phase was in error, the theory certainly demonstrated the proper wave-like behavior alongside the ship except for near the bow. At the same time, T. F. Ogilvie (1973) was developing a modified slender-body theory for application to the region near the bow of the ship. This theory gave finite wave amplitudes at the bow and showed that the waves decayed like the inverse of the distance from the bow. This agreed with the Tuck theory which said that the wave amplitude grew like one over the distance from the bow as the bow was approached.

Another sequence of events also occurred around the same time. H. Adachi (1973) made available a number of his experimental results on an extremely long, slender model. These results showed that the rate of decay of the waves generated by a ship was much more rapid than the rate predicted by conventional theory. (Conventional theories show that the waves decay like the inverse square root of the distance from the bow.) These results are shown in Figure 1 for two model lengths. Model S-201 is 3.5 meters long. Model ES-201 is made by extending Model S-201 with 20 meters of parallel middle body. Adachi produced data for a single model, ES-201, to show that the wave elevation increased linearly in the transverse direction as one proceeded away from the model, Figure 2. These results are similar to results obtained by O. M. Faltinsen (1971) for a problem with similar geometry.

Faltinsen was solving the diffraction problem for a ship at rest in head seas. In his solution Faltinsen found that the waves were attenuated in the longitudinal direction by the presence of the ship and that the wave amplitude grew as one moved away from the ship. This led to the conclusion [since confirmed by Ursell (1973, 1974a, 1974b)], that the waves generated by the ship were refracted by the presence of the ship. In fact, Ursell showed that the waves should decay like $1/x^{3/2}$ as they move away from the bow, where x is the distance from the bow.

These results have led us to formulate the following problem: let a ship in a coordinate system as shown in Figure 3 be represented by the sum of two source distributions. The first source distribution is a slowly varying one intended primarily to represent flow in an infinite fluid about the ship and its double model, with the related wave motion appearing as high order effects. The second source distribution will be rapidly varying in the longitudinal direction and will represent the

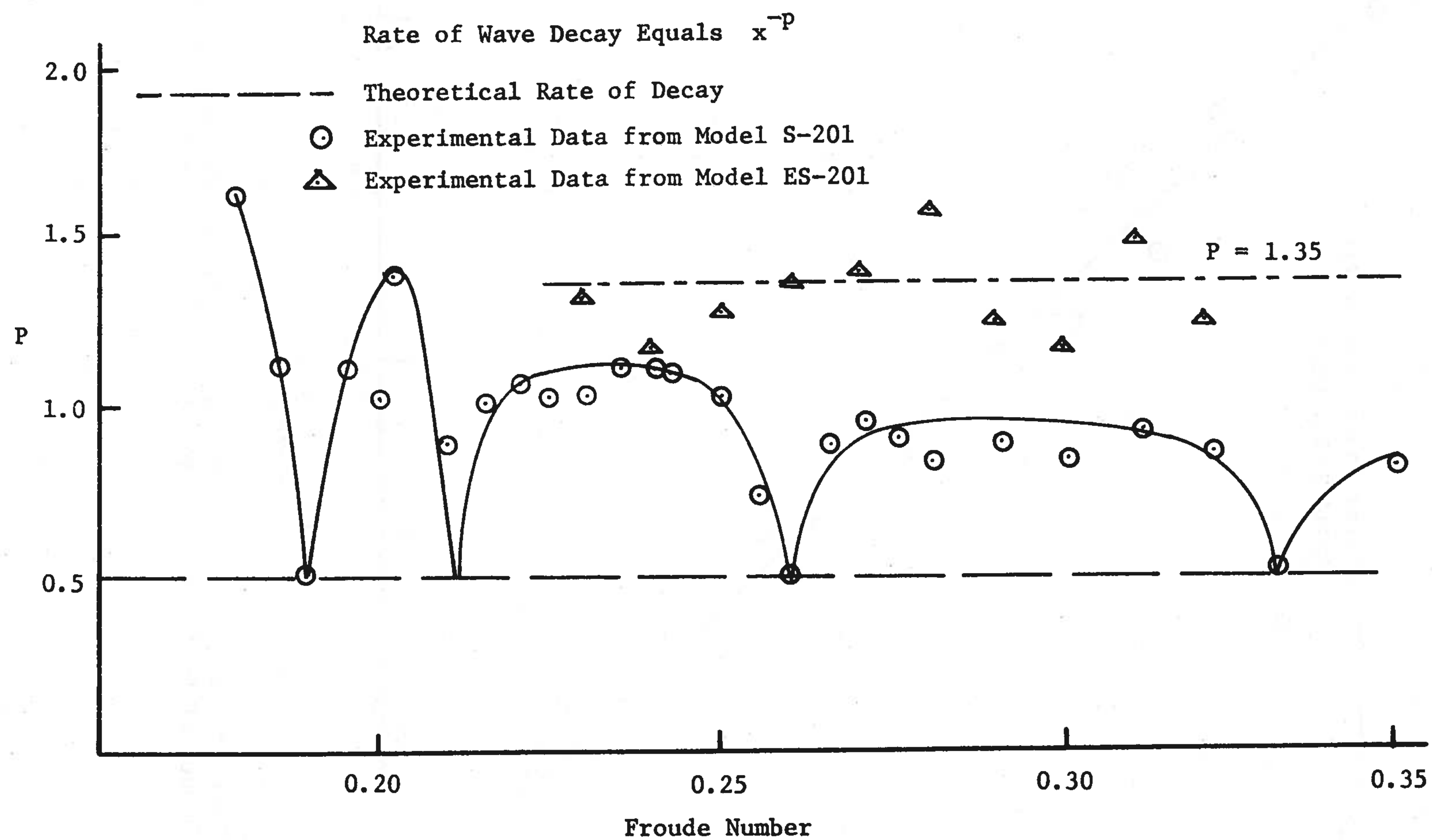

Figure 1: Comparison Between Experiments and Theory Showing Rate of Wave Decay as a Function of Speed [Adachi (1973)].

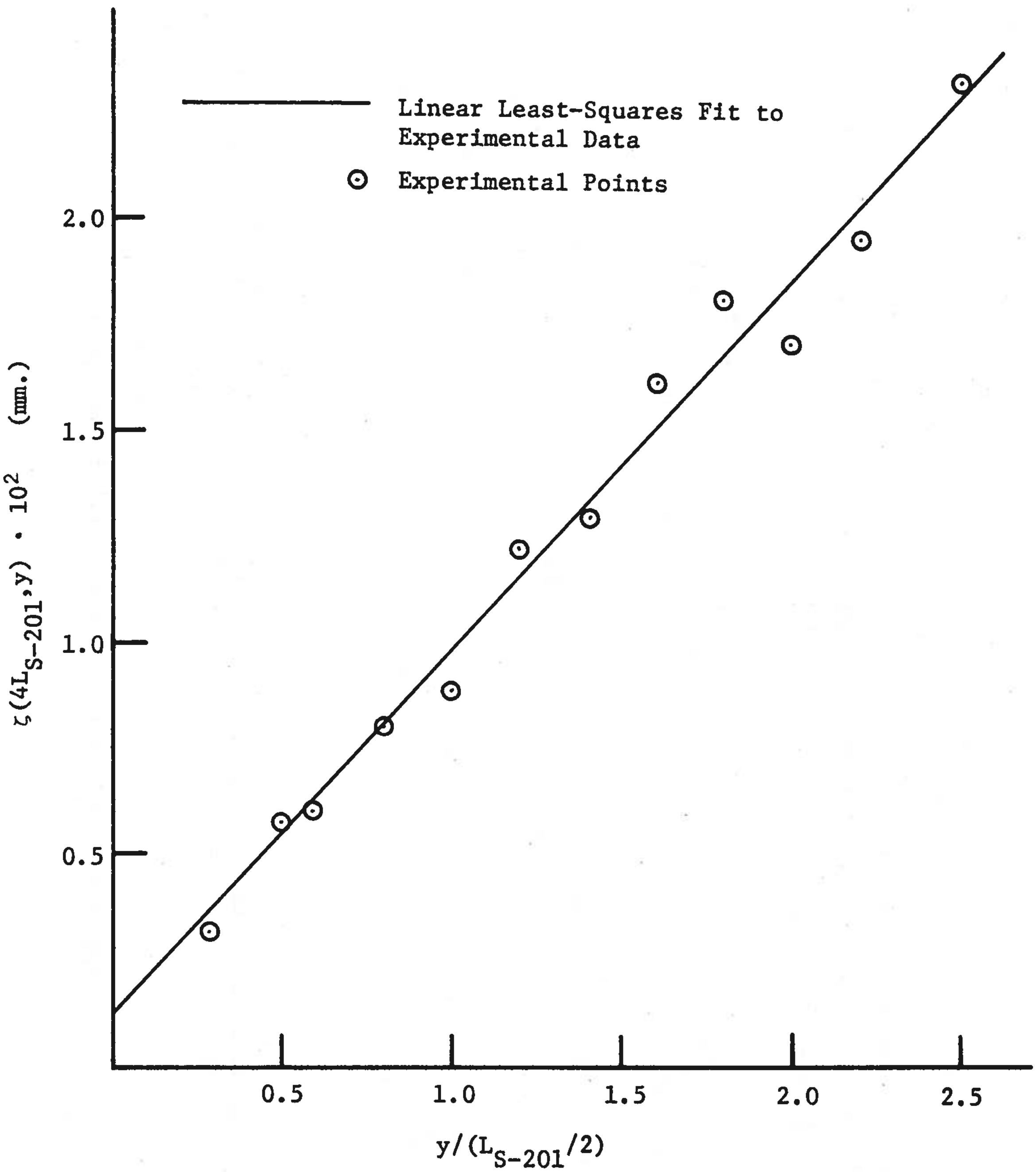

Figure 2: Rate of Transverse Wave Growth for Model ES-201 at a Froude Number of 0.25 [Adachi (1973)].

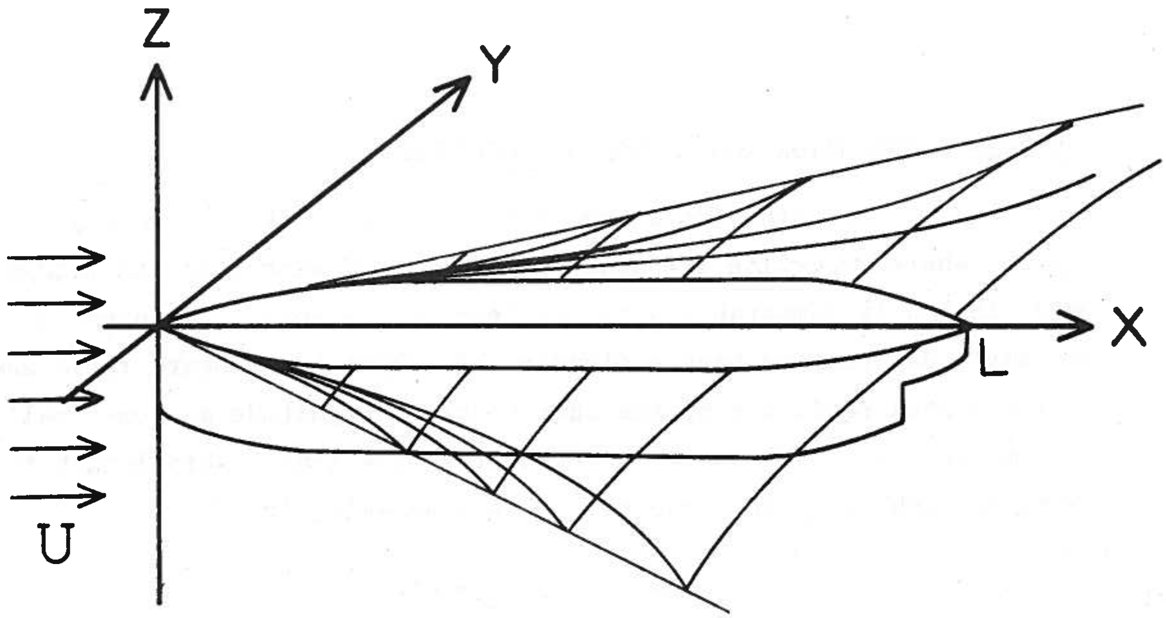

Figure 3: Ship with Typical Wave Pattern

effects of diffraction on the waves from the first source distribution. As we observe the flow near the ship due to the rapidly varying source distribution, we will find that the waves do indeed increase like $|y|$ as we move transversely away from the ship. Although we can not show the rate at which the waves decay as we move longitudinally away from the bow, we do obtain excellent correlation with experimental wave measurements alongside the parallel middle body of a ship model. These results are primarily due to the fact that we have now obtained a problem which allows us to have a wave-like free surface boundary condition near the ship. This is a concept which Ogilvie (1973, 1974) found has allowed him to obtain solutions to the bow problem. We shall now proceed to set up our problem precisely and to obtain a solution by employing the ideas just discussed.

PROBLEM FORMULATION AND METHOD OF SOLUTION

We wish to establish a theory for the wave making of a ship at low speed, where we define low speed to be a speed such that the fundamental wave length is comparable with the beam of the ship. To formalize this, we shall say that we have a slender ship whose beam-length ratio and draft-length ratio are of the same order of magnitude as some small parameter, ε. Now we can say that the wave length/ship length is comparable with ε. To state this more precisely, let

$$\frac{B}{L} = O(\varepsilon) \ , \quad \frac{D}{L} = O(\varepsilon) \ ,$$

and then

$$\frac{\lambda}{L} = \frac{2\pi U^2}{gL} = O(\varepsilon) \ ,$$

where B is the ship's beam, D the ship's draft, L the ship's length, λ the wave length, U the ship's speed, and g the gravitational constant.

The fact that we intend to develop a slender body theory implies some constraints on the geometry of the body. We define our body by the equation:

$$z = h(x,y) \ , \qquad 0 \le x \le L \ ,$$

where the body offsets are given by the points $[x,y,h(x,y)]$, and the function $h(x,y)$ is defined such that the ship is heading in the negative x-direction, with the origin at the bow, and with the z-axis vertically upwards. Now, defining slenderness, we can say that $h(x,y)$ is small and we shall also demand that derivatives of $h(x,y)$ with respect to x should also be small. Formalizing this, we can say:

$$h(x,y) = O(\varepsilon) \ ,$$

and

$$\frac{\partial h}{\partial x} = O(\varepsilon) \ .$$

An important fact to note is that in the limit as $\varepsilon \to 0$, the body cross-sectional dimensions shrink to zero, the wave length goes to zero, and we have an infinite number of wave lengths along the body. This implies that the speed is now a function of the small parameter, ε , and the speed goes to zero as $\varepsilon \to 0$, though not linearly. In fact, from the relationship associating λ/L to ship velocity and ε , we can see that:

$$U = O(\varepsilon^{1/2}) .$$

Now stating the problem which we intend to solve as a boundary value problem, we shall assume that the fluid is inviscid and incompressible and that the flow is irrotational. Given these assumptions, the flow can be described by a potential function. The partial differential equation governing this problem is the continuity equation,

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 , \tag{1}$$

in the fluid domain. The body boundary condition is that there shall be no flow through the body. This can be written as:

$$\frac{\partial \Phi}{\partial n} = 0 , \qquad \text{on} \qquad h(x,y) - z = 0 , \tag{2}$$

where $\partial/\partial n$ denotes the derivative in the direction of the three-dimensional outward normal vector to the body. On the free surface, two boundary conditions apply. The first is the dynamic free-surface condition,

$$g\zeta + \frac{1}{2}\{\Phi_x^2 + \Phi_y^2 + \Phi_z^2\} = \frac{1}{2} U^2 \qquad \text{on} \qquad z = \zeta(x,y) , \tag{3}$$

and the second is the kinematic free-surface condition,

$$\Phi_x\zeta_x + \Phi_y\zeta_y - \Phi_z = 0 , \qquad \text{on} \qquad z = \zeta(x,y) . \tag{4}$$

Finally, we must have a radiation condition which states that no waves should be propagating upstream of the ship.

Method of Solution

In order to solve the above boundary value problem, we must simplify it. To do this, we shall linearize the problem. By writing the potential function $\Phi(x,y,z)$ as follows:

$$\Phi(x,y,z) = Ux + U\phi(x,y,z) ,$$

and assuming that $U\phi(x,y,z)$ is higher order than Ux, we may rewrite the boundary value problem and its boundary conditions in terms of $\phi(x,y,z)$. Furthermore, we may expand the wave elevation in a Taylor series about $z = 0$ and discard all high order terms to obtain a linearized free-surface problem.* Rewriting the continuity equation (1), we obtain:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad \text{outside the body for } z < 0 . \tag{5}$$

Rewriting the body boundary condition, equation (2), we find that

$$\frac{\partial\phi}{\partial n} = - \frac{h_x}{\sqrt{1 + h_x^2 + h_y^2}} \qquad \text{on} \qquad h(x,y) - z = 0 . \tag{6}$$

By revising the dynamic free-surface condition, equation (3), and the kinematic free-surface condition, equation (4), we obtain:

$$g\zeta + \frac{1}{2} U^2\{2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2\} \qquad \text{on} \qquad z = 0 , \tag{7}$$

and

$$U\zeta_x + U\phi_x\zeta_x + U\phi_y\zeta_y - U\phi_z = 0 \qquad \text{on} \qquad z = 0 .$$

If we now neglect the product terms in both equations, and then differentiate the dynamic free-surface condition with respect to x and subtract the result from the kinematic condition, we obtain the linearized free-surface condition as follows:

$$\phi_{xx} + \kappa\phi_z = 0 \qquad \text{on} \qquad z = 0 , \tag{8}$$

where

$$\kappa = g/U^2 .$$

The radiation condition for this problem remains unchanged. Although the problem which we must now solve is linear, it is still an extremely difficult one to solve exactly. Therefore we shall make use of the method of matched asymptotic expansions to further facilitate finding a solution

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*We should linearize the problem separetely in the near field and far field of the body, because we might expect different orders of magnitude for the velocity components in these two regions. However, it turns out that for this problem the linearized problems in the near and far fields are independent of the initial linearization.

to the problem.

In the application of the method of matched asymptotic expansions to the linearized problem, we shall determine approximate solutions to the problem which are valid in the far field and near field respectively. In the far field, we shall assume that y and z are $O(1)$ as $\varepsilon \to 0$ and that the derivatives of the potential function with respect to x,y,z are all of the same order of magnitude. Because of the assumption about the order of magnitude of y and z, we will not see the body in any detail in the far field. In fact, in the limit as $\varepsilon \to 0$, we will represent the body as a line of singularities of unknown strength. In the near field, we will assume that y and z are $O(\varepsilon)$ and that derivatives of the potential with respect to y and z are $O(\varepsilon^{-1})$, while derivatives of the slowly varying potential with respect to x are $O(1)$. Because of these assumptions, we will find that, in the near field, the boundary value problem becomes two dimensional in the cross section of the ship. This two dimensionality allows us to formulate a problem involving the body geometry, which we can readily solve. In solving this problem, we will be able to determine the unknown singularity strengths in the far field.

The final step before starting the actual solution of the problem is to state our problem in terms of a perturbation series. In both the near and far fields, we shall assume that the potential function, $\phi(x,y,z)$, can be expanded in the following form:

$$\phi(x,y,z) = \phi^{(0)}(x,y,z;\varepsilon) + \phi^{(1)}(x,y,z;\varepsilon) + \cdots\cdot$$

For the far-field problem, we shall assume that each $\phi^{(n)}(x,y,z;\varepsilon)$ can be represented by a convolution integral of a source distribution and a Green's function over the length of the body. Let the source distributions for the first and second potentials be $\sigma(x)$ and $\Sigma(x)e^{i\kappa x}$ respectively, where both $\sigma(x)$ and $\Sigma(x)$ are slowly varying functions of x, that is to say:

$$\frac{\partial\sigma}{\partial x} = O(\sigma) \qquad \text{and} \qquad \frac{\partial\Sigma}{\partial x} = O(\Sigma) .$$

The second source distribution is chosen to be rapidly oscillating so that it can represent the diffraction effects which we know occur. Likewise, we shall assume that in the near field the second near-field potential is of the form $\phi^{(2)}(x,y,z;\varepsilon) = \mathrm{Re}[e^{i\kappa x}\psi^{(2)}(x,y,z;\varepsilon)]$, where again $\psi^{(2)}(x,y,z;\varepsilon)$ is a slowly varying function of x . Here we justify making $\phi^{(2)}(x,y,z;\varepsilon)$ a wave-like potential because we know that if we want $\phi^{(2)}(x,y,z)$ to represent the diffraction of waves, it must be rapidly varying in the x direction. We shall now set up and solve the first order problem.*

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*In the work which follows, we shall drop the superscripts without ambiguity.

FIRST ORDER PROBLEM

Far-Field Problem

In the far field we will represent the ship by a line of sources distributed on the free surface. First we need to determine what the governing equations should be for the far-field problem with a source distribution. Based on the continuity equation and the assumption that all derivatives are the same order of magnitude in the far field, the partial differential equation governing the far field becomes the Poisson equation. We have no body boundary condition to satisfy. This leaves only the free surface and the radiation conditions to be determined. Because we expect waves of length $\lambda = O(\varepsilon)$ in the far field, the derivative of the potential in the far field must be very large due to the rapid changes in the potential necessitated by wave-like behavior. In fact derivatives with respect to x,y,z must be $O(\varepsilon^{-1})$ in the far field. By employing this information, we determine that the orders of magnitude of the terms in the linearized free-surface condition, equation (8), remain unchanged. We can now write the corresponding boundary value problem as follows:

$$(\phi_{xx} + \phi_{yy} + \phi_{zz}) = \sigma(x)\delta(y)\delta(z-c) \; , \qquad z < 0$$

and

$$\phi_{xx} + \kappa\phi_z = 0 \qquad \text{on} \qquad z = 0 \; ,$$

where c is the vertical location of the source distribution $(c < 0)$ and $\delta(y)$ and $\delta(z - c)$ are Dirac delta functions. The solution to this problem must satisfy a radiation condition so that waves do not propagate upstream of the disturbance. Tuck (1963) has solved this problem and he gives the following results:

$$\phi^{**}(k,\ell;z) = -\frac{\sigma^*(k)}{2\sqrt{k^2+\ell^2}}\left\{\frac{\kappa\sqrt{k^2+\ell^2}+k^2}{\kappa\sqrt{k^2+\ell^2}-k^2}\, e^{\sqrt{k^2+\ell^2}\,(z+c)} + e^{-\sqrt{k^2+\ell^2}\,|z-c|}\right\}$$

where $\phi^{**}(k,\ell;z)$ is the double Fourier transform of $\phi(x,y,z)$ with respect to x and y. We can now let our line of sources approach the free surface by letting $c \to 0$. If we do this, $\phi^{**}(k,\ell;z)$ simplifies to:

$$\phi^{**}(k,\ell;z) = -\frac{\sigma^*(k)e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} \quad . \tag{9}$$

Applying the Fourier inversion theorem to $\phi^{**}(k,\ell;z)$, we find that the potential can be written as follows:*

$$\phi(x,y,z) = -\frac{1}{4\pi^2}\int_{\infty}^{\infty} dk\, e^{ikx}\sigma^*(k)\int_{-\infty}^{\infty} \frac{d\ell\, e^{i\ell y}e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} \quad . \tag{10}$$

This is the Fourier transform form for the first order potential in the far field.

If we are later going to match this solution to a near-field solution, it will be necessary to obtain an inner expansion of the outer expansion (IE-OE) . The IE-OE is an asymptotic form of the far field solution obtained by letting y and z become small $[O(\varepsilon)]$. The IE-OE of equation (10) is as follows:

$$\phi(x,y,z) \sim \underbrace{\frac{1}{\pi}\sigma(x)\log r}_{O(\sigma \log \varepsilon)} - \underbrace{\frac{1}{2\pi}\int_0^L d\xi\, \sigma'(\xi)\log 2|x-\xi|\,\mathrm{sgn}(x-\xi)}_{O(\sigma)} \tag{11a}$$

$$- \underbrace{\frac{e^{\kappa z}}{4}\int_0^L d\xi\, \sigma'(\xi)\{\mathbf{H}_0[\kappa(x-\xi)] + [2 + \mathrm{sgn}(x-\xi)]Y_0(\kappa|x-\xi|)\}}_{O(\sigma\varepsilon)} \tag{11b}$$

where $r = \sqrt{y^2+z^2}$ is the radial distance from the source distribution. $\mathbf{H}_0$ and Y_0 are the Struve function and the Bessel function of the second kind, respectively. The derivation of the IE-OE is given in Appendix A. The first two terms of the IE-OE , line (11a), are the same terms which we would obtain when finding the IE-OE for a line distribution of sources in an infinite fluid. This is not surprising considering that the usual first order approximation to the low speed problem is simply the problem in which the free-surface condition is $\phi_z = 0$. The third term, line (11b), is the term which contains the wave-like

*This integral has two Cauchy singularities for $|k| > \kappa$. The proper contour around the singularities is determined from the radiation condition (see Appendix B).

behavior, and it is higher order than the terms in line (11a).

Near-Field Problem

We shall now determine the first order near-field boundary value problem, and establish its behavior far from the body. It is at this stage that we shall use our slenderness assumptions to the utmost.

It is trivial to show that the zeroth order potential must be the same as the potential for the incident stream in the far-field problem, that is:

$$U\phi^{(0)}(x,y,z;\varepsilon) = Ux .$$

Therefore, we shall proceed with the first order near-field problem. The linearized boundary value problem was given in equations (5), (6), and (8) as follows:

$$\underset{O(\phi)}{\phi_{xx}} + \underset{O(\phi\varepsilon^{-2})}{\phi_{yy}} + \underset{O(\phi\varepsilon^{-2})}{\phi_{zz}} = 0 \quad \text{outside the body for } z < 0 \tag{12}$$

$$\frac{\partial\phi}{\partial n} = \frac{-h_x}{\sqrt{1+h_x^2+h_y^2}} \quad \text{on} \quad h(x,y) - z = 0 \tag{13}$$

and

$$\underset{O(\phi)}{\phi_{xx}} + \underset{O(\phi\varepsilon^{-2})}{\kappa\phi_z} = 0 \quad \text{on} \quad z = 0 . \tag{14}$$

In our previous discussion of slenderness, we assumed that:

$$h, \frac{\partial h}{\partial x} = O(\varepsilon) \qquad \text{and} \qquad \frac{\partial h}{\partial y} = O(1) ,$$

while in our discussion concerning the near field, we assumed that:

$$\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial r} = O(\phi\varepsilon^{-1}) .$$

Applying these assumptions to the body boundary condition [equation (13)], we obtain:

$$\underset{O(\phi\varepsilon^{-1})}{\frac{\partial\phi}{\partial n}} = \underset{O(\varepsilon)}{\frac{-h_x}{\sqrt{1+h_y^2}}} .$$

If we apply our assumptions concerning the orders of magnitude of the various derivatives of ϕ to equations (12) and (14), we obtain the orders of magnitude given beneath the terms of the above equations. Given these orders of magnitude, we can discard the high-order terms and simplify the near-field problem as follows:

$$\phi_{yy} + \phi_{zz} = 0 \qquad\qquad z < 0 \tag{16}$$

$$\phi_z = 0 \qquad \text{on} \qquad z = 0 \tag{17}$$

$$\frac{\partial\phi}{\partial N} = -\frac{h_x}{\sqrt{1 + h_y^2}} \qquad \text{on} \qquad h(x,y) - z = 0 \ , \tag{18}$$

where $\partial/\partial N$ denotes the derivative in the direction of the two-dimensional outward normal vector to the body in the cross plane. The body boundary condition also allows us to determine the order of magnitude of ϕ. From $O(\phi\varepsilon^{-1}) = O(\varepsilon)$ we have:

$$\phi = O(\varepsilon^2) \ .$$

This problem is a two-dimensional "rigid wall" problem, corresponding to the Neumann problem for a body and its mirror image, a problem which has been studied many times [e.g., Ward (1955)]. Ward has demonstrated that the solution far away from the body acts like:

$$\phi \sim \frac{1}{\pi} A'(x) \log r + b(x) \ ,$$

where $A(x)$ is the sectional area of the submerged part of the ship and $b(x)$ is an arbitrary function of x. This is the outer expansion of the inner expansion (OE-IE).

Matching

We are now in a position to match our inner expansion of the outer expansion with the outer expansion of the inner expansion. The process of matching involves equating terms of similar form from the IE-OE with those from the OE-IE, i.e., matching $f(x)\log r$ in the IE-OE with $g(x)\log r$ in the OE-IE. One way to facilitate the matching is to set up a table of the OE-IE alongside the IE-OE with the terms becoming higher order as you proceed down the table. Table 1 is such a table for the first order problem.

If we compare the two sides of this table, we see that the first line of the OE-IE is identical to that of the IE-OE, and therefore, they match trivially. In the second line, we see that the first terms will match if we set $\sigma(x) = A'(x) = O(\varepsilon^2)$. We can also match the second

TABLE 1

Matching of First-Order Solution

Outer Expansion of the Inner Expansion	Inner Expansion of the Outer Expansion	
Ux $O(\varepsilon^{1/2})$	Ux $O(\varepsilon^{1/2})$	(1)
$+ \frac{U}{\pi} A'(x) \log r + Ub(x)$ $O(\varepsilon^{5/2} \log \varepsilon)$	$+ \frac{U}{\pi}\sigma(x)\log r - \frac{U}{2\pi}\int_0^L d\xi\ \sigma'(\xi)\log(2\lvert x-\xi\rvert)\operatorname{sgn}(x-\xi)$ $O(\sigma\varepsilon^{1/2}\log \varepsilon)$ $\quad O(\sigma\varepsilon^{1/2})$	(2)
	$- \frac{U}{4} e^{\kappa z}\int_0^L d\xi\ \sigma'(\xi)\{\mathbf{H}_0[\kappa(x-\xi)] + [2 + \operatorname{sgn}(x-\xi)]Y_0(\kappa\lvert x-\xi\rvert)\}$ $O(\sigma\varepsilon^{3/2})$	(3)

terms of the second line if we set

$$b(x) = -\frac{1}{2\pi}\int_0^L d\xi\ \sigma'(\xi)\log(2|x-\xi|)\operatorname{sgn}(x-\xi) = O(\varepsilon^2) .$$

We have now matched all of the terms of the OE-IE to terms of the IE-OE . However, we have not matched the third line of the IE-OE with any other terms. This means that we have determined too many terms in the IE-OE and thus will not be able to utilize it until we have completed the next order of approximation.

This matching completes the solution of the first order problem. However, before we continue on with the second order problem, we should note the physical significance of the high order term which we neglected in the IE-OE . As stated earlier, this term contains all of the waves generated by the body. These waves were obtained from the far field and therefore, violate the body boundary condition. Thus we would expect that any higher order near-field solution must contain terms to match these wave-like terms. It is this fact which leads us to assume that our second order near-field solution will be rapidly varying in the x-direction and the corresponding far-field problem will have a rapidly varying source distribution. Using these ideas, let us now proceed with the second order solution.

SECOND ORDER PROBLEM

Far Field

As in the first order problem, we shall again write our far-field potential as the convolution integral of a line source distribution and a Green's function. This time, however, we shall take our source distribution to be of the form $\mathrm{Re}[e^{i\kappa x}\Sigma(x)]$, where $\Sigma(x)$ is a slowly varying function of x . As before, our governing equations are

$$(\phi_{xx} + \phi_{yy} + \phi_{zz}) = \mathrm{Re}[\Sigma(x)e^{i\kappa x}]\delta(y)\delta(z+c) \qquad \text{for} \qquad z < 0$$

and

$$\phi_{xx} + \kappa\phi_z = 0 \qquad \text{on} \qquad z = 0 \,,$$

along with a radiation condition which allows waves only downstream of the disturbance. Again we find the double Fourier transform of our solution (with $c = 0$) to be:

$$\phi^{**}(k,\ell;z) = -\frac{\Sigma^*(k-\kappa)\; e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} \,,$$

where $\Sigma^*(k-\kappa)$ is the Fourier transform of $\mathrm{Re}[\Sigma(x)e^{i\kappa x}]$ with respect to x . ($\Sigma^*(k-\kappa)$ is not strictly the Fourier transform of $\mathrm{Re}[\Sigma(x)e^{i\kappa x}]$. This matter is treated rigorously in Appendix B.) The corresponding potential may be written as follows:

$$\phi(x,y,z) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk\; e^{ikx}\Sigma^*(k-\kappa)\int_{-\infty}^{\infty} \frac{d\ell\; e^{i\ell y}\; e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} \,. \tag{19}$$

If we now let y and z become $O(\varepsilon)$ and obtain an asymptotic expansion of $\phi(x,y,z)$, we will find that our IE-OE is:

$$\phi(x,y,z) \sim \mathrm{Re}\left\{ \underbrace{-\kappa e^{\kappa z} e^{i\kappa x - i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}}}_{O(\Sigma\varepsilon^{-1/2})} \right.$$

$$\left. + \underbrace{\kappa|y| e^{\kappa z}\Sigma(x) e^{i\kappa x}}_{O(\Sigma)} - \underbrace{\frac{1}{2} e^{\kappa z}\Sigma(x) e^{i\kappa x}}_{O(\Sigma)} \right\} .$$

The details involved in obtaining the IE-OE are given in Appendix B.

Near Field Problem

In the near field, we shall now formulate a new boundary value problem and obtain its solution. As in the previous near-field problem, we shall assume that derivatives with respect to y and z are $O(\varepsilon^{-1})$. However, we shall also assume that $\phi(x,y,z) = e^{i\kappa x}\psi(x,y,z)$ where $\psi(x,y,z)$ is a slowly varying function of x. The linearized boundary value problem now needs to be rewritten in terms of $\psi(x,y,z)$. First we must determine the proper body boundary condition to be satisfied by the rapidly varying potential. We can restate the body boundary condition, equation (6), as follows:

$$\frac{\partial\phi}{\partial n} = -\frac{h_x}{\sqrt{1 + h_x^2 + h_y^2}} \qquad \text{on} \qquad h(x,y) - z = 0 .$$

If we observe the right-hand side of this equation, we see that it is a slowly varying function of x. No matter what method we use to further linearize the right-hand side, it will still be slowly varying. From this we may conclude that the potential which satisfies this body boundary condition must be slowly varying. Therefore, our rapidly varying potential must satisfy a zero normal velocity condition.

Using this we may rewrite the boundary value problem in terms of $\psi(x,y,z)$ as follows:

$$\underbrace{-\kappa^2\psi}_{O(\psi\varepsilon^{-2})} + \underbrace{2i\kappa\psi_x}_{O(\psi\varepsilon^{-1})} + \underbrace{\psi_{xx}}_{O(\psi)} + \underbrace{\psi_{yy}}_{O(\psi\varepsilon^{-2})} + \underbrace{\psi_{zz}}_{O(\psi\varepsilon^{-2})} = 0 \qquad \text{in the fluid domain,}$$

$$\underbrace{-\kappa^2\psi}_{O(\psi\varepsilon^{-2})} + \underbrace{2i\kappa\psi_x}_{O(\psi\varepsilon^{-1})} + \underbrace{\psi_{xx}}_{O(\psi)} + \underbrace{\kappa\psi_z}_{O(\psi\varepsilon^{-2})} = 0 \qquad \text{on} \qquad z = 0 ,$$

and

$$\underset{O(\psi)}{(i\kappa\psi h_x} + \underset{O(\psi\varepsilon)}{\psi_x h_x} + \underset{O(\psi\varepsilon^{-1})}{\psi_y h_y} - \underset{O(\psi\varepsilon^{-1})}{\psi_z)} \cdot \frac{1}{\sqrt{1 + h_y^2 + h_x^2}} = 0 \qquad \text{on } h(x,y) - z = 0 \,.$$

Employing our previous assumptions concerning the orders of magnitudes of derivatives with respect to x,y,z in the near field, we find that the terms of the boundary value problem for $\psi(x,y,z)$ have the orders of magnitude shown beneath them. Discarding the higher order terms, we obtain the following two-dimensional problem for $\psi(x,y,z)$:

$$\psi_{yy} + \psi_{zz} - \kappa^2\psi = 0 \qquad \text{in the fluid domain ,}$$

$$\psi_z - \kappa\psi = 0 \qquad \text{on } z = 0 \,,$$

and

$$\frac{\partial\psi}{\partial N} = 0 \qquad \text{on } h(x,y) - z = 0 \,.$$

This problem appears to have the trivial solution $\psi(y,z) = 0$ because of the fact that there is a zero normal derivative condition on the body. However, Ursell (1968b) gives a multipole expansion for the solution of the following problem:

$$\psi_{yy} + \psi_{zz} - \kappa^2\psi = 0 \qquad \text{in the fluid domain ,}$$

with

$$\psi_z - \kappa\psi = 0 \qquad \text{on } z = 0$$

which is valid for all r greater than some r_0 . Furthermore, Ursell (1968a) also shows that this solution is nonuniform at infinity. This nonuniformity causes no difficulty in the near field. In fact, it allows us to determine a nontrivial solution to the problem by supplying the terms which match the far-field solution.

The multipole expansion given by Ursell (1968b) is as follows:

$$\psi(y,z) = A_s S_0(y,z) + A^{(e)} e^{\kappa z} + \sum_{1}^{\infty} A_m^{(e)} O_m^{(e)}(y,z) \,.$$

The function $S_0(y,z)$ is the Helmholtz source, which is defined as:

$$S_0(y,z) = \frac{1}{4}\left\{ \int_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \right\} \frac{d\mu \cosh\mu}{\cosh\mu - 1}\, e^{\kappa z \cosh\mu + i\kappa y \sinh\mu} \,, \qquad (20)$$

where $\underset{\smile}{\int}$ and $\overset{\frown}{\int}$ are the integrals below and above the double pole at $\mu = 0$ respectfully. The function $O_m^{(e)}(y,z)$ is an even wave-free potential, which is given as follows:

$$O_m^{(e)}(y,z) = K_{2m-2}(\kappa r)\cos(2m-2)\theta + 2K_{2m-1}(\kappa r)\cos(2m-1)\theta + K_{2m}(\kappa r)\cos 2m\theta ,$$

where $K_m(\kappa r)$ is the modified Bessel function of the second kind.

The asymptotic expansion given by Ursell (1968a) is:

$$\psi(y,z) \sim \underset{O(A_s\varepsilon^{-1})}{- A_s\pi\kappa|y|e^{\kappa z}} + \underset{O(A_s)}{A^{(e)}e^{\kappa z}} \quad \text{as} \quad \kappa r\to\infty .$$

The orders of magnitude given beneath the terms are the orders of magnitude for the OE-IE which are based on the fact that A_s and $A^{(e)}$ must be the same order of magnitude in the near field. This now completes the determination of the OE-IE for the second order problem and we are ready to match our second order solutions.

Matching

In order to facilitate the matching process, we shall again construct a table containing the OE-IE and the IE-OE . As was done previously, we have placed similar terms on the same line, except where the orders of magnitude indicate that this is not valid. Lines (1), (2), and (3) are carried over from solving the first order problem, while lines (4) through (7) are from the second order problem which we have just solved.

When we compare the terms of the OE-IE to those of the IE-OE , we find that the first terms which match are the terms containing $\kappa|y|$ [line (5)]. This allows us to determine the value of $A_s(x)$ as follows:

$$\mathrm{Re}[A_s(x)e^{i\kappa x}] = -\frac{1}{\pi}\,\mathrm{Re}[\Sigma(x)e^{i\kappa x}] = O(\Sigma) .$$

If we now observe the next two lines [lines (6) and (7)], we see that these terms might match. However, based on the order of magnitude relation between $A_s(x)$ and $\Sigma(x)$, these terms cannot match. This is fortunate because we would not expect to be able to fix the relationship between $A_s(x)$ and $A^{(e)}(x)$ and still be able to satisfy the body

TABLE 2

Matching of Second-Order Solution

Outer Expansion of the Inner Expansion	Inner Expansion of the Outer Expansion	
Ux $O(\varepsilon^{1/2})$	Ux $O(\varepsilon^{1/2})$	(1)
$\frac{U}{\pi} A'(x)\log r + Ub(x)$ $O(\varepsilon^{5/2}\log\varepsilon)$ $O(\varepsilon^{5/2})$	$\frac{U}{\pi}\sigma(x)\log r - \frac{U}{2\pi}\int_0^L d\xi\ \sigma'(\xi)\log(2\lvert x-\xi\rvert)\mathrm{sgn}(x-\xi)$ $O(\sigma\varepsilon^{1/2}\log\varepsilon)$ $O(\sigma\varepsilon^{1/2})$	(2)
	$-\frac{U}{4}e^{\kappa z}\int_0^L d\xi\ \sigma'(\xi)\{\mathbf{H}_0[\kappa(x-\xi)] + [2+\mathrm{sgn}(x-\xi)]Y_0(\kappa\lvert x-\xi\rvert)\}$ $O(\sigma\varepsilon^{3/2})$	(3)
	$-U\kappa e^{\kappa z}\mathrm{Re}\left\{e^{i\kappa x-i\pi/4}\int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}}\right\}$ $O(\Sigma)$	(4)
$-U\pi\kappa\lvert y\rvert e^{\kappa z}A_s(x)e^{i\kappa x}$ $O(A_s\varepsilon^{-1/2})$	$+U\kappa\lvert y\rvert e^{\kappa z}\mathrm{Re}\{\Sigma(x)e^{i\kappa x}\}$ $O(\Sigma\varepsilon^{1/2})$	(5)
	$-\frac{U}{2}e^{\kappa z}\mathrm{Re}\{\Sigma(x)e^{i\kappa x}\}$ $O(\Sigma\varepsilon^{1/2})$	(6)
$+Ue^{\kappa z}A^{(e)}(x)e^{i\kappa x}$ $O(A_s\varepsilon^{1/2})$		(7)

boundary condition, $\partial\psi/\partial N = 0$, for arbitrary body shapes.

We have now completed the matching, except that the high order term of the first order IE-OE [line (3)] and the low order term of the second order IE-OE [line (4)] have not been matched. Also we still have not determined the second order source strength, $\Sigma(x)$. At this point we may now conclude that the sum of these two terms must be zero, which leads us to the following integral equation for $\Sigma(x)$:

$$\kappa \operatorname{Re}\left\{ e^{i\kappa x - i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}} \right\} = b_{fs}(x) \quad , \tag{21}$$

where

$$b_{fs}(x) = -\frac{1}{4} \int_0^L d\xi\ \sigma'(\xi)\{H_0[\kappa(x-\xi)] + [2 + \operatorname{sgn}(x-\xi)]Y_0(\kappa|x-\xi|)\} \quad . \tag{22}$$

This now completes the matching of our results and also completes the solution of the second order problem. We will now discuss the behavior of our solution to the wave making problem.

Summary of Results

In this section, we shall study the behavior of the second order source strength near the bow and develop equations for the velocity components and wave elevation in the near field. If we write the second order source strength as $\Sigma(x) = \Sigma_r(x) + i\Sigma_i(x)$ and write only the real part of the integral equation, equation (21), we obtain:

$$\kappa \cos(\kappa x - \pi/4) \int_0^x \frac{d\xi\ \Sigma_r(\xi)}{\sqrt{2\pi\kappa(x-\xi)}} - \kappa \sin(\kappa x - \pi/4) \int_0^x \frac{d\xi\ \Sigma_i(\xi)}{\sqrt{2\pi\kappa(x-\xi)}} = b_{fs}(x) \ .$$

The functions $\Sigma_r(x)$ and $\Sigma_i(x)$ are indeterminate because we have only one equation with two unknowns. Therefore we need to determine an integral equation in only one unknown. In Appendix B we have done this by developing the IE-OE in terms of $\operatorname{Re}[\Sigma(x)e^{i\kappa x}]$. If we return to this form of the IE-OE , equation (B27), we can rewrite our integral equation as:

$$2\kappa \int_0^x \frac{d\xi\ \operatorname{Re}[\Sigma(\xi)e^{i\kappa\xi}]\cos[\kappa(x-\xi) - \pi/4]}{\sqrt{2\pi\kappa(x-\xi)}} = b_{fs}(x) \ . \tag{23}$$

This is a single integral equation in one unknown, $\operatorname{Re}[\Sigma(x)e^{i\kappa x}]$. We will call this unknown $R(x)$ and shall investigate its behavior as

$\kappa x \to 0$, which is the limit as we approach the bow.

In studying the limiting behavior of the source strength as we approach the bow, we will let δ be some small distance $O(\varepsilon^2)$ from the bow. We can rewrite the integral equation in terms of δ as follows:

$$2\kappa \int_0^\delta \frac{d\xi\ R(\xi)\cos[\kappa(\delta-\xi) - \pi/4]}{\sqrt{2\pi\kappa(\delta-\xi)}} = b_{fs}(\delta) \ .$$

We may now expand the $\cos[\kappa(\delta-\xi) - \pi/4]$ and rewrite the integral equation yet again as:

$$2\kappa\ \cos(\kappa\delta-\pi/4) \int_0^\delta \frac{d\xi\ R(\xi)\cos(\kappa\xi)}{\sqrt{2\pi\kappa(\delta-\xi)}} + 2\kappa\ \sin(\kappa\delta-\pi/4) \int_0^\delta \frac{d\xi\ R(\xi)\sin(\kappa\xi)}{\sqrt{2\pi\kappa(\delta-\xi)}} = b_{fs}(\delta) \ .$$

Because we have chosen $\delta = O(\varepsilon^2)$, we know that $\kappa\delta, \kappa\xi = O(\varepsilon)$. Therefore we may neglect $\kappa\delta$ with respect to $\pi/4$ and set $\cos \kappa\xi = 1$ and $\sin \kappa\xi = 0$ over the interval of integration. This allows us to simplify our integral equation as follows:

$$2\kappa\ \cos(-\pi/4) \int_0^\delta \frac{d\xi\ R(\xi)}{\sqrt{2\pi\kappa(\delta-\xi)}} \sim b_{fs}(\delta) \ .$$

This is the classic Abel integral equation [Tricomi (1957)] which we can solve analytically. Writing the solution to this integral equation, we obtain:

$$R(\delta) = \frac{b_{fs}(0)}{\sqrt{\pi\kappa\delta}} + \frac{1}{\sqrt{\pi\kappa}} \int_0^\delta \frac{d\xi\ b'_{fs}(\xi)}{\sqrt{\delta-\xi}} \qquad \text{for} \qquad \delta = O(\varepsilon^2) \ .$$

From this we observe that $R(x)$ has a square root singularity at the bow unless $b_{fs}(0) = 0$, and that $R(x)$ is not bounded near the bow unless $b_{fs}(0)$ is finite.

We shall now develop the equations for the various velocity components and the wave elevation in the near field. In light of our change in the form of the integral equation for the second order source strength, the matching for $A_s(x)$ will also change. Making the change from $\Sigma(x)$ to $R(x)$, we obtain:

$$\mathrm{Re}[A_s(x)e^{i\kappa x}] = -R(x)/\pi \ .$$

In the same vein we shall redefine our multipole solution of the two-dimensional Helmholtz problem to be a series with $A_s = 1$ so that the entire solution of the problem can be multiplied by $-R(x)/\pi$.

Incorporating these changes into our formulas, we can write our total potential in the near field as:

$$\Phi(x,y,z) = \underset{O(\varepsilon^{1/2})}{Ux} + \underset{O(\varepsilon^{5/2})}{U\phi_{rw}(x,y,z)} - \underset{O(\varepsilon^{4})}{UR(x)\psi(y,z)/\pi} , \tag{24}$$

where $\phi_{rw}(x,y,z)$ is the solution to the "rigid" wall given by equations (16), (17), and (18). We can now differentiate this equation with respect to x,y,z and obtain the velocity components in these directions. We find that these velocity components are as follows:

$$\Phi_x(x,y,z) = \underset{O(\varepsilon^{1/2})}{U} + \underset{O(\varepsilon^{5/2})}{U\phi_{rw_x}(x,y,z)} - \underset{O(\varepsilon^{3})}{UR'(x)\psi(y,z)/\pi} , \tag{25}$$

$$\Phi_y(x,y,z) = \underset{O(\varepsilon^{3/2})}{U\phi_{rw_y}(x,y,z)} - \underset{O(\varepsilon^{3})}{UR(x)\psi_y(y,z)} , \tag{26}$$

and

$$\Phi_z(x,y,z) = \underset{O(\varepsilon^{3/2})}{U\phi_{rw_z}(x,y,z)} - \underset{O(\varepsilon^{3})}{UR(x)\psi_z(y,z)} . \tag{27}$$

If we return to equation (7), we find that we may write the non-dimensional wave elevation as follows:

$$\frac{g\zeta}{\frac{1}{2}U^2} = -\{[1 + \phi_{rw_x} - R'\psi/\pi]^2 + [\phi_{rw_y} - R\psi_y/\pi]^2 + [\phi_{rw_z} - R\psi_z/\pi]^2\} + 1 .$$

Squaring the terms in the above equation and keeping the low order terms, we obtain:

$$\frac{g\zeta}{\frac{1}{2}U^2} = -\{\underset{O(\varepsilon^{2})}{2\phi_{rw_x}(x,y,0)} - \underset{O(\varepsilon^{5/2})}{2R'(x)\psi(y,0)/\pi} + \underset{O(\varepsilon^{2})}{\phi^2_{rw_y}(x,y,0)}\} . \tag{28}$$

This completes our discussion of the solution of the second order problem and summarizes our results.

NUMERICAL ANALYSIS

In order to make use of the above equations describing the flow near the ship, it is necessary to resort to numerical calculations. While most of the computations are straightforward, there are two areas which pose some difficulty. One is the determination of the second order source strength and the other is the solution of the two-dimensional Helmholtz equation. We shall discuss these problems in the following sections.

Determining the Second Order Source Strength

In the previous subsection, we determined that the integral equation, equation (23), is as follows:

$$2\kappa \int_0^x \frac{d\xi \; R(\xi)\cos[\kappa(x-\xi) - \pi/4]}{\sqrt{2\pi\kappa(x-\xi)}} = b_{fs}(x) \quad ,$$

where $b_{fs}(x)$ is given by equation (22). Despite the logarithmic singularity in $Y_0(\kappa|x-\xi|)$, the evaluation of $b_{fs}(x)$ is a straightforward problem in numerical integration. Therefore, we shall concentrate on the solution of the integral equation.

Let the interval from 0 to 1 be broken up into N subintervals. (These intervals do not need to be uniform, and in fact, smaller intervals are desirable near the bow.) Then we can rewrite our integral equation as:

$$\sum_{i=1}^{n} 2\kappa \int_{x_{i-1}}^{x_i} \frac{d\xi \; R(\xi)\cos[\kappa(x_n-\xi) - \pi/4]}{\sqrt{2\pi\kappa(x_n-\xi)}} = b_{fs}(x_n)$$

for $n = 1,2,\cdots,N$. We may now apply a special form of the Mean Value Theorem for integrals, Rudin (1964), to the integrals over the subintervals. Doing this we obtain the following equation:

$$\sum_{i=1}^{n} 2\kappa \, R(x_i^*) \int_{x_{i-1}}^{x_i} \frac{d\xi \, \cos[\kappa\,(x_n-\xi) - \pi/4]}{\sqrt{2\pi\kappa(x_n-\xi)}} = b_{fs}(x_n) \ ,$$

where $x_{i-1} \le x_i^* \le x_i$. We may now make a change of variables, $u = \kappa(x_n-\xi)$, and apply a trigonometric identity, $\cos(u - \pi/4) = \cos u \cos \pi/4 + \sin u \sin \pi/4$. This yields:

$$\sum_{i=1}^{n} R(x_i^*) \left\{ \int_{\kappa(x_n-x_i)}^{\kappa(x_n-x_{i-1})} \frac{du \cos u}{\sqrt{2\pi u}} + \int_{\kappa(x_n-x_i)}^{\kappa(x_n-x_{i-1})} \frac{du \sin u}{\sqrt{2\pi u}} \right\} = \frac{b_{fs}(x_n)}{\sqrt{2}} \ . \tag{29}$$

These integrals are Fresnel integrals [see Abramowitz and Stegun (1964)], a fact which allows us to write our integral equation as follows:

$$\sum_{i=1}^{n} R(x_i^*)\{C_2[\kappa(x_n-x_{i-1})] - C_2[\kappa(x_n-x_i)] + S_2[\kappa(x_n-x_{i-1})]$$
$$- S_2[\kappa(x_n-x_i)]\} = \frac{b_{fs}(x_n)}{\sqrt{2}} \ . \tag{30}$$

The Fresnel integrals $C_2[\kappa(x_n-x_{i-1})]$ and $C_2[\kappa(x_n-x_i)]$ come from integrating the first integral in equation (29); the Fresnel integrals $S_2[\kappa(x_n-x_{i-1})]$ and $S_2[\kappa(x_n-x_i)]$ come from the second integral in equation (29).

Equation (30) is an exact equation, although we do not know the points x_i^* where the function $R(x)$ is evaluated. Therefore, we shall now make our only assumption in solving for $R(x)$. We shall assume that:

$$x_i^* = \frac{x_{i-1} + x_i}{2} \ .$$

If we set

$$R_i = R(x_i^*) \ ,$$

and

$$C_{ni} = C_2[\kappa(x_n-x_{i-1})] - C_2[\kappa(x_n-x_i)] + S_2[\kappa(x_n-x_{i-1})] - S_2[\kappa(x_n-x_i)] \ ,$$

we can rewrite the integral in the following abbreviated form:

$$\sum_{i=1}^{n} R_i C_{ni} = \frac{b_{fs}(x_n)}{\sqrt{2}} \qquad \text{for} \qquad n = 1,\cdots,N \ .$$

By first separating the term $R_n C_{nn}$ from the summation and then carrying the rest of the sum to the right-hand side, and finally by dividing both sides by C_{nn} , we obtain the following equation:

$$R_n = \left\{ \frac{b_{fs}(x_n)}{\sqrt{2}} - \sum_{i=1}^{n-1} R_i C_{ni} \right\} / C_{nn}, \quad n = 1, \cdots, N \ .$$

We may use this equation to determine $R(x_n^*)$ in a recursive manner, and if we have chosen the x_i close enough to each other, we may interpolate for values of $R(x)$ at additional points. Likewise, we may determine the derivative of $R(x)$ at arbitrary points by means of numerical differentiation.

Three curves of $R(x)$ for various wave numbers are given in Figure 4. The body to which these curves correspond is a body of revolution with pointed ends. (The details of the body geometry are given in the section COMPARISON WITH EXPERIMENTS.) These curves have been calculated using 114 points over the length of the ship. A spacing of 0.005 was used for $0.0 \leq x \leq 0.15$, and a spacing of 0.01 was used for $0.15 \leq x \leq 0.99$. Corresponding calculations with only 33 points over the length of the ship resulted in the same degree of accuracy for $x \leq 0.75$ at which point the results started to wander. This completes the discussion of the numerical solution of the integral equation for the second order source strength.

Solving the Two-Dimensional Helmholtz Equation

As we found previously, the second order near-field boundary value problem is as follows:

$$\psi_{yy} + \psi_{zz} - \kappa^2\psi = 0 \qquad \text{in the fluid domain,}$$

$$\psi_z - \kappa\psi = 0 \qquad \text{on } z = 0 \ ,$$

and

$$\frac{\partial\psi}{\partial N} = 0 \qquad \text{on } h(x,y) - z = 0 \ .$$

For the above boundary value problem, Ursell (1968b) provides the follow following multipole solution:

$$\psi(y,z) = A_s S_0(y,z) + A^{(e)} e^{\kappa z} + \sum_1^\infty A_m^{(e)} O_m^{(e)}(y,z) \ .$$

The matching at infinity provides the value of $A_s [A_s(x) e^{i\kappa x} = - R(x)/\pi]$.

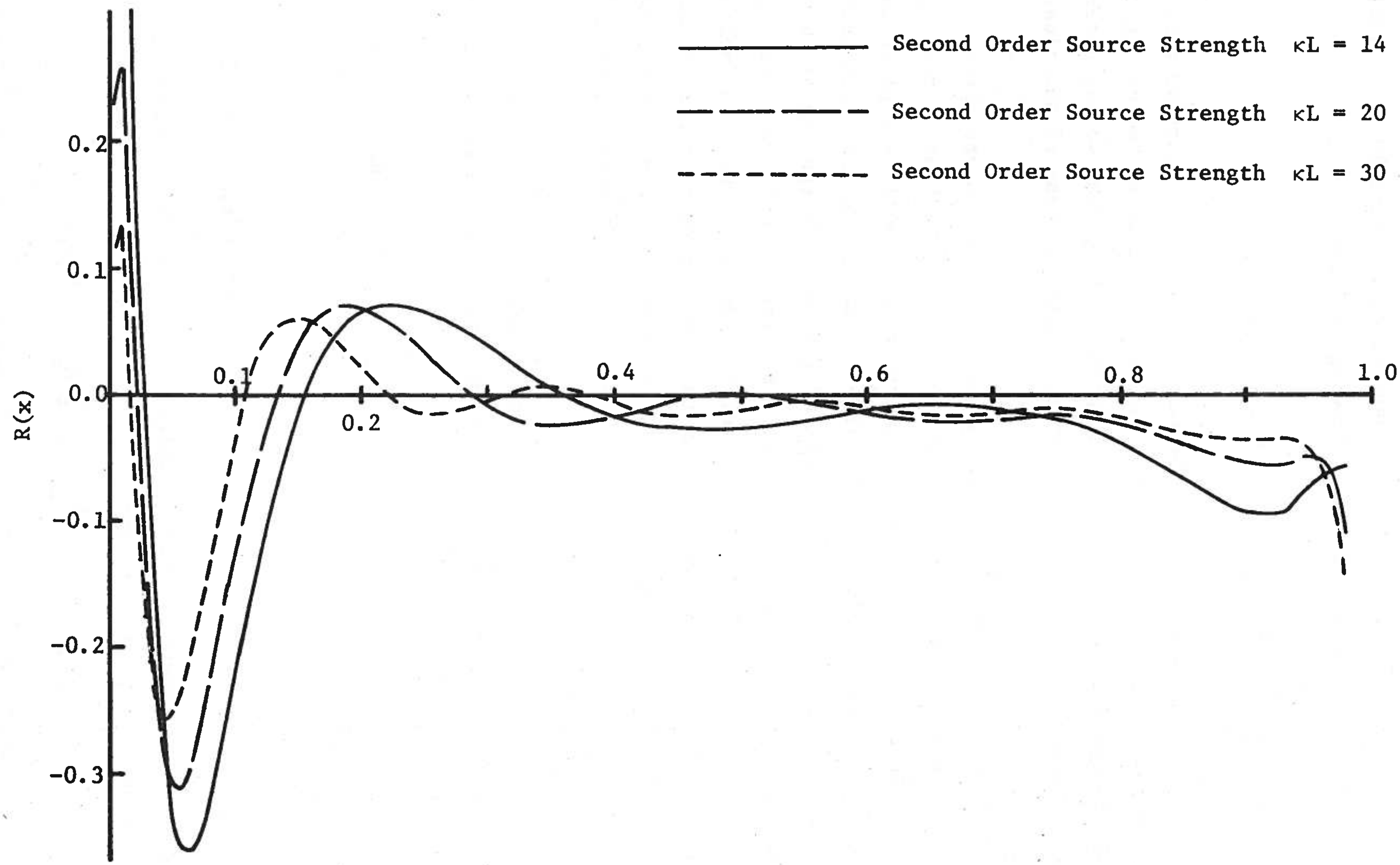

Figure 4: Rapidly Varying Source Strength from Second Order Slender-Body Theory at Three Non-Dimensional Speeds.

With A_s thus determined, it is only necessary to determine $A^{(e)}$ and $A_m^{(e)}$, such that the zero normal velocity condition is met on the body.

There are two methods which can be used to determine the $A^{(e)}$ and $A_m^{(e)}$. One could choose a number of points at which to satisfy the body boundary condition, and then determine an equal number of the unknown coefficients so as to satisfy the body boundary condition exactly at these points. Alternatively, one could choose to satisfy the body boundary conditions using fewer coefficients than points, and employ a least-squares technique to minimize the error. The success of either of these techniques relies on the coefficients $[A_m^{(e)}(x)]$ decreasing rapidly enough for increasing m so that the solution can be represented reasonably with a small number of terms.

The method we have chosen is the former because of its relative ease of computation and greater efficiency. The only danger in selecting this approach would be if the potential is not well behaved around the body, in which case the least-squares technique might give better results. As a check of this, a comparison was made with the solution of the same problem as calculated by Faltinsen using the least-squares technique, and it was found that the results agreed to within two to three per cent. Also, as a check of the rate of convergence of the series $\sum_1^\infty A_m^{(e)} O_m^{(e)}(r,\theta)$, the solution was calculated using both seven and ten terms and the solution changed by less than one per cent.

We shall now discuss the technique used to evaluate the potential due to a Helmholtz source and its corresponding normal velocity.

The Potential Due to a Helmholtz Source

For the following problem,

$$\nabla_{2D}^2\psi - \kappa^2\psi = 0 \qquad\qquad y < 0$$

and

$$\kappa\psi - \frac{\partial\psi}{\partial y} = 0 \qquad \text{on} \qquad y = 0 \;,$$

Ursell (1968b) gives the Green's function as:

$$S_0(\kappa x,\kappa y;\kappa\xi,\kappa\eta) = K_0[\kappa\sqrt{(x-\xi)^2 + (y-\eta)^2}]$$

$$+ \frac{1}{4}\left\{\int_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\right\}\frac{d\mu(\cosh\mu+1)}{\cosh\mu-1}\, e^{i\kappa(x-\xi)\sinh\mu + \kappa(y-\eta)\cosh\mu} \;,$$

where $\displaystyle\oint_{-\infty}^{\infty}$ and $\displaystyle\oint_{-\infty}^{\infty}$ denote integration below and above the double pole at $\mu = 0$ respectively. (Note: We have used y-positive upwards, as opposed to Ursell's y-positive downwards.) Faltinsen (1971) integrates $S_0(\kappa x, \kappa y; 0, 0)$ by means of contour integration, and obtains the following form for S_0 :

$$S_0(\kappa r, \theta) = -\pi\kappa r \sin\theta\, e^{-\kappa r \cos\theta} + \int_0^\infty \frac{du\, u^2 \cos(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}} , - \int_0^\infty \frac{du\, u \sin(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}} , \tag{30}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(x/y)$. However, it is possible to further simplify equation (30) by means of the following identity:

$$K_0(\kappa r) = \int_0^\infty \frac{du \cos(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{1/2}} = \int_0^\infty \frac{du\, (1+u^2) \cos(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}} . \tag{31}$$

If we now both add and subtract $K_0(\kappa r)$, equation (31), from the first integral of equation (30), we obtain:

$$S_0(\kappa r, \theta) = K_0(\kappa r) - \pi\kappa r \sin\theta\, e^{-\kappa r \cos\theta} - \int_0^\infty \frac{du \cos(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}} - \int_0^\infty \frac{du\, u \sin(u\kappa r \cos\theta) e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}} . \tag{32}$$

As a further check of our results, we can let $\theta = 0$ in equation

(32), which yields :

$$S_0(\kappa r,0) = K_0(\kappa r) - \int_0^\infty \frac{du\ \cos(u\kappa r)}{(1+u^2)^{3/2}} - \int_0^\infty \frac{du\ u\ \sin(u\kappa r)}{(1+u^2)^{3/2}} \ .$$

Using equation (9.6.25) from Abramowitz and Stegun (1964), we see that the first integral is equal to $\kappa r K_1(\kappa r)$. Faltinsen (1971) has shown that the second integral is equal to $\kappa r K_0(\kappa r)$. So for $\theta = 0$, we have:

$$S_0(\kappa r,0) = (1 - \kappa r)K_0(\kappa r) - \kappa r K_1(\kappa r) \ .$$

We can now obtain a simplified form for the radial velocity in a similar manner. Differentiating equation (32) with respect to κr , we obtain the radial velocity as:

$$\frac{\partial S_0}{\partial \kappa r}(\kappa r,\theta) = - K_1(\kappa r) - \pi \sin\theta\, e^{-\kappa r \cos\theta}(1 - \kappa r \cos\theta)$$

$$+ \int_0^\infty \frac{du\ \cos(u\kappa r\cos\theta)e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}}(\sqrt{1+u^2}\sin\theta - u^2\cos\theta)$$

$$+ \int_0^\infty \frac{du\ u\ \sin(u\kappa r\cos\theta)e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}}(\sqrt{1+u^2}\sin\theta + \cos\theta) \ . \tag{32}$$

If we now multiply $K_0(\kappa r)$, equation (31), by $\cos\theta$, and add and subtract it from the first integral of equation (33), we obtain:

$$\frac{\partial S_0}{\partial \kappa r}(\kappa r,\theta) = - K_1(\kappa r) - K_0(\kappa r)\cos\theta - \pi \sin\theta\, e^{-\kappa r \cos\theta}(1 - \kappa r \cos\theta)$$

$$+ \int_0^\infty \frac{du\ \cos(u\kappa r\cos\theta)e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}}(\sqrt{1+u^2}\sin\theta + \cos\theta)$$

$$+ \int_0^\infty \frac{du\ u\ \sin(u\kappa r\cos\theta)e^{-\kappa r \sin\theta\sqrt{1+u^2}}}{(1+u^2)^{3/2}}(\sqrt{1+u^2}\sin\theta + \cos\theta) \ . \tag{34}$$

We can again obtain an analytic integration for $\theta = 0$. If we set $\theta = 0$ in equation (34), we find that:

$$\frac{\partial S_0}{\partial \kappa r}(\kappa r,0) = -K_1(\kappa r) - K_0(\kappa r) + \int_0^\infty \frac{du \cos(u\kappa r)}{(1+u^2)^{3/2}} + \int_0^\infty \frac{du\ u \sin(u\kappa r)}{(1+u^2)^{3/2}} \ .$$

Evaluating these integrals in the same manner as for $S_0(\kappa r,0)$, we obtain:

$$\frac{\partial S_0}{\partial \kappa r}(\kappa r,0) = (\kappa r - 1)[K_0(\kappa r) + K_1(\kappa r)] \ .$$

The integral for $S_0(\kappa r,\theta)$, equation (32), and the integral for $\partial S_0(\kappa r,\theta)/\partial \kappa r$, equation (34), can both be evaluated by means of numerical integration. Figures 5 and 6 show the variation of $S_0(\kappa r,\theta)$ and $\partial S_0(\kappa r,\theta)/\partial \kappa r$ as functions of κr for several values of θ . The asymptotic curves shown on these figures are the functions $-\pi \kappa r \sin\theta\ e^{-\kappa r \cos\theta}$ and $-\pi \sin\theta\ e^{-\kappa r \cos\theta}(1 - \kappa r \cos\theta)$, respectively. These asymptotic limits are valid for $\kappa r \sin\theta \to \infty$. These calculations complete the section on numerical analysis.

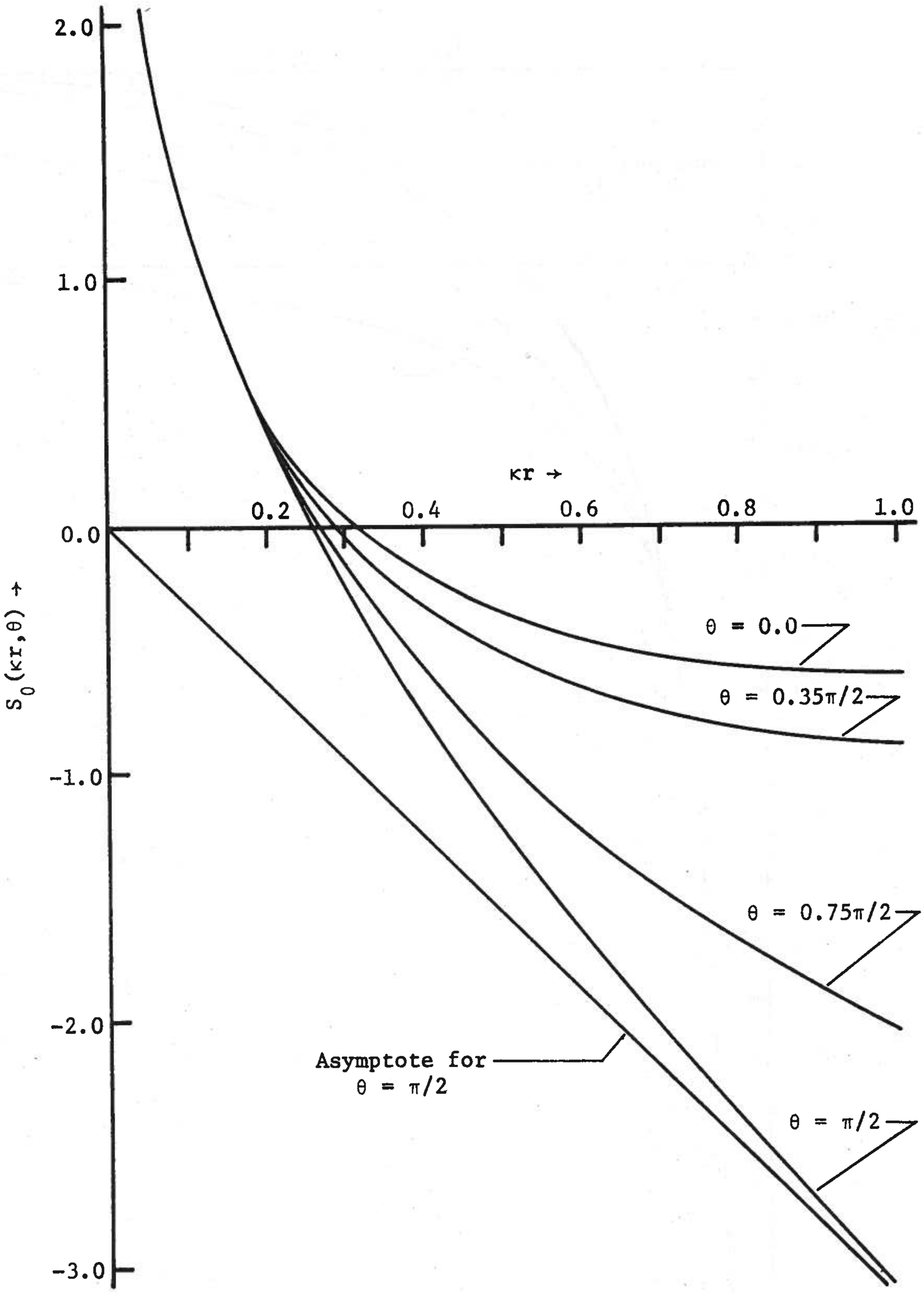

Figure 5: Contours of $S_0(\kappa r,\theta)$ Vs. κr for Various Values of θ .

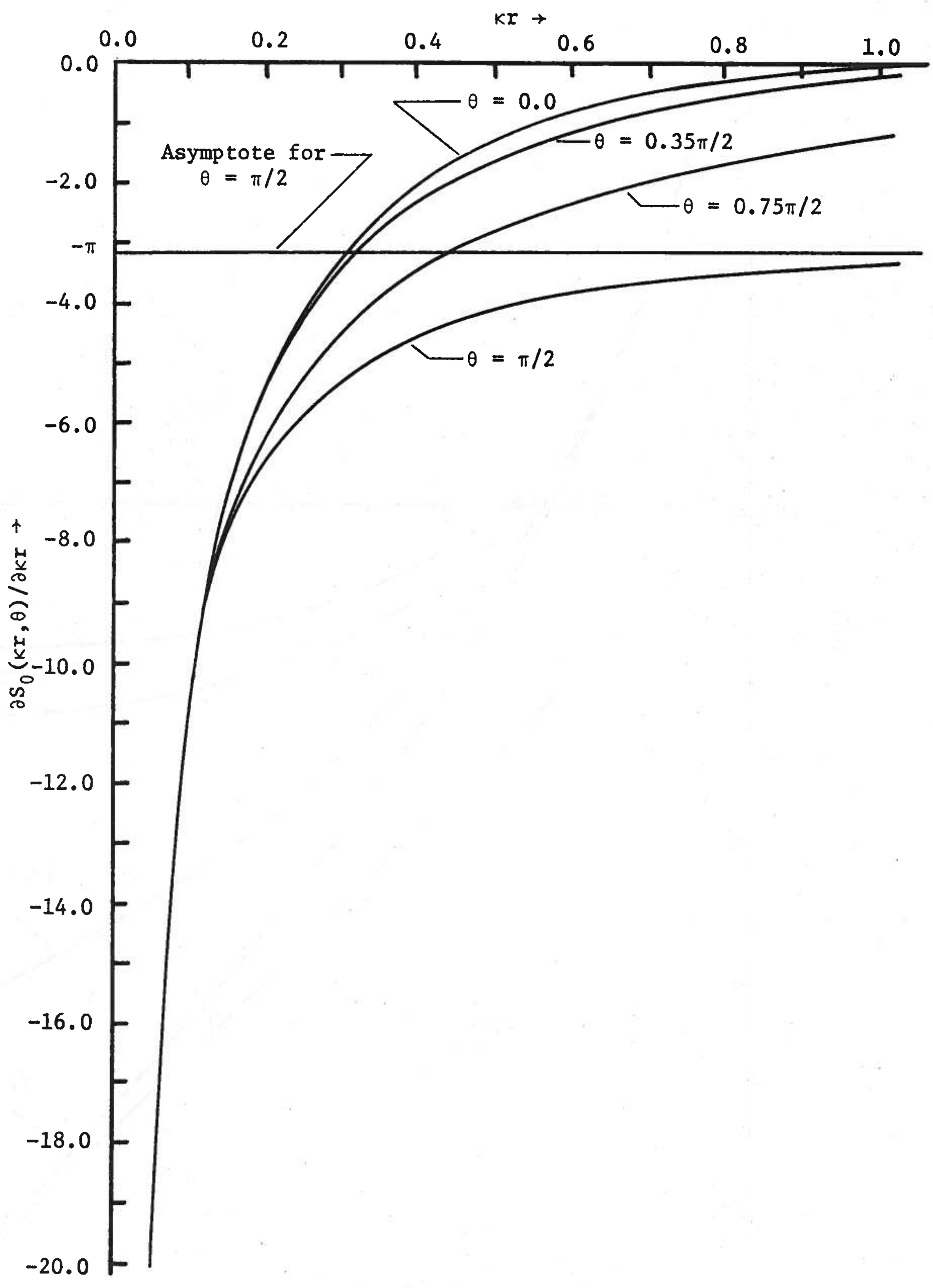

Figure 6: Contours of $\partial S_0(\kappa r,\theta)/\partial \kappa r$ Vs. κr for Various Values of θ .

COMPARISON WITH EXPERIMENTS

In order to obtain a measure of the validity of our results, it was decided that a comparison should be made with experimental data. As we saw earlier, the second order source strength behaves like:

$$R(x) = \frac{b_{fs}(0)}{\sqrt{\pi\kappa x}} + \frac{1}{\sqrt{\pi\kappa}} \int_0^x \frac{d\xi\, b'_{fs}(\xi)}{\sqrt{x-\xi}} \qquad \text{for} \qquad x = O(\varepsilon^2) \quad ,$$

and as we stated earlier, our solution depends critically on the value of $b_{fs}(0)$. If $b_{fs}(0)$ is infinite, then the source strength is infinite, at least near the bow. Experience with $b_{fs}(x)$ shows that for an arbitrary body, $b_{fs}(x)$ has a term which behaves like $\sigma(0)Y_0(\kappa x)$, which comes from the discontinuity of $\sigma(x)$ at $x = 0$; this term tends to infinity logarithmically as $x \to 0$. However, if we demand that the body be pointed, $\sigma(x)$ has no discontinuity at $x = 0$, and the corresponding $b_{fs}(0)$ is finite. With this information as a guide, it was decided to compare our results with experiments on a body of revolution.

After a brief investigation, it was determined that there was no existing data for bodies of revolution with pointed ends and a parallel middle body. Therefore, a decision was made to conduct our own experiments to obtain the necessary data. For a model we chose a parabola of degree 2n to represent the radius of the body as a function of longitudinal position. This function is as follows:

$$r(x) = r_0(1-x^{2n}) \qquad \text{for} \qquad -1 < x < 1 \ .$$

This representation has the advantage of allowing us to introduce an apparent parallel middle body (by choosing n large enough) and still maintain a continuous function throughout. We found that $n = 5$ would give about 60% of the body as parallel middle body. We chose $r_0 = 0.10$ to give us a beam-length ratio of one tenth $(B/L = 0.10)$.

The radius distribution given above was then scaled to give a model 15 feet in length with a beam of 1.5 feet. This model was built of fiberglass stretched over a plywood frame. The model, which was fixed to the carriage so that it could not sink or trim, was towed in the towing tank and wave measurements were taken 2 inches from the side of the model using a resistance wire wave probe. These wave records were recorded on a strip-chart recorder, digitized manually, and then non-dimensionalized by $g/\frac{1}{2}U^2$.

These non-dimensionalized points are plotted in Figures 7, 8, and 9 along with the wave elevations predicted by our second order slender-body theory and the Tuck Slender-Body Theory. The second order wave elevations are obtained by evaluating equation (28), while the Tuck calculations are obtained by evaluating the following equation:

$$\frac{g\zeta}{\frac{1}{2}U^2} = -\,[2\phi_{rw_x}(x,y,0) + 2b'_{fs}(x) + \phi^2_{rw_y}(x,y,0)] \quad ,$$

where $b'_{fs}(x)$ is the derivative of $b_{fs}(x)$ given by equation (22).

The curves of wave elevation from the second order theory have been truncated in the stern region because they start to oscillate rapidly with increasing amplitude. This oscillation, which is caused by $R(x)$ becoming irregular, probably has two causes. First, the values of $R(x)$ will tend to drift as we move toward the stern due to an accumulation of errors in the calculation of previous values. Second, as we approach the stern, $b_{fs}(x)$ starts to grow due to the waves generated by the stern section. These waves may be out of phase with the waves from the bow, which in turn results in $R(x)$ having to change phase. This is further complicated by the fact that the formula for $R(x)$ contains a memory term which includes information on the earlier phase. The combination of these two effects could well cause the irregularities found in the second order wave elevation.

If we disregard the stern region and compare our three sets of curves, we see that the phase of the waves from the second order theory lag behind the waves from the Tuck theory by about $\pi/2$. At the same time, we see that the second order phase agrees very well with the experimental one over the middle region of our curves. The amplitude of the second order waves over the midbody is much less than that due to the Tuck theory and agrees fairly well with the experimental curves. Both the second order theory and the Tuck theory grossly overestimate

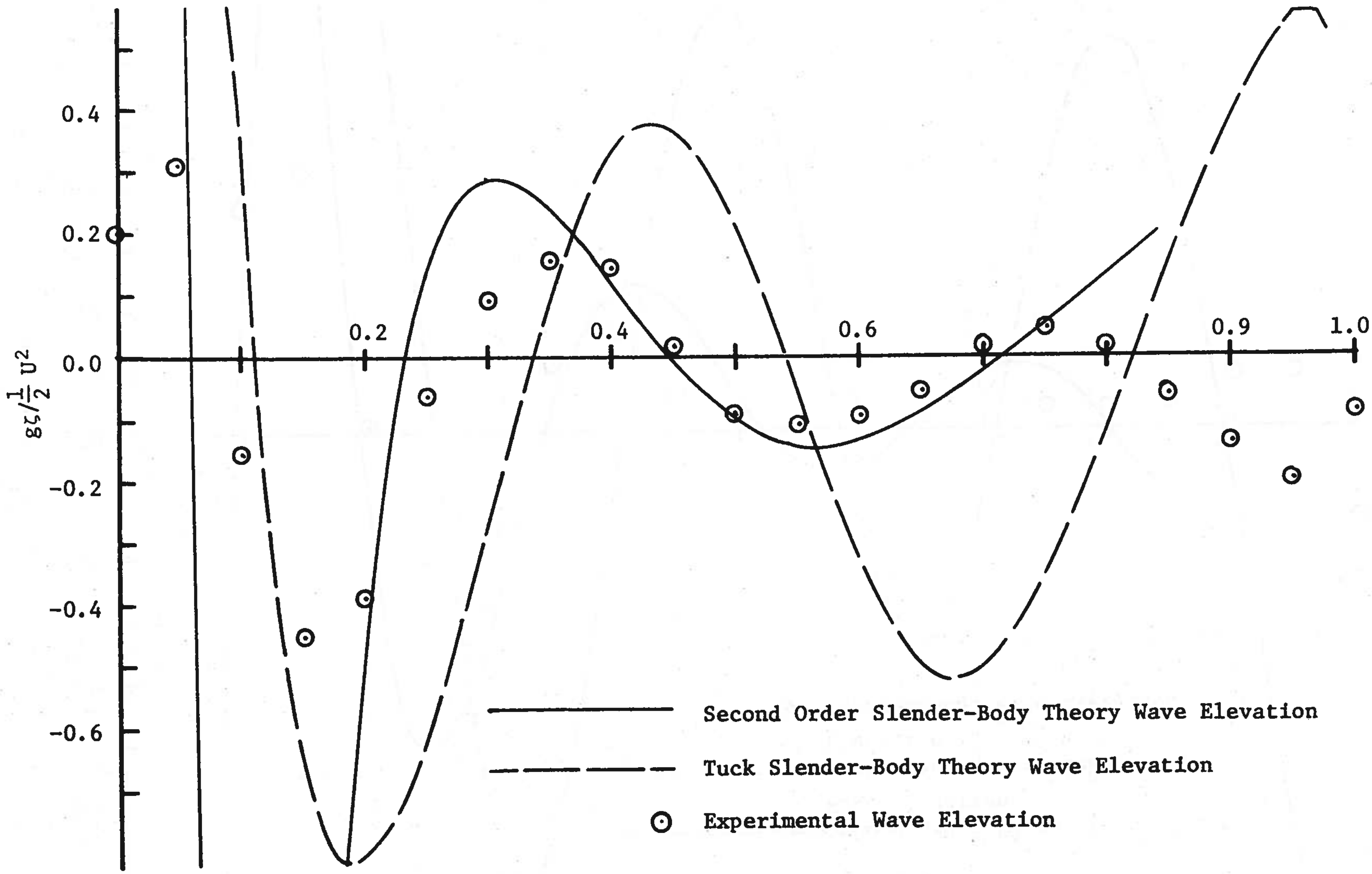

Figure 7: Comparison of Wave Elevations Between Second Order Slender-Body Theory, Tuck Slender-Body Theory, and Experimental Data at a Non-Dimensional Speed of $\kappa L = 14$.

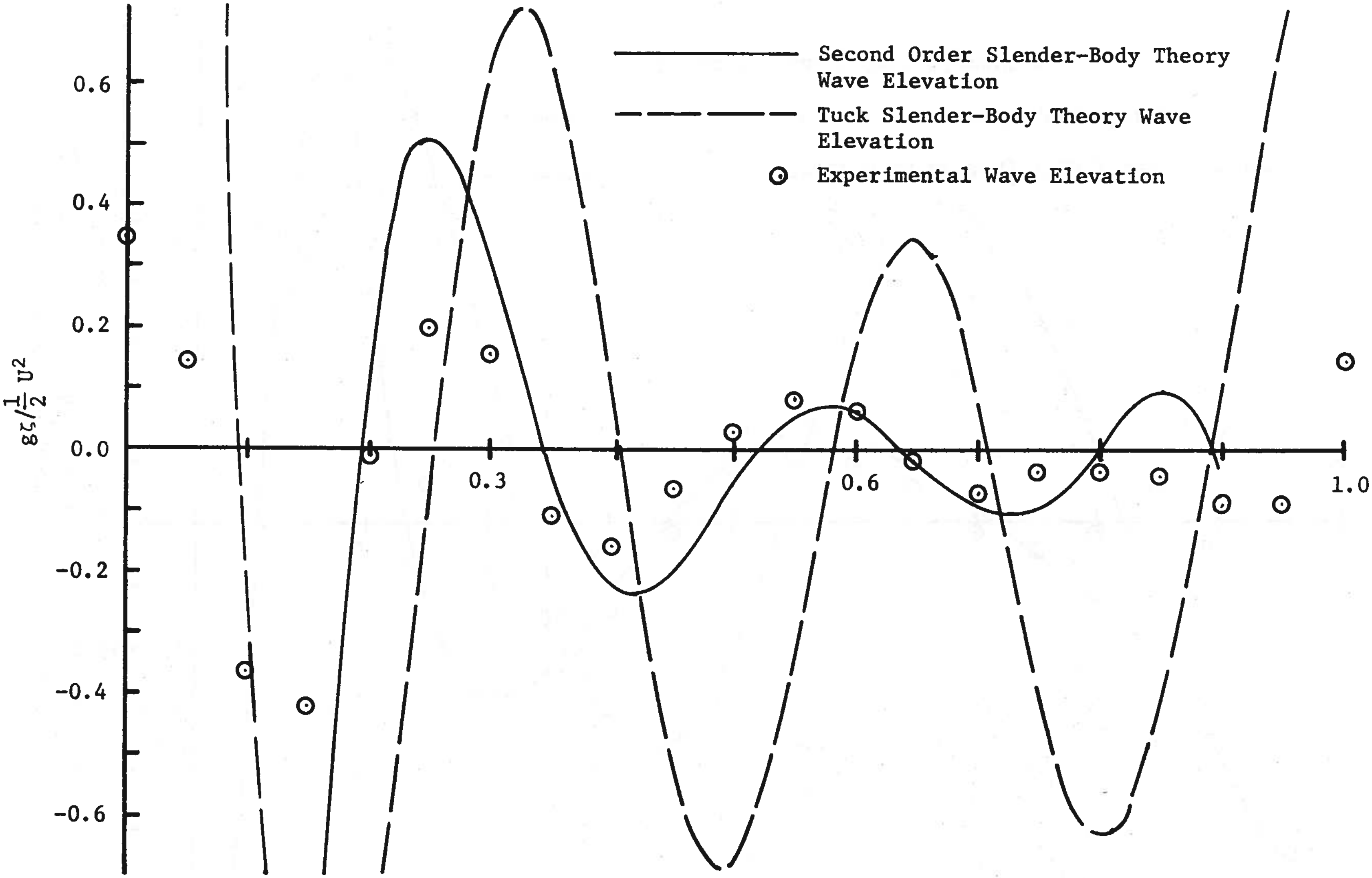

Figure 8: Comparison of Wave Elevations Between Second Order Slender-Body Theory, Tuck Slender-Body Theory, and Experimental Data at a Non-Dimensional Speed of $\kappa L = 20$.

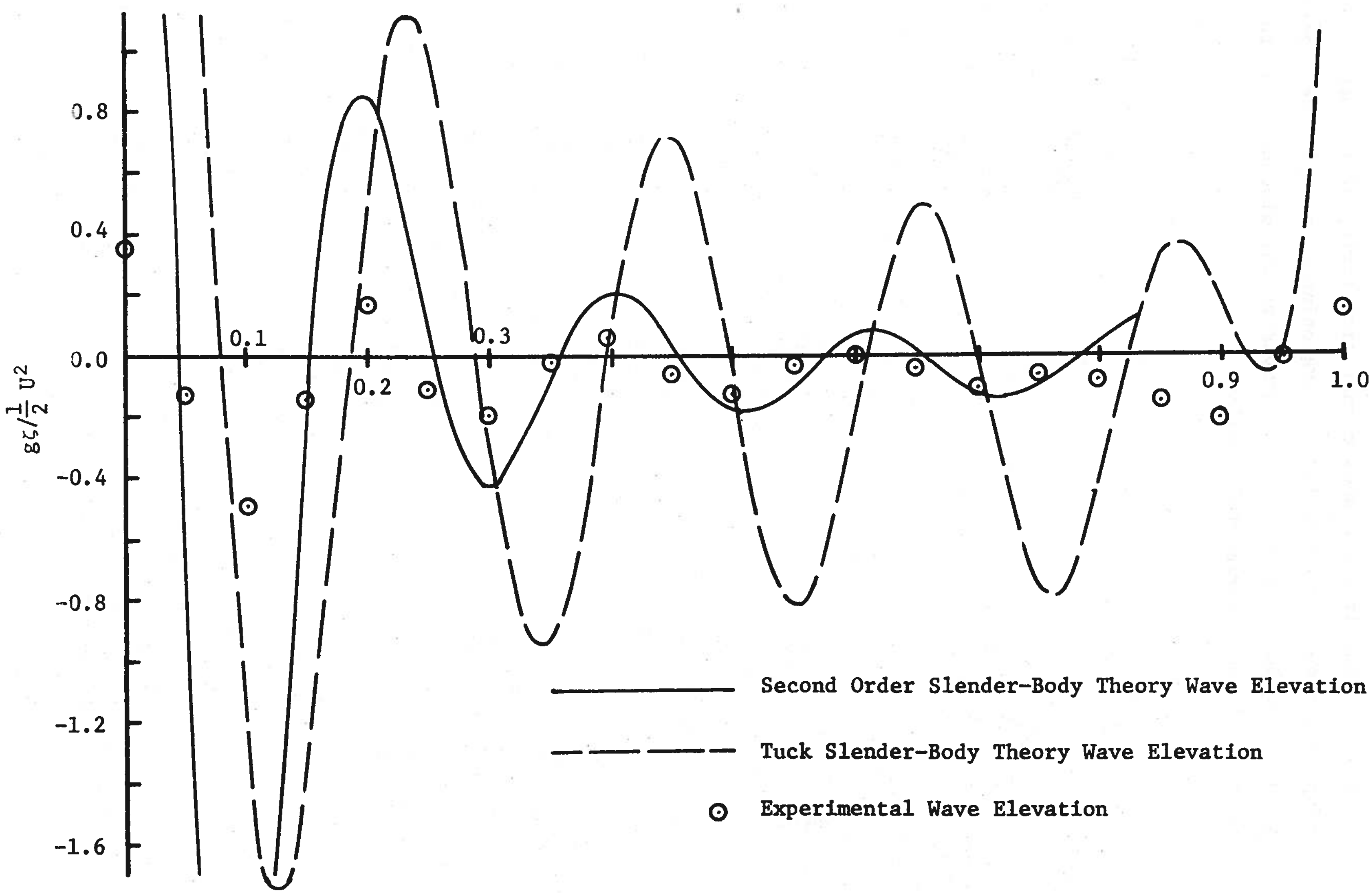

Figure 9: Comparison of Wave Elevations Between Second Order Slender-Body Theory, Tuck Slender-Body Theory, and Experimental Data at a Non-Dimensional Speed of $\kappa L = 30$.

wave amplitude in the bow region. This is not unexpected of the second order theory since it has a square root singularity at the bow. These findings point out the necessity of developing more fully the theory for flow about the bow of a ship. This completes our discussion of the computational and experimental results.

APPENDIX A: INNER EXPANSION OF THE OUTER EXPANSION FOR A SLOWLY VARYING SOURCE DISTRIBUTION

In this appendix we shall show how the inner expansion of the outer expansion equation (11a & b) for the potential due to the slowly varying source distribution, equation (10), is obtained. In the body of the paper, we gave this potential as:

$$\phi(x,y,z) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk\; e^{ikx}\sigma^*(k)\int_{-\infty}^{\infty}\frac{d\ell\; e^{i\ell y}e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2}-k^2/\kappa}\;. \tag{A1}$$

We shall do most of our work with the Green's function*—really its Fourier transform—which we can write as:

$$G^{**}(k,\ell;z) = -\frac{e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2}-k^2/\kappa}\;.$$

If we invert $G^{**}(k,\ell;z)$ with respect to y, we have

$$G^*(k;y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\ell\; e^{i\ell y}e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2}-k^2/\kappa}\;, \tag{A2}$$

and if we note that the integrand of $G^*(k;y,z)$ is even with respect to ℓ, we can rewrite the $e^{i\ell y}$ as $\cos(\ell y)$. By doing this and by making a change of variables,

$$\ell = |k|\sinh\mu$$
$$d\ell = |k|\cosh\mu\; d\mu\;,$$

we can rewrite equation (A2) as:

---

*By Green's function, we mean the fundamental solution to a partial differential equation. This solution may satisfy some of the boundary conditions of the problem. In our particular case, the Green's function satisfies both the linearized free-surface boundary condition and the radiation condition.

$$G^*(k;y,z) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\mu\ \cosh\mu}{\cosh\mu - |k|/\kappa} \cos(|k|y\ \sinh\mu) e^{|k|z\ \cosh\mu} ,$$

where this integral must be interpreted as a contour integral, and the proper path chosen depending on whether k is positive or negative. For small values of $|k|r$, Ursell (1962) gives the following asymptotic expansion of this integral:

$$\begin{aligned} G^*(k;y,z) = -\frac{1}{\pi} &\left\{ \begin{matrix} (-\pi i\ \text{sgn}\ k-\alpha)\text{ctnh}\ \alpha \\ (\pi-\beta)\text{ctn}\ \beta \end{matrix} \right\} \\ &\cdot \left[I_0(|k|r) + 2\sum_1^\infty (-1)^m I_m(|k|r)\cos m\theta \left\{ \begin{matrix} \cosh m\alpha \\ \cos m\beta \end{matrix} \right\} \right] \\ &- \frac{1}{\pi} K_0(|k|r) \qquad\qquad \text{(A3)} \\ &+ \frac{2}{\pi}\sum_1^\infty (-1)^m \left[\frac{\partial}{\partial\nu} I_\nu(|k|r)\cos\nu\theta \right]_{\nu=m} \left\{ \begin{matrix} \sinh m\alpha\ \text{ctnh}\ \alpha \\ \sin m\beta\ \text{ctn}\ \beta \end{matrix} \right\} , \end{aligned}$$

where the upper or lower expressions are valid as follows:

$$\frac{|k|}{\kappa} = \left\{ \begin{matrix} \cosh\alpha > 1 \\ \cos\beta < 1 \end{matrix} \right\} .$$

Using the identity below [Ursell (1973)],

$$I_0(|k|r) + 2\sum_1^\infty (-1)^m I_m(|k|r)\cos m\theta \left\{ \begin{matrix} \cosh m\alpha \\ \cos m\beta \end{matrix} \right\} = e^{\kappa z} \cos(y\sqrt{\kappa^2-k^2}) ,$$

where $\cos(y\sqrt{\kappa^2-k^2})$ becomes $\cosh(y\sqrt{k^2-\kappa^2})$ for $|k|>\kappa$, we can rewrite equation (A3) as:

$$\begin{aligned} G^*(k;y,z) = -\frac{1}{\pi} &\left\{ \begin{matrix} (-i\pi\ \text{sgn}\ k-\alpha)\text{ctnh}\ \alpha \\ (\pi-\beta)\text{ctn}\ \beta \end{matrix} \right\} e^{\kappa z} \cos(y\sqrt{\kappa^2-k^2}) \\ &- \frac{1}{\pi} K_0(|k|r) \qquad\qquad \text{(A4)} \\ &+ \frac{2}{\pi}\sum_1^\infty (-1)^m \left[\frac{\partial}{\partial\nu} I_\nu(|k|r)\cos\nu\theta \right]_{\nu=m} \left\{ \begin{matrix} \sinh m\alpha\ \text{ctnh}\ \alpha \\ \sin m\beta\ \text{ctn}\ \beta \end{matrix} \right\} . \end{aligned}$$

We shall now place this expression for $G^*(k;y,z)$ back in equation (A1). However, first let us discuss the expected behavior of $\sigma^*(k)$. According to our assumptions, $\sigma^*(k)$ is a slowly and smoothly varying function of x. Correspondingly, we can use the theory of Fourier transforms [e.g., Lighthill (1964)] to show that if $\sigma(x)$ is sufficiently smooth, $\sigma^*(k)$ must decay like an inverse power of k for large k. Using this information, we can conclude that $\sigma^*(k)$ is peaked near k equals zero and that it falls off rapidly away from the origin. Furthermore, we shall take advantage of the fact that we are interested in evaluating the potential in the near field, which implies that $r = O(\varepsilon)$.

Returning to the evaluation of the potential, we can now write:

$$\phi(x,y,z) = \int_{-\infty}^{\infty} d\xi\ \sigma(\xi)G(x-\xi,y,z)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\sigma^*(k)G^*(k;y,z)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}[-ik\sigma^*(k)]\left[\frac{G^*(k;y,z)}{-ik}\right] .$$

Now substituting the value of $G^*(k;y,z)$, equation (A4), into this equation, we find

$$\phi(x,y,z) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk\ e^{ikx}[-ik\sigma^*(k)]\begin{Bmatrix}(-i\pi\ \mathrm{sgn}\ k-\alpha)\mathrm{ctnh}\ \alpha\\ (\pi-\beta)\mathrm{ctn}\ \beta\end{Bmatrix} \cdot \frac{e^{\kappa z}\cos(y\sqrt{\kappa^2-k^2})}{ik} \tag{A5a}$$

$$-\frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk\ e^{ikx}\sigma^*(k)K_0(|k|r) \tag{A5b}$$

$$+\frac{1}{\pi^2}\sum_{1}^{\infty}(-1)^m\int_{-\infty}^{\infty} dk\ e^{ikx}[-ik\sigma^*(k)]\left[\frac{\partial}{\partial\nu}I_\nu(|k|r)\cos\nu\theta\right]_{\nu=m} \cdot \begin{Bmatrix}\sinh m\alpha\ \mathrm{ctnh}\ \alpha\\ \sin m\beta\ \mathrm{ctn}\ \beta\end{Bmatrix}\frac{1}{ik} . \tag{A5c}$$

Setting $\phi_1(x,y,z)$ equal to line (A5a), $\phi_2(x,y,z)$ equal to line (A5b), and $\phi_3(x,y,z)$ equal to line (A5c), we can work with one term at a time.

Evaluating $\phi_1(x,y,z)$ first, we can write

$$\phi_1(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}[ik\sigma^*(k)]\begin{Bmatrix}(-i\pi\ \mathrm{sgn}\ k-\alpha)\,\mathrm{ctnh}\ \alpha\\ (\pi-\beta)\,\mathrm{ctn}\ \beta\end{Bmatrix}\frac{e^{\kappa z}\cos(y\sqrt{\kappa^2-k^2})}{ik}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi\ \sigma'(\xi) f(x-\xi,y,z)$$

where

$$f(x,y,z) = \frac{e^{\kappa z}}{2\pi}\int_{-\infty}^{\infty}\frac{dk\ e^{ikx}}{-ik}\begin{Bmatrix}(-i\pi\ \mathrm{sgn}\ k-\alpha)\,\mathrm{ctnh}\ \alpha\\ (\pi-\beta)\,\mathrm{ctn}\ \beta\end{Bmatrix}\cos(y\sqrt{\kappa^2-k^2})\ .$$

By expanding $f(x,y,z)$ into three integrals,

$$f(x,y,z) = \frac{e^{\kappa z}}{2\pi}\int_{\kappa}^{\infty}\frac{dk\ e^{ikx}}{-ik}(-i\pi\ \mathrm{sgn}\ k-\alpha)\,\mathrm{ctnh}\ \alpha\ \cos(y\sqrt{\kappa^2-k^2})$$

$$+\frac{e^{\kappa z}}{2\pi}\int_{-\kappa}^{\kappa}\frac{dk\ e^{ikx}}{-ik}(\pi-\beta)\,\mathrm{ctn}\ \beta\ \cos(y\sqrt{\kappa^2-k^2})$$

$$+\frac{e^{\kappa z}}{2\pi}\int_{-\infty}^{-\kappa}\frac{dk\ e^{ikx}}{-ik}(-i\pi\ \mathrm{sgn}\ k-\alpha)\,\mathrm{ctnh}\ \alpha\ \cos(y\sqrt{\kappa^2-k^2})\ ,$$

and by combining terms and reducing the interval of integration to $0 \le \beta \le \infty$, we find that $f(x,y,z)$ reduces to:

$$f(x,y,z) = -e^{\kappa z}\int_{\kappa}^{\infty}\frac{dk\ \cos kx}{k}\,\mathrm{ctnh}\ \alpha\ \cosh(y\sqrt{k^2-\kappa^2}) \tag{A6a}$$

$$-\frac{e^{\kappa z}}{\pi}\int_{\kappa}^{\infty}\frac{dk\ \sin kx}{k}\,\alpha\ \mathrm{ctnh}\ \alpha\ \cosh(y\sqrt{k^2-\kappa^2}) \tag{A6b}$$

$$+\ e^{\kappa z}\int_{0}^{\kappa}\frac{dk\ \sin kx}{k}\,\mathrm{ctn}\ \beta\ \cos(y\sqrt{\kappa^2-k^2}) \tag{A6c}$$

$$-\frac{e^{\kappa z}}{\pi}\int_{0}^{\kappa}\frac{dk\ \sin kx}{k}\,\beta\ \mathrm{ctn}\ \beta\ \cos(y\sqrt{\kappa^2-k^2})\ . \tag{A6d}$$

If we now make the following change of variables, $k = \kappa\cosh\alpha$, $dk = \kappa\sinh\alpha\, d\alpha$ in lines (A6a) and (A6b), and $k = \kappa\cos\beta$, $dk = -\kappa\sin\beta\, d\beta$ in lines (A6c) and (A6d), we find that $f(x,y,z)$ can be written as follows:

$$f(x,y,z) = -e^{\kappa z}\int_0^\infty d\alpha\, \cos(\kappa x\cosh\alpha)\cosh(\kappa y\sinh\alpha)$$

$$-\frac{e^{\kappa z}}{\pi}\int_0^\infty d\alpha\, \sin(\kappa x\cosh\alpha)\alpha\cosh(\kappa y\sinh\alpha)$$

$$+\, e^{\kappa z}\int_0^{\pi/2} d\beta\, \sin(\kappa x\cos\beta)\cos(\kappa y\sin\beta)$$

$$+\frac{e^{\kappa z}}{\pi}\int_0^{\pi/2} d\beta\, \sin(\kappa x\cos\beta)\beta\cos(\kappa y\sin\beta)\;.$$

These integrals, although greatly simplified compared to those with which we started, are still difficult to evaluate. However, if we set y equal to zero, we find that the resulting integrals have been evaluated by Tuck (1963), and in fact,

$$f(x,0,z) = e^{\kappa z}\,\frac{\pi}{4}\{\mathbf{H}_0(\kappa x) + (2 + \operatorname{sgn} x)Y_0(\kappa|x|)\}\;.$$

Substituting $f(x,0,z)$ back into $\phi_1(x,y,z)$, we obtain

$$\phi_1(x,y,z) \sim -\frac{e^{\kappa z}}{4}\int_{-\infty}^{\infty} d\xi\, \sigma'(\xi)\{\mathbf{H}_0[\kappa(x-\xi)] + [2 + \operatorname{sgn}(x-\xi)]Y_0(\kappa|x-\xi|)\}. \qquad \text{(A7)}$$

Returning to line (A5b) for $\phi_2(x,y,z)$, we have

$$\phi_2(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\sigma^*(k)K_0(|k|r)\;.$$

This is the classical potential for a line distribution of sources in an infinite fluid. Following the analysis of Ward (1955), we can write the inner expansion of $\phi_2(x,y,z)$ as:

$$\phi_2(x,y,z) = \frac{\sigma(x)}{\pi} \log r - \frac{1}{2\pi} b(x) - \frac{\sigma''(x)}{4\pi} r^2 \log r + \frac{\sigma''(x)}{4\pi} r^2 + \frac{r^2 b''(x)}{8\pi} , \quad \text{(A8)}$$

where

$$b(x) = \int_{-\infty}^{\infty} d\xi \; \sigma'(\xi) \log(2|x-\xi|) \operatorname{sgn}(x-\xi) .$$

Finally, we need to obtain an inner expansion of $\phi_3(x,y,z)$, line (A5c),

$$\phi_3(x,y,z) = \frac{1}{\pi^2} \sum_1^{\infty} (-1)^m \int_{-\infty}^{\infty} dk \; e^{ikx} [-ik\sigma^*(k)] \left[\frac{\partial}{\partial\nu} I_\nu(|k|r) \cos \nu\theta\right]_{\nu=m}$$

$$\cdot \begin{Bmatrix} \sinh m\alpha \; \mathrm{ctnh} \; \alpha \\ \sin m\beta \; \mathrm{ctn} \; \beta \end{Bmatrix} \frac{1}{-ik}$$

$$= \frac{2}{\pi} \sum^{\infty} (-1)^m \int_{\infty}^{\infty} d\xi \; \sigma'(\xi) g_m(x-\xi,y,z) ,$$

where

$$g_m(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk \; e^{ikx}}{-ik} \left[\frac{\partial}{\partial\nu} I\nu(|k|r) \cos \nu\theta\right]_{\nu=m} \begin{Bmatrix} \sinh m\alpha \; \mathrm{ctnh} \; \alpha \\ \sin m\beta \; \mathrm{ctn} \; \beta \end{Bmatrix} . \quad \text{(A9)}$$

Breaking equation (A9) up into three integrals, we have:

$$g_m(x,y,z) = \frac{1}{2\pi} \left[\int_{-\infty}^{-\kappa} + \int_{\kappa}^{\infty}\right] \frac{dk \; e^{ikx}}{-ik} \left[\frac{\partial}{\partial\nu} I_\nu(|k|r) \cos \nu\theta\right]_{\nu=m} \sinh m\alpha \; \mathrm{ctnh} \; \alpha \quad \text{(A10a)}$$

$$+ \frac{1}{2\pi} \int_{-\kappa}^{\kappa} \frac{dk \; e^{ikx}}{ik} \left[\frac{\partial}{\partial\nu} I_\nu(|k|r) \cos \nu\theta\right]_{\nu=m} \sin m\beta \; \mathrm{ctn} \; \beta . \quad \text{(A10b)}$$

It is now possible to make the same change of variables, $k = \kappa \cosh \alpha$ in line (A10a) and $k = \kappa \cos \beta$ in line (A10b), as we did for $\phi_1(x,y,z)$. This results in

$$g_m(x,y,z) = + \frac{i}{2\pi} \int_{-\infty}^{\infty} d\alpha \; e^{i\kappa x \cosh \alpha} \left[\frac{\partial}{\partial \nu} I_\nu(\kappa r \cosh \alpha) \cos \nu\theta \right]_{\nu=m} \sinh m\alpha$$

$$- \frac{i}{2\pi} \int_{-\pi/2}^{\pi/2} d\beta \; e^{i\kappa x \cos \beta} \left[\frac{\partial}{\partial \nu} I_\nu(\kappa r \cos \beta) \cos \nu\theta \right]_{\nu=m} \sin m\beta \; .$$

We can now evaluate these two integrals by the method of stationary phase [Erdélyi (1956)]. To do this, we must find the stationary points of the exponential term in each integral. The stationary points, which are the zeros of the derivatives of the arguments of these terms, are the points $\alpha = 0$ and $\beta = 0$. At these points each of the integrands is identically zero, and from this we can assume that to leading order $\phi_3(x,y,z) = 0$.

Combining $\phi_1(x,y,z)$ [equation (A7)], $\phi_2(x,y,z)$ [equation (A8)], and $\phi_3(x,y,z)$, we can now write the inner expansion of the outer expansion as:

$$\phi(x,y,z) \sim \frac{1}{\pi} \sigma(x) \log r - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \; \sigma'(\xi) \log(2|x-\xi|) \operatorname{sgn}(x-\xi)$$

$$- \frac{e^{\kappa z}}{4} \int_{-\infty}^{\infty} d\xi \; \sigma'(\xi) \{ \mathbf{H}_0[\kappa(x-\xi)] + [2 + \operatorname{sgn}(x-\xi)] Y_0(\kappa|x-\xi|) \} \; .$$

This is the inner expansion of the outer expansion given in equation (10).

APPENDIX B: INNER EXPANSION OF THE OUTER EXPANSION FOR A RAPIDLY VARYING SOURCE DISTRIBUTION

In this appendix we shall obtain the inner expansion of the outer expansion for the potential due to a rapidly oscillating source distribution. This potential is given as:

$$\phi(x,y,z) = \int_{-\infty}^{\infty} d\xi \ \mathrm{Re}[\Sigma(\xi)e^{i\kappa\xi}]G(x-\xi,y,z). \tag{B1}$$

Alternatively, we may write $\phi(x,y,z)$ in the transform space, equation (19), as:

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \ e^{ikx}\Sigma^*(k-\kappa)G^*(k;y,z)$$

However, we must be careful of the fact that we can include only the real part of $\Sigma(x)e^{i\kappa x}$. We shall work in the transform space, which will allow us to take advantage of the fact that the peak of the source distribution's Fourier transform is at $k = \pm\kappa$. This will result in our finding a different inner expansion of the outer expansion of the potential from that obtained for a slowly varying source distribution.

Let us start this analysis by finding the Fourier transform of $\mathrm{Re}[\Sigma(x)e^{i\kappa x}]$. If we call our rapidly oscillating source distribution $R(x)$, which we will define as follows:

$$R(x) = \mathrm{Re}[\Sigma(x)e^{i\kappa x}] ,$$

then we can write its Fourier transform as:

$$R^*(k) = \int_{-\infty}^{\infty} dx \ e^{-ikx}\mathrm{Re}[\Sigma(x)e^{i\kappa x}] ,$$

which upon expansion yields:

$$R^*(k) = \int_{-\infty}^{\infty} dx \ e^{-ikx}[\Sigma_R(x)\cos\kappa x - \Sigma_I(x)\sin\kappa x] .$$

If we now expand the sine and cosine as complex exponentials, we obtain:

$$R^*(k) = \frac{1}{2}\int_{-\infty}^{\infty} dx[\Sigma_R(x)e^{-ix(k-\kappa)} + \Sigma_R(x)e^{-ix(k+\kappa)} + i\Sigma_I(x)e^{-ix(k-\kappa)}$$

$$-i\Sigma_I(x)e^{-ix(k+\kappa)}] \ .$$

We can now evaluate the Fourier transform of $R(x)$ as:

$$R^*(k) = \frac{1}{2}[\Sigma_R^*(k-\kappa) + \Sigma_R^*(k+\kappa) + i\Sigma_I^*(k-\kappa) - i\Sigma_I^*(k+\kappa)] \ .$$

Combining terms, we have as our final result:

$$R^*(k) = \frac{1}{2}[\Sigma^*(k-\kappa) + \overline{\Sigma^*(k+\kappa)}] \ ,$$

where $\overline{\Sigma^*(k)}$ is the complex conjugate of $\Sigma^*(k)$.

Rewriting $\phi(x,y,z)$, we obtain:

$$\phi(x,y,z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk[\Sigma^*(k-\kappa) + \overline{\Sigma^*(k+\kappa)}]G^*(k;y,z)e^{ikx} \ .$$

Writing $\phi(x,y,z)$ as $\phi(x,y,z) = \phi_1(x,y,z) + \phi_2(x,y,z)$, we have

$$\phi_1(x,y,z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk\ \Sigma^*(k-\kappa)G^*(k;y,z)e^{ikx} \tag{B2}$$

and

$$\phi_2(x,y,z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk\ \overline{\Sigma^*(k+\kappa)}G^*(k;y,z)e^{ikx} \tag{B3}$$

We shall now determine the outer expansion, valid for $y,z = O(1)$.

<u>Outer Expansion</u>

We must now investigate the behavior of $G^*(k;y,z)$, given as:

$$G^*(k;y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\ell\ e^{i\ell|y|}e^{\sqrt{k^2+\ell^2}\,z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} \ , \tag{B4}$$

from equation (19). First we note that this integral appears to be real. However, it has two singularities on the real axis for $|k| > \kappa$. These contribute a residue term to the integral which may give a complex contribution. In order to investigate whether $G(x,y,z)$ is real or complex, let us write this integral as a residue plus a principal value integral:

$$G^*(k;y,z) = -\frac{1}{2\pi} \fint_{-\infty}^{\infty} \frac{d\ell \; e^{i\ell|y|} e^{\sqrt{k^2+\ell^2}\, z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} - \frac{i\pi}{2\pi} \operatorname{Res}(\ell_0) \; .$$

Neglecting the residues for the moment, we note that the principal value integral is even in ℓ and therefore, we may rewrite it as:

$$G^*(k;y,z) = -\frac{1}{\pi} \fint_{0}^{\infty} \frac{d\ell \; \cos \ell|y| e^{\sqrt{k^2+\ell^2}\, z}}{\sqrt{k^2+\ell^2} - k^2/\kappa} - \frac{i}{2} \operatorname{Res}(\ell_0) \; .$$

We now see that the integral is pure real. Therefore, if we change the sign of k from plus to minus, it exhibits the property $G^*(-k;y,z) = \overline{G^*(k;y,z)}$, which is the property necessary for $G^*(k;y,z)$ to be the transform of a real quantity.

We shall now investigate the residues of $G^*(k;y,z)$. To do this, we must determine singularities of the integrand, which are the zeroes of the denominator. The zeroes are given by the values of ℓ for which

$$\sqrt{k^2+\ell^2} - k^2/\kappa = 0 \; .$$

This implies that

$$\sqrt{k^2+\ell^2} = k^2/\kappa$$

which in turn implies that

$$\ell^2 = k^2(k^2/\kappa^2 - 1) \; .$$

The poles of $G^*(k;y,z)$ are at the points $\ell_0 = \pm\, |k|\sqrt{k^2/\kappa^2 - 1}$. For $|k| \geq \kappa$, these poles lie on the real axis; for $|k| = \kappa$, the two poles converge to the origin resulting in a double pole; and for $|k| < \kappa$, the poles lie on the imaginary axis.

It is now necessary to determine which of the poles on the real axis to include in the residue. First, by observing the exponential [$e^{i\ell|y|}$ in equation (B4)] in the numerator of $G^*(k;y,z)$, we see that the contour should be closed in the upper half space. To obtain the direction in which the contour is perturbed, and therefore, the direction to go around the poles on the real axis, we shall use the fictitious viscosity. Rewriting the free surface condition as:

$$(U \frac{\partial}{\partial x} + \mu)^2 \phi + g\phi_z = 0 \qquad \text{on} \quad z = 0 ,$$

and, rewriting the solution, we obtain a new form for $G^*(k;y,z)$ as follows:

$$G^*(k;y,z) = \lim_{\mu \to 0} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\ell \; e^{i\ell|y|} e^{\sqrt{k^2+\ell^2}\, z}}{\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - i\mu)} .$$

For $|k| > \kappa$, we are interested in determining in which direction the poles are perturbed for $\mu > 0$. Again determining the locations of the poles of the integrand, we have that

$$k^2 + \tilde{\ell}_0^2 = \frac{1}{g^2}(Uk - i\mu)^4 ,$$

from which we can conclude that

$$\tilde{\ell}_0^2 = \frac{1}{g^2}(Uk - i\mu)^4 - k^2 .$$

Expanding this equation and determining the order of magnitude with respect to $\mu(\mu$ small$)$, we obtain:

$$\tilde{\ell}_0^2 = \underset{O(1)}{\frac{k^4}{\kappa^2}} - \underset{O(1)}{k^2} - \underset{O(\mu)}{4i\mu \frac{U^3k^3}{g^2}} - \underset{O(\mu^2)}{\frac{2\mu^2U^2k^2}{g^2}} - \underset{O(\mu^2)}{\frac{4\mu^2U^2k^2}{g^2}} + \underset{O(\mu^3)}{\frac{4i\mu^3Uk}{g^2}} + \underset{O(\mu^4)}{\frac{\mu^4}{g^2}} .$$

By neglecting the terms which are $O(\mu^2)$ and higher order, we find that:

$$\tilde{\ell}_0^2 \sim \frac{k^4}{\kappa^2} - k^2 - 4i\mu \frac{U^3k^3}{g^2} ,$$

and

$$\tilde{\ell}_0 \sim \pm |k| \sqrt{\frac{k^2}{\kappa^2} - 1 - 4i\mu \frac{U^3K}{g^2}} .$$

By factoring ℓ_0 out of the equation for $\tilde{\ell}_0$, we obtain:

$$\tilde{\ell}_0 \sim |k| \sqrt{\frac{k^2}{\kappa^2} - 1} \sqrt{1 - \frac{4i\mu U^3k}{g^2(\frac{k^2}{\kappa^2} - 1)}} .$$

Upon expanding the second square root term in a Taylor series, we have:

$$\tilde{\ell}_0 \sim \ell_0 \left[1 - \frac{2i\mu U^3k}{g^2(\frac{k^2}{\kappa^2} - 1)}\right] .$$

According to the sign of k, we can conclude that for $k > \kappa$ we take the following contour:

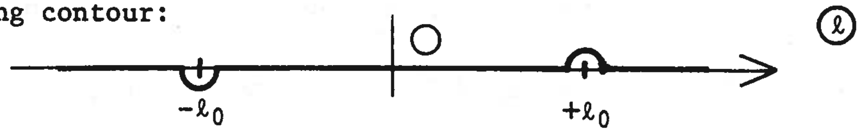

Figure B1a: Contour of Principal Value Integral for $k > \kappa$.

and for $k < -\kappa$ we take:

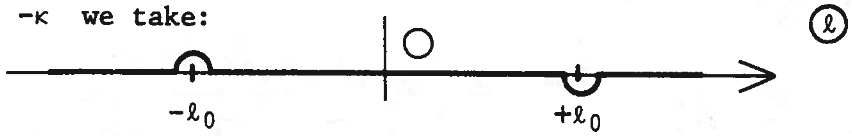

Figure B1b: Contour of Principal Value Integral for $k < -\kappa$.

We now need to determine the residues of $G^*(k;y,z)$, equation (B4), at the points $\pm\ell_0$. For $k > \kappa$ we must find $\mathrm{Res}(-\ell_0)$ and for $k < -\kappa$ we must determine $\mathrm{Res}(+\ell_0)$. The residue at $+\ell_0$ can be written as follows:

$$\mathrm{Res}(+\ell_0) = \lim_{\ell\to\ell_0} \frac{(\ell-\ell_0)e^{i\ell|y|}e^{\sqrt{k^2+\ell^2}\,z}}{(\sqrt{k^2+\ell^2} - k^2/\kappa)} .$$

Making a change of variables, $\varepsilon = \ell - \ell_0$, $\ell = \ell_0 + \varepsilon$, and $\ell^2 = \ell_0^2 + \varepsilon^2 + 2\ell_0\varepsilon$, and rewriting $\mathrm{Res}(+\ell_0)$, the residue becomes:

$$\mathrm{Res}(+\ell_0) = \lim_{\varepsilon\to 0} \frac{\varepsilon e^{i\ell_0|y|}e^{\sqrt{k^2+\ell_0^2}\,z}}{(\sqrt{k^2 + \ell_0^2 + \varepsilon^2 + 2\ell_0\varepsilon} - k^2/\kappa)}$$

If we apply L'Hôpital's rule and set ε equal to zero, we find that:

$$\mathrm{Res}(+\ell_0) = e^{i\ell_0|y|}e^{\sqrt{k^2+\ell_0^2}}\,\frac{\sqrt{k^2+\ell_0^2}}{\ell_0} .$$

Upon substituting the value for ℓ_0, we have our final result:

$$\mathrm{Res}(+\ell_0) = \frac{e^{i|k||y|\sqrt{k^2/\kappa^2 - 1} + k^2 z/\kappa}\,|k|}{\sqrt{k^2 - \kappa^2}} \qquad \text{for} \quad (k < -\kappa) . \qquad \text{(B5)}$$

Treating the case $k > \kappa$ in a similar manner, we obtain $\mathrm{Res}(-\ell_0)$ as follows:

$$\mathrm{Res}(-\ell_0) = -\frac{e^{-i|k||y|\sqrt{k^2/\kappa^2 - 1} + k^2 z/\kappa}\,|k|}{\sqrt{k^2 - \kappa^2}} \qquad \text{for} \quad (k > \kappa) . \qquad \text{(B6)}$$

We may now write $G^*(k;y,z)$, equation (B4), as:

$$G^*(k;y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\ell\, e^{i\ell|y|}e^{\sqrt{k^2+\ell^2}\, z}}{\sqrt{k^2+\ell^2} - k^2/\kappa}$$

$$-\frac{i}{2}\begin{cases} + \dfrac{e^{i|k||y|\sqrt{k^2/\kappa^2 - 1} + k^2 z/\kappa}\,|k|}{2\sqrt{k^2-\kappa^2}} & (k < -\kappa) \\[2ex] - \dfrac{e^{-i|k||y|\sqrt{k^2/\kappa^2 - 1} + k^2 z/\kappa}\,|k|}{2\sqrt{k^2-\kappa^2}} & (k > \kappa) \;. \end{cases} \tag{B7}$$

The crucial question now is, do the residue terms of equation (B7) obey the complex conjugate rule with respect to changes from positive to negative k ? We can show that this is the case by writing the imaginary part of the complex exponential in terms of sines and cosines; doing this, we obtain:

$$-i(-\cos A + i\sin A) = \sin A + i\cos A \qquad \text{for} \quad k > 0$$

and

$$-i(\cos A + i\sin A) = \sin A - i\cos A \qquad \text{for } k < 0 \;,$$

where $A = |k||y|\sqrt{k^2+\kappa^2}/\kappa$. The above proves that the residue term satisfies the conditions for $G(x,y,z)$ to be real.

We now want to transform this Green's function, equation (B7), into a form of the Green's function similar to that one given by Ursell (1968a) for rapidly oscillating disturbances. To do this, we shall write $G^*(k;y,z)$ as a contour integral:

$$G^*(k;y,z) = -\, i\,\mathrm{Res}(\ell_0) \tag{B8a}$$

$$+\frac{1}{2\pi}\int_{|k|}^{\infty} d\ell\, e^{-\ell|y|}\left\{\frac{e^{i\sqrt{\ell^2-k^2}\, z}}{\sqrt{\ell^2-k^2} + ik^2/\kappa} + \frac{e^{-i\sqrt{\ell^2-k^2}\, z}}{\sqrt{\ell^2-k^2} - ik^2/\kappa}\right\}, \tag{B8b}$$

where the contour chosen is shown in figure B2. (The contour of figure B2a is used for $k < -\kappa$, B2b for $|k| < \kappa$, and B2c for $k > \kappa$.) The integral term in equation (B8) comes from integrating in and out along the branch cut $\overline{CDE}$ in the above contours (Figures B2a, b, and c).

To evaluate $G^*(k;y,z)$ we shall need the residue on the imaginary axis for $|k| < \kappa$. For $|k| < \kappa$ we shall need the residue at:

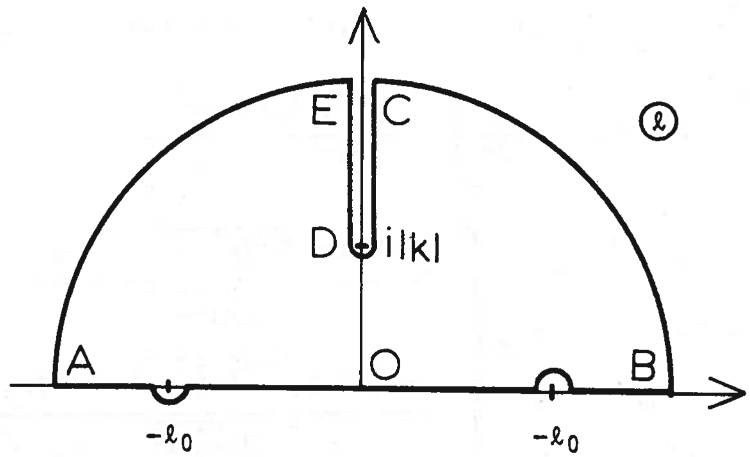

Figure B2a: Contour of Integration for $k > \kappa$.

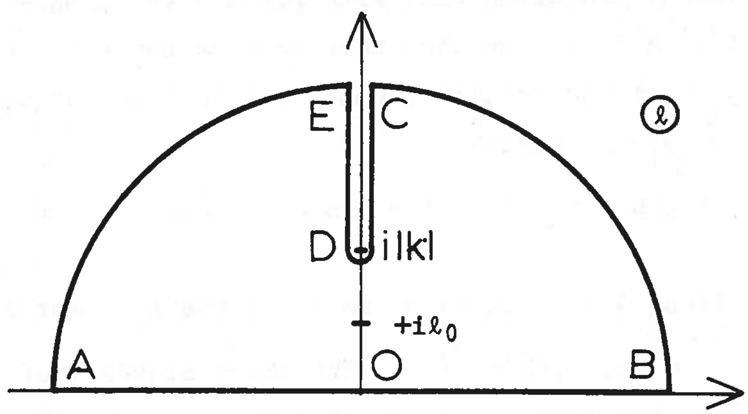

Figure B2b: Contour of Integration for $|k| < \kappa$.

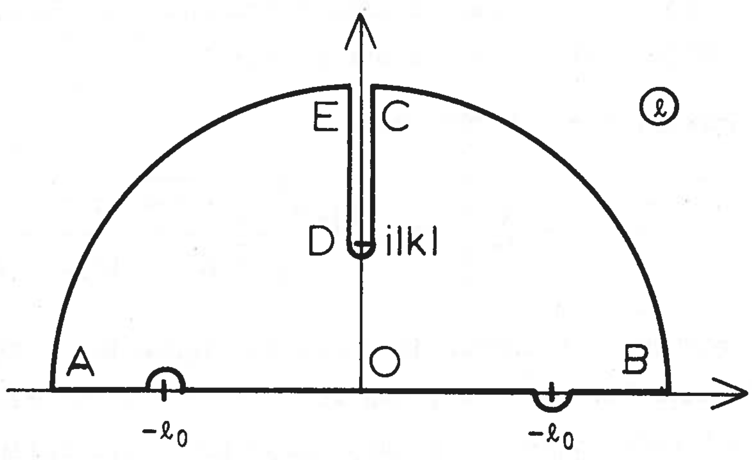

Figure B2c: Contour of Integration for $k < -\kappa$.

$$\ell_0 = + i|k|\sqrt{1 - k^2/\kappa^2} \ .$$

Writing the residue as we did previously, we obtain:

$$\mathrm{Res}(\ell_0) = + \lim_{\ell\to\ell_0} \frac{(\ell-\ell_0)e^{i\ell|y| + \sqrt{k^2+\ell^2}\ z}}{(\sqrt{k^2+\ell^2} - k^2/\kappa)} \ .$$

By making a change of variables, $\ell - \ell_0 = \varepsilon$, $\ell = \ell_0 + \varepsilon$, $\ell^2 = \ell_0^2 + \varepsilon^2 + 2\varepsilon\ell_0$, and by applying L'Hôpital's rule, we obtain:

$$\mathrm{Res}(\ell_0) = - \frac{ie^{-|k|\,|y|\sqrt{1 - k^2/\kappa^2} + k^2/\kappa^2\ z}|k|}{\sqrt{\kappa^2 - k^2}} \ . \tag{B9}$$

Letting $I(k;y,z)$ equal line (B8b),

$$I(k;y,z) = \int_{|k|}^{\infty} d\ell\ e^{-\ell|y|}\left\{\frac{e^{i\sqrt{\ell^2-k^2}\ z}}{\sqrt{\ell^2-k^2} + ik^2/\kappa} + \frac{e^{-i\sqrt{\ell^2-k^2}\ z}}{\sqrt{\ell^2-k^2} - ik^2/\kappa}\right\} ,$$

we will examine the integral around the branch cut, line (B8b), and determine how it behaves. By making a change of variables, $\ell = |k|\cosh\mu$ and $d\ell = |k|\sinh\mu\ d\mu$, we can rewrite $I(k;y,z)$ as:

$$I(k;y,z) = \int_0^{\infty} d\mu\ \sinh\mu\ e^{-|k|\,|y|\cosh\mu}\left\{\frac{e^{iz|k|\sinh\mu}}{\sinh\mu + i|k|/\kappa} + \frac{e^{-iz|k|\sinh\mu}}{\sinh\mu - i|k|/\kappa}\right\} .$$

Upon combining the two terms, we find that:

$$I(k;y,z) = 2\int_0^{\infty} d\mu\ e^{-|k|\,|y|\cosh\mu}$$

$$\cdot\left\{\frac{\sinh\mu[\sinh\mu\ \cos(z|k|\sinh\mu) + |k|/\kappa\ \sin(z|k|\sinh\mu)]}{\sinh\mu^2 + k^2/\kappa^2}\right\} .$$

The portion of the integrand in brackets is zero for $\mu = 0$, and goes to a maximum of ± 1 for $\mu > 0$. Therefore, $I(k;y,z)$ is bounded by $K_0(|k|\,|r|)$, as shown below:

$$|I(k;y,z)| < \int_0^{\infty} d\mu\ e^{-|k|\,|y|\cosh\mu} = k_0(|k|\,|r|) \ .$$

For $|k||y| \to \infty$ (as is the case in the far field), this integral is exponentially small, even for $z = 0$. From this we can conclude that for large $|k||y|$, $G^*(k;y,z)$ is given as follows:

$$G^*(k;y,z) \sim -i \operatorname{Res}[\ell_0(k)] .$$

Using this, we can now write $\phi(x,y,z)$, equation (B10), as:

$$\phi(x,y,z) \sim -\frac{i}{4\pi}\int_{-\infty}^{\infty} dk[\Sigma^*(k-\kappa) + \overline{\Sigma^*(k+\kappa)}]\operatorname{Res}[\ell_0(k)]e^{ikx} , \tag{B10}$$

and correspondingly from equations (B2) and (B3),

$$\phi_1(x,y,z) = -\frac{i}{4\pi}\int_{-\infty}^{\infty} dk\ \Sigma^*(k-\kappa)\operatorname{Res}[\ell_0(k)]e^{ikx} \tag{B11}$$

and

$$\phi_2(x,y,z) = -\frac{i}{4\pi}\int_{-\infty}^{\infty} dk\ \overline{\Sigma^*(k+\kappa)}\operatorname{Res}[\ell_0(k)]e^{ikx} . \tag{B12}$$

Working with just $\phi_1(x,y,z)$ and substituting the appropriate residue terms in equations (B5), (B6), and (B9) into equation (B11), we have:

$$\phi_1(x,y,z) = +\frac{i}{4\pi}\int_{\kappa}^{\infty} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-ik\frac{|y|}{\kappa}\sqrt{k^2-\kappa^2} + k^2z/\kappa}\ k}{\sqrt{k^2-\kappa^2}}$$

$$-\frac{1}{4\pi}\int_{-\kappa}^{\kappa} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-\frac{|k||y|}{\kappa}\sqrt{\kappa^2-k^2} + k^2z/\kappa}\ |k|}{\sqrt{\kappa^2-k^2}}$$

$$+\frac{i}{4\pi}\int_{-\infty}^{-\kappa} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-i\frac{k|y|}{\kappa}\sqrt{k^2-\kappa^2} + k^2z/\kappa}\ k}{\sqrt{k^2\ \kappa^2}}$$

Our Green's function has now been transformed into a form which is convenient for determining the outer expansion of the inner expansion. We shall next complete the task of determining a consistent outer expansion.

From our knowledge of Fourier transforms, we can see that $\Sigma^*(k-\kappa)$ has a peak at $k = \kappa$, and $G^*(k;y,z)$ has peaks (square root singularities) at $k = \pm\kappa$. Therefore, we would expect that the major contributions to $\phi_1(x;y,z)$ would come from near the singularities of $G^*(k;y,z)$. Because of the peak in $\Sigma^*(k-\kappa)$ at $k = \kappa$, we would expect the contribution from near $k = \kappa$ to be lower order than the contribution from near $k = -\kappa$. Splitting the interval of integration of equation (B13) in half, and expanding the integrand about the singularity in the interval of integration, we obtain:

$$\phi_1(x,y,z) = \frac{i}{4\pi}\int_{\kappa}^{\infty} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-i\frac{k|y|}{\kappa}\sqrt{k^2-\kappa^2} + k^2 z/\kappa}\ k}{\sqrt{k^2-\kappa^2}} \tag{B14a}$$

$$-\frac{1}{4\pi}\int_{0}^{\infty} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-\frac{|k||y|}{\kappa}\sqrt{\kappa^2-k^2} + k^2 z/\kappa}\ |k|}{\sqrt{\kappa^2-k^2}} \tag{B14b}$$

$$-\frac{1}{4\pi}\int_{-\kappa}^{0} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-\frac{|k||y|}{\kappa}\sqrt{\kappa^2-k^2} + k^2 z/\kappa}\ |k|}{\sqrt{\kappa^2-k^2}} \tag{B14c}$$

$$+\frac{i}{4\pi}\int_{-\infty}^{-\kappa} \frac{dk\ e^{ikx}\Sigma^*(k-\kappa)e^{-i\frac{k|y|}{\kappa}\sqrt{k^2-\kappa^2} + k^2 z/\kappa}\ k}{\sqrt{\kappa^2-k^2}} \tag{B14d}$$

If we let $m = k-\kappa$, $k = m+\kappa$, $dk = dm$, and $k^2 = m^2 + 2m\kappa + \kappa^2$ in lines (B14a and B14b), and if we let $m = k+\kappa$, $k = m-\kappa$, $dk = dm$, and $k^2 = m^2 - 2m\kappa + \kappa^2$ in lines (B14c and B14d), we find that equation (B14) becomes:

$$\phi_1(x,y,z) = \frac{ie^{i\kappa x + \kappa z}}{4\pi}\int_{0}^{\infty} \frac{dm\ e^{imx}\Sigma^*(m)(m+\kappa)e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}}{\sqrt{m(m+2\kappa)}}$$

$$-\frac{e^{i\kappa x + \kappa z}}{4\pi}\int_{-\kappa}^{0} \frac{dm\ e^{imx}\Sigma^*(m)|m+\kappa|e^{-\frac{|y|}{\kappa}|m+\kappa|\sqrt{-m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}}{\sqrt{-m(m+2\kappa)}}$$

$$-\frac{e^{-i\kappa x + \kappa z}}{4\pi}\int_{0}^{\kappa} \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)|m-\kappa|e^{-\frac{|y|}{\kappa}|m-\kappa|\sqrt{-m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} - \frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}}$$

$$+\frac{ie^{-i\kappa x + \kappa z}}{4\pi}\int_{-\infty}^{0} \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)(m-\kappa)e^{-i\frac{|y|}{\kappa}(m-\kappa)\sqrt{m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} - \frac{2m}{\kappa})}}{\sqrt{m(m-2\kappa)}}$$

We will next expand the integrands of the above integrals, but first we must divide the interval of integration further. We shall truncate the intervals of integration at $\pm\,\epsilon^{-1+\delta}$ and evaluate the two portions of the intervals separately. (We let δ be some small positive number, so that $\epsilon^{-1+\delta}$ is higher order than ϵ^{-1}.) Doing this, we obtain an extension of equation (B15).

$$\phi_1(x,y,z) = \frac{ie^{i\kappa x + \kappa z}}{4\pi}\int_{\epsilon^{-1+\delta}}^{\infty} \frac{dm\; e^{imx}\Sigma^*(m)(m+\kappa)e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}+\frac{2m}{\kappa})}}{\sqrt{m(m+2\kappa)}} \qquad \text{(B16a)}$$

$$+\frac{ie^{i\kappa x + \kappa z}}{4\pi}\int_{0}^{\epsilon^{-1+\delta}} \frac{dm\; e^{imx}\Sigma^*(m)(m+\kappa)e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}+\frac{2m}{\kappa})}}{\sqrt{m(m+2\kappa)}} \qquad \text{(B16b)}$$

$$-\frac{e^{i\kappa x + \kappa z}}{4\pi}\int_{-\epsilon^{-1+\delta}}^{0} \frac{dm\; e^{imx}\Sigma^*(m)|m+\kappa|e^{-\frac{|y|}{\kappa}|m+\kappa|\sqrt{-m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}+\frac{2m}{\kappa})}}{\sqrt{-m(m+2\kappa)}} \qquad \text{(B16c)}$$

$$-\frac{e^{i\kappa x + \kappa z}}{4\pi}\int_{-\kappa}^{-\epsilon^{-1+\delta}} \frac{dm\; e^{imx}\Sigma^*(m)|m+\kappa|e^{-\frac{|y|}{\kappa}|m+\kappa|\sqrt{-m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}+\frac{2m}{\kappa})}}{\sqrt{-m(m+2\kappa)}} \qquad \text{(B16d)}$$

$$-\frac{e^{-i\kappa x + \kappa z}}{4\pi}\int_{\epsilon^{-1+\delta}}^{\kappa} \frac{dm\; e^{imx}\Sigma^*(m-2\kappa)|m-\kappa|e^{-\frac{|y|}{\kappa}|m-\kappa|\sqrt{-m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}-\frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}} \qquad \text{(B16e)}$$

$$-\frac{e^{-i\kappa x + \kappa z}}{4\pi}\int_{0}^{\epsilon^{-1+\delta}} \frac{dm\; e^{imx}\Sigma^*(m-2\kappa)|m-\kappa|e^{-\frac{|y|}{\kappa}|m-\kappa|\sqrt{-m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}-\frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}} \qquad \text{(B16f)}$$

$$+\frac{ie^{-i\kappa x + \kappa z}}{4\pi}\int_{-\epsilon^{-1+\delta}}^{0} \frac{dm\; e^{imx}\Sigma^*(m-2\kappa)(m-\kappa)e^{-i\frac{|y|}{\kappa}(m-\kappa)\sqrt{m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}-\frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}} \qquad \text{(B16g)}$$

$$+\frac{ie^{-i\kappa x + \kappa z}}{4\pi}\int_{-\infty}^{-\epsilon^{-1+\delta}} \frac{dm\; e^{imx}\Sigma^*(m-2\kappa)(m-\kappa)e^{-i\frac{|y|}{\kappa}(m-\kappa)\sqrt{m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2}-\frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}} \qquad \text{(B16h)}$$

It is easy to show that the integrals of lines (B16a), (B16d), (B16e), and (B16h) are all exponentially small for $y,z = O(1)$, so we shall discard them, and our outer expansion becomes:

$\phi_1(x,y,z) =$

$$\frac{ie^{i\kappa x + \kappa z}}{4\pi} \int_0^{\varepsilon^{-1+\delta}} \frac{dm\ e^{imx}\Sigma^*(m)(m+\kappa)e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}}{\sqrt{m(m+2\kappa)}} \tag{B17a}$$

$$- \frac{e^{i\kappa x + \kappa z}}{4\pi} \int_{-\varepsilon^{-1+\delta}}^{0} \frac{dm\ e^{imx}\Sigma^*(m)|m+\kappa|e^{-\frac{|y|}{\kappa}|m+\kappa|\sqrt{-m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}}{\sqrt{-m(m+2\kappa)}} \tag{B17b}$$

$$- \frac{e^{-i\kappa x + \kappa z}}{4\pi} \int_0^{\varepsilon^{-1+\delta}} \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)|m-\kappa|e^{-\frac{|y|}{\kappa}|m-\kappa|\sqrt{-m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} - \frac{2m}{\kappa})}}{\sqrt{-m(m-2\kappa)}} \tag{B17c}$$

$$+ \frac{ie^{-i\kappa x + \kappa z}}{4\pi} \int_{-\varepsilon^{-1+\delta}}^{0} \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)(m-\kappa)e^{-i\frac{|y|}{\kappa}(m-\kappa)\sqrt{m(m-2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} - \frac{2m}{\kappa})}}{\sqrt{m(m-2\kappa)}} \tag{B17d}$$

This completes our outer expansion and all we need to do now is to find the appropriate inner expansion of the outer expansion.

<u>Inner Expansion of the Outer Expansion</u>

To find the inner expansion of the outer expansion, we shall let $y,z = O(\varepsilon)$ and we shall expand the exponential terms in y and z. Doing this for the integral in line (B17a), we obtain:

$$e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}$$

$$\sim \underset{O(1)}{1} - \underset{O(\varepsilon^{3\delta/2})}{i|y|\frac{m}{\kappa}\sqrt{m(m+2\kappa)}} - \underset{O(\varepsilon^{\delta/2})}{i|y|\sqrt{m(m+2\kappa)}} + \underset{O(\varepsilon^{2\delta})}{\kappa z\frac{m^2}{\kappa^2}} + \underset{O(\varepsilon^{\delta})}{\kappa z\frac{2m}{\kappa}} ,$$

where the given orders of magnitude correspond to the maximum value of m. By discarding all but the two lowest order terms and by dividing by $\sqrt{m(m+2\kappa)}$, we obtain:

$$\frac{e^{-i\frac{|y|}{\kappa}(m+\kappa)\sqrt{m(m+2\kappa)} + \kappa z(\frac{m^2}{\kappa^2} + \frac{2m}{\kappa})}}{\sqrt{m(m+2\kappa)}} \sim \frac{1}{\sqrt{m(m+2\kappa)}} - i|y|$$

to two terms. Repeating the same procedure with the integral in line (B17b), we obtain:

$$\frac{e^{-\frac{|y|}{\kappa}|m+\kappa|\sqrt{-m(m+2\kappa)}\ +\ \kappa z\left(\frac{m^2}{\kappa^2}+\frac{2m}{\kappa}\right)}}{\sqrt{-m(m+2\kappa)}} \sim \frac{1}{\sqrt{-m(m+2\kappa)}} - |y| \ .$$

In the two integrals in lines (B17c) and (B17d), the same procedure is followed. However, because the contribution due to these integrals is higher order, we need only keep the leading order term from each integral. Upon substituting these expansions into equation (B17) and expanding the square roots in a series, we find that to the leading order:

$$\begin{aligned}
\phi_1(x,y,z) \sim\ & \frac{i\kappa e^{i\kappa x + \kappa z}}{4\pi} \int_0^{\varepsilon^{-1+\delta}} dm\ e^{imx} \Sigma^*(m) \left(\frac{1}{\sqrt{2\kappa m}} - i|y|\right) \\
& - \frac{e^{i\kappa x + \kappa z}}{4\pi} \int_{-\varepsilon^{-1+\delta}}^{0} dm\ e^{imx} \Sigma^*(m) \left(\frac{1}{\sqrt{-2\kappa m}} - |y|\right) \\
& - \frac{e^{-i\kappa x + \kappa z}}{4\pi} \int_0^{\varepsilon^{-1+\delta}} dm\ e^{imx}\ \frac{\Sigma^*(m-2\kappa)}{\sqrt{2\kappa m}} \\
& - \frac{i\kappa e^{-i\kappa x + \kappa z}}{4\pi} \int_{-\varepsilon^{-1+\delta}}^{0} dm\ e^{imx}\ \frac{\Sigma^*(m-2\kappa)}{\sqrt{-2\kappa m}} \ .
\end{aligned} \qquad \text{(B18)}$$

We would like to use the theory of Fourier transforms to evaluate these integrals, and to do this, we must replace the limits of integration, $\pm\ \varepsilon^{-1+\delta}$, by $\pm\ \infty$. However, before we can extend the interval of integration, we must determine that the errors which we make by extending the limits of integration are higher order than the terms we keep in the expansion of $\phi_1(x,y,z)$. To do this, we need only to check the lowest order terms, the two integrals involving $\Sigma^*(m)$ of the form:

$$I = \int_0^{\varepsilon^{-1+\delta}} dm\ e^{imx}\ \frac{\Sigma^*(m)}{\sqrt{2\kappa m}} \ . \qquad \text{(B19)}$$

We can write equation (B19) as:

$$I = \int_0^\infty \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{2\kappa m}} - \int_{\varepsilon^{-1+\delta}}^\infty \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{2\kappa m}} \ . \tag{B20}$$

We now must show that the second integral is of sufficiently high order to be neglected. If we assume that $\Sigma(x)$ has a square root singularity at $x = 0$ and that $\Sigma(x)$ is identically zero for $x < 0$, by using the theory of generalized functions [Lighthill (1964)], we can say that:

$$\Sigma^*(k) = \frac{O(\Sigma)}{\sqrt{|k|}} \ ,$$

for large k. Substituting this into the error term in equation (B20), we can bound the error as follows:

$$\int_{\varepsilon^{-1+\delta}}^\infty \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{2\kappa m}} < \frac{c}{\sqrt{2\kappa}} \int_{\varepsilon^{-1+\delta}}^\infty \frac{dm}{m} = -\frac{2c}{\sqrt{2\kappa}}\varepsilon^{2-2\delta} \ ,$$

where c is a constant, $c = O(\Sigma)$. Based on this, we can say that:

$$I = \int_0^\infty \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{2\kappa m}} + O(\Sigma\varepsilon^{5/2}) \ ,$$

and thereby, extend the range of integration in equation (B18). However, we must note that this analysis is invalid if we find that the error in I is of the same order as the high order terms in equation (B18). Rewriting equation (B18) to include the extended range of integration, we obtain:

$$\phi_1(x,y,z) = \frac{\kappa e^{i\kappa x + \kappa z}}{4\pi}\left\{ i\int_0^\infty \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{2\kappa m}} - \int_{-\infty}^0 \frac{dm\ e^{imx}\Sigma^*(m)}{\sqrt{-2\kappa m}}\right\} \tag{B21a}$$

$$+ \frac{\kappa|y|e^{i\kappa x + \kappa z}}{4\pi}\int_{-\infty}^\infty dm\ e^{imx}\Sigma^*(m) \tag{B21b}$$

$$+ \frac{\kappa e^{-i\kappa x + \kappa z}}{4\pi}\left\{ -\int_0^\infty \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)}{\sqrt{2\kappa m}} - i\int_{-\infty}^0 \frac{dm\ e^{imx}\Sigma^*(m-2\kappa)}{\sqrt{-2\kappa m}}\right\} \tag{B21c}$$

By combining the integrals of line (B21a) and the integrals of line (B21c), and by applying the convolution theorem, we can rewrite $\phi_1(x,y,z)$ as:

$$\phi_1(x,y,z) = -\frac{\kappa}{2} e^{i\kappa x - i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}}$$

$$-\frac{\kappa}{2} e^{-i\kappa x + i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi) e^{2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}} \qquad \text{(B22)}$$

$$+\frac{\kappa|y|}{2} e^{i\kappa x} e^{\kappa z} \Sigma(x) \ .$$

Repeating the same procedures with $\phi_2(x,y,z)$ as we did with $\phi_1(x,y,z)$, we can write $\phi_2(x,y,z)$, equation (B12), in the transform space as:

$$\phi_2(x,y,z) = \frac{e^{i\kappa x + \kappa z}}{4\pi} \left\{ i \int_0^\infty \frac{dm\ e^{imx}\ \overline{\Sigma^*(m+2\kappa)}}{\sqrt{2\kappa m}} - \int_{-\infty}^0 \frac{dm\ e^{imx}\ \overline{\Sigma^*(m+2\kappa)}}{\sqrt{-2\kappa m}} \right\}$$

$$+\frac{\kappa e^{-\kappa x + \kappa z}|y|}{4\pi} \int_{-\infty}^{\infty} dm\ e^{imx}\ \overline{\Sigma^*(m)}$$

$$+\frac{\kappa e^{-i\kappa x + \kappa z}}{4\pi} \left\{ -\int_0^\infty \frac{dm\ e^{imx}\ \overline{\Sigma^*(m)}}{\sqrt{2\kappa m}} - \int_{-\infty}^0 \frac{dm\ e^{imx}\ \overline{\Sigma^*(m)}}{\sqrt{-2\kappa m}} \right\} .$$

Applying the convolution theorem to the above equation for $\phi_2(x,y,z)$, we can write it as:

$$\phi_2(x,y,z) = -\frac{\kappa}{2} e^{i\kappa x - i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \overline{\Sigma(\xi)} e^{-2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}}$$

$$-\frac{\kappa}{2} e^{-i\kappa x + i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \overline{\Sigma(\xi)}}{\sqrt{2\pi\kappa(x-\xi)}} \qquad \text{(B23)}$$

$$+\frac{\kappa|y|}{2} e^{-i\kappa x} e^{\kappa z}\ \overline{\Sigma(x)} \ .$$

We can now combine $\phi_1(x,y,z)$, equation (B22), and $\phi_2(x,y,z)$, equation (B23), to find $\phi(x,y,z)$ as:

$$\phi(x,y,z) = \phi_1(x,y,z) + \phi_2(x,y,z)$$

$$= -\frac{\kappa}{2} e^{i\kappa x - i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B24a}$$

$$-\frac{\kappa}{2} e^{-i\kappa x + i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \overline{\Sigma(\xi)}}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B24b}$$

$$-\frac{\kappa}{2} e^{-i\kappa x + i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi) e^{2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B24c}$$

$$-\frac{\kappa}{2} e^{i\kappa x - i\pi/4} e^{\kappa z} \int_0^x \frac{d\xi\ \overline{\Sigma(\xi)} e^{-2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B24d}$$

$$+\frac{\kappa|y|}{2} e^{i\kappa x} e^{\kappa z}\ \Sigma(x) \tag{B24e}$$

$$+\frac{\kappa|y|}{2} e^{-i\kappa x} e^{\kappa z}\ \overline{\Sigma(x)} \tag{B24f}$$

If we note that lines (B24b), (B24d) and (B24f) are the complex conjugates of lines (B24a), (B24c), and (B24e) respectively, we see that equation (B24) can be written as twice the real part of lines (B24a), (B24c) and (B24e). Thus, we can rewrite $\phi(x,y,z)$ as follows:

$$\phi(x,y,z) = \mathrm{Re} \left\{ -\kappa e^{\kappa z} e^{i\kappa x - i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}} \right. \tag{B25a}$$

$$-\kappa e^{\kappa z} e^{-i\kappa x + i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi) e^{2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B25b}$$

$$\left. +\kappa|y|\ e^{\kappa z} e^{i\kappa x}\ \Sigma(x) \right\} . \tag{B25c}$$

If we apply the methods of asymptotic expansions, the integral of line (B25b) can be simplified. However, first we shall rewrite lines (B25a) and (B25b) and show how $\phi(x,y,z)$ can be expressed in terms of $R(x)$. By multiplying line (B25a) by $e^{i\kappa\xi} e^{-i\kappa\xi}$ and by moving the $e^{i\kappa x - i\pi/4}$

inside the integral terms, we obtain:

$$\phi(x,y,z) = \mathrm{Re}\left\{ -\kappa e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi)e^{i\kappa\xi}\ e^{i\kappa(x-\xi) - i\pi/4}}{\sqrt{2\pi\kappa(x-\xi)}} \right.$$

$$- \kappa e^{\kappa z} \int_0^x \frac{d\xi\ \Sigma(\xi)e^{i\kappa\xi}\ e^{-i\kappa(x-\xi) + i\pi/4}}{\sqrt{2\pi\kappa(x-\xi)}} \tag{B26}$$

$$\left. + \kappa|y|\ e^{\kappa z}\ \Sigma(x)e^{i\kappa x} \right\} .$$

We may now combine the two integrals in equation (B26) to obtain:

$$\phi(x,y,z) = -2\kappa\ e^{\kappa z} \int_0^x \frac{d\xi\ \mathrm{Re}[\Sigma(\xi)e^{i\kappa\xi}]\cos[\kappa(x-\xi) - \pi/4]}{\sqrt{2\pi\kappa(x-\xi)}}$$

$$+ \kappa|y|e^{\kappa z}\mathrm{Re}[\Sigma(x)e^{i\kappa x}] \quad , \tag{B27}$$

where by definition $R(x) = \mathrm{Re}[\Sigma(x)e^{i\kappa x}]$.

Returning to equation (B25), we shall obtain an asymptotic expression of the integral in line (B25b). By employing the method for the asymptotic expansion of Fourier integrals given by Erdélyi (1956), we can show that:

$$\int_0^x \frac{d\xi\ \Sigma(\xi)e^{2i\kappa\xi}}{\sqrt{x-\xi}} = \sqrt{\frac{\pi}{2\kappa}}\ \Sigma(x)e^{2i\kappa x - i\pi/4} + O(\Sigma\kappa^{-1}) \quad ;$$

upon substituting this into line (B25b), we find that:

$$-\kappa e^{\kappa z}e^{-i\kappa x + i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi)e^{2i\kappa\xi}}{\sqrt{2\pi\kappa(x-\xi)}} = -e^{\kappa z}\ \frac{\Sigma(x)e^{i\kappa x}}{2} + O(\Sigma\kappa^{-1/2}) \quad .$$

We may now substitute this value into equation (B25) as shown below:

$$\phi(x,y,z) \sim \mathrm{Re}\left\{ \underbrace{-\kappa e^{\kappa z}e^{i\kappa x - i\pi/4} \int_0^x \frac{d\xi\ \Sigma(\xi)}{\sqrt{2\pi\kappa(x-\xi)}}}_{O(\Sigma\varepsilon^{-1/2})} \underbrace{- \frac{1}{2}e^{\kappa z}\Sigma(x)e^{i\kappa x}}_{O(\Sigma)} + \underbrace{\kappa|y|e^{\kappa z}\Sigma(x)e^{i\kappa x}}_{O(\Sigma)} \right\} .$$

This completes the inner expansion of the outer expansion for a rapidly oscillating source distribution.

BIBLIOGRAPHY

Abramowitz, M. and Stegun, I. 1964. Handbook of mathematical functions. Washington, D.C.: U.S. Government Printing Office.

Adachi, H. 1973. On some experimental results of a ship with extremely long parallel middle body. Translated by Hiroyuki Adachi. Unpublished report of the Ship Research Institute, Tokyo, Japan.

Erdélyi, E. 1956. Asymptotic expansions. New York: Dover Publications, Inc.

Faltinsen, O. M. 1971. A rational strip theory of ship motions: part 2, Report No. 113, Department of Naval Architect and Marine Engineering, University of Michigan, Ann Arbor, Michigan. 137 pp.

Froude, W. E. 1868. Observations and suggestions on the subject of determining by experiment the resistance of ships. The papers of William Froude, 1810-1879. London: The Institution of Naval Architects, 1955, pp. 120-27.

Lewison, G. R. G. 1963. Determination of the wave resistance of a partly immersed axisymmetric body. In International seminar on theoretical wave resistance, ed. F. C. Michelsen, pp. 585-97. Ann Arbor, Michigan: University of Michigan.

Lighthill, J. J. 1964. Introduction to Fourier analysis and generalized functions. London: Cambridge University Press.

Michell, J. H. 1898. The wave resistance of a ship. Philosophical Magazine 45:106-23.

Ogilvie, T. F. 1973. The wave generated by a fine ship bow, Report No. 127, Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, Michigan. 40 pp.

———. 1974. Workshop on slender body theory part I: free surface effects, Report No. 163, Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, Michigan.

Rudin, W. 1964. Principles of mathematical analysis. 2d ed. New York: McGraw-Hill Book Company.

Tricomi, F. G. 1957. Integral equations. New York: Wiley-Interscience.

Tuck, E. O. 1963. The steady motion of a slender ship. Ph.D. dissertation, Trinity College, University of Cambridge. 190 pp.

Ursell, F. 1962. Slender oscillating ships at zero forward speed. _Journal of fluid mechanics_ 19:496-516.

———. 1968a. On head seas traveling along a horizontal cylinder. _Journal of the Institute of Mathematics and its Applications_ 4:414-27.

———. 1968b. The expansion of water-wave potentials at great distances. _Proceedings of the Cambridge Philosophical Society_ 64:811-26.

———. 1973. On the refraction of head seas by a long ship. Unpublished report presented at Workship on Slender-Body Theory, The University of Michigan, Ann Arbor, Michigan.

———. 1974a. Note on the refraction of head seas by long ships. _Proceedings of the Tenth Symposium on Naval Hydrodynamics_. Washington, D.C.: Office of Naval Research, forthcoming.

———. 1974b. The refraction of head seas by a long ship. Unpublished.

Ward, G. N. 1955. _Linearized theory of steady high-speed flow_. London: Cambridge University Press