Symbolic Powers and other Contractions of Ideals in Noetherian Rings

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2012

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To my wife Shweta, my parents – Swati and Ajay More and my brother Aditya

ACKNOWLEDGEMENTS

I am grateful to Mel Hochster for giving me this opportunity to work under his mentorship. Coming from very little background in mathematics, I don't think I could have asked for a better advisor during the last few years. Mel Hochster's incredible intuition, infectious passion and infinite patience have been a huge influence in shaping my mathematical process. I am thankful to him for supporting me on his grant for several semesters. I am also thankful for his course notes which very often have very non-standard proofs of standard results.

I would like to thank the members of the 2006 graduate admissions committee for giving me a chance to pursue a Ph.D. at University of Michigan, Ann Arbor despite having not having an undergraduate degree in this field. I am especially thankful to Peter Scott and John Eric Fornaess for the smooth transition from the Applied Mathematics program to the Pure Mathematics program.

I would like to thank all the professors I have interacted with during the course of this program. I would especially like to thank Karen Smith for her exciting Algebraic Geometry class and for her suggestion of working with Mel Hochster. I would like to thank Gavin LaRose for all his help with teaching. I would like to thank the faculty members on my thesis committee – Karen Smith, Michael Zieve, Wenliang Zhang and Jun Zhang for their time and feedback. I am also thankful for the staff in the administrative office whose prompt responses have made every non-academic task substantially easy. In particular, a special thank you to Tara McQueen, Stephanie

Caroll and Bert Ortiz for all their support.

I would like to thank Emily Witt for all her help during the past couple of years. I would like to thank Stephanie Jakus for all the mathematical and non-mathematical conversations. Special thanks to Anirudh RR for being a great friend and helping me through some of the most difficult times in Ann Arbor. Special thanks to Geetesh Gadkari for his friendship and all the interesting discussions over the years.

I would like to thank my parents – Swati and Ajay More, for an endless list of reasons. In particular, I would like to thank them for their unwavering support of all my decisions and for providing me the best possible education despite our economic conditions. I would like to thank my brother Aditya for his support and friendship.

Finally and most importantly, I would like to thank my wife Shweta for her limitless support and for believing in me during my moments of despair. I will be forever grateful to her for the life we have together which is richer, happier and much more meaningful than the sum of its parts.

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CHAPTER 1

Introduction

This thesis focuses on questions about the properties of symbolic powers of ideals in Noetherian rings and some related problems. All rings in this thesis are commutative rings with identity.

Given a commutative ring R and a prime ideal P, for a positive integer n, the nth *symbolic power* of P (denoted $P^{(n)}$) is defined to be the contraction to R of the expansion of P^n to R_P, *i.e.,* $P^{(n)} = P^n R_P \cap R$. An equivalent definition is $P^{(n)} = \{r \in R : \exists w \in R \setminus P, wr \in P^n\}$. $P^{(n)}$ is the smallest P-primary ideal containing P^n. Further, if P^n has a primary decomposition (which is true if R is Noetherian), then, the P-primary ideal that must be used in any irredundant primary decomposition is $P^{(n)}$.

The motivation for the work in this thesis comes from the following four questions each of which is the basis of an individual chapter.

1. (Eisenbud-Mazur conjecture, chapter 3) Given a regular local ring $(R, \mathfrak{m})$ and a prime ideal $P \subset R$, when is $P^{(2)} \subseteq \mathfrak{m}P$? Eisenbud and Mazur [EM97] have constructed examples in every positive characteristic p when R contains a field, to show that the statement does not hold. They also conjecture that if R contains a field of characteristic zero, then, the statement is true.

2. (Integral closedness of $\mathfrak{m}I$, chapter 2) Given a regular local ring $(R, \mathfrak{m})$ and a radical ideal $I \subset R$, when is $\mathfrak{m}I$ integrally closed? We will illustrate that this question is closely related to the Eisenbud-Mazur conjecture.

3. (Uniform bounds on symbolic powers, chapter 4) Given a Noetherian complete local domain R, is there a positive integer k such that for any prime ideal $P \subset R$, $P^{(kn)} \subseteq P^n$ for all positive integers n?

4. (General contractions of powers of ideals, chapter 5) Given an extension of Noetherian rings $R \subseteq S$ and an ideal J in S what can be said about the behavior of $I_n := J^n \cap R$ as n varies over positive integers? In particular, when is $\oplus_{i=0}^{\infty} I_n$ a Noetherian ring?

1.1 Eisenbud-Mazur conjecture

Eisenbud and Mazur [EM97] studied symbolic powers in connection with the question of existence of non-trivial evolutions.

Definition 1.1.1. Let R be a ring and S be a local R-algebra essentially of finite type. An *evolution* of S over R consists of the following data:

- A local R-algebra T essentially of finite type.
- A surjection $T \to S$ of R-algebras such that if $\Omega_{T/R}$ and $\Omega_{S/R}$ denote the modules of Kähler differentials of T over R and of S over R respectively, then, the induced map $\Omega_{T/R} \otimes_T S \to \Omega_{S/R}$ is an isomorphism.

The evolution is said to be *trivial* if $T \to S$ is an isomorphism.

The question of existence of non-trivial evolutions leads to the Eisenbud-Mazur conjecture via theorem 1.1.5. We first need a more general definition of symbolic powers.

Definition 1.1.2. Let R be a ring and I an ideal in R. For a positive integer n, the *nth symbolic power* of I is defined to be

$$I^{(n)} := \{r \in R : \frac{r}{1} \in I^n R_P \text{ for all } P \text{ such that } P \text{ is a minimal prime of } I\}.$$

Definition 1.1.3. Let R be a ring and I an ideal in R. We say I is an *unmixed ideal* if every associated prime ideal of I is isolated. In other words, an unmixed ideal has no embedded prime ideals.

Remark 1.1.4. If I is an unmixed ideal in a ring R, then, $I^{(1)} = \cap_P(IR_P \cap R)$ by definition (where the intersection is taken over the minimal primes P of I) and the right hand side of this equation is a minimal primary decomposition for I and hence $I^{(1)} = I$.

Theorem 1.1.5. *[EM97] Let R be a regular ring. Let $(P, \mathfrak{m})$ be a localization of a polynomial ring in finitely many variables over R. Let I be an ideal of P. If P/I is reduced and generically separable over R, then, every evolution of P/I is trivial if and only if $I^{(2)} \subseteq \mathfrak{m}I$.*

We now state a slightly more general version of the Eisenbud-Mazur conjecture.

Conjecture 1.1.6. *(Eisenbud-Mazur) Given a regular local ring $(R, \mathfrak{m})$ containing a field of characteristic zero and an unmixed ideal I in R, $I^{(2)} \subseteq \mathfrak{m}I$.*

The hypothesis that R be regular is necessary. If R is not regular, Huneke-Ribbe [HR98] show that there exists a prime ideal P in R for which $P^{(2)} \not\subseteq \mathfrak{m}P$.

Example 1.1.7. [HR98] Let $R = k[x, y, z]/(x^2 - yz)$, $P = (x, y)R$. Then $z \in R \setminus P$ and $zy = x^2 \in P^2$. So that, $y \in P^{(2)}$. However, $y \notin (x, y, z)P$.

Eisenbud and Mazur construct examples in every positive characteristic p to show that the corresponding statement of conjecture 1.1.6 does not hold.

Example 1.1.8. [EM97] Let k be a field of characteristic $p > 0$ and let P be the kernel of the map

$$k[[x_1, ..., x_4]] \to k[[t]]$$

given by $x_1 \to t^{p^2}, x_2 \to t^{p(p+1)}, x_3 \to t^{p^2+p+1}, x_4 \to t^{(p+1)^2}$. Let $f = x_1^{p+1}x_2 - x_2^{p+1} - x_1x_3^p + x_4^p$. Then $f \in P^{(2)}$ but $f \notin \mathfrak{m}P$, where $\mathfrak{m} = (x_1, ..., x_4)R$ and $R = k[[x_1, ..., x_4]]$.

These counterexamples tend to focus attention on the statement of conjecture 1.1.6 in equal characteristic zero and mixed characteristic. However, it may be possible that the statement also holds in equal characteristic p with some auxiliary hypothesis on the ring. We finish this section with some known affirmative results for the conjecture discussed in [EM97] and [HR98].

Definition 1.1.9. An ideal I in a ring R is said to be *generically a complete intersection* if I is an unmixed ideal of height h and IR_P is generated by a regular sequence of length h in R_P for every minimal prime ideal P of I.

Theorem 1.1.10. *[EM97] Suppose that I is an ideal in a Noetherian ring R generated by the $(n-1) \times (n-1)$ minors of an $n \times (n-1)$ matrix M. Suppose that the depth of I on R is 2 and I is generically a complete intersection in R. If J is the ideal generated by the entries of any column of the matrix M, then, $I^{(2)} \subset JI$. In particular if R is local with maximal ideal $\mathfrak{m}$ and the entries of M are contained in $\mathfrak{m}$, then, $I^{(2)} \subset \mathfrak{m}I$.*

Theorem 1.1.11. *[EM97] Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Suppose that I is a monomial ideal in R . If P is a monomial prime ideal such that $I \subseteq P$, then, for all positive integers d, $I^{(d)} \subseteq PI^{(d-1)}$. In particular, if I is unmixed, with $d = 2$ and $P = \mathfrak{m} = (x_1, ..., x_n)R$, we get that $I^{(2)} \subseteq \mathfrak{m}I$.*

We make the following definition following [Vas04].

Definition 1.1.12. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let I, J be ideals in R. I and J are said to be *linked* if there is an R-sequence $\underline{x} = x_1, ..., x_n$ contained in $I \cap J$ such that $J = (\underline{x})R : I$ and $I = (\underline{x})R : J$. We denote this by $I \sim J$. I, J are said to be in the same *linkage class* if there exists a sequence of ideals $I_1, ..., I_m$ such that

$$I \sim I_1 \sim ... \sim I_m \sim J.$$

An ideal I that lies in the linkage class of a complete intersection ideal is said to be *licci.*

Theorem 1.1.13. *[EM97] Suppose that $(R, \mathfrak{m})$ is a regular local ring and $I \subset R$ is a perfect ideal that is generically a complete intersection. Let J be the ideal generated by the elements in a row of some presentation matrix over R (or over R/I) of the canonical module $\omega_{R/I}$ of R. If I is licci, then, $I^{(2)} \subset JI$. In particular, if the entries of J are in $\mathfrak{m}$, we have that $I^{(2)} \subseteq \mathfrak{m}I$.*

Theorem 1.1.14. *(Kunz) [EM97] Let $(R, \mathfrak{m})$ be a regular local ring and let I be a proper, unmixed ideal that is generically a complete intersection. If I can be generated by $ht(I) + 1$ elements, then, $I^{(2)} \subseteq \mathfrak{m}I$.*

Theorem 1.1.15. *(Huneke-Ribbe) [HR98] Let $(R, \mathfrak{m})$ be a regular local ring and I a proper ideal of R such that R/I is normal. If I can be generated by $ht(I) + 2$ elements, then, $I^{(2)} \subseteq \mathfrak{m}I$.*

Theorem 1.1.16. *[EM97] Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k and let $\mathfrak{m} = (x_1, ..., x_n)R$. Suppose that I is a quasihomogeneous ideal. If $char(k) = 0$, then, $I^{(d)} \subseteq \mathfrak{m}I^{(d-1)}$ for all positive integers d. In particular, if I is unmixed, with $d = 2$, we get that $I^{(2)} = \mathfrak{m}I$. Further, if I is radical, then, every evolution of R/I is trivial.*

Theorem 1.1.17. *[HR98] Let I be an unmixed ideal of height h in a regular local ring $(R, \mathfrak{m})$ of equicharacteristic 0. Assume that the minimal number of generators of I is $h+2$ and $I_P^2 = I_P^{(2)}$ for every prime P with height $h+1$ such that $I \subset P$. Then $I^{(2)} \subseteq \mathfrak{m}I$.*

Definition 1.1.18. Let R be a regular local ring and I an ideal of R such that R/I is Cohen-Macaulay. Then R/I is said to have *minimal multiplicity* if the multiplicity of R/I is exactly equal to $\dim(R)-\dim(R/I)+1$. R/I is said to have *almost minimal multiplicity* if the multiplicity of R/I is equal to $\dim(R) - \dim(R/I) + 2$.

Theorem 1.1.19. *[HR98] Let $(R, \mathfrak{m})$ be a regular local ring of equicharacteristic 0 and dimension d. Suppose that I is an unmixed integrally closed ideal of height $d-1$. Assume that $I \subseteq \mathfrak{m}^2$. Then*

1. if R/I has minimal multiplicity, $I^{(2)} \subseteq \mathfrak{m}I$.

2. if R/I is Gorenstein and has almost minimal multiplicity, $I^{(2)} \subseteq \mathfrak{m}I$.

1.2 Integral closedness of $\mathfrak{m}I$

Eisenbud and Mazur obtain the following result which raises a related question.

Theorem 1.2.1. *There exists a reduced, local $\mathbb{C}$-algebra of finite type whose localization at the origin has a nontrivial evolution if and only if there exists a polynomial $f \in \mathbb{C}[[x_1, ..., x_n]] = R$ such that $f(0) = 0$ and $f \notin \mathfrak{m}\sqrt{(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$, where $\mathfrak{m} = (x_1, ..., x_n)R$.*

The following question is then raised [EM97]:

Question 1.2.2. *Is there a power series $f \in \mathbb{C}[[x_1, ..., x_n]] = R$ such that $f \notin \overline{\mathfrak{m}(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$, where $\mathfrak{m} = (x_1, ..., x_n)R$ (where overline denotes integral closure in R).*

Since for any ideal I in a ring R, we have that $\overline{I} \subseteq \sqrt{I}$ [HS06], a negative answer to the above question would prove the existence of a non-trivial evolution by theorem 1.2.1 in the characteristic 0 case. However, it should be noted that $f \in \overline{\mathfrak{m}(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$ as shown in the next proposition.

Proposition 1.2.3. *Let $R = S[x_1, ..., x_n]$, where S is a Noetherian domain. Let $f \in R$. Then $f \in \overline{\mathfrak{m}(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$, where $\mathfrak{m} = (x_1, ..., x_n)R$.*

Proof. Let $I = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R$. Assume $f \neq 0$ for the statement is trivially true when $f = 0$ (for $I = 0$ in this case, and thus, $\overline{\mathfrak{m}I} = \overline{0} = 0$ since R is a domain). By the valuative criterion of integral closure (theorem 6.8.3., page 135, [HS06]), it suffices to show that $f \in (\mathfrak{m}I)V$ for every discrete valuation ring V such that for some minimal prime ideal P of R, $R/P \subseteq V \subseteq \mathrm{Frac}(R/P)$. Suppose that $f = a_1 x_1^{i_{11}} ... x_n^{i_{1n}} + ... + a_m x_1^{i_{m1}} ... x_n^{i_{mn}}$, where $a_1, ..., a_m \in R$, $i_{jk} \in \mathbb{Z}_{\geq 0}$ for $j \in \{1, ..., m\}$ and $k \in \{1, ..., n\}$. Let $t \in V$ be a uniformizing parameter. Suppose that $x_i = b_i t^{c_i}$, where $b_i \in V$ is a unit and $c_i \in \mathbb{Z}_{\geq 0}$. We may assume without loss of generality that $i_{11}c_1 + ... + i_{1n}c_n \leq i_{j1}c_1 + ... + i_{jn}c_n$ for $j \in \{1, ..., m\}$. Then $f = bt^{i_{11}c_1 + ... + i_{1n}c_n}$ for some unit $b \in V$. Now since V is a discrete valuation ring, every ideal is principal. Assume that $\mathfrak{m}I = x_r \frac{\partial f}{\partial x_s}$, where $r, s \in \{1, ..., n\}$. Now we may write that $\frac{\partial f}{\partial x_s} = dt^{\hat{i_{l1}}c_1 + ... + \hat{i_{ln}}c_n}$ for some $l \in \{1, ..., m\}$, where $\hat{i_{lj}} = i_{lj}$ if $j \neq s$ and $\hat{i_{ls}} = i_{ls} - 1$ and $d \in V$ is a unit. Then $(\mathfrak{m}I)V = t^{c_r} t^{\hat{i_{l1}}c_1 + ... + \hat{i_{ln}}c_n} V$. Now, if $\hat{i_{l1}}c_1 + ... + \hat{i_{ln}}c_n + c_r > i_{11}c_1 + ... + i_{1n}c_n$, then, suppose without loss of generality that $i_{11} > 0$ (at least one of $i_{11}, ..., i_{1n}$ must be positive since $f = bt^{i_{11}c_1 + ... + i_{1n}c_n}$ and $f \neq 0$). Then $(x_1 \frac{\partial f}{\partial x_1}) = t^{i_{11}c_1 + ... + i_{1n}c_n} V \supsetneq (\mathfrak{m}I)V$, which is a contradiction. So $\hat{i_{l1}}c_1 + ... + \hat{i_{ln}}c_n + c_r \leq i_{11}c_1 + ... + i_{1n}c_n$. Thus, $f = bt^{i_{11}c_1 + ... + i_{1n}c_n} \in t^{c_r} t^{\hat{i_{l1}}c_1 + ... + \hat{i_{ln}}c_n} V = (\mathfrak{m}I)V$. □

In light of the above discussion, the following question is considered in this thesis.

Question 1.2.4. *In a regular local ring $(R, \mathfrak{m})$ with an ideal I when is $\overline{\mathfrak{m}I} \subseteq \mathfrak{m}\sqrt{I}$?*

Let J be the radical of I. Then it suffices to prove $\mathfrak{m}J$ is integrally closed: for if the latter is true, then, $\overline{\mathfrak{m}I} \subseteq \overline{\mathfrak{m}J} = \mathfrak{m}J = \mathfrak{m}\sqrt{I}$. The question of integral closure of $\mathfrak{m}J$, where J is a radical ideal has been explored by Hübl-Huneke in [HH01]. In general the ideal $\mathfrak{m}I$ fails to be integrally closed even when I is prime. Huneke shows that if $R = \mathbb{C}[[x, y, z, w]]$ and $f_1 = x^3 - yz, f_2 = yz^2 - w^2, f_3 = xy^5 - z^3w, f_4 = y^6 - x^2y^2z, f_5 = z^5 - xy^4w, f_6 = x^2z^4 - y^5w$. Then $P = (f_1, ..., f_6)R$ is a prime ideal in R and $\mathfrak{m}P$ is not an integrally closed ideal, where $\mathfrak{m} = (x, y, z, w)R$ (example 4.4, [Hüb99]).

Question 1.2.4 also relates to the question of the existence of non-trivial evolutions via the following theorem.

Theorem 1.2.5. *[Hüb99] Let $(R, \mathfrak{m})$ be a regular local ring and let k be a field such that char$(k) = 0$. Suppose that R/k is essentially of finite type and $R/\mathfrak{m}$ is a finite extension of k. If I is an ideal such that $\mathfrak{m}I$ is integrally closed, then, R/I is has no non-trivial evolutions.*

Hübl and Huneke [HH01] obtained results for the following cases.

Theorem 1.2.6. *[HH01] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d > 1$ and let I be an unmixed ideal. Suppose that I is minimally generated by n elements and let $S = R[x_1, ..., x_n] \to \mathcal{R}(I)$ be the standard map onto the Rees ring of I with kernel $\mathfrak{a}$. Let $\mathfrak{a}_m$ denote the ideal generated by all homogeneous forms in $\mathfrak{a}$ of degree at most m. Assume there exists an integer $m \geq 2$ such that:*

1. $\mathfrak{a}_{m+1} \subseteq \mathfrak{m}S$.

2. I^m and I^{m+1} are integrally closed.

3. $\operatorname{depth}(R/I^m) = 0$.

Then $I \cap \overline{\mathfrak{m}I} \subseteq \mathfrak{m}I$. In particular, if I is integrally closed, then, $\overline{\mathfrak{m}I} = \mathfrak{m}I$.

Corollary 1.2.7. *[HH01] Let $(R, \mathfrak{m})$ be a regular local ring of dimension 3 and let I be an ideal of height 2 such that I is normal, R/I is Cohen-Macaulay and I is generically a complete intersection. If $\mathfrak{a}_3 \subseteq \mathfrak{m}I$, then, $\mathfrak{m}I$ is integrally closed.*

For a Noetherian local ring $(R, \mathfrak{m})$, and an ideal I of R define the *fiber cone* of I as follows

$$F_{\mathfrak{m}}(I) := \oplus_{n \in \mathbb{Z}_{\geq 0}} I^n/\mathfrak{m}I^{n+1} = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I).$$

Theorem 1.2.8. *[HH01] Let $(R, \mathfrak{m})$ be a Noetherian normal local domain of dimension d such that $R/\mathfrak{m}$ is infinite and I be a normal ideal of R. Suppose that I has analytical spread d. If $F_{\mathfrak{m}}(I)$ is equidimensional without embedded components, then, $\mathfrak{m}I^n = \overline{\mathfrak{m}I^n}$ for all positive integers n.*

Corollary 1.2.9. *[HH01] Let $(R, \mathfrak{m})$ be a normal local Cohen-Macaulay domain such that $R/\mathfrak{m}$ is infinite and let I be a normal $\mathfrak{m}$-primary ideal with reduction number at most 1. Then $\mathfrak{m}I^n$ is integrally closed for all positive integers n.*

Corollary 1.2.10. *[HH01] Let $(R, \mathfrak{m})$ be a normal local Cohen-Macaulay domain of dimension d and let I be a normal unmixed syzegetic ideal of height $d-1$ and analytic spread d. If I is generically a complete intersection and if I has reduction number 2, then, $\mathfrak{m}I^n = \overline{\mathfrak{m}I^n}$ for all positive integers n.*

Corollary 1.2.11. *[HH01] Let $(R, \mathfrak{m})$ be a regular local ring of dimension 3 and let I be an ideal of height 2 having analytic spread 3. If $I$is generically a complete intersection, unmixed and $\mathcal{R}(I)$ is normal and Cohen-Macaulay. Then $\mathfrak{m}I^n = \overline{\mathfrak{m}I^n}$ for all positive integers n.*

1.3 Uniform bounds on symbolic powers of prime ideals

The question of equivalence of symbolic and adic topologies has generated considerable interest in the past two decades. For an unmixed ideal I in $\mathbb{C}[x_1, ..., x_d]$, Ein-Lazarsfeld-Smith (theorem 2.2, [ELS01]) proved that if h is the largest height of an associated prime ideal of I, then, $I^{(hn)} \subseteq I^n$ for all positive integers n. Soon after, Hochster and Huneke (theorem 1.1, [HH02]) improved this result to show that for a regular ring R containing a field and ideal I of R, if h is the largest height of an associated prime ideal of I, then, $I^{(hn)} \subseteq I^n$ for all positive integers n. In particular this implies that there is a uniform bound for the growth of symbolic powers of ideals in a regular local ring of dimension d in equal characteristic, *viz.,* $I^{(dn)} \subseteq I^n$.

Theorem 1.3.1. *[HH02]) Let R be a Noetherian ring containing a field. Let I be an ideal of R and h be the largest height of any associated prime ideal of I.*

1. If R is regular, then, $I^{(hn+kn)} \subseteq (I^{(k+1)})^n$ for all positive integers n and non-negative integers k.

2. If I has finite projective dimension $I^{(hn)} \subseteq (I^n)^*$ for all positive integers n (where $\mathfrak{a}^*$ denotes the tight closure of an ideal $\mathfrak{a}$).

Earlier Swanson [Swa00] had proved that in a Noetherian local ring R, for every prime ideal P such that the P-adic and P-symbolic topologies are equivalent, there exists a positive integer h such that $P^{(hn)} \subseteq P^n$ for all positive integers n. The value of h, *a priori* depends on the prime ideal. Huneke, Katz and Validashti obtain the following uniform result in this direction.

Theorem 1.3.2. *[HKV09] Let $(R, \mathfrak{m})$ be an equicharacteristic local domain such that R is an isolated singularity. Assume either that R is essentially of finite type over* *a field of characteristic zero, or that R has positive characteristic, is F-finite and analytically irreducible. Then there exists an integer $h \geq 1$ such that for all prime ideals $P \subsetneq \mathfrak{m}$, $P^{(hn)} \subseteq P^n$, for all positive integers n*

1.4 General contractions of powers of ideals

In the final chapter we study more general contractions of powers of ideals. Given an extension of Noetherian rings $R \subseteq S$, and an ideal J in S, the goal is to understand what can be said about the behavior of $I_n := J^n \cap R$ as n varies over positive integers. Note that if R is a domain, P is a prime ideal in R, $S = R_P$ and $J = PR_P$ and $I_n = P^{(n)}$.

Note that in general, $R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ and $I_m I_n = (J^m \cap R)(J^n \cap R) \subseteq J^{m+n} \cap R = I_{m+n}$. So we can form a graded ring $\oplus_{i=0}^{\infty} I_n$ and we would specifically like to study when $\oplus_{i=0}^{\infty} I_n$ is a Noetherian ring. In the case when $I_n = P^{(n)}$ for some prime ideal P in R, the algebra $\oplus_{i=0}^{\infty} I_n$ is not necessarily Noetherian. Counterexamples are constructed in [Ree58, Nag60, Rob85, Rob90].

We discuss the case when $S = R[x]$, where x is an indeterminate over R in chapter 5.

1.5 Outline

The main results of this thesis are divided into four chapters, each dedicated to one of the questions listed at the beginning of this chapter. In chapter 2 we consider the question of integral closedness of $\mathfrak{m}I$. Specifically we obtain a positive result for the case where I is an ideal generated by a single binomial and several monomials in a polynomial ring over a field, where $\mathfrak{m}$ denotes the unique homogeneous maximal ideal. We also obtain positive results in a number of other cases. In chapter 3 we consider the Eisenbud-Mazur conjecture for the case of certain prime ideals in certain subrings of a formal power series ring over a field and discuss some computational results. In chapter 4 we explore the question of uniform bounds on symbolic powers of prime ideals. Finally in chapter 5 we raise some questions about contractions of powers of ideals from an overring and obtain some partial results to those questions for the case of polynomial extensions.

CHAPTER 2

Integral closedness of $\mathfrak{m}I$

Let $(R, \mathfrak{m})$ be a Noetherian, local ring with maximal ideal $\mathfrak{m}$ and I be an ideal. In this chapter, we study the question of integral closedness of $\mathfrak{m}I$. We show that if R is a polynomial ring over a field, $\mathfrak{m}$ the homogeneous maximal ideal of R and I an ideal generated by one binomial and several monomials, then, $\mathfrak{m}I$ is integrally closed in R. We also obtain several results on the structure of such ideals. One of the main results of this thesis is to show that in a Noetherian local ring $(R, \mathfrak{m})$, if $I = (a_1, ..., a_d)R$ is an integrally closed ideal such that $\overline{\mathfrak{m}I_i} \subseteq I_i$ for $1 \leq i \leq d$, where $I_i = (a_1, ..., \hat{a}_i, ..., a_d)R$, then, $\mathfrak{m}I$ is integrally closed in R. We also prove a graded analog of this result. Moreover, we define a notion of monomial ideals over a fixed regular system of parameters for a regular local ring $(R, \mathfrak{m})$ and show that if I is a monomial ideal of this type such that I is radical, then, $\overline{\mathfrak{m}I} = \mathfrak{m}I$.

2.1 Monomial Ideals

In this section, we show that for a polynomial ring $R = k[x_1, ..., x_n]$ and a radical monomial ideal I in R, $(x_1, ..., x_n)I$ is integrally closed.

We define a monomial in R to be an element of the form $x_1^{\alpha_1}...x_n^{\alpha_n}$ with $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, ..., n$. Unless otherwise stated, we will assume that $\alpha_i > 0$ for at least one $i \in \{1, ..., n\}$, *i.e.,* the monomial under consideration is different from 1. We fix the following notation. For a monomial $\mu = x_1^{\alpha_1}...x_n^{\alpha_n}$ with $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, ..., n$, in R, let $\mu^{\#} = x_1^{\delta_1}...x_n^{\delta_n}$, where $\delta_i = 1$ if $\alpha_i > 0$ and $\delta_i = 0$ if $\alpha_i = 0$ for $i = 1, ..., n$. Thus, $\mu^{\#}$ is the squarefree part of μ.

Theorem 2.1.1. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k and let I be a monomial ideal in R. Let $\mathfrak{m} = (x_1, ..., x_n)R$. Then $\overline{\mathfrak{m}I} \subseteq \mathfrak{m}\sqrt{I}$. In particular, if I is radical, $\mathfrak{m}I$ is integrally closed.*

Proof. Let I be generated by the monomials $\mu_1, ..., \mu_d$. Let $J = (\mu_1^{\#}, ..., \mu_d^{\#})R$ be the ideal generated by the squarefree parts of the monomials generating I. Then we have that $\sqrt{I} = J$ (proposition 4, page 41, [Frö97]). So we have that $\mathfrak{m}I \subseteq \mathfrak{m}J$ and hence, $\overline{\mathfrak{m}I} \subseteq \overline{\mathfrak{m}J}$ (remark 1.1.3, page 2, [HS06]). So it suffices to prove, $\overline{\mathfrak{m}J}$ is integrally closed, since then, $\overline{\mathfrak{m}I} \subseteq \overline{\mathfrak{m}J} = \mathfrak{m}J = \mathfrak{m}\sqrt{I}$.

Suppose that $\mathfrak{m}J$ is not integrally closed. Now $\mathfrak{m}J$ is a monomial ideal generated by $B := \{x_i\mu_j^{\#} : i = 1, ..., n \text{ and } j = 1, ..., d\}$. Then $\overline{\mathfrak{m}J}$ is also a monomial ideal (proposition 1.4.2, page 9, [HS06]). Let μ be a monomial such that $\mu \in \overline{\mathfrak{m}J} \setminus \mathfrak{m}J$. For a monomial β in R, let e_i^{β} denote the exponent of x_i in β. Also, let us rename the elements of B by $\beta_1, ..., \beta_t$, where $t = nd$. Then, by equation 1.4.5, [HS06], there exist rational numbers c_j for $j = 1, ..., t$, such that, $c_j \geq 0$, $\Sigma_{j=1}^{t} c_j = 1$ and

$$e_i^{\mu} \geq \Sigma_{j=1}^{t}(c_j e_i^{\beta_j}). \tag{2.1}$$

We may reindex the monomials β_j such that $c_j \neq 0$ for $j = 1, ..., r$ and $c_j = 0$ for $j = r+1, ..., t$. We may also relabel the indeterminates so that $e_i^{\mu} \neq 0$ for $i = 1, ..., s$ and $e_i^{\mu} = 0$ for $i = s+1, ..., n$. Then we claim that $e_i^{\beta_j} \geq 1$ for $i = 1, ..., s$ and $j = 1, ..., r$.

First we note that if $x_i|\beta_j$ for some $j \in \{1, ..., r\}$, then, $x_i|\mu$. For by the inequality 2.1, if $x_i|\beta_j$, then, $e_i^{\beta_j} \geq 1$ and thus, $e_i^{\mu} \geq c_j e_i^{\beta_j} > 0$. In other words, if $x_i|\beta_j$ then,

$i \in \{1, ..., s\}$.

This implies that β_j is not squarefree for $j \in \{1, ..., r\}$ for otherwise we have that μ is a multiple of β_j. Suppose that $\mu = \nu\beta_j = \nu x_{i'} \mu_{j'}^{\#}$ for some $i' \in \{1, ..., n\}$, $j' \in \{1, ..., d\}$ and some monomial ν. Thus, $\mu \in \mathfrak{m}J$, contrary to the assumption.

Now suppose we have that $e_k^{\beta_j} = 0$ for some $k \in \{1, ..., s\}$ and some $j \in \{1, ..., r\}$. We write that $\beta_j = x_1^{e_1^{\beta_j}} ... x_s^{e_s^{\beta_j}}$. Note that the monomials in B are such that the exponent of every indeterminate is either 0 or 1, with the exception of at most one indeterminate, which may have an exponent of 2. Since β_j is not squarefree, we must have, $e_{l(j)}^{\beta_j} = 2$ for some $l(j) \in \{1, ..., s\}$. Then if $\tilde{\beta}_j = x_1^{e_1^{\beta_j}} ... x_{l-1}^{e_{l-1}^{\beta_j}} x_l^{e_l^{\beta_j} - 1} x_{l+1}^{e_{l+1}^{\beta_j}} ... x_s^{e_s^{\beta_j}}$, *i.e.*, $\tilde{\beta}_j$ is the monomial such that $x_l \tilde{\beta}_j = \beta_j$ and is thus squarefree, then, $\tilde{\beta}_j = \mu_w^{\#}$ for some $w \in \{1, ..., d\}$, for the only monomials in B which have an indeterminate with an exponent 2 are those that are product of an indeterminate x_u (and hence an element $\mathfrak{m}$) and a squarefree monomial $\mu_v^{\#}$ such that $x_u | \mu_v^{\#}$. However, this implies that $x_k \tilde{\beta}_j = x_k \mu_w^{\#}$ is an element of $\mathfrak{m}J$ such that $x_i \nmid x_k \tilde{\beta}_j$ for $i \in \{s+1, ..., n\}$ and hence, $x_k \tilde{\beta}_j = \beta_y$ for some $y \in \{1, ..., t\}$. Moreover, β_y is squarefree since $\tilde{\beta}_j$ is squarefree and $e_k^{\tilde{\beta}_j} = e_k^{\beta_j} = 0$. Thus, μ is a multiple of β_y and hence $\mu \in \mathfrak{m}J$ again contradicting the supposition.

This proves the claim that $e_i^{\beta_j} \geq 1$ for $i = 1, ..., s$ and $j = 1, ..., r$. Moreover, the above arguments also show that $\{\beta_1, ..., \beta_r\} = \{x_1 ... x_{k-1} x_k^2 x_{k+1} ... x_s : 1 \leq k \leq s\}$ and in particular, that $r = s$.

It follows that $\mu = x_1 ... x_s$ for since $x_i | \beta_j$ for $i = 1, ..., s$ and $j = 1, ..., r$, $x_1 ... x_s | \mu$ and $x_i \nmid \mu$ for $i = s+1, ..., n$ by choice of s. If $e_l^{\mu} > 1$ for some $l \in \{1, ..., s\}$, then, μ is a multiple of β_l, and consequently, $\mu \in \mathfrak{m}J$ contradicting the assumption.

Rewriting the inequality 2.1 for $\mu = x_1 ... x_s$, there exist positive rational numbers

$c_1, ..., c_s$ such that $c_1 + c_2 + ... + c_s = 1$ and

$$1 = e_i^{\mu} \geq c_1 e_i^{\beta_1} + ... + c_s e_i^{\beta_s}$$

for $i = 1, ..., s$. Now $e_i^{\beta_j} = 2$ if $i = j$ and $e_i^{\beta_j} = 1$ if $i \neq j$. Adding the above set of inequalities as i varies from 1 to s we get that

$$s \geq c_1(s+1) + ... + c_s(s+1) = (c_1 + ... + c_s)(s+1) = s+1.$$

This is a contradiction. Consequently, there is no monomial μ such that $\mu \in \overline{\mathfrak{m}J} \setminus \mathfrak{m}J$ and thus, $\overline{\mathfrak{m}J} = \mathfrak{m}J$. $\square$

2.2 Ideals generated by monomials and one binomial

Suppose that $R = k[x_1, ..., x_n]$ is a polynomial ring over a field k. Consider the ideal $I = (\beta, \mu_1, ..., \mu_d)R$ of R such that β is a binomial and $\mu_1, ..., \mu_d$ are monomials. Then we show that if I is radical, $\overline{\mathfrak{m}I} = \mathfrak{m}I$, where as before $\mathfrak{m} = (x_1, ..., x_n)R$. Such ideals have been studied in a different context (to determine their arithmetic rank) in [Bar07].

Remark 2.2.1. We can focus our attention on the case where β is a pure difference binomial, *i.e.,* a difference of two monomials, for we can reduce the general problem to this case. Suppose that $I = (a\mu + b\nu, \mu_1, ..., \mu_d)R$, where $a, b \in k \setminus \{0\}$ and $\mu, \nu, \mu_1, ..., \mu_d$ are monomials in R. Let k^* denote the algebraic closure of k and let $S = k^*[x_1, ..., x_n]$. Now $IS = (a\mu + b\nu, \mu_1, ..., \mu_d)S = (\mu + a^{-1}b\nu, \mu_1, ..., \mu_d)S$. Without loss of generality we may assume that $\mu \neq \nu$ else I is monomial ideal and this case was discussed in the preceding section. So some indeterminate appears with different exponents in μ and ν. Without loss of generality, assume this indeterminate to be x_1. We may write that $\mu = x_1^u \mu'$ and $\nu = x_1^v \nu'$, where u, v are the integers such that $x_1^u | \mu$ but $x_1^{u+1} \nmid \mu$ and $x_1^v | \nu$ but $x_1^{v+1} \nmid \nu$ while μ', ν' are quotients of μ, ν

by the corresponding powers of x_1. By assumption, $u \neq v$. Again, without loss of generality, assume that $v > u$. Then consider the isomorphism $f : S \to S$ such that $f(x_1) = \sqrt[(v-u)]{-a^{-1}b}x_1$, $f(x_i) = x_i$ for $i = 2, .., n$ and $f(w) = w$ for $w \in k$. Then $f(IS) = (\mu - \nu, \mu_1, ..., \mu_d)S$. Now suppose that I is an ideal such that $(\mathfrak{m}S)f(IS)$ is integrally closed. Then, since f is an isomorphism on S and $f(\mathfrak{m}S) = \mathfrak{m}S$, we have that $(\mathfrak{m}S)IS = (\mathfrak{m}I)S$ is integrally closed. Now, since k^* is algebraic over k, $k \subseteq k^*$ is an integral ring extension and hence $R = k[x_1, ..., x_n] \subseteq k^*[x_1, ..., x_n] = S$ is also integral (exercise 9, page 68, [AM94]). Then, for any ideal J in R, $\overline{JS} \cap R = \overline{J}$ (proposition 1.6.1, page 15, [HS06]). Thus, $\overline{\mathfrak{m}I} = \overline{\mathfrak{m}IS} \cap R = \mathfrak{m}IS \cap R = \mathfrak{m}I$. So it is sufficient to consider the integral closedness of $\mathfrak{m}I$, where $\mathfrak{m}$ is the homogenous maximal ideal of a polynomial ring over an algebraically closed field and I is an ideal generated by a pure difference binomial and several monomials.

We now consider ideals in R generated by a pure difference binomial and a set of monomials. We will characterize ideals of this type that are radical. We first obtain a necessary and sufficient condition for the principal ideal generated by a pure difference binomial to be prime.

Proposition 2.2.2. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k and let $I = (\mu - \nu)R$, where $\mu = x_1^{u_1}...x_n^{u_n}$ and $\nu = x_1^{v_1}...x_n^{v_n}$ are monomials in R. Then I is prime if and only if $gcd(\mu, \nu) = 1$ and $gcd(u_1, ..., u_n, v_1, ..., v_n) = 1$.*

Proof. We first show that the condition is necessary. Suppose that $\alpha \neq 1$ is a monomial in R such that $\alpha = \gcd(\mu, \nu)$. Then $\alpha + I$ and $\frac{\mu-\nu}{\alpha} + I$ are non-zero elements in R/I, but $(\alpha + I)(\frac{\mu-\nu}{\alpha} + I) = (\mu - \nu) + I = 0 + I$. Thus, R/I is not a domain, so I is not prime. Next, if $c = \gcd(u_1, ..., u_n, v_1, ..., v_n)$. Then $x_1^{u_1/c}...x_n^{u_n/c} - x_1^{v_1/c}...x_n^{v_n/c}$ is a non-zero element in R/I, which is a zero divisor. So again, I is not prime.

Now we prove sufficiency. We may assume that there is no $i \in \{1, ..., n\}$ such that $u_i, v_i > 0$ else $\gcd(\mu, \nu) = x_i^{|u_i - v_i|}$ (where $|\cdot|$ denotes the usual absolute value of a real number) contradicting the assumption that $\gcd(\mu, \nu) = 1$. We may further assume that there is no $i \in \{1, ..., n\}$ such that $u_i = v_i = 0$, else we may just work in $k[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$ (for I is prime if and only if $\mu - \nu$ is irreducible and the irreducibility is not affected by working in a polynomial subring in the indeterminates occurring in μ and ν). In other words, we have that for every $i \in \{1, ..., n\}$, $x_i | \mu$ or $x_i | \nu$ but not both. Set $h_i = u_i - v_i$ for $1 \leq i \leq n$. Then $\gcd(h_1, ..., h_n) = 1$ since each h_i is equal to either u_i or v_i in absolute value and $\gcd(u_1, ..., u_n, v_1, ..., v_n) = 1$. Consider the $(n-1) \times n$ matrix M given by

$$M = \begin{bmatrix} -h_2 & h_1 & 0 & . & . & . & 0 & 0 \\ 0 & -h_3 & h_2 & 0 & . & . & . & 0 \\ 0 & 0 & -h_4 & h_3 & & & & . \\ . & . & 0 & & & & & . \\ . & . & . & & & & & . \\ . & . & . & & & & & 0 \\ 0 & 0 & . & . & . & 0 & -h_n & h_{n-1} \end{bmatrix}$$

Since by assumption $h_i \neq 0$ for $1 \leq i \leq n$, M has rank $n-1$. Also, if we let $H = (h_1, ..., h_n)^T$, then, MH is the $(n-1) \times 1$ zero matrix. Denote the jith entry of M by $M_{j,i}$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1$. Consider the Laurent polynomial ring $S = k[y_1, ..., y_{n-1}, y_1^{-1}, ..., y_{n-1}^{-1}]$. Consider the map $\phi : R \to S$ given by $\phi(x_i) = y_1^{M_{1,i}} ... y_{n-1}^{M_{n-1,i}}$. Then $\ker(\phi)$ is generated by all pure difference binomials $x_1^{a_1} ... x_n^{a_n} - x_1^{b_1} ... x_n^{b_n}$ such that $(a_1 - b_1, ..., a_n - b_n)^T$ lies in the kernel of M considered as a linear map $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$, *i.e.,* $\ker(\phi)$ is the toric ideal associated to the map ϕ (lemma 4.1, page 31, [Stu95])

In this case, M has rank $n-1$, so its kernel is a 1 dimensional $\mathbb{Z}$-module by the rank-nullity theorem. Now $H \in \ker(M)$ and H is not the zero vector. Further, the entries of H have no common factor. So every element in $\ker(M)$ is a multiple of H. Suppose that $K = (k_1, ..., k_n)^T$ and $K \in \ker(M)$. If we write $K = (k_1^+, ..., k_n^+)^T - (k_1^-, ..., k_n^-)^T$, where for an integer z, we define

$$z^+ = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{else} \end{cases},$$

$$z^- = \begin{cases} -z & \text{if } z < 0 \\ 0 & \text{else} \end{cases},$$

then, since K is a multiple of H, the binomial $x_1^{k_1^+}...x_n^{k_n^+} - x_1^{k_1^-}...x_n^{k_n^-}$ is a multiple of $\mu - \nu$. So $\ker(\phi)$ is generated by $\mu - \nu$. In other words, $I = \ker(\phi)$. Finally, $R/I = R/\ker(\phi) = S$. Since S is a domain, I is prime. □

Corollary 2.2.3. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k and let $I = (a\mu + b\nu)R$, where $\mu = x_1^{u_1}...x_n^{u_n}$ and $\nu = x_1^{v_1}...x_n^{v_n}$ are monomials in R and $a, b \in k \setminus \{0\}$. Then I is prime if $gcd(\mu, \nu) = 1$ and $gcd(u_1, ..., u_n, v_1, ..., v_n) = 1$. Further, if k is algebraically closed, then, I is prime if and only if $gcd(\mu, \nu) = 1$ and $gcd(u_1, ..., u_n, v_1, ..., v_n) = 1$*

Proof. We outline an argument similar to remark 2.2.1. Let k^* denote the algebraic closure of k and let $S = k^*[x_1, ..., x_n]$. Then $IS = (a\mu + b\nu)S = (\mu + a^{-1}b\nu)S$. Since $\gcd(\mu, \nu) = 1$, some indeterminate appears with different exponents in μ and ν, say x_1. We may write that $\mu = x_1^u\mu'$ and $\nu = x_1^v\nu'$, where u, v are the integers such that $x_1^u|\mu$ but $x_1^{u+1} \nmid \mu$ and $x_1^v|\nu$ but $x_1^{v+1} \nmid \nu$ while μ', ν' are quotients of μ, ν by the corresponding powers of x_1. By assumption, $u \neq v$. Again, without loss of generality assume that $v > u$. Consider the isomorphism, $f : S \to S$ such that

$f(x_1) = \sqrt[(v-u)]{-a^{-1}b}x_1$, $f(x_i) = x_i$ for $i = 2, .., n$ and $f(w) = w$ for $w \in k$. We have that $f(IS) = (\mu - \nu)S$. By proposition 2.2.2, $f(IS)$ is a prime ideal in S if and only if $\gcd(\mu, \nu) = 1$ and $\gcd(u_1, ..., u_n, v_1, ..., v_n) = 1$ (this proves the proposition in the case when k is algebraically closed). Then, since f is an isomorphism, IS is prime. Now R is a direct summand of S and hence we have that $IS \cap R = I$ since every ideal of R is a contracted ideal with respect to the inclusion $R \subseteq S$ (proposition 1, [Hoc73a]). Thus, since contraction of a prime ideal is prime, I is prime. □

We also obtain a criterion for the principal ideal generated by a binomial to be radical.

Proposition 2.2.4. *Let $R = k[x_1, ..., x_n]$, where k is a field. Let μ, ν be distinct monomials in R and let $a, b \in k \setminus \{0\}$. Let $I = (a\mu + b\nu)R$. Then I is radical if and only if $x_i^2 \nmid (a\mu + b\nu)$ for $1 \leq i \leq n$.*

Proof. To prove the condition is necessary, suppose without loss of generality that $x_1^2 | (a\mu + b\nu)$. Then, if we denote $\mu^{'} = \frac{\mu}{x_1^2}, \nu^{'} = \frac{\nu}{x_1^2}$, we have that $x_1(a\mu^{'} + b\nu^{'}) \in \sqrt{I}$ as $(x_1(a\mu^{'} + b\nu^{'}))^2 = (a\mu^{'} + b\nu^{'})(a\mu + b\nu) \in I$. However, $x_1(a\mu^{'} + b\nu^{'}) \notin I$ as $x_1(a\mu^{'} + b\nu^{'})$ is an element of a smaller degree than the generator of I.

Now suppose that $x_i^2 \nmid (a\mu + b\nu)$ for $1 \leq i \leq n$. We show that $(a\mu + b\nu)$ has no irreducible factor with multiplicity greater than 1. By factoring out the indeterminates dividing both μ, ν we may write that $(a\mu + b\nu) = x_{i_1}...x_{i_j}(a\mu^{'} + b\nu^{'})$, where $\mu^{'} = \frac{\mu}{x_{i_1}...x_{i_j}}$ and $\nu^{'} = \frac{\nu}{x_{i_1}...x_{i_j}}$. Now $\gcd(\mu^{'}, \nu^{'}) = 1$. It suffices to show that $(a\mu^{'} + b\nu^{'})$ has no irreducible factor of multiplicity greater than 1. Suppose that $f \in R$ is an irreducible element such that $f^2 | (a\mu^{'} + b\nu^{'})$. Suppose that $\{x_{l_1}, ..., x_{l_t}\}$ is the subset of indeterminates such that x_{l_h} divides at least one term of f for $1 \leq h \leq t$ and no indeterminate in $\{x_1, ..., x_n\} \setminus \{x_{l_1}, ..., x_{l_t}\}$ divides any term in f.

Then $f|\frac{\partial(a\mu^{'}+b\nu^{'})}{\partial x_{l_h}}$ for $1 \leq h \leq t$ and none of the partial derivatives $\frac{\partial(a\mu^{'}+b\nu^{'})}{\partial x_{l_h}}$ are zero by choice of x_{l_h}. However, $\frac{\partial(a\mu^{'}+b\nu^{'})}{\partial x_i}$ is either 0 or a monomial (since $\gcd(\mu^{'}, \nu^{'}) = 1$) for $1 \leq i \leq n$. Consequently, the only possibility is f is a monomial. However, if x_i is an indeterminate such that $x_i|f$, then, since $f^2|(a\mu^{'} + b\nu^{'})$, $x_i^2|(a\mu^{'} + b\nu^{'})$ and hence $x_i^2|(a\mu + b\nu)$ contradicting the hypothesis. So $(a\mu + b\nu)$ has no irreducible factor with multiplicity greater than 1.

Finally, suppose that $f_1, ..., f_m$ be the distinct irreducible factors of $(a\mu+b\nu)$. Then we have that $I = f_1R \cap ... \cap f_mR$. Since $(a\mu+b\nu)$ is a multiple of each of $f_1, ..., f_m$, we have that $I \subseteq f_1R \cap ... \cap f_mR$. Conversely, since R is a unique factorization domain and since $f_1, ..., f_m$ are irreducible, every element in $f_1R \cap ... \cap f_mR$ which is a multiple of each of $f_1, ..., f_m$ must be a multiple of $f_1...f_m = (a\mu+b\nu)$. Thus, $f_1R \cap ... \cap f_mR \subseteq I$. Now, since $f_1, ..., f_m$ are irreducible and R is a unique factorization domain, $f_1R, ..., f_mR$ are prime ideals, and thus, I is an intersection of prime ideals and hence radical. □

Definition 2.2.5. Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let μ, ν be monomials in R and let $a, b \in k \setminus \{0\}$. Then we will say that $a\mu + b\nu$ is a *squarefree binomial* if $x_i^2 \nmid (a\mu + b\nu)$ for $1 \leq i \leq n$.

Lemma 2.2.6. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let $a, b \in k \setminus \{0\}$ and let $\mu, \nu, \mu_1, ..., \mu_d$ be monomials in R. Let $I = (\mu_1, ..., \mu_d, a\mu + b\nu)R$. Let J be the ideal generated by the monomials in I, i.e., $J = (I \cap \mathcal{M})R$, where $\mathcal{M} := \{$monomials in $R\}$. Then $(J : \mu) = (J : \nu)$.*

Proof. Since J is a monomial ideal and μ, ν are monomials, $(J : \mu), (J : \nu)$ are monomial ideals. In fact they are generated by squarefree monomials since J is generated by squarefree monomials (section 2.3 [Frö97]). Now suppose that $\lambda \in$

$(J:\mu)\backslash(J:\nu)$. We may assume that λ is a monomial, for if f is a polynomial in $(J:\mu)\backslash(J:\nu)$, then, at least one term of f must be in $(J:\mu)\backslash(J:\nu)$ since $(J:\mu)$ is a monomial ideal (and hence homogeneous with respect to the standard multigrading on R). Then $\lambda\mu \in J$. Also, $(a\mu + b\nu) \in I \implies \lambda(a\mu + b\nu) = a\lambda\mu + b\lambda\nu \in I$. So $b\lambda\nu \in I \implies \lambda\nu \in I$ and since λ, ν are monomials, $\lambda\nu \in J$. Hence, $\lambda \in (J:\nu)$, a contradiction. Consequently, $(J:\mu) \subseteq (J:\nu)$. By the same token, $(J:\nu) \subseteq (J:\mu)$. Thus, $(J:\mu) = (J:\nu)$. □

Lemma 2.2.7. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let $\mu, \nu, \mu_1, ..., \mu_d$ be monomials in R and let $a, b \in k \setminus \{0\}$. Let $I = (\mu_1, ..., \mu_d, a\mu + b\nu)R$. Let J is the ideal generated by the monomials in I, i.e., $J = (I \cap \mathcal{M})R$, where $\mathcal{M} := \{$monomials in $R\}$. Suppose that $\mu, \nu \notin J$ and $\mu \neq \nu$. Then, if $f \in R$ such that $f|(a\mu + b\nu)$ and $\frac{a\mu+b\nu}{f}$ is not a unit in R, we have that $f \notin I$.*

Proof. Suppose that f is a proper factor of $a\mu + b\nu$ and $f \in I$. Then $f = r_1\mu_1 + ... r_d\mu_d + r(a\mu + b\nu)$ for some $r_1, ..., r_d, r \in R$. Now, since $\frac{a\mu+b\nu}{f}$ is not a unit in R, f is of smaller degree than $a\mu + b\nu$ while $r(a\mu + b\nu)$ has degree at least equal to $a\mu + b\nu$. So terms in $r(a\mu + b\nu)$ are not terms in f. Thus, every term of f is a term in the first d summands in the expression for f above and hence $f \in J$. Then, since $a\mu + b\nu$ is a multiple of f, we have that $a\mu + b\nu \in J$. Finally, since J is a homogeneous ideal with respect to the standard multigrading and μ, ν are distinct monomials, we have that $a\mu, b\nu \in J$ and hence $\mu, \nu \in J$ (since $a, b \neq 0$), contradicting the hypothesis. So $f \notin I$. □

We now give a criterion for determining whether an ideal generated by a single binomial and several monomials is radical. We first give a criterion for the case when the binomial is a pure difference of monomials and the underlying field of the polynomial ring is algebraically closed.

Theorem 2.2.8. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over an algebraically closed field k. Let $\mu, \nu, \mu_1, ..., \mu_d$ be monomials in R. Let $I = (\mu_1, ..., \mu_d, \mu - \nu)R$. Let J be the ideal generated by the monomials in I, i.e., $J = (I \cap \mathcal{M})R$, where $\mathcal{M} := \{$monomials in $R\}$. Suppose that $\mu, \nu \notin J$, $\mu \neq \nu$. Then I is radical if and only if J is generated by squarefree monomials and $\mu - \nu$ is a squarefree binomial (in other words I is radical if and only if J and $(\mu - \nu)R$ are radical).*

Proof. Suppose that I is radical. If $\chi = x_{i_1}^{\delta_1} ... x_{i_k}^{\delta_k} \in I$ with $\delta_j \in \mathbb{Z}_{>0}$ and $\delta_{\max} := \max\{\delta_1, ..., \delta_k\}$, then, $(x_{i_1} ... x_{i_k})^{\delta_{max}} = \chi(x_{i_1}^{\delta_{max}-\delta_1} ... x_{i_k}^{\delta_{max}-\delta_k}) \in I$. Then, since I is radical, $x_{i_1} ... x_{i_k} \in I$. So any monomial in I is a multiple of a squarefree monomial in I. Hence, J is generated by squarefree monomials. A monomial ideal is radical if and only if it is generated by squarefree monomials (lemma 3, proposition 4, page 41, [Frö97]), so J is radical. To show that $\mu - \nu$ is a squarefree binomial, suppose without loss of generality that $x_1^2 | (\mu - \nu)$. Then, if we denote $\mu' = \frac{\mu}{x_1^2}, \nu' = \frac{\nu}{x_1^2}$, we have that $x_1(\mu' - \nu') \in \sqrt{I}$ as $(x_1(\mu' - \nu'))^2 = (\mu' - \nu')(\mu - \nu) \in I$. However, $x_1(\mu' - \nu') \notin I$ by lemma 2.2.7. Hence, $\mu - \nu$ must be squarefree.

Now we proceed to prove the converse. Since I is a binomial ideal, $\sqrt{I}$ is also binomial (here, we need the assumption that k is algebraically closed) (theorem 3.1, [ES96]). Let $a\lambda + b\eta \in \sqrt{I}$, where λ, η are monomials in R and $a, b \in k$. Say $(a\lambda + b\eta)^m \in I$ for some nonnegative integer m. We may assume that $a, b \neq 0$ for otherwise if say $a = 0$, then, $b = 0$ and there is nothing to prove. Further, if $b \neq 0$, then, $b^m \eta^m \in I$ and hence $\eta^m \in I$. Then, since η^m is a monomial $\eta^m \in J$ and since J is radical, $\eta \in J \subseteq I$. So $a\lambda + b\eta = b\eta \in I$. Similarly, $a\lambda + b\eta \in I$ if $a \neq 0$ and $b = 0$.

We consider two cases: (1) $\lambda\eta \notin J$ and (2) $\lambda\eta \in J$.

Consider the first case: $\lambda\eta \notin J$. we have that $(a\lambda + b\eta)^m = r(\mu - \nu) + s$, where $r \in R$, $s \in J$. We may assume that no term of r is in $(J : \mu)$, else if $r^{'}$ is the sum of terms of r such that each term of $r^{'}$ is in $(J : \mu)$, then, we can write that $(a\lambda + b\eta)^m = (r - r^{'})(\mu - \nu) + (s + r^{'}(\mu - \nu))$ so that $s + r^{'}(\mu - \nu) \in J$ as $r^{'} \in (J : \mu) = (J : \nu)$ (where the equality follows from 2.2.6). We claim that no term of $(a\lambda + b\eta)^m$ is in J, for if $\lambda^r\eta^{m-r} \in J$, then, $(\lambda^{m-r}\eta^r)(\lambda^r\eta^{m-r}) = \lambda^m\eta^m \in J$. However, J is a radical ideal since it is generated by squarefree monomials. So $\lambda^m\eta^m \in J \implies \lambda\eta \in J$, a contradiction. So no term of $s = (a\lambda + b\eta)^m - r(\mu - \nu)$ is in J, so $s = 0$. So $(a\lambda + b\eta)^m = r(\mu - \nu)$. Now, by proposition 2.2.4, the ideal $(\mu - \nu)R$ is radical. So $(a\lambda + b\eta)^m \in (\mu - \nu)R \implies a\lambda + b\eta \in (\mu - \nu)R \subseteq I$.

Now consider the second case. Since $(a\lambda + b\eta)^m \in I$ and $\lambda\eta \in J$, we have that $(a\lambda)^m + (b\eta)^m \in I$. Again we can write that $(a\lambda)^m + (b\eta)^m = r(\mu - \nu) + s$ with $r \in R$, $s \in J$ and no term of r is in $(J : \mu)$. Suppose that $\lambda \in J$. Then, if $\eta \in J$, we have that $a\lambda + b\eta \in J$ and hence $a\lambda + b\eta \in I$. Otherwise, $\eta \notin J$ and since J is radical, $\eta^m \notin J$ and thus, $(b\eta)^m \notin J$. Now we can rewrite the above equation as $(a\lambda)^m - s = r(\mu - \nu) + (b\eta)^m$. Then no term of right hand side is in J and left hand side is an element of J. Then, since J is a monomial ideal and hence homogeneous under the standard multigrading, we must have, $(a\lambda)^m - s = 0$. So $(b\eta)^m = -r(\mu - \nu) \in (\mu - \nu)R$. Since $(\mu - \nu)R$ is a radical ideal and $b \neq 0$, we have that $\eta \in (\mu - \nu)R$. So $a\lambda + b\eta \in J + (\mu - \nu)R = I$. Similarly, if $\lambda \notin J$ and $\eta \in J$, we have that $a\lambda + b\eta \in I$. Finally, suppose that $\lambda, \eta \notin J$. Then $\lambda^m, \eta^m \notin J$. So no term of $s = (a\lambda)^m + (b\eta)^m - r(\mu - \nu)$ is in J, but $s \in J$ by assumption, hence, $s = 0$. So we have that $(a\lambda)^m + (b\eta)^m = r(\mu - \nu)$. Let ζ_i denote an m^{th} root of -1 for $1 \leq i \leq m$. Then $(a\lambda)^m + (b\eta)^m = \Pi_i(a\lambda + \zeta_i b\eta) = r(\mu - \nu)$ (we use the assumption that k is algebraically closed). Since $\mu - \nu$ divides $\Pi_i(a\lambda + b\zeta_i\eta)$ and R is a unique factorization domain, we have that $\mu - \nu = \Pi_{j\in A}(a\lambda + \zeta_j b\eta)$, where $A \subseteq \{1, ..., m\}$. Say, $|A| = t$. So $r(\mu - \nu) = (a\lambda)^t + \zeta(b\eta)^t + c$, where $\zeta = \Pi_{j\in A}\zeta_j$ and c is a multiple of $\lambda\eta$. We consider two subcases: (a) $\lambda \in (J : \mu)$ and (b) $\lambda \notin (J : \mu)$. In case (a) consider, $a^t\lambda^{t+1} = r\lambda(\mu - \nu) - \zeta\lambda(b\eta)^t - c\lambda$. Since $(J : \mu) = (J : \nu)$ and $\lambda\eta \in J$, each term on the right hand side of this equation is in J. So $a^t\lambda^{t+1} \in J$ and since J is radical and $a \neq 0$, we have that $\lambda \in J$, which contradicts the assumption. For case (b), again consider $a^t\lambda^{t+1} = r\lambda(\mu - \nu) - \zeta\lambda(b\eta)^t - c\lambda$. If $r\lambda \in (J : \mu)$, then, as in case (a), $\lambda \in J$, which is a contradiction. So $r\lambda \notin (J : \mu)$. Rewrite the preceding equation as $a^t\lambda^{t+1} - r\lambda(\mu - \nu) = \zeta\lambda(b\eta)^t - c\lambda$. Thus, each side of the equation must be zero else the left hand side is not an element of J and right hand side is an element of J. So $a^t\lambda^{t+1} = r\lambda(\mu - \nu)$. Now the left hand side of this equation is a k-multiple of a monomial while the right hand side has at least two terms, which is impossible. So cases (a) and (b) don't occur and hence we cannot have, $\lambda, \eta \notin J$, $\lambda\eta \in J$ and $a\lambda + b\eta \in \sqrt{I}$.

Thus, in all possible cases we have that $a\lambda + b\eta \in I$.

So $\sqrt{I} = I$ as required. □

Corollary 2.2.9. *Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let $\mu, \nu, \mu_1, ..., \mu_d$ be monomials in R and let $a, b \in k \setminus \{0\}$. Let $I = (\mu_1, ..., \mu_d, a\mu + b\nu)R$. Let J be the ideal generated by the monomials in I, i.e., $J = (I \cap \mathcal{M})R$, where $\mathcal{M} :=$ {monomials in R}. Suppose that $\mu, \nu \notin J$ and $\mu \neq \nu$. Then I is radical if and only if J is generated by squarefree monomials and $a\mu + b\nu$ is a squarefree binomial.*

Proof. The necessity of the condition follows by the same argument as in the first paragraph of the proof of theorem 2.2.8.

For sufficiency, we will mimic the proof of corollary 2.2.3. Let k^* denote the algebraic closure of k and let $S = k^*[x_1, ..., x_n]$. Then $IS = (\mu_1, ..., \mu_d, a\mu + b\nu)S =$

$(\mu_1, ..., \mu_d, \mu + a^{-1}b\nu)S$. We may assume that $\mu \neq \nu$ otherwise I is a monomial ideal and the result follows from the analogous result on monomial ideals. Then some indeterminate appears with different exponents in μ and ν, say x_1. We may write that $\mu = x_1^u\mu^{'}$ and $\nu = x_1^v\nu^{'}$, where u, v are the integers such that $x_1^u|\mu$ but $x_1^{u+1} \nmid \mu$ and $x_1^v|\nu$ but $x_1^{v+1} \nmid \nu$ while $\mu^{'}, \nu^{'}$ are quotients of μ, ν by the corresponding powers of x_1. By assumption, $u \neq v$. Again, without loss of generality, assume that $v > u$. Then consider the isomorphism, $f : S \to S$ such that $f(x_1) = \sqrt[(v-u)]{-a^{-1}b}x_1$, $f(x_i) = x_i$ for $i = 2, .., n$ and $f(w) = w$ for $w \in k$. Then $f(IS) = (\mu_1, ..., \mu_d, \mu - \nu)S$. As before, since R is a direct summand of S, we have that $IS \cap R = I$ since every ideal of R is a contracted ideal with respect to the inclusion $R \subseteq S$ (proposition 1, [Hoc73a]). Then, if α is a monomial in IS, since it is an element of R, it is a monomial in $I = IS \cap R$. Thus, the ideal generated by monomials in IS is precisely JS and JS is generated by squarefree monomials since J is generated by squarefree monomials. Further, since f is an isomorphism that takes monomials to k-multiples of monomials, the ideal generated by monomials in $f(IS)$ is $f(JS)$, and thus, it is generated by squarefree monomials. Also, $\mu - \nu$ is a squarefree binomial if and only if $a\mu + b\nu$ is a squarefree binomial by definition. Thus, by theorem 2.2.8, $f(IS)$ is radical. Since f is an isomorphism, IS is radical. Finally, since $IS \cap R = I$ and contraction of a radical ideal is radical, I is a radical ideal. $\square$

Note 2.2.10. Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Let $\mu, \nu, \mu_1, ..., \mu_d$ be monomials in R and let $a, b \in k \setminus \{0\}$. Let $I = (\mu_1, ..., \mu_d, a\mu + b\nu)R$. Suppose that J is the ideal generated by the monomials in I, *i.e.,* $J = (I \cap \mathcal{M})R$, where $\mathcal{M} := \{\text{monomials in } R\}$. Suppose that J is generated by squarefree monomials and $a\mu + b\nu$ is a squarefree binomial. Note that by corollary 2.2.9, under these conditions, I is radical.

Since J is generated by squarefree monomials, we can write that $J = \cap_j P_j$ with $1 \leq j \leq r$, where P_j are the prime ideals in R generated by subsets of the indeterminates $\{x_1, ..., x_n\}$ (lemma 3, page 41, [Frö97]). So we can write that $I = (\cap_j P_j, a\mu + b\nu)R$. We may further assume that the decomposition of J as intersections of primes of R is irredundant, *i.e.*, $P_l \not\supseteq \cap_{j \neq l} P_j$ for all l such that $1 \leq l \leq r$. We claim: $I = \cap_j (P_j, a\mu + b\nu)R$.

We have that $(\cap_j P_j, a\mu + b\nu)R \supseteq (\Pi_j P_j, a\mu + b\nu)R \supseteq \Pi_i (P_i, a\mu + b\nu)R$.

So $I = \sqrt{I} = \sqrt{(\cap_j P_j, a\mu + b\nu)R} \supseteq \sqrt{\Pi_j (P_j, a\mu + b\nu)R} = \sqrt{\cap_j (P_j, a\mu + b\nu)R}$ (where the last equality follows from exercise 1.13.iii, page 9, [AM94]).

Further, $(\cap_j P_j, a\mu + b\nu)R \subseteq (P_j, a\mu + b\nu)R$ for $1 \leq j \leq r \implies (\cap_j P_j, a\mu + b\nu)R \subseteq \cap_j (P_j, a\mu + b\nu)R \implies \sqrt{(\cap_j P_j, a\mu + b\nu)R} \subseteq \sqrt{\cap_j (P_j, a\mu + b\nu)R}$.

Hence, $I = \sqrt{(\cap_j P_j, a\mu + b\nu)R} = \sqrt{\cap_j (P_j, a\mu + b\nu)R} = \cap_j \sqrt{(P_j, a\mu + b\nu)R}$ (where the last equality follows from exercise 1.13.iii, page 9, [AM94]).

For $(P_j, a\mu + b\nu)R$ the following cases can occur:

1. $\mu \notin P_j$, $\nu \notin P_j$: Now $R/P_j = k[x_{i_1}, ..., x_{i_j}]$, where $x_{i_1}, ..., x_{i_j}$ are those indeterminates from $x_1, ..., x_n$ that are not part of the generating set of P_j. Then $(a\mu + b\nu)R/P_j$ is radical in R/P_j by proposition 2.2.4. So $(P_j, a\mu + b\nu)R$, which is the contraction of $(a\mu + b\nu)R/P_j$ in R under the canonical map $R \to R/P_j$ is radical.

2. $\mu, \nu \in P_j$: In this case, $(P_j, a\mu + b\nu)R = P_j$ is prime and hence radical.

3. $\mu \in P_j$ and $\nu \notin P_j$ or $\mu \notin P_j$ and $\nu \in P_j$.

We show that the last case cannot occur if J is squarefree. Suppose, without loss of generality, we have that $\mu \in P_1$ and $\nu \notin P_1$. We first show that there exists a (squarefree) monomial λ such that $P_1 = (J : \lambda)$. Now $J = \cap_j P_j \subsetneq \cap_{j \neq 1} P_j$. So we have a squarefree monomial, say λ such that $\lambda \in \cap_{j \neq 1} P_j - \cap_j P_j$ (note that $\cap_{j \neq 1} P_j$ also defines a squarefree monomial ideal). Then $P_1 \lambda \subseteq P_1(\cap_{j \neq 1} P_j) \subseteq P_1 \cap (\cap_{j \neq 1} P_j) =$

$\cap_j P_j = J$. So $P_1 \subseteq (J : \lambda)$. Now $(J : \lambda)$ is a monomial ideal (section 2.3 [Frö97]). Suppose, there is a monomial $\lambda^{'} \in (J : \lambda)$. Then $\lambda\lambda^{'} \in J \subseteq P_1$. Since $\lambda \notin P_1$ (or else $\lambda \in P_1 \cap (\cap_{j \neq 1} P_j) = \cap_j P_j$, a contradiction) and P_1 is prime, we have that $\lambda^{'} \in P_1$. So $(J : \lambda) \subseteq P_1$ and hence by the earlier inclusion, $P_1 = (J : \lambda)$. Now, since $\mu \in P_1$, we have that $\lambda\mu \in \lambda P_1 \subseteq J$. Since $a\mu + b\nu \in I$, we have that $\lambda(a\mu + b\nu) = a\lambda\mu + b\lambda\nu \in I$. So $b\lambda\nu \in J \implies \lambda\nu \in J \implies \nu \in (J : \lambda) = P_1$, which contradicts the hypothesis. Thus, the last case above cannot occur.

Thus, we have proved $(P_j, a\mu + b\nu)R$ is radical for all j. So $I = \cap_j \sqrt{(P_j, a\mu + b\nu)} = \cap_j (P_j, a\mu + b\nu)$ proving the claim.

The ideal generated by the monomials in the ideal generated by a binomial and several monomials can be computed as expressed in the following proposition.

Proposition 2.2.11. *Let $R = k[x_1, .., x_n]$ with k a field. Let $I = (\mu_1, .., \mu_d, a\mu + b\nu)R$, where $\mu_1, .., \mu_d, \mu, \nu$ are monomials in R with $\mu \neq \nu$ and $a, b \in k \setminus \{0\}$. Let J be the ideal generated by monomials in I and let $J^{'} = (\mu_1, ..., \mu_d)R$. Then*

$$J = J^{'} + (J^{'} : \mu)\nu + (J^{'} : \nu)\mu.$$

Proof. Suppose that $\lambda \in I$ is a monomial. Then $\lambda = r_1\mu_1 + ... + r_d\mu_d + r(a\mu + b\nu)$. Then λ must be a k-multiple of a term occurring in one of the summands on the right hand side of this equation. If λ is a k-multiple of a term occurring in one of the first d summands, then, λ is a multiple of μ_i for some $i \in \{1, ..., d\}$ so that $\lambda \in J^{'}$. Otherwise, λ is a k-multiple of a term occurring in $r(a\mu + b\nu)$. Since μ, ν are distinct monomials, if s is a term in r, then, at most one of $s\mu$ and $s\nu$ is a k-multiple of λ. Suppose, without loss of generality that $s\mu$ is a k-multiple of λ. Then, since $s\nu$ is not a k-multiple of λ, the coefficient of the underlying monomial of $s\nu$ in the above expression for λ must be zero and hence, $s\nu \in J^{'}$. In other words, $s \in (J^{'} : \nu)$. Then, since λ is a k-multiple of $s\mu$, we have that $\lambda \in (J^{'} : \nu)\mu$. By the same analysis, if $s\nu$ is a k-multiple of λ, then, $\lambda \in (J^{'} : \mu)\nu$. Thus, any monomial in I must be a monomial in $J^{'}$ or $(J^{'} : \nu)\mu$ or $(J^{'} : \mu)\nu$. Then, since J is the ideal generated by monomials in I, we have that $J \subseteq J^{'} + (J^{'} : \mu)\nu + (J^{'} : \nu)\mu$.

For the converse we first note that since $J^{'}$ is an ideal generated by a set of monomials in I, by definition, $J^{'} \subseteq J$. Now let $r \in (J^{'} : \mu)$ be a monomial. Then $r\mu \in J^{'}$. Since $a\mu + b\nu \in I$, we have that $r(a\mu + b\nu) \in I$. So $br\nu \in I$ and hence $r\nu \in I$. Since r, ν are monomials, $r\nu \in J$. So every monomial in $(J^{'} : \mu)\nu$ is in J. Further, $(J^{'} : \mu)$ is a monomial ideal (section 2.3 [Frö97]) and hence so is $(J^{'} : \mu)\nu$. Thus, $(J^{'} : \mu)\nu \subseteq J$. Similarly, $(J^{'} : \nu)\mu \subseteq J$. So $J^{'} + (J^{'} : \mu)\nu + (J^{'} : \nu)\mu \subseteq J$.

Thus, $J = J^{'} + (J^{'} : \mu)\nu + (J^{'} : \nu)\mu$. □

2.3 Main lemma

We now present one of the main results of this thesis.

2.3.1 Local and graded versions of the main lemma

Lemma 2.3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local domain. Let $I = (a_1, ..., a_d)R$ be an integrally closed ideal. Let $I_i = (a_1, ..., \hat{a}_i, ..., a_d)R$. If $\overline{\mathfrak{m}I_i} \subseteq I_i$ for $1 \leq i \leq d$, then, $\mathfrak{m}I$ is integrally closed in R.*

Proof. We first observe that, $\overline{\mathfrak{m}I} \subseteq I$, since $\overline{\mathfrak{m}I} \subseteq \overline{I}$ (which follows from the fact that $\mathfrak{m}I \subseteq I$ and remark 1.1.3(2), page 2, [HS06]) and I is integrally closed. In other words, every element of R that is integral over $\mathfrak{m}I$ lies in I. We will show that no minimal generator of I is integral over $\mathfrak{m}I$.

Since $(R, \mathfrak{m})$ is a Noetherian local domain, for any ideal $\mathfrak{a}$ of R, $r \in \overline{\mathfrak{a}}$ if and only if for every discrete valuation ring V such that $R \subseteq V \subseteq \mathrm{Frac}(R)$ and $\mathfrak{m}_V \cap R = \mathfrak{m}$, where $\mathfrak{m}_V$ is the maximal ideal of V, we have that $r \in \mathfrak{a}V$ (proposition 6.8.4, page

135, [HS06]). For the purposes of this proof, we will call a discrete valuation ring V such that $R \subseteq V \subseteq \text{Frac}(R)$ and $\mathfrak{m}_V \cap R = \mathfrak{m}$, where $\mathfrak{m}_V$ is the maximal ideal of V an R-special discrete valuation ring.

Without loss of generality, we may assume that $\{a_1, ..., a_d\}$ is a minimal set of generators for I. We first show that no generator a_i of I is integral over $\mathfrak{m}I$ for $i \in \{1, ..., d\}$. Suppose that a_i is integral over $\mathfrak{m}I$ for some $i \in \{1, ..., d\}$. Then, for every R-special discrete valuation ring V, we have that $a_i \in (\mathfrak{m}I)V$. We may write that $I = I_i + a_i R$. So that, $(\mathfrak{m}I)V = (\mathfrak{m}I_i)V + (\mathfrak{m}a_i)V$. Since V is a discrete valuation ring containing R there is a discrete valuation on $\text{Frac}(R)$, say $\mathbf{v}$, such that for any $s \in \text{Frac}(R)$, $s \in V \iff \mathbf{v}(s) \geq 0$. For any ideal J of V define $\mathbf{v}(J) = \min\{\mathbf{v}(s) : s \in J\}$. Then $a_i \in (\mathfrak{m}I)V \implies \mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I)V) = \min\{\mathbf{v}((\mathfrak{m}I_i)V), \mathbf{v}((\mathfrak{m}a_i)V)\}$. We will show that $\mathbf{v}(a_i) < \mathbf{v}((\mathfrak{m}a_i)V)$, which would imply that $\mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I_i)V)$. Suppose that $\mathfrak{m} = (b_1, ..., b_t)R$. Then $(\mathfrak{m}a_i)V = (b_1 a_i, ..., b_t a_i)V$. Since V is a discrete valuation ring, the ideals of V are totally ordered under inclusion. So after relabeling if necessary, we may assume that $(b_t a_i)V \subseteq (b_{t-1} a_i)V \subseteq ... \subseteq (b_1 a_i)V$. So that, $(\mathfrak{m}a_i)V = (b_1 a_i)V$. Then $(\mathfrak{m}a_i)V = (b_1 a_i)V = \{s \in V : \mathbf{v}(s) \geq \mathbf{v}((b_1 a_i))\}$. If $\mathbf{v}(a_i) \geq \mathbf{v}(b_1 a_i)$, then, since $\mathbf{v}(b_1 a_i) = \mathbf{v}(b_1) + \mathbf{v}(a_i)$, we have that $\mathbf{v}(b_1) \leq 0$. We cannot have, $\mathbf{v}(b_1) < 0$ since $b_1 \in V$. So $\mathbf{v}(b_1) = 0$. Then b_1 must be a unit in V so that $\mathfrak{m}V = (b_1, ..., b_t)V = V$. This is a contradiction, since if $\mathfrak{m}_V$ is the maximal ideal of V, we have that $(\mathfrak{m}_V \cap R)V \subseteq \mathfrak{m}_V$ (proposition 1.17(i), page 10, [AM94]) but $\mathfrak{m}_V \cap R = \mathfrak{m}$ since V is R-special and this implies that $(\mathfrak{m}_V \cap R)V = \mathfrak{m}V = V \subseteq \mathfrak{m}_V$. So $\mathbf{v}(a_i) < \mathbf{v}(b_1 a_i) = \mathbf{v}((\mathfrak{m}a_i)V)$. Hence, $\mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I_i)V)$ and thus, $a_i \in (\mathfrak{m}I_i)V$. Since this statement is true for every R-special discrete valuation ring V, we have that $a_i \in \overline{\mathfrak{m}I_i}$. By hypothesis, $\overline{\mathfrak{m}I_i} \subseteq I_i$. So $a_i \in I_i$. However, this contradicts the minimality of the generating set $\{a_1, ..., a_d\}$ for I for if $a_i \in I_i$, then,

$I_i = (a_1, ..., \hat{a_i}, ..., a_d)R = (a_1, ..., \hat{a_i}, ..., a_d)R + a_iR = I$. This shows that a_i is not integral over $\mathfrak{m}I$ for any i such that $1 \leq i \leq d$.

Now assume that $\mathfrak{m}I$ is not integrally closed. Suppose that $a = r_1a_1 + ... + r_da_d \in I$ (where $r_1, ..., r_d \in R$) such that $a \in \overline{\mathfrak{m}I} - \mathfrak{m}I$. If r_j is not a unit for some $j \in \{1, ..., d\}$, then, since R is a local ring, $r_j \in \mathfrak{m}$. Hence, $r_ja_j \in \mathfrak{m}I$. So that, $a - r_ja_j \in \overline{\mathfrak{m}I} - \mathfrak{m}I$. So we may assume that $r_1, ..., r_e$ are units for some e such that $1 \leq e \leq d$ and $r_{e+1}, ..., r_d \in \mathfrak{m}$. Then $b = r_1a_1 + ... + r_ea_e \in \overline{\mathfrak{m}I} - \mathfrak{m}I$. Moreover, $\{b, a_2, a_3, ..., a_d\}$ is also a minimal set of generators for I. For we have that $a_1 = r_1^{-1}b - (r_1^{-1}(r_2a_2 + ... + r_ea_e)) \in (b, a_2, a_3, ..., a_d)R$, so $I \subseteq (b, a_2, a_3, ..., a_d)R$. Conversely, $b \in I$, so $(b, a_2, a_3, ..., a_d)R \subseteq I$. So $\{b, a_2, a_3, ..., a_d\}$ is a generating set for I and since it has size d, it is a minimal generating set, since in a local ring all minimal generating sets of an ideal have the same size by Nakayama's lemma. Then, by the arguments in the first paragraph, b is not integral over $\mathfrak{m}I$, which is a contradiction.

Thus, $\mathfrak{m}I$ is integrally closed in R. □

We will need a graded version of lemma 2.3.1. We first make a few definitions following Huneke and Swanson [HS06].

Definition 2.3.2. Let G be an abelian monoid. A ring R is said to be G-*graded* if the following conditions are satisfied:

1. $R = \oplus_{g \in G} R_g$, where R_g is a subgroup of R under addition.
2. For each $g, g' \in G$, $R_gR_{g'} \subseteq R_{g+g'}$.

An element $r \in R$ will be said to be G-*homogeneous* of *degree* (denoted *deg*) g if $r \in R_g$ for some $g \in G$. An ideal I of R is said to be G-*homogeneous* if I is generated by G-*homogeneous* elements in R.

Remark 2.3.3. Note that if G is an abelian monoid and R is a G-graded ring, then, R_0 is a subring of R. Further, if I is a G-homogeneous ideal and $f \in I$, then, if $f = f_1 + ... + f_t$, where $f_1, ..., f_t$ are homogeneous, then, $f_1, ..., f_t \in I$. For suppose that $a_1, ..., a_d$ are G-homogeneous generators of ISo $f = r_1a_1 + ... + r_da_d$, where $r_1, ..., r_d \in R$. Writing $r_i = \Sigma_{j=1}^{m_i} r_{ji}$, where r_{ji} is G-homogeneous for $1 \leq j \leq m_i$ and $1 \leq i \leq d$, we get that $f_1 + ... + f_t = \Sigma_{i=1}^{d} \Sigma_{j=1}^{m_i} r_{ji} a_i$. Expanding the right hand side and equating the G-homogeneous components we get that $f_1, ..., f_t$ are each an R-linear combination of $a_1, ..., a_d$ and hence $f_1, ..., f_t \in I$.

We will be interested in the case where $G = \mathbb{Z}^q \times \mathbb{Z}^r_{\geq 0}$, $q, r \in \mathbb{Z}_{\geq 0}$, R is a G-graded k-algebra, where k is a field, $R_0 = k$ and $\mathfrak{m} = \oplus_{g \in G \setminus \{0\}} R_g$ is the unique G-homogeneous maximal ideal[1] of R. We quote a result needed for the graded version of the main lemma.

Theorem 2.3.4. *Let $G = \mathbb{Z}^q \times \mathbb{Z}^r_{\geq 0}$, $q, r \in \mathbb{Z}_{\geq 0}$. Let I be a G-homogeneous ideal in a G-graded ring R. Then $\overline{I}$ is G-homogeneous. Further, if R is Noetherian, then, the associated primes of $\overline{I}$ are G-homogeneous and it has a G-homogeneous primary decomposition (corollary 5.2.3, page 97 and corollary A.3.2, page 395, [HS06]).*

Lemma 2.3.5. *Let $G = \mathbb{Z}^q \times \mathbb{Z}^r_{\geq 0}$, $q, r \in \mathbb{Z}_{\geq 0}$. Let R be a G-graded Noetherian domain with $R_0 = k$, a field and such that $\mathfrak{m} = \oplus_{g \in G \setminus \{0\}} R_g$ is the unique G-homogeneous maximal ideal of R. Let $I = (a_1, ..., a_d)R$ be an integrally closed G-homogeneous ideal such that $\{a_1, ..., a_d\}$ is a G-homogeneous set of generators for I. Let $I_i = (a_1, ..., \hat{a}_i, ..., a_d)R$. If $\overline{\mathfrak{m}I}_i \subseteq I_i$ for $1 \leq i \leq d$, then, $\mathfrak{m}I$ is integrally closed in R.*

Proof. As in the proof of lemma 2.3.1, we have that $\overline{\mathfrak{m}I} \subseteq I$. We will show that no

[1] Note that in general $\oplus_{g \in G \setminus \{0\}} R_g$ need not be an ideal of R. For example, let $R = k[x, y, y^{-1}]$ and let $\deg(x) = (1, 0)$, $\deg(y) = (0, 1)$ while $\deg(\alpha) = 0$ for $\alpha \in k$, then, $y \in \oplus_{g \in G \setminus \{0\}} R_g$ and if $\oplus_{g \in G \setminus \{0\}} R_g$ were an ideal, it is closed under multiplication, so $y^{-1}y = 1 \in \oplus_{g \in G \setminus \{0\}} R_g$, which is a contradiction.

homogeneous minimal generator of I is integral over $\mathfrak{m}I$.

Since R is a Noetherian domain, for any ideal $\mathfrak{a}$ of R, $r \in \overline{\mathfrak{a}}$ if and only if for every discrete valuation ring V such that $R \subseteq V \subseteq \mathrm{Frac}(R)$ and $\mathfrak{m}_V \cap R$ is a maximal ideal of R (where $\mathfrak{m}_V$ is the maximal ideal of V), we have that $r \in \mathfrak{a}V$ (proposition 6.8.4, page 135, [HS06]). We claim that when R is G-graded, where $G = \mathbb{Z}^q \times \mathbb{Z}^r_{\geq 0}$, $q, r \in \mathbb{Z}_{\geq 0}$, with $R_0 = k$, a field and such that $\mathfrak{m} = \oplus_{g \in G \setminus \{0\}} R_g$ is the unique G-homogeneous maximal ideal of R, if $\mathfrak{a}$ is a G-homogeneous ideal, we only need to let the discrete valuation rings in the above collection vary over those centered on the G-homogeneous maximal ideal of R, *i.e.,* over those discrete valuation rings whose maximal ideal contracts to the homogeneous maximal ideal of R. For in this case, $\overline{\mathfrak{a}}$ is also G-homogeneous by theorem 2.3.4. Suppose that $\overline{\mathfrak{a}} = \cap_{i=1}^{l} \mathfrak{q}_{\mathfrak{i}}$ is an irredundant primary decomposition for $\overline{\mathfrak{a}}$. Then $\mathfrak{q}_i$ is also G-homogeneous for $1 \leq i \leq l$ and is thus contained in the unique G-homogeneous maximal ideal $\mathfrak{m}$. Setting $S = R \setminus \mathfrak{m}$, we get that $S^{-1}(\overline{\mathfrak{a}}) = S^{-1}(\cap_{i=1}^{l} \mathfrak{q}_{\mathfrak{i}}) = \cap_{i=1}^{l} S^{-1} \mathfrak{q}_{\mathfrak{i}}$ (proposition 3.11.v, page 42, [AM94]). Also, since $\mathfrak{q}_i \subseteq \mathfrak{m}$, $S^{-1}\mathfrak{q}_i$ is primary for $1 \leq i \leq l$ (proposition 4.8.ii, page 52, [AM94]) and $\overline{S^{-1}(\mathfrak{a})} = S^{-1}(\overline{\mathfrak{a}})$ (proposition 1.1.4, page 3, [HS06]). Now $\overline{S^{-1}(\mathfrak{a})} \cap R = S^{-1}(\overline{\mathfrak{a}}) \cap R = (\cap_{i=1}^{l} S^{-1} \mathfrak{q}_{\mathfrak{i}}) \cap R = \cap_{i=1}^{l} (S^{-1} \mathfrak{q}_{\mathfrak{i}} \cap R) = \cap_{i=1}^{l} \mathfrak{q}_{\mathfrak{i}} = \overline{\mathfrak{a}}$, where the second equality follows from exercise 1.18, page 10, [AM94] and the third equality follows from (proposition 4.8.ii, page 52, [AM94]). Thus, $r \in \overline{\mathfrak{a}}$ if and only if $r \in \overline{\mathfrak{a}R_{\mathfrak{m}}}$. Further, since $R_{\mathfrak{m}}$ is a Noetherian local domain, $r \in \overline{\mathfrak{a}R_{\mathfrak{m}}}$ if and only if for every discrete valuation ring V such that $R_{\mathfrak{m}} \subseteq V \subseteq \mathrm{Frac}(R_{\mathfrak{m}})$ and $\mathfrak{m}_V \cap R_{\mathfrak{m}} = \mathfrak{m}R_{\mathfrak{m}}$, where $\mathfrak{m}_V$ is the maximal ideal of V, we have that $r \in (\mathfrak{a}R_{\mathfrak{m}})V$ (proposition 6.8.4, page 135, [HS06]). Finally, since R is a domain, $\mathrm{Frac}(R_{\mathfrak{m}}) = \mathrm{Frac}(R)$ and by the previous two statements, $r \in \overline{\mathfrak{a}}$ if and only if for every discrete valuation ring V such that $R \subseteq V \subseteq \mathrm{Frac}(R)$ and $\mathfrak{m}_V \cap R = \mathfrak{m}R_{\mathfrak{m}} \cap R = \mathfrak{m}$, where $\mathfrak{m}_V$ is the maximal ideal of V, we have that $r \in (\mathfrak{a}R_{\mathfrak{m}})V = \mathfrak{a}V$. This proves the claim. For the purposes of this proof, we will call a discrete valuation ring V such that $R \subseteq V \subseteq \mathrm{Frac}(R)$ and $\mathfrak{m}_V \cap R = \mathfrak{m}$, where $\mathfrak{m}_V$ is the maximal ideal of V an R-special discrete valuation ring.

The next set of arguments parallel those in the proof of 2.3.1.

We may assume that $\{a_1, ..., a_d\}$ is a minimal G-homogeneous set of generators for I without loss of generality. We first show that no generator $a_1, ..., a_d$ of I is integral over $\mathfrak{m}I$. Suppose that a_i is integral over $\mathfrak{m}I$ for some $i \in \{1, ..., d\}$. Then, for every R-special discrete valuation ring V, we have that $a_i \in (\mathfrak{m}I)V$. We may write that $I = I_i + a_iR$. So that, $(\mathfrak{m}I)V = (\mathfrak{m}I_i)V + (\mathfrak{m}a_i)V$. Since V is a discrete valuation ring containing R there is a discrete valuation on $\mathrm{Frac}(R)$, say $\mathbf{v}$, such that for any $s \in \mathrm{Frac}(R)$, $s \in V \iff \mathbf{v}(s) \geq 0$. For any ideal J of V define $\mathbf{v}(J) = \min\{\mathbf{v}(s) : s \in J\}$. Then $a_i \in (\mathfrak{m}I)V \implies \mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I)V) = \min\{\mathbf{v}((\mathfrak{m}I_i)V), \mathbf{v}((\mathfrak{m}a_i)V)\}$. We will show that $\mathbf{v}(a_i) < \mathbf{v}((\mathfrak{m}a_i)V)$, which would imply that $\mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I_i)V)$. Suppose that $\mathfrak{m} = (b_1, ..., b_t)R$. Then $(\mathfrak{m}a_i)V = (b_1a_i, ..., b_ta_i)V$. Since V is a discrete valuation ring, the ideals of V are totally ordered under inclusion. So after relabeling if necessary, we may assume that $(b_ta_i)V \subseteq (b_{t-1}a_i)V \subseteq ... \subseteq (b_1a_i)V$. So that, $(\mathfrak{m}a_i)V = (b_1a_i)V$. Then $(\mathfrak{m}a_i)V = (b_1a_i)V = \{s \in V : \mathbf{v}(s) \geq \mathbf{v}((b_1a_i))\}$. If $\mathbf{v}(a_i) \geq \mathbf{v}(b_1a_i)$, then, since $\mathbf{v}(b_1a_i) = \mathbf{v}(b_1) + \mathbf{v}(a_i)$, we have that $\mathbf{v}(b_1) \leq 0$. We cannot have, $\mathbf{v}(b_1) < 0$ since $b_1 \in V$. So $\mathbf{v}(b_1) = 0$. Then b_1 must be a unit in V so that $\mathfrak{m}V = (b_1, ..., b_t)V = V$. This is a contradiction since, if $\mathfrak{m}_V$ is the maximal ideal of V, we have that $(\mathfrak{m}_V \cap R)V \subseteq \mathfrak{m}_V$ (proposition 1.17(i), page 10, [AM94]) but $\mathfrak{m}_V \cap R = \mathfrak{m}$ since V is R-special and this implies that $(\mathfrak{m}_V \cap R)V = \mathfrak{m}V = V \subseteq \mathfrak{m}_V$. So $\mathbf{v}(a_i) < \mathbf{v}(b_1a_i) = \mathbf{v}((\mathfrak{m}a_i)V)$. Hence, $\mathbf{v}(a_i) \geq \mathbf{v}((\mathfrak{m}I_i)V)$, *i.e.*, $a_i \in (\mathfrak{m}I_i)V$. Since this statement is true for every R-special discrete valuation ring V and since $\mathfrak{m}, I_i$, and consequently, $\mathfrak{m}I_i$ are G-homogeneous, we have that $a_i \in \overline{\mathfrak{m}I_i}$. By hypothesis, $\overline{\mathfrak{m}I_i} \subseteq I_i$. So $a_i \in I_i$. However, this contradicts the minimality of the generating set $\{a_1, ..., a_d\}$ for I for if $a_i \in I_i$, then, $I_i = (a_1, ..., \hat{a}_i, ..., a_d)R = (a_1, ..., \hat{a}_i, ..., a_d)R + a_iR = I$. This shows that a_i is not integral over $\mathfrak{m}I$ for any i such that $1 \leq i \leq d$.

Now assume that $\mathfrak{m}I$ is not integrally closed. Suppose that $a = r_1a_1 + ... + r_da_d \in I$ (where $r_1, ..., r_d \in R$) such that $a \in \overline{\mathfrak{m}I} - \mathfrak{m}I$. Since $\mathfrak{m}, I$ are G-homogeneous, so is $\mathfrak{m}I$ and hence $\overline{\mathfrak{m}I}$ is G-homogeneous by theorem 2.3.4. By remark 2.3.3, at least one G-homogeneous part of a must be in $\overline{\mathfrak{m}I}$ but not in $\mathfrak{m}I$. So without loss of generality, we may assume that a is G-homogeneous. Now, by definition, $R = \oplus_{g \in G} R_g = R_0 \oplus (\oplus_{g \in G \setminus \{0\}} R_g) = k \oplus \mathfrak{m}$. So we can write that $r_i = r_i^{'} + s_i$, where $r_i^{'} \in \mathfrak{m}$ and $s_i \in k$ for $1 \leq i \leq d$. Now suppose that $r_ia_i \neq 0$ for some $i \in \{1, ..., d\}$. Then $(r_i^{'}a_i + s_ia_i)$ is a non-zero summand of a. Since a is G-homogeneous, and since $r_i^{'}, s_i$ must have different degrees, exactly one of $r_i^{'}a_i$ and s_ia_i is non-zero. If $s_ia_i = 0$ for $1 \leq i \leq d$, then, $a = r_1^{'}a_1 + ... + r_d^{'}a_d \in \mathfrak{m}I$, which is a contradiction. So there exists some $i \in \{1, .., d\}$ such that $s_ia_i \neq 0$. Without loss of generality, assume that $s_1a_1 \neq 0$. Then $\{b, a_2, a_3, ..., a_d\}$ is also a minimal G-homogeneous set of generators for I. For we have that $a_1 = s_1^{-1}b - (s_1^{-1}(r_2a_2 + ... + r_da_d)) \in (b, a_2, a_3, ..., a_d)R$, so $I \subseteq (b, a_2, a_3, ..., a_d)R$. Conversely, $b \in I$, so $(b, a_2, a_3, ..., a_d)R \subseteq I$. So $\{b, a_2, a_3, ..., a_d\}$ is a generating set for I and each generator is G-homogeneous. Further, $(a_2, ..., a_d)R \subsetneq I$ by minimality of $\{a_1, ..., a_d\}$. Finally, if for some $l \in \{2, ..., d\}$, $a_l \in (b, a_2, ..., \hat{a}_l, ..., a_d)R$, then, $a_l = \alpha_1 b + \alpha_2 a_2 + ... + \alpha_{l-1}a_{l-1} + \alpha_{l+1}a_{l+1} + ... + \alpha_d a_d$ for some $\alpha_1, ..., \alpha_{l-1}, \alpha_l, ..., \alpha_d \in R$. Since α_l is a G-homogeneous, we may assume that each of the summands in the preceding expression is G-homogeneous of the same degree. Then, arguing as earlier in the paragraph, either $\alpha_j \in k$ or $\alpha_j \in \mathfrak{m}$ for $j \in \{1, ..., l-1, l+1, ..., d\}$. If α_1 is a unit, we can multiply the above equation by $\alpha_1^{-1}s_1^{-1}$ to get,

$$a_1 = \alpha_1^{-1}s_1^{-1}(a_l - \alpha_1(b - s_1a_1) - (\alpha_2a_2 + ... + \alpha_{l-1}a_{l-1} + \alpha_{l+1}a_{l+1} + ... + \alpha_da_d))$$

Then the right hand side of the preceding equation is an $R-$linear combination of $a_2, ..., a_d$, contradicting the minimality of $\{a_1, .., a_d\}$. If α_1 is not a unit, $\alpha_1 \in \mathfrak{m}$. Then we can write that

$$a_l = \alpha_1(b - r_la_l) + \alpha_1r_la_l + \alpha_2a_2 + ... + \alpha_{l-1}a_{l-1} + \alpha_{l+1}a_{l+1} + ... + \alpha_da_d$$

Since by assumption this is a G-homogeneous expression for a_l and since $\alpha_1r_l \in \mathfrak{m}$, we must have, $\alpha_1r_l = 0$. So the right hand side of the above equation is a $R-$linear combination of $a_1, ..., a_{l-1}, a_{l+1}, ..., a_d$, again contradicting the minimality of $\{a_1, ..., a_d\}$.

Thus, $\{b, a_2, ..., a_d\}$ is a minimal set of generators for I and by the arguments in the first paragraph, b is not integral over mI, which is a contradiction.

Thus, $\mathfrak{m}I$ is integrally closed in R. □

2.3.2 Consequences of the main lemma

We will use lemma 2.3.5 to show that if I is a radical ideal in a polynomial ring R generated by one binomial and several monomials, then, $\mathfrak{m}I$ is integrally closed, where $\mathfrak{m}$ is the standard homogeneous maximal ideal in the ring. First we will define a non-standard grading on the polynomial ring, which will make such ideals homogeneous with respect to the grading.

Note 2.3.6. Let $R = k[x_1, ..., x_n]$, where k is a field. Let $\mu = x_1^{u_1}...x_n^{u_n}$ and $\nu = x_1^{v_1}...x_n^{v_n}$ such that $\mu \neq \nu$. Let $d_i = u_i - v_i$ for $1 \leq i \leq n-1$ and let $h = v_n - u_n$. We may assume that $h > 0$ after relabeling indeterminates if necessary. Define a $\mathbb{Z}^{n-1}$-grading on R such that $\deg(\alpha) = (0, ..., 0)$ for $\alpha \in k$, $\deg(x_i) = \underbrace{(0, ..., h, ..., 0)}_{i\text{th position}}$

for $1 \leq i \leq n-1$ and $\deg(x_n) = (d_1, ..., d_{n-1})$. Then, under this grading, we show that $\deg(\mu) = \deg(\nu)$.

Let $\deg(\lambda)_i$ denote the ith component of $\deg(\lambda)$ for any monomial λ. Then, for $1 \leq i \leq n-1$,

$\deg(\mu)_i = u_i h + u_n d_i = u_i(v_n - u_n) + u_n(u_i - v_i) = u_i v_n - u_i u_n + u_n u_i - u_n v_i = u_i v_n - u_n v_i = v_n v_i - u_n v_i + u_i v_n - v_n v_i = v_i(v_n - u_n) + v_n(u_i - v_i) = v_i h + v_n d_i = \deg(\nu)_i$

Proposition 2.3.7. *With notation as in note 2.3.6, suppose that gcd$(h, d_j) = 1$ for some j such that $1 \leq j \leq n-1$. Then a pair of distinct monomials have the same degree if and only if they are of the form $\lambda\mu^q, \lambda\nu^q$, where λ is a monomial and q is some positive integer.*

Proof. Clearly, $\lambda\mu^q$ and $\lambda\nu^q$ have the same degree since μ and ν have the same degree.

Now suppose that $\mu' = x_1^{u_1'} ... x_n^{u_n'}$ and $\nu' = x_1^{v_1'} ... x_n^{v_n'}$ are distinct monomials in R such that $\deg(\mu') = \deg(\nu')$.

Equating components of the degrees, we have that $u_i' h + u_n' d_i = v_i' h + v_n' d_i$. So $(u_i' - v_i')h = (v_n' - u_n')d_i$. Now there exists an integer j such that $1 \leq j \leq n-1$ and $\gcd(h, d_j) = 1$. So $(u_j' - v_j')h = (v_n' - u_n')d_j$ implies that h divides $v_n' - u_n'$.

Hence, there exists a nonnegative integer q such that

$$v_n' - u_n' = hq = (v_n - u_n)q_n.$$

Thus,

$$v_n' - qv_n = u_n' - qu_n = r_n \text{ (say)}.$$

Therefore, $u_n' = qu_n + r_n$ and $v_n' = qv_n + r_n$.

In fact, q must be positive, for if $q = 0$, then, $v_n' - u_n' = 0$ by the above equation and hence, $(u_i' - v_i')h = (v_n' - u_n')d_i = 0d_i = 0$. Since $h > 0$ by assumption, we have that $u_i^{'} - v_i^{'} = 0$ for $1 \leq i \leq n$. Hence, $\mu^{'} = \nu^{'}$, which is a contrary to the supposition that $\mu^{'}, \nu^{'}$ are distinct.

Also, $(u_i^{'} - v_i^{'})h = (v_n^{'} - u_n^{'})d_i = hqd_i$, then, since $h \neq 0$, for $1 \leq i \leq n-1$, we have that

$$u_i^{'} - v_i^{'} = d_i q = (u_i - v_i)q.$$

Thus,

$$u_i^{'} - qu_i = v_i^{'} - qv_i = r_i \text{ (say)}.$$

Hence, $u_i^{'} = qu_i + r_i$ and $v_i^{'} = qv_i + r_i$. The above arguments show that these equations hold for $1 \leq i \leq n$.

Then $\mu^{'} = x_1^{qu_1+r_1}...x_n^{qu_n+r_n} = (x_1^{qu_1}...x_n^{qu_n})(x_1^{r_1}...x_n^{r_n}) = \mu^q\lambda$, where $\lambda = x_1^{r_1}...x_n^{r_n}$ and $\nu^{'} = x_1^{qv_1+r_1}...x_n^{qv_n+r_n} = (x_1^{qv_1}...x_n^{qv_n})(x_1^{r_1}...x_n^{r_n}) = \nu^q\lambda$. □

Corollary 2.3.8. *With notation as in note 2.3.6, suppose that* $gcd(h, d_1, ..., d_{n-1}) = 1$*. Then a pair of distinct monomials have the same degree if and only if they are of the form* $\lambda\mu^q, \lambda\nu^q$*, where* λ *is a monomial and* q *is some positive integer.*

Proof. If $\gcd(h, d_1, ..., d_{n-1}) = 1$, then, there exists an integer j such that. $1 \leq j \leq n-1$ and $\gcd(h, d_j) = 1$. Then, by proposition 2.3.7, we get the desired conclusion. □

Definition 2.3.9. Let $R = k[x_1, .., x_n]$ with k a field. Let $\mu = x_1^{u_1}...x_n^{u_n}$ and $\nu = x_1^{v_1}...x_n^{v_n}$ with $\mu \neq \nu$. Then we define a (μ, ν)-*special grading* on R to be the $\mathbb{Z}^{n-1}$-grading on R as in note 2.3.6.

Theorem 2.3.10. *Let* $R = k[x_1, .., x_n]$ *with* k *a field. Let* $I = (\mu_1, .., \mu_d, a\mu + b\nu)R$*, where* $\mu_1, .., \mu_d, \mu, \nu$ *are monomials in* R *and* $a, b \in k$*. If* I *is radical, then,* $\mathfrak{m}I$ *is integrally closed.*

Proof. For an ideal $\mathfrak{a}$ in R let $J_{\mathfrak{a}}$ denote the ideal generated by the monomials in $\mathfrak{a}$. We may assume that $a, b \neq 0$, $\mu \neq \nu$, $\mu, \nu \notin J_I$. For if any of these conditions don't hold, then, I is a radical monomial ideal and the statement of the theorem is true by theorem 2.1.1.

Now $G = \mathbb{Z}^{n-1}$ under the usual operation of addition is an abelian monoid. Then the (μ, ν)-special grading on R is a G-grading with $R_0 = R_{(0,\ldots,0)} = k$. Further, R is a Noetherian domain and $\mathfrak{m} := \oplus_{g \in G\setminus\{0\}} R_g = (x_1, \ldots, x_n)R$ is the unique G-homogeneous maximal ideal of R. To see the latter claim, note that if $x_1^{a_1} \ldots x_n^{a_n} \neq 1$ is a monomial in R, then, using the notation in note 2.3.6, $\deg(x_1^{a_1} \ldots x_n^{a_n}) = (a_1 h + a_n d_1, \ldots, a_{n-1} h + a_n d_{n-1})$. Now, since $d_i > 0$ for at least one $i \in \{1, \ldots, n-1\}$, say $i = j$, at least one coordinate in $\deg(x_1^{a_1} \ldots x_n^{a_n})$ must be positive, for if $a_n = 0$, then, $a_i h + a_n d_i = a_i h > 0$ for $a_i > 0$ and if $a_n > 0$, then, $a_j h + a_n d_j > 0$. Thus, $\oplus_{g \in G\setminus\{0\}} R_g$ contains all monomials other than 1 and hence contains all linear combinations of monomials. Thus, $\oplus_{g \in G\setminus\{0\}} R_g = (x_1, \ldots, x_n)R$ as sets and hence $\mathfrak{m} = \oplus_{g \in G\setminus\{0\}} R_g$ is an ideal. Further, since $\mathfrak{m}$ is generated by monomials, it is G-homogeneous and since it contains all monomials other than 1, it contains all G-homogeneous ideals. Thus, $\mathfrak{m} = \oplus_{g \in G\setminus\{0\}} R_g = (x_1, \ldots, x_n)R$ is the unique G-homogeneous maximal ideal in R.

Moreover, by note 2.3.6, $\deg(\mu) = \deg(\nu)$ under this grading. Hence, $a\mu + b\nu$ is a G-homogeneous element and I is a G-homogeneous ideal. Since I is a radical ideal, by corollary 2.2.9, the ideal generated by the monomials in I is a radical ideal and $a\mu + b\nu$ is a squarefree binomial. So we may write that $I = (\eta_1, \ldots, \eta_t, a\mu + b\nu)R$, where $\eta_1, .., \eta_t$ are squarefree monomials and $J_I = (\eta_1, \ldots, \eta_t)R$. Let $I_{\eta_i} = (\eta_1, \ldots, \hat{\eta}_i, \ldots, \eta_t, a\mu + b\nu)R$. If I_{η_i} is radical for $1 \leq i \leq t$, then, since a radical ideal is integrally closed (remark 1.1.3.(4), page 2, [HS06]), we have that $I_{\eta_1}, \ldots, I_{\eta_t}, J_I$ are integrally closed ideals and hence $\overline{\mathfrak{m} I_{\eta_i}} \subseteq \overline{I}_{\eta_i} = I_{\eta_i}$ and $\overline{\mathfrak{m} J_I} \subseteq \overline{J_I} = J_I$ (remark 1.1.3(2), page 2, [HS06]). Then, by lemma 2.3.5, $\mathfrak{m}I$ is integrally closed in R.

Now suppose that I_{η_i} is not radical for all i. After relabeling if necessary, we may assume that I_{η_i} is not radical for $1 \leq i \leq s_1 \leq t$ and I_{η_i} is radical for $s_1 + 1 \leq i \leq t$. Then we must have that $J_{I_{\eta_i}} \neq (\eta_1, ..., \hat{\eta_i}, ..., \eta_t)R$ by 2.2.9. Let $J_{i,1} = (\eta_1, ..., \hat{\eta_i}, ..., \eta_t)R$. Then, by proposition 2.2.11, $J_{I_{\eta_i}} = J_{i,1} + (J_{i,1} : \mu)\nu + (J_{i,1} : \nu)\mu$. We append generators of $(J_{i,1} : \mu)\nu, (J_{i,1} : \nu)\mu$ for $1 \leq i \leq s_1$, say $\{\delta_{1,1}, ..., \delta_{m,1}\}$ to the list of generators $\{\eta_1, ..., \eta_t\}$ of J_i and rewrite $I = (\eta_1, ..., \eta_t, \delta_{1,1}, ..., \delta_{m,1}, a\mu + b\nu)R$, since $(J_{i,1} : \mu)\nu \subseteq J_{I_{\eta_i}} \subseteq J_I$ and $(J_{i,1} : \nu)\mu \subseteq J_{I_{\eta_i}} \subseteq J_I$. We may assume that $\delta_{1,1}, ..., \delta_{m,1}$ are squarefree since I is radical. Also, $I_{\delta_{j,1}} = (\eta_1, ..., \eta_t, \delta_{1,1}, ..., \hat{\delta_{j,1}}, ..., \delta_{m,1}, a\mu + b\nu)R = I$. Then if

$$I_{\eta_i,1} = (\eta_1, ..., \hat{\eta_i}, ..., \eta_t, \delta_{1,1}, ..., \delta_{m,1}, a\mu + b\nu)R$$

are radical for $1 \leq i \leq s_1$, the hypothesis of lemma 2.3.5 is satisfied and we have that $\overline{\mathfrak{m}I} = \mathfrak{m}I$ (note that $I_{\eta_i,1} = I_{\eta_i}$ for $s + 1 \leq i \leq t$, which are already assumed to be radical). If $I_{\eta_i,1}$ are not all radical, then, we repeat the process in this paragraph. Explicitly, we may assume that $I_{\eta_i,1}$ is not radical for $1 \leq i \leq s_2 \leq s_1$ and $I_{\eta_i,1}$ is radical for $s_2 + 1 \leq i \leq t$. We define $J_{i,2}$ similarly and append generators of $(J_{i,2} : \mu)\nu, (J_{i,2} : \nu)\mu$ for $1 \leq i \leq s_2$, to the list of generators of I. We note that $J_{i,1} \subseteq J_{i,2}$. If the ideals obtained by deleting one generator from the specified list of generators of I are not all radical, we repeat the above process of augmenting generators. Since R is Noetherian, the chain of ideals $J_{i,1} \subseteq J_{i,2} \subseteq J_{i,3} \subseteq ...$ eventually stabilizes, say at step N, so that $J_{i,N}$ are the ideals generated by the monomials in $J_{\eta_i,N}$. Then, since $J_{i,N}$ are squarefree by construction, $I_{\eta_i,N}$ must be all radical by 2.2.9. Then, applying lemma 2.3.5, we get that $\mathfrak{m}I$ is integrally closed. $\square$

We will end this section with a couple of further applications of the main lemma from the previous section.

Proposition 2.3.11. *Let $R = k[[x_1, ..., x_n]]$, where k is a field. Let $\mathfrak{m} = (x_1, ..., x_n)R$. Suppose that $I = (f_1, ..., f_d)R$ is such that either:*

- $f_1, ..., f_d$ are homogeneous linear polynomials.
- $f_1, ..., f_d$ are irreducible elements of R such that f_i and $f_1, ..., f_{i-1}, f_{i+1}, ..., f_d$ can be written in terms of distinct sets of indeterminates for all i such that $1 \leq i \leq d$.

Then $\mathfrak{m}I$ is integrally closed.

Proof. Here I and $I_i = (f_1, ..., \hat{f_i}, ..., f_n)R$ are prime ideals in R and hence integrally closed. Then $\overline{\mathfrak{m}I_i} \subseteq \overline{I_i} = I_i$ (remark 1.1.3(2), page 2, [HS06]). So lemma 2.3.1 applies and hence $\mathfrak{m}I$ is integrally closed in R. □

Proposition 2.3.12. *Let $R = k[x_1, ..., x_n]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in $k[x_1, ..., x_n]$ and k is an infinite field. Suppose that I is an ideal in R that can be minimally generated by depth(I) elements and that I is integrally closed. Then, for every prime ideal $P \in$ Supp(I), we have that $(PI)R_P$ is integrally closed in R_P.*

Proof. Davis proves that any ideal J in R has a minimal basis such that any subset of this basis of size less than depth(J) generates a prime ideal in R (theorem 2, [Dav78]). Let $I = (y_1, ..., y_d)R$, where $d =$ depth(I) and assume that $\{y_1, ..., y_d\}$ is a basis such that every subset of this basis (of size necessarily less than depth(I)) generates a prime ideal. Then the ideals $I_i = (y_1, ..., \hat{y_i}, ..., y_d)R$ are prime for $1 \leq i \leq d$ and hence integrally closed (remark 1.1.3.(4), page 2, [HS06]). Since $P \in$ Supp(I), we have that $I \subseteq P$. Then $IR_P = (\frac{y_1}{1}, ..., \frac{y_d}{1})R_P$. Now IR_P is integrally closed since I is integrally closed (proposition 1.1.4, page 3, [HS06]). Further, the ideals

$(IR_P)_i = (\frac{y_1}{1}, ..., \hat{\frac{y_i}{1}}, ..., \frac{y_d}{1})R_P$ are prime (since I_i are prime and $I_i \subseteq P$) and hence integrally closed. Finally, we have that $\overline{(PR_P)(IR_P)_i} \subseteq \overline{(IR_P)_i} = (IR_P)_i$. Then, by lemma 2.3.1, $(PR_P)(IR_P) = (PI)R_P$ is integrally closed in R_P. □

2.4 Monomial type ideals in regular local rings

Let $(R, \mathfrak{m})$ be a regular local ring. Let $\underline{x} = x_1, ..., x_d$ be a fixed regular system of parameters of R, where $d = \dim(R)$. By a monomial over $\underline{x}$, we mean an element of R of the form $x_1^{a_1}...x_d^{a_d}$ with a_i non-negative integers for $i = 1, ..., d$. We deem a monomial over $\underline{x}$ to be squarefree if $0 \leq a_i \leq 1$ for $i = 1, ..., d$. By a (squarefree) monomial ideal over $\underline{x}$, we mean an ideal generated by (squarefree) monomials over $\underline{x}$. For a monomial $\mu = x_1^{a_1}...x_d^{a_d}$ over $\underline{x}$, define the squarefree part of μ, $\mu^{\#} = x_1^{b_1}...x_d^{b_d}$, where $b_i = 1$ if $a_i \geq 1$ and $b_i = 0$ if $a_i = 0$. We shall show that for a squarefree monomial ideal I over $\underline{x}$, $\mathfrak{m}I$ is integrally closed in R.

Such ideals were considered in [KS03]. In that paper, monomial ideals over regular sequences contained in the Jacobson radical in a Noetherian ring were defined analogously. Clearly, our definition above is a special case of this. Among the results proved in that paper include: the sum, product and colon of monomial ideals over regular sequences are monomial and under some mild assumptions the integral closure of a monomial ideal over regular sequences is monomial.

Hübl-Swanson [HS08] define an analogous notion of monomial ideals over permutable regular sequences in a regular domain such that every subsequence generates a prime ideal in the ring. Again our definition in the first paragraph is a special case of this since a regular local ring is a domain and a regular system of parameters is a permutable regular sequence such that every subsequence generates a prime ideal. Hübl-Swanson show that the integral closure of these generalized monomial ideals is also monomial and in fact it can be described in terms of the Newton polygon in a manner analogous to the monomial ideals in polynomial rings. Note that, the sequence of indeterminates in a polynomial ring over a field in finitely many indeterminates, is a special case of the kind of regular sequences considered in [KS03] and [HS08]. So the results in these papers can be considered as generalizations of the results on monomial ideals in polynomial rings.

We first prove the following lemma to show that squarefree monomial ideals over a regular system of parameters is radical.

Lemma 2.4.1. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\underline{x} = x_1, ..., x_d$ be a fixed regular system of parameters of R, where $d = dim(R)$. Let $I = (\mu_1, ..., \mu_t)R$ be a monomial ideal over $\underline{x}$ in R, where $\mu_1, ..., \mu_t$ are squarefree monomials over $\underline{x}$. Then I is radical.*

Proof. Let $\mu_j = x_{j_1} ... x_{j_{k(j)}}$, where $1 \leq j_1 < ... < j_{k(j)} \leq d$ and $k(j) \in \{1, ..., d\}$. Kiyek-Stückrad prove (proposition 1, [KS03]) that for monomial ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ over regular sequences contained in the Jacobson radical of a Noetherian ring $(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + (\mathfrak{b} \cap \mathfrak{c})$. This property is known as the modular law.

Consider $J_1 = (x_{1_1}, \mu_2, ..., \mu_t)R \cap (x_{1_2}, \mu_2, ..., \mu_t)R \cap ... \cap (x_{1_{k(1)}}, \mu_2, ..., \mu_t)R$. Denote $I_1 = (\mu_2, ..., \mu_t)R$. Then, by the modular law,

$$\begin{aligned}
(x_{1_1}R + I_1) \cap (x_{1_2}R + I_1) &= ((x_{1_1}R + I_1) \cap (x_{1_2}R)) + ((x_{1_1}R + I_1) \cap I_1) \\
&= ((x_{1_1}R + I_1) \cap (x_{1_2}R)) + I_1 \\
&= ((x_{1_1}R) \cap (x_{1_2}R)) + (I_1 \cap x_{1_2}R) + I_1 \\
&= ((x_{1_1}R) \cap (x_{1_2}R)) + I_1
\end{aligned}$$

Now for any principal ideals of R, say $\mathfrak{a} = aR, \mathfrak{b} = bR$, we have that $\mathfrak{a} \cap \mathfrak{b} = (\mathrm{lcm}(a, b))R$ (proposition 1, [KS03])[2]. Thus, $((x_{1_1}R) \cap (x_{1_2}R)) = x_{1_1}x_{1_2}R$. So that,

[2]Note that R is a unique factorization domain here so the notion of lcm is well defined. The usual definition

$(x_{1_1}R + I_1) \cap (x_{1_2}R + I_1) = x_{1_1}x_{1_2}R + I_1$. Proceeding inductively, we have that, $J_1 = (x_{1_1}R + I_1) \cap (x_{1_2}R + I_1) \cap ... \cap (x_{1_{k(1)}}R + I_1) = x_{1_1}...x_{1_{k(1)}}R + I_1 = I$. Carrying out an analogous process for the other monomials in the generating set of I, we can write that $I = \cap(x_{1_{l(1)}}, ..., x_{t_{l(t)}})R$, where $l(j) \in \{1, .., k(j)\}$. Now $x_{1_{l(1)}}, ..., x_{t_{l(t)}}$ is part of a regular system of parameters in R for any choice of subscripts and thus, $R/(x_{1_{l(1)}}, ..., x_{t_{l(t)}})R$ is a domain. Consequently, the ideals $(x_{1_{l(1)}}, ..., x_{t_{l(t)}})R$ are prime. Then I is an intersection of prime ideals is hence radical. □

Proposition 2.4.2. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\underline{x} = x_1, ..., x_d$ be a fixed regular system of parameters of R, where $d = dim(R)$. Let $I = (\mu_1, ..., \mu_t)R$, where $\mu_1, ..., \mu_t$ are monomials over $\underline{x}$. Then $\sqrt{I} = (\mu_1^{\#}, ..., \mu_t^{\#})R$.*

Proof. Let $\mu_i = x_1^{u_{1i}}...x_d^{u_{di}}$. Let $u_i = \max\{u_{ji} | 1 \le j \le d\}$. Then $(\mu_i^{\#})^{u_i} \in I$ since $(\mu_i^{\#})^{u_i}$ is a multiple of μ_i for $1 \le i \le t$. Set $N = u_1 + ... + u_t - t + 1$. Then $((\mu_1^{\#}, ..., \mu_t^{\#})R)^N \subseteq I$ by the pigeonhole principle. Taking radicals, we have that $\sqrt{((\mu_1^{\#}, ..., \mu_t^{\#})R)^N} = \sqrt{(\mu_1^{\#}, ..., \mu_t^{\#})R} = (\mu_1^{\#}, ..., \mu_t^{\#})R \subseteq \sqrt{I}$, where the second equality follows from lemma 2.4.1. Conversely, since μ_i is a multiple of $\mu_i^{\#}$, $I \subseteq (\mu_1^{\#}, ..., \mu_t^{\#})R$. So taking radicals we have that $\sqrt{I} \subseteq \sqrt{(\mu_1^{\#}, ..., \mu_t^{\#})R} = (\mu_1^{\#}, ..., \mu_t^{\#})R$. Thus, $\sqrt{I} = (\mu_1^{\#}, ..., \mu_t^{\#})R$. □

Now we are ready to prove the main theorem of this section.

Theorem 2.4.3. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\underline{x} = x_1, ..., x_d$ be a fixed regular system of parameters, where $d = dim(R)$. Let I be a squarefree monomial ideal over $\underline{x}$. Then $\mathfrak{m}I$ is integrally closed.*

Proof. Let $I = (\mu_1, ..., \mu_t)R$, where $\mu_1, ..., \mu_t$ are squarefree monomials over $\underline{x}$. By lemma 2.4.1, I is radical and hence integrally closed (remark 1.1.3.(4), page 2,

of lcm for a unique factorization domain coincides with the definition of lcm in [KS03] for the kind of ideals they consider.

[HS06]). Further, by the same lemma, $I_i = (\mu_1, ..., \hat{\mu_i}, ..., \mu_t)R$ is also radical (and hence integrally closed) since I_i is a squarefree monomial ideal. Then $\overline{mI_i} \subseteq \overline{I_i} = I_i$. Then, by lemma 2.3.1, mI is integrally closed in R. □

Proposition 2.4.4. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\underline{x} = x_1, ..., x_d$ be a fixed regular system of parameters of R, where $d = dim(R)$. Let I be a monomial ideal over $\underline{x}$. Then $\overline{\mathfrak{m}I} \subseteq \mathfrak{m}\sqrt{I}$.*

Proof. Since $I \subseteq \sqrt{I}$, $\mathfrak{m}I \subseteq \mathfrak{m}\sqrt{I}$ and hence, $\overline{\mathfrak{m}I} \subseteq \overline{\mathfrak{m}\sqrt{I}}$. Now lemma 2.4.1 shows that $\sqrt{I}$ is a squarefree monomial ideal over $\underline{x}$. Then, by theorem 2.4.3, $\overline{\mathfrak{m}\sqrt{I}} = \mathfrak{m}\sqrt{I}$. Thus, $\overline{\mathfrak{m}I} \subseteq m\sqrt{I}$. □

Note the similarity between the above proposition and theorem 2.1.1. Using the characterization of the integral closure of monomial ideals over a regular system of parameters in a regular local ring due to Hübl-Swanson ([HS08]), we can adapt the proof of theorem 2.1.1 to obtain the preceding two results. Also, we can use lemma 2.3.1 to obtain another proof of theorem 2.1.1.

2.5 Hübl's conjecture

In [Hüb99], Hübl makes the following conjecture.

Conjecture 2.5.1. *Let $(R, \mathfrak{m})$ be a regular local ring. Let I be a radical ideal in R. Let $f \in I$ such that $f^n \in I^{n+1}$. Then $f \in \mathfrak{m}I$.*

Hübl [Hüb99] remarks that a positive answer to the above conjecture implies that any reduced local algebra R/k, essentially of finite type over a field k of characteristic 0 is evolutionarily stable. Note that it is necessary to assume that I is radical in conjecture 2.5.1, otherwise we have the following counterexample (remark 1.5.(i), [Hüb99]) .

Example 2.5.2. Let $R = k[[x, y]]$ be a formal power series ring over a field k. Let $I = (x^3, x^2y^2, y^3)R$. Then, if $f := x^2y^2$, we have that $f^3 = x^6y^6 = (x^3)(x^3)(y^3)(y^3) \in I^4$ but $f \notin \mathfrak{m}I$, where $\mathfrak{m} = (x, y)R$.

Epstein and Hochster (theorem 5.4, [EH11]) define the inner integral closure of an ideal in a Noetherian ring as follows (among other equivalent ways).

Definition 2.5.3. Let R be a Noetherian ring and I an ideal of R. If $f \in I$ and there is a positive integer n such that $f^n \in I^{n+1}$, then, we say f lies in the *inner integral closure* of I (denoted $I_{>1}$).

Then conjecture 2.5.1 says that if $(R, \mathfrak{m})$ is a regular local ring and I a radical ideal in R, then, $I_{>1} \subseteq \mathfrak{m}I$.

We make the earlier remark due to Hübl connecting this conjecture to the question of existence of non-trivial evolutions more precise.

Remark 2.5.4. (Hübl) [Hüb99] Let k be a field of characteristic 0. A reduced local algebra R/k, essentially of finite type has no non-trivial evolutions if and only if it has a presentation $R = S/I$ such that S/k is smooth and $I_{>1} \subseteq \mathfrak{m}I$.

We now state two positive results for this conjecture due to Hübl.

Theorem 2.5.5. *[Hüb99] Let $(R, \mathfrak{m})$ be a regular local ring and let I be an equidimensional radical ideal, which is a complete intersection on the punctured spectrum of R and that depth$(R) \geq 1$. Then $I_{>1} \subseteq \mathfrak{m}I$.*

Theorem 2.5.6. *[Hüb99] Let $(R, \mathfrak{m})$ be a regular local ring and let I be an equidimensional radical ideal such that R/I^2 is Cohen-Macaulay. Then $I_{>1} \subseteq \mathfrak{m}I$.*

Definition 2.5.7. Let $R = S[x_1, ..., x_n]$, where S is a commutative ring. A polynomial $f \in R$ is said to be *quasihomogeneous* if there exists an n-tuple of non-negative integers $(i_1, ..., i_n)$ such that $f(\lambda^{i_1}x_1, ..., \lambda^{i_n}x_n) = \lambda^m f(x_1, ..., x_n)$ for some positive integer m.

We will show that if R is the polynomial ring over a field, I a radical ideal of R and if f is a quasihomogeneous element such that $f \in I_{>1}$, then, $f \in \mathfrak{m}I$. We first need the following lemma.

Lemma 2.5.8. *Let $R = k[x_1, ..., x_d]$, where k is a field and let $x_1, ..., x_d$ be indeterminates over k. Let $I \subset R$ be a radical ideal. Let $f \in R$ satisfy $f^n \in I^{n+1}$ for some positive integer n. Then*

1. If $n > 1$ and $\mathrm{char}(k) \nmid n$, then, $f^{n-1}(\frac{\partial f}{\partial x_i}) \in I^n$ and hence $f(\frac{\partial f}{\partial x_i}) \in I_{>1}$.

2. If $\mathrm{char}(k) \nmid n!$, then, $\frac{\partial f}{\partial x_i} \in I$ for $1 \leq i \leq d$.

Proof. We have that $f^n \in I^{n+1}$. So $\frac{\partial(f^n)}{\partial x_i} = nf^{n-1}\frac{\partial f}{\partial x_i} \in I^n$. Since k is a field and $\mathrm{char}(k) \nmid n$, we have that $f^{n-1}\frac{\partial f}{\partial x_i} \in I^n$. Then $(f(\frac{\partial f}{\partial x_i}))^{n-1} \in I^n$ and hence $f(\frac{\partial f}{\partial x_i}) \in I_{>1}$.

For the second assertion, we observe that, $\frac{\partial^n(f^n)}{\partial x_i^n} \in I$. Now $\frac{\partial^n(f^n)}{\partial x_i^n} = (n!)(\frac{\partial f}{\partial x_i})^n + fg$, where $g = g(f, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i^2}, ..., \frac{\partial^n f}{\partial x_i^n})$. Since $f^n \in I^{n+1} \subseteq I$ and I is radical, we have that $f \in I$, so that $fg \in I$. Hence, $(n!)(\frac{\partial f}{\partial x_i})^n \in I$. Since $\mathrm{char}(k) \nmid n!$, $(\frac{\partial f}{\partial x_i})^n \in I$. Again, since I is radical, $\frac{\partial f}{\partial x_i} \in I$ for $1 \leq i \leq d$. $\square$

Proposition 2.5.9. *Let $R = k[x_1, ..., x_d]$, where k is a field and let $x_1, ..., x_d$ be indeterminates over k. Let $I \subset R$ be a radical ideal and $\mathfrak{m} = (x_1, ..., x_d)R$. Let $f \in R$ be a quasihomogeneous element such that $f \in I_{>1}$. If char$(k) \nmid (n!)$deg(f), then, $f \in \mathfrak{m}I$.*

Proof. Suppose that n is a positive integer such that $f^n \in I^{n+1}$. Since f is quasihomogeneous, by Euler's formula, $(\deg(f))f = \Sigma_{i=1}^d (\deg(x_i))x_i\frac{\partial f}{\partial x_i}$. As shown in lemma

2.5.8, $\frac{\partial f}{\partial x_i} \in I$ for $i = 1, ..., d$. Then, since $\text{char}(k) \nmid \deg(f)$ and k is a field $\deg(f)$ is invertible in k, we have that $f \in \mathfrak{m}I$. $\square$

CHAPTER 3

Eisenbud-Mazur conjecture

In this chapter we obtain affirmative results for the Eisenbud-Mazur conjecture in some special cases.

3.1 Prime ideals in certain subrings of formal power series rings.

In this section we will consider the Eisenbud-Mazur conjecture for the following case.

Let $R = k[[t, x_1, ..., x_m]]$ be the formal power series ring over a field k of characteristic 0, in $m+1$ indeterminates. Let $S = k[[t^2, x_1, ..., x_m]]$. Let $f_1(t), ..., f_m(t) \in k[[t]]$. Let $Q_1 = (x_1 - f_1(t), ..., x_m - f_m(t))R$ and $Q_2 = (x_1 - f_1(-t), ..., x_m - f_m(-t))R$. Then Q_1, Q_2 are prime ideals that are conjugate under the action of the automorphism on R given by $\sigma : R \to R$, where $\sigma(t) = -t$ and $\sigma(x_i) = x_i$ for $1 \leq i \leq m$. Further, Q_1, Q_2 contract to the same prime ideal, say P in S, *i.e.,* $P = Q_1 \cap S = Q_2 \cap S = (Q_1 \cap Q_2) \cap S$. Let $\mathfrak{m} = (t^2, x_1, ..., x_n)S$. We will deem the above set of conditions as hypothesis (*). Under hypothesis (*), we will show that $P^{(2)} \subseteq \mathfrak{m}P$.

This situation is not as special as it may seem. We show, in proposition 3.1.4 below, that for an equicharacteristic, complete local ring S, in order to prove the Eisenbud-Mazur conjecture for prime ideals, it is sufficient to restrict to prime ideals P such that $\dim(S/P) = 1$. Further, we show that if $S = k[[t, x_1, ..., x_m]]$, where

k is an algebraically closed field of characteristic 0 and P is a prime ideal in S such that $\dim(S/P) = 1$, then, there exists a positive integer n such that for $R = k[[t^{\frac{1}{n}}, x_1, ..., x_m]]$ there exists a prime ideal $Q = (x_1 - f_1(t^{\frac{1}{n}}), ..., x_m - f_m(t^{\frac{1}{n}}))R$ such that $P = Q \cap S$. The case discussed in the preceding paragraph is special of this set-up with $n = 2$.

3.1.1 Motivation

We first show that if $(R, \mathfrak{m})$ is an equicharacteristic complete local ring and if the Eisenbud-Mazur conjecture holds for height unmixed ideals I such that $\dim(R/I) = 1$, then, it holds for all height unmixed ideals in R. We first need a few preparatory results starting with an irreducibility criterion for formal power series (page 164, [Kun05]).

Theorem 3.1.1. *[Kun05] Consider the grading on $k[x, y]$ (where k is a field) in which deg$(x) = p > 0$ and deg$(y) = q > 0$. Let $R = k[[x, y]]$ and let $f \in R \setminus \{0\}$. Let $l(f)$ denote the homogeneous polynomial of smallest degree (with respect to the above grading) occurring in f. If for some choice of p, q, $l(f)$ is an irreducible polynomial in $k[x, y]$, then, f is irreducible in R.*

Lemma 3.1.2. *Let $R = k[[x_1, ..., x_n]]$, where $n > 2$ and k is a field. If r, s are positive integers such that gcd$(r, s) = 1$, then, the ideal $P = (x_1^r - x_2^s)R$ is prime in R.*

Proof. By theorem 2.2.2, $x_1^r - x_2^s$ generates a prime ideal in $k[x_1, x_2]$ or equivalently it is irreducible when $\gcd(r, s) = 1$. Then, by theorem 3.1.1, $x_1^r - x_2^s$ is irreducible in $k[[x_1, x_2]]$ and hence irreducible in R. Then, since R is a unique factorization domain, ideal $P = (x_1^r - x_2^s)R$ is prime in R. $\square$

Proposition 3.1.3. *Let $(R, \mathfrak{m})$ be an equicharacteristic Noetherian complete local domain of dimension $d \geq 2$. Then, for every positive integer n, there exists a prime ideal $P_n \neq 0$ in R such that $P_n \subseteq \mathfrak{m}^n$.*

Proof. Since $(R, \mathfrak{m})$ is an equicharacteristic complete local domain, it is module finite over $S = k[[x_1, ..., x_d]]$, where k is a field (theorem 4.3.3, page 61, [HS06]). Since $d \geq 2$, by lemma 3.1.2, $\mathfrak{p}_n = (x_1^n - x_2^{n+1})S$ is a prime ideal in S and $\mathfrak{p}_n \subseteq (x_1, ..., x_d)^n S$. Let P_n be a prime ideal in R lying over $\mathfrak{p}_n$. Then $P_n \neq 0$ and $P_n \subseteq \mathfrak{m}^n$. □

Proposition 3.1.4. *Let $(R, \mathfrak{m})$ be an equicharacteristic Noetherian complete local ring. Let I be an ideal of R such that dim$(R/P) > 1$ for every associated prime ideal P of R. If there exists an element $r \in R$ such that $r \in I^{(2)} \setminus \mathfrak{m}I$, then, there exists an ideal J such that $I \subsetneq J$, $r \in J^{(2)} \setminus \mathfrak{m}J$ and dim$(R/J) <$ dim(R/I). Moreover, if I is height unmixed J can be chosen to be a height unmixed ideal. If I is radical, J can be chosen to be radical.*

Proof. Let $I = \mathfrak{p}_1 \cap ... \cap \mathfrak{p}_\mathfrak{n}$, where $\mathfrak{p}_\mathfrak{i}$ is a P_i-primary ideal, be the primary decomposition of I.

By hypothesis $\dim(R/P_i) > 1$. Then, by proposition 3.1.3, R/P_i has a non-zero prime ideal, say $\mathcal{Q}_{i,t}$ such that $\mathcal{Q}_{i,t} \subseteq (\mathfrak{m}/P_i)^t$ for all positive integers t. Without loss of generality we may choose $\mathcal{Q}_{i,t}$ such that $\text{ht}(\mathcal{Q}_{i,}) = 1$. Fix a positive integer t and let $Q_{i,t}$ denote the preimage of $\mathcal{Q}_{i,t}$ in R. Set $Q_i = Q_{i,t}$ for brevity of notation. Then Q_i are prime ideals in R such that $P_i \subsetneq Q_i \subseteq P_i + \mathfrak{m}^t$ for $1 \leq i \leq n$. Then we claim that $\mathfrak{q}_{i,m} := (\mathfrak{p}_\mathfrak{i} + Q_i^m) : (R \setminus Q_i) = \{r \in R : rs \in (\mathfrak{p}_\mathfrak{i} + Q_i^m) \text{ for some } s \in (R \setminus Q_i)\}$ are Q_i-primary ideals. Firstly, note that $\sqrt{(\mathfrak{p}_\mathfrak{i} + Q_i^m)} = \sqrt{(\sqrt{\mathfrak{p}_\mathfrak{i}} + \sqrt{Q_i^m})} = \sqrt{P_i + Q_i} = \sqrt{Q_i} = Q_i$ (exercise 1.13, page 9, [AM94]). Now, if $x \in \mathfrak{q}_{i,m}$, then, there exists $s \in R \setminus Q_i$ such that $sx \in (\mathfrak{p}_\mathfrak{i} + Q_i^m) \subseteq Q_i$. Since $s \notin Q_i$ we must have $x \in Q_i$. Further,

$(\mathfrak{p}_{\mathfrak{i}} + Q_i^m) \subseteq \mathfrak{q}_{i,m} \subseteq Q_i$. Taking radicals, we get that $Q_i = \sqrt{(\mathfrak{p}_{\mathfrak{i}} + Q_i^m)} \subseteq \sqrt{\mathfrak{q}_{i,m}} \subseteq \sqrt{Q_i} = Q_i$. Thus, $\sqrt{\mathfrak{q}_{i,m}} = Q_i$. Suppose that $xy \in \mathfrak{q}_{i,m}$, then, there exists $s \notin Q_i$ such that $sxy \in (\mathfrak{p}_{\mathfrak{i}} + Q_i^m)$. If $x \notin Q_i$, then, $sx \notin Q_i$. Thus, $y \in (\mathfrak{p}_{\mathfrak{i}} + Q_i^m) : (R \setminus Q_i) = \mathfrak{q}_{i,m}$. Hence, $\mathfrak{q}_{i,m}$ is Q_i-primary.

Next we claim that $\cap_{m \in \mathbb{Z}_{>0}} \mathfrak{q}_{i,m} = \mathfrak{p}_i$. Let $r \in \cap_{m \in \mathbb{Z}_{>0}} \mathfrak{q}_{i,m}$. Then there exists a $w_m \in R \setminus Q$ such that $rw_m \in (\mathfrak{p}_{\mathfrak{i}} + Q_i^m)$ for all $m \in \mathbb{Z}_{>0}$. Thus, $r \in \cap_{m \in \mathbb{Z}_{>0}} (\mathfrak{p}_{\mathfrak{i}} + Q_i^m) R_{Q_i} = \cap_{m \in \mathbb{Z}_{>0}} (\mathfrak{p}_{\mathfrak{i}} R_{Q_i} + Q_i^m R_{Q_i})$. By Krull's intersection theorem applied to R_{Q_i} we have that $\cap_{m \in \mathbb{Z}_{>0}} (Q_i^m R_{Q_i}) = 0$. So $r \in \mathfrak{p}_{\mathfrak{i}} R_{Q_i} \cap R = \mathfrak{p}_i$, where the last equality follows since $\mathfrak{p}_i$ is primary (proposition 4.8.ii, page 53, [AM94]).

Now, by Chevalley's theorem applied to $R/\mathfrak{p}_i$, there exists a function $b_i : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\mathfrak{q}_{i,b_i(N)} \subseteq \mathfrak{p}_i + \mathfrak{m}^N$ for all positive integers N. Let $J_N = \mathfrak{q}_{1,b_1(N)} \cap \ldots \cap \mathfrak{q}_{n,b_n(N)}$. Then $J_N \subseteq (\mathfrak{p}_1 + \mathfrak{m}^N) \cap \ldots \cap (\mathfrak{p}_n + \mathfrak{m}^N)$. We claim that $J_N \subseteq I + \mathfrak{m}^{N-c}$ for $N \gg 0$ and some positive integer $c < N$. We prove the claim by induction on n. Suppose that $r \in (\mathfrak{p}_1 + \mathfrak{m}^N) \cap (\mathfrak{p}_2 + \mathfrak{m}^N)$. Then we can write that $r = p_1 + m_1 = p_2 + m_2$ for some $p_i \in \mathfrak{p}_i$ for $i = 1, 2$ and $m_1, m_2 \in \mathfrak{m}^N$. Then $m_1 - m_2 = p_2 - p_1 \in \mathfrak{m}^N \cap (\mathfrak{p}_1 + \mathfrak{p}_2)$. By the Artin-Rees lemma, there exists a positive integer c_{12} such that $\mathfrak{m}^N \cap (\mathfrak{p}_1 + \mathfrak{p}_2) = \mathfrak{m}^{N-c_{12}}(\mathfrak{m}^{c_{12}} \cap (\mathfrak{p}_1 + \mathfrak{p}_2)) \subseteq \mathfrak{m}^{N-c_{12}}(\mathfrak{p}_1 + \mathfrak{p}_2) = \mathfrak{m}^{N-c_{12}}\mathfrak{p}_1 + \mathfrak{m}^{N-c_{12}}\mathfrak{p}_2$. So $p_2 - p_1 \in \mathfrak{m}^{N-c_{12}}\mathfrak{p}_1 + \mathfrak{m}^{N-c_{12}}\mathfrak{p}_2$. Write $p_2 - p_1 = m_1' + m_2'$, where $m_i' \in \mathfrak{m}^{N-c_{12}}\mathfrak{p}_i$ for $i = 1, 2$. Then $p_1 + m_1' = p_2 - m_2'$. Note that the left hand side of this equation is an element of $\mathfrak{p}_1$ and the right hand side is an element of $\mathfrak{p}_2$. Thus, each side is an element of $\mathfrak{p}_1 \cap \mathfrak{p}_2$. Now $r = p_1 + m_1 = (p_1 + m_1') - (m_1' - m_1) \in (\mathfrak{p}_1 \cap \mathfrak{p}_2) + \mathfrak{m}^{N-c_{12}}$. Thus, $(\mathfrak{p}_1 + \mathfrak{m}^N) \cap (\mathfrak{p}_2 + \mathfrak{m}^N) \subseteq (\mathfrak{p}_1 \cap \mathfrak{p}_2) + \mathfrak{m}^{N-c_{12}}$ for $N \geq c_{12}$. Proceeding inductively, we can show that there exists a positive integer c such that for $N \geq c$, $(\mathfrak{p}_1 + \mathfrak{m}^N) \cap \ldots \cap (\mathfrak{p}_n + \mathfrak{m}^N) \subseteq (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{\mathfrak{n}}) + \mathfrak{m}^{N-c}$. Consequently, $J_N \subseteq I + \mathfrak{m}^{N-c}$. Fix one such $N \gg c$ and set $J = J_N$.

By construction, $\dim(R/J) < \dim(R/I)$. Choose $u \in I^{(2)}$. Then there exists $v \in R$ such that v is not contained in any minimal prime of I and $uv \in I^2$. Since for all associated primes Q_i of J, $Q_i \subseteq P_i + \mathfrak{m}^t$, by choosing $t \gg 0$, we can ensure that v is not contained in any minimal prime of J. Then $vu \in J^2$ and $u \in J^{(2)}$. Further, if $u \notin \mathfrak{m}I$, we claim that $u \notin \mathfrak{m}J$. Suppose that $u \in \mathfrak{m}J$. Then, by the preceding paragraph, $u \in \mathfrak{m}(I + \mathfrak{m}^{N-c}) = \mathfrak{m}I + \mathfrak{m}^{N+1-c}$. Write $u = v + w$, where $v \in \mathfrak{m}I$ and $w \in \mathfrak{m}^{N+1-c}$. Since $u \in I^{(2)} \subseteq I$ and $v \in \mathfrak{m}I \subseteq I$, $w \in I$. So $w \in \mathfrak{m}^{N+1-c} \cap I$. By the Artin-Rees lemma there exists a positive integer $c^{'}$ such that for $N+1-c \geq c^{'}$, $\mathfrak{m}^{N+1-c} \cap I = \mathfrak{m}^{N+1-c-c^{'}}(\mathfrak{m}^{c^{'}} \cap I) \subseteq \mathfrak{m}^{N+1-c-c^{'}} I \subseteq \mathfrak{m}I$. Thus, for $N \gg 0$, $u = v + w \in \mathfrak{m}I$, which contradicts the choice of u. Hence, $u \in \mathfrak{m}J$. This proves the first assertion in the proposition.

If I is height unmixed, then, all associated prime ideals of I are minimal and have the same height. By choice of Q_i all associated prime ideals of J will also have the same height, which is 1 higher than the height of I. So J is height unmixed. If I is radical, we choose the primary decomposition of I as an intersection of it's minimal primes and choose the primary decomposition of J as the intersection of the minimal Q_i. So J is radical and by the arguments in the preceding paragraph we obtain the desired conclusion. $\square$

Now we show that if $S = k[[t, x_1, ..., x_m]]$, where k is a field under some additional hypothesis explained below and P is a prime ideal in S such that $\dim(S/P) = 1$, then, there exists a positive integer n such that for $R = k[[t^{\frac{1}{n}}, x_1, ..., x_m]]$ there exists a prime ideal $Q = (x_1 - f_1(t^{\frac{1}{n}}), ..., x_m - f_m(t^{\frac{1}{n}}))R$ such that $P = Q \cap S$.

Since S/P is a one dimensional Noetherian complete local domain, its integral closure, $\overline{S/P}$, is a one dimensional normal Noetherian complete local domain (theorem 2.2.5, page 31 and theorem 4.3.4, page 62, [HS06]) and hence regular (theorem 14.1, page 198, [Kem10]). Thus, we can identify, $\overline{S/P} = k[[y]]$ for some indeterminate y (theorem 15, [Coh46]). Thus, we have an inclusion $F : S/P \hookrightarrow K[[y]]$. Under this inclusion, let $t \mapsto y^n u$ and $x_i \mapsto g_i(y)$ for $1 \leq i \leq m$, where n is a positive integer and u is a unit in $k[[y]]$. Write $u = u_0 + f(y)$, where $f(y)$ is a power series in y with no constant term and $u_0 \in k$. Suppose that u_0 has an nth root in k (in particular, this is true if k is algebraically closed and n is invertible in k). Then u has an nth root in $k[[y]]$, say, $v^n = u$. Consider the automorphism $V : k[[y]] \to k[[y]]$ given by $V(y) = yv^{-1}$. Then we have an injective map $V \circ F : S/P \hookrightarrow k[[y]]$, where $t \mapsto y^n$ and $x_i \mapsto g_i(yv^{-1})$. Set $f_i(y) := g_i(yv^{-1})$. This induces a surjective map $G : R = k[[t^{\frac{1}{n}}, x_1, ..., x_m]] \to k[[y]]$, where $t^{\frac{1}{n}} \mapsto y$ and $x_i \mapsto f_i(y)$. Suppose that the kernel of G is Q. Note that Q is prime in R since $k[[y]]$ is a domain. Further, since the restriction of G to S is the map sending $t \mapsto y^n$ and $x_i \mapsto f_i(y)$, the kernel of $G|_S$ is P. Thus, $Q \cap S = P$. Finally, we have that $Q = (x_1 - f_1(t^{\frac{1}{n}}), ..., x_m - f_m(t^{\frac{1}{n}}))R$ since this ideal is clearly in the kernel of G by definition and the quotient of R modulo this ideal is precisely $k[[t]]$. This proves our claim.

3.1.2 Problem set-up

Let $R = k[[t, x_1, ..., x_m]]$ be the formal power series ring over a field k of characteristic 0 in $m+1$ indeterminates. Then the fraction field of R, say K, is $k((t, x_1, ..., x_m))$, the ring of formal Laurent series over the same indeterminates. Let $n > 1$ be a positive integer and let $S = k[[t^n, x_1, ..., x_m]]$, where $n > 1$ is a positive integer. The fraction field of S can be identified with $L = k((t^n, x_1, ..., x_m))$. Now R, S are regular local rings and hence unique factorization domains and hence normal. We show that the integral closure of S in L is R. An element $\alpha \in L$ integral over S is also integral over R as $S \subset R$. However, since R is normal and $L \subseteq K$, we must have $\alpha \in R$. So the integral closure of S in L is contained in R. For the converse, we first observe that R is a module-finite extension of S. For if $f \in R$, then, we may write that $f = \Sigma_{i=0}^{n-1} g_i(t^n, x_1, ..., x_m)t^i$, where $g_i \in S$ for $0 \leq i \leq n-1$. Thus, R is integral over S (proposition 5.1, page 59, [AM94]). Thus, the integral closure of S in L is R.

Now assume that k is algebraically closed if $n > 2$ .We show that K/L is a Galois extension and compute the Galois group of K/L. K/L is a finite extension since K is generated over L by $1, t, ..., t^{n-1}$. So K/L is algebraic. Consider $p(x) = x^n - t^n \in L[x]$. Then $p(x)$ splits completely in $K[x]$ as $p(x) = \Pi_{j=1}^{n}(x - \zeta^{j-1}t)$. Also, if $p(x)$ splits in a subfield of K, say E, then, since $x - t$ is a factor of $p(x)$, $t \in E$ and hence $K \subseteq E$. So $K = E$. Thus, K is a splitting field for $p(x)$. Further, since K/L is a finite extension and K is a splitting field of a polynomial in L, K/L is normal. Since the fields have characteristic 0, K/L is separable. So K/L is Galois. Consider the automorphisms σ_j of K, where $\sigma_j(t) = \zeta^{j-1}t$ for $j = 1, ..., n$, where ζ is a primitive nth root of unity and $\sigma_j(x_i) = x_i$ for $1 \leq i \leq m$. Any automorphism of K that fixes L must fix each x_i and must map an nth root of t^n to another nth root of t^n and hence must map t to $\zeta^{j-1}t$ for some $j \in \{1, ..., n\}$. Thus, the Galois group G of K/L is $\{\sigma_1, ..., \sigma_n\}$[1].

Given any prime ideal $\mathfrak{p}$ in S. Suppose that $\mathfrak{Q} = \{\mathfrak{q}_1, ..., \mathfrak{q}_l\}$ is the set of prime ideals of R lying over $\mathfrak{p}$. Then G acts transitively on $\mathfrak{Q}$ (proposition VII.2.1, page 340, [Lan02]).

Now let $f_1(t), ..., f_m(t) \in k[[t]]$ and let $Q_1 = (x_1 - f_1(t), ..., x_m - f_m(t))R$. Then Q_1 is a prime ideal as $R/Q_1 = k[[t]]$, which is a domain. If $P = Q_1 \cap S$, the set of primes lying over P are $Q_j = (x_1 - f_1(\zeta^{j-1}t), ..., x_m - f_m(\zeta^{j-1}t))R$ for $1 \leq j \leq n$ since G acts transitively on the set of primes lying over P. It follows that $Q_j \cap S = P = (Q_1 \cap S) \cap ... \cap (Q_n \cap S) = (Q_1 \cap ... \cap Q_n) \cap S$ for $1 \leq j \leq n$.

[1]Note that we could have alternately argued that since the automorphism group of K/L has size n, which is the degree of the field extension, K/L is a Galois extension.

We are now ready to show that under the conditions of hypothesis (*) (which is a special case of the above set up for $n = 2$), $P^{(2)} \subseteq (Q_1^2 \cap Q_2^2) \cap S$.

Proposition 3.1.5. *Let k be a field, $R = k[[t, x_1, ..., x_m]]$, $S = k[[t^n, x_1, ..., x_m]]$. Let $f_1(t), ..., f_m(t) \in k[[t]]$. If $n > 2$, assume that k is algebraically closed and let $Q_j = (x_1 - f_1(\zeta^{j-1}t), ..., x_m - f_m(\zeta^{j-1}t))R$, where ζ is a primitive nth root of unity and $j \in \{1, ..., n\}$. Then $P^{(l)} \subseteq (Q_1^{l_1} \cap ... \cap Q_n^{l_n}) \cap S$, where $P = (Q_1 \cap ... \cap Q_n) \cap S$ and $l \geq max\{l_1, ..., l_n\}$.*

Proof. The sequences $X_j = x_1 - f_1(\zeta^{j-1}t), ..., x_m - f_m(\zeta^{j-1}t)$ are regular sequences in R for $1 \leq j \leq n$, since $R/(x_1 - f_1(\zeta^{j-1}t), ..., x_i - f_i(\zeta^{j-1}t))R \cong k[[t, x_{i+1}, ..., x_m]]$. Being a domain, the latter ring has no non-zero zero-divisors and the class of $x_{i+1} - f_{i+1}(\zeta^{j-1}t)$ is not zero in this ring for $0 \leq i \leq m-1$. Also, by the same token, the ideals Q_j are prime for $1 \leq j \leq n$. Then we have that $Q_j^{(r)} = Q_j^r$ for every positive integer r (result 2.1, [Hoc73b]). Thus, the ideals Q_j^r are primary for every positive integer r and $1 \leq j \leq n$. Now the contraction of a primary ideal is primary (proposition 4.8, page 53, [AM94]). Consequently, the ideals $Q_j^r \cap S$ are primary in S.

Now $\sqrt{Q_j^r \cap S} = \sqrt{Q_j^r} \cap S = Q_j \cap S = P$ (exercise 1.13, page 9 and exercise 1.18 page 10, [AM94]). Thus, the ideals $Q_j^{l_j} \cap S$ are all P-primary. Hence $(Q_1^{l_1} \cap ... \cap Q_n^{l_n}) \cap S$ is P-primary (lemma 4.3, page 51, [AM94]). Further, $P^l = ((Q_1 \cap ... \cap Q_n) \cap S)^l \subseteq (Q_1 \cap ... \cap Q_n)^l \cap S \subseteq (Q_1^l \cap ... \cap Q_n^l) \cap S \subseteq (Q_1^{l_1} \cap ... \cap Q_n^{l_n}) \cap S$. Let $\mathfrak{q} = (Q_1^{l_1} \cap ... \cap Q_n^{l_n}) \cap S$.

Finally, for any irredundant primary decomposition of P^l, the P-primary ideal that must be used is $P^{(l)}$. Suppose that $P^l = P^{(l)} \cap P_1 \cap ... \cap P_r$ be an irredundant primary decomposition, where the $P_1, ..., P_r$ are primary ideals. Then $\sqrt{P_1}, ..., \sqrt{P_r}$ are all distinct and are distinct from P. Further, $P^{(l)} \not\supseteq \cap_{i=1}^r P_i$ and $P_{i'} \not\supseteq P^{(l)} \cap_{i=1, i \neq i'}^r P_i$ for $1 \leq i' \leq r$. We claim that $P^l = (P^{(l)} \cap \mathfrak{q}) \cap P_1 \cap P_2 \cap ... \cap P_r$ is also an irredundant primary decomposition (note that $P^l = P^l \cap \mathfrak{q}$ as $P^l \subseteq \mathfrak{q}$ by the preceding paragraph). For $P^{(l)}$ and $\mathfrak{q}$ are both P-primary and hence so is $P^{(l)} \cap \mathfrak{q}$. So the radicals of all ideals appearing in the decomposition are all distinct. Also, $P^{(l)} \cap \mathfrak{q} \not\supseteq \cap_{i=1}^{r} P_i$ follows from $P^{(l)} \not\supseteq \cap_{i=1}^{r} P_i$. Suppose that $P_{i'} \supseteq (P^{(l)} \cap \mathfrak{q}) \cap_{i=1, i \neq i'}^{r} P_i = (P^{(l)} \cap_{i=1, i\neq i'}^{r} P_i) \cap \mathfrak{q}$. Then, since $P_{i'} \not\supseteq P^{(l)} \cap_{i=1, i\neq i'}^{r} P_i$, we must have $P_{i'} \supseteq \mathfrak{q}$ (proposition 1.11.(ii), page 8, [AM94]). However, taking radicals, we get that $\sqrt{\mathfrak{q}} = P \subseteq \sqrt{P_{i'}} = P_{i'}$, which is a contradiction. Thus, $P_{i'} \not\supseteq (P^{(l)} \cap \mathfrak{q}) \cap_{i=1, i\neq i'}^{r} P_i$. So the new primary decomposition is indeed irredundant. Now, since the P-primary component in any primary decomposition of P^l must be $P^{(l)}$, we have that $P^{(l)} = P^{(l)} \cap \mathfrak{q}$. So $P^{(l)} \subseteq \mathfrak{q} = (Q_1^{l_1} \cap ... \cap Q_n^{l_n}) \cap S$. □

Corollary 3.1.6. *Let k be a field, $R = k[[t, x_1, ..., x_m]]$, $S = k[[t^2, x_1, ..., x_m]]$. Let $f_1(t), ..., f_m(t) \in k[[t]]$. Let $Q_1 = (x_1 - f_1(t), ..., x_m - f_m(t))R$ and $Q_2 = (x_1 - f_1(-t), ..., x_m - f_m(-t))R$. Then $P^{(2)} \subseteq (Q_1^2 \cap Q_2^2) \cap S$, where $P = (Q_1 \cap Q_2) \cap S$.*

Proof. This is a direct consequence of proposition 3.1.5 with $n = 2$, $l_0 = l_1 = l = 2$. □

We will prove the stronger containment $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ in the next sections, which will imply the Eisenbud-Mazur conjecture in this case by corollary 3.1.6.

3.1.3 Computing generators of P

Let the notation be as in hypothesis (*). We may assume that at least one of the power series $f_1(t), ..., f_m(t)$ is not even in t, *i.e.*, $f_i(t) \neq f_i(-t)$ for some $i \in \{1, .., m\}$. For suppose that $f_1(t), ..., f_m(t)$ are even power series. Then

$$Q_2 = (x_1 - f_1(-t), ..., x_m - f_m(-t))R = (x_1 - f_1(t), ..., x_m - f_m(t))R = Q_1.$$

Also, since $t^2 \in S$, $f_1(t), ..., f_m(t) \in S$. Thus, $P = Q_1 \cap S = (x_1 - f_1(t), ..., x_m - f_m(t))S$. Also, $(Q_1^2 \cap Q_2^2) \cap S = Q_1^2 \cap S = (\{(x_i - f_i(t))(x_j - f_j(t)) : i, j \in \{1, ..., m\}\})R \cap$

$S = (\{(x_i - f_i(t))(x_j - f_j(t)) : i, j \in \{1, ..., m\}\})S$. Now $(x_i - f_i(t))(x_j - f_j(t)) \in \mathfrak{m}(Q_1 \cap S)$ since $(x_i - f_i(t)) \in \mathfrak{m}$ and $(x_j - f_j(t)) \in (Q_1 \cap S)$. So $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ in this case. Then, using corollary 3.1.6, we get that $P^{(2)} \subseteq \mathfrak{m}P$. So at least one of $f_1(t), ..., f_m(t)$ is not even.

Without loss of generality, we may assume that $(f_1(t) - f_1(-t))|(f_i(t) - f_i(-t))$ and set $g_i(t) = \frac{(f_i(t)-f_i(-t))}{(f_1(t)-f_1(-t))}$ for $i = 1, ..., m$ (else we may renumber so that the leading term of $f_1(t) - f_1(-t)$ has the least non-zero degree). Set $a_i = x_i - f_i(t)$ and $b_i = x_i - f_i(-t)$ for $i = 1, ..., m$.

Proposition 3.1.7. *With the notation as in the preceding paragraph,* $P = (Q_1 \cap Q_2) \cap S = (\{(-b_i + b_1 g_i(t)) : i = 2, ..., m\})S + (\{tr((x_i - f_i(t))(x_j - f_j(-t))) : i, j \in \{1, ..., m\}\})S$ *(where* $tr(\cdot) : K \to L$ *is the usual trace map).*

Proof. We have that $b_1 - a_1 = f_1(t) - f_1(-t)$ and $b_i - a_i = f_i(t) - f_i(-t) = (f_1(t) - f_1(-t))g_i(t) = (b_1 - a_1)g_i(t)$ for $i = 1, ..., m$.

Let $u \in Q_1 \cap Q_2$. Then, for some $r_i, s_i \in R$, we may write that $u = \Sigma_{i=1}^m r_i a_i = \Sigma_{i=1}^m s_i b_i = \Sigma_{i=1}^m s_i(a_i + (b_i - a_i)) = \Sigma_{i=1}^m s_i(a_i + (b_1 - a_1)g_i(t))$. Hence, $\Sigma_{i=1}^m (r_i - s_i)a_i = (b_1 - a_1)\Sigma_{i=1}^m s_i g_i(t)$. Note the left hand side of the last equation lies in Q_1.

Thus, elements of $Q_1 \cap Q_2$ are determined by elements $s_i \in R$ such that $\Sigma_{i=1}^n s_i g_i(t) \in Q_1 : (b_1 - a_1)$, as given any s_i satisfying this condition, we may determine the r_i from the preceding equation, thus obtaining an element of $Q_1 \cap Q_2$.

Since Q_1 is a prime and $(b_1 - a_1) \notin Q_1$, we have that $Q_1 : (b_1 - a_1) = Q_1$. So elements of $Q_1 \cap Q_2$ are determined by elements $s_i \in R$ such that $\Sigma_{i=1}^m s_i g_i(t) \in Q_1$. Modulo Q_1, these are the preimages of the elements defining the relations between $g_i(t)$ in $R/Q_1 = k[[t]]$. Thus, $Q_1 \cap Q_2$ is generated by elements $\Sigma_{i=1}^m s_i b_i$, where either $s_i \in Q_1$ for $i = 1, ..., m$ or $\Sigma_{i=1}^m s_i g_i(t)$ represents the zero element in R/Q_1. Further, every element $\Sigma_{i=1}^m s_i b_i$, where $s_i \in Q_1$ for $i = 1, ..., m$ lies in $Q_1 Q_2$ since $b_i \in Q_2$

for $i = 1, ..., m$. Then, since $Q_1Q_2 \subseteq Q_1 \cap Q_2$, we have that $Q_1 \cap Q_2$ is generated over Q_1Q_2 by elements $\Sigma_{i=1}^m s_i b_i$ such that $\Sigma_{i=1}^m s_i g_i(t)$ represents the zero element in R/Q_1.

Given elements $w_1, ..., w_d \in k[[t]]$ we define a *relation* among these elements to be a d-tuple $(\alpha_1, ..., \alpha_d) \in k[[t]]^d$ such that $\alpha_1 w_1 + ... + \alpha_d w_d = 0$. The set of such elements is a submodule of $k[[t]]^n$, which we shall call the *module of relations.* Note that since $k[[t]]$ is a principal ideal domain, the module of relations is a free $k[[t]]$-module. Now the module of relations between the $g_i(t)$ in $R/Q_1 = k[[t]]$ is generated by the following m-tuples (note that $g_1(t) = 1$):

1. $(g_i(t), 0, 0, ..., \underbrace{-1}_{i^{\text{th}} \text{ position}}, 0, 0, ..., 0)$ (where $i = 2, ..., m$).
2. $(0, 0, ..., \underbrace{g_j(t)}_{i^{\text{th}} \text{ position}}, 0, 0, ..., 0, \underbrace{-g_i(t)}_{j^{\text{th}} \text{ position}}, 0, 0, ..., 0)$ (where $i \neq j$and $i, j \in \{2, ..., m\}$).

Let, $h_i = -b_i + b_1 g_i(t) = -a_i + a_1 g_i(t) \in Q_1 \cap Q_2$ (where the equality follows from $b_i - a_i = (b_1 - a_1) g_i(t)$). The set of relations in (1) correspond to the elements $g_i(t) b_1 + 0b_2 + 0b_3 + ... + (-1) b_i + 0b_{i+1} + 0b_{i+2} + ... + 0b_m = b_1 g_i(t) - b_i = h_i$ in $Q_1 \cap Q_2$. The set of relations in (2) correspond to the elements $h_{ij} = 0b_1 + 0b_2 + ... + g_j(t) b_i + 0b_{i+1} + 0b_{i+2} + ... + 0b_{j-1} + (-g_i(t)) b_j + 0b_{j+1} + ... + 0b_m = g_j(t) b_i - g_i(t) b_j$. Thus, $Q_1 \cap Q_2 = (h_2, h_3, ..., h_m, h_{23}, h_{24}, ..., h_{2m}, ..., h_{m-1,m}) R + Q_1Q_2$ (note $h_1 = 0$ and $h_{1j} = h_j$).

However, $g_i(t) h_j - g_j(t) h_i = g_i(t)(-b_j + b_1 g_j(t)) - g_j(t)(-b_i + b_1 g_i(t)) = g_j(t) b_i - g_i(t) b_j = h_{ij}$. Thus, $Q_1 \cap Q_2 = (h_2, h_3, ..., h_m) R + Q_1Q_2$.

Now we compute $(Q_1 \cap Q_2) \cap S = \operatorname{tr}(Q_1 \cap Q_2)$. The trace map applied to the generators of $Q_1 \cap Q_2$ yields, $\operatorname{tr}(h_i) = \operatorname{tr}(-b_i + b_1 g_i(t)) = \operatorname{tr}(-x_i + f_i(-t) + (x_1 -$

$f_1(-t))g_i(t))$. Now $g_i(-t) = \frac{(f_i(-t)-f_i(t))}{(f_1(-t)-f_1(t))} = \frac{(f_i(t)-f_i(-t))}{(f_1(t)-f_1(-t))} = g_i(t)$. Hence,

$$\begin{aligned}
\mathrm{tr}(h_i) &= \frac{1}{2}((-x_i + f_i(-t) + (x_1 - f_1(-t))g_i(t)) + (-x_i + f_i(t) + (x_1 - f_1(t))g_i(t))) \\
&= \frac{1}{2}((-b_i + b_1 g_i(t)) + (-a_i + a_1 g_i(t))) \\
&= \frac{1}{2}(2(-b_i + b_1 g_i(t))) \\
&= -b_i + b_1 g_i(t) \\
&= h_i
\end{aligned}$$

Next, we have that $Q_1Q_2 = (\{(x_i - f_i(t))(x_j - f_j(-t)) : i, j \in \{1, ..., m\}\})R$. Therefore, $(Q_1 \cap Q_2) \cap S = (h_2, h_3, ..., h_m)S + (\{\mathrm{tr}((x_i - f_i(t))(x_j - f_j(-t))) : i, j \in \{1, ..., m\}\})S$. □

3.1.4 Computing generators of $Q_1^2 \cap Q_2^2$ (special case)

Again, assume hypothesis (*). For the purpose of computing generators of $Q_1^2 \cap Q_2^2$ we show that it is sufficient to consider power series $f_1(t), ..., f_m(t)$ containing only odd powers of t, *i.e.*, those $f_i(t)$ that satisfy $f_i(-t) = -f_i(t)$ for $1 \leq i \leq m$. Suppose that $f_i(t) = \Sigma_{h=0}^{\infty} a_{i,h} t^h$. Then we may write that $f_i(t) = \Sigma_{h=0}^{\infty} a_{i,2h} t^{2h} + \Sigma_{h=0}^{\infty} a_{i,2h+1} t^{2h+1}$. Set $f_{i,e} = \Sigma_{h=0}^{\infty} a_{i,2h} t^{2h}$ and $f_{i,o} = \Sigma_{h=0}^{\infty} a_{i,2h+1} t^{2h+1}$ for the even and odd parts of f_i respectively. Consider the automorphism σ of R: $\sigma(x_i) = x_i - f_{i,e}(t)$ and $\sigma(t) = t$. We have that $Q_1 = (x_1 - f_1(t), ..., x_m - f_m(t))R = (x_1 - f_{1,e}(t) - f_{1,o}(t), ..., x_m - f_{m,e}(t) - f_{m,o}(t))R$. So that $\sigma(Q_1) = (x_1 - f_{1,o}(t), x_2 - f_{2,o}(t), ..., x_m - f_{m,o}(t))R$. Similarly, we have that $Q_2 = (x_1 - f_1(-t), ..., x_m - f_m(-t))R = (x_1 - f_{1,e}(t) + f_{1,o}(t), ..., x_m - f_{m,e}(t) + f_{m,o}(t))R$. So that $\sigma(Q_2) = (x_1 + f_{1,o}(t), x_2 + f_{2,o}(t), ..., x_m + f_{m,o}(t))R$. So the problem reduces to the case where $f_i(t)$ are odd power series.

We rewrite the result of proposition 3.1.7 under this reduction. We have that $g_i(t) = \frac{(f_i(t)-f_i(-t))}{(f_1(t)-f_1(-t))} = \frac{f_i(t)}{f_1(t)}$. Also, $-b_i + b_1 g_i(t) = -(x_i + f_i(t)) + (x_1 + f_1(t))g_i(t) =$

$-x_i - f_i(t) + x_1 g_i(t) + f_i(t) = -x_i + x_1 g_i(t)$. Thus, $P = (Q_1 \cap Q_2) \cap S = (\{(-b_i + b_1 g_i(t)) : i = 2, ..., m\})S + (\{\mathrm{tr}((x_i - f_i(t))(x_j - f_j(-t))) : i, j \in \{1, ..., m\}\})S = (\{(-x_i + x_1 g_i(t)) : i = 2, ..., m\})S + (\{(x_i x_j - f_i(t) f_j(t)) : i, j \in \{1, ..., m\}\})S$.

For the sake of notational sanity we first illustrate the method for the case when $m = 3$ and discuss the generalization after that.

We have that $Q_1 = (x_1 - f_1(t), x_2 - f_2(t), x_3 - f_3(t))R$ and $Q_2 = (x_1 + f_1(t), x_2 + f_2(t), x_3 + f_3(t))R$. Denote $f_i(t) = f_i$, $a_i = x_i - f_i$ and $b_i = x_i + f_i$ for $i = 1, 2, 3$. Then $Q_1^2 = (a_1^2, a_2^2, a_3^2, a_2 a_3, a_1 a_3, a_1 a_2)R$ and $Q_2^2 = (b_1^2, b_2^2, b_3^2, b_2 b_3, b_1 b_3, b_1 b_2)R$. If $u \in Q_1^2 \cap Q_2^2$, then, we can write that $u = r_1 a_1^2 + r_2 a_2^2 + r_3 a_3^2 + s_1 a_2 a_3 + s_2 a_1 a_3 + s_3 a_1 a_2$ for some $r_h, s_h \in R$, $h \in \{1, 2, 3\}$. Similarly, $u = r_1^{'} b_1^2 + r_2^{'} b_2^2 + r_3^{'} b_3^2 + s_1^{'} b_2 b_3 + s_2^{'} b_1 b_3 + s_3^{'} b_1 b_2$ for some $r_h^{'}, s_h^{'} \in R$, $h \in \{1, 2, 3\}$. Now $b_h^2 - a_h^2 = 4x_i f_i$ and $b_h b_{h'} - a_h a_{h'} = 2x_h f_h + 2x_{h'} f_{h'}$ with $h, h^{'} \in \{1, 2, 3\}$ and $h \neq h^{'}$. Equating the two expressions for u, we have that $(r_1 - r_1^{'})a_1^2 + (r_2 - r_2^{'})a_2^2 + (r_3 - r_3^{'})a_3^2 + (s_1 - s_1^{'})a_2 a_3 + (s_2 - s_2^{'})a_1 a_3 + (s_3 - s_3^{'})a_1 a_2 = r_1^{'}(4x_1 f_1) + r_2^{'}(4x_2 f_2) + r_3^{'}(4x_3 f_3) + s_1^{'}(2x_2 f_3 + 2x_3 f_2) + s_2^{'}(2x_1 f_3 + 2x_3 f_1) + s_3^{'}(2x_1 f_2 + 2x_2 f_1)$. The left hand side of this equation lies in Q_1^2, hence so does the right hand side. The set of coefficients $\{r_1^{'}, r_2^{'}, r_3^{'}, s_1^{'}, s_2^{'}, s_3^{'}\}$, which determine elements of $Q_1^2 \cap Q_2^2$ are completely determined by the relations on $\{4x_1 f_1, 4x_2 f_2, 4x_3 f_3, 2x_2 f_3 + 2x_3 f_2, 2x_1 f_3 + 2x_3 f_1, 2x_1 f_2 + 2x_2 f_1\}$ over R/Q_1^2 along with any 6-tuple of elements in Q_1^2. We now proceed to find the relations on $\{4x_1 f_1, 4x_2 f_2, 4x_3 f_3, 2x_2 f_3 + 2x_3 f_2, 2x_1 f_3 + 2x_3 f_1, 2x_1 f_2 + 2x_2 f_1\}$ over R/Q_1^2.

In R/Q_1^2 we have the following equations

$$x_1^2 = 2x_1 f_1 - f_1^2, x_2^2 = 2x_2 f_2 - f_2^2, x_3^2 = 2x_3 f_3 - f_3^2 \tag{3.1}$$

$$x_1 x_2 = x_1 f_2 + x_2 f_1 - f_1 f_2, x_2 x_3 = x_2 f_3 + x_3 f_2 - f_2 f_3, x_1 x_3 = x_1 f_3 + x_3 f_1 - f_1 f_3 \tag{3.2}$$

Thus, any element of R/Q_1^2 can be represented as $F_0(t) + F_1(t)x_1 + F_2(t)x_2 +$

$F_3(t)x_3$, where $F_i(t) \in k[[t]]$ for $i = 0, 1, 2, 3$. Thus, there is a one-one correspondence between the relations on $\{4x_1f_1, 4x_2f_2, 4x_3f_3, 2x_2f_3 + 2x_3f_2, 2x_1f_3 + 2x_3f_1, 2x_1f_2 + 2x_2f_1\}$ over R/Q_1^2 and the relations between

$$E = \{4x_1f_1, (4x_1f_1)x_1, (4x_1f_1)x_2, (4x_1f_1)x_3,$$

$$4x_2f_2, (4x_2f_2)x_1, (4x_2f_2)x_2, (4x_2f_2)x_3,$$

$$4x_3f_3, (4x_3f_3)x_1, (4x_3f_3)x_2, (4x_3f_3)x_3,$$

$$2x_2f_3 + 2x_3f_2, (2x_2f_3 + 2x_3f_2)x_1, (2x_2f_3 + 2x_3f_2)x_2, (2x_2f_3 + 2x_3f_2)x_3,$$

$$2x_1f_3 + 2x_3f_1, (2x_1f_3 + 2x_3f_1)x_1, (2x_1f_3 + 2x_3f_1)x_2, (2x_1f_3 + 2x_3f_1)x_3,$$

$$2x_1f_2 + 2x_2f_1, (2x_1f_2 + 2x_2f_1)x_1, (2x_1f_2 + 2x_2f_1)x_2, (2x_1f_2 + 2x_2f_1)x_3\}$$

over $k[[t]]$. We rewrite these elements as $k[[t]]$-linear combinations of x_1, x_2, x_3 and represent the coefficients of x_1, x_2, x_3 and the term independent of these in a matrix as follows. We abuse notation and denote the equivalence classes of elements in R modulo Q_1^2 by the same symbols as the elements themselves.

1. $4x_1f_1$

   (a) $4x_1f_1 = 0 + 4f_1x_1 + 0x_2 + 0x_3$.

   (b) $(4x_1f_1)x_1 = 4f_1x_1^2 = 4f_1(2x_1f_1 - f_1^2) = -4f_1^3 + 8f_1^2x_1 + 0x_2 + 0x_3$.

   (c) $(4x_1f_1)x_2 = 4f_1x_1x_2 = 4f_1(x_1f_2 + x_2f_1 - f_1f_2) = -4f_1^2f_2 + 4f_1f_2x_1 + 4f_1^2x_2 + 0x_3$.

   (d) $(4x_1f_1)x_3 = 4f_1x_1x_3 = 4f_1(x_1f_3 + x_3f_1 - f_1f_3) = -4f_1^2f_3 + 4f_1f_3x_1 + 0x_2 + 4f_1^2x_3$.

   We represent this data in the following matrix:

$$\begin{array}{l|cccc|}
\text{coefficients in} & 4x_1f_1 & (4x_1f_1)x_1 & (4x_1f_1)x_2 & (4x_1f_1)x_3 \\
\hline
\text{term independent of } x_1,x_2,x_3 & 0 & -4f_1^3 & -4f_1^2f_2 & -4f_1^2f_3 \\
\text{coefficient of } x_1 & 4f_1 & 8f_1^2 & 4f_1f_2 & 4f_1f_3 \\
\text{coefficient of } x_2 & 0 & 0 & 4f_1^2 & 0 \\
\text{coefficient of } x_3 & 0 & 0 & 0 & 4f_1^2
\end{array}$$

We capture the above data in matrix M_1 below,

$$M_1 = \begin{bmatrix} 0 & -4f_1^3 & -4f_1^2f_2 & -4f_1^2f_3 \\ 4f_1 & 8f_1^2 & 4f_1f_2 & 4f_1f_3 \\ 0 & 0 & 4f_1^2 & 0 \\ 0 & 0 & 0 & 4f_1^2 \end{bmatrix}.$$

2. $4x_2f_2$: We represent elements $4x_2f_2, (4x_2f_2)x_1, (4x_2f_2)x_2, (4x_2f_2)x_3$ as $k[[t]]$-linear combinations of x_1, x_2, x_3 and collect the coefficients in a matrix form in an analogous fashion. The associated matrix is

$$M_2 = \begin{bmatrix} 0 & -4f_1f_2^2 & -4f_2^3 & -4f_2^2f_3 \\ 0 & 4f_2^2 & 0 & 0 \\ 4f_2 & 4f_1f_2 & 8f_2^2 & 4f_2f_3 \\ 0 & 0 & 0 & 4f_2^2f_3 \end{bmatrix}.$$

3. $4x_3f_3$: We repeat the above process for $4x_3f_3, (4x_3f_3)x_1, (4x_3f_3)x_2, (4x_3f_3)x_3$ and the associated matrix is

$$M_3 = \begin{bmatrix} 0 & -4f_1f_3^2 & -4f_2f_3^2 & -4f_3^3 \\ 0 & 4f_3^2 & 0 & 0 \\ 0 & 0 & 4f_3^2 & 0 \\ 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 \end{bmatrix}.$$

4. $2x_2f_3 + 2x_3f_2$:

(a) $2x_2f_3 + 2x_3f_2 = 0 + 0x_1 + 2f_3x_2 + 2f_2x_3$.

(b) $(2x_2f_3+2x_3f_2)x_1 = 2f_3x_1x_2+2f_2x_1x_3 = 2f_3(x_1f_2+x_2f_1-f_1f_2)+2f_2(x_1f_3+x_3f_1 - f_1f_3) = -4f_1f_2f_3 + 4f_2f_3x_1 + 2f_1f_3x_2 + 2f_1f_2x_3$.

(c) $(2x_2f_3 + 2x_3f_2)x_2 = 2f_3x_2^2 + 2f_2x_2x_3 = 2f_3(2x_2f_2 - f_2^2) + 2f_2(x_2f_3 + x_3f_2 - f_2f_3) = -4f_2^2f_3 + 0x_1 + 6f_2f_3x_2 + 2f_2^2x_3$.

(d) $(2x_2f_3+2x_3f_2)x_3 = 2f_3x_2x_3+2f_2x_3^2 = 2f_3(x_2f_3+x_3f_2-f_2f_3)+2f_2(2x_3f_3 - f_3^2) = -4f_2f_3^2 + 0x_1 + 2f_3^2x_2 + 6f_2f_3x_3$.

So the associated matrix is

$$M_{23} = \begin{bmatrix} 0 & -4f_1f_2f_3 & -4f_2^2f_3 & -4f_2f_3^2 \\ 0 & 4f_2f_3 & 0 & 0 \\ 2f_3 & 2f_1f_3 & 6f_2f_3 & 2f_3^2 \\ 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 \end{bmatrix}.$$

5. Working as in the preceding case, the associated matrix for $2x_1f_3+2x_3f_1, (2x_1f_3+2x_3f_1)x_1, (2x_1f_3 + 2x_3f_1)x_2, (2x_1f_3 + 2x_3f_1)x_3$ is

$$M_{13} = \begin{bmatrix} 0 & -4f_1^2f_3 & -4f_1f_2f_3 & -4f_1f_3^2 \\ 2f_3 & 6f_1f_3 & 2f_2f_3 & 2f_3^2 \\ 0 & 0 & 4f_1f_3 & 0 \\ 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 \end{bmatrix}.$$

6. Finally, the associated matrix for $2x_1f_2 + 2x_2f_1, (2x_1f_2 + 2x_2f_1)x_1, (2x_1f_2 + 2x_2f_1)x_2, (2x_1f_2 + 2x_2f_1)x_3$ is

$$M_{12} = \begin{bmatrix} 0 & -4f_1^2f_2 & -4f_1f_2^2 & -4f_1f_2f_3 \\ 2f_2 & 6f_1f_2 & 2f_2^2 & 2f_2f_3 \\ 2f_1 & 2f_1^2 & 6f_1f_2 & 2f_1f_3 \\ 0 & 0 & 0 & 4f_1f_2 \end{bmatrix}.$$

We now consider the matrix $M = [M_1|M_2|M_3|M_{23}|M_{13}|M_{23}]$.

$$M = \left[\begin{array}{cccccccccccccccccccccccc} 0 & -4f_1^3 & -4f_1^2f_2 & -4f_1^2f_3 & 0 & -4f_1f_2^2 & -4f_2^3 & -4f_2^2f_3 & 0 & -4f_1f_3^2 & -4f_2f_3^2 & -4f_3^3 & 0 & -4f_1f_2f_3 & -4f_2^2f_3 & -4f_2f_3^2 & 0 & -4f_1^2f_3 & -4f_1f_2f_3 & -4f_1f_3^2 & 0 & -4f_1^2f_2 & -4f_1f_2^2 & -4f_1f_2f_3 \\ 4f_1 & 8f_1^2 & 4f_1f_2 & 4f_1f_3 & 0 & 4f_2^2 & 0 & 0 & 0 & 4f_3^2 & 0 & 0 & 0 & 4f_2f_3 & 0 & 0 & 2f_3 & 6f_1f_3 & 2f_2f_3 & 2f_3^2 & 2f_2 & 6f_1f_2 & 2f_2^2 & 2f_2f_3 \\ 0 & 0 & 4f_1^2 & 0 & 4f_2 & 4f_1f_2 & 8f_2^2 & 4f_2f_3 & 0 & 0 & 4f_3^2 & 0 & 2f_3 & 2f_1f_3 & 6f_2f_3 & 2f_3^2 & 0 & 0 & 4f_1f_3 & 0 & 2f_1 & 2f_1^2 & 6f_1f_2 & 2f_1f_3 \\ 0 & 0 & 0 & 4f_1^2 & 0 & 0 & 0 & 4f_2^2 & 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 & 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 & 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 & 0 & 0 & 0 & 4f_1f_2 \end{array}\right].$$

Each column of M encodes the $k[[t]]$-coefficients of the elements of E. We will compute the $k[[t]]$-relations on the columns of matrix M and these will correspond to the $k[[t]]$-relations on the elements of E. From these we can recover the relations $\{4x_1f_1, 4x_2f_2, 4x_3f_3, 2x_2f_3 + 2x_3f_2, 2x_1f_3 + 2x_3f_1, 2x_1f_2 + 2x_2f_1\}$ over R/Q_1^2 .

Multiplying the first row of M by -1, we get that

$$\left[\begin{array}{cccccccccccccccccccccccc} 0 & 4f_1^3 & 4f_1^2f_2 & 4f_1^2f_3 & 0 & 4f_1f_2^2 & 4f_2^3 & 4f_2^2f_3 & 0 & 4f_1f_3^2 & 4f_2f_3^2 & 4f_3^3 & 0 & 4f_1f_2f_3 & 4f_2^2f_3 & 4f_2f_3^2 & 0 & 4f_1^2f_3 & 4f_1f_2f_3 & 4f_1f_3^2 & 0 & 4f_1^2f_2 & 4f_1f_2^2 & 4f_1f_2f_3 \\ 4f_1 & 8f_1^2 & 4f_1f_2 & 4f_1f_3 & 0 & 4f_2^2 & 0 & 0 & 0 & 4f_3^2 & 0 & 0 & 0 & 4f_2f_3 & 0 & 0 & 2f_3 & 6f_1f_3 & 2f_2f_3 & 2f_3^2 & 2f_2 & 6f_1f_2 & 2f_2^2 & 2f_2f_3 \\ 0 & 0 & 4f_1^2 & 0 & 4f_2 & 4f_1f_2 & 8f_2^2 & 4f_2f_3 & 0 & 0 & 4f_3^2 & 0 & 2f_3 & 2f_1f_3 & 6f_2f_3 & 2f_3^2 & 0 & 0 & 4f_1f_3 & 0 & 2f_1 & 2f_1^2 & 6f_1f_2 & 2f_1f_3 \\ 0 & 0 & 0 & 4f_1^2 & 0 & 0 & 0 & 4f_2^2 & 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 & 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 & 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 & 0 & 0 & 0 & 4f_1f_2 \end{array}\right].$$

Using notation introduced earlier, we write that $f_2 = f_1g_2$ and $f_3 = f_1g_3$. Denote the ith column of the matrix under consideration by C_i. We perform the following column operations on the preceding matrix:

$C_2 - 2f_1C_1$, $C_3 - f_2C_1$, $C_4 - f_3C_1$, $C_6 - f_1g_2^2C_1$, $C_{10} - f_1g_3^2C_1$, $C_{14} - f_1g_2g_3C_1$, $C_{17} - \frac{1}{2}g_3C_1$, $C_{18} - \frac{3}{2}f_3C_1$, $C_{19} - \frac{1}{2}f_1g_2g_3C_1$, $C_{20} - \frac{1}{2}f_1g_3^2C_1$, $C_{21} - \frac{1}{2}g_2C_1$, $C_{22} - \frac{3}{2}f_2C_1$, $C_{23} - \frac{1}{2}f_1g_2^2C_1$, $C_{24} - \frac{1}{2}f_1g_2g_3C_1$. We get that

$$\left[\begin{array}{cccccccccccccccccccccccc} 0 & 4f_1^3 & 4f_1^2f_2 & 4f_1^2f_3 & 0 & 4f_1f_2^2 & 4f_2^3 & 4f_2^2f_3 & 0 & 4f_1f_3^2 & 4f_2f_3^2 & 4f_3^3 & 0 & 4f_1f_2f_3 & 4f_2^2f_3 & 4f_2f_3^2 & 0 & 4f_1^2f_3 & 4f_1f_2f_3 & 4f_1f_3^2 & 0 & 4f_1^2f_2 & 4f_1f_2^2 & 4f_1f_2f_3 \\ 4f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4f_1^2 & 0 & 4f_2 & 4f_1f_2 & 8f_2^2 & 4f_2f_3 & 0 & 0 & 4f_3^2 & 0 & 2f_3 & 2f_1f_3 & 6f_2f_3 & 2f_3^2 & 0 & 0 & 6f_1f_3 & 0 & 2f_1 & 2f_1^2 & 6f_1f_2 & 2f_1f_3 \\ 0 & 0 & 0 & 4f_1^2 & 0 & 0 & 0 & 4f_2^2 & 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 & 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 & 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 & 0 & 0 & 0 & 4f_1f_2 \end{array}\right].$$

We further perform the following column operations on the preceding matrix:

$C_3 - g_2C_2$, $C_4 - g_3C_2$, $C_6 - g_2^2C_2$, $C_7 - g_2^3C_2$, $C_8 - g_2^2g_3C_2$, $C_{10} - g_3^2C_2$, $C_{11} - g_2g_3^2C_2$, $C_{12} - g_3^3C_2$, $C_{14} - g_2g_3C_2$, $C_{15} - g_2^2g_3C_2$, $C_{16} - g_2g_3^2C_2$, $C_{18} - g_3C_2$, $C_{19} - g_2g_3C_2$, $C_{20} - g_3^2C_2$, $C_{22} - g_2C_2$, $C_{23} - g_2^2C_2$, $C_{24} - g_2g_3C_2$. We get that

$$\begin{bmatrix} 0 & 4f_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4f_1^2 & 0 & 4f_2 & 4f_1f_2 & 8f_2^2 & 4f_2f_3 & 0 & 0 & 4f_3^2 & 0 & 2f_3 & 2f_1f_3 & 6f_2f_3 & 2f_3^2 & 0 & 0 & 4f_1f_3 & 0 & 2f_1 & 2f_1^2 & 6f_1f_2 & 2f_1f_3 \\ 0 & 0 & 0 & 4f_1^2 & 0 & 0 & 0 & 4f_2^2 & 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 & 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 & 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 & 0 & 0 & 0 & 4f_1f_2 \end{bmatrix}.$$

The following column operations are performed on the preceding matrix:

$C_3 - 2f_1C_{21}$, $C_5 - 2g_2C_{21}$, $C_6 - 2f_2C_{21}$, $C_7 - 4f_1g_2C_{21}$, $C_8 - 2f_1g_2g_3C_{21}$, $C_{11} - 2f_1g_3^2C_{21}$, $C_{13} - g_3C_{21}$, $C_{14} - f_3C_{21}$, $C_{15} - 3f_1g_2g_3C_{21}$, $C_{16} - f_1g_3^2C_{21}$, $C_{19} - 2f_3C_{21}$, $C_{22} - f_1C_{21}$, $C_{23} - 3f_2C_{21}$, $C_{24} - f_3C_{21}$. We get that

$$\begin{bmatrix} 0 & 4f_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2f_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4f_1^2 & 0 & 0 & 0 & 4f_2^2 & 4f_3 & 4f_1f_3 & 4f_2f_3 & 8f_3^2 & 2f_2 & 2f_1f_2 & 2f_2^2 & 6f_2f_3 & 2f_1 & 2f_1^2 & 2f_1f_2 & 6f_1f_3 & 0 & 0 & 0 & 4f_1f_2 \end{bmatrix}.$$

Finally, the following column operations are performed on the preceding matrix:

$C_4 - 2f_1C_{17}$, $C_8 - 2f_1g_2^2C_{17}$, $C_9 - 2g_3C_{17}$, $C_{10} - 2f_3C_{17}$, $C_{11} - 2f_1g_2g_3C_{17}$, $C_{12} - 4f_1g_3^2C_{17}$, $C_{13} - g_2C_{17}$, $C_{14} - f_2C_{17}$, $C_{15} - f_1g_2^2C_{17}$, $C_{16} - 3f_1g_2g_3C_{17}$, $C_{18} - f_1C_{17}$, $C_{19} - f_2C_{17}$, $C_{20} - 3f_3C_{17}$, $C_{24} - 2f_2C_{17}$. We get that

$$\begin{bmatrix} 0 & 4f_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2f_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns with all entries zero correspond to the relations among the columns. We list them here:

1. $C_3 + 2f_2C_1 - g_2C_2 - 2f_1C_{21} = 0.$

2. $C_4 + 2f_3C_1 - 2f_1C_{17} - g_3C_2 = 0.$

3. $C_5 - 2g_2C_{21} + g_2^2C_1 = 0.$

4. $C_6 + 2f_2g_2C_1 - g_2^2C_2 - 2f_2C_{21} = 0.$

5. $C_7 - g_2^3C_2 + 4f_1g_2^3C_1 - 4f_1g_2^2C_{21} = 0.$

6. $C_8 - g_2^2g_3C_2 + 4f_1g_2^2g_3C_1 - 2f_1g_2g_3C_{21} - 2f_1g_2^2C_{17} = 0.$

7. $C_9 - 2g_3C_{17} + g_3^2C_1 = 0.$

8. $C_{10} - g_3^2C_2 + 2f_1g_3^2C_1 - 2f_3C_{17} = 0.$

9. $C_{11} - g_2g_3^2C_2 + 4f_1g_2g_3^2C_1 - 2f_1g_3^2C_{21} - 2f_1g_2g_3C_{17} = 0.$

10. $C_{12} - g_3^3C_2 - 4f_1g_3^2C_{17} + 4f_1g_3^3C_1 = 0.$

11. $C_{13} - g_3C_{21} + g_2g_3C_1 - g_2C_{17} = 0.$

12. $C_{14} - g_2g_3C_2 + 2f_1g_2g_3C_1 - f_3C_{21} - f_2C_{17} = 0.$

13. $C_{15} - g_2^2g_3C_2 - 3f_1g_2g_3C_{21} + 4f_1g_2^2g_3C_1 - f_1g_2^2C_{17} = 0.$

14. $C_{16} - g_2g_3^2C_2 + 4f_1g_2g_3^2C_1 - f_1g_3^2C_{21} - 3f_1g_2g_3C_{17} = 0.$

15. $C_{18} - g_3C_2 - f_1C_{17} + f_3C_1 = 0.$

16. $C_{19} + 3f_1g_2g_3C_1 - g_2g_3C_2 - 2f_3C_{21} - f_2C_{17} = 0.$

17. $C_{20} + 3f_1g_3^2C_1 - g_3^2C_2 - 3f_3C_{17} = 0.$

18. $C_{22} - g_2C_2 - f_1C_{21} + f_1g_2C_1 = 0.$

19. $C_{23} + 3f_1g_2^2C_1 - g_2^2C_2 - 3f_2C_{21} = 0.$

20. $C_{24} - g_2g_3C_2 + 3f_1g_2g_3C_1 - f_3C_{21} - 2f_2C_{17} = 0.$

Now consider the first relation $C_3 + 2f_2C_1 - g_2C_2 - 2f_1C_{21} = 0$. The first column corresponds to coefficients of $4x_1f_1$, the second corresponds to coefficients of $(4x_1f_1)x_1$, the third corresponds to coefficients of $(4x_1f_1)x_2$, and finally, the twenty-first column corresponds to coefficients of $2x_1f_2+2x_2f_1$. Now this relation over the columns corresponds to a relation on $\mathcal{E} = \{4x_1f_1, 4x_2f_2, 4x_3f_3, 2x_2f_3+2x_3f_2, 2x_1f_3+2x_3f_1, 2x_1f_2+2x_2f_1\}$ over R/Q_1^2. If $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ is a relation, then,

$$\alpha_i = (\text{coefficient of column } (4(i-1)+1)) + (\text{coefficient of column } (4(i-1)+2))x_1$$
$$+ (\text{coefficient of column } (4(i-1)+3))x_2 + (\text{coefficient of column } 4i)x_3$$

for $i \in \{1, 2, ..., 6\}$. The relation over R/Q_1^2 corresponding to the first column relation is then $(2f_2 - g_2x_1 + x_2, 0, 0, 0, 0, -2f_1)$. Proceeding similarly, we get that the relations corresponding to all column relations. The set of columns of M generate a submodule of $k[[t]]^{24}$ considered as a $k[[t]]$-module. Since $k[[t]]$ is a principal ideal domain, every submodule is in fact free. Therefore, the notion of rank is well defined. From the final step after performing the above column operations on M we can see that M has rank 4 and hence the twenty relations above generate the module of relations on $\{4x_1f_1, 4x_2f_2, 4x_3f_3, 2x_2f_3 + 2x_3f_2, 2x_1f_3 + 2x_3f_1, 2x_1f_2 + 2x_2f_1\}$. We indicate the corresponding relations over R/Q_1^2 in the following table.

Relation over the columns	Relation on $\mathcal{E}$ over R/Q_1^2
$C_3 + 2f_2C_1 - g_2C_2 - 2f_1C_{21} = 0$	$(2f_2 - g_2x_1 + x_2, 0, 0, 0, 0, -2f_1)$
$C_4 + 2f_3C_1 - 2f_1C_{17} - g_3C_2 = 0$	$(2f_3 - g_3x_1 + x_3, 0, 0, 0, -2f_1, 0)$
$C_5 - 2g_2C_{21} + g_2^2C_1 = 0$	$(g_2^2, 1, 0, 0, 0, -2g_2)$
$C_6 + 2f_2g_2C_1 - g_2^2C_2 - 2f_2C_{21} = 0$	$(2f_2g_2 - g_2^2x_1, x_1, 0, 0, 0, -2f_2)$
$C_7 - g_2^3C_2 + 4f_1g_2^3C_1 - 4f_1g_2^2C_{21} = 0$	$(4f_1g_2^3 - g_2^3x_1, x_2, 0, 0, 0, -4f_1g_2^2)$
$C_8 - g_2^2g_3C_2 + 4f_1g_2^2g_3C_1 - 2f_1g_2g_3C_{21} - 2f_1g_2^2C_{17} = 0$	$(4f_1g_2^2g_3 - g_2^2g_3x_1, x_3, 0, 0, -2f_1g_2^2, -2f_1g_2g_3)$
$C_9 - 2g_3C_{17} + g_3^2C_1 = 0$	$(g_3^2, 0, 1, 0, -2g_3, 0)$
$C_{10} - g_3^2C_2 + 2f_1g_3^2C_1 - 2f_3C_{17} = 0$	$(2f_1g_3^2 - g_3^3x_1, 0, x_1, 0, -2f_3, 0)$
$C_{11} - g_2g_3^2C_2 + 4f_1g_2g_3^2C_1 - 2f_1g_3^2C_{21} - 2f_1g_2g_3C_{17} = 0$	$(4f_1g_2g_3^2 - g_2g_3^2x_1, 0, x_2, 0 - 2f_1g_2g_3, -2f_1g_3^2)$
$C_{12} - g_2^3C_2 - 4f_1g_3^2C_{17} + 4f_1g_3^3C_1 = 0$	$(4f_1g_3^3 - g_3^3x_1, 0, x_3, 0, -4f_1g_3^2, 0)$
$C_{13} - g_3C_{21} + g_2g_3C_1 - g_2C_{17} = 0$	$(g_2g_3, 0, 0, 1, -g_2, -g_3)$
$C_{14} - g_2g_3C_2 + 2f_1g_2g_3C_1 - f_3C_{21} - f_2C_{17} = 0$	$(2f_1g_2g_3 - g_2g_3x_1, 0, 0, x_1, -f_2, -f_3)$
$C_{15} - g_2^2g_3C_2 - 3f_1g_2g_3C_{21} + 4f_1g_2^2g_3C_1 - f_1g_2^2C_{17} = 0$	$(4f_1g_2^2g_3 - g_2^2g_3x_1, 0, 0, x_2, -f_1g_2^2, -3f_1g_2g_3)$
$C_{16} - g_2g_3^2C_2 + 4f_1g_2g_3^2C_1 - f_1g_3^2C_{21} - 3f_1g_2g_3C_{17} = 0$	$(4f_1g_2g_3^2 - g_2g_3^2x_1, 0, 0, x_3, -3f_1g_2g_3, -f_1g_3^2)$
$C_{18} - g_3C_2 - f_1C_{17} + f_3C_1 = 0$	$(f_3 - g_3x_1, 0, 0, 0, -f_1 + x_1, 0)$
$C_{19} + 3f_1g_2g_3C_1 - g_2g_3C_2 - 2f_3C_{21} - f_2C_{17} = 0$	$(3f_1g_2g_3 - g_2g_3x_1, 0, 0, 0, -f_2 + x_2, -2f_3)$
$C_{20} + 3f_1g_3^2C_1 - g_3^2C_2 - 3f_3C_{17} = 0$	$(3f_1g_3^2 - g_3^2x_1, 0, 0, 0, -3f_3 + x_3, 0)$
$C_{22} - g_2C_2 - f_1C_{21} + f_1g_2C_1 = 0$	$(f_1g_2 - g_2x_1, 0, 0, 0, 0, x_1 - f_1)$
$C_{23} + 3f_1g_2^2C_1 - g_2^2C_2 - 3f_2C_{21} = 0$	$(3f_1g_2^2 - g_2^2x_1, 0, 0, 0, 0, -3f_2 + x_2)$
$C_{24} - g_2g_3C_2 + 3f_1g_2g_3C_1 - f_3C_{21} - 2f_2C_{17} = 0$	$(3f_1g_2g_3 - g_2g_3x_1, 0, 0, 0, -2f_2, -f_3 + x_3)$

Now the relation $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ corresponds to the generator of $Q_1^2 \cap Q_2^2$ given by $\alpha_1(x_1 + f_1)^2 + \alpha_2(x_2 + f_2)^2 + \alpha_3(x_3 + f_3)^2 + \alpha_4(x_2 + f_2)(x_3 + f_3) + \alpha_5(x_1 + f_1)(x_3 + f_3) + \alpha_6(x_1 + f_1)(x_2 + f_2)$. Corresponding to the above relations we obtain generators of $Q_1^2 \cap Q_2^2$ as follows:

1. $\gamma_1 = (2f_2 - g_2x_1 + x_2)(x_1 + f_1)^2 - 2f_1(x_1 + f_1)(x_2 + f_2)$.

2. $\gamma_2 = (2f_3 - g_3x_1 + x_3)(x_1 + f_1)^2 - 2f_1(x_1 + f_1)(x_3 + f_3)$.

3. $\gamma_3 = g_2^2(x_1 + f_1)^2 + (x_2 + f_2)^2 - 2g_2(x_1 + f_1)(x_2 + f_2)$.

4. $\gamma_4 = (2f_2g_2 - g_2^2x_1)(x_1 + f_1)^2 + x_1(x_2 + f_2)^2 - 2f_2(x_1 + f_1)(x_2 + f_2)$.

5. $\gamma_5 = (4f_1g_2^3 - g_2^3x_1)(x_1 + f_1)^2 + x_2(x_2 + f_2)^2 - 4f_1g_2^2(x_1 + f_1)(x_2 + f_2)$.

6. $\gamma_6 = (4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1 + f_1)^2 + x_3(x_2 + f_2)^2 - 2f_1g_2^2(x_1 + f_1)(x_3 + f_3) - 2f_1g_2g_3(x_1 + f_1)(x_2 + f_2)$.

7. $\gamma_7 = g_3^2(x_1 + f_1)^2 + (x_3 + f_3)^2 - 2g_3(x_1 + f_1)(x_3 + f_3)$.

8. $\gamma_8 = (2f_1g_3^2 - g_3^2x_1)(x_1 + f_1)^2 + x_1(x_3 + f_3)^2 - 2f_3(x_1 + f_1)(x_3 + f_3)$.

9. $\gamma_9 = (4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1 + f_1)^2 + x_2(x_3 + f_3)^2 - 2f_1g_2g_3(x_1 + f_1)(x_3 + f_3) - 2f_1g_3^2(x_1 + f_1)(x_2 + f_2)$.

10. $\gamma_{10} = (4f_1g_3^3 - g_3^3x_1)(x_1 + f_1)^2 + x_3(x_3 + f_3)^2 - 4f_1g_3^2((x_1 + f_1)(x_3 + f_3)$.

11. $\gamma_{11} = g_2g_3(x_1+f_1)^2+(x_2+f_2)(x_3+f_3)-g_2(x_1+f_1)(x_3+f_3)-g_3(x_1+f_1)(x_2+f_2)$.

12. $\gamma_{12} = (2f_1g_2g_3 - g_2g_3x_1)(x_1 + f_1)^2 + x_1(x_2 + f_2)(x_3 + f_3) - f_2(x_1 + f_1)(x_3 + f_3) - f_3(x_1 + f_1)(x_2 + f_2)$.

13. $\gamma_{13} = (4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1 + f_1)^2 + x_2(x_2 + f_2)(x_3 + f_3) - f_1g_2^2(x_1 + f_1)(x_3 + f_3) - 3f_1g_2g_3(x_1 + f_1)(x_2 + f_2)$.

14. $\gamma_{14} = (4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1 + f_1)^2 + x_3(x_2 + f_2)(x_3 + f_3) - 3f_1g_2g_3(x_1 + f_1)(x_3 + f_3) - f_1g_3^2((x_1 + f_1)(x_2 + f_2)$.

15. $\gamma_{15} = (f_3 - g_3x_1)(x_1 + f_1)^2 + (x_1 - f_1)(x_1 + f_1)(x_3 + f_3)$.

16. $\gamma_{16} = (3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 + (x_2-f_2)(x_1+f_1)(x_3+f_3) - 2f_3(x_1+f_1)(x_2+f_2)$.

17. $\gamma_{17} = (3f_1g_3^2 - g_3^2x_1)(x_1+f_1)^2 + (x_3-3f_3)(x_1+f_1)(x_3+f_3)$.

18. $\gamma_{18} = (f_1g_2 - g_2x_1)(x_1+f_1)^2 + (x_1-f_1)(x_1+f_1)(x_2+f_2)$.

19. $\gamma_{19} = (3f_1g_2^2 - g_2^2x_1)(x_1+f_1)^2 + (x_2-3f_2)(x_1+f_1)(x_2+f_2)$.

20. $\gamma_{20} = (3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 - 2f_2((x_1+f_1)(x_3+f_3) + (x_3-f_3)(x_1+f_1)(x_2+f_2)$.

Since these generators correspond to relations on $\mathcal{E} =\{4x_1f_1, 4x_2f_2, 4x_3f_3, 2x_2f_3 + 2x_3f_2, 2x_1f_3 + 2x_3f_1, 2x_1f_2 + 2x_2f_1\}$ over R/Q_1^2, we need to include the set of generators of $Q_1^2Q_2^2$ to get a complete set of generators for $Q_1^2 \cap Q_2^2$. Thus, $Q_1^2 \cap Q_2^2 = (\{\gamma_i : i = 1, ..., 20\})R + Q_1^2Q_2^2$.

3.1.5 $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ (special case)

We obtain the contraction of $Q_1^2 \cap Q_2^2$ to S by applying the trace map to each of the generators of $Q_1^2 \cap Q_2^2$. We show that each of these generators $(Q_1^2 \cap Q_2^2) \cap S$ lies in $\mathfrak{m}P$, thus proving the Eisenbud-Mazur conjecture in this case. Note that we showed earlier $P = (\{(-x_i + x_1g_i(t)) : i = 2, ..., m\})S + (\{(x_ix_j - f_i(t)f_j(t)) : i, j \in \{1, ..., m\}\})S$. In the case of three generators, we have that $P = (x_1g_2 - x_2, x_1g_3 - x_3, x_1x_2 - f_1f_2, x_1x_3 - f_1f_3, x_2x_3 - f_2f_3, x_1^2 - f_1^2, x_2^2 - f_2^2, x_3^2 - f_3^2)S$. We express each generator of $(Q_1^2 \cap Q_2^2) \cap S$ as a linear combination of these generators of P with coefficients in $\mathfrak{m}$.

1.

$$\begin{aligned}
& \operatorname{tr}((2f_2 - g_2x_1 + x_2)(x_1+f_1)^2 - 2f_1(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2 + x_1^2x_2 - 2x_1^2f_1g_2 + 2x_1^2f_2 - x_1f_1^2g_2 - x_2f_1^2 + 2x_1f_1f_2 \\
= \;& -x_1^3g_2 + x_1^2x_2 - 2x_1^2f_1g_2 + 2x_1^2f_1g_2 - x_1f_1^2g_2 - x_2f_1^2 + 2x_1f_1^2g_2 \\
= \;& -x_1^2(g_2x_1 - x_2) + f_1^2(g_2x_1 - x_2) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

Note that f_1 is a multiple of t being an odd power series, so that f_1^2 is a multiple of t^2 and hence, $f_1^2 \in \mathfrak{m}$.

2.

$$\begin{aligned}
& \operatorname{tr}((2f_3 - g_3x_1 + x_3)(x_1+f_1)^2 - 2f_1(x_1+f_1)(x_3+f_3)) \\
= \;& -x_1^3g_3 + x_1^2x_3 - x_1f_1^2g_3 + x_3f_1^2 - 2x_1x_3f_1 + 2x_1^2f_3 \\
= \;& -x_1^2(x_1g_3 - x_3) - f_1^2(x_1g_3 - x_3) - 2x_1f_1(x_3 - x_1g_3) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

3.

$$\begin{aligned}
& \operatorname{tr}(g_2^2(x_1+f_1)^2 + (x_2+f_2)^2 - 2g_2(x_1+f_1)(x_2+f_2)) \\
= \;& x_1^2g_2^2 + f_1^2g_2^2 - 2x_1x_2g_2 - 2f_1f_2g_2 + x_2^2 + f_2^2 \\
= \;& x_1^2g_2^2 + f_1^2g_2^2 - 2x_1x_2g_2 - 2f_1^2g_2^2 + x_2^2 + f_1^2g_2^2 \\
= \;& x_1^2g_2^2 - 2x_1x_2g_2 + x_2^2 \\
= \;& x_1^2g_2^2 - x_1x_2g_2 - x_1x_2g_2 + x_2^2 \\
= \;& x_1g_2(x_1g_2 - x_2) - x_2(x_1g_2 - x_2) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

4.

$$
\begin{aligned}
& \operatorname{tr}((2f_2g_2 - g_2^2x_1)(x_1+f_1)^2 + x_1(x_2+f_2)^2 - 2f_2(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2^2 - x_1f_1^2g_2^2 + 2x_1^2f_2g_2 + 2f_1^2f_2g_2 + x_1x_2^2 - 2x_1x_2f_2 + x_1f_2^2 - 2f_1f_2^2 \\
= \;& -x_1^3g_2^2 - x_1f_1^2g_2^2 + 2x_1^2f_1g_2^2 + 2f_1^3g_2^2 + x_1x_2^2 - 2x_1x_2f_1g_2 + x_1f_1^2g_2^2 - 2f_1^3g_2^2 \\
= \;& -x_1^3g_2^2 + x_1x_2^2 - 2x_1x_2f_1g_2 + 2x_1^2f_1g_2^2 \\
= \;& -x_1(x_1g_2+x_2)(x_1g_2-x_2) - 2x_1f_1g_2(x_2-x_1g_2) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

5.

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_2^3 - g_2^3x_1)(x_1+f_1)^2 + x_2(x_2+f_2)^2 - 4f_1g_2^2(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2^3 + 4x_1^2f_1g_2^3 - x_1f_1^2g_2^3 + 4f_1^3g_2^3 - 4x_1x_2f_1g_2^2 - 4f_1^2f_2g_2^2 + x_2^3 + x_2f_2^2 \\
= \;& -x_1^3g_2^3 + 4x_1^2f_1g_2^3 - x_1f_1^2g_2^3 + 4f_1^3g_2^3 - 4x_1x_2f_1g_2^2 - 4f_1^3g_2^3 + x_2^3 + x_2f_1^2g_2^2 \\
= \;& x_2^3 - x_1^3g_2^3 - x_1f_1^2g_2^3 + x_2f_1^2g_2^2 + 4x_1^2f_1g_2^3 - 4x_1x_2f_1g_2^2 \\
= \;& (x_2-x_1g_2)(x_2^2 + x_1x_2g_2 + x_1^2g_2^2) - f_1^2g_2^2(x_1g_2-x_2) + 4x_1f_1g_2^2(x_1g_2-x_2) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

6.

$$
\begin{aligned}
&\mathrm{tr}((4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1+f_1)^2 + x_3(x_2+f_2)^2 \\
&-2f_1g_2^2(x_1+f_1)(x_3+f_3) - 2f_1g_2g_3(x_1+f_1)(x_2+f_2)) \\
= &-x_1^3g_2^2g_3 + 4x_1^2f_1g_2^2g_3 - x_1f_1^2g_2^2g_3 \\
&+4f_1^3g_2^2g_3 - 2x_1x_3f_1g_2^2 - 2f_1^2f_3g_2^2 - 2x_1x_2f_1g_2g_3 \\
&-2f_1^2f_2g_2g_3 + x_2^2x_3 + x_3f_2^2 \\
= &-x_1^3g_2^2g_3 + 4x_1^2f_1g_2^2g_3 - x_1f_1^2g_2^2g_3 \\
&+4f_1^3g_2^2g_3 - 2x_1x_3f_1g_2^2 - 2f_1^3g_3g_2^2 - 2x_1x_2f_1g_2g_3 \\
&-2f_1^3g_2^2g_3 + x_2^2x_3 + x_3f_1^2g_2^2 \\
= &-x_1^3g_2^2g_3 + x_2^2x_3 - x_1f_1^2g_2^2g_3 + x_3f_1^2g_2^2 \\
&+2x_1^2f_1g_2^2g_3 - 2x_1x_3f_1g_2^2 + 2x_1^2f_1g_2^2g_3 - 2x_1x_2f_1g_2g_3
\end{aligned}
$$

Now $-x_1^3g_2^2g_3 + x_2^2x_3 = -x_1^2g_2g_3(x_1g_2 - x_2) - x_1x_2g_3(x_1g_2 - x_2) - x_2^2(x_1g_3 - x_3)$.

So that,

$$
\begin{aligned}
&\mathrm{tr}((4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1+f_1)^2 + x_3(x_2+f_2)^2 \\
&-2f_1g_2^2(x_1+f_1)(x_3+f_3) - 2f_1g_2g_3(x_1+f_1)(x_2+f_2)) \\
= &-x_1^2g_2g_3(x_1g_2 - x_2) - x_1x_2g_3(x_1g_2 - x_2) - x_2^2(x_1g_3 - x_3) \\
&-f_1^2g_2^2(x_1g_3 - x_3) + 2x_1f_1g_2^2(x_1g_3 - x_3) \\
&+2x_1f_1g_2g_3(x_1g_2 - x_2) \\
\in\ &\mathfrak{m}P
\end{aligned}
$$

7.

$$\begin{aligned}
& \operatorname{tr}(g_3^2(x_1+f_1)^2+(x_3+f_3)^2-2g_3(x_1+f_1)(x_3+f_3)) \\
= \;& x_1^2g_3^2+f_1^2g_3^2-2x_1x_3g_3-2f_1f_3g_3+x_3^2+f_3^2 \\
= \;& x_1^2g_3^2+f_1^2g_3^2-2x_1x_3g_3-2f_1^2g_3^2+x_3^2+f_1^2g_3^2 \\
= \;& x_1^2g_3^2-2x_1x_3g_3+x_3^2 \\
= \;& x_1^2g_3^2-x_1x_3g_3+x_3^2-x_1x_3g_3 \\
= \;& x_1g_3(x_1g_3-x_3)+x_3(x_3-x_1g_3) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

8.

$$\begin{aligned}
& \operatorname{tr}((2f_1g_3^2-g_3^2x_1)(x_1+f_1)^2+x_1(x_3+f_3)^2-2f_3(x_1+f_1)(x_3+f_3)) \\
= \;& -x_1^3g_3^2+2x_1^2f_1g_3^2-x_1f_1^2g_3^2+2f_1^3g_3^2+x_1x_3^2-2x_1x_3f_3+x_1f_3^2-2f_1f_3^2 \\
= \;& -x_1^3g_3^2+2x_1^2f_1g_3^2-x_1f_1^2g_3^2+2f_1^3g_3^2+x_1x_3^2-2x_1x_3f_1g_3+x_1f_1^2g_3^2-2f_1^3g_3^2 \\
= \;& -x_1^3g_3^2+x_1x_3^2+2x_1^2f_1g_3^2-2x_1x_3f_1g_3 \\
= \;& -x_1(x_1g_3+x_3)(x_1g_3-x_3)+2x_1f_1g_3(x_1g_3-x_3) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

9.

$$
\begin{aligned}
& \mathrm{tr}((4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1+f_1)^2 + x_2(x_3+f_3)^2 \\
& -2f_1g_2g_3(x_1+f_1)(x_3+f_3) - 2f_1g_3^2(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2g_3^2 + 4x_1^2f_1g_2g_3^2 - x_1f_1^2g_2g_3^2 + 4f_1^3g_2g_3^2 \\
& -2x_1x_3f_1g_2g_3 - 2f_1^2f_3g_2g_3 - 2x_1x_2f_1g_3^2 - 2f_1^2f_2g_3^2 \\
& +x_2x_3^2 + x_2f_3^2 \\
= \;& -x_1^3g_2g_3^2 + 4x_1^2f_1g_2g_3^2 - x_1f_1^2g_2g_3^2 + 4f_1^3g_2g_3^2 \\
& -2x_1x_3f_1g_2g_3 - 2f_1^3g_2g_3^2 - 2x_1x_2f_1g_3^2 - 2f_1^3g_2g_3^2 \\
& +x_2x_3^2 + x_2f_1^2g_3^2 \\
= \;& -x_1^3g_2g_3^2 + x_2x_3^2 + x_2f_1^2g_3^2 - x_1f_1^2g_2g_3^2 \\
& +2x_1^2f_1g_2g_3^2 - 2x_1x_3f_1g_2g_3 + 2x_1^2f_1g_2g_3^2 - 2x_1x_2f_1g_3^2
\end{aligned}
$$

Now $-x_1^3g_2g_3^2 + x_2x_3^2 = -x_1^2g_2g_3(x_1g_3 - x_3) - x_1x_3g_2(x_1g_3 - x_3) - x_3^2(x_1g_2 - x_2)$.

So that

$$
\begin{aligned}
& \mathrm{tr}((4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1+f_1)^2 + x_2(x_3+f_3)^2 \\
& -2f_1g_2g_3(x_1+f_1)(x_3+f_3) - 2f_1g_3^2(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^2g_2g_3(x_1g_3 - x_3) - x_1x_3g_2(x_1g_3 - x_3) - x_3^2(x_1g_2 - x_2) \\
& +x_1f_1^2g_3^2(x_2 - x_1g_2) + 2x_1f_1g_2g_3(x_1g_3 - x_3) \\
& +2x_1f_1g_3^2(x_1g_2 - x_2) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

10.

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_3^3 - g_3^3x_1)(x_1+f_1)^2 + x_3(x_3+f_3)^2 - 4f_1g_3^2((x_1+f_1)(x_3+f_3)) \\
= \;& -x_1^3g_3^3 + 4x_1^2f_1g_3^3 - x_1f_1^2g_3^3 + 4f_1^3g_3^3 - 4x_1x_3f_1g_3^2 - 4f_1^2f_3g_3^2 + x_3^3 + x_3f_3^2 \\
= \;& -x_1^3g_3^3 + 4x_1^2f_1g_3^3 - x_1f_1^2g_3^3 + 4f_1^3g_3^3 - 4x_1x_3f_1g_3^2 - 4f_1^3g_3^3 + x_3^3 + x_3f_1^2g_3^2 \\
= \;& x_3^3 - x_1^3g_3^3 + 4x_1^2f_1g_3^3 - 4x_1x_3f_1g_3^2 - x_1f_1^2g_3^3 + x_3f_1^2g_3^2 \\
= \;& (x_3^2 + x_3x_1g_3 + x_1^2g_3^2)(x_3 - x_1g_3) + 4x_1f_1g_3^2(x_1g_3 - x_3) - f_1^2g_3^2(x_1g_3 - x_3) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

11.

$$
\begin{aligned}
& \operatorname{tr}(g_2g_3(x_1+f_1)^2 + (x_2+f_2)(x_3+f_3) - g_2(x_1+f_1)(x_3+f_3) \\
& -g_3(x_1+f_1)(x_2+f_2)) \\
= \;& x_1^2g_2g_3 + f_1^2g_2g_3 - x_1x_3g_2 - f_1f_3g_2 - x_1x_2g_3 - f_1f_2g_3 + x_2x_3 + f_2f_3 \\
= \;& x_1^2g_2g_3 + f_1^2g_2g_3 - x_1x_3g_2 - f_1^2g_3g_2 - x_1x_2g_3 - f_1^2g_2g_3 + x_2x_3 + f_1^2g_2g_3 \\
= \;& x_1^2g_2g_3 - x_1x_3g_2 - x_1x_2g_3 + x_2x_3 \\
= \;& x_1g_2(x_1g_3 - x_3) - x_2(x_1g_3 - x_3) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

12.

$$
\begin{aligned}
& \operatorname{tr}((2f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 + x_1(x_2+f_2)(x_3+f_3) \\
& -f_2(x_1+f_1)(x_3+f_3) - f_3(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2g_3 + 2x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 2f_1^3g_2g_3 + x_1x_2x_3 \\
& -x_1x_3f_2 - x_1x_2f_3 + x_1f_2f_3 - 2f_1f_2f_3 \\
= \;& -x_1^3g_2g_3 + 2x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 2f_1^3g_2g_3 + x_1x_2x_3 \\
& -x_1x_3f_1g_2 - x_1x_2f_1g_3 + x_1f_1^2g_2g_3 - 2f_1^3g_2g_3 \\
= \;& -x_1^3g_2g_3 + x_1x_2x_3 + x_1^2f_1g_2g_3 - x_1x_2f_1g_3 \\
& +x_1^2f_1g_2g_3 - x_1x_3f_1g_2
\end{aligned}
$$

Now $-x_1^3g_2g_3 + x_1x_2x_3 = -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3)$. So that,

$$
\begin{aligned}
& \operatorname{tr}((2f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 + x_1(x_2+f_2)(x_3+f_3) \\
& -f_2(x_1+f_1)(x_3+f_3) - f_3(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3) + x_1f_1g_3(x_1g_2 - x_2) \\
& +x_1f_1g_2(x_1g_3 - x_3) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

13.

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1+f_1)^2 + x_2(x_2+f_2)(x_3+f_3) \\
& -f_1g_2^2(x_1+f_1)(x_3+f_3) - 3f_1g_2g_3(x_1+f_1)(x_3+f_3)) \\
= \; & -x_1^3g_2^2g_3 + 4x_1^2f_1g_2^2g_3 - x_1f_1^2g_2^2g_3 + 4f_1^3g_2^2g_3 \\
& -x_1x_3f_1g_2^2 - f_1^2f_3g_2^2 - 3x_1x_2f_1g_2g_3 - 3f_1^2f_2g_2g_3 \\
& +x_2^2x_3 + x_2f_2f_3 \\
= \; & -x_1^3g_2^2g_3 + 4x_1^2f_1g_2^2g_3 - x_1f_1^2g_2^2g_3 + 4f_1^3g_2^2g_3 \\
& -x_1x_3f_1g_2^2 - f_1^3g_3g_2^2 - 3x_1x_2f_1g_2g_3 - 3f_1^3g_2^2g_3 \\
& +x_2^2x_3 + x_2f_1^2g_2g_3 \\
= \; & x_2^2x_3 - x_1^3g_2^2g_3 + x_1^2f_1g_2^2g_3 - x_1x_3f_1g_2^2 \\
& +3x_1^2f_1g_2^2g_3 - 3x_1x_2f_1g_2g_3 + x_2f_1^2g_2g_3 - x_1f_1^2g_2^2g_3
\end{aligned}
$$

Now $x_2^2x_3 - x_1^3g_2^2g_3 = -x_1^2g_3g_2(x_1g_2 - x_2) - x_1x_2g_3(x_1g_2 - x_2) - x_2^2(x_1g_3 - x_3)$.

So that,

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_2^2g_3 - g_2^2g_3x_1)(x_1+f_1)^2 + x_2(x_2+f_2)(x_3+f_3) \\
& -f_1g_2^2(x_1+f_1)(x_3+f_3) - 3f_1g_2g_3(x_1+f_1)(x_3+f_3)) \\
= \; & (-x_1^2g_3g_2 - x_1x_2g_3)(x_1g_2 - x_2) - x_2^2(x_1g_3 - x_3) \\
& +x_1f_1g_2^2(x_1g_3 - x_3) + 3x_1f_1g_2g_3(x_1g_2 - x_2) \\
& +f_1^2g_2g_3(x_2 - x_1g_2) \\
\in \; & \mathfrak{m}P
\end{aligned}
$$

14.

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1+f_1)^2 + x_3(x_2+f_2)(x_3+f_3) \\
& -3f_1g_2g_3(x_1+f_1)(x_3+f_3) - f_1g_3^2((x_1+f_1)(x_2+f_2)) \\
= \ & -x_1^3g_2g_3^2 + 4x_1^2f_1g_2g_3^2 - x_1f_1^2g_2g_3^2 + 4f_1^3g_2g_3^2 \\
& -3x_1x_3f_1g_2g_3 - 3f_1^2f_3g_2g_3 - x_1x_2f_1g_3^2 \\
& -f_1^2f_2g_3^2 + x_2x_3^2 + x_3f_2f_3 \\
= \ & -x_1^3g_2g_3^2 + 4x_1^2f_1g_2g_3^2 - x_1f_1^2g_2g_3^2 + 4f_1^3g_2g_3^2 \\
& -3x_1x_3f_1g_2g_3 - 3f_1^3g_2g_3^2 - x_1x_2f_1g_3^2 \\
& -f_1^3g_2g_3^2 + x_2x_3^2 + x_3f_1^2g_2g_3 \\
= \ & x_2x_3^2 - x_1^3g_2g_3^2 + 3x_1^2f_1g_2g_3^2 - 3x_1x_3f_1g_2g_3 \\
& +x_1^2f_1g_2g_3^2 - x_1x_2f_1g_3^2 + x_3f_1^2g_2g_3 - x_1f_1^2g_2g_3^2
\end{aligned}
$$

Now $x_2x_3^2 - x_1^3g_2g_3^2 = -x_1^2g_2g_3(x_1g_3 - x_3) - x_1x_3g_2(x_1g_3 - x_3) - x_3^2(x_1g_2 - x_2)$.

So that,

$$
\begin{aligned}
& \operatorname{tr}((4f_1g_2g_3^2 - g_2g_3^2x_1)(x_1+f_1)^2 + x_3(x_2+f_2)(x_3+f_3) \\
& -3f_1g_2g_3(x_1+f_1)(x_3+f_3) - f_1g_3^2((x_1+f_1)(x_2+f_2)) \\
= \ & (-x_1^2g_2g_3 - x_1x_3g_2)(x_1g_3 - x_3) - x_3^2(x_1g_2 - x_2) \\
& 3x_1f_1g_2g_3(x_1g_3 - x_3) + x_1f_1g_3^2(x_1g_2 - x_2) \\
& +f_1^2g_2g_3(x_3 - x_1g_3) \\
\in \ & \mathfrak{m}P
\end{aligned}
$$

15.

$$\begin{aligned}
& \operatorname{tr}((f_3 - g_3x_1)(x_1+f_1)^2 + (x_1-f_1)(x_1+f_1)(x_3+f_3)) \\
= \; & -x_1^3g_3 + x_1^2f_1g_3 - x_1f_1^2g_3 + f_1^3g_3 + x_1^2x_3 - x_1x_3f_1 + x_1f_1f_3 - f_1^2f_3 \\
= \; & -x_1^3g_3 + x_1^2f_1g_3 - x_1f_1^2g_3 + f_1^3g_3 + x_1^2x_3 - x_1x_3f_1 + x_1f_1^2g_3 - f_1^3g_3 \\
= \; & -x_1^3g_3 + x_1^2x_3 + x_1^2f_1g_3 - x_1x_3f_1 \\
= \; & -x_1^2(x_1g_3 - x_3) + x_1f_1(x_1g_3 - x_3) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

16.

$$\begin{aligned}
& \operatorname{tr}((3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 \\
& +(x_2-f_2)((x_1+f_1)(x_3+f_3) - 2f_3(x_1+f_1)(x_2+f_2)) \\
= \; & -x_1^3g_2g_3 + 3x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 3f_1^3g_2g_3 - x_1x_3f_1g_2 \\
& -f_1^2f_3g_2 - 2x_1x_2f_1g_3 - 2f_1^2f_2g_3 + x_1x_2x_3 + x_2f_1f_3 \\
= \; & -x_1^3g_2g_3 + 3x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 3f_1^3g_2g_3 - x_1x_3f_1g_2 \\
& -f_1^3g_3g_2 - 2x_1x_2f_1g_3 - 2f_1^3g_2g_3 + x_1x_2x_3 + x_2f_1^2g_3 \\
= \; & -x_1^3g_2g_3 + x_1x_2x_3 + x_1^2f_1g_2g_3 - x_1x_3f_1g_2 \\
& +2x_1^2f_1g_2g_3 - 2x_1x_2f_1g_3 + x_2f_1^2g_3 - x_1f_1^2g_2g_3
\end{aligned}$$

Now $-x_1^3g_2g_3 + x_1x_2x_3 = -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3)$. So that,

$$\begin{aligned}
& \operatorname{tr}((3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 \\
& +(x_2-f_2)((x_1+f_1)(x_3+f_3) - 2f_3(x_1+f_1)(x_2+f_2)) \\
= \; & -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3) + x_1f_1g_2(x_1g_3 - x_3) \\
& +2x_1f_1g_3(x_1g_2 - x_2) + f_1^2g_3(x_2 - x_1g_2) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

17.

$$
\begin{aligned}
& \operatorname{tr}((3f_1g_3^2 - g_3^2x_1)(x_1+f_1)^2 + (x_3-3f_3)(x_1+f_1)(x_3+f_3)) \\
= \;& -x_1^3g_3^2 + 3x_1^2f_1g_3^2 - x_1f_1^2g_3^2 + 3f_1^3g_3^2 - 3x_1x_3f_1g_3 - 3f_1^2f_3g_3 + x_1x_3^2 + x_3f_1f_3 \\
= \;& -x_1^3g_3^2 + 3x_1^2f_1g_3^2 - x_1f_1^2g_3^2 + 3f_1^3g_3^2 - 3x_1x_3f_1g_3 - 3f_1^3g_3^2 + x_1x_3^2 + x_3f_1^2g_3 \\
= \;& -x_1^3g_3^2 + x_1x_3^2 + 3x_1^2f_1g_3^2 - 3x_1x_3f_1g_3 + x_3f_1^2g_3 - x_1f_1^2g_3^2 \\
= \;& -x_1(x_1g_3+x_3)(x_1g_3-x_3) + 3x_1f_1g_3(x_1g_3-x_3) + f_1^2g_3(x_3-x_1g_3) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

18.

$$
\begin{aligned}
& \operatorname{tr}((f_1g_2 - g_2x_1)(x_1+f_1)^2 + (x_1-f_1)(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2 + x_1^2f_1g_2 - x_1f_1^2g_2 + f_1^3g_2 + x_1^2x_2 - x_1x_2f_1 + x_1f_1f_2 - f_1^2f_2 \\
= \;& -x_1^3g_2 + x_1^2f_1g_2 - x_1f_1^2g_2 + f_1^3g_2 + x_1^2x_2 - x_1x_2f_1 + x_1f_1^2g_2 - f_1^3g_2 \\
= \;& -x_1^3g_2 + x_1^2x_2 + x_1^2f_1g_2 - x_1x_2f_1 \\
= \;& -x_1^2(x_1g_2-x_2) + x_1f_1(x_1g_2-x_2) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

19.

$$
\begin{aligned}
& \operatorname{tr}((3f_1g_2^2 - g_2^2x_1)(x_1+f_1)^2 + (x_2-3f_2)(x_1+f_1)(x_2+f_2)) \\
= \;& -x_1^3g_2^2 + 3x_1^2f_1g_2^2 - x_1f_1^2g_2^2 + 3f_1^3g_2^2 - 3x_1x_2f_1g_2 - 3f_1^2f_2g_2 + x_1x_2^2 + x_2f_1f_2 \\
= \;& -x_1^3g_2^2 + 3x_1^2f_1g_2^2 - x_1f_1^2g_2^2 + 3f_1^3g_2^2 - 3x_1x_2f_1g_2 - 3f_1^3g_2^2 + x_1x_2^2 + x_2f_1^2g_2 \\
= \;& -x_1^3g_2^2 + x_1x_2^2 + 3x_1^2f_1g_2^2 - 3x_1x_2f_1g_2 + x_2f_1^2g_2 - x_1f_1^2g_2^2 \\
= \;& -x_1(x_1g_2+x_2)(x_1g_2-x_2) + 3x_1f_1g_2(x_1g_2-x_2) + f_1^2g_2(x_2-x_1g_2) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

20.

$$
\begin{aligned}
& tr((3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 - 2f_2((x_1+f_1)(x_3+f_3) \\
& +(x_3-f_3)((x_1+f_1)(x_2+f_2)) \\
= \quad & -x_1^3g_2g_3 + 3x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 3f_1^3g_2g_3 - 2x_1x_3f_1g_2 \\
& -2f_1^2f_3g_2 - x_1x_2f_1g_3 - f_1^2f_2g_3 + x_1x_2x_3 + x_3f_1f_2 \\
= \quad & -x_1^3g_2g_3 + 3x_1^2f_1g_2g_3 - x_1f_1^2g_2g_3 + 3f_1^3g_2g_3 - 2x_1x_3f_1g_2 \\
& -2f_1^3g_3g_2 - x_1x_2f_1g_3 - f_1^3g_2g_3 + x_1x_2x_3 + x_3f_1^2g_2 \\
= \quad & -x_1^3g_2g_3 + x_1x_2x_3 + 2x_1^2f_1g_2g_3 - 2x_1x_3f_1g_2 \\
& +x_1^2f_1g_2g_3 - x_1x_2f_1g_3 + x_3f_1^2g_2 - x_1f_1^2g_2g_3
\end{aligned}
$$

Now $-x_1^3g_2g_3 + x_1x_2x_3 = -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3)$. So that,

$$
\begin{aligned}
& \mathrm{tr}((3f_1g_2g_3 - g_2g_3x_1)(x_1+f_1)^2 - 2f_2((x_1+f_1)(x_3+f_3) \\
& +(x_3-f_3)((x_1+f_1)(x_2+f_2)) \\
= \quad & -x_1^3g_3(x_1g_2 - x_2) - x_1x_2(x_1g_3 - x_3) + 2x_1f_1g_2(x_1g_3 - x_3) \\
& +x_1f_1g_3(x_1g_2 - x_2) + f_1^2g_2(x_3 - x_1g_3) \\
\in \quad & \mathfrak{m}P
\end{aligned}
$$

Finally, we need to show that the generators of $(Q_1^2 \cap Q_2^2) \cap S$ arising by applying the trace map to $Q_1^2Q_2^2$ also lie in $\mathfrak{m}P$. We have that $Q_1^2Q_2^2 = \{(x_i - f_i)(x_j - f_j)(x_{i'} + f_{i'})(x_{j'} + f_{j'}) : i, i^{'}, j, j^{'} = 1, 2, 3\}$.

We have that

$$
\begin{aligned}
& \operatorname{tr}((x_i - f_i)(x_j - f_j)(x_{i'} + f_{i'})(x_{j'} + f_{j'})) \\
= \;& x_i x_{i'} x_j x_{j'} + f_i f_{i'} x_j x_{j'} + x_i x_{i'} f_j f_{j'} - f_i x_{i'} f_j x_{j'} \\
& - f_i x_{i'} x_j f_{j'} - x_i f_{i'} f_j x_{j'} - x_i f_{i'} x_j f_{j'} + f_i f_{i'} f_j f_{j'} \\
= \;& x_i x_{i'} x_j x_{j'} - f_i x_{i'} f_j x_{j'} + f_i f_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - f_i x_{i'} x_j f_{j'} - x_i f_{i'} x_j f_{j'} + f_i f_{i'} f_j f_{j'} \\
= \;& x_{i'} x_{j'}(x_i x_j - f_i f_j) + f_i f_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - f_i x_{i'} x_j f_{j'} - f_{i'} f_{j'}(x_i x_j - f_i f_j) \\
= \;& x_{i'} x_{j'}(x_i x_j - f_i f_j) + f_i f_{i'} x_j x_{j'} - x_i x_{i'} x_j x_{j'} + x_i x_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - x_i x_{i'} x_j x_{j'} + x_i x_{i'} x_j x_{j'} - f_i x_{i'} x_j f_{j'} - f_{i'} f_{j'}(x_i x_j - f_i f_j \\
= \;& x_{i'} x_{j'}(x_i x_j - f_i f_j) + x_j x_{j'}(f_i f_{i'} - x_i x_{i'}) + x_i x_{j'}(x_{i'} x_j - f_{i'} f_j) \\
& + x_i x_{i'}(f_j f_{j'} - x_j x_{j'}) + x_{i'} x_j(x_i x_{j'} - f_i f_{j'}) - f_{i'} f_{j'}(x_i x_j - f_i f_j) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

Note, $f_{i'} f_{j'}$ are is a multiple of t^2 and hence in $\mathfrak{m}$ as $f_{i'}, f_{j'}$ are odd power series.

Thus, every generator of $(Q_1^2 \cap Q_2^2) \cap S$ is in $\mathfrak{m}P$, showing $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ as promised.

3.1.6 Computing generators of $Q_1^2 \cap Q_2^2$ (general case)

We extend the methods of section 3.1.4 to compute the generators of $Q_1^2 \cap Q_2^2$ for any number m of generators of Q_1. In this case, we seek relations on

$$
\begin{aligned}
\mathcal{E} = (& 4x_1 f_1, ..., 4x_m f_m, 2x_1 f_2 + 2x_2 f_1, ..., 2x_1 f_m + 2x_m f_1, 2x_2 f_3 + 2x_3 f_2, ..., \\
& 2x_2 f_m + 2x_m f_2, ..., 2x_{m-1} f_m + 2x_m f_{m-1})
\end{aligned}
$$

over R/Q_1^2. Denote a typical relation by the vector

$$(\alpha_1, ..., \alpha_m, \alpha_{12}, ..., \alpha_{1m}, \alpha_{23}, ..., \alpha_{2m}, ..., \alpha_{m-1,m}).$$

As before, we can express every element of R/Q_1^2 as a $k[[t]]$-linear combination of $x_1, ..., x_m$. We associate the matrix M of coefficients in $k[[t]]$ with the set $\mathcal{E}$ as we did in the case when $m = 3$. Corresponding to each element in $\mathcal{E}$, we will have $m + 1$ columns of coefficients. Now $|\mathcal{E}| = m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$. So we have that $\frac{m(m+1)(m+1)}{2}$ columns in M. Performing column operations as before, we can show that M has rank equal to the number of rows, which is $m + 1$, giving rise to $\frac{m(m+1)^2}{2} - (m+1) = \frac{(m-1)(m+1)(m+2)}{2}$ relations. We indicate these relations below.

1. The $m-1$ relations with $\alpha_1 = g_i^2, \alpha_i = 1, \alpha_{1i} = -2g_i$, where $1 < i \leq m$.

2. The $m-1$ relations with $\alpha_1 = g_i(f_1 - x_1), \alpha_{1i} = x_1 - f_1$, where $1 < i \leq m$.

3. The $(m-1)(m-2)$ relations with $\alpha_1 = g_i g_j(3f_1 - x_1), \alpha_{1i} = x_j - f_j, \alpha_{1j} = -2f_i$, where $1 < i \neq j < m$.

4. The $(m-1)(m-2)/2$ relations with $\alpha_1 = g_i g_j(2f_1 - x_1), \alpha_{1i} = -f_j, \alpha_{1j} = -f_i, \alpha_{ij} = x_1$, where $1 < i \neq j < m$.

5. The $(m-1)(m-2)/2$ relations with $\alpha_1 = g_i g_j, \alpha_{1i} = -g_j, \alpha_{1j} = -g_i, \alpha_{ij} = 1$, where $1 < i \neq j < m$.

6. The $(m-1)(m-2)$ relations with $\alpha_1 = g_i^2 g_j(4f_1 - x_1), \alpha_{1i} = -3f_1 g_i g_j, \alpha_{1j} = -f_1 g_i^2, \alpha_{ij} = x_i$, where $1 < i \neq j < m$.

7. The $m-1$ relations with $\alpha_1 = g_i^2(3f_1 - x_1), \alpha_{1i} = x_i - 3f_i$, where $1 < i \leq m$.

8. The $m-1$ relations with $\alpha_1 = g_i(2f_1 - x_1) + x_i, \alpha_{1i} = -2f_1$, where $1 < i \leq m$.

9. The $m-1$ relations with $\alpha_1 = g_i^3(4f_1 - x_1), \alpha_i = x_i, \alpha_{1i} = -4f_1g_i^2$, where $1 < i \leq m$.

10. The $m-1$ relations with $\alpha_1 = g_i^2(2f_1 - x_1), \alpha_i = x_1, \alpha_{1i} = -2f_i$, where $1 < i \leq m$.

11. The $(m-1)(m-2)$ relations with $\alpha_1 = g_i^2 g_j(4f_1 - x_1), \alpha_i = x_j, \alpha_{1i} = -2f_1g_ig_j, \alpha_{1j} = -2f_1g_i^2$, where $1 < i \neq j < m$.

12. The $(m-1)(m-2)(m-3)/2$ relations with

$$\begin{cases} \alpha_1 = g_ig_jg_k(2f_1 - x_1), \alpha_{ij} = x_k, \alpha_{ki} = -f_1g_j, \alpha_{kj} = -f_1g_i & \text{where } 1 < k < i < j \leq n \\ \alpha_1 = g_ig_jg_k(2f_1 - x_1), \alpha_{ij} = x_k, \alpha_{ik} = -f_1g_j, \alpha_{kj} = -f_1g_i & \text{where } 1 < i < k < j \leq n \cdot \\ \alpha_1 = g_ig_jg_k(2f_1 - x_1), \alpha_{ij} = x_k, \alpha_{ik} = -f_1g_j, \alpha_{jk} = -f_1g_i & \text{where } 1 < i < j < k \leq n \end{cases}$$

So we have accounted for all the $6(m-1) + 4(m-1)(m-2) + \frac{(m-1)(m-2)(m-3)}{2} = \frac{(m-1)(m+1)(m+2)}{2}$ relations.

The corresponding generators of $Q_1^2 \cap Q_2^2$ are as follows.

1. The $m-1$ generators $g_i^2(x_1 + f_1)^2 + (x_i + f_i)^2 - 2g_i(x_1 + f_1)(x_i + f_i)$, where $1 < i \leq m$.

2. The $m-1$ generators $g_i(f_1 - x_1)(x_1 + f_1)^2 + (x_1 - f_1)(x_1 + f_1)(x_i + f_i)$, where $1 < i \leq m$.

3. The $(m-1)(m-2)$ generators $g_ig_j(3f_1 - x_1)(x_1 + f_1)^2 + (x_j - f_j)(x_1 + f_1)(x_i + f_i) + -2f_i(x_1 + f_1)(x_j + f_j)$, where $1 < i \neq j < m$.

4. The $(m-1)(m-2)/2$ generators $g_ig_j(2f_1 - x_1)(x_1 + f_1)^2 - f_j(x_1 + f_1)(x_i + f_i) - f_i(x_1 + f_1)(x_j + f_j) + x_1(x_i + f_i)(x_j + f_j)$, where $1 < i \neq j < m$.

5. The $(m-1)(m-2)/2$ generators $g_ig_j(x_1 + f_1)^2 - g_j(x_1 + f_1)(x_i + f_i) - g_i(x_1 + f_1)(x_j + f_j) + (x_i + f_i)(x_j + f_j)$, where $1 < i \neq j < m$.

6. The $(m-1)(m-2)$ generators $g_i^2 g_j(4f_1 - x_1)(x_1+f_1)^2 - 3f_1 g_i g_j(x_1+f_1)(x_i + f_i) - f_1 g_i^2(x_1+f_1)(x_j+f_j) + x_i(x_i+f_i)(x_j+f_j)$, where $1 < i \neq j < m$.

7. The $m-1$ generators $g_i^2(3f_1 - x_1)(x_1+f_1)^2 + (x_i - 3f_i)(x_1+f_1)(x_i+f_i)$, where $1 < i \leq m$.

8. The $m-1$ generators $(g_i(2f_1 - x_1) + x_i)(x_1+f_1)^2 - 2f_1(x_1+f_1)(x_i+f_i)$, where $1 < i \leq m$.

9. The $m-1$ generators $g_i^3(4f_1 - x_1)(x_1+f_1)^2 + x_i(x_i+f_i)^2 - 4f_1 g_i^2(x_1+f_1)(x_i+f_i)$, where $1 < i \leq m$.

10. The $m-1$ generators $g_i^2(2f - x_1)(x_1+f_1)^2 + x_1(x_i+f_i)^2 - 2f_i(x_1+f_1)(x_i+f_i)$, where $1 < i \leq m$.

11. The $(m-1)(m-2)$ generators $g_i^2 g_j(4f_1 - x_1)(x_1+f_1)^2 + x_j(x_i+f_i)^2 - 2f_1 g_i g_j(x_1 + f_1)(x_i+f_i) - 2f_1 g_i^2(x_1+f_1)(x_j+f_j)$, where $1 < i \neq j < m$.

12. The $(m-1)(m-2)(m-3)/2$ generators $g_i g_j g_k(2f_1 - x_1)(x_1+f_1)^2 + x_k(x_i + f_i)(x_j+f_j) - f_1 g_j(x_k+f_k)(x_i+f_i) - f_1 g_j(x_k+f_k)(x_j+f_j)$.

3.1.7 $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ (general case)

As in the special case of three generators, we apply the trace map to each of the generators of $Q_1^2 \cap Q_2^2$ above along with the generators of $Q_1^2 Q_2^2$ to get the generators for $(Q_1^2 \cap Q_2^2) \cap S$ and show that each of these generators lies in $\mathfrak{m}P$. Since the computations are essentially identical to the special case of three generators discussed earlier, we relegate these computations to the appendix. This completes the proof that $P^{(2)} \subseteq \mathfrak{m}P$ in the general case.

3.2 Some computational results using Macaulay2

In this section we verify a few specific cases of the Eisenbud-Mazur conjecture using the computational algebra package Macaulay2.

We consider prime ideals in polynomial rings over $\mathbb{Q}$ (since Macaulay2 allows direct computations in such rings). We will describe these prime ideals as kernels of certain maps to another polynomial ring. Let R be a polynomial ring over $\mathbb{Q}$, let $\mathfrak{m}$ be the homogeneous maximal ideal of R and let P be a prime ideal in R. The symbolic square of P, $P^{(2)}$ is computed by computing a primary decomposition of P^2 and then finding the P-primary ideal. Then we use a Macaulay2 subroutine for checking whether $P^{(2)} \subseteq \mathfrak{m}P$. We first describe an illustrative Macaulay2 code below with a polynomial ring in 3 variables. Statements followed by -- are explanatory comments and are not part of the code.

```
R=QQ[x1,x2,x3]   -- Polynomial ring over the rationals in 3 variables.

M=ideal(x1,x2,x3)  -- Homogeneous maximal ideal in A.

S=QQ[t]  -- Target ring.

(a,b,c,d)=(5,7,9,11)

f=map(S,R,{t^a,t^b,t^c+t^d}) -- Homomorphism R->S given by x1->t^a,x2->t^b,x3->t^c+t^d.

P=kernel f -- Prime ideal for which the Eisenbud-Mazur conjecture is to be verified.

Q=primaryDecomposition P^2 -- Set of primary ideals in a primary decomposition of P^2.

m=#Q -- Number of elements in the primary decomposition of P^2.

i=0; while i<m do (
if P==radical Q#i then PS2=Q#i;
i=i+1;)

-- Loop to determine the symbolic square of P, assigned to the variable PS2.

conj=isSubset(PS2,M*P)  -- Assigns 'true' to 'conj' if the symbolic square is contained in M*P.
```

The Eisenbud-Mazur conjecture holds in the given instance if the boolean variable "conj" has value "true". We illustrate below the cases for which the program was used. The Eisenbud-Mazur conjecture was verified to be true in all cases.

1. The conjecture was verified in the following case for $5 \leq n \leq 16$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+1}, f(x_3) =$

$t^{n+2} + t^{n+3}$.

2. The conjecture was verified in the following case for $5 \le n \le 51$, n odd: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n+4} + t^{n+6}$.

3. The conjecture was verified in the following case for $3 \le n \le 20$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n-2} + t^{n+1}$.

4. The conjecture was verified in the following case for $3 \le n \le 20$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n-2} + t^{n+7}$.

5. The conjecture was verified in the following case for $3 \le n \le 20$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n-2} + t^{2n+7}$.

6. The conjecture was verified in the following case for $3 \le n \le 20$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n-2} + t^{3n+1}$.

7. The conjecture was verified in the following case for $3 \le n \le 20$: $R = \mathbb{Q}[x_1, x_2, x_3]$, $S = \mathbb{Q}[t]$, $f : R \to S$ such that $f(x_1) = t^n, f(x_2) = t^{n+2}, f(x_3) = t^{n-2} + t^{3n+7}$.

3.3 An alternative version of the Eisenbud-Mazur conjecture

In the introduction we discussed another result of Eisenbud and Mazur about the existence of non-trivial evolutions. We repeat it here for convenience.

Theorem 3.3.1. *[EM97] There exists a reduced local $\mathbb{C}$-algebra T of finite type whose localization at the origin has a non-trivial evolution if and only if there exists a polynomial $f \in R = \mathbb{C}[[x_1, ..., x_n]]$ without constant term such that $f \notin \mathfrak{m}I$, where $\mathfrak{m} = (x_1, ..., x_n)R$, $I = \sqrt{(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$.*

This theorem motivates the following version of the Eisenbud-Mazur conjecture.

Conjecture 3.3.2. *Let $R = \mathbb{C}[[x_1, ..., x_n]]$, $\mathfrak{m} = (x_1, ..., x_n)R$. If $f \in \mathfrak{m}$, then, $f \in \mathfrak{m}I$, where $I = \sqrt{(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R}$.*

The conjecture is true if f is regular, for in this case $I = R$. It is also true if f has an isolated singularity (for in this case $f \in \mathfrak{m}^2$ and $I = \mathfrak{m}$) or if f is quasihomogeneous (by Euler's formula, $f \in \mathfrak{m}I$). Eisenbud and Mazur show that the conjecture holds when the singular locus of f is a curve. More generally the conjecture is shown to hold when the embedding dimension of the reduced singular locus of f is less than 4. In case of embedding dimension 4, the conjecture has been shown to hold if the reduced singular locus of f is Gorenstein or licci [EM97].

We show that the conjecture holds for a family of polynomials.

Proposition 3.3.3. *Let $R = k[[x_1, ..., x_n]]$ be a ring of formal power series over a field k (or alternately let $R = k[x_1, ..., x_n]$ be a polynomial ring over k). Let $f = x_1^{a_1}x_2^{b_2} + x_2^{a_2}x_3^{b_3} + ... + x_{n-1}^{a_{n-1}}x_n^{b_n} + x_n^{a_n}x_1^{b_1}$. Then, if $(a_1...a_n + (-1)^{n-1}b_1...b_n)$ is invertible in k, $f \in \mathfrak{m}I$, where $\mathfrak{m} = (x_1, ..., x_n)R$ and $I = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})R$.*

Proof. Denote $\frac{\partial f}{\partial x_i}$ by f_i. Then $f_i = b_i x_{i-1}^{a_{i-1}} x_i^{b_i - 1} + a_i x_i^{a_i - 1} x_{i+1}^{b_{i+1}}$ for $1 \leq i \leq n$, where we define $x_{n+1} := x_1, a_{n+1} := a_1, b_{n+1} := b_1$ and $x_0 := x_n, a_0 := a_n, b_0 := b_n$ for brevity.

Then we have that

$$(a_1...a_n + (-1)^{n-1}b_1...b_n)x_1^{a_1}x_2^{b_2} = a_2...a_n x_1 f_1 - b_1 a_2...a_{n-1} x_n f_n +$$

$$b_1 a_2...a_{n-2} b_{n-1} x_{n-1} f_{n-1} - ... + (-1)^{n-1} b_1 b_3...b_n x_2 f_2.$$

Then, if $(a_1...a_n + (-1)^{n-1}b_1...b_n)$ is invertible in k, we have $x_1^{a_1}x_2^{b_2} \in \mathfrak{m}I$. Expressing other terms similarly, we get that $x_i^{a_i}x_{i+1}^{b_{i+1}} \in \mathfrak{m}J$ for $1 \leq i \leq n$. Hence, $f \in \mathfrak{m}I$. $\square$

CHAPTER 4

Uniform bounds on symbolic powers of prime ideals

In this section we consider the question of uniform bounds on the growth symbolic powers of prime ideals.

An important stepping stone in proofs of the results in [Swa00] and [HKV09] is to show that, under the corresponding assumptions on the local ring $(R, \mathfrak{m})$, there exists a positive integer h (uniform in the latter case), such that for every prime ideal P of R, $P^{(hn)} \subseteq \mathfrak{m}^n$. We shall prove that for a Noetherian complete local domain $(R, \mathfrak{m})$ there exists a function $\beta : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ (depending on the ring but independent of any prime ideal in the ring), such that for any prime ideal P in the ring, we have that $P^{(\beta(n))} \subseteq \mathfrak{m}^n$ for all positive integers n. The main ingredient in the proof is the strong approximation theorem, which first appeared in [Art69] (without this name) and was later generalized first by Popescu and Pfister [PP75] and then again by Popescu [Pop86].

Another case in which we are able to get a uniform bound on the growth of symbolic powers is the following: suppose that $R \subseteq S$ is a module-finite extension of domains and R is normal while S is regular, equicharacteristic. Then, under mild conditions on R, S, there exists a positive integer c such that for any prime ideal P in R we have that $P^{(cn)} \subseteq P^n$ for all positive integers n.

4.1 Strong approximation

In the paper [Art69] where Artin proved his famous approximation theorem, the following stronger theorem was also proven.

Theorem 4.1.1. *[Art69] Let* n, N, d, c be non-negative integers. *Let* k *be a field and let* $\underline{F} = (f_1, ..., f_m)$ *be polynomials in* $R = k[x_1, ..., x_n, y_1, ..., y_N]$. *Let* d *be an upper bound for the degrees of all the polynomials in* $\underline{F}$. *Then there is a non-negative integer valued function* $\beta(n, N, d, c)$ with the following property: *given polynomials* $\overline{y} = (\overline{y_1}, ..., \overline{y_N}) \in k[x_1, ..., x_n]$ *such that*

$$\underline{F}(x_1, ..., x_n, \overline{y}) \equiv 0 \text{ mod } (x_1, ..., x_n)^{\beta} R$$

there are elements $\overline{z} = (z_1, ..., z_N) \in k[x_1, ..., x_n]^{\sim}$ *(where* $^{\sim}$ *denotes the henselization at the homogeneous maximal ideal) solving the system of equations*

$$\underline{F} = 0$$

and such that $z_i \equiv \overline{y_i}$ *mod* $(x_1, ..., x_n)^c R$.

This result later was generalized by Pfister and Popescu [PP75]. We first make the following definition used in their article.

Definition 4.1.2. Let R be a Noetherian ring and let I be an ideal in R. Then the pair (R, I) is said to satisfy the *strong approximation property* if the following condition holds:

For every system of polynomials $\underline{F}$ over R there exists a function $\beta_{\underline{F}} : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for every positive integer c and for every system of elements $\overline{y}$ of R such that $\underline{F}(\overline{y}) \equiv 0 \text{ mod } I^{\beta(c)}$, there exists a system of elements of $\overline{z}$ in R such that $\underline{F}(\overline{z}) = 0$ and $\overline{z} \equiv \overline{y} \text{ mod } I^c$.

Now we can state the result of Pfister and Popescu. We will call $\beta_{\underline{F}}$ the *Artin function* for the system of polynomials $\underline{F}$ for the pair (R, I).

Theorem 4.1.3. *[PP75] If $(R, \mathfrak{m})$ is a Noetherian complete local ring, then, the pair $(R, \mathfrak{m})$ satisfies the strong approximation property.*

Popescu obtained a further generalization (corollary 4.5, [Pop86]) stated in the following theorem.

Theorem 4.1.4. *[Pop86] Let $(R, \mathfrak{m})$ be an excellent Henselian local ring. Then the pair $(R, \mathfrak{m})$ satisfies the strong approximation property.*

We use the strong approximation property to obtain a uniform bound for symbolic powers of prime ideals in certain Noetherian rings.

We make another definition.

Definition 4.1.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{d}(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ (the embedding dimension of R). For a Noetherian ring A, not necessarily local, define $\delta(A) = \max\{\mathfrak{d}(A_P) : P \text{ a prime ideal in } A\}$.

The following proposition due to Hochster (proposition 3.8, [Hoc71]) illustrates some properties of δ.

Proposition 4.1.6. *[Hoc71] Let A be a Noetherian ring. If B is a residue class ring or localization of A, then, $\delta(B) \leq \delta(A)$. If B is a regular local ring, $\delta(B) = dim(B)$. If B is an extension of A generated by r elements, then, $\delta(B) \leq \delta(A) + r$ and equality holds if $B = A[x_1, ..., x_r]$, where $x_1, ..., x_r$ are indeterminates over A.*

From the above proposition it immediately follows that if R is a ring essentially of finite type over a homomorphic image of a regular local ring, then, $\delta(R) < \infty$.

We quote another result due to Hochster (proposition 3.10, [Hoc71]), which will be needed for the proof of the subsequent theorem.

Theorem 4.1.7. *[Hoc71] Let R be a normal Noetherian domain such that $\delta(R) < \infty$. Let $K = Frac(R)$ and L be a finite Galois extension field of K. Let S be the integral closure of R in L. Then there exists $s \in \mathbb{Z}_{>0}$ such that for each prime ideal Q of S and each $n \in \mathbb{Z}_{>0}$, $a \in Q^{(n)} \cap R \implies a^s \in (Q \cap R)^{(n)}$.*

Theorem 4.1.8. *Let $(R, \mathfrak{m})$ be an excellent Henselian regular local ring of dimension d. Let $K = Frac(R)$ and let L be a finite Galois extension field of K. Let S be the integral closure of R in L. Then there exists a function $\nu : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, such that for any prime ideal Q in S, we have that $Q^{(\nu(c))} \subseteq \mathfrak{m}^c S$ for all positive integers c.*

Proof. Let $P = Q \cap R$, which is a prime ideal. There are a finite number of prime ideals of S that lie over P say $Q_1, ..., Q_r$ with $Q_1 = Q$. The Galois group G of L/K acts transitively on this set of prime ideals (proposition VII.2.1, page 340, [Lan02]).

Suppose that $a \in Q^{(N)} \backslash \{0\}$. If $G = \{g_1, ..., g_m\}$, the norm function $\mathcal{N} : S \to R$ acts on $s \in S$ as follows: $\mathcal{N}(s) = g_1(s) \cdot ... \cdot g_m(s)$. Then $\mathcal{N}(a) \in Q^{(N)} \cap R$. Since R is a regular local ring, it is a unique factorization domain and hence normal (corollary 2.2.20, page 70, [BH93]). Further, since R is a regular local ring, $\delta(R) = d < \infty$. Then, by theorem 4.1.7, there exists a positive integer s (independent of Q) such that $\mathcal{N}(a)^s \in P^{(N)}$.

Now, since R is a regular local ring, given an ideal I of R, we have that $\overline{I^{n+d-1}} \subseteq I^n$ by a version of the Briançon-Skoda theorem (theorem 2.1, [LT81]). In particular, $\overline{\mathfrak{m}^{n+d-1}} \subseteq \mathfrak{m}^n$ for all positive integers n. Further, since R is a regular local ring, for every prime ideal $\mathfrak{p}$ we have that $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$ (page 9, [Hoc71]).

Since R is a excellent, Henselian, local ring, the pair $(R, \mathfrak{m})$ satisfies the strong approximation property by theorem 4.1.4. Let $\beta_{\underline{F}}$ be the Artin function for the pair $(R, \mathfrak{m})$ for a system of polynomials $\underline{F}$. Set $N = s(\beta_{\underline{F}}(c) + d - 1)$ for some positive integer c. We have that $P^{(N)} = P^{(s(\beta_{\underline{F}}(c)+d-1))} \subseteq \mathfrak{m}^{s\beta_{\underline{F}}(c)+s(d-1)}$. So $\mathcal{N}(a)^s \in$

$\mathfrak{m}^{s\beta_{\underline{F}}(c)+s(d-1)}$. Now, if $b^l \in I^l$ for some ideal I, then, b satisfies the monic equation $x^l - b^l = 0$ and is thus integral over I. Thus, $\mathcal{N}(a) \in \overline{\mathfrak{m}^{\beta_{\underline{F}}(c)+d-1}}$. Then, by the Briançon-Skoda theorem quoted in the preceding paragraph, we get that $\mathcal{N}(a) \in \overline{\mathfrak{m}^{\beta_{\underline{F}}(c)+d-1}} \subseteq \mathfrak{m}^{\beta_{\underline{F}}(c)}$.

Now S is a finitely generated R-module (corollary V.4.1, page 265, [ZS75]). Let α_1,...,α_t be a set of generators for S as an R-module. Suppose that $a = r_1\alpha_1 + ... + r_t\alpha_t$ for some $r_1, ..., r_t \in R$. Then $g_i(a) = r_1 g_i(\alpha_1) + ... + r_t g_i(\alpha_t)$ for $i = 1, ..., m$. Now we can write that $g_i(\alpha_j) = s_{1,i,j}\alpha_1 + ... + s_{t,i,j}\alpha_t$ for $1 \leq i \leq m$, $1 \leq j \leq t$ and $s_{k,i,j} \in R$ for $1 \leq i \leq m$, $1 \leq j, k \leq t$. Further, since $\alpha_j\alpha_k$ are elements in S for $1 \leq j, k \leq t$ we may express each of these products as R-linear combinations of α_1,...,α_t. Using the expressions for $g_i(\alpha_j)$ and repeatedly using the expressions for $\alpha_j\alpha_k$, we can express $\mathcal{N}(a) = g_1(a) \cdot ... \cdot g_m(a) = F_1\alpha_1 + ... + F_t\alpha_t$, where $F_1, ..., F_t$ are polynomials in $r_1, ..., r_t$ with coefficients in R (the coefficients of $F_1, ..., F_t$ are functions of $s_{k,i,j}$ and the coefficients of $\alpha_1, ..., \alpha_t$ in the linear expressions for $\alpha_j\alpha_k$). By the preceding paragraph, $\mathcal{N}(a) \in \mathfrak{m}^{\beta_{\underline{F}}(c)}$. Set $\underline{F} = \{F_1, ..., F_t\}$ and denote $\beta_{\underline{F}}$ by β. Therefore,

$$F_1\alpha_1 + ... + F_t\alpha_t \in \mathfrak{m}^{\beta(c)} \subseteq \mathfrak{m}^{\beta(c)}S.$$

So we must have, $F_1, ..., F_t \in \mathfrak{m}^{\beta(c)}S$. Since R is a regular local ring, all powers of the maximal ideal are integrally closed in R (section 1, page 142, [HRW05]). Then, since $F_1, ..., F_t \in R$, we have that $F_1, ..., F_t \in \mathfrak{m}^{\beta(c)}S \cap R \subseteq \overline{\mathfrak{m}^{\beta(c)}S} \cap R = \overline{\mathfrak{m}^{\beta(c)}} = \mathfrak{m}^{\beta(c)}$ (where the first equality follows from proposition 1.6.1, page 16, [HS06]).

We shall now show that $a \in \mathfrak{m}^c S$. We consider two cases: (1) $r_1, ..., r_t \in \mathfrak{m}^c$ and (2) $r_j \notin \mathfrak{m}^c$ for some $j \in \{1, ..., t\}$. In case (1), $a = r_1\alpha_1 + ... + r_t\alpha_t \in \mathfrak{m}^c S$. Now consider case (2) and suppose that $r_j \notin \mathfrak{m}^c$ for some $j \in \{1, ..., t\}$. Then, since $F_j(r_1, ..., r_t) \equiv 0 \bmod \mathfrak{m}^{\beta(c)}$ for $1 \leq j \leq t$, by the strong approximation property for the pair $(R, \mathfrak{m})$ (theorem 4.1.4), there exists a set of elements, say, $s_1, ..., s_t \in R$

such that $F_j(s_1, ..., s_t) = 0$ for $1 \le j \le t$ and $s_k \equiv r_k \bmod \mathfrak{m}^c$. Since $\mathcal{N}(a) = F_1\alpha_1 + ... + F_t\alpha_t$, we get that $\mathcal{N}(s_1\alpha_1 + ... + s_t\alpha_t) = 0$. Set $b = s_1\alpha_1 + ... + s_t\alpha_t$. Then, since $\mathcal{N}(b)$ is the product of conjugates of b and S is a domain, $\mathcal{N}(b) = 0$ if and only if $b = 0$. So $b = 0$. Now $a \equiv b \bmod \mathfrak{m}^c S$, so $a \in \mathfrak{m}^c S$. This proves the claim.

Finally, let $\nu(c) = ds(\beta(c) + d - 1)$. Note that the system of polynomials $F_1, ..., F_t$ depends only on the coefficients in the expressions $g_i(\alpha_j)$ for $1 \le i \le m$, $1 \le j \le t$ and the coefficients in the expressions for $\alpha_j\alpha_j$ for $1 \le j, k \le t$. Fixing the choice of these expressions, the function β is independent of any prime ideal in S (note that d, s only depend on R). Then, since $a \in Q^{(\nu(c))} \implies a \in \mathfrak{m}^c S$, we have that $Q^{(\nu(c))} \subseteq \mathfrak{m}^c S$. $\square$

Corollary 4.1.9. *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain. Then there exists a function $\eta : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, such that for any prime ideal P in R, we have that $P^{(\eta(c))} \subseteq \mathfrak{m}^c$ for all positive integers c.*

Proof. Let $\dim(R) = d$. If R is equicharacteristic, then, R is a module-finite extension of the formal power series ring $k[[x_1, ..., x_d]]$,, where k is a field. In mixed characteristic R is a module-finite extension of $V[[x_1, ..., x_{d-1}]]$, where V is a complete Noetherian discrete valuation domain (theorem 4.3.3, page 61, [HS06]). In the former case set $T = k[[x_1, ..., x_d]]$ and in the latter case set $T = V[[x_1, ..., x_{d-1}]]$. Let $K = \mathrm{Frac}(T)$ and $L = \mathrm{Frac}(R)$.

Since R is a module-finite over T, L/K is a finite extension. If L/K is a separable field extension[1], then, it can be extended to a finite, Galois extension (for example we may take the normal closure of L/K) (theorem 1.6.13, page 69, [Lev08]). Say $K \subseteq L \subseteq M$, where M is a finite, Galois extension field of K. Let S be the integral closure of T in M. Then, by theorem 4.1.8, there exists a function $\nu : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such

[1]This will necessarily be the case in equal characteristic zero and mixed characteristic.

that for any prime ideal Q in S we have that $Q^{(\nu(c))} \subseteq \mathfrak{M}^c S$ for all positive integers c, where $\mathfrak{M} = (x_1, ..., x_d)T$. Note that since $T \subseteq R$ is a module-finite extension, it is integral and hence $R \subseteq S$ (lemma 2.1.9, proposition 2.1.10, page 26, [HS06]). Thus, $R \subseteq S$ is an integral extension (corollary 2.1.12, page 27, [HS06]). Let P be any prime ideal in R. Since $R \subseteq S$ is integral there exists a prime ideal $\mathfrak{p}$ in S such that $\mathfrak{p} \cap R = P$ (theorem 5.10, page 62, [AM94]). Then $P^{(\nu(c))} \subseteq \mathfrak{p}^{(\nu(c))} \subseteq \mathfrak{M}^c S$. Thus, $P^{(\nu(c))} \subseteq \mathfrak{M}^c S \cap R$. Now $\mathfrak{M} \subseteq \mathfrak{m}$ (theorem 4.3.3, page 61, [HS06]). So $P^{(\nu(c))} \subseteq \mathfrak{m}^c S \cap R$. Further, S is finitely generated as an R-module since S is module-finite over T (corollary V.4.1, page 265, [ZS75]) and $T \subseteq R$. Thus, by the Artin-Rees lemma, there exists a positive integer a such that $\mathfrak{m}^c S \cap R = \mathfrak{m}^{c-a}(\mathfrak{m}^a S \cap R) \subseteq \mathfrak{m}^{c-a}$ for all positive integers $c \geq a$. Set $\eta(c) = \nu(c + a)$. Then $P^{(\eta(c))} \subseteq \mathfrak{m}^c$.

Suppose that L/K is not a separable field extension. Then K, L must have prime characteristic, say p and R is a complete local ring in equal characteristic p. Then there exists an F-finite, local ring $(R^\Gamma, \mathfrak{m}_\Gamma)$ faithfully flat over R such that $\mathfrak{m}R^\Gamma = \mathfrak{m}_\Gamma$. Further, since R is module-finite over $T = k[[x_1, ..., x_d]]$, R^Γ is module-finite over $k^\Gamma[[x_1, ..., x_d]]$. Set $\tilde{T} = k^\Gamma[[x_1, ..., x_d]]$[2]. Let $y_i = x_i^{\frac{1}{p^e}}$ for $1 \leq i \leq d$, where e is a positive integer. Set $R' = R^\Gamma[y_1, ..., y_d]$ and $T' = k'[[y_1, ..., y_d]]$, where $k' = (k^\Gamma)^{\frac{1}{p^e}}$. Then R' is module-finite over T' for some $e \gg 0$ (proof of theorem 4.3, page 23, [Hoc75]). Fix one such e. Then, if we let $K' = \mathrm{Frac}(T')$ and $L' = \mathrm{Frac}(R')$, then, L'/K' is a finite, separable field extension. This can be extended to a Galois field extension as before. Say $K' \subseteq L' \subseteq M'$, where M' is a finite, Galois extension field of K'. Let S' be the integral closure of T' in M'. Then, by theorem 4.1.8, there exists a function $\nu : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for any prime ideal Q in S' we have that $Q^{(\nu(c))} \subseteq \mathfrak{M}'^c S'$ for all positive integers c, where $\mathfrak{M}' = (y_1, ..., y_d)T'$. Now

[2]For a discussion of the construction and properties of R^Γ, consult section 6.11 in [Hoc94].

$\mathfrak{M}'^{dp^e} \subseteq \mathfrak{M}$, where $\mathfrak{M} = (x_1, ..., x_d)\tilde{T}$. Then $Q^{(\nu(dp^e c))} \subseteq \mathfrak{M}^c S'$. Let $\mathcal{P}$ be a prime in R^Γ. Then $\mathcal{P}R^{'}$ is prime in $R^{'}$. Since $S^{'}$ is the integral closure of $T^{'}$ in $M^{'}$ and $R^{'}$ is module-finite (hence, integral) over $T^{'}$, the extension $R^{'} \subseteq S^{'}$ is also integral. Consequently, there exists a prime ideal $Q_{\mathcal{P}}$ in $S^{'}$ such that $\mathcal{P}R^{'} = Q_{\mathcal{P}} \cap R^{'}$. Then $\mathcal{P}^{(\nu(dp^e c))} \subseteq (\mathcal{P}R^{'})^{(\nu(dp^e c))} \subseteq Q_{\mathcal{P}}^{(\nu(dp^e c))} \subseteq \mathfrak{M}^c S'$. Thus, $\mathcal{P}^{(\nu(dp^e c)} \subseteq \mathfrak{M}^c S' \cap R^\Gamma$. Now, since k^Γ is F-finite, $\tilde{T}$ is F-finite (example 2.1, [BMS08]). Then $T^{'}$ is module-finite over $\tilde{T}$. Further, $S^{'}$ is module-finite over $T^{'}$ (corollary V.4.1, page 265, [ZS75]). Thus, $S^{'}$ is module-finite over $\tilde{T}$. Finally, $\tilde{T} \subseteq R^\Gamma$ and hence $S^{'}$ is module-finite over R^Γ. Now $\mathcal{P}^{(\nu(dp^e c))} \subseteq \mathfrak{M}^c S^{'} \cap R^\Gamma = \mathfrak{m}_\Gamma^c S^{'} \cap R$, where $\mathfrak{m}_\Gamma$ is the maximal ideal in R^Γ. By the Artin-Rees lemma, there exists a positive integer b such that $\mathfrak{m}_\Gamma^c S^{'} \cap R^\Gamma = \mathfrak{m}_\Gamma^{c-b}(\mathfrak{m}_\Gamma^b S^{'} \cap R) \subseteq \mathfrak{m}_\Gamma^{c-b}$ for all positive integers $c \geq b$. Thus, $\mathcal{P}^{(\nu(dp^e c))} \subseteq \mathfrak{m}_\Gamma^{c-b}$. Set $\eta(c) = \nu(dp^e(c+b))$. Then $\mathcal{P}^{(\eta(c))} \subseteq \mathfrak{m}_\Gamma^c$. Finally, given a prime ideal P in R, there exists a prime ideal, say P_Γ, lying over P in R^Γ (proposition B.1.1, page 399, [HS06]). Then $P^{(\eta(c))} \subseteq P_\Gamma^{(\eta(c))} \subseteq \mathfrak{m}_\Gamma^c = \mathfrak{m}^c R^\Gamma$. Thus, $P^{(\eta_\Gamma(c))} \subseteq \mathfrak{m}^c R^\Gamma \cap R$. Then, since R^Γ is faithfully flat over R, $\mathfrak{m}^c R^\Gamma \cap R = \mathfrak{m}^c$ (theorem 4.74.(2), page 150, [Lam98]). Hence, $P^{(\eta(c))} \subseteq \mathfrak{m}^c$. $\square$

A modification of the argument in the proof of theorem 4.1.8 shows the failure of the strong approximation property in excellent, Henselian, regular local rings in general (hence, in particular, for complete regular local rings). We explore this in the next theorem.

In [Pop86] Popescu asks the following question: Let R be a Noetherian ring and I an ideal such that R is complete with respect to the I-adic topology. Does the pair (R, I) necessarily satisfy the strong approximation property? A counterexample is shown in [Spi94]. We present the following theorem in the same spirit.

Theorem 4.1.10. *Let $(R, \mathfrak{m})$ be an excellent Henselian regular local ring. Let P be a prime ideal of R such that $P \neq \mathfrak{m}$. Then the pair (R, P) does not satisfy the strong approximation property.*

Proof. Let $K = \mathrm{Frac}(R)$ and let L be a finite non-trivial Galois extension of K. Let G be the Galois group of L/K. Let S be the integral closure of R in L. Suppose that $Q_1, ..., Q_r$ are the set of distinct prime ideals of S lying over P. Since R is a regular local ring, it is a unique factorization domain, hence, normal (corollary 2.2.20, page 70, [BH93]). So G acts transitively on the set $\{Q_1, ..., Q_r\}$ (proposition VII.2.1, page 340, [Lan02]).

We may choose $a \in Q_1^{(N)} \setminus \cup\{Q_2, ..., Q_n\}$. Such an element exists because by the prime avoidance theorem if $Q_1^{(N)} \subseteq \cup\{Q_2, ..., Q_n\}$, then, $Q_1^{(N)} \subseteq Q_i$ for some $i \in \{2, ..., n\}$. Then $Q_1^N \subseteq Q_i$ and taking radicals we get that $Q_1 \subseteq Q_i$. Since Q_1, Q_i both lie over P, by the lying over theorem, we get that $Q_1 = Q_i$, which is a contradiction. If $G = \{g_1, ..., g_m\}$, the norm function $\mathcal{N} : S \to R$ acts on $s \in S$ as follows: $\mathcal{N}(s) = g_1(s) \cdot ... \cdot g_m(s)$. Then $\mathcal{N}(a) \in Q_1^{(N)} \cap R$. Since R is a regular local ring, it is a unique factorization domain and hence normal (corollary 2.2.20, page 70, [BH93]). Further, since R is a regular local ring, $\delta(R) = d < \infty$. Then, by theorem 4.1.7, there exists a positive integer s such that $\mathcal{N}(a)^s \in P^{(N)}$.

Now the suppose that the pair (R, P) does satisfy the strong approximation property with the Artin function $\beta_{\underline{F}}$ for a system of polynomials $\underline{F}$. Set $N = s(\beta_{\underline{F}}(c)+d-1)$ for some positive integer c. We have that $P^{(N)} = P^{(s(\beta_{\underline{F}}(c)+d-1))} \subseteq \mathfrak{m}^{s\beta_{\underline{F}}(c)+s(d-1)}$ (where the containment follows from the fact that $P^{(N)} \subseteq \mathfrak{m}^N$ as R is a regular local ring). So $\mathcal{N}(a)^s \in \mathfrak{m}^{s\beta_{\underline{F}}(c)+s(d-1)}$. Now, if $b^l \in I^l$ for some ideal I, then, b satisfies the monic equation $x^l - b^l = 0$ and is thus integral over I. Thus, $\mathcal{N}(a) \in \overline{P^{\beta_{\underline{F}}(c)+d-1}}$.

Now S is a finitely generated R-module (corollary V.4.1, page 264, [ZS75]). Let

α_1,...,α_t be a set of generators for S as an R-module. Suppose that $a = r_1\alpha_1+...+r_t\alpha_t$ for some $r_1, ..., r_t \in R$. Then $g_i(a) = r_1 g_i(\alpha_1) + ... + r_t g_i(\alpha_t)$ for $i = 1, ..., m$. Thus, we can write that $g_i(\alpha_j) = s_{1,i,j}\alpha_1 + ... + s_{t,i,j}\alpha_t$ for $1 \leq i \leq m$, $1 \leq j \leq t$ and $s_{k,i,j} \in R$ for $1 \leq i \leq m$, $1 \leq j, k \leq t$. Further, since $\alpha_j\alpha_k$ are elements in S for $1 \leq j, k \leq t$, we may express each of these products as an R-linear combination of α_1,...,α_t. Using the expressions for $g_i(\alpha_j)$ and repeatedly using the expressions for $\alpha_j\alpha_k$, we can express $\mathcal{N}(a) = g_1(a) \cdot ... \cdot g_m(a) = F_1\alpha_1 + ... + F_t\alpha_t$, where $F_1, ..., F_t$ are polynomials in $r_1, ..., r_t$ with coefficients in R (the coefficients of $F_1, ..., F_t$ are functions of $s_{k,i,j}$ and the coefficients of $\alpha_1, ..., \alpha_t$ in the linear expressions for $\alpha_j\alpha_k$). By the preceding paragraph, $\mathcal{N}(a) \in \overline{P^{\beta_{\underline{F}}(c)+d-1}}$. Set $\underline{F} = \{F_1, ..., F_t\}$ and denote $\beta_{\underline{F}}$ by β. So $F_1\alpha_1 + ... + F_t\alpha_t \in \overline{P^{\beta(c)+d-1}} \subseteq \overline{P^{\beta(c)+d-1}S}$. So we must have, $F_1, ..., F_t \in \overline{P^{\beta(c)+d-1}S}$. Then, since $F_1, ..., F_t \in R$, we have that

$$F_1, ..., F_t \in \overline{P^{\beta(c)+d-1}S} \cap R \subseteq \overline{\overline{P^{\beta(c)+d-1}}S} \cap R = \overline{\overline{P^{\beta(c)+d-1}}} = \overline{P^{\beta(c)+d-1}}$$

(where the first containment follows from persistence of integral closure and the first equality follows from proposition 1.6.1, page 16, [HS06]). Further, since R is a regular local ring, given an ideal I of R, we have that $\overline{I^{n+d-1}} \subseteq I^n$ by a version of the Briançon-Skoda theorem (theorem 2.1, [LT81]). In particular, $\overline{P^{n+d-1}} \subseteq P^n$ for all positive integers n. So $F_1, ..., F_t \in P^{\beta(c)}$.

Then we claim that $a \in P^cS$. We consider two cases: (1) $r_1, ..., r_t \in P^c$ and (2) $r_j \notin P^c$ for some $j \in \{1, ..., t\}$. In case (1), $a = r_1\alpha_1 + ... + r_t\alpha_t \in P^cS$. Now consider case (2) and suppose that $r_j \notin P^c$ for some $j \in \{1, ..., t\}$. Then, since $F_j(r_1, ..., r_t) \equiv 0 \bmod P^{\beta(c)}$ for $1 \leq j \leq t$, by the strong approximation property for the pair (R, P) (theorem 4.1.4), there exists a set of elements, say, $s_1, ..., s_t \in R$ such that $F_j(s_1, ..., s_t) = 0$ for $1 \leq j \leq t$ and $s_k \equiv r_k \bmod P^c$. Since $\mathcal{N}(a) = F_1\alpha_1 + ... + F_t\alpha_t$, we get that $\mathcal{N}(s_1\alpha_1 + ... + s_t\alpha_t) = 0$. Set $b = s_1\alpha_1 + ... + s_t\alpha_t$. Then, since $\mathcal{N}(b)$ is the product of conjugates of b and S is a domain, $\mathcal{N}(b) = 0$ if and only if $b = 0$. So $b = 0$. Now $a \equiv b \mod P^cS$, so $a \in P^cS$. This proves the claim.

However, this contradicts the choice of a since $P^cS \subseteq Q_1 \cap ... \cap Q_r$ and by the choice of a, $a \notin \cup\{Q_2, ..., Q_r\}$. So the pair (R, P) cannot satisfy the strong approximation property. □

4.2 Uniform bounds for an isolated singularity

We are able to obtain a linear bound for the growth of contractions of symbolic powers of prime ideals from certain integral extensions of an isolated singularity. We will need a couple of results from literature before we can state this proposition.

Huneke-Katz-Validashti obtain a linear bound for the growth of symbolic powers of prime ideals in an isolated singularity (corollary 3.10, [HKV09]).

Theorem 4.2.1. *[HKV09] Let $(R, \mathfrak{m})$ be an equicharacteristic local domain such that R is an isolated singularity. Assume that R is either essentially of finite type over a field of characteristic zero or R has positive characteristic, is F-finite and analytically irreducible. Then there exists a positive integer $h \geq 1$ with the following property: for every prime ideal P of R such that $P \neq \mathfrak{m}$, $P^{(hn)} \subseteq P^n$ for all positive integers n.*

Note 4.2.2. Note that we can disregard the condition $P \neq \mathfrak{m}$ in theorem 4.2.1 since $\mathfrak{m}^{(hn)} = \mathfrak{m}^{hn} \subseteq \mathfrak{m}^n$ for all positive integers n.

We will also need a uniform Briançon-Skoda theorem due to Huneke (theorem 4.13, [Hun92]).

Theorem 4.2.3. *[Hun92] Let R be a Noetherian reduced ring. Suppose that R satisfies one of the following:*

1. R is essentially of finite type over an excellent Noetherian local ring.

2. R is characteristic p and F-finite.

3. R is essentially of finite type over $\mathbb{Z}$.

Then there exists a positive integer k such that for all ideals I of R, $\overline{I^n} \subseteq I^{n-k}$ for all positive integers $n \geq k$.

We can now state and prove our result.

Proposition 4.2.4. *Let* $(R, \mathfrak{m})$ *be an equicharacteristic local domain such that* R *is an isolated singularity. Assume that* R *is either essentially of finite type over a field of characteristic zero or* R *has positive characteristic, is F-finite and analytically irreducible. Let* $K = Frac(R)$ *and* L *be a finite Galois extension field of* K*. Suppose that* S *is the integral closure of* R *in* L*. Then there exist positive integers* r, k *such that for any prime ideal* Q *of* S *we have that* $Q^{(rn)} \cap R \subseteq (Q \cap R)^{n-k}$ for all positive integers $n \geq k$.

Proof. Suppose that $a \in Q^{(N)} \cap R$. Then, by theorem4.1.7, there exists a positive integer s independent of Q such that $a^s \in P^{(N)}$, where $P = Q \cap R$. Set $N = hsn$ for some positive integer n, where h is as in theorem 4.2.1. Then $P^{(N)} = P^{(hsn)} \subseteq P^{sn}$ (where the containment follows from theorem 4.2.1 and note 4.2.2). So $a^s \in P^{sn}$. Now, if $b^l \in I^l$ for some ideal I, then, b satisfies the monic equation $x^l - b^l = 0$ and is thus integral over I. Thus, $a \in \overline{P^n}$. Finally, since R satisfies the hypothesis of theorem 4.2.3, there exists a positive integer, say k such that for all ideals I of R $\overline{I^n} \subseteq I^{n-k}$. So $a \in \overline{P^n} \subseteq P^{n-k}$. Thus, $Q^{(hsn)} \cap R \subseteq (Q \cap R)^{n-k}$ for all positive integers $n \geq k$. Setting $r = hs$, we get that $Q^{(rn)} \cap R \subseteq (Q \cap R)^{n-k}$ for all positive integers $n \geq k$. $\square$

Corollary 4.2.5. *Let $(R, \mathfrak{m})$ be an equicharacteristic local domain such that R is an isolated singularity. Assume that R is either essentially of finite type over a field of characteristic zero or R has positive characteristic, is F-finite and analytically irreducible. Let $K = Frac(R)$ and L be a finite Galois extension field of K. Suppose that S is the integral closure of R in L. Then there exists a positive integer c such that for any prime ideal Q of S, we have that $Q^{(cn)} \cap R \subseteq (Q \cap R)^n$* for all positive integers n.

Proof. By proposition 4.2.4, there exist positive integers r, k such that for any prime ideal Q of S, we have that $Q^{(rn)} \cap R \subseteq (Q \cap R)^{n-k}$ all positive integers n. Thus, $Q^{(r(n+k))} \cap R \subseteq (Q \cap R)^n$. Since $nk \geq k$, we have that $Q^{(r(n+nk))} \subseteq Q^{(hs(n+k))}$. Set $c = r(k+1)$ to get $Q^{(cn)} \cap R \subseteq (Q \cap R)^n$. $\square$

4.3 Uniform bounds for normal subrings of equicharacteristic, regular rings

In the previous sections, we obtained uniform bounds for symbolic powers of prime ideals in integral extensions of certain rings using the uniform bounds for symbolic powers of prime ideals in the base ring. In this section, we explore the other direction. We obtain bounds for symbolic powers of prime ideals in normal subrings of equicharacteristic, regular rings using the bounds in the overring. We first need a couple of lemmas.

Lemma 4.3.1. *Let $R \subseteq S$ be an integral extension of domains. Let P be a prime ideal in R. Let $J = \{s \in S : s^n + p_1 s^{n-1} + ... + p_n = 0$ for some n and some $p_i \in P, 1 \leq i \leq n\}$ (i.e. J is the set of elements of S that satisfy a monic polynomial with non-leading coefficients in P). Then J is a radical ideal of S and in fact, $\sqrt{PS} = J$.*

Proof. Let $u \in J$ and $v \in S$. Then u satisfies a monic polynomial with non-leading coefficients in P. Say, $u^n + p_1u^{n-1} + ... + p_n = 0$ with $p_i \in P$ for $1 \leq i \leq n$. Also, since S is integral over R, v satisfies a monic polynomial with coefficients in R. Say, $v^n + r_1v^{n-1} + ... + r_m = 0$ with $r_j \in R$ for $1 \leq j \leq m$. Set $f(x) = x^n + p_1x^{n-1} + ... + p_n$ and $g(x) = x^n + r_1x^{n-1} + ... + r_m$. Then $f(x)$ is the characteristic polynomial of the $n \times n$ matrix $U \in R^{n \times n}$ given by

$$U = \begin{bmatrix} -p_1 & -p_2 & . & . & . & -p_{n-1} & -p_n \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & & & & 0 & 0 \\ . & . & . & & & . & . \\ . & . & & . & & . & . \\ . & . & & & . & . & . \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}$$

and $g(x)$ is the characteristic polynomial of the $m \times m$ matrix $V \in R^{m \times m}$ given by

$$V = \begin{bmatrix} -r_1 & -r_2 & . & . & . & -r_{n-1} & -r_m \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & & & & 0 & 0 \\ . & . & . & & & . & . \\ . & . & & . & & . & . \\ . & . & & & . & . & . \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}.$$

Then u is an eigenvalue for U and v is an eigenvalue for V. So uv is an eigenvalue for $U \otimes_R V$. For if μ, ν are eigenvectors corresponding to u and v respectively, then, $(U \otimes_R V)(\mu \otimes_R \nu) = U\mu \otimes_R V\nu = u\mu \otimes_R v\nu = uv(\mu \otimes_R \nu)$.

Then $U \otimes_R V$ is as follows.

$$
U \otimes_R V = \left[\begin{array}{*{25}{c}}
p_1r_1 & p_1r_2 & . & . & . & p_1r_m & p_2r_1 & . & . & . & & p_2r_m & . & . & & & & & & & p_nr_1 & . & . & . & p_nr_m \\
-p_1 & 0 & . & . & . & 0 & -p_2 & 0 & . & . & & . & & & & & & & & & -p_n & & & & 0 \\
0 & -p_1 & 0 & . & . & . & 0 & -p_2 & . & . & & . & & & & & & & & & & -p_n & & & 0 \\
. & . & . & & & & . & . & . & & & & & & & & & & & & & & & & . \\
. & . & & . & & & . & . & & . & & & & & & & & & & & & & & & . \\
. & . & & & . & & . & . & & & & . & & & & & & & & & & & & & . \\
0 & . & . & . & 0 & -p_1 & 0 & . & . & . & 0 & -p_2 & & & & & & & & & 0 & . & . & . & -p_n \\
-r_1 & -r_2 & . & . & . & -r_m \\
1 & 0 & . & . & . & 0 \\
0 & 1 & & & & 0 \\
. & . & . & & & . \\
. & . & & . & & . \\
. & . & & & . & . \\
0 & 0 & . & . & . & 0 \\
& & & & & & -r_1 & -r_2 & . & . & . & -r_m \\
& & & & & & 1 & 0 & . & . & & 0 \\
& & & & & & 0 & 1 & & & & 0 \\
& & & & & & . & . & . & & & . \\
& & & & & & . & . & & . & & . \\
& & & & & & . & . & & & & . \\
& & & & & & 0 & 0 & . & . & . & 0 \\
\\
& & & & & & & & & & & & & & -r_1 & -r_2 & . & . & . & -r_m & 0 & . & . & . & 0 \\
& & & & & & & & & & & & & & 1 & 0 & . & . & & 0 & 0 & . & . & . & 0 \\
& & & & & & & & & & & & & & 0 & 1 & & & & 0 & . & & & & . \\
& & & & & & & & & & & & & & . & . & . & & & . & . & & & & . \\
& & & & & & & & & & & & & & . & . & & . & & . & . & & & & . \\
& & & & & & & & & & & & & & . & . & & & & . \\
& & & & & & & & & & & & & & 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0
\end{array}\right]
$$

So uv is the root of the characteristic polynomial (say $h(x)$) of $U \otimes_R V$. Thus, $h(x)$ is the determinant of $M = xI_{mn} - U \otimes_R V$ (where I_{mn} is the identity matrix of size mn). Expanding along the first row of M we observe the coefficient of every term of the characteristic polynomial arising from the product of the entries $(1, i)$ of M for $i > 1$ with the corresponding minors is an R-linear combination of $p_j r_k$ and hence lies in P. We need to account for the product of the $(1, 1)$ entry of M with the corresponding minor. To compute the $(1, 1)$ minor we can expand along the second row and observe that the contribution of the product of the $(1, 1)$ entry with the corresponding minor is the leading monic term x^{mn} and lower degree terms with coefficients in P. So uv is the root of a monic polynomial with non-leading coefficients in P. Thus, the set J is closed under multiplication by elements of S.

Now suppose that $u, v \in J$. Then considering P as a subring of S (without identity), we see that u, v are integral over P. So $P[u, v]$ is a finitely generated P-module. Hence, $P[u + v]$ is contained in a finitely generated P module and thus, $u + v \in J$. Thus, J is closed under addition[3].

Finally, if $u \in J$ and $u^n + p_1 u^{n-1} + ... + p_n = 0$ with $p_i \in P$ for $1 \leq i \leq n$, then, multiplying this equation by $(-1)^n$, we get that $(-u)^n + (-p_1)(-u)^{n-1} + ... + (-1)^n p_n = 0$, which is a monic equation for $-u$ with non-leading coefficients in P. So J is an ideal of S. Further, if $u^m \in J$ for some m, we have that $(u^m)^n + p_1(u^m)^{n-1} + ... + p_n = 0$, which may be rewritten as $u^{mn} + p_1 u^{m(n-1)} + ... + p_n = 0$. So $u \in J$, and thus, J is a radical ideal. Clearly any element in P satisfies a monic linear polynomial with the non-leading coefficient in P, so $P \subseteq J$ and thus, $PS \subseteq J$ and since J is radical, we have that $\sqrt{PS} \subseteq J$. Conversely, if $u \in J$ and $u^n + p_1 u^{n-1} + ... + p_n = 0$

[3] Alternately, we may prove that J is closed under addition as follows: suppose that $u, v \in J$, then, u, v satisfy monic polynomials with coefficients in P. So u, v are eigenvalues of matrices $U \in R^{n \times n}$ and $V \in R^{m \times m}$ as before with their first rows consisting of entries in P. Then $u + v$ is the eigenvalue of $N = U \otimes_R I_n + I_m \otimes_R V$, where I_m, I_n are identity matrices of size m, n respectively. Then $u + v$ is a root of $M = \det(xI_{mn} - N)$. Expanding M as before we get the desired result.

with $p_i \in P$ for $1 \leq i \leq n$, then, $u^n = -p_1u^{n-1} - ... - p_n \in PS$. So $u^n \in PS$ and hence, $u \in \sqrt{PS}$. Thus, $J = \sqrt{PS}$. $\square$

Lemma 4.3.2. *Let R be a ring and I, J be ideals of R such that there exists a positive integer n such that for any element $r \in I$, we have that $r^n \in J$. If $n!$ is invertible in R, then, $I^n \subseteq J$.*

Proof. Any element of I^n is a finite sum of products of n elements of I. So it suffices to prove that for any elements $r_1, ..., r_n \in I$, we have that $\Pi_{i=1}^n r_i \in J$. Now, by the multinomial theorem,

$$(r_1 + ... + r_n)^n = \Sigma_{k_1+...+k_n=n}(\frac{n!}{k_1!...k_n!})\Pi_{i=1}^n r_i^{k_i}$$

Then we can write that $n!\Pi_{i=1}^n r_i = (r_1 + ... + r_n)^n - \Sigma_{1\leq i\leq n}(r_1 + ... + r_{i-1} + r_i + ... + r_n)^n + \Sigma_{1\leq i\neq j\leq n}(r_1 + ...r_{i-1} + r_{i+1} + ...r_{j-1} + r_{j+1}... + r_n)^n - ... + \Sigma_{1\leq i\leq n}(-1)^{n-1}r_i^n$. Now every term on the right hand side of the preceding equation lies in J since the nth power of every element of I lies in J. So $n!\Pi_{i=1}^n r_i \in J$. Assuming, $n!$ is invertible in R, we get that $\Pi_{i=1}^n r_i \in J$. Thus, $I^n \subseteq J$. $\square$

Proposition 4.3.3. *Let $R \subseteq S$ be a module-finite extension of domains. Let $L = Frac(S)$ and let $K = Frac(R)$. Then L/K is a finite extension and let $\delta = [L : K]$. Let P be any prime ideal in R. Then $(\sqrt{PS})^\delta \subseteq PS$.*

Proof. The arguments in this proof mimic those in proposition 3.10, [Hoc71]. Consider $J = \{s \in S : s^n + p_1s^{n-1} + ... + p_t = 0$ for some n and some $p_i \in P, 1 \leq i \leq t\}$. By lemma 4.3.1, $J = \sqrt{PS}$. If $s \in S$ is any element, its minimal monic polynomial over K, say $f_{min}(x)$, has all its coefficients in R (theorem 2.1.17, page 29, [HS06]). Further, if $s \in \sqrt{PS}$ it satisfies a monic polynomial with non-leading coefficients in P since $J = \sqrt{PS}$. Then $f_{min}(x)|f(x)$, and consequently, all the non-leading coefficients of $f_{min}(x)$ must be in P (lemma 8.31, page 111, [Pes96]). Suppose that $f_{min}(x) = x^l + p_1 x^{l-1} + ... + p_l$. Then $s^l + p_1 s^{l-1} + ... + p_l = 0$ and thus, $s^l = -p_1 s^{l-1} - ... - p_l \in PS$. Since $\delta = [L : K]$, the degree of the minimal monic polynomial of any element of L over K is at most δ. In particular, $l \leq \delta$. So $s^\delta \in PS$.

Thus, $\sqrt{PS}$ is an ideal such that for every element $s \in \sqrt{PS}$, we have that $s^\delta \in PS$. Then, by lemma 4.3.2, $(\sqrt{PS})^\delta \subseteq PS$. □

Theorem 4.3.4. *Let $R \subseteq S$ be an extension of domains such that S is finitely generated over R by r elements. Assume that R is normal and S is regular, equicharacteristic, dimension $d < \infty$. Assume that $r!$ is invertible in S. Then there exists a positive integer h such that for any prime ideal P of R, we have that $P^{(hn)} \subseteq (P^{n-d+1}S) \cap R$ and $P^{(hn)} \subseteq \overline{P^n}$ for all positive integers n.*

Proof. Let P be a prime ideal in R and let $Q_1, ..., Q_m$ be the prime ideals of S lying over P.

We claim that $Q_1, ..., Q_m$ is the set of minimal primes of PS. We have that $P \subseteq Q_i$ since Q_i lies over P and hence, $PS \subseteq Q_i$. If Q is a prime ideal of R such that $PS \subseteq Q \subsetneq Q_i$, then, $P \subseteq PS \cap R \subseteq Q \cap R \subseteq Q_i \cap R = P$. So we must have equality throughout and thus, $Q \cap R = P$. So by the lying over theorem, we have that $Q = Q_i$. So $Q_1, ..., Q_m$ are minimal over PS. Conversely, if $Q^{'}$ is any minimal prime ideal of PS, then, since $P \subseteq Q^{'}$, $Q^{'} \cap R = P^{'}$ is a prime ideal in R containing P. Then, by the going down theorem, there exists a prime ideal Q in S such that $Q \subseteq Q^{'}$ and $Q \cap R = P$. However, this contradicts the minimality of $Q^{'}$ as a prime ideal containing PS since $P \subseteq PS \subseteq Q \subseteq Q^{'}$. So the set of minimal prime ideals of PS is precisely $\{Q_1, ..., Q_m\}$. In particular, $\sqrt{PS} = Q_1 \cap ... \cap Q_m$.

Now for any prime ideal of $\mathfrak{p}$ of S with height h, we have that $\mathfrak{p}^{(hn)} \subseteq \mathfrak{p}^n$ for all positive integers n (theorem 2.1, [HH07]). In particular, we have that $\mathfrak{p}^{(dn)} \subseteq \mathfrak{p}^n$, and thus, $Q_i^{(dn)} \subseteq Q_i^n$ for $1 \leq i \leq m$. Suppose that $u \in P^{(dn)}$ for some positive integer n. Then there exists $w \in R \setminus P$ such that $wu \in P^{dn}$. Therefore, $w \in S \setminus Q_i$ for if $w \in Q_i$ then, $w \in Q_i \cap R = P$, which contradicts the choice of w. Hence, $wu \in Q_i^{dn}$ for $1 \leq i \leq m$. Thus, $u \in Q_i^{(dn)}$. Then $u \in Q_1^{(dn)} \cap ... \cap Q_m^{(dn)} \subseteq Q_1^n \cap ... \cap Q_m^n$. So $u^m \in Q_1^n ... Q_m^n$. Since S is module-finite over R, the number of prime ideals of S lying over a prime ideal of P is at most the number of generators of S as an R-module, *viz.*, r. In particular, $m \leq r$. Consequently, $u^r \in Q_1^n ... Q_m^n$. Now

$$Q_1^n ... Q_m^n = (Q_1 .. Q_m)^n \subseteq (Q_1 \cap ... \cap Q_m)^n = (\sqrt{PS})^n.$$

Thus, $u^r \in (\sqrt{PS})^n$.

Thus, if $u \in P^{(dm)}$, then, $u^r \in (\sqrt{PS})^m$ for every positive integer m. Replacing m by $r^2 n$, we get that $u \in P^{(dr^2 n)}$ implies that $u^r \in (\sqrt{PS})^{r^2 n}$. Let $L = \text{Frac}(S)$ and let $K = \text{Frac}(R)$. Now, since $R \subseteq S$ is a module finite extension with S generated by r elements as an R-module, the extension L/K is finite and $[L : K] \leq r$. It follows from proposition 4.3.3 that $(\sqrt{PS})^{r^2 n} \subseteq (PS)^{rn}$. So $u^r \in (PS)^{rn}$. Thus, u satisfies the monic equation $x^r - u^r = 0$ over $(PS)^n$. Hence, $u \in \overline{(PS)^n}$. Now $\overline{(PS)^n} \subseteq (PS)^{n-d+1}$ for all $n \geq d$ (theorem 2.1, [LT81]). Thus, $u \in (PS)^{n-d+1}$. Also, since $u \in R$, we have that $u \in P^{n-d+1} S \cap R$. So $P^{(dr^2 n)} \subseteq P^{n-d+1} S \cap R$. Set $h = dr^2$ to get $P^{(hn)} \subseteq P^{n-d+1} \cap R$.

Finally, $u \in \overline{(PS)^n} \cap R = \overline{P^n S} \cap R = \overline{P^n}$ by (proposition 1.6.1, page 15, [HS06]). Hence, $P^{(dr^2 n)} \subseteq \overline{P^n}$, *i.e.*, $P^{(hn)} \subseteq \overline{P^n}$. □

Corollary 4.3.5. *Under the hypothesis of theorem 4.3.4, if R further satisfies the hypothesis of theorem 4.2.3, then, there exists a positive integer c, such that for any prime ideal P of R, we have that $P^{(cn)} \subseteq P^n$ for all positive integers n.*

Proof. By theorem 4.3.4, there exists a positive integer h, such that for any prime ideal P of R, we have that $P^{(hn)} \subseteq \overline{P^n}$. If R further satisfies the hypothesis of theorem 4.2.3, then, there exists a positive integer k independent of P such that $\overline{P^n} \subseteq P^{n-k}$ for all positive integers $n \geq k$. Thus, $P^{(hn)} \subseteq P^{n-k}$. Hence, $P^{(h(n+k))} \subseteq P^n$ for all positive integers n. Since $h, k \geq 1$, we have that $nhk \geq hk$. Consequently, $P^{(hn+hnk)} \subseteq P^{(hn+hk)} \subseteq P^n$. Setting $c = h(k+1)$, we get that $P^{(cn)} \subseteq P^n$ for all positive integers n. □

We end this section with a couple of results on uniform bounds in certain integral extensions of normal domains.

Proposition 4.3.6. *Let R be a normal Noetherian domain. Let $K = Frac(R)$ and L be a finite Galois extension field of K. Let S be the integral closure of R in L. Then there exists a positive integer δ such that if P is a prime ideal in R with a unique prime ideal, say Q, of S lying over P, then, $Q^{(n\delta)} \subseteq P^n S_P \cap S$ (where $S_P = (R \setminus P)^{-1}S$) for all positive integers n.*

Proof. In this set-up, S is a finitely generated R-module (Corollary V.4.1, page 265, [ZS75]). Further, $L = \mathrm{Frac}(S)$ (proof of theorem V.7, page 264, [ZS75]) and L/K is a finite extension. Let $[L : K] = \delta$ Then, by proposition 4.3.3, $(\sqrt{PS})^\delta \subseteq PS$.

We show that $Q = \sqrt{PS}$. Suppose that $\mathfrak{q}$ is a minimal prime of PS. Then $\mathfrak{q} \cap R = \mathfrak{p}$ and since $\mathfrak{q} \supseteq PS$, $\mathfrak{p} \supseteq P$. Then, by the going down theorem, there exists a prime ideal $\mathcal{Q}$ in S such that $\mathfrak{q} \supseteq \mathcal{Q}$ and $\mathcal{Q} \cap R = P$. Since Q is the only prime ideal lying over P, we have that $\mathcal{Q} = Q$. However, since $\mathfrak{q}$ is a minimal prime of PS and $Q \supseteq P$ and hence $Q \supseteq PS$, we must have, $\mathfrak{q} = Q$. Thus, Q is the only minimal prime of PS and hence, $Q = \sqrt{PS}$.

Thus, we have that $Q^\delta \subseteq PS$.

Let $R' = R_P$, $P' = PR_P$, $W = R \setminus P$, $S' = W^{-1}S$, $Q' = QS'$. We claim that S'

is a local ring with maximal ideal $Q^{'}$. For $Q^{'} \cap R^{'} = W^{-1}(Q \cap R) = W^{-1}P = P^{'}$. Hence, $Q^{'}$ is maximal (corollary 5.8, page 61, [AM94]). Further, $\mathfrak{m} \neq Q^{'}$ is a maximal ideal in $S^{'}$, which implies that $\mathfrak{m} \cap R^{'}$ is maximal (corollary 5.8, page 61, [AM94]), so that $\mathfrak{m} \cap R^{'} = P^{'}$. Then $\mathfrak{m} \cap S \neq Q$ (corollary 3.11.iv, page 41, [AM94]), and

$$(\mathfrak{m} \cap S) \cap R = \mathfrak{m} \cap R = (\mathfrak{m} \cap R^{'}) \cap R = P^{'} \cap R = P.$$

However, this contradicts the hypothesis that there is only one prime ideal of S lying over R. This proves the claim.

Let $a \in Q^{(n\delta)}$ for some positive integer n. Then there exists an element $w \in S \setminus Q$ such that $wa \in Q^{n\delta} \subseteq P^n S$. Then $wa \in P^n S_P \cap S$. Now $Q^{'} = QS^{'} = W^{-1}Q = W^{-1}(\sqrt{PS}) = \sqrt{W^{-1}(PS)} = \sqrt{P^{'}S^{'}}$. Further, $P^n S_P = P^{'n}S^{'}$, so that $\sqrt{P^n S_P} = \sqrt{P^{'n}S^{'}} = \sqrt{P^{'}S^{'}} = Q^{'}$. Since $\sqrt{P^n S_P} = Q^{'}$ is maximal in $S^{'}$, $P^n S_P$ is $Q^{'}$-primary (proposition 4.2, page 51, [AM94]) and hence, $P^n S_P \cap S$ is Q-primary (proposition 4.8.ii, page 53, [AM94]). Hence, $a \in P^n S_P \cap S$. Thus, $Q^{(n\delta)} \subseteq P^n S_P \cap S$. □

Proposition 4.3.7. *Let R be a normal Noetherian domain. Let $K = Frac(R)$ and L be a finite Galois extension field of K. Let S be the integral closure of R in L. Let G be the Galois group of L/K. Let Q be a prime ideal in S and let $P = Q \cap R$. Let H be the subgroup of G that stabilizes Q. Let $Q_1, ..., Q_t$ be the set prime ideals in S lying over P with $Q = Q_1$. Let L^H be the subfield of L consisting of elements of L fixed by every element in H. Let S^H be the integral closure of R in L^H. Then there exists a positive integer δ independent of Q such that $Q^{(n\delta)} \subseteq P_H^n S_{P_H} \cap S$, where $P_H = Q \cap S^H$ and $S_{P_H} = (S^H \setminus P_H)^{-1}S$. Further, $P_H^{(n)}$ is generated by those elements $P_H^{(n)}$ that do not lie in any of $Q_2 \cap S^H, ..., Q_t \cap S^H$.*

Proof. The extension L/L^H is Galois with Galois group H. We also have $\mathrm{Frac}(S^H) = L^H$ (proof of theorem V.7, page 264, [ZS75]). Since H acts transitively on the set of prime ideals in S lying over P_H ((proposition VII.2.1, page 340, [Lan02]) and H stabilizes Q, Q is the only prime ideal in S lying over P_H. Also, S is the integral closure of S^H in L, since every element of L integral over S^H is integral over R and hence lies in S by transitivity of integral dependence. Conversely every element of S is integral over R and hence over S^H. Then, by proposition 4.3.6, there exists a positive integer δ such that for every positive integer n we have that $Q^{(n\delta)} \subseteq P_H^n S_{P_H} \cap S$ (in fact from the proof of proposition 4.3.6 it follows that $\delta = [L : L^H]$).

Now let J be the ideal generated by those elements $P_H^{(n)}$ that do not lie in any of $Q_2 \cap S^H, ..., Q_t \cap S^H$. Then $P_H^{(n)} \subseteq J \cup (\cup_{i=2}^t (Q_i \cap S^H)$. Therefore, by the prime avoidance lemma, either $P_H^{(n)} \subseteq J$ or $P_H^{(n)} \subseteq Q_i \cap S^H$ for some $i \in \{2, ..., t\}$. The latter case cannot be true, for then, taking radicals we get that $P_H \subseteq Q_i \cap S^H$. Then, since Q, Q_i lie over P, we have that

$$Q \cap R = (Q \cap S^H) \cap R = P_H \cap R = P = Q_i \cap R = (Q_i \cap S^H) \cap R.$$

Then, since S^H is integral over S and $P_H, Q_i \cap S^H$ lie over the same prime with the former contained in the latter, by the lying over theorem, $P_H = Q_i \cap S^H$. Hence, Q, Q_i lie over the same prime ideal in S^H. This is a contradiction, since the Galois group of L/L^H is H and hence H acts transitively on the set of prime ideals lying over any prime ideal in S. However, by hypothesis, H stabilizes Q. Hence, we must have that $P_H^{(n)} \subseteq J$, and since $J \subseteq P_H^{(n)}$, by definition, we get that $P_H^{(n)} = J$. $\square$

CHAPTER 5

Results on general contractions of powers of ideals

In this chapter we examine a few questions on more general contractions of powers of ideals in a ring from an overring. Given an extension of Noetherian rings $R \subseteq S$, and an ideal I in S, we would like to understand the growth behavior of $I^n \cap R$ as n varies over positive integers. In particular, we would like to study when the ring $\oplus_{i=0}^{\infty}(I^n \cap R)$ is Noetherian. We obtain some results for the case when $S = R[x]$, where x is an indeterminate over R. We consider the following question: if I is an ideal in $R[x]$, when is it true that $I^n \cap R = (I \cap R)^n$? We show that this is false in general if R is polynomial ring over a field in more than 1 indeterminate. We also show that the rings $\oplus_{i=0}^{\infty}(I^n \cap R)$ are Noetherian for certain kinds of ideals I generated by one binomial and several monomials in polynomial rings R in several indeterminates over a field.

5.1 Contractions of powers of ideals versus powers of contractions of ideals

Question 5.1.1. *Let R be a Noetherian ring and x an indeterminate over R. If I is an ideal in $R[x]$, when is it true that $I^n \cap R = (I \cap R)^n$ for all positive integers n?*

We can find counterexamples to question 5.1.1 in all polynomial rings $K[x_1, ..., x_n]$, where $n > 1$ and K is a field of characteristic zero. For example, if $R = K[x, a, b]$,

$S = K[a,b]$, where K is a field of characteristic 0 and $I = (x^2+a, ax, bx)$, then, it can be shown that $I^2 \cap R = (a^2b, a^3)S$ and $I \cap R = (a^2, ab)S$. Then $a^3 \in (I^2 \cap R) \setminus (I \cap R)^2$. We prove a generalization of this example in the following proposition.

Proposition 5.1.2. *Let $R = K[x, y, y_1, ..., y_d]$, $S = K[y, y_1, ..., y_d]$, where K is a field and char$(K) = 0$. Let $I = (x^k+y, yx, \mu_1 x, ..., \mu_m x, \eta_1, ..., \eta_t)R$, where $\mu_1, ..., \mu_m, \eta_1, ..., \eta_t$ are monomials in S and k is a positive integer. Let $J_n = I^n \cap S$. Then*

$$J_n = (((y\sqrt[k]{y}, \mu_1\sqrt[k]{y}, ..., \mu_m\sqrt[k]{y}, \eta_1, ..., \eta_t)^n S[\sqrt[k]{y}]) \cap S) \cap ((y, \eta_1, ..., \eta_t)^n S).$$

Further, J_n is generated by monomials of the form $y^{u+\lceil \frac{u}{k} \rceil - i_1 - ... - i_m} \mu_1^{i_1} ... \mu_m^{i_m} \eta_1^{u_1} ... \eta_t^{u_t}$, where $i_1, ..., i_m, u, u_1, ..., u_t$ are non-negative integers such that $0 \le i_1 + ... + i_m \le \lceil \frac{u}{k} \rceil$ and $u + u_1 + ... + u_t = n$.

Proof. We first show that J_n is a monomial ideal. Consider a $\mathbb{Z}_{\ge 0}^{d+2}$ grading on R, where $\deg(x) = (1, 0, ..., 0)$, $\deg(y) = (k, 0, ..., 0)$ and

$$\deg(y_i) = (0, ..., 0, \underbrace{k}_{(i+2)\text{th position}}, 0, ..., 0).$$

Then all monomials are homogeneous and so is $x^k + y$ (of degree $(k, 0, ..., 0)$). Thus, I is a homogeneous ideal in R with respect to this grading. So the ideals I^n are also homogeneous for all positive integers n. Consider the induced grading on S. Then the ideals J_n are also homogeneous. However, the induced grading on S is the standard multigrading on S and under this grading the only homogeneous elements are monomials and hence the only homogeneous ideals are monomial. So J_n is monomial for all positive integers n.

We now show that each of the purported generators of J_n is actually an element of J_n. We may write that $(y, \eta_1, ..., \eta_t)^n S = (\{y^u \eta_1^{u_1} ... \eta_t^{u_t} : u + u_1 + ... + u_t = n\})R$. Fix non-negative integers $u, u_1, ..., u_t$ such that $u + u_1 + ... + u_t = n$ and let $\beta = \eta_1^{u_1} ... \eta_t^{u_t}$.

Fix a positive integer n and non-negative integers $i_1, ..., i_m$ such that $0 \leq i_1+...+i_m \leq \lceil \frac{u}{k} \rceil$. Let $\eta = \lceil \frac{u}{k} \rceil - i_1 - ... - i_m$. Finally, let $\eta + u = \alpha$.

We have that

$$\begin{aligned} y^{\alpha}\mu_1^{i_1}...\mu_m^{i_m}\beta &= (x^k+y)^{\alpha}\mu_1^{i_1}...\mu_m^{i_m}\beta \\ &\quad -x^{k\lceil \frac{u}{k} \rceil - u}\Sigma_{i=0}^{\alpha-1}\binom{\alpha}{i}(yx)^i(\mu_1 x)^{i_1}...(\mu_m x)^{i_m}(x^{k+1})^{\alpha-i-\lceil \frac{u}{k} \rceil}\beta. \end{aligned}$$

Note that $(x^k+y)^{\alpha} \in I^{\alpha} = I^{\eta+u}$ and $\eta = \lceil \frac{u}{k} \rceil - i_1 - ... - i_m \geq 0$ as $i_1 + ... + i_m \leq \lceil \frac{u}{k} \rceil$. Also, $\beta \in I^{u_1+...+u_t}$. Hence, $(x^k+y)^{\eta}\mu_1^{i_1}...\mu_m^{i_m}\beta \in I^{\eta+u+u_1+...+u_t} = I^{\eta+n} \subseteq I^n$. Also, $x^{k+1} = x(x^k+y) - (yx) \in I$. So $(yx)^i(\mu_1 x)^{i_1}...(\mu_m x)^{i_m}(x^{k+1})^{\alpha-i-\lceil \frac{u}{k} \rceil}\beta \in I^{i+i_1+...+i_m+(\eta+u-i-\lceil \frac{u}{k} \rceil)+u_1+...+u_t} = I^{u+u_1+...+u_t} = I^n$. So the right hand side of the above equation lies in I^n. So $y^{\alpha}\mu_1^{i_1}...\mu_m^{i_m}\beta \in I^n \cap S = J_n$.

Let $J_n' = (((y\sqrt[k]{y}, \mu_1\sqrt[k]{y}, ..., \mu_m\sqrt[k]{y}, \eta_1, ..., \eta_t)^n S[\sqrt[k]{y}]) \cap S) \cap ((y, \eta_1, ..., \eta_t)^n S)$. Note that each ideal in this intersection is a specialization of I^n (to $x = \sqrt[k]{y}$ and to $x = 0$). Consequently, $J_n \subseteq J_n'$. Now any monomial in J_n' is a multiple of a monomial in $S[\sqrt[k]{y}]$ of the form

$$(y\sqrt[k]{y})^j(\mu_1\sqrt[k]{y})^{j_1}...(\mu_m\sqrt[k]{y})^{j_m}\eta_1^{v_1}...\eta_t^{v_t}$$

where $j_1, ..., j_m, v_1, ..., v_t$ are non-negative integers such that $j+j_1+...+j_m+v_1+...+v_t = n$ and $j+\lceil \frac{j+j_1+...+j_m}{k} \rceil + v_1+...+v_t \geq n$. Let $v = j+j_1+...+j_m$. Then we have that $v+v_1+...+v_t = n$ and $j+\lceil \frac{v}{k} \rceil + v_1+...+v_t \geq n$. Equivalently, $j+\lceil \frac{v}{k} \rceil + n - v \geq n$ or $v - j \leq \lceil \frac{v}{k} \rceil$, *i.e.*, $j_1+...+j_m \leq \lceil \frac{v}{k} \rceil$. The exponent of y in such a typical monomial is $j + \lceil \frac{j+j_1+...+j_m}{k} \rceil = n - (v_1+...+v_t) - (j_1+...+j_m) + \lceil \frac{v}{k} \rceil = v - j_1 - ... - j_m + \lceil \frac{v}{k} \rceil$. In other words, every monomial in J_n' is a multiple of a monomial of the form

$$y^{v+\lceil \frac{v}{k} \rceil - j_1 - ... - j_m}\mu_1^{j_1}...\mu_m^{j_m}\eta_1^{v_1}...\eta_t^{v_t}$$

where $j_1, ..., j_m, v_1, ..., v_t$ are non-negative integers such that $j_1 + ... + j_m \leq \lceil \frac{v}{k} \rceil$ and $v + v_1 + ... + v_t = n$. Now, by the preceding paragraph, each of these monomials is in J_n. So we have that $J_n^{'} = J_n$. This completes the proof of the proof of the proposition. $\square$

We now prove another proposition about the behavior of ideals in polynomial rings under elimination of variables.

Proposition 5.1.3. *Let $R = k[x_1, ..., x_m, y_1, ..., y_n]$ be a polynomial ring over a field k. Let $f_1, ..., f_t$ be monomials in R such that $f_1, ..., f_t$ is a regular sequence (equivalently, any indeterminate $x_1, ..., x_m, y_1, ..., y_n$* divides at most one of *$f_1, ..., f_t$). Let $A = k[f_1, ..., f_t] \subseteq R$. Let $S = k[y_1, ..., y_n]$. We may assume, without loss of generality, $f_1, ..., f_j \notin S$ and $f_{j+1}, ..., f_t \in S$ for some $j \in \{1, ..., t\}$. Let $T = k[f_{j+1}, ..., f_t]$. Let I be an ideal in A. Then $IR \cap S = (IR \cap T)S$.*

Proof. Consider the lexicographic monomial order on R, denoted $\leq$, where $y_n < ... < y_1 < x_m < ... < x_1$ and impose the induced order on R, A, T. Let $I = (g_1, ..., g_d)$ be an ideal in A such that $g_1, ..., g_d$ is a Gröbner basis for I.

We claim that $g_1, ..., g_d$ is a Gröbner basis for IR. Let $\text{in}_{\mathfrak{R}}(f)$ denote the initial term of an element f in a ring $\mathfrak{R}$ with respect to a given monomial order. The set of monomials in A is a subset of monomials in R since $f_1, ..., f_t$ are monomials. Since the monomial ordering on A is induced from R, we have that $\text{in}_A(g) = \text{in}_R(g)$ for all $g \in A$. In particular, $\text{in}_A(g_i) = \text{in}_R(g_i)$ for $1 \leq i \leq d$. Further, since $f_1, ..., f_t$ is a regular sequence of monomials in R, $\text{gcd}_A(\mu, \nu) = \text{gcd}_R(\mu, \nu)$ for any monomials $\mu, \nu \in A$. Let $\Delta_{ij}^A = \text{gcd}_A(\text{in}_A(g_i), \text{in}_A(g_j))$ and $\Delta_{ij}^R = \text{gcd}_R(\text{in}_R(g_i), \text{in}_R(g_j))$. Then we have that $\Delta_{ij}^A = \Delta_{ij}^R$ for $1 \leq i, j \leq d$. Let $G_{ij} = \frac{\text{in}(g_j)}{\Delta_{ij}} g_i - \frac{\text{in}(g_i)}{\Delta_{ij}} g_j$, where $\Delta_{ij} = \Delta_{ij}^A = \Delta_{ij}^R$ for $1 \leq i, j \leq d$ and $\text{in}(g_i) = \text{in}_A(g_i) = \text{in}_R(g_i)$ for $1 \leq i \leq d$. Then, by the Buchberger criterion, there is a standard expression for G_{ij} with respect to $g_1, ..., g_d$ such that the remainder in the standard expression is zero for $1 \leq i \neq j \leq d$.

So we may write that

$$G_{ij} = \Sigma_{k=1}^{d} q_{ijk} g_k \ (*)$$

where $q_{ijk} \in A$ and if $q_{ijk}g_k \neq 0$, then, $\text{in}_A(q_{ijk}g_k) \leq \text{in}_A(G_{ij})$. Then (*) holds in R and since $\text{in}_A(g) = \text{in}_R(g)$ for all $g \in A$, we have that $\text{in}_R(q_{ijk}g_k) \leq \text{in}_R(G_{ij})$. So (*) is a standard expression for G_{ij} in R for $1 \leq i \neq j \leq d$. Thus, the G_{ij} have standard expressions with respect to $g_1, ..., g_d$ with zero remainder in R and hence by the Buchberger criterion, $g_1, ..., g_d$ is Gröbner basis for IR.

Now, by the elimination theorem, $IR \cap S$ is generated by $\{g_1, ..., g_d\} \cap S$. Also, by elimination theorem, $I \cap T$ is generated by $\{g_1, ..., g_d\} \cap T$. However, by construction, $\{g_1, ..., g_d\} \cap S = \{g_1, ..., g_d\} \cap T$. Thus, $(I \cap T)S = (\{g_1, ..., g_d\} \cap T)S = (\{g_1, ..., g_d\} \cap S)S = IR \cap S$.

Now, since $f_1, ..., f_t$ is a regular sequence, $A \subseteq R$ is flat (proposition A.73, page 313, [Vas04]). Further, since $(f_1, ..., f_t)R \subseteq (x_1, ..., x_m, y_1, ..., y_m)R \subsetneq R$, $A \subseteq R$ is faithfully flat. So for any ideal J in A, $J = JR \cap A$. Then $I \cap T = (IR \cap A) \cap T = IR \cap T$, where the last equality follows from $T \subseteq A$. Then, by the preceding paragraph, $IR \cap S = (I \cap T)S = (IR \cap T)S$. □

5.2 Finite generation of certain Rees rings with respect to contracted ideals

Let $R[x]$ be a polynomial ring in one indeterminate over a ring R. Let I be an ideal of R. Let $J_n = I^n \cap R$. Then $J_{n_1}J_{n_2} = (I^{n_1} \cap R)(I^{n_2} \cap R) \subseteq I^{n_1+n_2} \cap R = J_{n_1+n_2}$. Thus, the set of ideals $\{J_n\}_{n \in \mathbb{Z}_{\geq 0}}$ with $J_0 = R$ is a filtration. We now raise the following question.

Question 5.2.1. *Let $R[x]$ be the polynomial ring in one indeterminate over a ring R. Let I be an ideal of R. Let $J_n = I^n \cap R$. Is the ring $R \oplus J_1 \oplus J_2 \oplus ...$ Noetherian?*

We show that the answer to question 5.2.1 is yes for the ideals considered in proposition 5.1.2.

Proposition 5.2.2. *Let $R = K[x, y, y_1, ..., y_d]$, $S = K[y, y_1, ..., y_d]$, where K is a field of characteristic 0. Let $I = (x^k+y, yx, \mu_1 x, ..., \mu_m x, \eta_1, ..., \eta_t)R$, where $\mu_1, ..., \mu_m, \eta_1, ..., \eta_t$ are monomials in S and k is a positive integer. Let $J_n = I^n \cap S$. Then the ring $R \oplus J_1 \oplus J_2 \oplus ...$ is Noetherian.*

Proof. We show that $J_{nk}J_k = J_{(n+1)k}$ for all positive integers n. We only need to show that $J_{(n+1)k} \subseteq J_{nk}J_k$. By proposition 5.1.2,

$$J_n = (\{y^{u+\lceil \frac{u}{k} \rceil - i_1 - ... - i_m} \mu_1^{i_1} ... \mu_m^{i_m} \eta_1^{u_1} ... \eta_t^{u_t} : i_1 + ... + i_m \leq \left\lceil \frac{u}{k} \right\rceil, u + u_1 + ... + u_t = n\}).$$

Then

$$J_{nk} = (\{y^{v+\lceil \frac{v}{k} \rceil - j_1 - ... - j_m} \mu_1^{j_1} ... \mu_m^{j_m} \eta_1^{v_1} ... \eta_t^{v_t} : j_1 + ... + j_m \leq \left\lceil \frac{v}{k} \right\rceil, v + v_1 + ... + v_t = nk\})$$

and

$$J_k = (\{y^{w+\lceil \frac{w}{k} \rceil - l_1 - ... - l_m} \mu_1^{l_1} ... \mu_m^{l_m} \eta_1^{w_1} ... \eta_t^{w_t} : l_1 + ... + l_m \leq \left\lceil \frac{w}{k} \right\rceil, w + w_1 + ... + w_t = k\}).$$

So

$$J_{nk}J_k = (\{y^{(v+w)+(\lceil \frac{v}{k} \rceil + \lceil \frac{w}{k} \rceil)-(j_1+l_1)-...-(j_m+l_m)} \mu_1^{j_1+l_1} ... \mu_m^{j_m+l_m} \eta_1^{v_1+w_!} ... \eta_t^{v_t+w_t} :$$

$$\begin{aligned} J_{nk}J_k \quad = \quad & (\{y^{(v+w)+(\lceil \frac{v}{k} \rceil + \lceil \frac{w}{k} \rceil)-(j_1+l_1)-...-(j_m+l_m)} \mu_1^{j_1+l_1} ... \mu_m^{j_m+l_m} \eta_1^{v_1+w_!} ... \eta_t^{v_t+w_t} : \\ & (j_1 + l_1) + ... + (j_m + l_m) \leq \left\lceil \frac{v}{k} \right\rceil + \left\lceil \frac{w}{k} \right\rceil, \\ & (v + w) + (v_1 + w_1) + ... + (v_t + w_t) = (n + 1)k\}). \end{aligned}$$

So to prove $J_{(n+1)k} \subseteq J_{nk}J_k$ we need to show that the monomials $y^{u+\lceil \frac{u}{k} \rceil} \eta_1^{u_1} ... \eta_t^{u_t}$, where $u + u_1 + ... + u_t = (n + 1)k$ belong to $J_{nk}J_k$. We show this below.

First suppose that $u = ik$, where $i \in \{0, ..., n+1\}$. Let $\alpha = y^{(i-1)k+\lceil \frac{(i-1)k}{k} \rceil}\eta_1^{v_1}...\eta_t^{v_t} = y^{(i-1)k+(i-1)}\eta_1^{v_1}...\eta_t^{v_t}$, where $v_1 + ... + v_t = nk - (i-1)k$. Then $\alpha \in J_{nk}$ and $y^{k+1} = y^{k+\lceil \frac{k}{k} \rceil}\eta_1^0...\eta_t^0 \in J_k$. Thus, $\alpha y^{k+1} = y^{ik+i}\eta_1^{v_1}...\eta_t^{v_t} \in J_{nk}J_k$. Note $(ik) + v_1 + ... + v_t = ik + nk - (i-1)k = (n+1)k$. Thus, $\alpha y^{k+1} = y^{u+\lceil \frac{u}{k} \rceil}\eta_1^{u_1}...\eta_t^{u_t}$ with $u = ik$, $u_r = v_r$ for $1 \leq r \leq t$ and $u + u_1 + ... + u_t = (n+1)k$.

Now suppose that $u = ik + j$, where $i \in \{0, ..., n\}$ and $j \in \{1, ..., k-1\}$. Let $\alpha = y^{ik+\lceil \frac{ik}{k} \rceil}\eta_1^{v_1}...\eta_t^{v_t} = y^{ik+i}\eta_1^{v_1}...\eta_t^{v_t}$, where $v_1 + ... + v_t = nk - ik$. Then $\alpha \in J_{nk}$. Let $\beta = y^{j+\lceil \frac{j}{k} \rceil}\eta_1^{w_1}...\eta_t^{w_t} = y^{j+1}\eta_1^{w_1}...\eta_t^{w_t}$, where $w_1 + ... + w_t = k - j$. Then $\beta \in J_k$. Thus, $\alpha\beta = y^{u+\lceil \frac{u}{k} \rceil}\eta_1^{u_1}...\eta_t^{u_t}$ with $u = ik + j$, $u_r = v_r + w_r$ for $1 \leq r \leq t$ and $u+u_1+...+u_t = ik+j+(v_1+w_1)+...+(v_t+w_t) = ik+j+(v_1+...+v_t)+(w_1+...+w_t) = ik + j + nk - ik + k - j = (n+1)k$.

This complete the proof of the claim that $J_{(n+1)k} = J_{nk}J_k$ for all positive integers n. By induction on n, we get that $J_{nk} = (J_k)^n$ for all positive integers n. Then, by lemma 2.2, [Kur94], $R \oplus J_1 \oplus J_2 \oplus ...$ is Noetherian. $\square$

APPENDIX

APPENDIX A

A.1 $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ (general case) - computations

Here, we illustrate the computations discussed in section 3.1.7.

1.

$$
\begin{aligned}
& \operatorname{tr}(g_i^2(x_1+f_1)^2 + (x_i+f_i)^2 - 2g_i((x_1+f_1)(x_i+f_i))) \\
= \;& x_1^2g_i^2 + f_1^2g_i^2 - 2x_1x_ig_i - 2f_1f_2g_i + x_i^2 + f_i^2 \\
= \;& x_1^2g_i^2 + f_1^2g_i^2 - 2x_1x_ig_i - 2f_1^2g_i^2 + x_i^2 + f_1^2g_i^2 \\
= \;& x_1^2g_i^2 - 2x_1x_ig_i + x_i^2 \\
= \;& x_1^2g_i^2 - x_1x_ig_i - x_1x_ig_i + x_i^2 \\
= \;& x_1g_i(x_1g_i - x_i) - x_i(x_1g_i - x_i) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

2.

$$\begin{aligned}
& \operatorname{tr}(g_i(f_1 - x_1)(x_1 + f_1)^2 + (x_1 - f_1)(x_1 + f_1)(x_i + f_i)) \\
= \;& -x_1^3 g_i + x_1^2 f_1 g_i - x_1 f_1^2 g_i + f_1^3 g_i + x_1^2 x_i - x_1 x_i f_1 + x_1 f_1 f_i - f_1^2 f_i \\
= \;& -x_1^3 g_i + x_1^2 f_1 g_i - x_1 f_1^2 g_i + f_1^3 g_i + x_1^2 x_i - x_1 x_i f_1 + x_1 f_1^2 g_i - f_1^3 g_i \\
= \;& -x_1^3 g_i + x_1^2 x_i + x_1^2 f_1 g_i - x_1 x_i f_1 \\
= \;& -x_1^2 (x_1 g_i - x_i) + x_1 f_1 (x_1 g_i - x_i) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

3.

$$\begin{aligned}
& \operatorname{tr}(g_i g_j (3f_1 - x_1)(x_1 + f_1)^2 - 2f_i((x_1 + f_1)(x_j + f_j) \\
& +(x_j - f_j)((x_1 + f_1)(x_i + f_i)) \\
= \;& -x_1^3 g_i g_j + 3x_1^2 f_1 g_i g_j - x_1 f_1^2 g_i g_j + 3f_1^3 g_i g_j - 2x_1 x_j f_1 g_i \\
& -2f_1^2 f_j g_i - x_1 x_i f_1 g_j - f_1^2 f_i g_j + x_1 x_i x_j + x_j f_1 f_i \\
= \;& -x_1^3 g_i g_j + 3x_1^2 f_1 g_i g_j - x_1 f_1^2 g_i g_j + 3f_1^3 g_i g_j - 2x_1 x_j f_1 g_i \\
& -2f_1^3 g_j g_i - x_1 x_i f_1 g_j - f_1^3 g_i g_j + x_1 x_i x_j + x_j f_1^2 g_i \\
= \;& -x_1^3 g_i g_j + x_1 x_i x_j + 2x_1^2 f_1 g_i g_j - 2x_1 x_j f_1 g_i \\
& +x_1^2 f_1 g_i g_j - x_1 x_i f_1 g_j + x_j f_1^2 g_i - x_1 f_1^2 g_i g_j
\end{aligned}$$

Now, $-x_1^3 g_i g_j + x_1 x_i x_j = -x_1^3 g_j (x_1 g_i - x_i) - x_1 x_i (x_1 g_j - x_j)$. So that,

$$\begin{aligned}
& \operatorname{tr}(g_i g_j (3f_1 - x_1)(x_1 + f_1)^2 - 2f_i((x_1 + f_1)(x_j + f_j) \\
& +(x_j - f_j)((x_1 + f_1)(x_i + f_i)) \\
= \;& -x_1^3 g_j (x_1 g_i - x_i) - x_1 x_i (x_1 g_j - x_j) + 2x_1 f_1 g_i (x_1 g_j - x_j) \\
& +x_1 f_1 g_j (x_1 g_i - x_i) + f_1^2 g_i (x_j - x_1 g_j) \\
\in \;& \mathfrak{m}P
\end{aligned}$$

4.

$$\begin{aligned}
& \operatorname{tr}(g_i g_j(2f_1 - x_1)(x_1+f_1)^2 + x_1(x_i+f_i)(x_j+f_j) \\
& -f_i(x_1+f_1)(x_j+f_j) - f_j(x_1+f_1)(x_i+f_i)) \\
= \; & -x_1^3 g_i g_j + 2x_1^2 f_1 g_i g_j - x_1 f_1^2 g_i g_j + 2f_1^3 g_i g_j + x_1 x_i x_j \\
& -x_1 x_j f_i - x_1 x_i f_j + x_1 f_i f_j - 2f_1 f_i f_j \\
= \; & -x_1^3 g_i g_j + 2x_1^2 f_1 g_i g_j - x_1 f_1^2 g_i g_j + 2f_1^3 g_i g_j + x_1 x_i x_j \\
& -x_1 x_j f_1 g_i - x_1 x_i f_1 g_j + x_1 f_1^2 g_i g_j - 2f_1^3 g_i g_j \\
= \; & -x_1^3 g_i g_j + x_1 x_i x_j + x_1^2 f_1 g_i g_j - x_1 x_i f_1 g_j \\
& +x_1^2 f_1 g_i g_j - x_1 x_j f_1 g_i
\end{aligned}$$

Now, $-x_1^3 g_i g_j + x_1 x_i x_j = -x_1^3 g_j(x_1 g_i - x_i) - x_1 x_i(x_1 g_j - x_j)$. So that,

$$\begin{aligned}
& \operatorname{tr}(g_i g_j(2f_1 - x_1)(x_1+f_1)^2 + x_1(x_i+f_i)(x_j+f_j) \\
& -f_i(x_1+f_1)(x_j+f_j) - f_j(x_1+f_1)(x_i+f_i)) \\
= \; & -x_1^3 g_j(x_1 g_i - x_i) - x_1 x_i(x_1 g_j - x_j) + x_1 f_1 g_j(x_1 g_i - x_i) \\
& +x_1 f_1 g_i(x_1 g_j - x_j) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

5.

$$
\begin{aligned}
& \operatorname{tr}(g_i g_j (x_1+f_1)^2 + (x_i+f_i)(x_j+f_j) \\
& -g_i(x_1+f_1)(x_j+f_j) - g_j(x_1+f_1)(x_i+f_i)) \\
= \;& x_1^2 g_i g_j + f_1^2 g_i g_j - x_1 x_j g_i - f_1 f_j g_i - x_1 x_i g_j - f_1 f_i g_j + x_i x_j + f_i f_j \\
= \;& x_1^2 g_i g_j + f_1^2 g_i g_j - x_1 x_j g_i - f_1^2 g_j g_i - x_1 x_i g_j - f_1^2 g_2 g_j + x_i x_j + f_1^2 g_i g_j \\
= \;& x_1^2 g_i g_j - x_1 x_j g_i - x_1 x_i g_j + x_i x_j \\
= \;& x_1 g_i (x_1 g_j - x_j) - x_i (x_1 g_j - x_j) \\
\in \;& \mathfrak{m} P
\end{aligned}
$$

6.

$$
\begin{aligned}
& \operatorname{tr}(g_i^2 g_j (4f_1 - x_1)(x_1+f_1)^2 + x_i(x_i+f_i)(x_j+f_j) \\
& -f_1 g_i^2 (x_1+f_1)(x_j+f_j) - 3 f_1 g_i g_j (x_1+f_1)(x_j+f_j)) \\
= \;& -x_1^3 g_i^2 g_j + 4x_1^2 f_1 g_i^2 g_j - x_1 f_1^2 g_i^2 g_j + 4 f_1^3 g_i^2 g_j \\
& -x_1 x_j f_1 g_i^2 - f_1^2 f_j g_i^2 - 3x_1 x_i f_1 g_i g_j - 3 f_1^2 f_i g_i g_j \\
& +x_i^2 x_j + x_i f_i f_j \\
= \;& -x_1^3 g_i^2 g_j + 4x_1^2 f_1 g_i^2 g_j - x_1 f_1^2 g_i^2 g_j + 4 f_1^3 g_i^2 g_j \\
& -x_1 x_j f_1 g_i^2 - f_1^3 g_j g_i^2 - 3x_1 x_i f_1 g_i g_j - 3 f_1^3 g_i^2 g_j \\
& +x_i^2 x_j + x_i f_1^2 g_i g_j \\
= \;& x_i^2 x_j - x_1^3 g_i^2 g_j + x_1^2 f_1 g_i^2 g_j - x_1 x_j f_1 g_i^2 \\
& +3x_1^2 f_1 g_i^2 g_j - 3x_1 x_i f_1 g_i g_j + x_i f_1^2 g_i g_j - x_1 f_1^2 g_i^2 g_j
\end{aligned}
$$

Now, $x_i^2x_j - x_1^3g_i^2g_j = -x_1^2g_jg_i(x_1g_i - x_i) - x_1x_ig_j(x_1g_i - x_i) - x_i^2(x_1g_j - x_j)$. So that,

$$\begin{aligned}
& \operatorname{tr}(g_i^2g_j(4f_1 - x_1)(x_1 + f_1)^2 + x_i(x_i + f_i)(x_j + f_j) \\
& -f_1g_i^2(x_1 + f_1)(x_j + f_j) - 3f_1g_ig_j(x_1 + f_1)(x_j + f_j)) \\
= \; & (-x_1^2g_jg_i - x_1x_ig_j)(x_1g_i - x_i) - x_i^2(x_1g_j - x_j) \\
& +x_1f_1g_i^2(x_1g_j - x_j) + 3x_1f_1g_ig_j(x_1g_i - x_i) \\
& +f_1^2g_ig_j(x_i - x_1g_i) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

7.

$$\begin{aligned}
& \operatorname{tr}(g_i^2(3f_1 - x_1)(x_1 + f_1)^2 + (x_i - 3f_i)(x_1 + f_1)(x_i + f_i)) \\
= \; & -x_1^3g_i^2 + 3x_1^2f_1g_i^2 - x_1f_1^2g_i^2 + 3f_1^3g_i^2 - 3x_1x_if_1g_i - 3f_1^2f_ig_i + x_1x_i^2 + x_if_1f_i \\
= \; & -x_1^3g_i^2 + 3x_1^2f_1g_i^2 - x_1f_1^2g_i^2 + 3f_1^3g_i^2 - 3x_1x_if_1g_i - 3f_1^3g_i^2 + x_1x_i^2 + x_if_1^2g_i \\
= \; & -x_1^3g_i^2 + x_1x_i^2 + 3x_1^2f_1g_i^2 - 3x_1x_if_1g_i + x_if_1^2g_i - x_1f_1^2g_i^2 \\
= \; & -x_1(x_1g_i + x_i)(x_1g_i - x_i) + 3x_1f_1g_i(x_1g_i - x_i) + f_1^2g_i(x_i - x_1g_i) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

8.

$$\begin{aligned}
& \operatorname{tr}((g_i(2f_1 - x_1) + x_i)(x_1 + f_1)^2 - 2f_1((x_1 + f_1)(x_i + f_i))) \\
= \; & -x_1^3g_i + x_1^2x_i - 2x_1^2f_1g_i + 2x_1^2f_i - x_1f_1^2g_i - x_if_1^2 + 2x_1f_1f_i \\
= \; & -x_1^3g_i + x_1^2x_i - 2x_1^2f_1g_i + 2x_1^2f_1g_i - x_1f_1^2g_i - x_if_1^2 + 2x_1f_1^2g_i \\
= \; & -x_1^2(g_ix_1 - x_i) + f_1^2(g_ix_1 - x_i) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

9.

$$
\begin{aligned}
& \operatorname{tr}(g_i^3(4f_1 - x_1)(x_1 + f_1)^2 + x_i(x_i + f_i)^2 - 4f_1 g_i^2(x_1 + f_1)(x_i + f_i)) \\
= \;& -x_1^3 g_i^3 + 4x_1^2 f_1 g_i^3 - x_1 f_1^2 g_i^3 + 4f_1^3 g_i^3 - 4x_1 x_i f_1 g_i^2 - 4f_1^2 f_i g_i^2 + x_i^3 + x_i f_i^2 \\
= \;& -x_1^3 g_i^3 + 4x_1^2 f_1 g_i^3 - x_1 f_1^2 g_i^3 + 4f_1^3 g_i^3 - 4x_1 x_i f_1 g_i^2 - 4f_1^3 g_i^3 + x_i^3 + x_i f_1^2 g_i^2 \\
= \;& x_i^3 - x_1^3 g_i^3 - x_1 f_1^2 g_i^3 + x_i f_1^2 g_i^2 + 4x_1^2 f_1 g_i^3 - 4x_1 x_i f_1 g_i^2 \\
= \;& (x_i - x_1 g_i)(x_i^2 + x_1 x_i g_i + x_1^2 g_i^2) - f_1^2 g_i^2 (x_1 g_i - x_i) + 4x_1 f_1 g_i^2 (x_1 g_i - x_i) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

10.

$$
\begin{aligned}
& \operatorname{tr}(g_i^2(2f - x_1)(x_1 + f_1)^2 + x_1(x_i + f_i)^2 - 2f_i((x_1 + f_1)(x_i + f_i))) \\
= \;& -x_1^3 g_i^2 - x_1 f_1^2 g_i^2 + 2x_1^2 f_i g_i + 2f_1^2 f_i g_i + x_1 x_i^2 - 2x_1 x_i f_i + x_1 f_i^2 - 2f_1 f_i^2 \\
= \;& -x_1^3 g_i^2 - x_1 f_1^2 g_i^2 + 2x_1^2 f_1 g_i^2 + 2f_1^3 g_i^2 + x_1 x_i^2 - 2x_1 x_i f_1 g_i + x_1 f_1^2 g_i^2 - 2f_1^3 g_i^2 \\
= \;& -x_1^3 g_i^2 + x_1 x_i^2 - 2x_1 x_i f_1 g_i + 2x_1^2 f_1 g_i^2 \\
= \;& -x_1(x_1 g_i + x_i)(x_1 g_i - x_i) - 2x_1 f_1 g_i (x_i - x_1 g_i) \\
\in \;& \mathfrak{m}P
\end{aligned}
$$

11.

$$
\begin{aligned}
& \operatorname{tr}(g_i^2 g_j(4f_1 - x_1)(x_1 + f_1)^2 + x_j(x_i + f_i)^2 \\
& -2f_1 g_i^2(x_1 + f_1)(x_j + f_j) - 2f_1 g_i g_j(x_1 + f_1)(x_i + f_i)) \\
= \; & -x_1^3 g_i^2 g_j + 4x_1^2 f_1 g_i^2 g_j - x_1 f_1^2 g_i^2 g_j \\
& +4f_1^3 g_i^2 g_j - 2x_1 x_j f_1 g_i^2 - 2f_1^2 f_j g_i^2 - 2x_1 x_i f_1 g_i g_j \\
& -2f_1^2 f_i g_i g_j + x_i^2 x_j + x_j f_i^2 \\
= \; & -x_1^3 g_i^2 g_j + 4x_1^2 f_1 g_i^2 g_j - x_1 f_1^2 g_i^2 g_j \\
& +4f_1^3 g_i^2 g_j - 2x_1 x_j f_1 g_i^2 - 2f_1^3 g_j g_i^2 - 2x_1 x_i f_1 g_i g_j \\
& -2f_1^3 g_i^2 g_j + x_i^2 x_j + x_j f_1^2 g_i^2 \\
= \; & -x_1^3 g_i^2 g_j + x_i^2 x_j - x_1 f_1^2 g_i^2 g_j + x_j f_1^2 g_i^2 \\
& +2x_1^2 f_1 g_i^2 g_j - 2x_1 x_j f_1 g_i^2 + 2x_1^2 f_1 g_i^2 g_j - 2x_1 x_i f_1 g_i g_j
\end{aligned}
$$

Now, $-x_1^3 g_i^2 g_j + x_i^2 x_j = -x_1^2 g_i g_j(x_1 g_i - x_i) - x_1 x_i g_j(x_1 g_i - x_i) - x_i^2(x_1 g_j - x_j)$.

So that,

$$
\begin{aligned}
& \operatorname{tr}(g_i^2 g_j(4f_1 - x_1)(x_1 + f_1)^2 + x_j(x_i + f_i)^2 \\
& -2f_1 g_i^2(x_1 + f_1)(x_j + f_j) - 2f_1 g_i g_j(x_1 + f_1)(x_i + f_i)) \\
= \; & -x_1^2 g_i g_j(x_1 g_i - x_i) - x_1 x_i g_j(x_1 g_i - x_i) - x_i^2(x_1 g_j - x_j) \\
& -f_1^2 g_i^2(x_1 g_j - x_j) + 2x_1 f_1 g_i^2(x_1 g_j - x_j) \\
& +2x_1 f_1 g_i g_j(x_1 g_i - x_i) \\
\in \; & \mathfrak{m}P
\end{aligned}
$$

12.

$$\begin{aligned}
& \operatorname{tr}(g_ig_jg_k(2f_1-x_1)(x_1+f_1)^2+x_k(x_i+f_i)(x_j+f_j) \\
& -f_1g_j(x_k+f_k)(x_i+f_i)-f_1g_j(x_k+f_k)(x_j+f_j)) \\
= \; & -x_1^3g_ig_jg_k+2x_1^2f_1g_1g_jg_k-x_1f_1^2g_ig_jg_k+2f_1^3g_ig_jg_k \\
& +x_ix_jx_k-f_ix_jx_k-f_jx_ix_k+f_if_jx_k-2f_if_jf_k \\
= \; & -x_1^3g_ig_jg_k+2x_1^2f_1g_1g_jg_k-x_1f_1^2g_ig_jg_k+2f_1^3g_ig_jg_k \\
& +x_ix_jx_k-f_1g_ix_jx_k-f_1g_jx_ix_k+f_1^2g_ig_jx_k-2f_1^3g_ig_jg_k \\
= \; & -x_1^3g_ig_jg_k+x_ix_jx_k-x_1f_1^2g_ig_jg_k+f_1^2g_ig_jx_k \\
& +x_1^2f_1g_1g_jg_k-f_1g_ix_jx_k+x_1^2f_1g_1g_jg_k-f_1g_jx_ix_k
\end{aligned}$$

Now, $-x_1^3g_ig_jg_k+x_ix_jx_k=-x_1^2g_ig_j(x_1g_k-x_k)-x_1x_kg_i(x_1g_j-x_j)-x_jx_k(x_1g_i-x_i)$. Also, $x_1^2f_1g_1g_jg_k-f_1g_ix_jx_k=f_1g_ig_jx_1(x_1g_k-x_k)+f_1g_ix_k(x_1g_j-x_j)$. Finally, $x_1^2f_1g_1g_jg_k-f_1g_jx_ix_k=f_1g_ig_jx_1(x_1g_k-x_k)+f_1g_jx_k(x_1g_i-x_i)$. So that,

$$\begin{aligned}
& \operatorname{tr}(g_ig_jg_k(2f_1-x_1)(x_1+f_1)^2+x_k(x_i+f_i)(x_j+f_j) \\
& -f_1g_j(x_k+f_k)(x_i+f_i)-f_1g_j(x_k+f_k)(x_j+f_j)) \\
= \; & -x_1^2g_ig_j(x_1g_k-x_k)-x_1x_kg_i(x_1g_j-x_j)-x_jx_k(x_1g_i-x_i) \\
& -f_1^2g_ig_j(x_1g_k-x_k)+f_1g_ig_jx_1(x_1g_k-x_k)+f_1g_ix_k(x_1g_j-x_j) \\
& +f_1g_ig_jx_1(x_1g_k-x_k)+f_1g_jx_k(x_1g_i-x_i) \\
\in \; & \mathfrak{m}P
\end{aligned}$$

Lastly, we need to show the generators of $(Q_1^2\cap Q_2^2)\cap S$ arising by applying the trace map to $Q_1^2Q_2^2$ also lie in $\mathfrak{m}P$. We have, $Q_1^2Q_2^2=\{(x_i-f_i)(x_j-f_j)(x_{i'}+f_{i'})(x_{j'}+f_{j'}):i,i^{'},j,j^{'}\in\{1,...,m\}\}$.

We have,

$$
\begin{aligned}
& \operatorname{tr}((x_i - f_i)(x_j - f_j)(x_{i'} + f_{i'})(x_{j'} + f_{j'})) \\
= \; & x_i x_{i'} x_j x_{j'} + f_i f_{i'} x_j x_{j'} + x_i x_{i'} f_j f_{j'} - f_i x_{i'} f_j x_{j'} \\
& - f_i x_{i'} x_j f_{j'} - x_i f_{i'} f_j x_{j'} - x_i f_{i'} x_j f_{j'} + f_i f_{i'} f_j f_{j'} \\
= \; & x_i x_{i'} x_j x_{j'} - f_i x_{i'} f_j x_{j'} + f_i f_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - f_i x_{i'} x_j f_{j'} - x_i f_{i'} x_j f_{j'} + f_i f_{i'} f_j f_{j'} \\
= \; & x_{i'} x_{j'} (x_i x_j - f_i f_j) + f_i f_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - f_i x_{i'} x_j f_{j'} - f_{i'} f_{j'} (x_i x_j - f_i f_j) \\
= \; & x_{i'} x_{j'} (x_i x_j - f_i f_j) + f_i f_{i'} x_j x_{j'} - x_i x_{i'} x_j x_{j'} + x_i x_{i'} x_j x_{j'} - x_i f_{i'} f_j x_{j'} \\
& + x_i x_{i'} f_j f_{j'} - x_i x_{i'} x_j x_{j'} + x_i x_{i'} x_j x_{j'} - f_i x_{i'} x_j f_{j'} - f_{i'} f_{j'} (x_i x_j - f_i f_j \\
= \; & x_{i'} x_{j'} (x_i x_j - f_i f_j) + x_j x_{j'} (f_i f_{i'} - x_i x_{i'}) + x_i x_{j'} (x_{i'} x_j - f_{i'} f_j) \\
& + x_i x_{i'} (f_j f_{j'} - x_j x_{j'}) + x_{i'} x_j (x_i x_{j'} - f_i f_{j'}) - f_{i'} f_{j'} (x_i x_j - f_i f_j) \\
\in \; & \mathfrak{m} P
\end{aligned}
$$

Thus, every generator of $(Q_1^2 \cap Q_2^2) \cap S$ is in $\mathfrak{m}P$, showing $(Q_1^2 \cap Q_2^2) \cap S \subseteq \mathfrak{m}P$ as before. So we have shown, using corollary 3.1.6 that under hypothesis (*), $P^{(2)} \subseteq \mathfrak{m}P$.

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