FINITENESS PROPERTIES OF LOCAL COHOMOLOGY

by

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CHAPTER I

Introduction

This thesis focuses on finding finiteness structural properties of local cohomology modules over a ring and their use in the study of singularity. This work contains results obtained in [NB12c, NB12b, NB12a, NB13, NBW12a, NBW12b, NBP13, HNBW13]. Of these, [HNBW13] is in collaboration with Daniel J. Henández and Emily E. Witt, [NBP13] with Juan F. Pérez, and finally [NBW12a, NBW12b] with Emily E. Witt. In addition, these projects have been under the supervision and advice of Mel Hochster. All results stated formally in this introduction appear in a paper of the author unless otherwise indicated.

I.1 Algebra and geometry

It is well know that the equation for a conic section takes the following quadratic form:

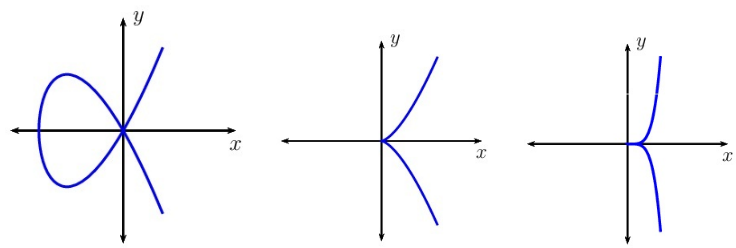
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

One can classify the geometry of the section in terms of the constants in the equation. For instance, if $B^2 - 4AC = 0$ it is a parabola. Following this example further, we can ask about the geometry of the set of points in a euclidean space $\mathbb{R}^n$ defined as the zeros of several polynomial equations $f_1, \ldots, f_\ell$. We denote that set by $\mathcal{V}(f_1, \ldots, f_\ell)$, and called it an algebraic variety. Using the defining equations of $\mathcal{V}(f_1, \ldots, f_\ell)$, we can make sense of the concepts of dimension, irreducibility, and smoothness. The mathematical area that studies this interaction is Algebraic Geometry, and its algebraic side is dominated by Commutative Algebra.

Keeping in mind the close relationship between algebra and geometry, we consider $f(x)$, a polynomial in n variables and with real coefficients. We say the hypersurface given by the points that satisfy $f(x) = 0$ is smooth at $x = a$ if at least one partial derivative $\frac{\partial f}{\partial x_i}(a)$ is not zero. Otherwise, we say that f has a singularity at $x = a$.

f smooth at $(0,0)$ $\qquad$ f singular at $(0,0)$

Not every singular point has the same type of singularity; for instance, the figures below show three different singular curves. Measuring singularity of a curve, surface or any variety at a point has been an object of study in geometry.



$y^2 - x^3 - x^2 = 0$ $\qquad$ $y^2 - x^3 = 0$ $\qquad$ $y^2 - x^9 = 0$

Now instead of the real numbers, we consider the integers modulo p, a prime number. Then every integer is equivalent to its remainder while applying the division algorithm. For instance, if $p = 5$, then $5 \equiv 0, 6 \equiv 1$ and $12 \equiv 2$. When we take a polynomial with coefficients over the integers modulo p, we can use the Frobenius map, $r \mapsto r^p$, to classify singularities. Two important types of singularities are the F-regular and the F-pure singularities. The F-regular singularities behave similarly to a smooth point from the perspective of tight closure theory (see Chapter II.6 and [HH90, HH94a]). Similarly, the F-purity of singularities simplifies computations for cohomology groups and implies vanishing properties of these groups (see Chapter II.5). These properties are very important and much-studied in algebraic geometry and commutative algebra [BMS08, BMS09, Har68, HL90, Fed87, Ogu73, PS73]. The relations among these properties are the following:

$$\left\{\begin{matrix}\text{Any}\\ \text{point}\end{matrix}\right\} \supset \left\{\begin{matrix}F-\text{pure}\\ \text{singularity}\end{matrix}\right\} \supset \left\{\begin{matrix}F-\text{Regular}\\ \text{singularity}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{Smooth}\\ \text{point}\end{matrix}\right\}.$$

The study of algebraic invariants that measure a singularity has an important role in the study of its geometric properties. In particular, this work focuses on those invariants that measure the difference between the types of singularities previously described.

I.2 Local cohomology

In order to study singularities, we first introduce the main algebraic object investigated in this thesis: local cohomology (see Chapter II.3). These modules were first introduced by Grothendieck in the 60's (see [Har67]). If M is an R-module and $I \subset R$ is an ideal, we denote the i-th local cohomology of M with support in I by $H^i_I(M)$. These modules capture several algebraic and geometric properties of a ring, R, an ideal, I, and an R-module, M, for instance, Cohen-Macaulayness of R, depth of I, and dimension of M.

In a basic course in complex analysis one studies the difficulty of extending a holomorphic function to a region where it is undefined. For instance, a removable singularity is that in which it is possible to define the function at that point. A pole of order m is a singularity that could be "controlled" with a polynomial of order m. Therefore, the higher the order of the pole, the harder to extend the function. One of the many strong connections between local cohomology of modules and cohomology of sheaves is that the elements of $H^1_I(M)$ give the obstruction to extending sections of M supported off $\mathcal{V}(I)$ to all $\operatorname{Spec}(R)$. Here, as before, $\mathcal{V}(I)$ denotes the closed set defined by the vanishing of elements in I.

The local cohomology modules are usually not finitely generated; however, they satisfy finiteness properties over regular local rings containing a field and over unramified regular local rings [HS93, Lyu93, Lyu00b, NB12b]: The set of associated primes of $H^i_I(R)$ is finite (see Chapter II.1), the Bass numbers of $H^i_I(R)$ are finite (see Chapter II.2), and $\operatorname{inj.dim} H^i_I(R) \leq \dim_S \operatorname{Supp} H^i_I(R)$ (see Chapter II.2).

Lyubeznik approached these problems using modules over a ring of differential operators (see Chapter II.4 and [Lyu93]). Given two commutative rings A and R such that $A \subset R$, the ring of A-linear differential operators of R, $D(R, A)$, is defined as the subring of $\operatorname{Hom}_A(R, R)$ obtained inductively as follows. The differential operators of order zero are morphisms induced by multiplication by elements in R ($\operatorname{Hom}_R(R, R) = R$). An element $\theta \in \operatorname{Hom}_A(R, R)$ is a differential operator of order less than or equal to $k+1$ if $\theta \cdot r - r \cdot \theta$ is a differential operator of order less than or equal to k for every $r \in R$. In particular, if $R = \mathbb{C}[x_1, \dots, x_n]$, then

$$D(R, \mathbb{C}) = R\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle.$$

If M is a $D(R, A)$-module, then M_f has the structure of a $D(R, A)$-module such that, for every $f \in R$, the natural morphism $M \to M_f$ is a morphism of $D(R, A)$-

modules. As a consequence, $H^{i_1}_{I_1} \cdots H^{i_\ell}_{I_\ell}(R)$ is also a $D(R, A)$-module [Lyu93].

The only cases of regular rings for which these important structural properties have not been shown are the regular local rings of ramified mixed characteristic $p > 0$. A proof or a counter-example for the remaining case would give us a better understanding of rings of mixed characteristic. Thus, the main conjecture that motivates this research is:

Conjecture I.2.1. Let (R, m, K) be a regular local ring of ramified mixed characteristic $p > 0$. Then

(a) the set of associated primes of $H^i_I(R)$ is finite;

(b) the Bass numbers of $H^i_I(R)$ are finite;

(c) $\operatorname{inj.dim} H^i_I(R) \leq \dim_S \operatorname{Supp} H^i_I(R)$.

for every ideal I and every $i \in \mathbb{N}$.

One of the main results of this thesis is the finiteness of the set formed by certain associated primes of local cohomology over any regular ring of mixed characteristic:

Theorem I.2.2 (see Theorem III.3.5 and [NB13])**.** *Let (R, m, K) be a regular commutative Noetherian local ring of mixed characteristic $p > 0$. Then the set of associated primes of $H^i_I(R)$ that do not contain p is finite for every $i \in \mathbb{N}$ and every ideal $I \subset R$.*

The main tool developed to prove the previous theorem is an extension to a greater generality of results about rings of differential operators (cf. [Bjö79, Bjö72, MNM91]):

Theorem I.2.3 (see Theorem III.2.13 and [NB13])**.** *Let R be a regular commutative Noetherian ring with unity that contains a field, F, of characteristic 0 satisfying the following conditions:*

(1) *R is equidimensional of dimension n;*

(2) *every residual field with respect to a maximal ideal is an algebraic extension of F;*

(3) $\operatorname{Der}_F(R)$ *is a finitely generated projective R-module of rank n such that for every maximal ideal $m \subset R$, $R_m \otimes_R \operatorname{Der}_F(R) = \operatorname{Der}_F(R_m)$.*

Then the ring of F-linear differential operators $D(R, F)$ is a ring of differentiable type of weak global dimension equal to $\dim(R)$. Moreover, the Bernstein class of $D(R, F)$ is closed under localization at one element.

Using D-modules over a polynomial or power series ring with coefficients over a ring of small dimension, we extend some of Lyubeznik's results to these rings:

Theorem I.2.4 (see Theorem IV.2.6 and [NB12b])**.** *Let A be a zero dimensional commutative Noetherian ring. Let R be either $A[x_1, \dots, x_n]$ or $A[[x_1, \dots, x_n]]$. Then*

(i) *the set associated primes of $H^i_I(R)$ is finite, and*

(ii) *the Bass numbers of $H^i_I(R)$ are finite*

for every ideal I and every $i \in \mathbb{N}$.

Theorem I.2.5 (see Theorem IV.2.10 and [NB12b])**.** *Let A be a one-dimensional ring, and let R be either $A[x_1, \dots, x_n]$ or $A[[x_1, \dots, x_n]]$. Let $\pi \in A$ denote an element such that $\dim(A/\pi A) = 0$. Then, the set of associated primes over R of $H^i_I(R)$ that contain π is finite for every ideal I and every $i \in \mathbb{N}$. Moreover, if A is Cohen-Macaulay and π is a nonzero divisor then the Bass numbers of $H^i_I(R)$, with respect to a prime ideal P that contains π, are finite.*

The previous two theorems recover results of Lyubeznik [Lyu00b] for regular local rings of unramified mixed characteristic, but his proofs use a different approach. The motivation behind these theorems is to find techniques to prove Conjecture I.2.1 for

$$\frac{V[[x, y, z_1, \dots, z_n]]}{(\pi - xy)V[[x, y, z_1, \dots, z_n]]},$$

where $(V, \pi V, K)$ is a complete DVR of mixed characteristic. This is, to the best of our knowledge, the simplest example of a regular local ring of ramified mixed characteristic in which the claims of Conjecture I.2.1 are unknown. Motivated by this example and previous results [HS93, Lyu93, NB12b, Rob12], Hochster raised the following related question:

Question I.2.6 (see Question V.0.4)**.** Let (A, m, K) be a local ring and R be a flat extension with regular closed fiber. Is

$$\operatorname{Ass}_R H^0_{mR} H^i_I(R) = \mathcal{V}(mR) \cap H^i_I(R)$$

finite for every ideal $I \subset R$ and $i \in \mathbb{N}$?

We answer this question affirmatively for some cases:

Theorem I.2.7 (see Theorem V.4.3 and [NB12a]). *Let $(A, m, K) \to (R, \eta, L)$ be a flat extension of local rings with regular closed fiber such that A contains a field. Let $I \subset R$ be an ideal such that $\dim(A/I \cap A) \leq 1$. Suppose that the morphism induced in the completions $\widehat{A} \to \widehat{R}$ maps a coefficient field of A into a coefficient field of R. Then*

$$\operatorname{Ass}_R H^0_{mR} H^i_I(R)$$

is finite for every $i \in \mathbb{N}$. Moreover, if R is either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$, then $\operatorname{Ass}_R H^j_{mR} H^i_I(R)$ is finite for every ideal $I \subset R$ such that $mR \subset \sqrt{I}$ and every $j \in \mathbb{N}$.

When A is not a zero-dimensional ring the local cohomology modules are not necessarily of finite length or finitely generated as D-modules. To avoid this difficulty, we introduce Σ-finite D-modules, which are D-modules that behave similarly to D-modules of finite length.

Definition I.2.8 (see Definition V.1.2 and [NB12a]). Let M be a D-module supported at mR and $\mathcal{M}$ be the set of all $D(R, A)$-submodules of M that have finite length. For $N \in \mathcal{M}$, let $\mathcal{C}(N)$ denote the composition series of N as $D(R, S)$-module. We say that M is Σ-*finite* if:

(i) $\bigcup_{N \in \mathcal{M}} N = M$,

(ii) $\bigcup_{N \in \mathcal{M}} \mathcal{C}(N)$ is finite, and

(iii) for every $N \in \mathcal{M}$ and $L \in \mathcal{C}(N)$, $L \in C(R/mR, A/mA)$.

This work also includes results on the finiteness of local cohomology over direct summands of regular rings. An example of this is given when S is a polynomial ring over a field and R is the invariant ring of an action of a linearly reductive group over S [DK02]. Another example is when $R \subset K[x_1, \ldots, x_n]$ is an integrally closed ring that is finitely generated as a K-algebra by monomials. This is because such a ring is a direct summand of a possibly different polynomial ring (cf. [Hoc72, Proposition 1 and Lemma 1]). Another case in which an inclusion splits is when $R \to S$ is a module finite extension of rings containing a field of characteristic zero such that S has finite projective dimension as an R-module. Moreover, such a splitting exists when Koh's conjecture holds (cf. [Koh83, Vél95, VF00]). The results in this direction are:

Theorem I.2.9 (see Theorem VI.1.5 and [NB12c]). *Let $R \to S$ be a homomorphism of Noetherian rings that splits. If $\operatorname{Ass}_S H^i_{IS}(S)$ is finite, then $\operatorname{Ass}_R H^i_I(R)$ is finite for every ideal $I \subset R$.*

Theorem I.2.10 (see Theorem VI.2.4 and [NB12c])**.** *Let $R \to S$ be a homomorphism of Noetherian rings that splits such that S is finitely generated as R-module. Suppose that S is a Cohen-Macaulay ring such that the Bass numbers of $H^i_{IS}(S)$ are finite for every ideal $I \subset R$. Then the Bass numbers of $H^i_I(R)$ are finite.*

Corollary I.2.11 (see Corollary VI.1.7 and [NB12c])**.** *There exists a Gorenstein F-regular UFD, R, that is not a pure subring of any regular ring. In particular, R is not direct summand of any regular ring.*

We point out that the property about injective dimension does not hold for direct summands of regular rings, even in the finite extension case. A counterexample is $R = K[x^3, x^2y, xy^2, y^3] \subset S = K[x, y]$, where S is the polynomial ring in two variables with coefficients in a field K. The splitting of the inclusion is the map $\theta : S \to R$ defined in the monomials by $\theta(x^\alpha y^\beta) = x^\alpha y^\beta$ if $\alpha + \beta \in 3\mathbb{Z}$ and as zero otherwise. We have that the dimension of $\operatorname{Supp}(H^2_{(x^3,x^2y,xy^2,y^3)}(R))$ is zero, but it is not an injective module, because R is not a Gorenstein ring, since $R/(x^3, y^3)R$ has a two dimensional socle.

I.3 Applications to measure of singularity

The finiteness structural properties of local cohomology over regular rings have been applied to define algebro-geometric invariants [ÀM04, Lyu93, NBW12a, NBW12b, NBWZ13, Zha11a] or to find properties for certain kind of rings [Kaw02, Mar01, NB12c, Zha07]. Using the result obtained for local cohomology modules, we expand these applications to study singularity of local rings.

I.3.1 F-Jacobian ideals

Suppose that $S = K[x_1, \dots, x_n]$ is a polynomial ring over a perfect field K, and $f \in S$. The Jacobian ideal is defined by $\operatorname{Jac}(f) = (f, \frac{\partial f}{\partial_{x_1}}, \dots, \frac{\partial f}{\partial_{x_n}})$. This ideal plays a fundamental role in the study of singularity in zero and positive characteristic. In this case, $\operatorname{Jac}(f) = R$ if and only if R/fR is a regular ring. Another important property, given by the Leibniz rule, is that $\operatorname{Jac}(fg) \subset f \operatorname{Jac}(g) + g \operatorname{Jac}(f)$ for $f, g \in S$. The equality in the previous containment holds only in specific cases [Fab13, Proposition 8] and it is used to study transversality of singular varieties [Fab13, FA12].

Let R be an F-finite regular local ring. We define the F-Jacobian ideal, $J_F(f)$, to be the pull back of the intersection of $(R/fR) \subset H^1_f(R)$ with the sum of the simple F-submodules in the local cohomology module $H^1_f(R)$. The F-Jacobian ideal behaves similarly to the Jacobian ideal of a polynomial. Like the Jacobian ideal, they determine singularity:

- if R/fR is F-regular, then $J_F(f) = R$ (see Corollary VII.2.11);
- if R/fR is F-pure, then R/fR is F-regular if and only if $J_F(f) = R$ (see Corollary VII.2.13).
- If f has an isolated singularity and R/fR is F-pure, then $J_F(f) = R$ if R/fR is F-regular, and $J_F(f) = m$ otherwise (see Proposition VII.3.1).

In particular, we have that the submodules of the local cohomology module $H^1_f(R)$ give information about the singularity of R/fR or $\mathcal{V}(f)$. In addition, the F-Jacobian ideal also satisfies a Leibniz rule: $J_F(fg) = fJ_F(g) + gJ_F(f)$ for relatively prime elements $f, g \in R$ (Proposition VII.1.14). The Leibniz rule in characteristic zero is important in the study of transversality of singular varieties and free divisors over the complex numbers [Fab13, FA12].

The F-Jacobian ideals behave well with p^e-th powers $J_F(f^{p^e}) = J_F(f)^{[p^e]}$ (Proposition VII.1.19). This is a technical property that was essential in several proofs. This contrasts with how the Jacobian ideal changes with p^e-th powers: $\operatorname{Jac}(f^{p^e}) = f^{p^e}R$.

Furthermore, we define the F-Jacobian ideal for a regular F-finite UFD such that R_f/R has finite length as D-module (Section VII.1) and for an algebra essentially of finite type over an F-finite local ring (Chapter VII.2).

I.3.2 Generalized Lyubeznik numbers

Lyubeznik introduced a set of invariants, now called Lyubeznik numbers, to study rings of equal characteristic [Lyu93]. Suppose that (R, m, K) is a local ring admitting a surjection from an n-dimensional local regular local ring (S, η, K) containing a field. If I is the kernel of this surjection, the Lyubeznik numbers of R, depending on two nonnegative integers i and j, are defined as $\lambda_{i,j}(R) := \dim_K \operatorname{Ext}^i_S\left(K, H^{n-j}_I(S)\right)$. Remarkably, these numbers only depend on the ring R and on i and j, not on S, nor even on the choice of surjection from S [Lyu93, Theorem 4.1]. Moreover, if R is *any* local ring containing a field, letting $\lambda_{i,j}(R) := \lambda_{i,j}(\widehat{R})$ extends the original definition, making the Lyubeznik numbers well defined for every such ring (see [NBWZ13] for a survey on this subject).

For R containing a field, the Lyubeznik numbers of R provide essential information about the ring, and have extensive connections with geometry and topology, including étale cohomology and the connected components of certain punctured spectra (see, for example, [BB05, GLS98, Kaw00, Wal01, Zha07]).

The generalized Lyubeznik numbers were introduced by the author and Witt [NBW12a] with the aim to extend the the study of rings via local cohomology. To prove that these generalized Lyubeznik numbers are well defined, we formalize and develop the theory of a functor that Lyubeznik utilized to show that his original invariants are well defined [Lyu93]. In particular, the definition of these new invariants relies heavily on the fact that this functor gives a category equivalence with a certain category of D-modules, which somehow mirrors Kashiwara's equivalence [Cou95].

Theorem I.3.1 (see Theorem VIII.0.10 and [NBW12a])**.** *Let R be a Noetherian ring, and let $S = R[[x]]$. Let $\mathcal{C}$ denote the category of R-modules and $\mathcal{D}$ the category of $D(S,R)$-modules that are supported on $\mathcal{V}(xS)$, the Zariski closed subset of* $\operatorname{Spec}(S)$ *given by xS. Then the functor*

$$\begin{aligned} G : \mathcal{C} &\to \mathcal{D} \\ M &\mapsto M \otimes_R S_x/S \end{aligned}$$

is an equivalence of categories, with inverse functor $\widetilde{G} : \mathcal{D} \to \mathcal{C}$ given by $\widetilde{G}(N) =$ $\operatorname{Ann}_N(xS)$.

Moreover, if $R = K[[y_1, \ldots, y_n]]$, K a field, then $S = K[[y_1, \ldots, y_n, x]]$, and G is an equivalence of categories between the category of $D(R,K)$-modules and the category of $D(S,K)$-modules supported on $\mathcal{V}(xS)$.

The definition of the generalized Lyubeznik numbers depends on the fact that certain local cohomology modules have finite length as $D(S,K)$-modules, where K is a field and $S = K[[x_1, \ldots, x_n]]$ for some n [Lyu00a, Corollary 6]. These new invariants depend on the local ring R containing a field, a coefficient field $K' \subseteq \widehat{R}$, a collection of ideals $I_1, \ldots, I_s$ of R, as well as $j_1, \ldots, j_s \in \mathbb{N}$. The definition is as follows:

Definition I.3.2 (see Definition IX.1.4 and [NBW12a])**.** Let (R, m, K) be a local ring containing a field, and $\widehat{R}$ its completion at m. Let K' be a coefficient field of $\widehat{R}$. Then $\widehat{R}$ admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where $S = K[[x_1, \ldots, x_n]]$ for some $n \in \mathbb{N}$, and $\pi(K) = K'$. For $1 \leq i \leq s$, fix $j_i \in \mathbb{N}$ and ideals $I_i \subseteq R$, and let $J_i = \pi^{-1}(I_i\widehat{R}) \subseteq S$. The *generalized Lyubeznik number of R with respect to K', $I_1, \ldots, I_s$ and $j_1, \ldots, j_s$,*

$$\lambda^{j_s,\ldots,j_1}_{I_s,\ldots,I_1}(R; K') := \operatorname{length}_{D(S,K')} H^{i_s}_{J_s} \cdots H^{i_2}_{J_2} H^{n-i_1}_{J_1}(S),$$

is finite and depends only on R, K', $I_1, \dots, I_s$ and $j_1, \dots, j_s$, but neither on S nor on π.

Although the generalized Lyubeznik numbers a priori depend on the choice of a coefficient field of $\widehat{R}$, there are some cases where only one such field exists. For example, this happens when K is a perfect field of characteristic $p > 0$. Whether it is possible to avoid the dependence of

$$\operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S) = \operatorname{length}_{D(S,L)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S)$$

on the choice of coefficient field of S is, to the best our knowledge, an open question.

These invariants include the original Lyubeznik numbers. As a consequence of this new approach, our work also gives a different proof that the original Lyubeznik numbers are well defined.

Proposition I.3.3 (see Proposition IX.1.8 and [NBW12a]). *If (R, m, K) is a local ring containing a field, then*

$$\lambda_{i,j}(R) = \lambda^{i,j}_{m,0}(R; K')$$

for any coefficient field K' of $\widehat{R}$.

Generalized Lyubeznik numbers behave similarly to the original Lyubeznik numbers. In particular, we have the following properties:

Proposition I.3.4 (see Proposition IX.1.10 and [NBW12a]). *Given ideals $I_1 \subseteq \ldots \subseteq I_s$ of a local ring (R, m, K) containing a field, $i_j \in \mathbb{N}$ for $1 \leq j \leq s$, and a coefficient field K' of $\widehat{R}$, we have that*

(i) $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R; K') = 0$ *for* $i_1 > \dim(R/I_1)$,

(ii) $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R; K') = 0$ *for* $i_j > \dim(R/I_{j-1})$ *and* $2 \leq j \leq \ell$,

(iii) $\lambda^{i_2,i_1}_{I_2,I_1}(R; K') = 0$ *for* $i_2 > i_1$,

(iv) $\lambda^{i_1}_{I_1}(R; K') \neq 0$ *for* $i_1 = \dim(R/I_1)$, *and*

(v) $\lambda^{i_2,i_1}_{I_2,I_1}(R; K') \neq 0$ *if* $i_2 = \dim(R/I_1) - \dim(R/I_2)$ *and* $i_1 = \dim(R/I_1)$.

We also introduce a new invariant, *the Lyubeznik characteristic,* which is inspired by the the definition of the Euler characteristic using the Betti numbers [Eis05].

Definition I.3.5 (see Definition IX.1.16 and [NBW12a])**.** Let (R, m, K) be a local ring containing a field such that $\dim(R) = d$. We define the *Lyubeznik characteristic* of R by

$$\chi_\lambda(R) = \sum_{i=0}^{d} (-1)^i \lambda_0^i(R).$$

In addition, results of Blickle [Bli04a] enable characterizations of F-regularity and F-rationality in terms of certain generalized Lyubeznik numbers.

Proposition I.3.6 (see Proposition X.1.2 and [NBW12a])**.** *Let (R, m, K) be a complete local domain of characteristic $p > 0$ and of dimension d, such that K is F-finite. The following hold:*

(i) *If $\lambda_0^d(R) = 1$, then $0^*_{H^d_m(R)}$ is F-nilpotent.*

(ii) *If R is F-injective and $\lambda_0^d(R) = 1$, then R is F-rational.*

In addition, if K is perfect:

(iii) *$\lambda_0^d(R) = 1$ if and only if $0^*_{H^d_m(R)}$ is F-nilpotent.*

(iv) *If R is F-rational, then $\lambda_0^d(R) = 1$.*

(v) *If R is F-injective, then $\lambda_0^d(R) = 1$ if and only if R is F-rational.*

Moreover, if R is one-dimensional:

(vi) *If $\lambda_0^d(R) = 1$, then R is unibranch.*

(vii) *If K is perfect, then $\lambda_0^d(R) = 1$ if and only if R is unibranch.*

The previous theorem motivates the idea of generalized Lyubeznik numbers to study singularity in positive characteristic. In order to make statements about this idea we recall some results for test ideals (see Chapter II.6). We assume that (S, m, K) is a complete regular local ring of characteristic $p > 0$. We fix a radical ideal $I \subset S$ and define $R = S/I$. We set $n = \dim(S)$, $d = \dim(R)$ and $c = n - d$. The test ideal of R, $\tau(R)$, plays a crucial role in tight closure theory. For instance, this ideal determines whether R is strongly F-regular.

If (R, m, K) is an F-finite regular local ring and $I \subset R$ is an ideal such that R/I is F-pure, there exists an strictly ascending chain of ideals

$$I = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

such that $(\tau_i^{[p]} : \tau_i)) \subset (\tau_{i+1}^{[p]} : \tau_{i+1})$ and τ_{i+1} is the pullback of the test ideal of R/τ_i [Vas98] (see Chapter II.6).

We first confirm that the Lyubeznik numbers measure singularity for hypersurfaces [NBP13]:

Theorem I.3.7 ([NBP13])**.** *Let (R, m, K) be an F-finite complete regular local ring, and $f \in R$ such that R/fR is reduced. If R/fR is F-pure and*

$$0 \subset fR = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

is the flag of ideals defined above, then

$$\ell \leq \lambda_0^{\dim(R/fR)}(R/fR; K').$$

for every K' coefficient field of R.

When $R = K[[x_1, \ldots, x_n]]$ the previous theorem implies that, when R/fR is F-pure, ℓ is a lower bound the generalized Lyubeznik numbers, $\lambda^{\dim(K';R/fR)}(R/fR)$. The previous theorem says that $\lambda^{\dim(R/fR)}(R/fR)$ is measuring how far is an F-pure hypersurface from being F-regular.

Developing some techniques in [NBP13], we extend the previous theorem to Gorenstein rings:

Theorem I.3.8 (see Theorem X.2.9 and [HNBW13])**.** *Suppose that R is Gorenstein and F-pure. Let*

$$I = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

be the flag of test ideals defined by Vassilev. Then, $\ell \leq \lambda_0^d(R; K')$ for every coefficient field K'.

Smith [Smi97] proved that an F-pure Cohen-Macaulay ring R is F–rational if and only if $H_m^d(R)$ is a simple left $R\langle F\rangle$ module (see Chapter II.9). We have that, for Cohen-Macaulay rings, $\text{length}_{R\langle F\rangle} H_m^d(R)$ gives a measure of how far R is from being F–rational. Using results of Lyubeznik on F-modules [Lyu97], of Blickle on intersection homology [Bli04b] and of Ma on $R\langle F\rangle$-modules [Ma12], we prove that the highest generalized Lyubeznik number $\lambda_0^d(R; K')$ is an upper bound for $\text{length}_{R\langle F\rangle} H_m^d(R)$. This results holds for all F-finite rings even if they are not Cohen-Macaulay.

Theorem I.3.9 (see Theorem X.3.1 and [HNBW13])**.** *Suppose that R is an F-pure ring. Then*

$$\text{length}_{R\langle F\rangle} H_m^d(R) \leq \lambda_0^d(R; K')$$

for every coefficient field K'.

Furthermore, the study of the generalized Lyubeznik numbers of a Stanley-Reisner ring, R, give connections with the simplicial complex that gives rise to R. In addition, we find a connection with categories related to simplicial complexes (Chapter IX.3).

Theorem I.3.10 (see Theorem IX.3.10 and [NBW12a]). *Let K be a field, $S = K[x_1, \ldots, x_n]$, and $\widehat{S} = K[[x_1, \ldots, x_n]]$. Let $I_1, \ldots, I_s \subseteq S$ be ideals generated by square-free monomials. Then*

$$\begin{aligned}\lambda^{i_1,\ldots,i_s}_{I_1,\ldots,I_s}(\widehat{S}) &= \operatorname{length}_{D(\widehat{S},K)} H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(\widehat{S}) \\ &= \operatorname{length}_{\mathbf{Str}} H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(\omega_S) \\ &= \sum_{\alpha \in \{0,1\}^n} \dim_k \left[H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(\omega_S) \right]_{-\alpha}\end{aligned}$$

Moreover, if $\operatorname{char}(K) = 0$, *then*

$$\lambda^{i_1,\ldots,i_s}_{I_1,\ldots,I_s}(\widehat{S}) = e(H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(S)),$$

where $e(-)$ denotes $D(S,K)$-module multiplicity.

It is well know that there is a bijective correspondence between simplicial complexes and square-free monomial ideals [MS05]. Using this bijection, we relate the Lyubeznik characteristic of a Stanley-Reisner ring with its simplicial complex associated to it.

Theorem I.3.11 (see Theorem IX.4.10 and [NBW12a]). *Take a simplicial complex Δ on the vertex set $[n]$. Let R be the Stanley-Reisner ring of Δ, and let m be its maximal homogeneous ideal. Then*

$$\chi_\lambda(R_m) = \sum_{i=-1}^{n} (-2)^{i+1} |F_i(\Delta)|.$$

The previous theorem says, in particular, that the Lyubeznik characteristic does not depend on the chosen field. This contrasts how the original and the generalized Lyubeznik numbers behave with respect to the characteristic of the fields (see Example IX.4.9 and [ÀMV])

I.3.3 Lyubeznik numbers in mixed characteristic

We define a new family of invariants associated to *any* local ring whose residue field has prime characteristic. These numbers are again defined using local cohomology modules over a regular ring. The introduction of these invariants has the objective of studying all rings of mixed characteristic through regular rings of unramified mixed characteristic, whose local cohomology have finiteness properties.

If S is a regular local ring of unramified mixed characteristic, the Bass numbers of local cohomology modules of the form $H^i_I(S)$ are finite (see Theorem IV.3.1 and [Lyu00b, NB12b]). Using the theory of p-bases, and explicit constructions used in the Cohen Structure Theorems, we prove that the Lyubeznik numbers in mixed characteristic are well-defined:

Definition I.3.12 (see Definition XI.1.7 and [NBW12b])**.** Let (R, m, K) be a local ring such that $\operatorname{char}(K) = p > 0$, and let $\widehat{R}$ denote its completion. By the Cohen Structure Theorems, $\widehat{R}$ admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where S is an n-dimensional unramified regular local ring of mixed characteristic. Let $I = \operatorname{Ker}(\pi)$ and take $i, j \in \mathbb{N}$. Then the *Lyubeznik number of R in mixed characteristic with respect to i and j* is defined as

$$\widetilde{\lambda}_{i,j}(R) := \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S)).$$

This number is finite and depends only on R, i, and j, but not on S, nor on π.

We have again that these new invariants behave similar to the original.

Proposition I.3.13 (see Proposition XI.1.11 and [NBW12b])**.** *Let (R, m, K) be a local ring such that $\operatorname{char}(K) = p > 0$ and $d = \dim(R)$. Then*

(i) $\widetilde{\lambda}_{i,j}(R) = 0$ *if $j > d$ or $i > j + 1$, and*

(ii) $\widetilde{\lambda}_{d,d}(R) \neq 0$.

Since the structure of the local cohomology over regular rings of mixed characteristic is not as nice as in equal characteristic, we need to overcome some technical difficulties by studying further the injective dimension. In particular, we find vanishing theorems for these local cohomology modules; therefore, for the Lyubeznik numbers in mixed characteristic.

Theorem I.3.14 (see Theorem XI.2.10 and [NBW12b])**.** *Let (S, m, K) be either a regular local ring of unramified mixed characteristic, or a regular local ring containing* *a field. Let* $n = \dim(S)$*, and let* I *be an ideal of* S *such that* $\dim(S/I) = d$*. Then*

$$\operatorname{inj.dim}(H_I^{n-d}(S)) = d.$$

In particular, if $d = \dim R$, $\widetilde{\lambda_{d,d}}(R) \neq 0$, and $\widetilde{\lambda}_{i,j}(R) = 0$ if either $i > d$ or $j > d$, so the "highest" Lyubeznik number exists.

When R is a ring of equal characteristic $p > 0$, we have two notions of Lyubeznik numbers: the original defined by Lyubeznik [Lyu93] and the new one introduced in [NBW12b]. We give some conditions for which these two definitions agree.

Proposition I.3.15 (see Corollary XI.3.4 and [NBW12b])**.** *Let* (R, m, K) *be a local ring of characteristic* $p > 0$ *such that either* $\dim(R) \leq 2$ *or* R *is Cohen-Macaulay. Then*

$$\widetilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R).$$

In addition, we present an example, inspired by an the triangularization of the projective real plane, in which these invariants disagree.

Theorem I.3.16 (see Theorem XI.4.12 and [NBW12b])**.** *There exists a regular local ring* (S, m, K) *of unramified mixed characteristic* $p > 0$*, and an ideal* $I \subseteq S$*, such that* S/pS *is a regular ring,* $p \in I$ *and*

$$\widetilde{\lambda}_{i,j}(S/I) = \dim_K \operatorname{Ext}_S^j(K, H_I^i(S)) \neq \dim_K \operatorname{Ext}_{S/pS}^j(K, H_{IS/pS}^{i-1}(S/pS)) = \lambda_{i,j}(S/I).$$

CHAPTER II

Background

In this chapter we introduce the concepts and tools that we need to prove the results obtained in this work. We refer to [AM69, Eis95, Mat80] for details about associated primes, to [Bas63] for injective modules and Bass numbers, to [BS98, ILL+07] for local cohomology, to [Bjö79, Bjö72, Cou95, Lyu93, MNM91] for D-modules in characteristic zero, to [Lyu97, Lyu00a, Smi95a, Yek92] for D-modules in positive characteristic, to [Fed87, Smi95a] for F-split, F-pure and F-injective rings, to [HH90, HH94a, HH94b] for tight closure to [BMS08, BMS09, HY03] for generalized test ideals. to [Lyu97] for F-modules to [Bli03] for $R\langle F\rangle$-modules.

II.1 Associated primes

A prime ideal $P \subset R$ is an *associated prime* of an R-module, M, if one of the following equivalent conditions holds

- There is an injection $R/ \hookrightarrow M$;
- there exists an element $u \in M$ such that $P = \operatorname{Ann}_R u$.

the set of associated primes of M is denoted by $\operatorname{Ass}_R(M)$. Every zerodivisor of M belong to an associated primes. In other words, the union of the associated primes form the set of the zerodivisors for M. A prime ideal $P \subset R$ is in the *support of* M if $M_P \neq 0$, and we take $\operatorname{Supp}_R(M) = \{P \in \operatorname{Spec}(R) \mid M_P \neq 0\}$. If M is a finitely generated module, we have that $\operatorname{Supp}_R(M)$ is Zariski closed subset of $\operatorname{Spec}(R)$; moreover, $\operatorname{Supp}_R(M) = \mathcal{V}(\operatorname{Ann}_R(M))$. The minimal elements of $\operatorname{Supp}_R(M)$ are the same as the minimal elements of $\operatorname{Ass}_R(M)$.

If $W \subset R$ is a multiplicative system, we have that

$$\operatorname{Ass}_{W^{-1}R} W^{-1}M = \{PW^{-1}R \mid P \in \operatorname{Ass}_R M \text{ and } R \cap W = \varnothing\}$$

If S is a flat R-algebra, we have that $\mathrm{Ass}_S(M \otimes_R S) = \bigcup_{P \in \mathrm{Ass}_R M} \mathrm{Ass}_S(S/PS)$. In particular, if S is a faithfully flat algebra, we have that

$$\mathrm{Ass}_S(M \otimes_R S) \text{ is finite} \Leftrightarrow \mathrm{Ass}_R(M) \text{ is finite.}$$

This property allows to pass to the completion of R to study associated primes, when R is a local ring.

II.2 Injective modules and Bass numbers

An R-module E is injective if the functor $\mathrm{Hom}_R(-, E)$ is exact (it is always left exact). The category of R-modules have enough injectives, this is, for every R-module there exist an injective R-module E and an injection $M \hookrightarrow E$.

An essential extension of M is an R-module with an injection $M \hookrightarrow N$ such that every nonzero submodule of N non-trivially intersects the image of M. By Zorn's Lemma every R-module have a maximal essential extension. We have that a an R-module is injective if and only if it has no proper essential extension. If $M \subset E$, where E is injective, we have that the maximal essential extension of M in E is an injective R-module. Moreover, it is a maximal essential extension of M in an absolute sense: it is not a properly contained in any module that is an essential extension of M. It is called *the injective hull of* M and denoted by $E_R(M)$. Every injective module is a direct sum of injective hulls of the form $E_R(R/P)$, where P is a prime ideal.

Given a module M over a ring S, we build a complex $E^\bullet$ as follows, We take $E^0 = E_R(M)$ and $N_1 = \mathrm{Coker}(M \hookrightarrow E^0)$. Then, we take $E^1 = E_R(N_1)$. By countinuing this process, we obtain a minimal injective resolution, $E^\bullet$, of M. The number of copies of $E_R(k)_S(S/P)$ in E^i is the *i-th Bass number of M with respect to P*, denoted $\mu_i(P, M)$ and as well equal to $\dim_{S_P/PS_P} \mathrm{Ext}^i_S(S_P/PS_P, M_P)$. If R is a Gorenstein ring, we have that $\mathrm{Ext}^i_S(S_P/PS_P, M_P) = 1$ if $\mathrm{ht}(P) = i$ and zero otherwise.

If (R, m, K) is a complete local ring, the injective hull of the residue field, $E_R(K)$, plays an important role in the study of R-module. For instance, the Matlis duality, $^\vee = \mathrm{Hom}_R(-, E_R(K)$, is defined using this injective module. Moreover, we have that

$$M = (M^\vee)^\vee.$$

The functor $\mathrm{Hom}_R(M, E_R(K))$ is called the *Matlis dual* of M and $\mathrm{Hom}_R(-, E_R(K))$ gives an anti-equivalence between the category of Noetherian R-modules and the category of Artinian R-modules.

II.3 Local cohomology

Let R be a ring, $I \subset R$ an ideal, and M an R-module. If I is generated by $f_1, \ldots, f_\ell \in R$, *the Čech complex,* $\check{\mathrm{C}}(\underline{f}; M)$, is defined as

$$0 \to M \to \oplus_j M_{f_j} \to \ldots \to M_{f_1 \cdots f_\ell} \to 0,$$

Here, $\check{\mathrm{C}}^i(\underline{f}; S) = \bigoplus_{j_1 < \ldots < j_i} S_{f_{j_1} \cdots f_{j_i}}$, and each morphism $\check{\mathrm{C}}^i(\underline{f}; M) \to \check{\mathrm{C}}^{i+1}(\underline{f}; M)$ is a localization map with an appropriate sign. For instance, if $\ell = 2$, the complex is

$$0 \to M \to M_{f_1} \oplus M_{f_2} \to M_{f_1 f_2} \to 0,$$

where $M \to M_{f_1} \oplus M_{f_2}$ sends $v \mapsto (\frac{v}{1}, \frac{v}{1})$ and $M_{f_1} \oplus M_{f_2} \to M_{f_1 f_2}$ sends $\left(\frac{v}{f_1^\alpha}, \frac{w}{f_2^\beta}\right) \mapsto \frac{v}{f_1^\alpha} - \frac{w}{f_2^\beta}$.

We define the *i-th local cohomology of M with support in I* as the i-th cohomology of the complex $\check{\mathrm{C}}^\bullet(\underline{f}; S) \otimes_S M$; i.e.,

$$H_I^i(M) := H^i(\check{\mathrm{C}}^\bullet(\underline{f}; M) = \frac{\mathrm{Ker}\left(\check{\mathrm{C}}^i(\underline{f}; M) \to \check{\mathrm{C}}^{i+1}(\underline{f}; M)\right)}{\mathrm{Im}\left(\check{\mathrm{C}}^{i-1}(\underline{f}; M) \to \check{\mathrm{C}}^i(\underline{f}; M)\right)}$$

There are several ways to define local cohomology. In fact, the definition we chose is not the most natural, although it will be advantageous for us due to the interactions between the cited complex $\check{\mathrm{C}}^\bullet(\underline{f}; S)$ and D-modules (see Chapter II.4). The local cohomology module $H_I^i(M)$ can also be defined as the direct limit, $\varinjlim_t \mathrm{Ext}_S^i(S/I^t, M)$, or as the i-th right derived functor of $\Gamma_I(M) = \{v \in M \mid I^j v = 0 \text{ for some } j \in \mathbb{N}\}$.

Let $\mathcal{K}(f_1, \ldots, f_s; M)$ denote the Koszul complex associated to the sequence $\underline{f} = f_1, \ldots, f_\ell$. In Figure II.3 there is a direct limit involving $\mathcal{K}(f_i^t; M)$, whose limit is $\check{\mathrm{C}}(f_i; M)$.

$$\begin{array}{ccccccccccc}
M & \to & M & \to & M & \to & M & \to & M & \to & \ldots \\
\downarrow & & f_i \downarrow & & f_i^2 \downarrow & & f_i^3 \downarrow & & f_i^4 \downarrow & & \\
M & \xrightarrow{f_i} & M & \xrightarrow{f_i} & M & \xrightarrow{f_i} & M & \xrightarrow{f_i} & M & \xrightarrow{f_i} & \ldots
\end{array}$$

Figure II.3.0.1: Direct limit of Koszul complexes

Let $\underline{f}^t$ denote the sequence $f_1^t, \ldots, f_s^t$. Since

$$\mathcal{K}(\underline{f}; M) = \mathcal{K}(f_1; M) \otimes_R \ldots \otimes_R \mathcal{K}(f_\ell; M),$$

we have that

$$\begin{aligned}\check{C}(\underline{f}; M) &= \check{C}(f_1; M) \otimes_S \ldots \otimes_S \check{C}(f_s; M) \\ &= \lim_{\to t} \mathcal{K}(f_1^t; M) \otimes_S \ldots \otimes_S \lim_{\to} \mathcal{K}(f_s^t; M) \\ &= \lim_{\to t} \mathcal{K}(f_1^t; M) \otimes_S \ldots \otimes_S \mathcal{K}(f_s^t; M).\end{aligned}$$

Hence, $H^i_I(M) = \lim_{\overrightarrow{t}} H^i(\mathcal{K}(\underline{f}; M))$.

The modules $H^i_I(M)$ are usually not finitely generated, even when M is. For instance, if $(S, \mathfrak{m}, K)$ is an n-dimensional regular local ring, then $H^d_m(S) \cong E_S(K)$, the injective hull of K over R, which is not finitely generated unless S is a field.

We define the *cohomological dimension of* I by

$$\mathrm{cd}_R I = \mathrm{Max}\{i \mid H^i_I(R) \neq 0\}.$$

By the definition of local cohomology using the Čech complex, we have that $\mathrm{cd}_R I$ is smaller or equal that the number of minimal set of generator of I; moreover, smaller or equal that the number of the minimal set of generator for any ideal whose radical is $\sqrt{I}$.

Local cohomology characterizes some properties of the ring. For instance, if (R, m, K) is a local ring of dimension d, we have that R is Cohen-Macaulay if and only if

$$H^i_m(R) = 0 \iff i \neq d.$$

In addition, R is Gorenstein if and only if it is Cohen-Macaulay and $H^d_m(R) = E_K(R)$.

There are strong connections between local cohomology and sheaf cohomology: Let M be a finitely generated graded R-module, and let $\widetilde{M}$ be the sheaf on $\mathbb{P}^n$ associated to M. Then there are a functorial isomorphisms (see [Eis95, A4.1])

$$H^t_m(M) \cong \bigoplus_{\ell \in \mathbb{Z}} H^{t-1}(\mathbb{P}^n, \widetilde{M}(\ell)) \text{ when } t \geq 2,$$

and an exact sequence (functorial in M) of degree-preserving maps

$$0 \to H^0_m(M) \to M \to \bigoplus_{\ell \in \mathbb{Z}} H^0(\mathbb{P}^n, \widetilde{M}(\ell)) \to H^1_m(M) \to 0.$$

Among the structural properties obtained for local cohomology is that the set of associated primes of $H^i_I(R)$ is finite for certain regular rings. Huneke and Sharp proved this for characteristic $p > 0$ [HS93]. Lyubeznik showed this finiteness property for regular local rings of equal characteristic zero and finitely generated regular algebras over a field of characteristic zero [Lyu93]. We point out that this property does not necessarily hold for ring that are not regular [Kat02, SS04]. Huneke and Sharp [HS93] proved that if S is a regular ring of characteristic $p > 0$ and I is an ideal of S, then the Bass numbers of the local cohomology modules of the form $H^j_I(S)$, $j \in \mathbb{N}$, are finite, raising the analogous question in the characteristic zero case. Utilizing D-module theory, Lyubeznik proved the same statement for regular local rings of characteristic zero containing a field [Lyu93]. In these cases we also have that $\operatorname{inj.dim}(H^i_I(S)) \leq \dim \operatorname{Supp}(H^i_I(S))$ [HS93, Lyu93].

For regular rings of unramified characteristic we also have that the associated primes and the Bass numbers of local cohomology are finite (see Chapter IV and [Lyu00b, NB12b]). In this case the inequality about injective dimension is weaker: $\operatorname{inj.dim}(H^i_I(S)) \leq \dim \operatorname{Supp}(H^i_I(S)) + 1$ [Zho98].

Many of these properties holds for a family of functors introduced by Lyubeznik [Lyu93]. If $Z \subset \operatorname{Spec}(R)$ is a closed subset and M is an R-module, we denote by $H^i_Z(M)$ the i-th local cohomology module of M with support in Z. This can be calculated via the Čech complex as follows:

$$0 \to M \to \oplus_i M_{f_i} \to \ldots \to \oplus_i M_{f_1 \cdots \hat{f}_1 \cdots f_\ell} \to M_{f_1 \cdots f_\ell} \to 0 \tag{II.3.0.1}$$

where $Z = \mathcal{V}(f_1, \ldots, f_\ell) = \{P \in \operatorname{Spec}(R) : (f_1, \ldots, f_\ell) \subset P\}$

For two closed subsets of $\operatorname{Spec}(R)$, $Z_1 \subset Z_2$, there is a long exact sequence of functors. In particular, $H^i_Z(M) = H^i_I(M)$.

$$\ldots \to H^i_{Z_1} \to H^i_{Z_2} \to H^i_{Z_1/Z_2} \to \ldots \tag{II.3.0.2}$$

Definition II.3.1. We say that $\mathcal{T}$ is a *Lyubeznik functor* if has the form $\mathcal{T} = \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_t$, where every functor $\mathcal{T}_j$ is either H_{Z_1}, $H^i_{Z_1 \setminus Z_2}$, or the kernel, image or cokernel of some arrow in the previous long exact sequence for closed subsets Z_1, Z_2 of $\operatorname{Spec}(R)$ such that $Z_2 \subset Z_1$.

II.4 D-modules

Given rings $A \subseteq S$, we define *the ring of A-linear differential operators of S*, $D(S, A)$, as the subring of $\mathrm{Hom}_A(S, S)$ defined inductively as follows: the differential operators of order zero are induced by multiplication by elements in S. An element $\theta \in \mathrm{Hom}_A(S, S)$ is a differential operator of order less than or equal to $k + 1$ if, for every $r \in S$, $[\theta, r] := \theta \cdot r - r \cdot \theta$ is a differential operator of order less than or equal to k. From the definition, if B is a subring A, then $D(S, A) \subseteq D(S, B)$.

If M is a $D(S, A)$-module, then M_f has the structure of a $D(S, A)$-module such that, for every $f \in S$, the natural morphism $M \to M_f$ is a morphism of $D(S, A)$-modules. As a result, since S is a $D(S, A)$-module, for all ideals $I_1, \ldots, I_s \subseteq S$, and all $i_1, \ldots i_s \in \mathbb{N}$, $H^{i_\ell}_{I_\ell} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(S)$ is also a $D(S, A)$-module [Lyu93, Example 2.1(iv)].

If $S = A[[x_1, \ldots, x_n]]$, then $D(S, A) = S \left\langle \frac{1}{t!} \frac{\partial^t}{\partial {x_i}^t} \mid t \in \mathbb{N}, 1 \leq i \leq n \right\rangle \subseteq \mathrm{Hom}_A(S, S)$ [Gro67, Theorem 16.12.1]. Moreover, if $A = K$ is a field, then S_f has finite length in the category of $D(S, K)$-modules for every $f \in S$. Consequently, every module of the form $H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(S)$ also has finite length in this category [Lyu00a, Corollary 6].

Remark II.4.1. Let (R, m, K) be a local ring. Let S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$, then

$$D(S, R) = R \left[\frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \mid t \in \mathbb{N}, 1 \leq i \leq n \right] \subset \mathrm{Hom}_R(S, S)$$

[Gro67, Theorem 16.12.1]. Then, there is a natural surjection

$$\rho : D(S, R) \to D(S/IS, R/IR)$$

for every ideal $I \subset R$. Moreover,

(i) If M is a $D(S, R)$-module, then IM is a $D(S, R)$-submodule and the structure of M/IM as a $D(S, R)$-module is given by ρ, i.e., $\delta \cdot v = \rho(\delta) \cdot v$ for all $\delta \in D(S, R)$ and $v \in M/IM$.

(ii) If R contains the rational numbers $D(S, R)$ is a Noetherian ring. Let $\Gamma_i = \{\delta \in D(S, R) \mid \mathrm{ord}(\delta) \leq i\}$. We have that $\mathrm{gr}^\Gamma D = S[y_1, \ldots, y_n]$, which is Noetherian and then so D is.

We recall a subcategory of $D(S, R)$-modules introduced by Lyubeznik [Lyu00a]. We denote by $C(S, R)$ the smallest subcategory of $D(S, R)$-modules that contains S_f

for all $f \in S$ and that is closed under subobjects, extensions and quotients. In particular, the kernel, image and cokernel of a morphism of $D(S,R)$-modules that belongs to $C(S,R)$ are also objects in $C(S,R)$. We note that if M is an object in $C(S,R)$, then $H^{i_1}_{I_1}\cdots H^{i_\ell}_{I_\ell}(M)$ is also an object in this subcategory; in particular, $H^{i_1}_{I_1}\cdots H^{i_\ell}_{I_\ell}(S)$ belongs to $C(S,R)$ [Lyu00a, Lemma 5].

A $D(S,R)$-module, M, is *simple* if its only $D(S,R)$-submodules are 0 and M. We say that a $D(S,R)$-module, M, has *finite length* if there is a strictly ascending chain of $D(S,R)$-modules, $0 \subset M_0 \subset M_1 \subset \ldots \subset M_h = M$, called *a composition series*, such that M_{i+1}/M_i is a nonzero simple $D(S,R)$-module for every $i = 0,\ldots,h$. In this case, h is independent of the filtration and it is called the *length* of M. Moreover, the *composition factors*, M_{i+1}/M_i, are the same, up to permutation and isomorphism, for every filtration.

Notation II.4.2. If M is a $D(S,R)$-module of finite length, we denote the set of its composition factors by $\mathcal{C}(M)$.

Remark II.4.3. (i) If M is a nonzero simple $D(S,R)$-module, then M has only one associated prime. This is because $H^0_P(M)$ is a $D(S,R)$-submodule of M for every prime ideal $P \subset S$. As a consequence, if M is a $D(S,R)$-module of finite length, then $\operatorname{Ass}_S M \subset \bigcup_{N\in\mathcal{C}(M)} \operatorname{Ass}_S N$, which is finite.

(ii) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $D(S,R)$-modules of finite length, then $\mathcal{C}(M) = \mathcal{C}(M')\bigcup\mathcal{C}(M'')$.

Hypothesis II.4.4. *Throughout the rest of Section II.4, we will assume that S is either or $K[x_1,\ldots,x_n]$ or $K[[x_1,\ldots,x_n]]$, where K is a field of characteristic 0. Let $D = D(S,K)$.*

We recall some relevant definitions and properties of D-modules, and refer the reader to [Bjö79, Bjö72, Cou95, MNM91] for details. Under Hypothesis II.4.4, we know that $D = S\left\langle \frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right\rangle \subseteq \operatorname{Hom}_K(S,S)$, and there is an ascending filtration

$$\Gamma_i := \{\delta \in D \mid \operatorname{ord}(\delta) \geq i\} = \bigoplus_{\alpha_1+\ldots+\alpha_n \leq i} R\cdot\frac{\partial^\alpha}{\partial x_i^\alpha}.$$

Moreover, $\operatorname{gr}^\Gamma(D) \cong S[y_1,\ldots,y_n]$, a polynomial ring over S. A filtration $\Omega = \{\Omega_j\}$ of S-modules on a D-module M is a *good filtration* if $\Omega_j \subseteq \Omega_{j+1}$, $\bigcup\limits_{j\in\mathbb{N}} \Omega_j = M$,

$\Gamma_i \Omega_j \subseteq \Omega_{i+j}$, and $\mathrm{gr}^{\Omega}(M) = \bigoplus_{j \in \mathbb{N}} \Omega_{j+1}/\Omega_i$ is a finitely generated $\mathrm{gr}^{\Gamma}(D(S,K))$-module.

If Γ is a good filtration, neither $\dim_{\mathrm{gr}^{\Gamma}(D)} \mathrm{gr}^{\Omega}(M)$ nor $\mathrm{Rad}(\mathrm{Ann}_{\mathrm{gr}^{\Gamma}(D)} \mathrm{gr}^{\Omega}(M))$ depend on the choice of good filtration. For the sake of clarity, we will omit the filtration when referring to the associated graded ring or module.

A finitely generated D-module M is *holonomic* if either $M = 0$ or $\dim_{\mathrm{gr}(D)} \mathrm{gr}(M) = n$. The holonomic D-modules form a full abelian subcategory of the category of D-modules, and every holonomic D-module has finite length as a D-module. Moreover, if M is holonomic, then M_f is also holonomic for every $f \in S$. As a consequence, since S is holonomic, every module of the form $H^{i_\ell}_{I_\ell} \cdots H^{i_2}_{I_2} H^{i_1}_{I_1}(S)$ is also.

Definition II.4.5 (Characteristic variety, characteristic cycle, characteristic cycle multiplicity)**.** Given a holonomic D-module, the *characteristic variety of* M is

$$C(M) = \mathcal{V}\left(\mathrm{Rad}\left(\mathrm{Ann}_{\mathrm{gr}(D(S,K))} \mathrm{gr}(M)\right)\right) \subseteq \mathrm{Spec}\, \mathrm{gr}(D),$$

and its *characteristic cycle* is $CC(M) = \sum m_i V_i$, where the sum is taken over all the irreducible components V_i of $C(M)$, and m_i is the corresponding multiplicity. We define the (*characteristic cycle*) *multiplicity of* M by $e(M) = \sum m_i$.

Remark II.4.6. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of holonomic D-modules, then $CC(M) = CC(M') + CC(M'')$; as a consequence, $e(M) = e(M') + e(M'')$. In addition, $CC(M) = 0$ if and only if $M = 0$, so that $e(M) = 0$ if and only if $M = 0$ as well.

Now let $S = K[x_1, \dots, x_n]$, and take $f \in S$. Let $N[s]$ be the free $S_f[s]$-module generated by a symbol $\mathbf{f^s}$. We give $N[s]$ a left $D_f[s]$-module structure as follows: $\frac{\partial}{\partial x_i} \cdot \frac{g}{f^\ell}\mathbf{f^s} = \left(\frac{1}{f^\ell}\frac{\partial g}{\partial x_i} - s\frac{g}{f}\frac{\partial f}{\partial x_i}\right)\mathbf{f^{-s}}$. There exist a polynomial $0 \neq b(s) \in \mathbb{Q}[s]$ and an operator $\delta(s) \in D[s]$ that satisfy

$$\delta(s) f \cdot (1 \otimes \mathbf{f^s}) = b(s)(1 \otimes \mathbf{f^s}) \tag{II.4.6.1}$$

in $N[s]$ [Cou95, Chapter 10].

Given $\ell \in \mathbb{Z}$, we define the specialization map $\phi_\ell : N[s] \to R_f$ by $\phi_\ell(vs^i \otimes \mathbf{f^s}) = \ell^i v \mathbf{f}^\ell$. Thus, $\phi_\ell(\delta(s)v) = \delta(\ell)\phi_\ell(v)$. Then, by applying this morphism to the result, we have

$$\delta(\ell) f^{\ell+1} = b(\ell) f^\ell.$$

The set of all polynomials $h(s) \in \mathbb{Q}[s]$ that satisfy Equation II.4.6.1 forms an ideal of $\mathbb{Q}[s]$. We call the minimal monic polynomial satisfying it the *Bernstein-Sato* *polynomial of f*, and denote it $b_f(s)$.

If R is a reduced F-finite ring of characteristic $p > 0$, we have that $D_R = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R,R)$ [Yek92]. We denote $\operatorname{Hom}_{R^{p^e}}(R,R)$ by $D_R^{(e)}$. Moreover, if R is an F-finite domain, then R is an strongly F-regular ring if and only if R is F-split and a simple D_R-module [Smi95a, Theorem 2.2].

If R is an F-finite reduced ring, $W \subset R$ a multiplicative system and M a simple D_R-module, then $W^{-1}M$ is either zero or a simple $D_{W^{-1}R}$-module. As a consequence, for every D_R-module of finite length, N,

$$\operatorname{length}_{D_{W^{-1}R}} W^{-1}N \leq \operatorname{length}_{D_R} N.$$

II.5 F-pure, F-split and F-injective rings

Throughout this section, R is a ring of characteristic $p > 0$ and $F : R \to R$ denotes the Frobenius morphism, $r \mapsto r^p$. If R is reduced, we define $R^{1/q}$ as the ring of formal q^{th}-roots of S. A ring R is *F-finite* if $R^{1/p}$ is a finitely generated R-module.

We say that R is *F-pure* if for every R-module M, the morphism induced by the inclusion of $R \hookrightarrow R^{1/p}$, $M \otimes_R R \to M \otimes_R R^{1/p}$, is injective. If M is an R-module, then F acts naturally on it. If (R, m, K) is local, we say that a ring is F-injective if the induced Frobenius map $F : H^i_m(R) \to H^i_m(R)$ is injective for every $i \in \mathbb{N}$. F-purity implies F-injectivity, and in a Gorenstein ring, these properties are equivalent [Fed87, Lemma 3.3].

II.6 Tight closure

If R is a reduced F-finite ring, then $D(R,\mathbb{Z}) = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R,R)$. Moreover, if K is a perfect field and $R = K[[x_1,\ldots,x_n]]$, then $D(R,\mathbb{Z}) = D(R,K)$.

If I is an ideal of R, the *tight closure* I^* of I is the ideal of R consisting of all those elements $z \in R$ for which there exists some $c \in R$, c not in any minimal prime of R, such that $cz^q \in I^{[q]}$ for all $q = p^e \gg 0$, where $I^{[q]}$ denotes the ideal of R generated by q^{th} powers of elements in I.

We say that R is *weakly F-regular* if $I = I^*$ for every ideal I of R. If every localization of R is weakly F-regular, then R is *F-regular*. In general, tight closure does not commute with localization, and it is unknown whether the localization of a weakly F-regular ring must again be weakly F-regular; this explains the use of the adjective "weakly." If R is a local ring, we say that the ring is *F-rational1* if for every parameter ideal I, $I = I^*$.

A ring R is *strongly F-regular* if for all $c \in R$ not in any minimal prime, there exists some $q = p^e$ such that the R-module map $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits. Strong F-regularity is preserved under localization. In a Gorenstein ring, F-rationality, strong F-regularity, and weak F-regularity are equivalent.

We define *the test ideal of R* by

$$\tau(R) = \bigcap_{I \subset R} (I : I^*).$$

If R is a Gorenstein ring, we have that

$$\tau(R) = \bigcap_{I \text{ parameter ideal}} (I : I^*).$$

[HH90, Theorem 8.23] [Mat80, Theorem 18.1].

Remark II.6.1. Let R be a reduced ring essentially of finite type over an excellent local ring of prime characteristic. Let $\tau(R)$ denote the test ideal of R. We know that for every multiplicative system $W \subset R$, $W^{-1}\tau(R) = \tau(W^{-1}R)$ [Smi94, Proposition 3.3] [LS01, Theorem 2.3]. It is worth pointing out that, in this case, $\tau(R)$ contains a nonzerodivisor [HH94a, Theorem 6.1].

II.7 Generalized test ideals

Test ideals were generalized by Hara and Yoshida [HY03] in the context of pairs (R, I^c), where I is an ideal in R and c is a real parameter. Blickle, Mustaţă, and Smith [BMS08] gave an elementary description of these ideals in the case of a regular F-finite ring R. We give the definition introduced by them,

Given an ideal I in R we denote by $I^{[1/p^e]}$ the smallest ideal J such that $I \subseteq J^{[p^e]}$ [BMS08, Definition 2.2]. The existence of a smallest such ideal is a consequence of the flatness of the Frobenius map in the regular case.

We recall some properties that we will use often

$$(IJ)^{1/p^e} \subset I^{[1/p^e]} \cdot J^{[1/p^e]}$$

and

$$\left(I^{[p^e]}\right)^{1/p^s} = I^{[p^e/p^s]} \subset \left(I^{[p^s]}\right)^{1/p^e}$$

[BMS08, Proposition 2.4]. In addition, $\left((f)^{[1/p^e]}\right)^{[p^e]} = D^{(e)}f$ [ÀMBL05, Proposition 3.1], where $D^{(e)} = \mathrm{Hom}_{R^{p^e}}(R, R)$.

Given a non-negative number c and a nonzero ideal I, we define the *generalized test ideal with exponent* c by

$$\tau(I^c) = \bigcup_{e>0} (I^{\lceil cp^e \rceil})^{[1/p^e]},$$

where $\lceil c \rceil$ stands for the smallest integer $\geq c$.

The ideals in the union above form an increasing chain of ideals; therefore, as R is Noetherian, they stabilize. Hence for e large enough, $\tau(I^c) = (I^{\lceil cp^e \rceil})^{[1/p^e]}$. In particular, $\tau(f^{\frac{s}{p^e}}) = (f^s)^{[1/p^e]}$ [BMS09, Lemma 2.1].

An important property of test ideals is given by Skoda's Theorem [BMS08, Theorem 2.25]: if I is generated by s elements and $c \leq s$, then $\tau(I^c) = I \cdot \tau(I^{c-1})$.

For every nonzero ideal I and every non-negative number c, there exists $\epsilon > 0$ such that $\tau(I^c) = \tau(I^{c'})$ for every $c < c' < c + \epsilon$ [BMS08, Corollary 2.16].

A positive real number c is an *F-jumping number* for I, if $\tau(I^c) \neq \tau(I^{c-\epsilon})$ for all $\epsilon > 0$

All F-jumping numbers of an ideal I are rational and they form a discrete set, that is, there are no accumulation points of this set [BMS08, Theorem 3.1].

Let α be a positive number. Since the set of F-jumping numbers of f is discrete and it is form by rational numbers, there is a positive rational number $\beta < \alpha$ such that $\tau(f^\beta) = \tau(f^\gamma)$ for every $\gamma \in (\beta, \alpha)$. We denote $\tau(f^\beta)$ by $\tau(f^{\alpha-\epsilon})$.

II.8 F-modules

In this section, we recall some definitions and properties of the Frobenius functor introduced by Peskine and Szpiro [PS73]. We assume that R is regular. This allows us to use the theory of F-modules introduced by Lyubeznik [Lyu97].

Every morphism of rings $\varphi : R \to S$ defines a functor from R-modules to S-modules, where $\varphi^* M = S \otimes_R M$. If $S = R$ and φ is the Frobenius morphism, $F_R M$ denote $\varphi^* M$. If R is a regular ring, F_R is an exact functor. We denote the e-th iterated Frobenius functor by F_R^e.

Example II.8.1. If M is the cokernel of a matrix $(r_{i,j})$, then $F_R(M)$ is the cokernel of $(r_{i,j}^p)$. In particular, if $I \subset R$ is an ideal, then $F(R/I) = R/I^{[p]}$.

We say that an R-module, $\mathcal{M}$, is an F-Module if there exists an isomorphism of R-modules $\nu : \mathcal{M} \to F\mathcal{M}$.

If M is an R-module and $\beta : M \to FM$ is a morphism of R-modules, we consider

$$\mathcal{M} = \lim_{\to}(M \xrightarrow{\beta} FM \xrightarrow{F\beta} F^2M \xrightarrow{F^2\beta} \ldots).$$

Then, $\mathcal{M}$ is an F^e-module and $\mathcal{M} \xrightarrow{\beta} \mathcal{M}$ is the structure isomorphism. In this case, we say that $\mathcal{M}$ is generated by $\beta : M \to F^e_R M$. If M is a finitely generated R-module, we say that $\mathcal{M}$ is an *F-finite F-module*. If β is an injective map, then M injects into $\mathcal{M}$. In this case, we say that β is a *root morphism* and that M is a *root* for $\mathcal{M}$.

Example II.8.2. (i) Since $FR = R$, we have that R is an F-module, where the structure morphism $\nu : R \to R$ is the identity.

(ii) For every element $f \in R$, we take $\alpha = \frac{r}{p-1}$ and take the F-module structure on R_f that is generated by

$$R \xrightarrow{f^{p-1}} R \xrightarrow{f^{p(p-1)}} R \xrightarrow{f^{p^2(p-1)}} \ldots.$$

We say that $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of F-modules if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\ \downarrow{\scriptstyle \nu_{\mathcal{M}}} & & \downarrow{\scriptstyle \nu_{\mathcal{N}}} \\ F_R\mathcal{M} & \xrightarrow{F_R\phi} & F_R\mathcal{N} \end{array}$$

The F-modules form an Abelian category, and the F-finite F-modules form a full Abelian subcategory. Moreover, if $\mathcal{M}$ is F-finite then $\mathcal{M}_f$ is also an F-finite F-module for every $f \in R$. In adition, if R is a local ring, every F finite F-module has finite length as F^e-module and has a minimal root [Bli04a, Lyu97].

Example II.8.3. The localization map $R \to R_f$ is a morphism of F_R-modules for every $f \in R$.

Example II.8.4. The quotient of localization map $R \to R_f$ is an F_R-finite F_R-module for every $f \in R$. R_f/R is generated by $R/fR \xrightarrow{f^{p-1}} F_R(R/fR) = R/f^pR$.

We recall that every F^e-submodule $M \subset R_f/R$ is a D-module [Lyu97, Examples 5.2]. We have that R_f/R has finite length as F-module because R_f/R has finite length as D-module. Let R be an F-finite regular ring. If R_f/R has finite length as

D_R-module, then R_f/R has finite length as F_R-module for every $f \in R$. Therefore, if R_f/R has finite length as D_R-module, then R_f/R has finitely many F-submodules [Hoc07].

II.9 $R[F]$-modules

$R\langle F\rangle$ is defined as the associative R-algebra with one generator F, with the relations $F^e a = a^q F^e$ for every $r \in R$.

Having an $R\langle F\rangle$-module is equivalent to a morphism of R-modules

$$\nu : F(M) \to M.$$

By adjointness, $\nu \in \operatorname{Hom}(FM, M)$ corresponds to a map $F_\nu \operatorname{Hom}(M, F_* M)$ where $F_\nu(m) = \nu(1 \otimes m)$.

If R is regular, every F-module is an $R\langle F\rangle$-module and it is often called a unit $R\langle F\rangle$-module.

An element $u \in M$ of an $R[F]$-module (M, ν) is called *F-nilpotent* if $F^\ell(u) = 0$ for some $\ell \in \mathbb{N}$; M is called *F-nilpotent* if $F^\ell(M) = 0$.

Definition II.9.1. [EH08] M is *anti-nilpotent* if for every $R\langle F\rangle$-submodule, $N \subset M$, F acts injectively on M/N.

Lemma II.9.2 ([Ma12])**.** *An $R\langle F\rangle$-module is anti-nilpotent if and only if every $R\langle F\rangle$-submodule is F–Full.*

Definition II.9.3 ([Bli04a, Lyu97])**.** We define a functor $\mathcal{D}$ from the category of cofinite $S\langle F\rangle$-modules to the category of F_S-modules as

$$\mathcal{D}(M) := \lim_{to} \left(M^* \xrightarrow{\beta^*} F^e M^* \xrightarrow{F^e \beta^*} \dots \right)$$

Theorem II.9.4 ([Lyu97])**.** *$\mathcal{D}$ satisfies the following properties for $R\langle F\rangle$-modules that are Artinian as R-modules:*

- *$\mathcal{D}(M) = 0$ if and only if M is F-nilpotent*
- *For every F^e-submodule $N' \subset \mathcal{D}(M)$, there exist a $S\langle F\rangle$-module N such that $\mathcal{D}(N) = N'$.*
- *if N and M are cofinite, $\mathcal{D}(N) \cong_{F-mod} \mathcal{D}(M)$ if and only if $N_{red} \cong_{S\langle F\rangle} M_{red}$*

Theorem II.9.5 ([Ma12]). *If R is an F-pure ring, then $H^i_m(R)$ is anti-nilpotent for every $i \in \mathbb{N}$.*

Since $R = S/I$, we have that every $R\langle F\rangle$-module has a natural structure of $S\langle F\rangle$-module. In particular, $H^d_m(R)$ is an $S\langle F\rangle$-module.

Proposition II.9.6 ([Bli04a]). $\mathcal{D}(H^d_m(R)) = H^c_I(S)$.

CHAPTER III

Rings of differentiable type and rings of mixed characteristic

In this chapter we develop the theory of ring of differentiable over regular rings of characteristic zero. In particular, we do not assume that the base ring is local, complete or have global variables. Instead, we assume that the module differential over the ring is a projective module (see Hypothesis III.1.3). Then, we prove that the localization of any regular local ring, R, of mixed characteristic $p > 0$ at the characteristic satisfies this hypothesis. We emulate Lyubeznik proof for regular local rings to conclude that the associated primes of the local cohomology modules over $R[1/p]$ is finite.

The results presented in this section appear in [NB13].

III.1 Rings of differentiable type

We start by recalling a couple of theorems from Matsumura's book [Mat80]:

Theorem III.1.1 (Theorem 98 [Mat80])**.** *Let (A, m, K) be a Noetherian local domain containing the rational numbers. Suppose that A contains a field, F, such that K is an algebraic extension of F. Then,*

$$\operatorname{rank}(\operatorname{Der}_F(A)) \leq \dim A.$$

Theorem III.1.2 (Theorem 99 in [Mat80])**.** *Let (R, m, F) be a regular local commutative Noetherian ring with unity of dimension n containing a field F_0. Suppose that F is an algebraic separable extension of F_0. Let $\hat{R}$ denote the completion of R with respect to m. Let $x_1, \ldots, x_n$ be a regular system of parameters of R. Then, $\widehat{R} = F[[x_1, \ldots, x_n]]$ is the power series ring with coefficients in F, and $\operatorname{Der}_F \hat{R}$ is a* *free $\widehat{R}$-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Moreover, the following conditions are equivalent:*

- *$\partial/\partial x_i$ $(i = 1, \ldots, n)$ maps R into R, i.e. $\partial/\partial x_i \in Der_{F_0}(R)$;*
- *there exist derivation $D_1, \ldots, D_n \in \mathrm{Der}_{F_0}(R)$ and elements $a_1, \ldots, a_n \in R$ such that $D_i a_j = 1$ if $i = j$ and 0 otherwise;*
- *there exist derivations $D_1, \ldots, D_n \in \mathrm{Der}_{F_0}(R)$ and elements $a_1 \ldots, a_n \in R$ such that $\det(D_i a_j) \notin m$;*
- *$\mathrm{Der}_{F_0}(R)$ is a free module of rank n (with basis $D_1, \ldots, D_n$);*
- *$\mathrm{rank}(\mathrm{Der}_{F_0}(R)) = n$.*

Hypothesis III.1.3. *From now on, we will consider a commutative Noetherian regular ring R with unity that contains a field, F, of characteristic zero satisfying:*

(1) *R is equidimensional of dimension n;*

(2) *every residual field with respect to a maximal ideal is an algebraic extension of F;*

(3) *$\mathrm{Der}_F(R)$ is a finitely generated projective R-module of rank n such that $R_m \otimes_R \mathrm{Der}_F(R) = \mathrm{Der}_F(R_m)$.*

Remark III.1.4. In property (3), we require that $R_m \otimes_R \mathrm{Der}_F(R) = \mathrm{Der}_F(R_m)$ because we are not assuming that the module of Kahler differential, $\Omega_{F/R}$, is a finitely generated R-module. In addition, property (3) and Theorem III.1.1 say that $\mathrm{Der}_F(R_m)$ has the maximum rank possible.

This hypothesis is inspired by the properties (i), (ii) and (iii) (1.1.2) in [MNM91]. There, R is a commutative Noetherian regular ring that contains a field, F, of characteristic zero satisfying (1), (2), but instead of (3) in Hypothesis III.1.3, there exist F-linear derivations $\partial_1, \ldots, \partial_n \in \mathrm{Der}_{F_0}(R)$ and $a_1 \ldots, a_n \in R$ such that $\partial_i a_j = 1$ if $i = j$ and 0 otherwise. In our hypothesis, part (3) includes more rings; for instance, Remark III.1.8 gives an example of a ring that satisfies Hypothesis III.1.3 but not (1.1.2) in [MNM91]. However, when R is a local ring the properties are the same by Theorem III.1.2.

Remark III.1.5. Every regular finitely generated algebra over the complex numbers, R, satisfies Hypothesis III.1.3. This is because, by Theorem 8.8 [Har77], $\mathrm{Der}_{\mathbb{C}}(R) = \mathrm{Hom}_R(\Omega_{R/\mathbb{C}}, R)$ and $\Omega_{R/\mathbb{C}}$ is a projective module such that $\mathrm{rank}(\Omega_{R_m/\mathbb{C}}) = \dim(R)$ for every maximal ideal $m \subset R$.

Proposition III.1.6. *Let R be a commutative Noetherian regular ring that contains a field, F, of characteristic zero satisfying* (1), (2), *and such that there exist F-linear derivations $\partial_1, \ldots, \partial_n \in \mathrm{Der}_{F_0}(R)$ and $a_1 \ldots, a_n \in R$ such that $\partial_i a_j = 1$ if $i = j$ and* 0 *otherwise. Then, R satisfies Hypothesis III.1.3.*

Proof. Theorem III.1.2 implies that $\mathrm{Der}_{F_0}(R) = R\partial_1 \oplus \ldots \oplus R\partial_n$ and that $\mathrm{Der}_{F_0}(R_m) = R_m\partial_1 \oplus \ldots \oplus R_m\partial_n$ for every maximal ideal $m \subset R$, which concludes the proof of property (3) in Hypothesis III.1.3. □

A proof of Proposition III.1.6, along with several consequences, is contained in Remark 2.2.5 in [MNM91].

Theorem III.1.7. *Let S be a commutative Noetherian regular domain that contains a field, F, of characteristic zero satisfying Hypothesis III.1.3. If there is an element $f \in S$ such that $R = S/fS$ is a regular ring, then R satisfies Hypothesis III.1.3.*

Proof. We have that property (1) holds because $\dim S - 1 = \dim S_\eta - 1 = \dim R_m$ for every maximal ideal $m = \eta R \subset R$, where $\eta \subset S$ is a maximal ideal of S containing fS. In addition, property (2) holds because every residual field of R is a residual field of S.

We only need to prove property (3). Let $n = \dim(S)$. For every maximal ideal $\eta \subset S$ containing fS, we may pick a regular system of parameters, $y_1, \ldots, y_n$ for S_η such that $y_1 = f$. Then, by Theorem III.1.2, there exist $\delta_i \in \mathrm{Der}_F(S_\eta)$ such that $\delta_i(y_j) = 1$ if $i = j$ and zero otherwise; moreover, $\mathrm{Der}_F(S_\eta)$ is a free S_η-module of rank n generated by $\delta_1, \ldots \delta_n$.

Let $\varphi_f : \mathrm{Der}_F(S) \to R$ be the morphism defined by $\partial \to [\partial(f)]$, where $[\partial(f)]$ represents the class of $\partial(f)$ in R. Then, $S_\eta \otimes_S \mathrm{Ker}(\varphi_f)$ is isomorphic to $\{\delta \in \mathrm{Der}_F(S_\eta) : \delta(f) \in f \cdot S_\eta\} = S_\eta f\delta_1 \oplus S_\eta\delta_2 \oplus \ldots \oplus S_\eta\delta_n$

Noticing that $f \cdot \mathrm{Der}_F(S) \subset \mathrm{Ker}\,\varphi_f$, we define $N = \mathrm{Ker}\,\varphi_f/(f \cdot \mathrm{Der}_F(S))$ and point out that it is a finitely generated R-module. Let $m = \eta R$. Then, $R_m \otimes_R N = R_m\delta_2 \oplus \ldots \oplus R_m\delta_n = \mathrm{Der}_F(R_m)$, where the last equality uses Theorem III.1.2.

We have a morphism $\psi : N \to \mathrm{Der}_F(R)$ defined by taking $\psi[\partial](r) = [\partial(r)]$, which is well defined by the definition of N. For every maximal ideal $m \subset R$, there is a natural morphism $i_m : R_m \otimes_R \mathrm{Der}_F(R) \to \mathrm{Der}_F(R_m)$. We notice that

$(i_m \circ 1_{R_m} \otimes \psi)$ is an isomorphism between $R_m \otimes_R N$ and $\mathrm{Der}_F(R_m)$ for all maximal $m \subset R$. Therefore, $N_m \cong R_m \otimes \mathrm{Der}_F(R) \cong \mathrm{Der}_F(R_m)$ for all maximal $m \subset R$. Hence, ψ is an isomorphism. □

Remark III.1.8. It is worth pointing out that there are examples were R satisfies Hypothesis III.1.3 but $\mathrm{Der}_F(R)$ is not free. Let $S = \mathbb{R}[x, y, z]$ be the polynomial ring in three variables and coefficients over $\mathbb{R}$. Let $f = x^2 + y^2 + z^2 - 1$. Then, $R = S/fS$, the coordinate ring associated to the sphere, satisfies Hypothesis III.1.3 but $\mathrm{Der}_{\mathbb{R}}(R)$ is not free. Let $\phi : R^3 \to R$ be the morphism given by $(a, b, c) \to (ax, by, cz)$. Thus, $\mathrm{Der}_{\mathbb{R}}(R) = Ker(\phi)$ by the proof of Theorem III.1.7. Therefore, $\mathrm{Der}_{\mathbb{R}}(R)$ is the projective module corresponding to the tangent bundle of the sphere, and so it is not free. This example also shows that the conclusion of Theorem III.1.7 does not hold for properties (i), (ii) and (iii) (1.1.2) in [MNM91]. In that sense, Hypothesis III.1.3 behaves better under regular subvarieties.

Main Example III.1.9. Let $(V, \pi V, K)$ be a DVR of mixed characteristic $p > 0$, and let F denote its fraction field. Let $S = V[[x_1, \ldots, x_{n+1}]] \otimes_V F$ be the tensor product of the power series ring with coefficients in V and F. Let $R = S/(f)S$ be a regular ring where $f = \pi - h$ for an element h in the square of maximal ideal of $V[[x_1, \ldots, x_{n+1}]]$. Then, R satisfies Hypothesis III.1.3.

Proof. Since S is as in Proposition III.1.6 (cf. pages $5880 - 5881$ in [Lyu00b]) and $\pi - h \in S$ is a regular element, we have that R satisfies Hypothesis III.1.3 by Theorem III.1.7. □

Definition III.1.10. We say that an associative ring A is filtered if there exists an ascending filtration

$$\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \ldots$$

of additive subgroups such that $1 \in \Sigma_0, \bigcup \Sigma_i = A$ and $\Sigma_i \Sigma_j \subset \Sigma_{i+j}$ for every $i, j \in \mathbb{N}$. We denote by $\mathrm{gr}^{\Sigma}(A)$ the associated graded ring

$$\Sigma_0 \oplus \Sigma_1/\Sigma_0 \oplus \Sigma_2/\Sigma_1 \oplus \ldots.$$

Let F be a field of characteristic 0 and R a commutative Noetherian ring with unity containing F. We denote by $D(R, F)$ the ring of F-linear differential operators of R. This is a subring of $\mathrm{Hom}_F(R, R)$ defined inductively as follows. The differential operators of order zero are the morphisms induced by multiplying by elements in R. An element $\theta \in \mathrm{Hom}_F(R, R)$ is a differential operator of order less than or equal to

$j+1$ if $[\theta, r] := \theta \cdot r - r \cdot \theta$ is a differential operator of order less than or equal to j. We have an induced filtration $\Gamma = (\Gamma^j)$ on $D(R,F)$ given by $\Gamma^j = \{\theta \in D(R,F) \mid \operatorname{ord}(\theta) \leq j\}$. As a consequence of the definition, we have that $\Gamma_j \Gamma_i \subset \Gamma_{j+i}$ and that $\operatorname{gr}^{\Gamma}(D(R,F)) = \oplus_{j=0}^{\infty} \Gamma^j / \Gamma^{j-1}$ is a commutative ring.

An example is given by a commutative Noetherian regular ring R with unity that contains a field, F, of characteristic 0, as in Proposition III.1.6. In this case, $D(R,F) = R[\partial_1, \ldots, \partial_n] \subset \operatorname{Hom}_F(R,R)$; moreover, $\operatorname{gr}^{\Gamma}(D(R,F)) = R[y_1, \ldots, y_n]$ and $\operatorname{w.gl.dim}(D(R,F)) = \dim(R)$ (cf. Main Theorem in [Bjö72], (1.1.3) and Theorem 1.1.4 in [MNM91], and Theorem 2.17 in [NM09]). We would like to have similar properties for $D(R,F)$ and $\operatorname{gr}^{\Gamma}(D(R,F))$ when R satisfies Hypothesis III.1.3.

We will denote by D the subalgebra of $\operatorname{Hom}_F(R,R)$ generated by R and $\operatorname{Der}_F(R)$, where $R = \operatorname{Hom}_R(R,R) \subset \operatorname{Hom}_F(R,R)$. We define an ascending filtration Γ'_j of R-modules in D inductively as follows. $\Gamma'_0 = R$. Given Γ'_j, we take Γ'_{j+1} as the Abelian additive group generated by $\{\Gamma'_j, \operatorname{Der}_F(R) \cdot \Gamma'_j\}$. Since Γ'_j is generated by multiplying derivations, we have that for every $\delta \in \Gamma'_i$ and $f \in R$, $[\delta, f] = f\delta - \delta f \in \Gamma'_{j-1}$. Therefore, Γ'_j is an R-submodule of D with respect to the structures induced by multiplication by the left or by the right. Additionally, $D \subset D(R,F)$ and $\Gamma'_j \subset \Gamma_j$ because $\operatorname{Der}_F(R) \subset \Gamma_1$.

We have that for every $s \in R$, $\operatorname{Adj}_s : D(R,A) \to D(R,A)$, defined by $\operatorname{Adj}_s(\delta) = s\delta - \delta s$, is nilpotent. Let $m \subset R$ be a maximal ideal and $S = R \setminus m$ be the induced multiplicative system. Then, S is a multiplicative set satisfying the Ore condition on the left and on the right in $D(R,A)$ and, as a consequence, in D. Hence, $S^{-1}D(R,F)$ and $S^{-1}D$ exist as filtered rings.

Proposition III.1.11. *With the same notation as above, $D(R,F) = D$ as filtered rings.*

Proof. Let $m \subset R$ be a maximal ideal and $S = R \setminus m$ be the induced multiplicative system. We have that $S^{-1}\Gamma_j = S^{-1}\Gamma'_j$ by condition (3) in Hypothesis III.1.3. Therefore, $S^{-1}D = R_m[\Omega_{R_m,F}] = D(R_m,F) = S^{-1}D(R,F)$ as filtered rings. □

For simplicity, we will denote $D(R,F)$ by D and $(R \setminus m)^{-1}D(R,F)$ by D_m for a maximal ideal $m \subset R$. We note that the inclusion $D \to D_m$ induces an inclusion $\operatorname{gr}^{\Gamma}(D) \to \operatorname{gr}^{\Gamma_m}(D_m)$ of rings, and $\operatorname{gr}^{\Gamma_m}(D_m) = R_m \otimes_R \operatorname{gr}^{\Gamma}(D)$. If M is a left or right finitely generated D-module with a good filtration Π, then $D_m \otimes_D M$ or $M \otimes_D D_m$, respectively, has a filtration given by $R_m \otimes_R \Pi$ and $\operatorname{gr}^{\Pi_m}(D_m \otimes_D M) = R_m \otimes_R \operatorname{gr}^{\Pi}(M)$.

We have that D_m is a left and right flat module over D, and $D_m \otimes_D D_m \cong R_m \otimes_R D \cong D \otimes_R R_m \cong D_m$. If M is a left or right finitely generated D-module, there exist a canonical isomorphism $\operatorname{Ext}^i_{D_m}(M_m, D_m) \cong S^{-1}\operatorname{Ext}^i_D(M,D) \cong R_m \otimes_R \operatorname{Ext}^i_D(M,D)$ for every $i \in \mathbb{N}$.

We have, by Theorem III.1.2, that for every maximal ideal $m \subset R$ there exist elements $x_1, \ldots, x_d \in R_m$ and F-linear derivations $\partial_1, \ldots, \partial_d \in \operatorname{Der}_F(R_m)$ such that $\partial_i(x_j) = 1$ if $i = j$ and zero otherwise. Therefore, $\operatorname{w.gl.dim}(D_m) = n$ and $gr^{\Gamma_m}(D_m) = R[y_1, \ldots, y_n]$ is the polynomial ring with n variables and coefficients in R_m [Bjö72].

We recall the definition of a ring of differentiable type (cf. (1.1) in [MNM91]).

Definition III.1.12. A filtered ring A is a *ring of differentiable type* if its associated graded ring is commutative Noetherian regular with unity and pure graded dimension.

Theorem III.1.13. *(D, Γ) is a ring of differentiable type such that $\operatorname{gr}^\Gamma(D)$ is a regular ring of pure graded dimension $2n$.*

Proof. Let $\operatorname{gr}^\Gamma(D)$ be the associated graded ring. We will prove the proposition by parts.
$\operatorname{gr}^\Gamma(D)$ *is commutative:* This follows from the definition of the filtration Γ on $D = D(R, F)$.
$\operatorname{gr}^\Gamma(D)$ *is Noetherian:* Let $\partial_1, \ldots, \partial_m$ be a set of generators for $\operatorname{Der}_F(R)$. Let $\phi : R[z_1, \ldots, z_m] \to \operatorname{gr}^\Gamma(D)$ be the morphism of commutative R-algebras defined by $z_i \to [\partial_i]$. We have, by the definition of $\Gamma' = \Gamma$, that ϕ is surjectve. Hence $\operatorname{gr}^\Gamma(D)$ is Noetherian.
$\operatorname{gr}^\Gamma(D)$ *is regular:* Let $Q \subset \operatorname{gr}^\Gamma(D)$ be a prime ideal and $m \subset R$ be a maximal ideal that contains $Q \cap R$. Then $\operatorname{gr}^\Gamma(D)_Q = (\operatorname{gr}^\Gamma(D)_m)_Q$ which is regular because $\operatorname{gr}^\Gamma(D)_m$ is a polynomial ring over R_m.
$\operatorname{gr}^\Gamma(D)$ *has pure graded dimension* $2n$*:* Let η be a maximal homogeneous ideal of $\operatorname{gr}^\Gamma(D)$. We claim that $m = \eta \cap R$ is a maximal ideal of R. If not, there exist a maximal ideal $m' \subset R$ strictly containing m. Then, $m' + \eta$ would be a proper ideal of $\operatorname{gr}^\Gamma(D)$ that strictly contains η. Hence, $\operatorname{gr}^\Gamma(D)_\eta$ is the localization of $\operatorname{gr}^\Gamma(D)_m$ at a maximal homogeneous ideal, then, $\dim(\operatorname{gr}^\Gamma(D)_\eta) = 2n$ because $\operatorname{gr}^\Gamma(D_m)$ is a ring of pure graded dimension $2n$. □

Remark III.1.14. Narváez-Macarro [NM09] showed that if S is a ring containing a field, F, of characteristic 0 and $\operatorname{Der}_F(S)$ is a projective S-modules of finite rank, then $\operatorname{gr}(D(S,F)) \cong \operatorname{Sym}(\operatorname{Der}_F(S))$. Hence, we have that $\operatorname{gr}(D) \cong \operatorname{Sym}(\operatorname{Der}_F(R))$ by Hypothesis III.1.3.

Corollary III.1.15. *D is left and right Noetherian.*

Proof. This follows from Proposition 6.1 in [Bjö79]. □

Proposition III.1.16. $\operatorname{w.gl.dim}(D) = \dim(R)$

Proof. Since D is left and right Noetherian, $\operatorname{w.gl.dim}(D) = \operatorname{l.pd}(D) = \operatorname{r.pd}(D)$ by Theorem 8.27 in [Rot09]. The value to this dimension is equal to the maximum integer j such that $\operatorname{Ext}^j_D(M, R) \neq 0$ for some finitely generated D-module M because D is of differentiable type. As $R_m \otimes_R \operatorname{Ext}^j_D(M, D) = \operatorname{Ext}^n_{D_m}(M_m, D_m) = 0$ for every maximal ideal $m \subset R$ and integer $j > n$, we have that $\operatorname{Ext}^j_D(M, D) = 0$ for every D-module M and for $j > n$. Hence, $\operatorname{w.gl.dim}(D) \leq n$. Likewise,

$$R_m \otimes_R \operatorname{Ext}^n_D(R, D) = \operatorname{Ext}^n_{D_m}(R_m, D_m) \neq 0$$

for any $m \subset R$, so, $\operatorname{Ext}^n_D(R, D) \neq 0$. Hence $\operatorname{w.gl.dim}(D) \geq n$. □

III.2 The theory of the Bernstein-Sato polynomial and the Bernstein class of D

Throughout this section we are adapting the results of Mebkhout and Narváez-Macarro to R and D [MNM91]. In particular, we show that the existence of the Bernstein-Sato polynomial and that the Bernstein class of D is closed under localization at one element.

Definition III.2.1. Let A be a ring of differentiable type. Let $M \neq 0$ be a finitely generated left or right A-module. We define

$$\operatorname{grade}_A(M) = \inf\{j : \operatorname{Ext}^j_A(M, A) \neq 0\}.$$

Proposition III.2.2. *Let A be a ring of differentiable type. Let $M \neq 0$ be a finitely generated left or right A-module. Then,*

$$\dim(M) + \operatorname{grade}_A(M) = \dim(\operatorname{gr}^\Gamma(A)).$$

In particular, $\dim(M) \geq \dim(\operatorname{gr}(A)) - \operatorname{w.gl.dim}(A)$. *Moreover, we have that*

$$\operatorname{codim}_A(\operatorname{Ext}^i_A(M, A)) \geq i$$

for all $i \geq 0$ *such that* $\operatorname{codim}_A(\operatorname{Ext}^i_A(M, A)) \neq 0$.

Proof. This is a generalized form of Theorem 7.1 of section 2 in [Bjö79] given by Gabber [Gab13]. The proposition is stated in this form in Mebkhout and Narváez-Macarro's article as Theorem 1.2.2 [MNM91]. □

Definition III.2.3. Let A be a ring of differentiable type. Let M be a finitely generated left or right A-module. We say that M is in the left or right *Bernstein class* if it has minimal dimension, i.e. $\dim(M) = \dim(\mathrm{gr}(A)) - \mathrm{w.gl.dim}(A)$.

This class is closed under submodules, quotients and extensions. Let d denote $\mathrm{w.gl.dim}(A)$. The functor that sends M to $\mathrm{Ext}^d_A(M, A)$ is an exact contravariant functor that interchanges the left Bernstein class and the right Bernstein class. Moreover, $M = \mathrm{Ext}^d_A(\mathrm{Ext}^d_A(M, A), A)$ naturally if M is in either the left or the right Bernstein class, so that we have an anti-equivalence of categories. In consequence, the modules in the Bernstein class have finite length as A-modules because it is a left and right Noetherian ring (cf. Proposition 1.2.7[MNM91]).

Proposition III.2.4 (Prop. 1.2.7 in [MNM91])**.** *Let A be a ring of differentiable type and let f be an element in A_0. Let M be an A_f-module finitely generated, such that $\mathrm{Ext}^i_{A_f}(M, A_f) = 0$ if $i \neq \mathrm{w.gl.dim}(A)$. Then, there exists a submodule $M' \subset M$ over A such that M' is finitely generated with minimal dimension and $M'_f = M$.*

Through the rest of this section, $F(s)$ denotes the fraction field of the polynomial ring $F[s]$ over the field F, and $D(s)$ denotes the ring $F(s) \otimes_F D$ with the filtration given by $F(s) \otimes_F \Gamma^i$. By $R(s)$, we mean the $F(s)$-algebra $F(s) \otimes_F R$. Similarly, $D[s]$ denotes $F[s] \otimes_F D$ and $R[s]$ denotes $F[s] \otimes_F R$.

Proposition III.2.5. *$R(s)$ is an $F(s)$-algebra equidimensional of dimension* $\dim(R)$*.*

Proof. This is an immediate consequence of Theorem 2.1.1 in [MNM91]. □

Proposition III.2.6. *$D(s)$ is a ring of differentiable type with the filtration $F(s) \otimes_F \Gamma$ such that $\mathrm{gr}^{F(s)\otimes_F\Gamma}(D(s))$ is a ring of pure graded dimension $2\dim(R)$.*

Proof. Since D is a ring of differentiable type,

$$\mathrm{gr}^{F(s)\otimes_F\Gamma}(D(s)) = F(s) \otimes_{F[s]} F[s] \otimes_F \mathrm{gr}^{\Gamma}(D) = F(s) \otimes_{F[s]} \otimes_F \mathrm{gr}^{\Gamma}(D)[s]$$

is commutative, Noetherian and regular. For the sake of simplicity, we will omit the filtration. We claim that $\mathrm{gr}(D(s))$ has pure graded dimension $2\dim(R) = 2n$. Let $\eta \subset \mathrm{gr}(D(s))$ be a maximal homogeneous ideal and $P = \eta \cap R$. Let $m \subset R$ be a maximal ideal containing P. We have that the ideal η_m, induced by η, is a maximal homogeneous ideal of

$$(R \setminus m)^{-1}\mathrm{gr}(D(s)) = F(s) \otimes_F \mathrm{gr}(D_m) = (F(s) \otimes_F R_m)[y_1, \dots, y_n],$$

the polynomial ring with coefficients on $F(s) \otimes_F R_m$ and variables $y_1, \dots, y_n$. Then, $\mathrm{ht}(\eta) = \mathrm{ht}(\eta_m) = 2n$ because $F(s) \otimes_F R_m$ is equidimensional of dimension n by Theorem 2.1.4 in [MNM91]. □

Remark III.2.7. Theorem 3.4 in [NM91] gives an alternative proof for rings that satisfies the hypotheses of Proposition III.1.6.

Let M be a left $D(s)$-module in the Bernstein class of $D(s)$. Let N be a D-module in the Bernstein class of D such that $F(s) \otimes_F N = M$. For every $\ell \in F$, the D-module $M_\ell := N/(s-\ell)N$ is the Bernstein class of D.

Proposition III.2.8. *With the same notation as above, we have that*

$$\dim_{D(s)}(M) \geq \dim_D(N_\ell),$$

for all but finitely many $\ell \in F$.

Proof. This is analogous to the proof of Theorem 2.2.1 in [MNM91]. □

Proposition III.2.9. $\mathrm{w.gl.dim}(D(s)) = \dim(R) = n$.

Proof. This is analogous to the proof of Theorem 2.2.3 in [MNM91]. □

Let $N[s]$ be the free $R_f[s]$-module generated by a symbol $\mathbf{f^s}$ and let $N(s) = F(s) \otimes_F N[s]$. We give to $N[s]$ (resp. $N(s)$) a structure of a left $D_f[s]$-module (resp. $D_f(s)$-module) as follows,

$$\partial g\mathbf{f^s} = (\partial g + sg\partial(f)f^{-1})\mathbf{f^s}$$

for every $\partial \in \mathrm{Der}_F(R)$ and every $g \in R_f[s]$ (resp. $g \in R_f(s)$). If M is a left D-module, we define $M_f[s]\mathbf{f^s} := M_f[s] \otimes_{R_f[s]} N[s] = M \otimes_R N[s]$ and $M_f(s)\mathbf{f^s} := N(s) \otimes_{R_f[s]} M_f(s) = M \otimes_R N(s)$. This is a left $D_f[s]$-module ($D_f(s)$-module), and clearly, $M_f[s]\mathbf{f^s}$ ($M_f(s)\mathbf{f^s}$) is a finitely generated $D_f[s]$-module ($D_f(s)$-module) if M is.

Proposition III.2.10. *Let M be a left D-module in the Bernstein class and let $u \in M$. Then, there exists a nonzero polynomial $b(s) \in F[s]$ and an operator $P(s) \in D[s]$ that satisfies the equation*

$$b(s)(u \otimes \mathbf{f^s}) = P(s)f(u \otimes f^s)$$

in $M[s]\mathbf{f^s}$.

Proof. This is analogous to 3.1.1 in [MNM91]. □

Corollary III.2.11. *If M is a left D-module in the Bernstein class, the M_f is a finitely generated D-module.*

Proof. For $\ell \in \mathbb{Z}$, we define a morphism of specialization $\phi_\ell : M_f[s]\mathbf{f^s} \to M_f$ by $\phi_\ell(us^i \otimes \mathbf{f^s}) = \ell^i f^\ell u$, such that $\phi_\ell(P(s)v) = P(\ell)\phi_\ell(v)$. Then, by applying this morphism to the result of Proposition III.2.10, we have

$$b(\ell)f^\ell u = P(\ell)f^\ell u$$

and the conclusion follows. □

Corollary III.2.12. *Let M be a left D-module in the Bernstein class. Then, M_f is also in the Bernstein class for all $f \in R$.*

Proof. Since M_f is a finitely generated D-module by Corollary III.2.11, it suffices to show that $\dim_{\mathrm{gr}(D)}(\mathrm{gr}(M_f)) = n$. Since $R_m \otimes_R M$ is in the Bernstein class of D_m, we have that M_f is in the Bernstein class of D_m for every maximal ideal $m \subset R$ by Theorem 2.2.3 in [MNM91]. Thus, $\dim_{\mathrm{gr}(D_m)}(\mathrm{gr}((M_m)_f)) = n$ and, therefore $\dim_{\mathrm{gr}(D)}(\mathrm{gr}(M_f)) = n$. □

Theorem III.2.13. *Let R be a regular commutative Noetherian ring with unity that contains a field, F, of characteristic 0 satisfying the following conditions:*

(1) *R is equidimensional of dimension n;*

(2) *every residual field with respect to a maximal ideal is an algebraic extension of F;*

(3) *$\mathrm{Der}_F(R)$ is a finitely generated projective R-module of rank n such that for every maximal ideal $m \subset R$, $R_m \otimes_R \mathrm{Der}_F(R) = \mathrm{Der}_F(R_m)$.*

Then, the ring of F-linear differential operators $D(R,F)$ is a ring of differentiable type of weak global dimension equal to $\dim(R)$. Moreover, the Bernstein class of $D(R,F)$ is closed under localization at one element.

Proof. This is a consequence of Theorem III.1.13, Proposition III.1.16 and Corollary III.2.12. □

The previous theorem generalizes some of the results of Mebkhout and Narváez-Macarro about certain rings of differentiable type [MNM91]. There, R is a commutative Noetherian regular ring that contains a field, F, of characteristic zero satisfying (1), (2), but instead of (3) in Hypothesis III.1.3, there exist F-linear derivations $\partial_1,\ldots,\partial_n \in \mathrm{Der}_{F_0}(R)$ and $a_1 \ldots, a_n \in R$ such that $\partial_i a_j = 1$ if $i=j$ and 0 otherwise.

III.3 Local cohomology

Lemma III.3.1. *Let M be a left D-module in the Bernstein class. Then, $\mathcal{T}(M)$ has a natural structure of D-module such that it belongs to the Bernstein class for every functor $\mathcal{T}$ as in Definition XI.4.2.*

Proof. M_f has the structure of D-module given by

$$\partial \cdot m/f^{\ell} = (f^{\ell}\delta \cdot m - \delta(f^{\ell})m)/f^{2\ell}$$

for every $\delta \in \mathrm{Der}_F(R)$. Then, $\mathcal{T}(M)$ is a D-module by Examples 2.1 in [Lyu93]. Since M is in the Bernstein class, M_f is in the Bernstein class and $M \to M_f$ is a morphism of D-modules by Corollary III.2.12. Since the Bernstein class is closed under extension, submodules and quotients, every element in the complexes (II.3.0.1) and (II.3.0.2) as well as the kernels, images and homology groups are in the same class, and the result follows. □

Lemma III.3.2. *Let M be a left D-module in the Bernstein class. Then, $\mathrm{Ass}_R(M)$ is finite.*

Proof. Suppose $M \neq 0$. Let $M_1 = M$ and P_1 be a maximal element in the set of the associated primes of M_1. Then, $N_1 = H^0_{P_1}(M_1)$ a nonzero D-submodule of M_1, and it has only one associated prime. Given N_j and M_j, set $M_{j+1} = M_j/N_j$. If $M_{j+1} \neq 0$, let P_{j+1} be a maximal element in the set of the associated primes of M_{j+1}. Then $N_{j+1} = H^0_{P_j}(M_{j+1})$ has only one associated prime. If $M_{j+1} = 0$, set $N_{j+1} = 0$. Since $M_1 = M$ has finite length as a $D(R,A)$-module, there exist $\ell \in \mathbb{N}$ such that $M_j = 0$ for $j \geq \ell$, and then $\mathrm{Ass}(M) \subset \{P_1,\ldots,P_\ell\}$. □

Theorem III.3.3. *Let R be a ring that satisfies Hypothesis III.1.3 and let M be a D-module in its left Bernstein class. Then, $\mathrm{Ass}_R(\mathcal{T}(M))$ is finite for every functor $\mathcal{T}(-)$ as in Definition XI.4.2. In particular, this holds for $H^i_I(R)$ for every $i \in \mathbb{N}$ and ideal $I \subset R$.*

Proof. This follows immediately form Lemmas III.3.1 and III.3.2. □

Corollary III.3.4. *Let (R, m, K) be a regular local ring of mixed characteristic $p > 0$. Then, the set of associated primes of $\mathcal{T}(R)$ that does not contain p is finite for every functor $\mathcal{T}$ as in Definition XI.4.2.*

Proof. Let $\hat{R}$ be the completion of R with respect to the maximal ideal. Then, the set of associated primes of $\mathcal{T}(R)$ that does not contain p is finite if the set of associated primes of $\mathcal{T}(\hat{R}) = \hat{R} \otimes_R \mathcal{T}(R)$ that does not contain p is finite. We can assume without loss of generality that R is complete. Thus, $R = V[[x_1, \ldots, x_{n+1}]]$ or

$$R = V[[x_1, \ldots, x_{n+1}]]/(p - h)V[[x_1, \ldots, x_{n+1}]],$$

where (V, pV, K) is a DVR of unramified mixed characteristic $p > 0$ and h is an element in the square of maximal ideal of $V[[x_1, \ldots, x_{n+1}]]$, by the Cohen Structure Theorems. Let F be the fraction field of V. It suffices to show that

$$\mathrm{Ass}_R(F \otimes_V \mathcal{T}(R)) = \mathrm{Ass}_R(\mathcal{T}(F \otimes_V R)$$

is finite, which follows from our main example and Theorem III.3.3. □

Theorem III.3.5. *Let (R, m, K) be a regular commutative Noetherian local ring of mixed characteristic $p > 0$. Then the set of associated primes of $H^i_I(R)$ that do not contain p is finite for every $i \in \mathbb{N}$ and every ideal $I \subset R$.*

Proof. This is an immediate consequence of Corollary III.3.4. □

CHAPTER IV

Polynomial rings and power series rings over a ring of small dimension

Throughout this chapter, A, and R denote commutative Noetherian rings with unity such that R is either a polynomial ring, $A[x_1, \ldots, x_n]$, or a power series ring, $A[[x_1, \ldots, x_n]]$. The main aim in this chapter is to prove that the set of associated primes and the Bass number of $H^i_I(R)$ are finite for every ideal $I \subset R$. In particular, these rings include regular local complete rings of unramified mixed characteristic, which gives a different proof for the properties in that case [Lyu00b]. In addition, we study the injective dimension of these modules and extend previous results of Zhou [Zho98].

The results presented in this section appear in [NB12b].

IV.1 Associated primes

Lemma IV.1.1. *Let A and R be Noetherian rings such that $A \subset R$. Let M be a $D(R, A)$-module of finite length. Then, $\operatorname{Ass}_R M$ is finite.*

Proof. Suppose $M \neq 0$. Let $M_1 = M$ and P_1 be a maximal element in the set of the associated primes of M_1. Then, $N_1 = H^0_{P_1}(M_1)$ is a nonzero $D(R, A)$-submodule of M_1, and it has only one associated prime. Given N_j and M_j, set $M_{j+1} = M_j/N_j$. If $M_{j+1} \neq 0$, let P_{j+1} be a maximal element in the set of the associated primes of M_{j+1}. Then $N_{j+1} = H^0_{P_j}(M_{j+1})$ has only one associated prime. If $M_{j+1} = 0$, set $N_{j+1} = 0$ and $P_{j+1} = 0$. Since $M_1 = M$ has finite length as a $D(R, A)$-module, there exist $\ell \in \mathbb{N}$ such that $M_j = 0$ for $j \geq \ell$ and then $\operatorname{Ass}(M) \subset \{P_1, \ldots, P_\ell\}$. □

Lemma IV.1.2. *Let A be a zero-dimensional Noetherian ring. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Then, R_f has finite length as a $D(R, A)$-module for every $f \in R$.*

Proof. Since A has finite length as a A-module, there is a finite filtration of ideals $0 = N_0 \subset N_1 \subset \ldots \subset N_\ell = A$ such that N_{j+1}/N_j is isomorphic to a field. Then, we have an induced filtration of $D(R, A)$-modules, $0 = N_0 R_f \subset N_1 R_f \subset \ldots \subset N_\ell R_f = R_f$. It suffices to prove that $N_{j+1}R_f/N_j R_f$ has finite length for $j = 1, \ldots, \ell$. We note that $N_{j+1}R_f/N_j R_f$ is zero or isomorphic to $(R/m)_f$ for some maximal ideal $m \subset A$. Since $N_{j+1}R_f/N_j R_f$ has finite length as a $D(R/mR, A/mA)$-module, it has finite length as a $D(R, A)$-module, which concludes the proof. □

Proposition IV.1.3. *Let A be a zero-dimensional commutative Noetherian ring. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Then, $\operatorname{Ass}_R M$ is finite for every object in $M \in C(R, A)$; in particular, this holds for $\mathcal{T}(R)$ for every functor $\mathcal{T}$.*

Proof. By Lemma IV.1.2, R_f has finite length in the category of $D(R, A)$-modules for every $f \in R$. If M is an object of $C(R, A)$, then M has finite length as a $D(R, A)$-module, because length is additive. □

Lemma IV.1.4. *Let A be a one-dimensional ring, $\pi \in A$ be an element such that $\dim(A/\pi A) = 0$, and R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Then, $R_f/\pi R_f$ has finite length as a $D(R, A)$-module for every $f \in R$.*

Proof. The length of $R_f/\pi R_f$ as a $D(R, A)$-module or as a $D(R/\pi R, A/\pi A)$-module is the same. Since $A/\pi A$ has dimension zero and $R/\pi R$ is either $(A/\pi A)[x_1, \ldots, x_n]$ or $(A/\pi A)[[x_1, \ldots, x_n]]$, the result follows from Lemma IV.1.1 and Lemma IV.1.2. □

Lemma IV.1.5. *Let A be a one-dimensional ring, $\pi \in A$ be an element such that $\dim(A/\pi A) = 0$, and R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let $\overline{A}$ and $\overline{R}$ denote $A/\pi A$ and $R/\pi A$ respectively. Let M be a $D(R, A)$-module, such that $\operatorname{Ann}_M(\pi)$ and $M \otimes_R \overline{R}$ are objects in $C(\overline{R}, \overline{A})$. Then, $\operatorname{Ann}_{\mathcal{T}(M)}(\pi)$ and $\mathcal{T}(M) \otimes_R \overline{R}$ are objects in $C(\overline{R}, \overline{A})$ for every functor $\mathcal{T}$.*

Proof. We recall that $\mathcal{T}$ has the form $\mathcal{T} = \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_t$, where every functor $\mathcal{T}_j$ is either $H^i_{Z_1}$, $H^i_{Z_1 \setminus Z_2}$, or the kernel, image or cokernel of some arrow in the long exact sequence

$$\text{(IV.1.5.1)} \qquad \ldots \xrightarrow{\alpha_i} H^i_{Z_1}(M) \xrightarrow{\beta_i} H^i_{Z_2}(M) \xrightarrow{\gamma_i} H^i_{Z_1/Z_2}(M) \to \ldots$$

for closed subsets Z_1, Z_2 of $\operatorname{Spec}(R)$ such that $Z_2 \subset Z_1$.

It suffices to prove the claim for $t = 1$ by an inductive argument. Suppose that $\mathcal{T} = H_Z(-)$ where $Z = Z_1 \setminus Z_2$ for closed subsets $Z_1, Z_2 \subset \operatorname{Spec}(R)$ such that $Z_2 \subset Z_1$.

We note that $H^i_Z(-) = H^i_{Z_1}(-)$, if we choose $Z_2 = \emptyset$. The exact sequences

$$0 \to \mathrm{Ann}_M(\pi) \to M \xrightarrow{\cdot\pi} \pi M \to 0,$$

and

$$0 \to \pi M \to M \to M \otimes_R \overline{R} \to 0,$$

induce two long exact sequences,

$$\ldots \to H^i_Z(\mathrm{Ann}_M(\pi)) \xrightarrow{\phi_i} H^i_Z(M) \xrightarrow{\varphi_i} H^i_Z(\pi M) \to \ldots$$

and

$$\ldots \to H^i_Z(\pi M) \xrightarrow{\phi'_i} H^i_Z(M) \xrightarrow{\varphi'_i} H^i_Z(M \otimes_R \overline{R}) \to \ldots .$$

Since the composition of $\phi'_i \circ \varphi_i$ is the multiplication by π on $H^i_Z(M)$, we obtain the exact sequences

$$0 \to \mathrm{Ker}(\varphi_i) \to \mathrm{Ann}_{H^i_Z(M)}(\pi) \xrightarrow{\varphi_i} \mathrm{Ker}(\phi'_i),$$

and

$$\mathrm{Coker}(\varphi_i) \xrightarrow{\phi'_i} H^i_Z(M) \otimes_R \overline{R} \to \mathrm{Coker}(\phi'_i) \to 0.$$

Then, $\mathrm{Ann}_{H^i_Z(M)}(\pi)$ and $H^i_Z(\pi M) \otimes_R \overline{R}$ are objects in $C(\overline{R}, \overline{A})$, because $\mathrm{Ker}(\varphi_i)$, $\mathrm{Ker}(\phi'_i), \mathrm{Coker}(\varphi_i)$ and $\mathrm{Coker}(\phi'_i)$ belong to $C(\overline{R}, \overline{A})$ and this category is closed under sub-objects, extensions and quotients.

If $\mathcal{T}$ is a kernel, image or cokernel of a morphism in the long exact sequence (IV.1.5.1), there exists an injection, $\mathcal{T}(M) \to H^{i_1}_{Z_{j_1}}(M)$, and a surjection $H^{i_2}_{Z_{j_2}}(M) \to \mathcal{T}(M)$ for some $i_1, i_2 \geq 0$ and $j_1, j_2 \in \{1, 2\}$. Then,

$$0 \to \mathrm{Ann}_{\mathcal{T}(M)}(\pi) \to \mathrm{Ann}_{H^{i_1}_{Z_{j_1}}(M)}(\pi)$$

and

$$H^{i_2}_{Z_{j_2}}(M) \otimes_R \overline{R} \to \mathcal{T}(M) \otimes_R \overline{R} \to 0$$

are exact. Therefore, $\mathrm{Ann}_{\mathcal{T}(M)}(\pi)$ and $\mathcal{T}(M) \otimes_R \overline{R}$ belong to $C(\overline{R}, \overline{A})$. □

Proposition IV.1.6. *Let A be a one-dimensional ring, $\pi \in A$ be an element such that $\dim(A/\pi A) = 0$, and R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Then, the set of associated primes over R of $\mathcal{T}(R)$ that contain π is finite for every functor $\mathcal{T}$.*

Proof. The set of associated primes of $\mathcal{T}(R)$ that contain π is equal to $\mathrm{Ass}_R \mathrm{Ann}_{\mathcal{T}(R)}(\pi)$. Since $\mathrm{Ann}_{\mathcal{T}}(\pi)$ is a $D(R, A)$-module of finite length by Lemma IV.1.5, $\mathrm{Ass}_R \mathrm{Ann}_{\mathcal{T}(R)}(\pi)$

is finite by Lemma IV.1.1. □

Corollary IV.1.7. *Let A be a one-dimensional local ring, and let $R = A[x_1, \dots, x_n]$. Then,* $\operatorname{Ass}_R \mathcal{T}(R)$ *is finite.*

Proof. Let π be a parameter for A. Then, the set of associated primes over R of $\mathcal{T}(R)$ that contain π is finite by Corollary IV.1.6. Since $R_\pi = A_\pi[x_1, \dots, x_n]$ and $\dim(A_\pi) = 0$, the set of associated primes over R of $\mathcal{T}(R)$ that does not contain π, which is in correspondence with $\operatorname{Ass}_{R_\pi} \mathcal{T}(R_\pi)$, is finite by Corollary IV.1.3. □

Corollary IV.1.8. *Let (A, m, K) be a one-dimensional local domain, and let $R = A[[x_1, \dots, x_n]]$. Then,* $\operatorname{Ass}_R \mathcal{T}(R)$ *is finite.*

Proof. Let π be a parameter for A. Then, the set of associated primes over R of $\mathcal{T}(R)$ that contain π is finite by Corollary IV.1.6. It remains to show that the set of the associated primes not containing π is finite.

We will proceed by cases. If A is a ring of characteristic $p > 0$. We have that R_π is a regular ring by Theorem 5.1.2 in [Gro67] because R_π is the fiber at the zero prime ideal of A. Then, $\operatorname{Ass}_{R_\pi} \mathcal{T}(R_\pi)$ is finite by Corollary 2.14 in [Lyu97].

If A is not a ring of characteristic $p > 0$. We have again that R_π is a regular ring by Theorem 5.1.2 in [Gro67]. Let $F = A_\pi$ be the fraction field of A and $S = F \otimes_A R = R_\pi$. Then, F is a field of characteristic 0 and S is an F-algebra. We claim that S and F satisfy the properties:

(i) S is equidimensional of dimension n;

(ii) every residual field with respect to a maximal ideal is an algebraic extension of F;

(iii) there exist F-linear derivations $\partial_1, \dots, \partial_n \in \operatorname{Der}_F(S)$ and elements $z_1 \dots, z_n \in R$ such that $\partial_i a_j = 1$ if $i = j$ and 0 otherwise.

We will proceed following the ideas of Lyubeznik in [Lyu00b]. Let $\eta \subset S$ be a maximal ideal and let $Q = \eta \cap R$. Then Q is a prime ideal of R not containing f and it is maximal among the ideals of R not containing π. By induction on n, it suffices to show that if P is a nonzero prime ideal of R not containing π, then there exist elements $y_1, \dots, y_n \in R$ such that $R = A[[y_1, \dots, y_n]]$ and R/P is a finitely generated $R_{n-1}/P \cap R_{n-1}$-module, where $R_{n-1} = A[[y_1, \dots, y_{n-1}]]$. Then, the finiteness implies that $P \cap R_{n-1} = \operatorname{ht} P - 1$, the prime ideal P is maximal among all ideals of R not containing π if and only if $P \cap R_n$ is maximal among all ideals of R_n not containing

π. In addition, $S/PS = (R/P) \otimes_A F$ is an algebraic extension of F if and only if $F \otimes_A R_{n-1}/P \cap R_{n-1}$ is an algebraic extension of F.

Let $\overline{P}$ be the image of P in $\overline{R} = R/mR = k[[x_1, \dots, x_n]]$. There exist new variables $y_1, \dots, y_n$ such that $\overline{R}/\overline{P}$ is finite over $\overline{R}_{n-1}/\overline{P} \cap \overline{R}_{n-1}$, where $\overline{R}_{n-1} = K[[y_1, \dots y_{n-1}]]$. Let $r_1, \dots, r_s \in \overline{R}/\overline{P}$ be a set of generators over $\overline{R}_{n-1}/\overline{P} \cap \overline{R}_{n-1}$. Lifting these variables to R, we get that $R = A[[y_1, \dots, y_n]]$. For every $f \in R/P$ there exist a finite number of elements $g_{1,1}, \dots, g_{1,s}, v_{1,j} \in R_{n-1}$ and $h_{1,j} \in m$ with

$$f = g_{1,1} r_1 + \ldots + g_{1,s} s r_s + \sum_j h_{1,j} v_{1,j}.$$

We can apply the same idea to $v_{i,j}$ inductively to obtain a finite number of elements $g_{t,1}, \dots, g_{t,s}, v_{t,j} \in R_{n-1}$ and $h_{t,j} \in m^t$ such that

$$f = \left(\sum_{k=1}^{t} \sum_i h_{k-1,i} g_{k,1} \right) r_1 + \ldots + \left(\sum_{k=1}^{t} \sum_i h_{k-1,i} g_{k,s} \right) r_s + \sum_j h_{t,j} v_{t,j}.$$

Since R/P is m-adically separated and complete, we can take

$$G_\ell = \sum_{k=1}^{\infty} \sum_i h_{k-1,i} g_{k,\ell}.$$

Then $f = G_1 r_1 + \ldots G_s r_s$. This proves that $r_1, \dots, r_s$ is a finite system of generators of R/P as an R_{n-1}/PR_{n-1}-module and concludes the proof of the claim that $R \otimes_A F$ and F satisfy properties (i) and (ii). In addition, we have that $z_i = x_i$ and $\partial_i = \frac{\partial}{\partial x_i}$ satisfies (iii). Then, we have that $\operatorname{Ass}_{R_\pi} \mathcal{T}(R_\pi)$ is finite by using the results of D-modules in [MNM91] as it was done in [Lyu93]. It is proven explicitly in Theorem III.3.5. ☐

IV.2 Bass numbers

IV.2.1 Facts about Bass numbers

Lemma IV.2.1. *Let (R, m, K) be a Noetherian Cohen-Macaulay ring and $\pi \in R$ be a nonzero divisor. Let M be an R-module annihilated by π. Then, $\dim_K \operatorname{Ext}^\ell_R(K, M)$ is finite for all $j \in \mathbb{N}$ if and only if $\dim_K \operatorname{Ext}^\ell_{R/\pi R}(K, M)$ is finite for all $\ell \in \mathbb{N}$.*

Proof. Let $g_i \in R$, such that $\pi, g_1, \dots, g_n$ form a system of parameters. Let J denote $(\pi, g_1, \dots, g_n)R$. Using the Koszul complex to compute the free resolution of R/J as an R-module and as an $R/\pi R$-module, we obtain that

$$\begin{aligned}\operatorname{length}(\operatorname{Ext}^{\ell}_R(R/J,M)) &= \operatorname{length}(\operatorname{Ext}^{\ell}_{R/\pi R}(R/J,M))\\ &\quad + \operatorname{length}(\operatorname{Ext}^{\ell-1}_{R/\pi R}(R/J,M)).\end{aligned}$$

The result follows from Lemma VI.2.1, because R/J has finite length. □

Lemma IV.2.2. *Let R be a Cohen-Macaulay local ring, M be an R-module and $\pi \in R$ be a nonzero divisor. Let $\overline{R}$ denote $R/\pi R$. Suppose that $\dim_K \operatorname{Ext}^j_{\overline{R}}(K, \operatorname{Ann}_M(\pi))$ and $\dim_K \operatorname{Ext}^j_{\overline{R}}(K, M \otimes_R \overline{R})$ are finite for all $j \in \mathbb{N}$. Then, $\dim_K \operatorname{Ext}^j_R(K,M)$ is finite for all $j \in R$.*

Proof. $\dim_K \operatorname{Ext}^j_R(K, \operatorname{Ann}_M(\pi))$ and $\dim_K \operatorname{Ext}^j_R(K, M \otimes_R \overline{R})$ are finite for all $j \in \mathbb{N}$ by Lemma IV.2.1. From the short exact sequences

$$0 \to \operatorname{Ann}_M(\pi) \to M \xrightarrow{\pi} \pi M \to 0$$

and

$$0 \to \pi M \to M \to M \otimes_R \overline{R} \to 0,$$

we get two long exact sequences induced by Ext:

$$\ldots \to \operatorname{Ext}^{\ell}_R(K, \operatorname{Ann}_M(\pi)) \xrightarrow{\alpha_\ell} \operatorname{Ext}^{\ell}_R(K,M) \xrightarrow{\beta_\ell} \operatorname{Ext}^{\ell}_R(K,\pi M) \to \ldots,$$

and

$$\ldots \to \operatorname{Ext}^{\ell}_R(K,\pi M) \xrightarrow{\gamma_\ell} \operatorname{Ext}^{\ell}_R(K,M) \xrightarrow{\theta_\ell} \operatorname{Ext}^{\ell}_R(K, M \otimes_R \overline{R}) \to \ldots.$$

Since $\Im(\theta_\ell)$ injects into $\operatorname{Ext}^{\ell}_R(K, M \otimes_R \overline{R})$, we have that $\dim_K \Im(\theta_\ell)$ is finite. Likewise, $\dim_K \operatorname{Coker}(\beta_\ell)$ is finite, because it injects into $\operatorname{Ext}^{\ell+1}_R(K, \operatorname{Ann}_M(\pi))$. We note that

$$\operatorname{Ext}^{\ell}_R(K,\pi M) = \operatorname{Ker}(\beta_\ell) \oplus \operatorname{Coker}(\beta_\ell).$$

Since

$$\gamma_\ell \circ \beta_\ell = \operatorname{Ext}^{\ell}_R(K,M) \xrightarrow{\pi} \operatorname{Ext}^{\ell}_R(K,M)$$

is the zero morphism for $\ell \in \mathbb{N}$, we have that $\Im(\gamma_\ell) = \gamma_\ell(\operatorname{Coker}(\beta_\ell))$. Therefore, $\gamma(\operatorname{Coker}(\beta_\ell)) \to \operatorname{Ext}^{\ell}_R(K,M) \to \Im(\theta_\ell) \to 0$ is exact, and then $\dim_K(\operatorname{Ext}^{\ell}_R(K,M))$ is finite. □

IV.2.2 Finiteness properties of Bass numbers of local cohomology modules

Definition IV.2.3. Let A be a zero dimensional Noetherian ring. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let M be an $D(R, A)$-module. We say that M is *C-filtered* if there exists a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_\ell = M$$

of $D(R, A)$-modules, such that M_{i+1}/M_i is either zero or

(1) M_{i+1}/M_i is annihilated by a maximal ideal $m_i \subset R$,

(2) M_{i+1}/M_i is an object in $C(R/m_iR, A/m_i)$, and

(3) M_{i+1}/M_i is a simple $D(R, A)$-module.

Lemma IV.2.4. *Let A be a zero dimensional Noetherian ring. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let M be an object in $C(R, A)$. Then, M is C-filtered.*

Proof. We first prove the claim for R_f for every $f \in R$. Since A has finite length as an A-module, there is a finite filtration of ideals,

$$0 = N_0 \subset N_1 \subset \ldots \subset M_\ell = A,$$

such that M_{i+1}/M_i is isomorphic to a field, $K_i = A/m_i$, where m_i is a maximal ideal of A. Then, we have an induced filtration of $D(R, A)$-modules,

$$0 = N_0R_f \subset N_1R_f \subset \ldots \subset N_\ell R_f = R_f.$$

Thus, $N_{i+1}R_f/N_iR_f = R_f/m_iR_f$, which is an object in $C(R/m_iR, A/m_i)$. Then, there exist a filtration,

$$N_i = M_{i,1} \subset \ldots \subset M_{i,j_i} = N_{i+1},$$

of objects in $C(R/m_iR, A/m_i)$, such that $M_{i,t+1}/M_{i,t}$ is a simple $D(R/m_iR, A/m_i)$-module. Therefore,

$$0 = M_{0,1} \subset \ldots \subset M_{0,j_1} \subset M_{1,1} \subset \ldots \subset M_{\ell,j_\ell} = R_f$$

is a filtration that makes R_f a C-filtered module. By the definition of $C(R,A)$, it suffices to show that if $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is a short exact sequence of objects in $C(R,A)$, then M is C-filtered if and only if M' and M'' are C-filtered. If M is C-filtered with a filtration M_i, we define a filtration M_i' in M' by $M_i' = \alpha^{-1}(M_i)$. Similarly, we define a filtration M_i'' in M'' by $M_i'' = \beta(M_i)$. Then, we have a short exact sequence of short exact sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_i' & \xrightarrow{\alpha} & M_{i+1}' & \xrightarrow{\beta} & M_{i+1}'/M_i' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_i & \xrightarrow{\alpha} & M_{i+1} & \xrightarrow{\beta} & M_{i+1}/M_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{\alpha} & M_i'' & \xrightarrow{\beta} & M_{i+1}'' & \to & M_{i+1}''/M_i'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since M_{i+1}/M_i is either zero or a simple $D(R,A)$-module, M_{i+1}'/M_i' is either M_{i+1}/M_i or zero. Likewise, M_{i+1}''/M_i'' is either M_{i+1}/M_i or zero. Thus, M_i' and M_i'' satisfy parts (1), (2) and (3) in Definition IV.2.3.

If M' and M'' are C-filtered modules with filtrations $M_0' \subset \ldots \subset M_{\ell'}'$ and $M_0'' \subset \ldots \subset M_{\ell''}''$, we define a filtration on M by $M_i = \alpha(M_i')$ for $i = 0, \ldots \ell'$ and $M_i = \beta^{-1}(M_{i-\ell'}')$ for $i = \ell'+1, \ldots, \ell'+\ell''$. Since $M_{i+1}/M_i = M_{i+1}'/M_i'$ for $i = 0, \ldots \ell'$ and $M_{i+1}/M_i = M_{i+1-\ell'}''/M_{i-\ell'}''$ for $i = \ell'+1, \ldots \ell'+\ell''$, M_i satisfies parts (1), (2) and (3) in Definition IV.2.3. Hence, every object in $C(R,A)$ is a C-filtered module. □

Proposition IV.2.5. *Let A be a zero-dimensional Noetherian ring. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let M be an object in $C(R,A)$. Then, all the Bass numbers of M are finite. In particular, this holds for $\mathcal{T}(R)$ for every Lyubeznik functor $\mathcal{T}$.*

Proof. We fix a prime ideal $P \subset R$ and denote R_P/PR_P by K_P. Since M is a C-filtered module by Lemma IV.2.4, we have a filtration $0 = M_0 \subset \ldots \subset M_\ell = M$ such that M_{i+1}/M_i is annihilated by a maximal ideal $m_i \subset R$, is an object in $C(R/m_iR, A/m_i)$, and is a simple $D(R,A)$-module. From the short exact sequences $0 \to M_j \to M_{j+1} \to M_{j+1}/M_j \to 0$, we get long exact sequences

$$\ldots \to \operatorname{Ext}^j_{R_P}(K_P, (M_i)_P) \to \operatorname{Ext}^j_{R_P}(K_P, (M_{i+1})_P)$$

$$\to \operatorname{Ext}^j_{R_P}(K_P, (M_{i+1}/M_i)_P) \to \operatorname{Ext}^{j+1}_{R_P}(K_P, (M_i)_P) \to \dots .$$

Then, it suffices to show the claim for M_{i+1}/M_i for $i = 0, \dots, \ell$. We fix an i and denote M_{i+1}/M_i by N. Let $m \subset A$ be the maximal ideal such that $mN = 0$. If $mR \not\subset P$, then $N \otimes R_P = 0$. We may assume that $mR \subset P$. Let $\overline{R} = R/mR$. We note that $\overline{R}$ is a regular ring containing A/m, a field. Let $g_1, \dots, g_d$ be a system of parameters for R_P and let $f_1, \dots, f_d$ be the class of $g_1, \dots, g_d$ in $\overline{R}_P$. Since A is a zero dimensional ring, we have that $f_1, \dots, f_d$ is a system of parameters for $\overline{R}_P$. Let $I = (g_1, \dots, g_d)R_P$. Using the Koszul complex $\mathcal{K}$, we obtain that,

$$\operatorname{Ext}^i_{R_P}(R_P/I, N_P) = H^i(\operatorname{Hom}_{R_P}(\mathcal{K}(\underline{g}), N_P))$$

$$= H^i(\operatorname{Hom}_{\overline{R}_P}(\mathcal{K}(\underline{f}), N_P)) = \operatorname{Ext}^i_{\overline{R}_P}(\overline{R}_P/I\overline{R}_P, N_P),$$

because R_P and $\overline{R}_P$ are Cohen-Macaulay rings of the same dimension. Using Lemma VI.2.1 several times, we obtain that

$$\begin{aligned}
\operatorname{length}_{\overline{R}_P} \operatorname{Ext}^i_{\overline{R}_P}(K_P, N_P) < \infty &\Leftrightarrow \operatorname{length}_{\overline{R}_P} \operatorname{Ext}^i_{\overline{R}_P}(\overline{R}_P/I\overline{R}_P, N_P) < \infty \\
&\Leftrightarrow \operatorname{length}_{\overline{R}_P} H^i \operatorname{Hom}_{\overline{R}_P}(\mathcal{K}(\underline{f}), N_P) < \infty \\
&\Leftrightarrow \operatorname{length}_{R_P} H^i \operatorname{Hom}_{R_P}(\mathcal{K}(\underline{g}), N_P) < \infty \\
&\Leftrightarrow \operatorname{length}_{R_P} \operatorname{Ext}^i_{R_P}(R_P/I, N_P) < \infty \\
&\Leftrightarrow \operatorname{length}_{R_P} \operatorname{Ext}^i_{R_P}(K_P, N_P) < \infty
\end{aligned}$$

Since $\operatorname{length}_{\overline{R}_P} \operatorname{Ext}^i_{\overline{R}_P}(K_P, N_P) < \infty$ by Corollary 8 in [Lyu11], we have that

$$\operatorname{length}_{R_P} \operatorname{Ext}^i_{R_P}(K_P, N_P) < \infty$$

Hence, all the Bass numbers of M are finite. □

Theorem IV.2.6. *Let A be a zero dimensional commutative Noetherian ring. Let R be either $A[x_1, \dots, x_n]$ or $A[[x_1, \dots, x_n]]$. Then,*

- $\operatorname{Ass}_R \mathcal{T}(R)$ *is finite for every functor* $\mathcal{T}$*, and*
- *the Bass numbers of* $\mathcal{T}(R)$ *are finite.*

In particular, these properties hold for $H^i_I(R)$ for every ideal $I \subset R$ and every integer $i \in \mathbb{N}$.

Proof. This is a consequence of Proposition IV.1.3 and Proposition IV.2.5. □

Proposition IV.2.7. *Let A be a Noetherian Cohen-Macaulay ring such that* $\dim(A) = 1$*, and let $\pi \in A$ be a nonzero divisor. Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Then all the Bass numbers of $\mathcal{T}(R)$, as an R-module, with respect to a prime ideal P containing π, are finite.*

Proof. Let $\overline{R}$ and $\overline{A}$ denote $R/\pi R$ and $A/\pi A$, respectively. We have that $\operatorname{Ann}_{\mathcal{T}(R)}(\pi)$ and $\mathcal{T}(R) \otimes \overline{R}$ are objects in $C(\overline{R}, \overline{A})$ by Lemma IV.1.5. Then, their Bass numbers, as $\overline{R}$-modules, with respect to P are finite by Proposition IV.2.5. Since R_P and $\overline{R}_P$ are Cohen-Macaulay rings, we have that the Bass numbers of $\mathcal{T}(R)$ with respect to P are finite by Lemma IV.2.1 for every functor $\mathcal{T}$. □

We claim that we cannot generalize Proposition IV.2.7 for Cohen-Macaulay rings of dimension higher than 3. Let $A = K[[s,t,u,w]]/(us+vt)$, where K is field. This is the ring given by Hartshorne's example [Har68]. Let $I = (s,t)A$. Hartshorne showed that $\dim_K \operatorname{Hom}_A(K, H^2_I(A))$ is not finite.

Let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let $P = mR$ be the prime ideal generated by m. Then,

$$\begin{aligned}\operatorname{Ext}^0_R(R/P, H^2_I(R)) &= \operatorname{Hom}_R(R/P, H^2_I(R)) \\ &= \operatorname{Hom}_A(K, H^2_I(A)) \otimes_A R = \oplus R/mR,\end{aligned}$$

where the direct sum in the last equality is infinite. Therefore,

$$\dim_{R_P/mR_P} \operatorname{Ext}^0_{R_P}(R_P/PR_P, H^2_I(R_P))$$

is not finite.

Corollary IV.2.8. *Let A be a one-dimensional local Cohen-Macaulay ring, and let $R = A[x_1, \ldots, x_n]$. Then, all the Bass numbers of $\mathcal{T}(R)$, as an R-module, are finite.*

Proof. Let π be a parameter for A. Then, the Bass numbers of $\mathcal{T}(R)$ with respect to a prime ideal containing π, are finite by Proposition IV.2.7. Since $R_\pi = A_\pi[x_1, \ldots, x_n]$ and $\dim(A_\pi) = 0$, the Bass numbers of $\mathcal{T}(R)$ with respect to a prime ideal that does not contain π, are finite by Proposition IV.2.5. □

Corollary IV.2.9. *Let A be a one-dimensional local domain, and $R = A[[x_1, \ldots, x_n]]$. Then, all the Bass numbers of $\mathcal{T}(R)$, as an R-module, are finite.*

Proof. Let π be a parameter for A. Then, the Bass numbers of $\mathcal{T}(R)$ with respect to a prime ideal P containing π, are finite by Proposition IV.2.7.

On the other hand, the Bass numbers of $\mathcal{T}(R)$ with respect to prime ideals not containing π, are in correspondence with the Bass numbers of R_π. We have that R_π is a regular ring that contains a field, A_π, by Theorem 5.1.2 in [Gro67] because R_π is the fiber at the zero prime ideal of A. Then the result follows from Theorem 2.1 in [HS93] and Theorem 3.4 in [Lyu93]. □

Theorem IV.2.10. *Let A be a one-dimensioal ring, and let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let $\pi \in A$ denote an element such that $\dim(A/\pi A) = 0$. Then, the set of associated primes over R of $\mathcal{T}(R)$ that contain π is finite for every functor $\mathcal{T}$. Moreover, if A is Cohen-Macaulay and π is a nonzero divisor, then the Bass numbers of $\mathcal{T}(R)$, with respect to a prime ideal P that contains π, are finite. In particular, these properties hold for $H^i_I(R)$ for every ideal $I \subset R$ and every integer $i \in \mathbb{N}$.*

Proof. This is a consequence of Proposition IV.1.6 and Proposition IV.2.7. □

IV.3 Local cohomology of unramified regular rings

As consequence of the results in Section 3 and 4, we are able to give a different proof for some parts of Theorem 1 in [Lyu00b].

Theorem IV.3.1. *Let (R, m, K) be an unramified regular local ring and $p = Char(K)$. Then:*

(i) *the Bass numbers of $\mathcal{T}(R)$ are finite, and*

(ii) *the set of associated primes of $\mathcal{T}(R)$ that contain p is finite*

for every Lyubeznik functor $\mathcal{T}$.

Proof. The finiteness of associated primes of $\mathcal{T}(R)$ that contain p is not affected by completion with respect to the maximal ideal. Since completion of R respect to m is a power series ring over a complete DVR of mixed characteristic, the result follows from Proposition IV.1.6.

In order to prove the finiteness of the Bass numbers, We need to show that $\dim_{R_P/PR_P} \operatorname{Ext}^j_{R_P}(R_P/PR_P, \mathcal{T}(R_P))$ is finite for every prime ideal $P \subset R$. There are two cases: $p \in P$ or not. If $p \notin P$ then R_P has equal characteristic 0 and the result follows from Theorem 3.4 in [Lyu93]. If $p \in P$, R_P is an unramified regular local ring and its completion of R_P respect to the maximal ideal is a power series ring over a complete DVR of mixed characteristic. Since the dimension of $\operatorname{Ext}^{\ell}_{R_P}(R_P/PR_P, \mathcal{T}(R_P))$ as R_P/PR_P-vector space is not affected by completion, the result follows from Corollary IV.2.7. □

IV.4 Injective Dimension

In this section, we recover and generalize some results of Zhou about injective dimension [Zho98].

Lemma IV.4.1. *Let (A, m, K) be a regular local ring and let R be either $A[[x_1, \dots, x_n]]$ or $A[x_1, \dots, x_n]$. Let $P \subset R$ be a prime ideal containing mR and let K_P denote the field R_P/PR_P. Let M be a $D(R, A)$-module. Then, $\operatorname{Ext}^{\ell}_{R}(K_P, M_P) = 0$ for $\ell > \dim(A) + \dim(\operatorname{Supp}_R(M))$.*

Proof. The proof will be by induction on the $d = \dim(A)$. If $d = 0$, then $A = K$ is a field and the proof follows from the first Theorem in [Lyu00c]. We assume that the claim is true for $d - 1$. Let $y_1, \dots, y_d$ denote a minimal set of generator for m. Let $\overline{A} = A/y_d A$, $\overline{R} = R/y_d R = \overline{A}[[x_1, \dots, x_n]]$ and $\overline{P} = P\overline{R}$. Let $\overline{y}_1, \dots, \overline{y}_{d-1}$ be the class of $y_1, \dots, y_{d-1}$ in $\overline{R}$. We note that $\overline{P} \subset \overline{R}$ is a prime ideal and it contains $m\overline{R}$. Let $f_1, \dots f_s \in P$ be such that $y_1, \dots, y_{d-1}, f_1, \dots f_s$ form a minimal set of generator for the maximal ideal PR_P. From the Koszul complex associated to $y_1, \dots, y_{d-1}, f_1, \dots f_s$ in R_P, we have that for every $\overline{R}_P$-module, N,

$$\dim_{K_P} \operatorname{Ext}^{\ell}_{R_P}(K_P, N) = \dim_{K_P} \operatorname{Ext}^{\ell}_{\overline{R}_P}(K_P, N) + \dim_{K_P} \operatorname{Ext}^{\ell-1}_{\overline{R}_P}(K_P, N).$$

In this case, we have that $\operatorname{Ann}_M(y_d)$ and $M/y_d M$ are $D(\overline{R}, \overline{A}))$-modules. By the induction hypothesis,

$$\operatorname{Ext}^{\ell}_{\overline{R}_{\overline{P}}}(K_P, \operatorname{Ann}_{M_P}(y_d)) = 0 \text{ and } \operatorname{Ext}^{\ell}_{\overline{R}_{\overline{P}}}(K, M_P/y_d M_P) = 0$$

for $\ell > d + \dim(\operatorname{Supp}_R(M)) - 1 = \dim(\overline{A}) + \dim(\operatorname{Supp}_R(M))$. Then,

$$\operatorname{Ext}^{\ell}_{R_P}(K_P, \operatorname{Ann}_{M_P}(y_d)) = 0 \text{ and } \operatorname{Ext}^{\ell}_{R_P}(K, M_P/y_d M_P) = 0$$

for $\ell > d + \dim(\operatorname{Supp}_R(M))$.

From the short exact sequences

$$0 \to \operatorname{Ann}_{M_P}(y_d) \to M_P \xrightarrow{y_d} y_d M_P \to 0$$

and

$$0 \to y_d M_P \to M_P \to M_P \otimes_R \overline{R} \to 0,$$

we get two long exact sequences induced by Ext:

$$\ldots \to \operatorname{Ext}^{\ell}_{R_P}(K_P, \operatorname{Ann}_{M_P}(y_d)) \to \operatorname{Ext}^{\ell}_{R_P}(K_P, M_P)$$
$$\xrightarrow{\rho_\ell} \operatorname{Ext}^{\ell}_{R_P}(K_P, y_d M_P) \to \ldots$$

and

$$\ldots \to \operatorname{Ext}^{\ell}_{R_P}(K_P, y_d M_P) \xrightarrow{\theta_\ell} \operatorname{Ext}^{\ell}_{R_P}(K_P, M_P)$$
$$\xrightarrow{\varrho_\ell} \operatorname{Ext}^{\ell}_{R_P}(K_P, M_P \otimes_R \overline{R}) \to \ldots$$

In this case, ρ_ℓ is an isomorphism and θ_ℓ is surjective for $\ell > d + \dim(\operatorname{Supp}_R(M))$. Then, $\theta_\ell \circ \rho$ is surjective for $\ell > d + \dim(\operatorname{Supp}_R(M))$. Since

$$\theta_\ell \circ \rho_\ell = \operatorname{Ext}^{j}_{R_P}(K_P, M_P) \xrightarrow{y_d} \operatorname{Ext}^{j}_{R_P}(K_P, M_P)$$

is the zero morphism, $\operatorname{Ext}^{j}_{R_P}(K_P, K_P) = 0$ for $\ell > d + \dim(\operatorname{Supp}_R M)$. □

Proposition IV.4.2. *Let A be a Noetherian ring and let $R = A[x_1, \ldots, x_n]$. Let $P \subset R$ be a prime ideal and let K_P denote the field R_P/PR_P. Let M be a $D(R, A)$-module. Then, $\operatorname{Ext}^{\ell}_{R}(K_P, M_P) = 0$ for $\ell > \dim(A) + \dim(\operatorname{Supp}_R M)$.*

Proof. Let $Q = P \cap A$. Then, PR_Q is a prime ideal in R_Q that contains QR_Q. Since, $R_Q = A_Q[x_1, \ldots, x_n]$ and M_Q is a $D(R_Q, A_Q)$-module and $(M_Q)_P = M_P$, we have that $\operatorname{Ext}^{\ell}_{R_P}(K_P, M_P) = 0$ is zero for $\ell > \dim(A) + \dim(\operatorname{Supp}_R(M_P))$ by Lemma IV.4.1. Hence, $\operatorname{Ext}^{\ell}_{R_P}(K_P, M_P) = 0$ is zero for $\ell > \dim(A) + \dim(\operatorname{Supp}_R(M))$, because $\dim(\operatorname{Supp}_R(M)) \geq \dim(\operatorname{Supp}_R(M_Q))$. □

Theorem IV.4.3. *Let (A, m, K) be a regular local Noetherian ring and let R be either $A[x_1, \ldots, x_n]$ or $A[[x_1, \ldots, x_n]]$. Let M be a $D(R, A)$-module supported only at mR. Then,*

$$\operatorname{inj.dim}(M) \leq \dim(A) + \dim(\operatorname{Supp} M).$$

In particular,

$$\operatorname{inj.dim}(H^j_\eta(\mathcal{T}(S))) \leq \dim(A),$$

where $\eta = (m, x_1, \ldots, x_n)S$ and $\mathcal{T}$ is a Lyubeznik functor. In addition, if $R = A[x_1, \ldots, x_n]$, then

$$\operatorname{inj.dim}(M) \leq \dim(A) + \dim(\operatorname{Supp} M).$$

for every $D(R, A)$*-module,* M*.*

Proof. This is a consequence of Proposition IV.4.2 and Proposition IV.4.2 □

CHAPTER V

Flat extensions with regular fibers

This chapter studies the following related question raised by Hochster:

Question V.0.4. Let (R, m, K) be a local ring and S be a flat extension with regular closed fiber. Is

$$\operatorname{Ass}_S H^0_{mS}(H^i_I(S)) = \mathcal{V}(mS) \cap H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

Question V.0.5. Let (R, m, K) be a local ring and S denote either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. Is

$$\operatorname{Ass}_S H^0_{mS}(H^i_I(S)) = \mathcal{V}(mS) \cap H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

It is clear that Question V.0.5 is a particular case of Question V.0.4. In Proposition V.4.2, we show that under minor additional hypothesis these questions are equivalent. Question V.0.5 has a positive answer when R is a ring of dimension 0 or 1 of any characteristic (see Theorem IV.2.6, IV.2.10, and [Lyu00b]). In her thesis [Rob12], Robbins answered Question V.0.5 positively for certain algebras of dimension smaller than or equal to 3 in characteristic 0. Namely:

Theorem V.0.6 ([Rob12]). *Let R be a domain finitely generated as an algebra over a field, K, of characteristic 0 and $S = R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. Suppose that R has a resolution of singularities, Y_0, which is the blowup of R along an ideal of depth at least two. If either*

- *Y_0 has an affine open cover by only U_1 and U_2 where $H^1(Y_0, O_{Y_0})$ has finite length over R,*

- *Y_0 has an open cover by only U_1, U_2, and U_3 where $H^1(Y_0, O_{Y_0})$ and $H^2(Y_0, O_{Y_0})$ have finite length over R, or*

- $\dim(R) = 2$ *or* $\dim(R) = 3$ *and R has an isolated singularity,*

then $\operatorname{Ass}_R H^i_I(R)$ is finite for any i and any ideal $I \subset S$.

In addition, several of her results can be obtained in characteristic $p > 0$, by working in the category $C(S, R)$ (see the discussion after Remark II.4.1).

A positive answer for Question V.0.4 would help to that the associated primes of local cohomology modules, $H^i_I(R)$, over certain regular local rings of mixed characteristic, R. For example,

$$\frac{V[[x, y, z_1, \ldots, z_n]]}{(\pi - xy)V[[x, y, z_1, \ldots, z_n]]} = \left(\frac{V[[x, y]]}{(\pi - xy)V[[x, y]]}\right)[[z_1, \ldots, z_n]],$$

where $(V, \pi V, K)$ is a complete DVR of mixed characteristic. This is, to the best of our knowledge, the simplest example of a regular local ring of ramified mixed characteristic that the finiteness of $\operatorname{Ass}_R H^i_I(R)$ is unknown.

The results presented in this section appear in [NB12a].

V.1 Σ-finite D-modules

Notation V.1.1. Thorough this section (R, m, K) denotes a local ring and S denotes either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. In addition, D denotes $D(S, R)$.

Definition V.1.2. Let M be a D-module supported at mS and $\mathcal{M}$ be the set of all D-submodules of M that have finite length. We say that M is Σ*-finite* if:

(i) $\bigcup_{N \in \mathcal{M}} N = M$,

(ii) $\bigcup_{N \in \mathcal{M}} \mathcal{C}(N)$ is finite, and

(iii) For every $N \in \mathcal{M}$ and $L \in \mathcal{C}(N)$, $L \in C(S/mS, R/mR)$.

We denote the set of *composition factors of* M, $\bigcup_{N \in \mathcal{M}} \mathcal{C}(N)$, by $\mathcal{C}(M)$.

Remark V.1.3. We have that

$$\operatorname{Ass}_S M \subset \bigcup_{N \in \mathcal{C}(M)} \operatorname{Ass}_S M$$

for every Σ-finite D-module, M. In particular, $\operatorname{Ass}_S M$ is finite.

Lemma V.1.4. *Let M be a Σ-finite D-module and N be a D-submodule of M. Then, N has finite length as D-module if and only if N is a finitely generated as D-module.*

Proof. Suppose that N is finitely generated. Let $v_1, \ldots, v_\ell$ be a set the generators of N. Since $\bigcup_{N\in\mathcal{M}} N = M$, there exists a finite length module N_i that contains v_i. Then, $N \subset N_1 + \ldots + N_\ell$ and it has finite length. It is clear that if N has finite length then it is finitely generated. □

Proposition V.1.5. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of D-modules. If M is Σ-finite, then M' and M'' are Σ-finite. Moreover, $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M')$.*

Proof. We first assume that M is Σ-finite. We have that

$$M' = \bigcup_{N\in\mathcal{M}} N \cap M' = \bigcup_{N'\in\mathcal{M}} N',$$

an so M is Σ-finite by Remark II.4.3. Let ρ denote the morphism $M \to M''$ and $N'' \in \mathcal{M}''$ and $\ell = \operatorname{length}_D N''$. There are $v_1, \ldots, v_\ell \in N''$ such that $N'' = D\cdot v_1 + \ldots + D\cdot v_\ell$. Let w_j be a preimage of v_j and N be the D-module generated by $w_1, \ldots, w_\ell$. We have that $N \to N''$ is a surjection, and that N has finite length by Lemma V.1.4. Therefore, $M'' = \bigcup_{N\in\mathcal{M}} \rho(N) = \bigcup_{N''\in\mathcal{M}''} N''$ and the result follows by Remark II.4.3. □

Proposition V.1.6. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of D-modules. Suppose that R contains the rational numbers. Then, M is Σ-finite if and only if M' and M'' are Σ-finite. Moreover, $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M')$.*

Proof. We first assume that M' and M'' are Σ-finite. Let $v \in M$. We have a short exact sequence

$$0 \to M' \cap D\cdot v \to D\cdot v \to D\cdot \overline{v} \to 0.$$

$M' \cap D\cdot v$ is finitely generated because D is Notherian by Remark II.4.1. Then, $M' \cap D\cdot v$ has finite length by Lemma V.1.4, and so $D\cdot v$ has finite length. Therefore, $M = \bigcup_{N\in\mathcal{M}} N$.

Let $N \in \mathcal{M}$. Then, $N \cap M' \in \mathcal{M}'$ and $\rho(N) \in \mathcal{M}''$. We have a short exact sequence

$$0 \to N \cap M' \to N \to \rho(N) \to 0$$

of finite length D-modules, and then result follows by Remark II.4.3.

The other direction follows from Proposition V.1.5 □

Proposition V.1.7. *Let M be a Σ-finite D-module. Then, M_f is Σ-finite for every $f \in S$.*

Proof. Let $N \subset M_f$ be a module of finite length. We have that N is a finitely generated D-module. Then there exists a finitely generated D-submodule N' of M such that $N \subset N'_f$. We have that N'_f has finite length and $\mathcal{C}(N'_f) = \bigcup_{V \in \mathcal{C}(N)} \mathcal{C}(V_f)$ because V_f is in $C(S/mS, R/m)$ [Lyu11]. Then,

$$M_f = \bigcup_{N \subset \mathcal{M}_f} N \subset \bigcup_{N \subset \mathcal{M}} N_f = M_f$$

and the result follows. □

Lemma V.1.8. *Let M and M' be Σ-finite D-modules. Then, $M \oplus M'$ is also Σ-finite.*

Proof. It is clear that $M \oplus M'$ is supported on mS. For every $(v, v') \in M \oplus M'$, there exist N and N', D-modules of finite length. such that $v \in N$ and $v' \in N'$. Then, $N \oplus N' \subset M \oplus M'$ has finite length and $(v, v') \in N \oplus N'$. Therefore,

$$\bigcup_{N \subset \mathcal{M}, N' \subset \mathcal{M}'} N \oplus N' = M \oplus M',$$

and the $M \oplus M'$ is union of its D-modules of finite length. The rest follows from Remark II.4.3. □

Corollary V.1.9. *Let M be a Σ-finite D-module. Then, $H^i_I(M)$ is Σ-finite for every ideal $I \subset S$ and $i \in \mathbb{N}$.*

Proof. Let $f_1, \ldots, f_\ell$ be generators for I. We have that $\check{\mathrm{C}}(\underline{f}; M)$ is Σ-finite by Lemma V.1.8. Then $H^i_I(M)$ is also Σ-finite by Proposition V.1.5. □

Proposition V.1.10. *Let M_t be an inductive direct system of Σ-finite D-modules. If $\bigcup_t \mathcal{C}(M_t)$ is finite, then $\lim_{\to t} M_t$ is Σ-finite and $\mathcal{C}(M) \subset \bigcup_t \mathcal{C}(M_t)$.*

Proof. Let $M = \lim_{\to t} M_t$ and $\varphi_t : M_t \to M$ the morphism induced by the limit. We have that $\phi_t(M_t)$ is a Σ-finite D-module by Proposition V.1.5. We may replace M_t by $\phi_t(M_t)$ by Remark II.4.3, and assume that $M = \bigcup M_t$ and $M_t \subset M_{t+1}$. If $N \subset M$ has finite length as D-module, then it is finitely generated and there exists a t such that $N \subset M_t$. Therefore, $M = \bigcup_t M_t = \bigcup_t \bigcup_{N \in \mathcal{M}_t} N$ and the result follows. □

V.2 Associated Primes

Notation V.2.1. Throughout this section (R, m, K) denotes a local ring and S denotes either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. In addition, D denotes $D(S, R)$.

Lemma V.2.2. *Let $J \subset S$ be an ideal and M be an R-module of finite length. Then, $H^i_J(M \otimes_R S)$ is a $D(S,R)$-module of finite length. Moreover, $\mathcal{C}(H^i_{JS}(M \otimes_R S)) \subset \bigcup_j \mathcal{C}(H^j_{JS}(S/mS))$.*

Proof. Our proof will be by induction on $h = \text{length}_R(M)$. If $h = 1$, we have that $H^i_{JS}(R/m \otimes_R S) = H^i_{JS}(S/mS)$, which has finite length as a $D(S,R)$-module [Lyu11, Theorem 2, Corollary 3] and by Remark II.4.1. Clearly,

$$\mathcal{C}(H^i_{JS}(M \otimes_K A)) = \mathcal{C}(H^i_{JS}(R/m \otimes_K A)) = \bigcup_j \mathcal{C}(H^j_{JS}(S/mS))$$

in this case. Suppose that the statement is true for h and $\text{length}_R(M) = h+1$. We have a short exact sequence of R-modules, $0 \to K \to M \to M' \to 0$, where $h = \text{length}_R(M')$. Since S is flat over R, we have that

$$0 \to K \otimes_R S \to M \otimes_R S \to M' \otimes_R S \to 0$$

is also exact. Then, we have a long exact sequence

$$\dots \to H^i_J(K \otimes_R S) \to H^i_J(M \otimes_R S) \to H^i_J(M' \otimes_R S) \to \dots$$

Then $H^i_J(M \otimes_R S)$ has finite length by the induction hypothesis and Remark II.4.3. In addition,

$$\begin{aligned}\mathcal{C}(H^i_J(M \otimes_R S)) &\subset \mathcal{C}(H^i_J(M' \otimes_R S)) \bigcup \mathcal{C}(H^i_J(K \otimes_R S)) \\ &\subset \bigcup_j \mathcal{C}(H^j_J(S/mS)).\end{aligned}$$

and the result follows by the induction hypothesis and Remark II.4.3. □

Proposition V.2.3. *Let $I \subset S$ be an ideal containing mS. Then $H^i_I(S)$ is Σ-finite for every $i \in \mathbb{N}$.*

Proof. Let $f_1, \dots, f_d$ be a system of parameters for R and $g_1, \dots, g_\ell$ be a set of generators for I. Let $\underline{f}^t$ denote the sequence $f^t_1, \dots, f^t_\ell$. Let $T_i = \{T^{p,q}_t\}$ be the double complex of $D(S,R)$-modules given by the tensor product $\mathcal{K}(\underline{f}; R) \otimes_R \check{\mathrm{C}}(\underline{g}; S)$.

The direct limit $\mathcal{K}(\underline{f}^t; R)$ introduced in Figure II.3, induces a direct limit of double complexes $\mathrm{Tot}(T_t) \to \mathrm{Tot}(T_{t+1})$. Since $\lim\limits_{\to t} \mathcal{K}(\underline{f}^t; R) = \check{\mathrm{C}}(\underline{f}; R)$, we have that $\lim\limits_{\to t} \mathrm{Tot}(T_t) = \check{\mathrm{C}}(\underline{f, g}; S)$. Let $E^{p,q}_{r,t}$ be the spectral sequence associated to T_t. We have that

$$E^{p,q}_{2,t} = H^p_I(H^q(\mathcal{K}(\underline{f}^t; S)) \Rightarrow E^{p,q}_{\infty,t} = H^{p+q}\mathrm{Tot}(T_t).$$

We notice that $H^q(\mathcal{K}(\underline{f}^t; S)) = H^q(\mathcal{K}(\underline{f}^t; R)) \otimes_R S$, because S is R-flat. Since $H^q(\mathcal{K}(\underline{f}^t; R))$ has finite length as an R-module, we have that $E^{p,q}_{2,t}$ is a $D(S,R)$-module of finite length for all $p, q \in \mathbb{N}$ and that $\mathcal{C}(E^{p,q}_{2,t}) = \bigcup_j \mathcal{C}(H^i_{JS}(S/mS))$ by Lemma V.2.2. Moreover, $E^{p,q}_{r,t}$ is a $D(S,R)$-module of finite length, and

$$\mathcal{C}(E^{p,q}_{r,t}) \subset \bigcup_{p,q} \mathcal{C}(E^{p,q}_{2,t}) = \bigcup_j \mathcal{C}(H^j_I(S/mS))$$

for $r > 2$. Then, $\mathcal{C}(H^i(\mathrm{Tot}(T_t))) \subset \bigcup_j \mathcal{C}(H^i_I(S/mS))$ for every $j, t \in \mathbb{N}$ by Remark II.4.3; in particular, $\bigcup_t \mathcal{C}(H^i\mathrm{Tot}(T_t))$ is finite and every element there belongs to $C(S/mS, R/mR)$. Therefore, $E^{p,q}_{r,t}$ is a Σ-finite $D(S,R)$-module. Moreover,

$$H^i_I(S) = H^i(\check{\mathrm{C}}(\underline{f, g}; S)) = H^i(\lim_{\to t} \mathrm{Tot}(T_t)) = \lim_{\to t} H^i(\mathrm{Tot}(T_t))$$

because the direct limit is exact. Hence, $H^i_I(S)$ is Σ-finite by Proposition V.1.10. □

Corollary V.2.4. *Let $I \subset S$ be an ideal containing mS and $J_1, \ldots, J_\ell \subset S$ be any ideals. Then $H^{j_1}_{J_1} \cdots H^{j_\ell}_{J_\ell} H^i_I(S)$ is Σ-finite.*

Proof. This is a consequence of Proposition V.2.3 and Corollary V.1.9. □

Theorem V.2.5. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Then, $\mathrm{Ass}_S\, H^0_{mS} H^i_I(S)$ is finite for every ideal $I \subset S$ such that $\dim R/I \cap R \leq 1$ and every $i \in \mathbb{N}$. Moreover, if $mS \subset \sqrt{I}$,*

$$\mathrm{Ass}_S\, H^{j_1}_{J_1} \cdots H^{j_\ell}_{J_\ell} H^i_I(S)$$

is finite for all ideals $J_1, \ldots, J_\ell \subset S$ and integers $j_1, \ldots, j_\ell \in \mathbb{N}$.

Proof. This is a consequence of Remark V.1.3 and Corollary V.2.4. □

Proposition V.2.6. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Let $I \subset S$ be an ideal, such that $\dim R/I \cap R \leq 1$. Then,*

$$\mathrm{Ass}_S\, H^0_m H^i_I(S)$$

is finite for every $i \in \mathbb{N}$.

Proof. Since $\dim R/(I \cap R) \leq 1$, there exists $f \in R$ such that $mS \subset \sqrt{I + fS}$. We have the exact sequence

$$\ldots \to H^i_{(I,f)S}(S) \xrightarrow{\alpha_i} H^i_I(S) \xrightarrow{\beta_i} H^i_I(S_f) \to \ldots.$$

Then,

$$\operatorname{Ass}_S H^i_I(S) \cap \mathcal{V}(mS) \subset (\operatorname{Ass}_S \operatorname{Im}(\alpha_i) \cap \mathcal{V}(mS)) \bigcup (\operatorname{Ass}_S \operatorname{Im}(\beta_i) \cap \mathcal{V}(mS))$$

Since $H^i_{(I,f)S}(S)$ is a Σ-finite $D(S,R)$-module by Proposition V.2.3, we have that $\operatorname{Im}(\alpha_1)$ is also Σ-finite by Proposition V.1.5, and so $\operatorname{Ass}_S \operatorname{Im}(\alpha_i)$ is finite. Since $\operatorname{Im}(\beta_i) \subset H^i_I(S_f)$, $\operatorname{Ass}_S \operatorname{Im}(\beta_i) \cap \mathcal{V}(mS) = \emptyset$. Therefore,

$$\operatorname{Ass}_S H^i_I(S) \cap \mathcal{V}(mS) = \operatorname{Ass}_S H^0_{mS} H^i_I(S)$$

is finite. □

Proposition V.2.7. *Suppose that* R *is a ring of characteristic* 0 *and that* $\dim R/(I \cap R) \leq 1$. *Then* $H^j_{mS} H^i_I(S)$ *is* Σ*-finite for every* $i, j \in \mathbb{N}$.

Proof. Since $\dim R/(I \cap R) \leq 1$, there exists $g \in R$, such that $mS \subset \sqrt{(I,g)S}$. We have the long exact sequence

$$\ldots \to H^i_{(I,g)S}(S) \to H^i_I(S) \to H^i_I(S_g) \to \ldots.$$

Let $M_i = \operatorname{Ker}(H^i_{(I,g)S}(S) \to H^i_I(S))$, $N_i = \operatorname{Im}(H^i_{(I,g)S}(S) \to H^i_I(S))$ and $W_i = \operatorname{Im}(H^i_I(S) \to H^i_I(S_g))$. We have the following short exact sequences:

$$0 \to M_i \to H^i_{(I,g)S}(S) \to N_i \to 0,$$

$$0 \to N_i \to H^i_I(S) \to W_i \to 0$$

and

$$0 \to W_i \to H^i_I(S_g) \to M_{i+1} \to 0.$$

Since $mS \subset \sqrt{(I,g)S}$, $H^i_{(I,g)S}(S)$ is Σ-finite by Proposition V.2.3. Then, M_i and N_i is Σ-finite for every $i \in \mathbb{N}$ by Proposition V.1.5. By the long exact sequences

$$\ldots \to H^j_{mS}(M_i) \to H^j_{mS} H^i_{(I,g)S}(S) \to H^j_{mS}(N_i) \to \ldots,$$

$$\ldots \to H^j_{mS}(N_i) \to H^j_{mS}H^i_I(S) \to H^j_{mS}(W_i)) \to \ldots$$

and

$$\ldots \to H^j_{mS}(W_i) \to H^j_{mS}H^i_I(S_g) \to H^j_{mS}(M_{i+1}) \to \ldots,$$

$H^j_{mS}(M_i)$, $H^j_{mS}H^i_{(I,g)S}(S)$ and $H^j_{mS}(M_i)$ are Σ-finite for every $i, j \in \mathbb{N}$. $H^j_{mS}(W_i) = H^j_{mS}(M_{i+1})$ because $H^j_{mS}H^i_I(S_g) = 0$. Then, $H^j_{mS}(W_i)$ is Σ-finite, and so $H^j_{mS}H^i_I(S)$ is Σ-finite by V.1.6. ☐

V.3 More examples of Σ-finite D-modules

In the previous section we gave a positive answer for specific cases for Question V.0.5. Our method consisted in proving that $H^j_{mS}H^i_I(S)$ is Σ-finite and then applying Remark V.1.3. This motivates the following question:

Question V.3.1. Is $H^j_{mS}H^i_I(S)$ Σ-finite for every ideal $I \subset S$ and $i, j \in \mathbb{N}$?

In this section, we provide positive examples for Question V.3.1.

Proposition V.3.2. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Let $I \subset S$ be an ideal such that $\operatorname{depth}_S I = \operatorname{cd}_S I$. Then,*

$$H^i_{mS}H^{\operatorname{depth}_S I}_I(S)$$

is Σ-finite for every $i \in \mathbb{N}$.

Proof. We have that the spectral sequence

$$E^{p,q}_2 = H^p_{mS}H^q_I(S) \Longrightarrow E^{p,q}_\infty = H^{p+q}_{(I,m)S}(S)$$

converges at the second spot, because $\operatorname{depth}_S I = \operatorname{cd}_S I$. Hence,

$$H^p_{mS}H^q_I(S) = H^{p+q}_{(I,m)S}(S)$$

and the result follows by Proposition V.2.3. ☐

Proposition V.3.3. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Let $I \subset S$ be an ideal such that $\operatorname{Ext}^i_S(S/mS, H^j_I(S))$ is a D-module in $C(R, S)$ for every $i \in \mathbb{N}$. Then, $H^i_{mS}H^j_I(S)$ is a Σ-finite $D(S, R)$-module for every $i, j \in \mathbb{N}$.*

Proof. We claim that $\operatorname{Ext}^i_S(N\otimes_R S, H^j_I(S))$ is a $D(S/mS,K)$-module in $C(S/mS,K)$ for every $i\in\mathbb{N}$ and every finite length R-module N. Moreover,

$$\mathcal{C}(\operatorname{Ext}^i_S(N\otimes_R S, H^j_I(S)))\subset\bigcup_i\mathcal{C}(\operatorname{Ext}^i_S(K\otimes_R S, H^j_I(S))).$$

The proof of our claim is analogous to Lemma V.2.2.

The direct system $\operatorname{Ext}^i(S/m^\ell S, H^j_I(S))\to\operatorname{Ext}^i(S/m^{\ell+1}S, H^j_I(S))$ satisfies the hypotheses of Proposition V.1.10. Hence,

$$H^i_{mS}H^j_I(S)=\lim_{\to\ell}\operatorname{Ext}^i(S/m^\ell S, H^j_I(S))$$

is a Σ-finite $D(S,R)$-module. ☐

Remark V.3.4. The condition that $\operatorname{Ext}^i_S(S/mS, H^j_I(S))$ be a $D(S/mS,K)$-module in $C(S/mS,K)$ for every $i\in\mathbb{N}$ is not necessary.

Let $R=K[[s,t,u,w]]/(us+vt)$, where K is a field. This is the ring given by Hartshorne's example [Har68]. He showed that $\dim_K\operatorname{Hom}_A(K,H^2_I(A))$ is not finite for $I=(s,t)A$. Let S be either $R[x_1,\dots,x_n]$ or $R[[x_1,\dots,x_n]]$. Therefore,

$$\operatorname{Ext}^0_S(S/mS,H^2_I(S))=\operatorname{Hom}_S(S/mS,H^2_I(S))$$

$$=\operatorname{Hom}_R(K,H^2_I(R))\otimes_R S=\oplus S/mS,$$

where the direct sum is infinite. Then, $\operatorname{Ext}^0_S(S/mS,H^2_I(S))$ does not belong to $C(S,R)$.

On the other hand, $H^0_mH^2_I(S)$ is a direct limit of finite direct sums of S/mS. This direct limit satisfies the hypotheses of Proposition V.1.10. Therefore, $H^0_mH^2_I(S)$ is a Σ-finite $D(S,R)$-module.

Proposition V.3.5. *Let (R,m,K) be any local ring and let S denote $R[x_1,\dots,x_n]$. Let $I\subset S$ be an ideal. Then, $H^i_{mS}H^0_I(S)$ is Σ-finite for every $i\in\mathbb{N}$. In addition, if $\operatorname{cd}_S I\le 1$, then $H^i_{mS}H^j_I(S)$ is Σ-finite for every $i,j\in\mathbb{N}$.*

Proof. We claim that there exists an ideal $J\subset R$ such that $H^0_I(S)=JS$. We have that $H^0_I(S)$ is a $D(S,R)$-module. For every $f=\sum_\alpha c_\alpha x^\alpha\in H^0_I(S)$ and $\partial\in D(S,R)$, $\partial f\in H^0_I(S)$. Therefore, $c_\alpha\in H^0_I(S)$. and $H^0_I(S)=JS$, where

$$J=\{c_\alpha\mid\sum_\alpha c_\alpha x^\alpha\in H^0_I(S)\}.$$

We have that

$$\begin{aligned}\operatorname{Ext}^i_S(S/mS, H^0_I(S)) &= \operatorname{Ext}^i_S(R/mR \otimes_R S, J \otimes_R S)\\ &= \operatorname{Ext}^i_R(K, J) \otimes_S S\\ &= \oplus^{\mu} S/mS, \text{ where } \mu = \dim_K \operatorname{Ext}^i_R(K, J),\end{aligned}$$

and it is a $D(S, R)$-module in $C(S, R)$ for every $i \in \mathbb{N}$. The first claim follows from Proposition V.3.3.

We have that $H^1_I(S) = H^1_{I(S/J)}(S/J)$ [BS98, Corollary 2.1.7]. In addition, $S/JS = (R/J)[x_1, \ldots, x_n]$ and

$$\operatorname{depth}_{I(S/JS)} = \operatorname{cd}_{I(S/JS)}(S/JS) = 1.$$

The second claim follows from Proposition V.3.2. □

V.4 Reduction to power series rings

Discussion V.4.1. Suppose that (R, m, K) and (S, η, L) are complete local rings and that $\varphi : R \to S$ is a flat extension of local rings with regular closed fiber. Assume that φ maps a coefficient field of R to a coefficient field of S. We pick such coefficient fields, and then $\varphi(K) \subset L$. Thus, $R = K[[x_1, \ldots, x_n]]/I$ for some ideal $I \subset K[[x_1, \ldots, x_n]]$. Let $A = L\widehat{\otimes}_K R = L[[x_1, \ldots, x_n]]/IL[[x_1, \ldots, x_n]]$. We note that A is a flat local extension of R, such that mA is the maximal ideal of A. Let $\theta : A \to S$ be the morphism induced by φ and our choice of coefficient fields.

We claim that S is a flat A-algebra. Let F_* be a free resolution of R/mR. Then, $A \otimes_R F_*$ is a free resolution for A/mA. We have that

$$\operatorname{Tor}^A_1(S, A/mA) = H_1(S \otimes_A A \otimes_R F_*)$$

$$= H_1(S \otimes_R F_*) = \operatorname{Tor}^R_1(S, R/mR) = 0$$

because S is a flat extension. Since mA is the maximal ideal of A, we have that S is a flat A-algebra by the local criterion of flatness [Eis95, Theorem 6.8].

Let $d = \dim(S/mS)$ and $z_1, \ldots, z_d \in S$ be preimages of a regular system of parameters for S/mS. Let $\phi : A[[y_1, \ldots, y_d]] \to S$ be the morphism given by sending A to S via θ and y_i to z_i. Since

$$(mA + (z_1, \ldots, z_d)A)S = \eta$$

and the morphism induced by ϕ in the quotient fields of A and S is an isomorphism. Hence, φ is an isomorphism.

Proposition V.4.2. *Questions V.0.4 and V.0.5 are equivalent when we restrict them to a local extensions, such that the induced morphism in the completions maps a coefficient field of the domain to a coefficient field of the target.*

Proof. Let $\varphi : (R, m, K) \to (S, \eta, L)$ be a flat extension of local rings with regular closed fiber. Suppose that $\widehat{\varphi} : \widehat{R} \to \widehat{S}$, the induced morphism in the completions, maps a coefficient field of the $\widehat{R}$ to a coefficient field of $\widehat{S}$. We have that $\operatorname{Ass}_R H^0_{mR} H^i_I(S)$ is finite if and only if $\operatorname{Ass}_R H^0_{m\widehat{R}} H^i_I(\widehat{S})$ is finite. Let A be as in the previous discussion and $d = \dim(S/mS)$. The result follows, because $\widehat{S} = A[[y_1, \ldots, y_d]]$ and $mS = (mA)S$. □

Theorem V.4.3. *Let $(R, m, K) \to (S, \eta, L)$ be a flat extension of local rings with regular closed fiber such that R contains a field. Let $I \subset S$ be an ideal such that* $\dim R/I \cap R \leq 1$. *Suppose that the morphism induced in the completions $\widehat{R} \to \widehat{S}$ maps a coefficient field of R into a coefficient field of S. Then,*

$$\operatorname{Ass}_S H^0_m H^i_I(S)$$

is finite for every $i \in \mathbb{N}$.

Proof. By Discussion V.4.1, we may assume that R is complete and S is a power series ring over R. The rest is a consequence of Proposition V.2.6. □

Remark V.4.4. In the previous proposition, the hypothesis that $\widehat{\varphi}$ maps a coefficient field of $\widehat{R}$ to a coefficient field of $\widehat{S}$ is satisfied when L is a separable extension of K [Mat89, Theorem 28.3].

In the previous theorem, the hypothesis that $\widehat{\varphi}$ maps a coefficient field of $\widehat{R}$ to a coefficient field of $\widehat{S}$ is not very restrictive. For instance, it is satisfied when L is a separable extension of K [Mat89, Theorem 28.3]. In particular, this holds when K is a field of characteristic 0 or a perfect field of characteristic $p > 0$.

CHAPTER VI

Direct summands

Our aim in this chapter is to prove the finiteness of associated primes and Bass numbers of local cohomology for direct summands. We need to make some observations. Let $R \to S$ be a homomorphism of Noetherian rings. For an ideal $I \subset R$, we have two functors associated with it, $H^i_I(-) : R\text{-mod} \to R\text{-mod}$ and $H^i_{IS}(-) : S\text{-mod} \to S\text{-mod}$, which are naturally isomorphic when we restrict them to S-modules. Moreover, for two ideals of R, $I_2 \subset I_1$, the natural morphism $H^i_{I_1}(-) \to H^i_{I_2}(-)$ is the same as the natural morphism $H^i_{I_1 S}(-) \to H^i_{I_2 S}(-)$ when we restrict the functors to S-modules. Thus, their kernel, cokernel and image are naturally isomorphic as S-modules. Hence, every Lyubeznik functor $\mathcal{T}$ for R is a functor of the same type for S when we restrict it to S-modules.

As per the previous discussion, for an S-module, M, we will make no distinction in the notation or meaning of $\mathcal{T}(M)$ whether it is induced by ideals of R or their extensions to S and, therefore, by the corresponding closed subsets of their respective spectra.

We wanto to point examples of direct sumands of regular rings. If S is a polynomial ring over a field and R is the invariant ring of an action of a linearly reductive group over S. It also holds when $R \subset K[x_1, \ldots, x_n]$ is an integrally closed ring that is finitely generated as a K-algebra by monomials. This is because such a ring is a direct summand of a possibly different polynomial ring (cf. Proposition 1 and Lemma 1 in [Hoc72]).

We would like to mention another case in which an inclusion splits. This is when $R \to S$ is a module finite extension of rings containing a field of characteristic zero such that S has finite projective dimension as an R-module. Moreover, such a splitting exists when Koh's conjecture holds (cf. [Koh83, Vél95, VF00]). Therefore, if Koh's conjecture applies to $R \to S$ and $\mathcal{T}(S)$ has finite associated primes or finite Bass numbers, so does $\mathcal{T}(R)$.

We point out that the property $\operatorname{inj.dim} H^i_I(R) \le \dim_S \operatorname{Supp} H^i_I(R)$. does not hold for direct summands of regular rings, even in the finite extension case. A counterexample is $R = K[x^3, x^2y, xy^2, y^3] \subset S = K[x,y]$, where S is the polynomial ring in two variables with coefficients in a field K. The splitting of the inclusion is the map $\theta : S \to R$ defined in the monomials by $\theta(x^\alpha y^\beta) = x^\alpha y^\beta$ if $\alpha + \beta \in 3\mathbb{Z}$ and as zero otherwise. We have that the dimension of $\operatorname{Supp}(H^2_{(x^3,x^2y,xy^2,y^3)}(R))$ is zero, but it is not an injective module, because R is not a Gorenstein ring, since $R/(x^3,y^3)R$ has a two dimensional socle.

The results presented in this section appear in [NB12c].

VI.1 Associated Primes

Lemma VI.1.1. *Let $R \to S$ be an injective homomorphism of Noetherian rings, and let M be an S-module. Then, $\operatorname{Ass}_R M \subset \{Q \cap R : Q \in \operatorname{Ass}_S M\}$.*

Proof. Let $P \in \operatorname{Ass}_R M$ and $u \in M$ be such that $\operatorname{Ann}_R u = P$. We have that $(\operatorname{Ann}_S u) \cap R = P$. Let $Q_1, \ldots, Q_t$ denote the minimal primes of $\operatorname{Ann}_S u$. We obtain that

$$P = \sqrt{P} = \sqrt{\operatorname{Ann}_S u} \cap R = (\cap_j Q_j) \cap R = \cap_j (Q_j \cap R),$$

so, there exists a Q_j such that $P = Q_j \cap R$. Since Q_j is a minimal prime for $\operatorname{Ann}_S u$, we have that $Q_j \in \operatorname{Ass}_S M$ and the result follows. □

Definition VI.1.2. We say that a homomorphism of Noetherian rings $R \to S$ is pure if $M = M \otimes_R R \to M \otimes_R S$ is injective for every R-module M. We also say that R is a pure subring of S.

Proposition VI.1.3 (Cor. 6.6 in [HR74])**.** *Suppose that $R \to S$ is a pure homeomorphism of Noetherian rings and that $\mathcal{G}$ is a complex of R-modules. Then, the induced map $j : H^i(\mathcal{G}) \to H^i(\mathcal{G} \otimes_R S)$ is injective.*

Proposition VI.1.4. *Let $R \to S$ be a pure homomorphism of Noetherian rings. Suppose that $\operatorname{Ass}_S H^i_{IS}(S)$ is finite for some ideal $I \subset R$ and $i \ge 0$. Then, $\operatorname{Ass}_S H^i_I(R)$ is finite.*

Proof. Since $H^i_I(R) \to H^i_{IS}(S)$ is injective by Proposition VI.1.3, $\operatorname{Ass}_R H^i_I(R) \subset \operatorname{Ass}_R H^i_{IS}(S)$ and the result follows by Lemma VI.1.1. □

Theorem VI.1.5. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Suppose that $\operatorname{Ass}_S \mathcal{T}(S)$ is finite for a functor $\mathcal{T}$ induced by extensions of ideals of R.*

Then, $\operatorname{Ass}_R \mathcal{T}(R)$ *is finite. In particular,* $\operatorname{Ass}_R H^i_I(R)$ *is finite for every ideal* $I \subset R$*, if* $\operatorname{Ass}_S H^i_{IS}(S)$ *is finite.*

Proof. The splitting between R and S makes $\mathcal{T}(R)$ into a direct summand of $\mathcal{T}(S)$; in particular, $\mathcal{T}(R) \subset \mathcal{T}(S)$. Therefore, $\operatorname{Ass}_R \mathcal{T}(R) \subset \operatorname{Ass}_R \mathcal{T}(S)$ and the result follows by Lemma VI.1.1. □

If R is a ring containing a field of characteristic $p > 0$, Theorem VI.1.5 gives a method for showing that R is not a direct summand of a regular ring. We used this method to prove that there exists a Gorenstein strongly F-regular UFD of characteristic $p > 0$ that is not a direct summand of any regular ring.

Theorem VI.1.6 (Thm. 5.4 in [SS04])**.** *Let* K *be a field, and consider the hypersurface*

$$R = \frac{K[r, s, t, u, v, w, x, y, z]}{(su^2x^2 + sv^2y^2 + tuxvy + rw^2z^2)}.$$

Then, R *is a unique factorization domain for which the local cohomology module* $H^3_{(x,y,z)}(R)$ *has infinitely many associated prime ideals. This is preserved if* R *is replaced by the localization at its homogeneous maximal ideal. The hypersurface* R *has rational singularities if* K *has characteristic zero, and it is* F*-regular if* K *has positive characteristic.*

Corollary VI.1.7. *Let* R *be as in the previous theorem taking* K *of positive characteristic. Then,* R *is a Gorenstein* F*-regular UFD that is not a pure subring of any regular ring. In particular,* R *is not direct summand of any regular ring.*

Proof. Since $H^3_{(x,y,z)}(R)$ has infinitely many associated prime ideals, it cannot be a direct summand or pure subring of a regular ring by Theorem VI.1.5, Proposition VI.1.3 and finiteness properties of regular rings of positive characteristic (cf. [Lyu97]). □

Theorem VI.1.8 (Thm. 1 in [Zha11b])**.** *Assume that* $S = K[x_1, \ldots, x_n]$ *is a polynomial ring in* n *variables over a field* K *of characteristic* $p > 0$*. Suppose that* $I = (f_1, \ldots, f_s)$ *is an ideal of* S *such that* $\sum_i \deg f_i < n$*. Then* $\dim S/Q \geq n - \sum_i \deg f_i$ *for all* $Q \in \operatorname{Ass}_S H^i_I(S)$*.*

Corollary VI.1.9. *Let* $S = K[x_1, \ldots, x_n]$ *be a polynomial ring in* n *variables over a field* K *of characteristic* $p > 0$*. Let* $R \to S$ *be a homomorphism of Noetherian rings that splits. Suppose that* $I = (f_1, \ldots, f_s)$ *is an ideal of* R *such that* $\sum_i \deg(f_i) < \dim R$*. If* S *is a finitely generated* R*-module, then* $\dim R/P \geq \dim R - \sum_i \deg f_i$ *for all* $P \in \operatorname{Ass}_R H^i_I(R)$*.*

Proof. Since $H^i_I(-)$ commutes with direct sum of R-modules, we have that a splitting of $R \hookrightarrow S$ over R induces an splitting of $H^i_I(R) \hookrightarrow H^i_I(S)$ over R. Then, by Lemma VI.1.1, for any $P \in \operatorname{Ass}_R H^i_I(R) \subset \operatorname{Ass}_R H^i_I(S)$ there exists $Q \in \operatorname{Ass}_R H^i_I(S)$ such that $P = Q \cap R$ and then $\dim R/P = \dim S/Q > n - \sum_i \deg f_i$, and the result follows. $\square$

VI.2 Bass Numbers

Lemma VI.2.1. *Let (R, m, K) be a local ring and M be an R-module. Then, the following are equivalent:*

a) $\dim_K(\operatorname{Ext}^j_R(K, M))$ *is finite for all $j \geq 0$;*

b) $\operatorname{length}(\operatorname{Ext}^j_R(N, M))$ *is finite for every finite length module N for all $j \geq 0$;*

c) there exists one module N of finite length such that $\operatorname{length}(\operatorname{Ext}^j_R(N, M))$ *is finite for all $j \geq 0$.*

Proof. *a) ⇒ b):* Our proof will be by induction on $h = \operatorname{length}(N)$. If $h = 1$, then $N = K$, and the proof follows from our assumption. We will assume that the statement is true for h and prove it when $\operatorname{length}(N) = h + 1$. In this case, there is a short exact sequence $0 \to K \to N \to N' \to 0$, where N' has length h. From the induced long exact sequence

$$\ldots \to \operatorname{Ext}^{j-1}_R(N', M) \to \operatorname{Ext}^j_R(K, M) \to \operatorname{Ext}^j_R(N, M) \to \ldots,$$

we see that $\operatorname{length}(\operatorname{Ext}^i_R(N, M))$ is finite for all $i \geq 0$.

b) ⇒ c): Clear.

c) ⇒ a): We will prove the contrapositive. Let j be the minimum non-negative integer such that $\dim_K(\operatorname{Ext}^j_R(K, M))$ is infinite. We claim that $\operatorname{length}(\operatorname{Ext}^i_R(N, M)) < \infty$ for $i < j$ and $\operatorname{length}(\operatorname{Ext}^j_R(N, M)) = \infty$ for any module N of finite length. Our proof will be by induction on $h = \operatorname{length}(N)$. If $h = 1$, then $N = K$ and it follows from our choice of j. We will assume that this is true for h and prove it when $\operatorname{length}(N) = h + 1$. We have a short exact sequence $0 \to K \to N \to N' \to 0$, where N' has length h. From the induced long exact sequence

$$\ldots \to \operatorname{Ext}^{j-1}_R(N', M) \to \operatorname{Ext}^j_R(K, M) \to \operatorname{Ext}^j_R(N, M) \to \ldots,$$

we have that $\operatorname{length}(\operatorname{Ext}^i_R(N, M)) < \infty$ for $i < j$ and that the map

$$\operatorname{Ext}^j_R(K, M)/\operatorname{Im}(\operatorname{Ext}^{j-1}_R(N', M)) \to \operatorname{Ext}^j_R(N, M)$$

is injective. Therefore, $\operatorname{length}(\operatorname{Ext}^j_R(N, M)) = \infty$. □

Lemma VI.2.2. *Let $R \to S$ be a pure homomorphism of Noetherian rings. Assume that S is a Cohen-Macaulay ring. If S is finitely generated as an R-module, then R is a Cohen-Macaulay ring.*

Proof. Let $P \subset R$ be a prime ideal. Let $\underline{x} = x_1, \ldots, x_d$ denote a system of parameters of R_P, where $d = \dim(R_P)$. It suffices to show that $H_i(\mathcal{K}(\underline{x}; R_P)) = 0$ for $i \neq 0$, where $\mathcal{K}$ is the Koszul complex with respect to $\underline{x}$. We notice that the natural inclusion $R_P \to S_P$ is a pure homeomorphism of rings. This induces an injective morphism of R-modules $H_i(\mathcal{K}(\underline{x}; R_P)) \to H_i(\mathcal{K}(\underline{x}; S_P))$ by Proposition VI.1.3. Thus, it is enough to show that $H_i(\mathcal{K}(\underline{x}; S_P)) = 0$ for $i \neq 0$. Since S_P is a module finite extension of R_P, we have that every maximal ideal $Q \subset S_P$ contracts to PR_P and $\underline{x}$ is a system of parameters for S_Q. Then, $H_i(\mathcal{K}(\underline{x}; S_Q)) = 0$ for $i \neq 0$ and every maximal ideal $Q \subset S_P$. Hence, $H_i(\mathcal{K}(\underline{x}; S_P)) = 0$ for $i \neq 0$ and the result follows.

□

Proposition VI.2.3. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Assume that S is a Cohen-Macaulay ring and S is finitely generated as an R-module. Let N be an R-module and M be an S-module. Let $N \to M$ be a morphism of R-modules that splits. If all the Bass numbers of M, as an S-module, are finite, then all the Bass numbers of N, as an R-module, are finite.*

Proof. Since $N \hookrightarrow M$ splits, we have that $\operatorname{Ext}^i_{R_P}(R_P/PR_P, N_P)$ is a direct summand of $\operatorname{Ext}^i_{R_P}(R_P/PR_P, M_P)$, so, we may assume that $N = M$.

Let P be a fixed prime ideal of R and let K_P denote R_P/PR_P. Since we want to show that $\dim_{K_P}(\operatorname{Ext}^i_{R_P}(K_P, M_P))$ is finite, we may assume without loss of generality that R is local and P is its maximal ideal. Let $\underline{x} = x_1, \ldots, x_n$ be a system of parameters for R. Since R is Cohen-Macaulay by Lemma VI.2.2, we have that the Koszul complex, $\mathcal{K}_R(\underline{x})$, is a free resolution for R/I, where $I = (x_1, \ldots, x_n)$. We also have that for every maximal ideal $Q \subset S$ lying over P, $\underline{x}$ is a system of parameters of S_Q because $\dim R = \dim S_Q$ and S_Q/IS_Q is a zero dimensional ring. From the Cohen-Macaulayness of S and the previous fact, we have that the Koszul complex $\mathcal{K}_S(\underline{x})$ is a free resolution for S/IS. Therefore, $\operatorname{Ext}^i_R(R/I, M) = H^i(\operatorname{Hom}_R(\mathcal{K}_R(\underline{x}), M)) =$

$H^i(\mathrm{Hom}_S(\mathcal{K}_S(\underline{x}), M)) = \mathrm{Ext}^i_S(S/IS, M)$. Since

$$\mathrm{Ext}^i_S(S/IS, M) = \oplus_Q \mathrm{Ext}^i_{S_Q}(S_Q/IS_Q, M_Q)$$

has finite length as an S-module by Lemma VI.2.1, we have that $\mathrm{Ext}^i_R(R/I, M)$ has finite length as an R-module because S is finitely generated. Then, we have that $\dim_{K_P}(\mathrm{Ext}^i_R(K_P, M))$ is finite by Lemma VI.2.1. □

Theorem VI.2.4. *Let $R \to S$ be a homomorphism of Noetherian rings that splits. Suppose that S is a Cohen-Macaulay ring such that all the Bass numbers of $\mathcal{T}(S)$, as an S-module, are finite for a functor $\mathcal{T}$ induced by extension of ideals of R. If S is a finitely generated R-module, then all the Bass numbers of $\mathcal{T}(R)$, as an R-module, are finite. In particular, for every ideal $I \subset R$ the Bass numbers of $H^i_I(R)$ are finite, if the Bass numbers of $H^i_{IS}(S)$ are finite.*

Proof. The splitting between R and S induces a splitting between $\mathcal{T}(R) \hookrightarrow \mathcal{T}(S)$. The rest follows from Proposition VI.2.3. □

CHAPTER VII

F-Jacobian ideals for hypersurfaces

Suppose that $f \in K[x_1, \ldots, x_n]$ is a polynomial over a perfect field K. We know that the Jacobian ideal $\operatorname{Jac}(f) = (f, \frac{\partial_1 f}{\partial x_1}, \ldots, \frac{\partial_n f}{\partial x_n})$ determines whether the hypersurface $\mathcal{V}(f) = \{v \in K^n \mid f(v) = 0\}$ is smooth or not. Moreover, $\operatorname{Jac}(f)$ defines the singular locus of $\mathcal{V}(f)$.

The aim of this chapter is to introduce the F-Jacobian ideal, $J_F(f)$, of an element in a regular ring R. This is an ideal that measures singularity in positive characteristic. Under suitable hypothesis, it defines the locus in which R/fR is not F-regular. $J_F(f)$ is connected with the sum of all simple F-submodules of the first local cohomology of R supported at f, $H^1_f(R)$. In this chapter we define the F-Jacobian ideal and deduce some of its properties.

The results presented in this chapter are part of joint work with Pérez [NBP13].

VII.1 Definition for unique factorization domains

Notation VII.1.1. Throughout this section R denotes an F-finite regular UFD of characteristic $p > 0$ such that R_f/R has finite length as $D(R, \mathbb{Z})$-module for every $f \in R$.

This hypothesis is satisfied for every F-finite regular local ring and for every F-finite polynomial ring [Lyu97, Theorem 5.6].

Lemma VII.1.2. *Let S be a UFD and $f \in S$ be an irreducible element. Then, $N \cap M \neq 0$ for any S-submodules $M, N \subset S_f/S$.*

Proof. Let $a/f^\beta \in M \setminus \{0\}$ and $b/f^\gamma \in N \setminus \{0\}$, where $\beta, \gamma \geq 1$. Since S is a UFD and f is irreducible, we may assume that $\gcd(a, f) = \gcd(b, f) = 1$. Then, $\gcd(ab, f) = 1$, and so $ab/f \neq 0$ in S_f/S. We have that $ab/f = bf^{\beta-1}(a/f^\beta) = af^{\gamma-1}(b/f^\gamma)$. Then, $ab/f \in N \cap M$ and it is not zero. □

Lemma VII.1.3. *Let S be a regular ring of characteristic $p > 0$, $f \in S$ an element and $\pi : S \to S/fS$ be the quotient morphism. Let*

$$\mathcal{I} : \{I \subset S \mid I \text{ is an ideal}, f \in I, (I^{[p]} : f^{p-1}) = I\}$$

and

$$\mathcal{N} = \{N \subset S_f/S | N \text{ is an } F\text{-submodule}\}.$$

Then, the correspondence given by sending N to $I_N = \pi^{-1}(N \cap R/fR)$ is bijective, with tinverse defined by sending the ideal $I \in \mathcal{I}$ to the F-module N_I generated by $I/fS \stackrel{f^{p-1}}{\to} F(I/fS) = I^{[p]}/f^pS$.

Proof. Since $\phi : R/fR \stackrel{f^{p-1}}{\to} R/f^pR$ is a root for R_f/R, its F-submodules are in correspondence with ideals $J \subset R/fR$ such that $\phi^{-1}(F(J)) = J$ [Lyu97, Corollary 2.6]. We have the following generating morphisms,

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
I/fR & \stackrel{f^{p-1}}{\to} & F(IR/fR) & \stackrel{f^{p^2-p}}{\to} & F^2(IR/fR) & \stackrel{f^{p^3-p^2}}{\to} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
R/fR & \stackrel{f^{p-1}}{\to} & R/f^pR & \stackrel{f^{p^2-p}}{\to} & R/f^{p^2}R & \stackrel{f^{p^3-p^2}}{\to} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
R/I & \stackrel{f^{p-1}}{\to} & R/I^{[p]} & \stackrel{f^{p^2-p}}{\to} & R/I^{[p^2]} & \stackrel{f^{p^3-p^2}}{\to} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Since J is a quotient I/fR of an ideal, $F(J) = I^{[p]}/f^pR$. Then,

$$\begin{aligned}
I/fR &= \phi^{-1}(I^{[p]}/f^p) \\
&= \{h \in R/fR \mid f^{p-1}h \in I^{[p]}/f^p\} \\
&= \{h \in R \mid f^{p-1}h \in I^{[p]}\}/fR \\
&= (I^{[p]} : f^{p-1})/f
\end{aligned}$$

and the result follows. □

Lemma VII.1.4. *Let $f \in R$ be an irreducible element. Then, there is a unique simple F-module in R_f/R.*

Proof. Since R_f/R is an F-module of finite length, there exists a simple F-submodule $M \subset R_f/R$. Let N be an F-submodule of R_f/R. Since $M \cap N \neq 0$ by Lemma VII.1.2 and M is a simple F-module, $M = M \cap N$. Hence, M is the only nonzero simple F-submodule of R_f/R. □

Proposition VII.1.5. *Let $g \in R$ be an irreducible element and $f = g^n$ for some integer $n \geq 1$. Then, there exists a unique ideal $I \subset R$ such that:*

(i) $f \in I$,

(ii) $I \neq fR$,

(iii) $(I^{[p]} : f^{p-1}) = I$, *and*

(iv) *I is contained in any other ideal satisfying* (i),(ii) and (iii).

Proof. We note that $R_f/R = R_g/R$. Let I be the ideal corresponding, under the bijection in Lemma IX.4.7, to the minimal simple F-submodule in given in Lemma VII.1.4. Then, it is clear from Lemma IX.4.7 that I satisfies (i)-(iv). □

Definition VII.1.6. Let $g \in R$ be an irreducible element and $f = g^n$ for some integer $n \geq 1$. We denote the minimal simple submodule of R_f/R by $\min_{F_R}(f)$, and we called it the *minimal F-module of f*. Let $\sigma : R/fR \to R_f/R$ be the morphism defined by $\sigma([a]) = a/f$ which is well defined because R is a domain. Since image of σ is $R\frac{1}{f}$, we will abuse notation and consider $R/fR \subset R_f/R$. We denote $(\phi\sigma)^{-1}(\min_F(f) \cap R\frac{1}{f})$ by $J_F(f)$, and we call it the *F-Jacobian ideal of f*. If f is a unit, we take $\min_F(f) = 0$ and $J_F(f) = R$.

Notation VII.1.7. If it is clear in which ring we are working, we write $J_F(f)$ instead of $J_{F_R}(f)$ and $\min_F(f)$ instead of $\min_{F_R}(f)$.

Proposition VII.1.8. *Let $f \in R$ be an irreducible element. Then $\min_F(f)$ is the only simple D-submodule of R_f/R.*

Proof. We claim that R_f/R has only one simple D_R-module. Since R_f/R has finite length as D-module, there is a simple D-submodule, M. It suffices to show that for any other D_R-submodule, $N \subset M$. We have that $M \cap N \neq 0$ by Lemma VII.1.2, and so $M = M \cap N \subset N$ because M is a simple D_R-module. Since $\min_{F_R}(f)$ is an D_R-module [Lyu97, Examples 5.1 and 5.2], we have that $M \subset \min_F(f)$.

It suffices to prove that M is an F-submodule of R_f/R. Since R/fR is a domain, we have that the localization morphism, $R/fR \to R_m/fR_m$, is injective. Then, $\mathrm{Supp}_R(R/fR) = \mathrm{Supp}_R(J)$ for every nonzero ideal $J \subset R/fR$. Then,

$$\mathrm{Supp}_R(R/fR) = \mathrm{Supp}_R(R/fR \cap N) \subset \mathrm{Supp}_R(N) \subset \mathrm{Supp}_R(R_f/R) = \mathrm{Supp}_R(R/f)$$

for every R-submodule of R_f/R by Lemma VII.1.2. Let m denote a maximal ideal such that $f \in m$. Thus, $M_m \neq 0$, and then, M_m is the only simple D_{R_m}-module of $(R_f)_m/R_m$. Since R_m is a regular local F-finite ring, we have that $\min_{F_{R_m}}(f)$ is a finite direct sum of simple D_{R_m}-modules [Lyu97, Theorem 5.6]. Therefore, $M_m = \min_{F_{R_m}}(f)$ by Lemma VII.1.2.

Let $\pi : R \to R/fR$ denote the quotient morphism, and $I = \pi^{-1}(R/fR \cap M)$. We note that $I \neq fR$ because $R/f \cap M \neq 0$ by Lemma VII.1.2. We claim that $(I_m^{[p]} : f) = I_m$ for every maximal ideal. If $f \in m$,

$$I_m/f = (R_m/fR_m) \cap M_m = (R_m/fR_m) \cap \min_{F_{R_m}}(f) = J_{F_{R_m}}(f)/f;$$

otherwise, $I_m = R_m = J_{F_{R_m}}(f)$ because f is a unit in R_m Then, $(I^{[p]} : f^{p-1}) = I$ and so I corresponds to an F_R-submodule of R_f/R, N_I by Lemma IX.4.7. Moreover,

$$N_I = \lim_{\to}(I/fR \stackrel{f^{p-1}}{\to} I^{[p]}/f^pR \stackrel{f^{p^2-p}}{\to} \ldots).$$

Since localization commutes with direct limit, we have that for every maximal ideal such that $f \in m$,

$$M_m = \min_{F_{R_m}}(f) = \lim_{\to}(I_m/fR_m \stackrel{f^{p-1}}{\to} I_m^{[p]}/f^pR_m \stackrel{f^{p^2-p}}{\to} \ldots) = N_{I_m} = N_I \otimes_R R_m.$$

Therefore, $M = N_I$ because $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(R/f)$, and it is an F-submodule of R_f/R. Hence, $M = \min_{F_R}(f)$. □

Remark VII.1.9. If $f \in R$ is an irreducible element, then:

(i) $\min(f) = \min(f^n)$ for every $n \in \mathbb{N}$ because $R_{f^n}/R = R_f/R$,

(ii) $J_F(f)$ is the minimal of the family of ideals I containing properly fR such that $(I : f^{p-1}) = I$ by Proposition VII.1.5.

(iii) $J_F(f)$ is not the usual Jacobian ideal of f. If $S = \mathbb{F}_3[x, y, z, w]$ and $f = xy+zw$, we have that the Jacobian of f is $m = (x, y, z, w)S$. However, $m \neq (m^{[p]} : f^2)$.

(iv) $J_F(f) = R$ if and only if R_f/R is a simple F-module by the proof of Proposition VII.1.5 and Lemma IX.4.7.

(v) $J_F(f) = R$ if and only if R_f/R is simple D_R-module by Proposition VII.1.8.

Proposition VII.1.10. *Let $f_i, \ldots f_\ell \in R$ be irreducible relatively prime elements and $f = f_1 \cdots f_\ell$. Then $\min_F(f_i)$ is an F-submodule of R_f/R. Moreover, all the simple F-submodules of R_f/R are $\min_F(f_1), \ldots, \min_F(f_\ell)$.*

Proof. The morphism $R_{f_i}/R \to R_f/R$, induced by the localization map $R_{f_1} \to R_f$, is a morphism of F-finite F-modules given by the diagram:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
R/f_iR & \xrightarrow{f_i^{p-1}} & R/f_i^pR & \xrightarrow{f_i^{p^2-p}} & R/f_i^{p^2}R & \longrightarrow & \ldots \\
\downarrow {\scriptstyle f_1^p \cdots \hat{f}_i^p \cdots f_\ell^p} & & \downarrow {\scriptstyle f_1^{p^2} \cdots \hat{f}_i^{p^2} \cdots f_\ell^{p^2}} & & \downarrow {\scriptstyle f_1^{p^3} \cdots \hat{f}_i^{p^3} \cdots f_\ell^{p^3}} & & \\
R/fR & \xrightarrow{f^{p-1}} & R/f^pR & \xrightarrow{f^{p^2-p}} & R/f^{p^2}R & \longrightarrow & \ldots
\end{array}
$$

Then $\min_F(f_i)$ is a simple F-submodule of R_f/R. Let N be an F-submodule of R_f/R, and $a/f_1^{\beta_1} \cdots f_\ell^{\beta_\ell} \in N \setminus \{0\}$. Since f_i is irreducible, we may assume that $\gcd(a, f_i) = 1$ and $\beta_i \neq 0$ for some $i = 1 \ldots, \ell$. Thus, $a/f_i \in N \cap R_{f_i}/R$ and $a/f_i \neq 0$. Then, $\min_F(f_i) \subset N \cap R_{f_i}/R \subset N$. In particular, if N is a simple F-submodule, then $N = \min_F(f_i)$. □

Remark VII.1.11. As a consequence of Lemma VII.1.10, we have that

$$\min_F(f_1) \oplus \ldots \oplus \min_F(f_\ell) \in R_f/R$$

because $R_g \cap R_h = R$ for all elements $g, h \in R$ such that $\gcd(g, h) = 1$.

Definition VII.1.12. Let $f_i, \ldots f_\ell \in R$ be irreducible relatively prime elements, $f = f_1^{\beta_1} \cdots f_\ell^{\beta_\ell}$, and $\pi : R \to R/fR$ be the quotient morphism. We define $\min_F(f)$ by

$$\min_F(f_1) \oplus \ldots \oplus \min_F(f_\ell),$$

and we called it the *minimal F-module of f*. Let $\sigma : R/fR \to R_f/R$ be the morphism defined by $\sigma([a]) = a/f$ which is well defined because R is a domain. Since image of σ is $R\frac{1}{f}$, we will abuse notation and consider $R/fR \subset R_f/R$. We denote $(\phi\sigma)^{-1}(\min_F(f) \cap R\frac{1}{f})$ by $J_F(f)$, and we call it the *F-Jacobian ideal of f*.

Remark VII.1.13. In the local case, $\min_F(f)$ is the intersection homology D-modules $\mathcal{L}(R/f, R)$ previously defined by Blickle [Bli04a, Theorem 4.5].

Proposition VII.1.14. *Let $f, g \in R$ be relatively prime elements. Then,*

$$J_F(fg) = fJ_F(g) + gJ_F(f).$$

Moreover, $fJ_F(g) \cap gJ_F(f) = fgR$

Proof. We consider R_f/R and R_g/R as F-submodules of R_{fg}/R, where the inclusion is given by the localization maps, $\iota_f : R_f \to R_{fg}$ and $\iota_g : R_g \to R_{fg}$. Let $\pi : R \to R/fgR$ and $\rho : R \to R/fR$ be the quotient morphisms. The limit of the morphism induced by the diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
R/fR & \xrightarrow{f^{p-1}} & R/f^pR & \xrightarrow{f^{p^2-p}} & R/f^{p^2}R & \xrightarrow{f^{p^3-p^2}} & \dots \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^{p^2}} & & \downarrow{\scriptstyle g^{p^3}} & & \\
R/fgR & \xrightarrow{(fg)^{p-1}} & R/f^pg^pR & \xrightarrow{(fg)^{p^2-p}} & R/(fg)^{p^2}R & \xrightarrow{(fg)^{p^3-p^2}} & \dots
\end{array}$$

is ι_f. Moreover, under this correspondence

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
J_F(f)/fR & \xrightarrow{f^{p-1}} & J_F(f)^{[p^2]}/f^pR & \xrightarrow{f^{p^2-p}} & J_F(f)^{[p^2]}/f^{p^2}R & \longrightarrow & \dots \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^{p^2}} & & \downarrow{\scriptstyle g^{p^3}} & & \\
gJ_F(f)/fgR & \xrightarrow{(fg)^{p-1}} & g^pJ_F(f)^{[p]}/f^pg^pR & \xrightarrow{(fg)^{p^2-p}} & g^{p^2}J_F(f)^{[p^2]}/(fg)^{p^2}R & \longrightarrow & \dots
\end{array}$$

induces the isomorphism of F-modules, $\iota_f : \min_F(f) \to \iota_f(\min_F(f))$. We have that

$$g(J_F(f)) = \pi^{-1}(\min_F(f) \cap R/fgR) \subset \pi^{-1}(\min_F(f) \cap R/fgR) = J_F(fg).$$

In addition,

$$(g^pJ_F(f)^{[p]} : (fg)^{p-1}) = gJ_F(f),$$

and it defines $\min_F(f)$ as a F-submodule of R_{fg}/R. Likewise,

$$fJ_F(g) \subset J_F(fg),\ (f^pJ_F(g)^{[p]} : (fg)^{p-1}) = fJ_F(g),$$

and it defines $\min_F(g)$ as a F-submodule of R_{fg}/R. Then,

$$fJ_F(g) + gJ_F(f) \subset J_F(fg).$$

Since $\min_F(f) \cap \min_F(g) = 0$, we have that $fJ_F(g) \cap gJ_F(g) = fgR$.

We claim that

$$(f^p J_F(g)^{[p]} + g^p J_F(f)^{[p]} : f^{p-1}g^{p-1}) = fJ_F(g) + gJ_F(f).$$

To prove the first containment, take

$$h \in (f^p J_F(g)^{[p]} + g^p J_F(f)^{[p]} : f^{p-1}g^{p-1}).$$

Then $f^{p-1}g^{p-1}h = f^p v + g^p w$ for some $v \in (J_F(g))^{[p]}$ and $w \in J_F(g)^{[p]}$. Since f and g are relatively prime, f^{p-1} divides w and g^{p-1} divides v. Thus, there exist $a, b \in R$ such that $v = g^{p-1}a$ and $w = g^{p-1}b$. Then, $a \in (J_F(g)^{[p]} : g^{p-1}) = J_F(g)$ and $b \in (J_F(f)^{[p]} : f^{p-1}) = J_F(f)$. Since,

$$f^{p-1}g^{p-1}h = f^p v + g^p w = f^p g^{p-1}a + g^p g^{p-1}b,$$

$h = fa + gb \in fJ_F(g) + gJ_F(f)$.

For the other containment, it is straightforward to check that

$$fJ_F(g) + gJ_F(f) \subset (f^p J_F(g)^{[p]} + g^p J_F(f)^{[p]} : f^{p-1}g^{p-1}).$$

Since $N_{fJ_F(g)+gJ_F(f)}$, the F-module generated by $fJ_F(g) + gJ_F(f)$, contains $\min_F(f)$ and $\min_F(g)$,

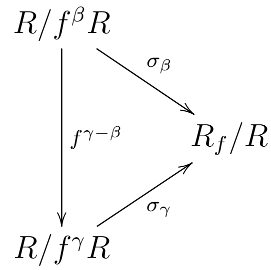
$$\min_F(f) \oplus \min_F(g) \subset N_{fJ_F(g)+gJ_F(f)}.$$

Therefore, $J_F(h) \subset fJ_F(g) + gJ_F(f)$ and the result follows. □

Proposition VII.1.15. *Let $\beta, \gamma \in \mathbb{N}$ be such that $\beta < \gamma$. Then,*

$$f^{\gamma-\beta} J_F(f^\beta) \subset J_F(f^\gamma) \subset J_F(f^\beta).$$

Proof. Let $\sigma_\ell : R/f^\ell \to R_f/R$ be the injection defined by sending $[a] \to a/f^\ell$. We note that the image of σ_ℓ is $R\frac{1}{f^\ell}$ We have that the following commutative diagram,



Then, $R\frac{1}{f^\beta} \cap \min_F(f) \subset R\frac{1}{f^\gamma} \cap \min_F(f)$, and this corresponds to

$$f^{\gamma-\beta} J_F(f^\beta)/f^\gamma R \subset J_F(f^\gamma)/f^\gamma R.$$

Hence, $f^{\gamma-\beta} J_F(f^\beta) \subset J_F(f^\gamma)$ because f^γ belongs to both ideals.

The morphism $R\frac{1}{f^\gamma} \cap \min_F(f^\beta) \stackrel{f^{\gamma-\beta}}{\to} R\frac{1}{f^\beta} \cap \min_F(f)$ is well defined and it is equivalent to the morphism $J_F(f^\gamma)/f^\gamma R \to J_F(f^\beta)/f^\beta$ given by the quotient morphism $R/f^\gamma R \to R/f^\beta R$. Then, $J_F(f^\gamma) + f^\beta R \subset J_F(f^\beta)$ and the result follows. ☐

Remark VII.1.16. There are examples in which the containment in Proposition VII.1.15 is strict. Let $R = \mathbb{F}_p[x]$ and $f = x$. In this case, R_f/R is a simple F-module. Then, $J_F(x^\beta) = R$ for every $\beta \in \mathbb{N}$ and $f^{\gamma-\beta} J_F(f^\beta) \subset J_F(f^\gamma)$ for every $\gamma > \beta$.

Corollary VII.1.17. *Let $f, g \in R$ be such that f divides g. Then, $J_F(g) \subset J_F(f)$.*

Proof. This follows from Propositions VII.1.15 and VII.1.14. ☐

Proposition VII.1.18. *Let $f \in R$ and $W \subset R$ be a multiplicative system. Then, $J_{F_{W^{-1}R}}(f) = W^{-1} J_{F_R}(f)$.*

Proof. By Proposition VII.1.14, it suffices to prove the claim for $f = g^n$, where g is an irreducible element. We note that g is either a unit or a irreducible element in $W^{-1}R$. We have that $\min_{F_{W^{-1}R}}(f) = \min_{F_{W^{-1}R}}(g)$ is either zero or a simple F-module by Lemma VII.1.8. Then, $\min_{F_{W^{-1}R}}(f) = W^{-1} \min_{F_R}(f)$, and so

$$\begin{aligned} J_{F_{W^{-1}R}}(f)/fW^{-1}R &= W^{-1}R/fW^{-1}R \cap \min_{F_{W^{-1}R}}(f) \\ &= W^{-1}(R/fR \cap \min_{F_R}(f)) \\ &= W^{-1} J_{F_R}(f)/fW^{-1}R, \end{aligned}$$

and the result follows because f belongs to both ideals. ☐

Proposition VII.1.19. *Let $f \in R$. Then, $J_{F_{R^{1/p^e}}}(f) = J_{F_R}(f)R^{1/p^e}$. Moreover, $J_{F_R}(f^{p^e}) = J_{F_R}(f)^{[p^e]}$.*

Proof. By Proposition VII.1.14, we may assume that $f = g^n$ where g is a irreducible. Let q denote p^e and h denote the length of R_f/R in the category of F-modules. Let $G : R^{1/q} \to R$ be the isomorphism defined by $r \to r^q$. Under the isomorphism G, $R_f^{1/q}/R^{1/q}$ corresponds to R_{f^q}/R. Then, the length of $R_f^{1/q}/R^{1/q}$ in the category of $F_{R^{1/q}}$-modules is h. Let $0 = M_0 \subset \ldots \subset M_h = R_f/R$ be a chain of F_R-submodules of R_f/R such that M_{i+1}/M_i is a simple F_R-module. Let $fR = J_0 \subset \ldots \subset J_h = R$ be the corresponding chain of ideals under the bijection given in Lemma IX.4.7. Since $f = g^n$ and g is irreducible, $M_1 = \min_{F_R}(f)$ and $J_1 = J_{F_R}(f)$. We note that $(J_i^p R^{1/q} : f^{p-1}) = J_i R^{1/q}$ and $J_i R^{1/q} \neq J_{i+1} R^{1/q}$ because $R^{1/q}$ is a faithfully flat R-algebra.

Then, we have a strictly ascending chain of ideals

$$fR^{1/q} = J_0 R^{1/q} \subset \ldots \subset J_h R^{1/p} = R^{1/q}$$

that corresponds to a strictly ascending chain of $F_{R^{1/q}}$-submodules of $R_f^{1/q}/R^{1/q}$.

Since $f = (g^{1/q})^{qn}$, $g^{1/q}$ is irreducible and the length of $R_f^{1/q}/R^{1/q}$ is h, we have that

$$J_{F_R}(f)R^{1/q} = J_1 R^{1/p} = J_{F_{R^{1/q}}}(f).$$

After applying the isomorphism G to the previous equality, we have that

$$J_{F_R}(f)^{[q]} = G(J_{F_R}(f)R^{1/q}) = G(J_{F_{R^{1/q}}}(f)) = J_{F_R}(f^q)$$

□

Proposition VII.1.20. *Let $R \to S$ be flat morphism of UFDs and let $f \in R$. If S is as in Notation VII.1.1, then $J_{F_S}(f) \subset J_{F_R}(f)S$.*

Proof. We may assume that $f = g^\beta$ where g is an irreducible element in R by Proposition VII.1.14. Since S is flat, $(J_{F_R}(f)^{[p]}S : f^{p-1}) = J_{F_R}(f)S$. Let M denote the F_S-submodule of S_f/S given by $J_{F_R}(f)S$ under the correspondence in Lemma IX.4.7. If f is a unit in S, then $J_F(f)S = S$ and the result is immediate. We may assume that f is not a unit in S. Since $J_F(f) \neq fR$, we can pick $a \in J_F(f) \setminus fR$. Then, $a = bg^\gamma$ for some $0 \le \gamma < \beta$ and $b \in R$ such that $\gcd(b, g) = 1$. Then, $R/g \xrightarrow{b} R/g$ is injective, and so $S/gS \xrightarrow{b} S/gS$ is also injective. Thus, $\gcd(b, g) = 1$ in S. Hence, b/g is not zero in S_g/S. Moreover, $b/g = g^{\beta-\gamma-1}a/f \in M$ and it is not zero. Let

$g_1, \dots, g_\ell \in S$ irreducible relatively prime elements such that $g = g_1^{\beta_1} \cdots g_1^{\beta_\ell}$. We have that $b/g_i = hb/g \in S_{g_i}/S \cap M \setminus \{0\}$, where $h = g_1^{\beta_1} \cdots g_i^{\beta_i - 1} \cdots g_1^{\beta_\ell}$. Then, $\min_{F_S}(g_i) \subset M$ and so $\min_{F_S}(f) \subset M$. Therefore, $J_{F_S}(f) \subset J_{F_R}(f)S$. □

Proposition VII.1.21. *Suppose that R is a local ring. Let $f \in R$. Then*

$$J_{F_{\widehat{R}}}(f) = J_{F_R}(f)\widehat{R},$$

where $\widehat{R}$ denotes the completion of R with respect to the maximal ideal.

Proof. We have that $\min_{F_{\widehat{R}}}(f) = \min_{F_R}(f) \otimes_R \widehat{R}$ [Bli04a, Theorem 4.6]. Then,

$$J_{F_{\widehat{R}}} = \left(\widehat{R}/f\widehat{R}\right) \cap \min_{F_{\widehat{R}}}(f) = ((R/fR) \cap \min_{F_R}(f)) \otimes_R \widehat{R} = J_{F_R}(f)\widehat{R}$$

□

Lemma VII.1.22. *Let $R = K[x_1, \dots, x_n]$, where K is a perfect field. Let $K \to L$ be an algebraic field extension of K, $S = L[x_1 \dots, x_n]$, and $R \to S$ the map induced by the extension. Then, $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We can assume that $f = g^\beta$ where g is an irreducible element in R by VII.1.14. It suffices to show that $J_{F_R}(R)S \subset J_{F_S}(S)$ by Proposition VII.1.20.

There is an inclusion $\phi : R_f/R \to S_f/S$, which is induced by $R \to S$. We take $M = (\min_{F_S}(f)) \cap R_f/R$. We claim that M is a D_R-module of R_f/R. Since K is perfect,

$$D_R = \bigcup_{e \in \mathbb{N}} \operatorname{Hom}_{S^{p^e}}(S, S) = D(R, K) = R[\frac{1}{t!}\frac{\partial^t}{\partial x_i^t}].$$

We note that $D_R \subset D_S$, and that for every $m \in R_f/R$, $\phi(\frac{\partial^t}{\partial x_i^t} m) = \frac{\partial^t}{\partial x_i^t}\phi(m)$. As a consequence, $\frac{\partial^t}{\partial x_i^t} m \in M$ for every $m \in M$. Therefore, M is a D_R-module.

Let $I = M \cap R/fR$. We note that

$$I = \min_{F_S}(f) \cap R/fR = (J_{F_S}(f)/fS) \cap R/fR$$

and that S/fS is an integral extension of R/fR because L is an algebraic extension of K. Let $r \in J_{F_S}(f)/fS$ not zero, and $a_j \in R/fR$ such that $a_0 \neq 0$

$$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0$$

in S/fS. Then,

$$r(a_{n-1}r^{n-1} + \ldots + a_1) = -a_0,$$

and so $a_0 \in I = (J_{F_S}(f)/fS) \cap R/fR$, and then $M \neq 0$. Therefore, $\min_{F_R}(f) \subset M$ and so $J_F(f)/f \subset I$. Let $\pi : R \to R/fR$ be the quotient morphism. Then,

$$J_{F_R}(f) \subset \pi^{-1}(I) = J_{F_S}(f) \cap R,$$

and

$$J_{F_R}(f)S \subset (J_{F_S}(f) \cap R)S \subset J_{F_S}(f).$$

□

Lemma VII.1.23. *Let $R = K[x_1, \ldots, x_n]$, where K is an F-finite field. Let $L = K^{1/p}$, $S = L[x_1 \ldots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We have that $R \subset S \subset R^{1/p}$. Then, by Proposition VII.1.20,

$$J_{F_{R^{1/p}}}(f) \subset J_{F_S}(f)R^{1/p} \subset (J_{F_R}(f)S)R^{1/p} = J_{F_R}(f)R^{1/p}.$$

Since $J_{F_{R^{1/p}}}(f) = J_{F_R}(f)R^{1/p}$ by Proposition VII.1.19,

$$0 = J_{F_S}(f)R^{1/p}/(J_{F_R}(f)S)R^{1/p} = (J_{F_S}(f)/J_{F_R}(f)S) \otimes_S R^{1/p}.$$

Therefore, $J_{F_S}(f) = J_{F_R}(f)S$ because $R^{1/p}$ is a faithfully flat S-algebra. □

Lemma VII.1.24. *Let $R = K[x_1, \ldots, x_n]$, where K is an F-finite field. Let L be the perfect closure of K, $S = L[x_1 \ldots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We may assume that $f = g^n$ for an irreducible $g \in R$ by Proposition VII.1.14. Let $S^e = K^{1/p^e}[x_1, \ldots, x_n]$. Let $h_1, \ldots, h_\ell$ denote a set of generators for $J_{F_S}(f)$. In this case, $(J_{F_S}(f)^{[p]} : f^{p-1}) = J_{F_S}(f)$. Then there exist $c_{i,j} \in S$ such that

$$f^{p-1}h_j = \sum c_{i,j}h_j^p.$$

Since $S = \bigcup_e S^e$, there exist an N such that $c_{i,j}, h_j \in S^N$. Let $I \subset R^N$ be the ideal generated by $h_1, \ldots, h_\ell$. We note that $IS = J_{F_S}(f)$; moreover, $J_{F_S}(f) \cap S^N = I$ because $S^e \to S$ splits for every $e \in \mathbb{N}$.

We claim that $(I^{[p]} : f^{p-1}) = I$. We have that $f^{p-1}h_\ell \in I^{[p]}$ by our choice of N and so $I \subset (I^{[p]} : f^{p-1})$. If $g \in (I^{[p]} : f^{p-1})$, then $f^{p-1}g \in I^{[p]} \subset J_{F_S}(f)^{[p]}$ and $g \in J_{F_S}(f) \cap S^N = I$.

As in the proof of Lemma VII.1.22, $(J_{F_S}(f)/fS) \cap (S^N/fS^N) \neq 0$ and then $J_{F_S}(f) \cap S^N = I \neq fS$. Therefore, $J_{F_{S^N}}(f) \subset I$ by Proposition VII.1.5. Hence,

$$J_{F_{S^N}}(f)S \subset IS = J_{F_S}(f) \subset J_{F_{S^N}}(f)S,$$

and the result follows because

$$J_{F_R}(f)S = (J_{F_R}(f)S^N)S = J_{F_{S^N}}(f)S.$$

□

Theorem VII.1.25. *Let $R = K[x_1, \ldots, x_n]$, where K is an F-finite field. Let L be an algebraic extension of K, $S = L[x_1 \ldots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. It suffices to show $J_{F_R}(f)S \subset J_{F_S}(f)$ by Proposition VII.1.20. Let K^* and L^* denote the perfect closure of K and L respectively. Let $R^* = K^*[x_1, \ldots, x_n]$ and $S^* = L^*[x_1, \ldots, x_n]$.

Then,

$$(J_{F_R}(f)S)S^* J_{F_R}(f)S^* = (J_{F_R}(f)R^*)S^* = J_{F_{R^*}}(f)S^* = J_{F_{S^*}}(f) = J_{F_S}(f)S^*$$

by Lemma VII.1.22 and VII.1.24. Therefore,

$$(J_{F_R}(f)S/J_{F_S}(f)) \otimes_S S^* = (J_{F_R}(f)S)S^*/(J_{F_S}(f))S^* = 0.$$

Hence $J_{F_R}(f)S/J_{F_S}(f) = 0$ because S^* is a faithfully flat S-algebra. □

Example VII.1.26. Let $R = \mathbb{F}_3[x, y]$, and $f = x^2 + y^2$ and $m = (x, y)$. We have that $(m^{[p]} : f^{p-1}) = m$. Then, $J_{F_R}(f) \subset m$. Let $\mathbb{F}_3[i]$ the extension of $\mathbb{F}_3$ by $\sqrt{-1}$, $S = L[x, y]$ and $\phi : R \to S$ be the inclusion given by the field extension. Then, $J_{F_S}(f) = (x, y)S$ by Proposition VII.1.14 because $x^2 + y^2 = (x + iy)(x - iy)$. Since ϕ is a flat extension, $J_{F_S}(f) \subset J_{F_R}(f)S$. Then, $m = R \cap J_{F_S}(f) \subset R \cap J_{F_R}(f)S$. Hence, $J_F(f) = m$.

Proposition VII.1.27. *Let $f \in R$ be an irreducible element. Then,*

$$J_F(f) = \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e \in \mathbb{N}} \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right) \right)$$

Proof. We have that $\min_F(f)$ is the intersection of all nonzero D-submodules of R_f/R by Proposition VII.1.8. In particular, $\min_F(f)$ is the intersection of all nonzero cyclic D-modules generated by elements $a/f \in R_f/R$. Hence,

$$
\begin{aligned}
J_F(f)/f &= \bigcap_{\gcd(a,f)=1} ((D \cdot a/f) \cap R/f) \\
&= \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e\in\mathbb{N}} \left(D^{(e)} \cdot a/f \cap R/f\right) \right)
\end{aligned}
$$

We have that $b \in J_F(f)$ if $b/f \in \bigcap_{\gcd(a,f)=1} \bigcup_{e\in\mathbb{N}} \left(D^{(e)} \cdot a/f\right)$, so, for every $a \in R$ such that $\gcd(a,f)=1$, there exists an $e \in \mathbb{N}$ such that $b/f \in D^{(e)} \cdot a/f$. Thus, there exists $\phi \in \operatorname{Hom}_{R^{p^e}}(R,R)$ such that

$$\phi(a/f) = 1/f^{p^e}\phi(f^{p^e-1}a) = b/f + r$$

Therefore, after multiplying by f^{p^e}, we have that

$$b \in \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e\in\mathbb{N}} \left(\left(\left(f^{p^e-1}a\right)^{[1/p^e]}, f\right)^{[p^e]} : f^{p^e-1} \right) \right)$$

because $\left((f^{p^e-1}a)^{[1/p^e]}\right)^{[p^e]} = D^{(e)}\left(f^{p^e-1}a\right)$.

On the other hand, if

$$b \in \bigcap_{\gcd(a,f)=1} \bigcup_{e\in\mathbb{N}} \left(\left(\left(f^{p^e-1}a\right)^{[1/p^e]}, f\right)^{[p^e]} : f^{p^e-1} \right),$$

then for every $a \in R$ such that $\gcd(a,f)=1$, there exists an $e \in \mathbb{N}$ and $\phi \in \operatorname{Hom}_{R^{p^e}}(R,R)$ such that

$$f^{p^e-1}b = \phi(f^{p^e-1}a) + f^{p^e}r$$

because $\left(\left(f^{p^e-1}a), f\right)^{[1/p^e]}\right)^{[p^e]} = D^{(e)}(f^{p^e-1}a)$. Therefore, after dividing by f^{p^e}, we have that $b/f \in \bigcap_{\gcd(a,f)=1} \bigcup_{e\in\mathbb{N}} \left(D^{(e)} \cdot a/f\right)$, and then $b \in J_F(f)$. $\square$

Theorem VII.1.28. *Let $f \in R$ be such that R/fR is a F-pure ring. If $J_F(f) = R$, then R/fR is strongly F-regular.*

Proof. We may assume that (R, m, K) is local because being reduced, F-pure, and F-regular are local properties for R/fR. Then, f is irreducible by Proposition VII.1.14.

Since $J_F(f)=R$, for every a such that $\gcd(a,f)=1$ there exits an $e\in\mathbb{N}$ such that $R=\left(\left(\left(f^{p^e-1}a\right)^{[1/p^e]},f\right)^{[p^e]}:f^{p^e-1}\right)$ by Lemma VII.1.27. Then,

$$f^{p^e-1}\in\left(\left(f^{p^e-1}a\right)^{[1/p^e]},f\right)^{[p^e]}.$$

Since $f^{p^e-1}\not\in m^{[p^e]}$ for every $e\in\mathbb{N}$ by Fedder's Criterion , $R=\left(f^{p^e-1}a\right)^{[1/p^e]}$; otherwise, $(\left(f^{p^e-1}a\right)^{[1/p^e]},f)\subset m$. Then, there exist a morphism $\phi:R\to R^{p^e}$ of R^{p^e}-modules such that $\phi(f^{p^e-1}a)=1$. Let $\varphi:R/fR\to R/fR$ be the morphism defined by $\varphi([x])=[\phi(f^{p^e-1}x)]$. We note that φ is a well defined morphism of $(R/fR)^p$-modules such that $\varphi([a])=1$. Then, R/fR is a simple $D(R/fR)$-module. Hence, R/fR is strongly F-regular [Smi95a, Theorem 2.2]. □

Remark VII.1.29.

- The result of the previous theorem is a consequence of a result of Blickle [Bli04a, Corollary 4.10], as R/fR is a Gorenstein ring. However, our proof is different from the one given there.
- $J_{F_R}(f)=R$ does not imply that R/fR is F-pure. Let $K=\mathrm{Frac}(\mathbb{F}_2[u])$ be the fraction field of the polynomial ring $\mathbb{F}_2[u]$, $R=K[[x,y]]$, and $f=x^2+uy^2$. Then, f is an irreducible element such that R/fR is not pure because $f\in(x,y)^{[2]}R$. Let $L=K^{1/2}$, $S=L[[x,y]]$ and $R\to S$ be the inclusion given by the extension $K\subset L$. Thus, $f=(x-u^{1/2}y)^2$ in S, and then $J_{F_S}(S)=S\subset J_{F_R}(f)S$. Then, $R=J_{F_S}(S)\cap R=J_{F_R}(f)S\cap R=J_{F_R}(f)$ because $R\to S$ splits. Hence, $J_{F_R}(f)=R$ and R/fR is not F-pure.

VII.2 Definition for rings essentially of finite type over an F–finite local ring.

Notation VII.2.1. Throughout this section R denotes a ring essentially of finite type over an F–finite local ring. Let $f\in R$, $\pi:R\to R/fR$ be the quotient morphism. If R/fR is reduced τ_f denotes $\pi^{-1}(\tau(R/fR))$, the pullback of the test ideal of R/fR.

Under the hypotheses on R in Notation VII.2.1, there is an F-module and D-module of R_f/R called the intersection homology $\mathcal{L}(R,R/fR)$ [Bli01, Bli04a]. We have that for every maximal ideal $m\subset R$, $(R\setminus m)^{-1}\mathcal{L}(R,R/fR)=\min_{F_{R_m}}(f)$.

Definition VII.2.2. Recall tht $R/fR\subset R_f/R$ be the inclusion morphism $1\mapsto\frac{1}{f}$. We define the F-Jacobian, $J_f(f)$ as the pullback to R of $(R/fR)\cap\mathcal{L}(R,R/fR)$.

Lemma VII.2.3. *Suppose that R/fR is reduced. Let $I^j(f) = (\tau_f^{[p^{j-1}]} : f^{p^{j-1}-1})$. Then $I^j(f) \subset I^{j+1}(f)$ and*

$$I^{j+1}(f) = (I^j(f)^{[p]} : f^{p-1}).$$

Proof. Since, in this case, the test ideal of R/fR commutes with localization, we may assume that R is a local ring. We have that τ_f/fR is the minimal root for $\min_F(f)$ [Bli04a, Theorem 4.6]. Then, $f^{p-1}I^1(f) = f^{p-1}\tau_f \subset \tau_f^{[p^e]} = I^1(f)^{[p^e]}$. Thus, $I^1 \subset I^2$ and $f^{p-1}\mathcal{I}^2 \subset \mathcal{I}_1^{[p]}$. Moreover, I^2/fR is also a root for $\min_F(f)$ because $I^2(f)/I^1(f)$ is the kernel of the map

$$R/I^1(f) \stackrel{f^{p-1}}{\to} R/I^1(f)^{[p]}.$$

Inductively, we obtain that $I^j \subset I^{j+1}$, $f^r I^{j+1} \subset I^j$ and that I^j/fR is a root for $\min_F(f)$ for every $j \in \mathbb{N}$ and the result follows. □

Proposition VII.2.4. *Suppose that R/fR is reduced. Then, $J_F(f) = \cup_j I^j_R(f)$.*

Proof. We have that $\tau_f \frac{1}{f}$ is the minimal root for $\mathcal{L}(R, R/fR)$. Moreover, any ideal $\mathcal{I}^j(f) \stackrel{f^{p-1}}{\to} \mathcal{I}^j(f)^{[p]}$ also generates $\mathcal{L}(R, R/fR)$ as F-module. Moreover, $\cup_j I^j_R(f) = \mathcal{L}(R, R/fR) \cap R/fR = J_F(f)$. □

Remark VII.2.5. In general, we do not have $\tau_f = J_F(f)$. Let $R = K[x]$, where K is any perfect field of characteristic $p > 0$. Let $f = x^2$. Then, $\tau_f = xR \neq R = J_F(f)$. In addition, Example VII.3.3, shows another situation where $\tau_f \neq J_F(f)$.

Remark VII.2.6. If R is an F-finite local ring, then

$$J_F(f) \stackrel{f^{p-1}}{\to} J_F(f)^{[p]}$$

is a generating morphism for $\min_F(f))$ because in this case $\min_F(f)) = \mathcal{L}(R, R/fR)$.

Corollary VII.2.7. *Let S be a ring that is as in Notation VII.1.1 and as in Notation VII.2.1. Let $f \in S$. Let J denote the F-Jacobian ideal of f as in Definition VII.1.12 and let J' the F-Jacobian ideal of f as in Definition VII.2.2. Then, $J = J'$.*

Proof. We have that in both contexts the F-Jacobian ideal commutes with localization. We may assume that R is a regular local F-finite ring. As $J_2 = (J_2^{[p]} : f^{p-1})$ and $J_2/fR \stackrel{f^{p-1}}{\to} J_2^{[p]}/f^pR$ is a root for $\min_F(f)$ by Lemma VII.2.6, we have that $J_1 = J_2$. □

Remark VII.2.8. As for every maximal ideal $m \subset R$, R_m is as in Notation VII.1.1, we have that

- $f^n J_F(f) \subset J_F(f^n)$,
- $J_F(f^{p^e}) = J_F(f)^{[p^e]}$, and
- if $\gcd(f, g) = 1$, $J_F(f) = fJ_F(g) + gJ_F(f)$.

because those properties can be checked locally.

Proposition VII.2.9. *Suppose that (R, m, K) is local. Let (S, η, L) denote a regular F-finite ring. Let $R \to S$ be a flat local morphism such that the closed fiber S/mS is regular L/K is separable. Then, $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. It suffices to proof that $\min_{F_R}(f)S = \min_{F_S}(f)$. We can assume without loss of generlization tat R/fR is reduced. We have that $J_{F_{\widehat{R}}}(f) = J_{F_R}(f)\widehat{R}$ and $J_{F_{\widehat{S}}}(f) = J_{F_S}(f)\widehat{S}$. In addition, the induced morphism in the completion $\widehat{R} \to \widehat{S}$ is still a flat local morphism. Since $J_{F_S}(f) \subset J_{F_R}(f)S$ and $J_{F_{\widehat{S}}}(f) \subset J_{F_{\widehat{R}}}(f)\widehat{S}$ by Proposition VII.1.20, $J_{F_{\widehat{R}}}(f)\widehat{S}/J_{F_{\widehat{S}}}(f) = (J_{F_R}(f)S/J_{F_S}(f)) \otimes_S \widehat{S}$. Therefore, we can assume that R and S are complete.

We note that $R/fR \to S/fS$ is again a flat local morphism such that the closed fiber S/mS is regular L/K is separable by flat base change. Then S/fS is reduced and $\tau(R/fR)S = \tau(S/fS)$ [HH94a, Theorem 7.2], and so $I^j_{F_S}(f) = I^j_{F_R}(f)S$. Hence, $J_{F_S}(f) = J_{F_R}(f)S$ by Proposition VII.2.4. □

Corollary VII.2.10. *Suppose that R is a $\mathbb{Z}^h$-graded ring. Let $f \in R$ be a homogeneous element. Then, $J_F(f)$ is a homogeneous ideal.*

Proof. It suffices to proof that $\min(f)$ is a $\mathbb{Z}^h$-graded submodule of R_f/R. We can assume that R/fR is reduced. We have that $\tau(R/fR)$ is a homogeneous ideal ideal [HH94b, Theorem 4.2]. This means that $I^j_R(f)$ is a homogeneous ideal for every j. Therefore, $J_F(f)$ is homogeneous and that $\min_F(f)$ $\mathbb{Z}^n$-graded submodule of R_f/R. □

Corollary VII.2.11. *Let S be a ring that is as in Notation VII.1.1 or as in Notation VII.2.1. Let $f \in S$ be such that R/fR is reduced. Then, $\mathcal{V}(J_F(f)) \subset \mathrm{Sing}_F(S/fS)$. Moreover, if S/fS is an F-pure ring, then $\mathcal{V}(J_F(f)) = \mathrm{Sing}_F(S/fS)$.*

Proof. For every prime ideal $P \in \mathcal{V}(J_F(f))$, $J_{F_{S_P}}(f) \neq S_P$. Since S_P is as in Notation VII.2.1, we have that $\tau(S_P/fS_P) \subset J_{F_{S_P}}(f) \subset PS_P$. Then, S_P is not F-regular and then $P \in \mathrm{Sing}_F(S/fS)$.

Now, we suppose that S/fS is F-pure. For every prime ideal $P \in \mathrm{Sing}_F(S/fS)$, S_P/pS_P is not F-regular. Then, $J_{F_{R_P}}(f) \neq R_P$ by Theorem VII.1.28. Then, $P \in \mathcal{V}(J_F(f))$. $\square$

Lemma VII.2.12. *Let S be a ring that is as in Notation VII.1.1 and as in Notation VII.2.1. Let $f \in S$ be an element and $Q \subset S$ be a prime ideal. If S_Q/fS_Q is F-pure, then $S_Q/J_{F_{S_Q}}(f)$ is F-pure.*

Proof. We may replace S by S_Q. Since S/fS is F-pure, we have that $f^{p-1} \not\in Q^{[p]}$ by Fedder's Criterion. We have that $f^{p-1} \in (J_F(f)^{[p]} : J_F(f))$, and so $(J_F(f)^{[p]} : J_F(f)) \not\subset Q^{[p]}$. Therefore, $S/J_F(f)$ is F-pure. $\square$

Corollary VII.2.13. *Let $f \in R$. If R/fR is an F-pure ring, then $J_F(f) = \tau_f$.*

Proof. We have that $\sqrt{J_F(f)} = \sqrt{\pi^{-1}(R/fR)}$ by Corollary VII.2.11 because

$$\mathrm{Sing}_F(R/fR) = \mathcal{V}(\tau(R/fR))$$

in this case. Since $R/J_F(f)$ is F-pure by Lemma VII.2.12, $J_F(f)$ is a radical ideal. In addition, $\tau(R/fR)$ is a radical ideal [FW89, Proposition 2.5]. Hence, $J_F(f) = \tau_f$. $\square$

VII.3 Examples

Proposition VII.3.1. *Let $f \in R$ be an element with an isolated singularity at the maximal ideal m. If R_m/fR_m is F-pure, then*

$$J_F(f) = \begin{cases} R & \text{if } R/fR \text{ is } F-\text{regular} \\ m & \text{otherwise} \end{cases}$$

Proof. Since R/fR has an isolated singularity at m, we have that $J_F(f)R_P = R_P$ for every prime ideal different from m. Then, $m \subset \sqrt{J_F(f)}$.

If R_m/fR_m is F-regular, then R/fR is F-regular, and so $J_F(R) = R$ by Theorem VII.1.28.

If R_m/fR_m is not F-regular, then $J_F(R) \neq R$ by Theorem VII.1.28. Then, $m = \sqrt{J_F(f)}$. Since R_m/fR_m is F-pure, we have that $R_m/J_F(f)R_m$ is F-pure by Lemma VII.2.12. Then, $R_m/J_F(f)R_m$ is a reduced ring. Hence, $J_F(f) = m$. $\square$

Example VII.3.2. Let K is an F-finite field. Let E be an elliptic curve over K. We choose a closed immersion of E in $\mathbb{P}^2_K$ and set $R = K[x, y, z]$, the completed homogeneous co-ordinate ring of $\mathbb{P}^2_K$. We take $f \in R$ as the cubic form defining E. We know that f has an isolated singularity at $m = (x, y, z)R$. If the elliptic curve is ordinary, then R/fR is F-pure [Har77, Proposition 4.21] [Bha12, Theorem 2.1]). We know that R/fR is never an F-regular ring [HH94b, Discussion 7.3b(b), Theorem 7.12]. Then, $J_F(f) = m$ by Proposition VII.3.1.

Example VII.3.3. Let $R = K[x, y, z]$, where is an F-finite field of characteristic $p > 3$. Let $f = x^3 + y^3 + z^3 \in R$, and $\pi : R \to R/fR$ be the quotient morphism and $m = (x, y, z)R$. We have that $\tau_f = m$ [Smi95b, Example 6.3]. Then, $m \subset J_F(f)$ by Proposition VII.2.4.

We have that R/fR is F-pure if and only if $p \equiv 1 \bmod 3$. We have that $(m^{[p]} : f^{p-1}) = m$ if $p \equiv 1 \bmod 3$, and $(m^{[p]} : f^{p-1}) = R$ if $p \equiv 1 \bmod 2$. Hence,

$$J_F(f) = \begin{cases} R & p \equiv 2 \bmod 3 \\ m & p \equiv 1 \bmod 3. \end{cases}$$

Example VII.3.4. Let $R = K[x_1, \ldots, x_n]$, where K is an F–finite field of characteristic $p > 0$. Let $f = a_1 x_1^{d_1} + \ldots + a_n x_n^{d_n}$, be such that $a \neq 0$. We have that R/fR has an isolated singularity at the maximal ideal $m = (x_1, \ldots, x_n)$.

If $\frac{1}{d_1} + \ldots + \frac{1}{d_n} = 1$ and $(p-1)/d_1$ is an integer for every i, then R/fR is F-pure for $p \gg 0$ [Her12, Theorem 3.1] and not F-regular [Gla96, Theorem 3.1] because f^{p-1} is congruent to $c(x_1 \cdots x_n)^{p^e-1}$ module $m^{[p^e]}$ for a nonzero element $c \in K$. Hence, $J_F(f) = R$ for $p \gg 0$ by Proposition VII.3.1.

Remark VII.3.5. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring and $f \in R$ be such that R/fR is reduced. We can obtain $J_F(f)$ from $\tau(R/fR)$ by Proposition VII.2.4. In the case where $n > 3$, $f = x_1^d + \ldots x_n^d$ and d is not divided by the characteristic of K, there is an algorithm to compute the test ideal of R/fR [McD03]. Therefore, there is an algorithm to compute $J_F(f)$.

Example VII.3.6. Let $R = K[x_1, \ldots, x_n]$, where K is a field of characteristic $p > 0$. Let $f = x_1^d + \ldots + x_n^d$. This examples are based in computations done by McDermott [McD03, Example 11, 12 and 13].

If $p = 2, n = 5$ and $d = 5$,

$$\tau_f = (x_i^2 x_j)_{1 \le i,j \le 5}.$$

Then, $(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2x_3x_4x_5)R = (\tau_f^{[2]} : f)$ and $R = (\tau_f^{[4]} : f^3)$. Hence, $J_F(f) = R$.

If $p=3, n=4$ and $d=7$,

$$\tau_f = (x_i^2 x_j^2)_{1\le i,j\le 4}.$$

Then $R = (\tau_f^{[3]} : f^2)$ and $J_F(f) = R$.

If $p=7, n=5$ and $d=4$,

$$\tau_f = (x_1, \ldots, x_5)R.$$

Then $R = (\tau_f^{[7]} : f^6)$ and $J_F(f) = R$.

CHAPTER VIII

A key functor

In this chapter, we study a functor utilized by Lyubeznik to prove that his original invariants are well defined (cf. [Lyu93, Lemma 4.3]). In order to define the generalized Lyubeznik numbers in Chapter IX, significant development of the theory of this functor is necessary. The fact that this functor gives, in fact, an equivalence with a certain category of D-modules is essential to the results here, as we will see in Theorem VIII.0.10. This theorem somehow mirrors Kashiwara's equivalence [Cou95] equivalence for any local ring.

The results presented in this chapter are part of joint work with Witt [NBW12a].

Definition VIII.0.7 (Key functor G)**.** Let R be a Noetherian ring, and let $S = R[[x]]$. Let $G : R\text{-mod} \to S\text{-mod}$ be the functor given by $G(-) = (-) \otimes_R S_x/S$.

We note that the functor G is reminiscent of the "direct image" functor utilized by Àlvarez Montaner, by differs due to the base ring in the tensor product [ÀM04].

Remark VIII.0.8. For every element in $u \in G(M)$ there exist $\ell, \alpha_1, \ldots, \alpha_\ell \in \mathbb{N}$, $m_1, \ldots, m_\ell \in M$, uniquely determined, such that $u = m_\ell \otimes x^{-\alpha_\ell} + \ldots + m_1 \otimes x^{-\alpha_1}$ and $m_\ell \neq 0$ because

(VIII.0.8.1) $$G(M) = M \otimes_R S_x/S = M \otimes_R \left(\bigoplus_{\alpha \in \mathbb{N}} Rx^{-\alpha} \right) = \bigoplus_{\alpha \in \mathbb{N}} \left(M \otimes Rx^{-\alpha} \right).$$

Moreover, G is an exact functor and commutes with local cohomology.

Remark VIII.0.9. In fact, G is a functor from the category R-modules to the category of $D(S,R)$-modules: Let M be a $D(S,R)$-module. Since $D(S,R) = S\langle \frac{1}{t!}\frac{\partial^t}{\partial x^t} \mid t \in \mathbb{N}\rangle \subseteq \operatorname{Hom}_K(S,S)$, it is enough to give an action of each $\frac{1}{t!}\frac{\partial^t}{\partial x^t}$ on $G(M)$. If $m \otimes x^{-\alpha} \in$

$G(M)$, we define

$$\left(\frac{1}{t!}\frac{\partial^t}{\partial x^t}\right) \cdot (m \otimes x^{-\alpha}) = \binom{\alpha+t-1}{t} \cdot \left((-1)^t m \otimes x^{-\alpha-t}\right).$$

In particular, taking α =1 and $t = \beta$, we see that, for every $\beta \in \mathbb{N}$,

(VIII.0.9.1) $$m \otimes x^{-\beta} = \frac{(-1)^{\beta-1}}{(\beta-1)!}\frac{\partial^{\beta-1}}{\partial x^{\beta-1}}(m \otimes x^{-1}).$$

Similarly, for every morphism of R-modules φ, $G(\varphi) = \varphi \otimes_R S_x/S$ is a morphism of $D(S,R)$-modules.

Moreover, G is an equivalence of certain categories:

Theorem VIII.0.10. *Let R be a Noetherian ring, and let $S = R[[x]]$. Let $\mathcal{C}$ denote the category of R-modules and $\mathcal{D}$ denote the category of $D(S,R)$-modules that are supported on $\mathcal{V}(xS)$, the Zariski closed subset of* Spec(S) *given by xS. Then $G : \mathcal{C} \to \mathcal{D}$ as in Definition VIII.0.7 is an equivalence of categories with inverse functor $\widetilde{G} : \mathcal{D} \to \mathcal{C}$ given by $\widetilde{G}(M) = \operatorname{Ann}_M(xS)$.*

Proof. It is clear that for every R-module M, $\widetilde{G}(G(M))$ is naturally isomorphic to M. It suffices to prove that for every $D(S,R)$-module N with support on $\mathcal{V}(xS)$, $G(\widetilde{G}(N))$ is naturally isomorphic to N. Let $M = \widetilde{G}(N) = \operatorname{Ann}_N(xS)$, and let $\phi : G(M) \to N$ be the morphism of R-modules defined on simple tensors by $m \otimes x^{-\alpha} \mapsto \frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}m$. We will prove, in steps, that ϕ is an isomorphism of $D(S,R)$-modules.

First, we will show that ϕ is a morphism of $D(S,R)$-modules. Since $D(S,R) = S\langle \frac{1}{t!}\frac{\partial^t}{\partial x^t} \mid t \in \mathbb{N}\rangle$, it is enough to show that ϕ commutes with multiplication by x and by any $\frac{1}{t!}\frac{\partial^t}{\partial x^t}$.

We first prove commutativity with $\frac{1}{t!}\frac{\partial^t}{\partial x^t}$. For any $t \in \mathbb{N}$,

$$\begin{aligned}
\phi\left(\frac{1}{t!}\frac{\partial^t}{\partial x^t}(m \otimes x^{-\alpha})\right) &= \phi\left(\binom{\alpha+t-1}{t}\left((-1)^t m \otimes x^{-\alpha-t}\right)\right) \\
&= \binom{\alpha+t-1}{t}\frac{(-1)^{\alpha-1}}{(\alpha+t-1)!}\frac{\partial^{\alpha+t-1}}{\partial x^{\alpha+t-1}}m \\
&= \frac{1}{t!}\frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha+t-1}}{\partial x^{\alpha+t-1}}m \\
&= \frac{1}{t!}\frac{\partial^t}{\partial x^t}\left(\frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}m\right) \\
&= \frac{1}{t!}\frac{\partial^t}{\partial x^t}\phi(m \otimes x^{-\alpha}),
\end{aligned}$$

which is sufficient.

We now prove that the morphism commutes with x. Note that

$$x\frac{1}{t!}\frac{\partial^t}{\partial x^t} - \frac{1}{t!}\frac{\partial^t}{\partial x^t}x = -\frac{1}{(t-1)!}\frac{\partial^{t-1}}{\partial x^{t-1}}$$

as differential operators for every $t \in \mathbb{N}$. We conclude that

$$\begin{aligned}
\phi(x(m \otimes x^{-\alpha})) &= \phi(m \otimes x^{-\alpha+1}) \\
&= \phi\left(m \otimes \frac{(-1)^{\alpha-2}}{(\alpha-2)!}\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}x^{-1}\right) \\
&= \frac{(-1)^{\alpha-2}}{(\alpha-2)!}\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\phi(m \otimes x^{-1}) \qquad \text{(VIII.0.10.1)} \\
&= x\frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\phi(m \otimes x^{-1}) - \frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}x\phi(m \otimes x^{-1}) \\
&= x\frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\phi(m \otimes x^{-1}) \\
&= x\phi(m \otimes \frac{(-1)^{\alpha-1}}{(\alpha-1)!}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}x^{-1}) \qquad \text{(VIII.0.10.2)} \\
&= x\phi(m \otimes x^{-\alpha}),
\end{aligned}$$

where (VIII.0.10.1) and (VIII.0.10.2) are due to the commutativity of $\frac{1}{t!}\frac{\partial^t}{\partial x^t}$.

It remains to prove that ϕ is bijective; we proceed by contradiction. Suppose that there exists $u = m_\ell \otimes x^{-\alpha_\ell} + \ldots + m_1 \otimes x^{-\alpha_1} \in \operatorname{Ker}(\phi)$ such that $m_\ell \neq 0$. Then $\phi(m_\ell \otimes x^{-1}) = \phi(x^{\ell-1}u) = x^{\ell-1}\phi(u) = 0$. Thus, $m_\ell = 0$ because $\phi|_{M\otimes Rx^{-1}}$ is bijective, and we get a contradiction.

We now see that $\phi(\operatorname{Ann}_{G(M)}(x^jS)) = \operatorname{Ann}_N(x^jS)$ for every $j \geq 1$ by induction, which will imply that ϕ is surjective (since N is supported on $\mathcal{V}(xS)$). Since $\phi(\operatorname{Ann}_{G(M)}(x^jS)) \subseteq \operatorname{Ann}_N(x^jS)$ for all j, we seek the opposite inclusion. For $j = 1$, take $n \in M = \operatorname{Ann}_N(xS)$; then $n \otimes x^{-1} \in G(M)$, so $\phi(n \otimes x^{-1}) = n$. Now take any $j \geq 1$ and assume the statement holds for $j-1$. For any $u \in \operatorname{Ann}_N(x^jS)$, $xu \in \operatorname{Ann}_N(x^{j-1}S)$, so $xu = \phi(v)$ for some $v = m_{j-1} \otimes x^{-j+1} + \ldots + m_1 \otimes x^{-1} \in G(M)$ by the inductive hypothesis. Let $w = m_{j-1}x^{-j} + \ldots + m_1 \otimes x^{-2}$. Thus, $x\phi(w) = \phi(xw) = \phi(v) = xu$. This means that $x(\phi(w) - u) = 0$, and so $\phi(w) - u \in \operatorname{Ann}_N(xS) = \phi(\operatorname{Ann}_{G(M)}(xS))$ and $\phi(m' \otimes x^{-1}) = \phi(w) - u$ for some $m' \in M$ by the base case. Therefore, $u = \phi(w - m \otimes x^{-1}) \in \phi(\operatorname{Ann}_{G(M)}(x^jS))$. □

Proposition VIII.0.11. *Let R be a Noetherian ring, and let $S = R[[x]]$. Then M is a finitely generated R-module if and only if $G(M)$ is a finitely generated $D(S,R)$-*

module.

Proof. Given $m_1, \dots, m_s \in M$, generators for M as R-module,

$$m_1 \otimes x^{-1}, \dots, m_s \otimes x^{-1}$$

generate $G(M)$ as a $D(S,R)$-module: by (VIII.0.9), for $\beta \in \mathbb{N}$,

$$m_i \otimes x^{-\beta} = \frac{(-1)^{\beta-1}}{(\beta-1)!} \frac{\partial^{\beta-1}}{\partial x^{\beta-1}} (m_i \otimes x^{-1}),$$

and the set $\{m_i \otimes x^{-\beta} \mid 1 \leq i \leq s, \beta \in \mathbb{N}\}$ generates $G(M)$ as an R-module.

If $u_1, \dots, u_s \in G(M)$ is a set generators for $G(M)$ as a $D(S,R)$-module, then each u_i can be written as $u_i = m_{i,1} \otimes x^{-1} + m_{i,2} \otimes x^{-2} + \ldots + m_{i,\ell_i} \otimes x^{-\ell_i}$ for some $\ell_i \in \mathbb{N}$ and $m_{i,j} \in M$. Then $\{m_{i,j} \otimes x^{-j} \mid 1 \leq i \leq s, 1 \leq j \leq \ell_i\}$ is also a set of generators for $G(M)$ as a $D(S,R)$-module. Since $m_{i,j} \otimes x^{-j} = \frac{(-1)^{j-1}}{(j-1)!} \frac{\partial^{j-1}}{\partial x^{j-1}} (m_{i,j} \otimes x^{-1})$, the decomposition in (VIII.0.8.1) implies that the $m_{i,j}$ must generate M. □

Corollary VIII.0.12. *Let R be a Noetherian ring, M an R-module, and $S = R[[x]]$. Then* $\operatorname{length}_R(M) = \operatorname{length}_{D(S,R)} G(M)$.

Proof. If M is a simple nonzero R-module, then $G(M)$ is a simple $D(S,R)$-module since the $D(S,R)$-submodules of $G(M)$ correspond precisely to R-submodules of M by Theorem VIII.0.10. Now say that $\operatorname{length}_R(M) = h < \infty$, so that we have a filtration of R-modules $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_h = M$ such that each M_{j+1}/M_j is a simple R-module. Then $0 = G(M_0) \subseteq G(M_1) \subseteq \ldots \subseteq G(M_h) = G(M)$ is a filtration of $D(S,R)$-modules such that $G(M_{j+1})/G(M_j) \cong G(M_{j+1}/M_j)$ is a simple $D(S,R)$-module for every j by our initial argument. Therefore, $\operatorname{length}_{D(S,R)}(G(M)) = h$. Similarly, if $\operatorname{length}_R(M) = \infty$, then $\operatorname{length}_{D(S,R)}(G(M)) = \infty$. □

Remark VIII.0.13. In the following work, we often make use of the following observation: for R a ring and $S = R[[x]]$, if P is a prime ideal of R, then $(P,x)S$ is a prime ideal of S since $S/(P,x)S = R/P$ is a domain.

Proposition VIII.0.14. *Let R be a Noetherian ring, M an R-module, and $S = R[[x]]$. Then* $\operatorname{Ass}_S G(M) = \{(P,x)S \mid P \in \operatorname{Ass}_R M\}$.

Proof. Let $Q \in \operatorname{Ass}_S G(M)$, so that $Q = \operatorname{Ann}_S u$ for some $u \in G(M)$. As $H^0_{xS}(G(M)) = G(M)$, $x \in Q$. Thus, $u \in \operatorname{Ann}_{G(M)} xS \cong M$ (the isomorphism is due to Theorem VIII.0.10). Moreover, we have the natural epimorphism $R \twoheadrightarrow S/Q$ with kernel $P = \operatorname{Ann}_R u \in \operatorname{Ass}_R M$. Thus, $Q = (P,x)S$.

Take $Q = (P, x)S$, where $P = \operatorname{Ann}_R u \in \operatorname{Ass}_R M$, $u \in M$. Therefore $Q = \operatorname{Ann}_S(u \otimes x^{-1})$. Hence, $Q \in \operatorname{Ass}_S G(M)$. □

Lemma VIII.0.15. *Let R be a Noetherian ring, M an R-module, and $S = R[[x]]$. Then for every ideal $I \subseteq R$ and all $j \in \mathbb{N}$, $G\left(H^j_I(M)\right) = H^{j+1}_{(I,x)S}(M \otimes_R S)$.*

Proof. Since S and S_x are flat R-algebras and S_x/S is a free R-module, we know that $H^j_I(M) \otimes_R S = H^j_{IS}(M \otimes_R S)$, $H^j_I(M) \otimes_R S_x = H^j_{IS}(M \otimes_R S_x)$ and $H^j_I(M) \otimes_R S_x/S = H^j_{IS}(M \otimes_R S_x/S)$. Moreover, the sequence

$$0 \to H^j_{IS}(M \otimes_R S) \to H^j_{IS}(M \otimes_R S_x) \to H^j_{IS}(M \otimes_R S_x/S) \to 0 \tag{VIII.0.15.1}$$

is exact, so $G(H^j_I(M)) = H^j_{IS}(M \otimes_R S_x)/H^j_{IS}(M \otimes_R S)$.

On the other hand, we have a long exact sequence

$$\cdots \to H^j_{(I,x)S}(M \otimes_R S) \to H^j_{IS}(M \otimes_R S) \to H^j_{IS}(M \otimes_R S_x) \to \cdots.$$

Since $H^j_{IS}(M \otimes_R S) \to H^j_{IS}(M \otimes_R S_x)$ is injective by (VIII.0.15.1), the long sequence splits into short exact sequences

$$0 \to H^j_{IS}(M \otimes_R S) \to H^j_{IS}(M \otimes_R S_x) \to H^{j+1}_{(I,x)S}(M \otimes_R S) \to 0.$$

Hence, $G\left(H^j_I(M)\right) = H^{j+1}_{(I,x)S}(M \otimes_R S)$. □

Proposition VIII.0.16. *Let (R, m, K) be a Noetherian local ring, M an R-module, and $S = R[[x]]$. Fix $I_1, \ldots, I_s$ ideals of R and $j_1, \ldots j_s \in \mathbb{N}$. Then*

$$G\left(H^{j_s}_{I_s} \cdots H^{j_2}_{I_2} H^{j_1}_{I_1}(M)\right) \cong H^{j_s}_{(I_s,x)S} \cdots H^{j_2}_{(I_2,x)S} H^{j_1+1}_{(I_1,x)S}(M \otimes_R S).$$

Proof. We proceed by induction on s. If $s = 1$, the statement follows from Lemma VIII.0.15. Suppose it holds for some $s \geq 1$. Let $N_\ell = H^{j_\ell}_{I_\ell} \ldots H^{j_2}_{I_2} H^{j_1}_{I_1}(M)$ for $1 \leq \ell \leq s+1$, so we need to prove that $G(N_{s+1}) \cong H^{j_{s+1}}_{(I_{s+1},x)S}(G(N_s))$. Now,

$$G(N_{s+1}) = H^{j_{s+1}}_{I_{s+1}}(N_s) \otimes_R S_x/S \cong H^{j_{s+1}}_{I_{s+1}S}(N_s \otimes_R S_x/S) = H^{j_{s+1}}_{I_{s+1}S}(G(N_s)).$$

Consider the long exact sequence of functors

$$\ldots \to H^{j_{s+1}}_{I_{s+1}S}(-) \to H^{j_{s+1}}_{(I_{s+1},x)S}(-) \to H^{j_{s+1}}_{I_{s+1}S}(- \otimes_S S_x) \to \ldots. \tag{VIII.0.16.1}$$

Since $G(N_s)$ is supported on $\mathcal{V}(xS)$, $H^i_{I_{s+1}S}(G(N_s) \otimes_S S_x) = 0$ for all $i \in \mathbb{N}$, and

$G(N_s) \otimes_S S_x = 0$. Moreover, $H^{j_{s+1}}_{I_{s+1}S}(G(N_s)) \cong H^{j_{s+1}}_{(I_{s+1},x)S}(G(N_s))$. Hence, $G(N_{s+1}) \cong H^{j_{s+1}}_{(I_{s+1},x)S}(G(N_s))$. □

As G is an equivalence of categories, $G(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_{D(S,R)}(G(N), G(M))$. Thus, M is an injective R-module if and only if $G(M)$ is an injective object in $\mathcal{D}$, the category of $D(S,R)$-modules supported at $\mathcal{V}(xS)$. We now characterize precisely when $G(M)$ is injective as an S-module:

Proposition VIII.0.17. *Let $S = R[[x]]$, where R is a Gorenstein ring. Given a prime ideal P of R, let $E_R(R/P)$ denote the injective hull of R/P over R. Then $G(E_R(R/P)) = E_S(S/(P,x)S)$. Moreover, M is an injective R-module if and only if $G(M)$ is an injective S-module.*

Proof. Let $d = \dim(R_P)$. Since R is a Gorenstein ring, S_x/S a flat R-module, and $G(H^d_P(R)) \cong H^{d+1}_{(P,x)S}(S)$ by Lemma VIII.0.15, we have that

$$G(E_R(R/P)) \cong G(H^d_{PR_P}(R_P)) \cong G(H^d_P(R) \otimes_R R_P) \cong G(H^d_P(R)) \otimes_R R_P \cong H^{d+1}_{(P,x)S}(S_P).$$

As $S_P/(P,x)S_P \cong R_P/PR_P$, $(P,x)S_P$ is a maximal ideal of the Gorenstein ring S_P, so

$$H^{d+1}_{(P,x)S}(S_P) = E_{S_P}\left(S_P/(P,x)S_P\right) = E_S\left(S/(P,x)S\right).$$

Therefore, $G\left(E_R(R/P)\right) = E_S\left(S/(P,x)S\right)$. Moreover, G sends injective R-modules to injective S-modules because every injective R-module is a direct sum of injective hulls of prime ideals.

It remains to prove that if $G(M)$ is an injective S-module, then M is an injective R-module. This follows because $M = \mathrm{Ann}_{G(M)}(xS)$ by Theorem VIII.0.10: any injection of R-modules $\iota : N \hookrightarrow N'$ is also an injection of S-modules, where x acts by zero. Then any S-module map $f : N \to G(M)$ is an R-module map and must have image in $\mathrm{Ann}_{G(M)}(xS) = M$, so the induced map $g : N \to M$ is a map of R-modules such that $f = g \circ \iota$. □

Proposition VIII.0.18. *Let R be a Gorenstein ring, and let $S = R[[x]]$. Since $R = S/xS$, every R module has an structure of S-module via extension of scalars. For R-modules M, N and $i, j \in \mathbb{N}$,*

$$\mathrm{Ext}^i_S(M, G(N)) = \mathrm{Ext}^i_R(M, N).$$

Proof. Let $E^* = E^0 \to E^1 \to \ldots \to E^i \to \ldots$ be an injective R-resolution of N. Then $G(E^*)$ is an injective S-resolution for $G(N)$ by Proposition XI.2.9. We notice that $\operatorname{Hom}_S(M, -) = \operatorname{Hom}_S(M, \operatorname{Hom}_S(R, -))$ as functors. Then

$$\operatorname{Hom}_S(M, G(E^*)) = \operatorname{Hom}_S(M, \operatorname{Hom}_S(R, G(E^*)) = \operatorname{Hom}_S(M, E^*) = \operatorname{Hom}_R(M, E^*),$$

and the result follows. □

Corollary VIII.0.19. *Let (R, m, K) be a Gorenstein local ring, and let S denote $R[[x_1, \ldots, x_n]]$. For every ideal I of R and all $i, j \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^i_S(K, H^{j+n}_{(I,x_1,\ldots,x_n)S}(S)) = \dim_K \operatorname{Ext}^i_R(K, H^j_I(R)).$$

Proof. Using Lemma XI.1.1, apply induction on n. □

CHAPTER IX

Generalized Lyubeznik numbers

The aim of this chapter is to define and study a family of invariants of a local ring containing a field. This family includes the Lyubeznik numbers, but captures finer information. These new invariants are defined in terms of lengths of certain local cohomology modules in a category of D-modules.

To prove that these *generalized Lyubeznik numbers* are well defined, we formalize and develop the theory of a functor that Lyubeznik utilized to show that his original invariants are well defined [Lyu93]. In particular, the definition of these new invariants relies heavily on the fact that this functor gives a category equivalence with a certain category of D-modules. As a consequence of this new approach, our work also gives a different proof that the original Lyubeznik numbers are well defined.

Some properties analogous to those of the original invariants hold for the generalized Lyubeznik numbers; however, results on curves and on hypersurfaces show that, unlike the original invariants, the generalized Lyubeznik numbers can differentiate one-dimensional rings, and complete intersection rings.

We compute the generalized Lyubeznik numbers associated to monomial ideals as certain lengths in a category of straight modules, and in characteristic zero, with characteristic cycle multiplicities as well. The study of the generalized Lyubeznik numbers associated to certain determinantal ideals provides further examples of these new invariants, some striking.

The results presented in this chapter are part of joint work with Witt [NBW12a].

IX.1 Definitions and first properties

Theorem IX.1.1. *Let K be a field, let $R = K[[x_1, \ldots, x_n]]$, and let $S = R[[x_{n+1}]]$. Let $\mathcal{C}$ denote the category of $D(R, K)$-modules, and let $\mathcal{D}$ denote the category of $D(S, K)$-modules that are supported on $\mathcal{V}(xS)$. Then*

(i) *$G : \mathcal{C} \to \mathcal{D}$ given by $G(M) = M \otimes_R S_{x_{n+1}}/S$ is an equivalence of categories with inverse $\widetilde{G} : \mathcal{D} \to \mathcal{C}$, where $\widetilde{G}(N) = \operatorname{Ann}_N(xS)$,*

(ii) *M is a finitely generated $D(R, K)$-module if and only if $G(M)$ is a finitely generated $D(S, K)$-module, and*

(iii) $\operatorname{length}_{D(R,K)} M = \operatorname{length}_{D(S,K)} G(M)$.

Proof. The proofs of the statements are analogous to the those of Theorem VIII.0.10, Proposition VIII.0.11, and Corollary VIII.0.12, respectively. □

Remark IX.1.2. For a local ring (R, m, K), we say that a field K' is a *coefficient field* of R if K' contained in R, and the composition $K' \hookrightarrow R \twoheadrightarrow R/m = K$ is an isomorphism of fields. Every complete local ring containing a field has a coefficients field by the Cohen Structure Theorems [Coh46].

Theorem IX.1.3. *Let (R, m, K) be a local ring containing a field, and $\widehat{R}$ its completion at m. Let K' be a coefficient field of $\widehat{R}$. Then $\widehat{R}$ admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where $S = K[[x_1, \ldots, x_n]]$ for some $n \in \mathbb{N}$, and $\pi(K) = K'$. For $1 \leq i \leq s$, fix $j_i \in \mathbb{N}$ and ideals $I_i \subseteq R$, and let $J_i = \pi^{-1}(I_i\widehat{R}) \subseteq S$. Then*

$$\operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S)$$

is finite and depends only on R, K', $I_1, \ldots, I_s$ and $j_1, \ldots, j_s$, but neither on S nor on π.

Proof. We may assume without loss of generality that R is complete. We know that $\operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \ldots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S)$ is finite by [Lyu00a, Corollary 6]. Let $\pi' : S' \to R$ be another surjection, where $S' = K[[y_1, \ldots, y_{n'}]]$. Let $J'_1, \ldots, J'_s$ be the corresponding preimages of $I_1, \ldots, I_s$ in S'.

Let $S'' = K[[z_1, \ldots, z_{n+n'}]]$. Let $\pi'' : S'' \to R$ be the surjection defined by $\pi''(K) = K'$, $\pi''(z_j) = \pi(x_j)$ for $0 \leq j \leq n$ and $\pi''(z_j) = \pi'(y_{j-n})$ for $n + 1 \leq j \leq n + n'$. Let $J''_1, \ldots, J''_s$ be the corresponding preimages of $I_1, \ldots, I_s$ in S'' under π''. Let $\alpha : S \to S''$ be the map defined by $\alpha(x_j) = z_j$. We note that $\pi''\alpha = \pi$. There exist $f_1, \ldots, f_{n'} \in S$ such that $\pi''(z_{n+j}) = \pi(f_j)$ for $j \leq n'$. Then $z_{n+j} - \alpha(f_j) \in \operatorname{Ker}(\pi'')$. We note that $\beta : S'' \to S$ defined by sending $z_j \to x_j$ for $j \leq n$ and $z_{n+j} \to f_j$ for $j \leq n'$ is an splitting of α. Then $J''_i = (\alpha(J_i), z_{n+1} - \alpha(f_1), \ldots, z_{n'+n} - \alpha(f_{n'}))S''$. Since

$$z_1, \ldots, z_n, z_{n+1} - \alpha(f_1), \ldots, z_{n'+n} - \alpha(f_{n'})$$

form a regular system of parameters, we obtain that

$$\operatorname{length}_{D(S'',K)} H^{j_s}_{J''_s} \dots H^{j_2}_{J''_2} H^{n'+n-j_1}_{J''_1}(S'') = \operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \dots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S)$$

by Proposition VIII.0.16 and Theorem IX.1.1. Similarly,

$$\operatorname{length}_{D(S'',K)} H^{j_s}_{J''_s} \dots H^{j_2}_{J''_2} H^{n'+n-j_1}_{J'_1}(S'') = \operatorname{length}_{D(S',K)} H^{j_s}_{J'_s} \dots H^{j_2}_{J'_2} H^{n'-j_1}_{J'_1}(S'),$$

and the result follows. □

Definition IX.1.4 (Generalized Lyubeznik numbers)**.** Let (R, m, K) be a local ring containing a field, and $\widehat{R}$ its completion at m. Let K' be a coefficient field of $\widehat{R}$. Then $\widehat{R}$ admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where $S = K[[x_1, \dots, x_n]]$ for some $n \in \mathbb{N}$, and $\pi(K) = K'$. For $1 \leq i \leq s$, fix $j_i \in \mathbb{N}$ and ideals $I_i \subseteq R$, and let $J_i = \pi^{-1}(I_i\widehat{R}) \subseteq S$. Then the *generalized Lyubeznik number of R with respect to K', $I_1, \dots, I_s$ and $j_1, \dots, j_s$,*

$$\lambda^{j_s,\dots,j_1}_{I_s,\dots,I_1}(R; K') := \operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-i_1}_{J_1}(S),$$

is finite and depends only on R, K', $I_1, \dots, I_s$ and $j_1, \dots, j_s$, but neither on S nor on π (by Theorem IX.1.3).

If $\widehat{R}$ contains only one coefficient field, or if the election of coefficient field is clear in the context, we simply use $\lambda^{i_s,\dots,i_1}_{I_s,\dots,I_1}(R)$ to denote this invariant.

Remark IX.1.5. In Definition IX.1.4 (and Theorem IX.1.3), we rely on a choice of coefficient field $K' \subseteq \widehat{R}$. In some cases there is only one of such field; for instance, if K is a perfect field of characteristic $p > 0$.

In general, to decide whether it is possible to avoid the generalized Lyubeznik numbers' dependence on the choice of coefficient field of $\widehat{R}$, we would need to answer the following question asked by Lyubeznik.

Question IX.1.6 (Lyubeznik)**.** Let S be a complete regular local ring of equal characteristic. For $1 \leq i \leq s$, fix $j_i \in \mathbb{N}$ and ideals $J_i \subseteq S$. Given any two coefficient fields of S, K and L, is

$$\operatorname{length}_{D(S,K)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S) = \operatorname{length}_{D(S,L)} H^{j_s}_{J_s} \cdots H^{j_2}_{J_2} H^{n-j_1}_{J_1}(S)?$$

The answer is currently unknown even when $s = 1$.

Remark IX.1.7. In Definition IX.1.4, we may assume that $I_1 \subseteq \ldots \subseteq I_s$, because if an R-module M is such that $H^0_I(M) = M$ for some ideal I of R, then $H^i_J(M) = H^i_{I+J}(M)$ for every ideal J of S. In addition, $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R, K') = \lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_2,0}(R/I_1, K')$.

Proposition IX.1.8. *If (R, m, K) is a local ring containing a field, then $\lambda_{i,j}(R) = \lambda^{i,j}_{m,0}(R; K')$ for any coefficient field K' of $\widehat{R}$.*

Proof. Since completion is flat and the Bass numbers are not affected by completion, we may assume that R is complete. Take $S = K[[x_1, \ldots, x_n]]$ such that there exist a surjective ring map $\pi : S \twoheadrightarrow R$ such that $\pi(K) = K'$. Set $I = Ker(\pi)$, the preimage of the zero ideal in R. We notice that the maximal ideal, η, of S is the preimage of the maximal ideal, m, of R. By [Lyu93, Lemma 1.4],

$$\lambda_{i,j}(R) = \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S)) = \dim_K \operatorname{Hom}_S(K, H^i_\eta H^{n-j}_I(S)).$$

Since $H^i_\eta H^{n-j}_I(S)$ is isomorphic to a finite direct sum of copies of $E_S(K)$ by [Lyu93, Corollary 3.6], and $E_S(K)$ is a simple $D(S, K)$-module (cf. [Lyu00c]), we obtain that

$$\dim_K \operatorname{Hom}_S(K, H^i_\eta H^{n-j}_I(S)) = \operatorname{length}_{D(S,K)} H^i_\eta H^{n-j}_I(S) = \lambda^{i,j}_{m,0}(R; K'),$$

and we are done. □

Remark IX.1.9. In characteristic zero, Àlvarez Montaner introduced a family of invariants using the multiplicities of the characteristic cycle of local cohomology modules [ÀM04]. Like ours, this family includes the original Lyubeznik numbers; however, this definition does not include rings of prime characteristic.

Proposition IX.1.10. *Given ideals $I_1 \subseteq \ldots \subseteq I_s$ of a local ring (R, m, K) containing a field, $i_j \in \mathbb{N}$ for $1 \le j \le s$, and a coefficient field K' of $\widehat{R}$, we have that*

(i) $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R; K') = 0$ *for* $i_1 > \dim(R/I_1)$,

(ii) $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R; K') = 0$ *for* $i_j > \dim(R/I_{j-1})$ *and* $2 \le j \le \ell$,

(iii) $\lambda^{i_2,i_1}_{I_2,I_1}(R; K') = 0$ *for* $i_2 > i_1$,

(iv) $\lambda^{i_1}_{I_1}(R; K') \neq 0$ *for* $i_1 = \dim(R/I_1)$, *and*

(v) $\lambda^{i_2,i_1}_{I_2,I_1}(R; K') \neq 0$ *if* $i_2 = \dim(R/I_1) - \dim(R/I_2)$ *and* $i_1 = \dim(R/I_1)$.

Proof. We may assume that R is complete, so that it admits a surjective ring map $\pi : S \twoheadrightarrow R$, where $S = K[[x_1, \ldots, x_n]]$ for some n and $\pi(K) = K'$.

Let $J_j = \pi^{-1}(I_j)$ for $1 \le j \le s$.

As S is Cohen-Macaulay, $\operatorname{depth}_{J_1}(S) = \operatorname{codim}(S/J_1) = n - \dim(S/J_1) = n - \dim(R/I_1)$. We have that (i) and (iv) hold because $H^{i_1}_{J_1}(S) = 0$ if $i < \operatorname{depth}_{J_1}(S)$ and $H^{\operatorname{depth}_I(S)}_{J_1}(S) \neq 0$.

To see (ii), note that

$$\begin{aligned}\operatorname{inj.dim} H^{i_{j-1}}_{J_{j-1}} \dots H^{i_2}_{J_2} H^{n-i_1}_{J_1}(S) &\leq \dim(\operatorname{Supp} H^{i_{j-1}}_{J_{j-1}} \dots H^{i_2}_{J_2} H^{n-i_1}_{J_1}(S)) \\ &\leq \dim(S/J_{j-1}) = \dim(R/I_{j-1})\end{aligned}$$

by [Lyu00a]. Similarly, (iii) follows because

$$\operatorname{inj.dim} H^{n-i_1}_{J_1}(S) \leq \dim(\operatorname{Supp} H^{n-i_1}_{J_1}(S)) \leq i_1.$$

To prove (v), choose a minimal prime P of J_2. Now, $\operatorname{Rad}(J_1 S_P) = PS_P$ in S_P. Then $H^p_{PS_P} H^{\dim(S_P)-q}_{J_1 S_P}(S_P) \neq 0$ when $p = q = \dim(S_P/J_1 S_P)$ by [Lyu93, Property 4.4(iii)]. Noting that

$$\begin{aligned}&\dim(S_P) = \dim(S) - \dim(S/P) = \dim(S) - \dim(S/J_1) = n - \dim(R/I_1), \text{ and} \\ &\dim(S_P/J_1 S_P) = \dim(S/J_1) - \dim(S/J_2) = \dim(R/I_1) - \dim(R/I_2),\end{aligned}$$

we see that $H^{i_2}_{J_2} H^{i_1}_{J_1}(S) \otimes_S S_P \neq 0$ if $i_2 = \dim(R/I_1) - \dim(R/I_2)$ and $i_1 = \dim(R/I_1)$. □

Lemma IX.1.11. *Given an extension of fields $K \subseteq L$, let $R = K[[x_1, \dots, x_n]]$ and $S = L[[x_1, \dots, x_n]]$. Via $R \hookrightarrow S$, the map induced by the field extension, if M is a simple $D(R, K)$-module, then $M \otimes_R S$ is a simple $D(S, L)$-module.*

Proof. We have that $S = R \otimes_K L$ because the field extension is finite. Then $M \otimes_R S = M \otimes_K L$ and the action of $\partial \in D(S, L)$ is given by $\partial(v \otimes a) = \partial(v) \otimes a$. Let $e_1, \dots, e_h$ be a basis for L as K-vector space. If $v \in M \otimes_K L$ is not zero, then $v = w_1 \otimes e_1 + \ldots + w_h \otimes e_h$ for some $w_i \in M$, where at least one w_j is not zero. We assume that $w_1 \neq 0$, an there exist operators $\delta_j \in D(R, K)$ such that $w_j = \delta_j w_1$ because M is simple. Let $\delta = \delta_1 + \ldots \delta_h$ and $u = e_1 \dots e_h$. Then $v = \delta(w_1 \otimes a) = a\delta(w_1 \otimes 1)$. Since $v \neq 0$, $\delta(w_1) \neq 0$ and there exist $\partial \in D(S, L)$ such that $\partial \delta w_1 = w_1$. Then $u^{-1}\partial v = w_1 \otimes 1$. Therefore for every $v \in M \otimes_K L$ not zero, $v \in D(S, L) \cdot w_1 \otimes 1$ and $w_1 \otimes 1 \in D(S, L) \cdot v$. Hence, $M \otimes_K L$ is a simple $D(S, L)$-module. □

Proposition IX.1.12. *Let $K \subseteq L$ be a finite field extension, $R = K[[x_1, \dots, x_n]]$,* *and* $S = L[[x_1, \ldots, x_n]]$*. Then for all ideals* $I_1, \ldots, I_s$ *of* R *and all* $i_1, \ldots, i_s \in \mathbb{N}$*,*

$$\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R) = \lambda^{i_s,\ldots,i_1}_{I_sS,\ldots,I_1S}(S).$$

Proof. We have that $S = R \otimes_K L$ because the field extension is finite. Let

$$0 = M_1 \subsetneq \ldots \subsetneq M_\ell = H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(S)$$

be a filtration of $D(R, K)$-modules such that M_{i+1}/M_i is a simple $D(R, K)$-module. Since S is a faithfully flat R-algebra, $M_{i+1}/M_i \otimes_K L \cong (M_{i+1} \otimes_K L)/(M_i \otimes_K L)$ is a simple $D(S, L)$-module. Thus, $\lambda^{i_s,\ldots,i_1}_{I_s,\ldots,I_1}(R) = \ell = \lambda^{i_s,\ldots,i_1}_{I_sS,\ldots,I_1S}(S)$.

□

Proposition IX.1.13. *Let* $I_1, \ldots, I_\ell$ *be ideals of* $S = K[[x_1, \ldots, x_n]]$*, where* K *is a field of characteristic zero. Then* $\lambda^{i_\ell,\ldots,i_1}_{I_\ell,\ldots,I_1}(S) \leq e\left(H^{i_\ell}_{I_\ell} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(S)\right)$.

Proof. Since $H^{i_\ell}_{I_\ell} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(S)$ is a holonomic $D(S, K)$-module, the claim follows from Remark II.4.6. □

For R a one-dimensional or complete intersection ring, $\lambda_{i,j}(R) = 1$ if $i = j = \dim R$, and vanishes otherwise. However, Propositions IX.1.14 and IX.1.15 will show that the generalized Lyubeznik numbers capture finer information that can distinguish these cases.

Proposition IX.1.14. *Let* (R, m, K) *be a complete local ring containing a field such that* $\dim(R) = 1$*. Fix a coefficient field* K' *of* R *and let* $P_1, \ldots P_\ell$ *be all the minimal primes of* R*. Then*

$$\lambda^1_0(R; K') = \lambda^1_0(R/P_1; K') + \ldots + \lambda^1_0(R/P_\ell; K') + \ell - 1.$$

Proof. We proceed by induction on ℓ. Suppose $\ell = 1$, and take a surjection $\pi : S = K[[x_1, \ldots, x_n]] \twoheadrightarrow R \cong S/I$ where $I = \mathrm{Ker}(\pi)$ and $\pi(K') = K$. If P is the minimal prime of R, then $\pi^{-1}(P) = \mathrm{Rad}(I)$ is the only minimal prime of I. Then

$$\lambda^1_0(R) = \mathrm{length}_{D(S,K)}\, H^{n-1}_I(S) = \mathrm{length}_{D(S,K)}\, H^{n-1}_{\pi^{-1}(P)}(S) = \lambda^1_0(R/P).$$

Now suppose that the formula holds for $\ell - 1$. Take a surjection $\pi : S \twoheadrightarrow R \cong S/I$, where $S = K[[x_1, \ldots, x_n]]$ and $\pi(K') = K$. Let η denote the maximal ideal of S. Let $Q_i = \pi^{-1}(P_i)$, so that $\mathrm{Rad}(I) = Q_1 \cap \ldots \cap Q_\ell$. Let $J = Q_1 \cap \cdots \cap Q_{\ell-1}$.

Since $\mathrm{Rad}(J + Q_\ell) = \eta$, the Mayer-Vietoris sequence in local cohomology with respect to J and Q_ℓ gives the following exact sequence:

$$0 \to H^{n-1}_J(S) \oplus H^{n-1}_{Q_\ell}(S) \to H^{n-1}_I(S) \to H^n_\eta(S) \to 0.$$

where $H^n_{I+J}(S) \cong E_S(K)$, a simple $D(S,K)$-module (cf. [Lyu00c]). Then $\lambda^1_0(R;K')$ equals

$$\begin{aligned}
\mathrm{length}_{D(S,K)} H^{n-1}_I(S) &= \mathrm{length}_{D(S,K)} H^{n-1}_J(S) + \mathrm{length}_{D(S,K)} H^{n-1}_{Q_\ell}(S) + 1 \\
&= \lambda^1_0(S/J;K') + \lambda^1_0(S/Q_\ell;K') + 1, \text{ and inductively,} \\
&= \big(\lambda^1_0(S/Q_1;K') + \ldots + \lambda^1_0(S/Q_\ell;K') + \ell - 2\big) + \lambda^1_0(S/Q_\ell;K') + 1 \\
&= \lambda^1_0(R/P_1;K') + \ldots + \lambda^1_0(R/P_\ell;K') + \ell - 1, \text{ as } R/P_i \cong S/Q_i.
\end{aligned}$$

□

Proposition IX.1.15. *Let $S = K[[x_1, \ldots, x_n]]$, where K is a field. Let $f_1, \ldots, f_\ell \in S$ be irreducible, and $f = f_1^{\alpha_1} \cdots f_\ell^{\alpha_\ell}$, where each $\alpha_i \in \mathbb{N}$. Then*

$$\lambda^{n-1}_0(S/f) \geq \lambda^{n-1}_0(S/f_1) + \ldots + \lambda^{n-1}_0(S/f_\ell) + \ell - 1.$$

Proof. Since $H^i_I(S) = H^i_{\sqrt{I}}(S)$ for every ideal $I \subseteq S$, we may assume that $\alpha_1 = \ldots = \alpha_\ell = 1$. Our proof will be by induction on ℓ. If $\ell = 1$, it is clear. We suppose that the formula holds for $\ell - 1$ and we will prove it for ℓ. Let $g = f_1 \cdots f_{\ell-1}$. Since $f_\ell^{\alpha_\ell}, g$ form a regular sequence, we obtain the exact sequence

$$0 \to H^1_{gS}(S) \oplus H^1_{f_\ell S}(S) \to H^1_{fS}(S) \to H^2_{(g,f_\ell)S}(S) \to 0$$

by the Mayer-Vietoris sequence. Since $H^2_{(g,f_\ell)S}(S) \neq 0$, we have that $\mathrm{length}_{D(S,K)} H^2_{(g,f_\ell)S}(S) \geq 1$. Moreover,

$$\begin{aligned}
\lambda^{n-1}_0(S/fS) &= \mathrm{length}_{D(S,K)} H^1_{fS}(S) \\
&\geq \mathrm{length}_{D(S,K)} H^1_{gS}(S) + \mathrm{length}_{D(S,K)} H^1_{f_\ell S}(S) + 1 \\
&= \lambda^{n-1}_0(S/gS) + \lambda^{n-1}_0(S/f_\ell) + 1, \text{ and inductively,} \\
&\geq \lambda^{n-1}_0(S/f_1) + \ldots + \lambda^{n-1}_0(S/f_{\ell-1}S) + \ell - 2 + \lambda^{n-1}_0(S/f_\ell S) + 1 \\
&= \lambda^{n-1}_0(S/f_1S) + \ldots + \lambda^{n-1}_0(S/f_\ell S) + \ell - 1.
\end{aligned}$$

□

Definition IX.1.16 (Lyubeznik characteristic)**.** Let (R, m, K) be a local ring containing a field such that $\dim(R) = d$.

Fix a coefficient field K' of $\widehat{R}$. The *Lyubeznik characteristic of R (with respect to K')* is defined as

$$\chi_\lambda(R; K') := \sum_{i=0}^{d} (-1)^i \lambda_0^i(R; K').$$

If the choice of the coefficient field is clear, we write $\chi_\lambda(R)$.

Proposition IX.1.17. *Let I and J be ideals of a local ring (R, m, K) containing a field. For any coefficient field K' of $\widehat{R}$,*

$$\chi_\lambda(R/I; K') + \chi_\lambda(R/J; K') = \chi_\lambda(R/(I+J); K') + \chi_\lambda(R/I \cap J; K').$$

Proof. This an immediate consequence of the Mayer-Vietoris associated sequence for local cohomology with respect to I and J. □

Proposition IX.1.18. *If $I = (f_1, \dots, f_\ell)$ an ideal of $S = K[[x_1, \dots, x_n]]$, where K is a field, then*

$$\chi_\lambda(S/I) = (-1)^n \sum_{j=0}^{\ell} \sum_{1 \le i_1 < \ldots < i_j \le \ell} (-1)^j \lambda_0^{n-1}\left(S/(f_{i_1} \cdot \ldots \cdot f_{i_j})\right).$$

In particular, if $f_1, \dots, f_\ell$ form a regular sequence or if $\operatorname{char}(k) = p > 0$ and S/I is a Cohen-Macaulay ring of dimension d, then $\lambda_0^{n-\ell}(S/(f_1, \dots, f_\ell)S)$, or $\lambda_0^d(S/I)$, respectively, equals $\sum_{j=0}^{\ell} \sum_{1 \le i_1 < \ldots < i_j \le \ell} (-1)^{n-d+j} \lambda_0^{n-1}\left(S/(f_{i_1} \cdot \ldots \cdot f_{i_j})\right)$.

Proof. For brevity, let $D = D(S, K)$. By the additivity of $\operatorname{length}_D(-)$ on short exact sequences and the Čech-like complex definition of local cohomology,

$$\sum_{j=0}^{\ell} (-1)^j \operatorname{length}_D H_I^j(S) = \sum_{j=0}^{\ell} (-1)^j \sum_{1 \le i_1 < \ldots < i_j \le \ell} \operatorname{length}_D S_{f_{i_1} \cdot \ldots \cdot f_{i_j}}.$$

Moreover, the short exact sequence $0 \to S \to S_g \to H^1_{(f_{i_1} \cdot \ldots \cdot f_{i_j})}(S) \to 0$ indicates that $\operatorname{length}_D S_{f_{i_1} \cdot \ldots \cdot f_{i_j}} = \operatorname{length}_D H^1_{(f_{i_1} \cdot \ldots \cdot f_{i_j})} + 1$. The first statement then follows from a straightforward calculation from the definition of Lyubeznik characteristic using these two observations.

The statement for a regular sequence is an immediate consequence, and the final statement follows since the only nonvanishing local cohomology module is $H_I^{n-d}(S)$ by [PS73, Proposition 4.1], since S/I is Cohen-Macaulay. □

IX.2 Generalized Lyubeznik numbers of ideals generated by maximal minors

Lemma IX.2.1. *Suppose that K is a field of characteristic zero, $R = K[x_1, \ldots, x_n]$, and $S = K[[x_1, \ldots, x_n]]$. Let $f \in R$ be homogeneous. Let D_R and D_S denote $D(S, K)$ and $D(S, K)$, respectively. If for some $N \in \mathbb{N}$, $D_S \frac{1}{f^N} = S_f$, then $D_R \frac{1}{f^N} = R_f$.*

Proof. For every $r \in \mathbb{N}$, there exists $\delta = \sum\limits_{\alpha} g_\alpha \frac{\partial^\alpha}{\partial x^\alpha} \in D_S = S\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$ such that $\delta \frac{1}{f^N} = \frac{1}{f^r}$. In addition, there exist $\mu \in \mathbb{N}$ and homogeneous $h_\alpha \in R$ such that $\mu > r$ and $\frac{\partial^\alpha}{\partial x^\alpha} \frac{1}{f^N} = \frac{h_\alpha}{f^\mu}$, so $\delta \frac{1}{f^N} = \sum_\alpha g_\alpha \frac{h_\alpha}{f^\mu} = \frac{1}{f^r}$.

We have that $\sum\limits_{\alpha} g_\alpha h_\alpha = f^{\mu - r}$, and there exist homogeneous $g_{\alpha,t} \in R$ of degree t such that $g_\alpha = \sum\limits_{t=0}^{\infty} g_{\alpha,t}$. If $t_\alpha = (\mu - r)\deg(f) - \deg(h_\alpha)$, then

$$f^{\mu - r} = \sum_\alpha g_\alpha h_\alpha = \sum_\alpha \sum_{t=0}^{\infty} g_{\alpha,t} h_\alpha = \sum_\alpha g_{\alpha,t_\alpha} h_\alpha$$

because f and h_α are homogeneous polynomials.

Let $\widetilde{\delta} = \sum\limits_{\alpha} g_{\alpha,t_\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \in D_R$. Then

$$\widetilde{\delta} \frac{1}{f^N} = \sum_\alpha g_{\alpha,t_\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \frac{1}{f^N} = \sum_\alpha g_{\alpha,t_\alpha} \frac{h_\alpha}{f^\mu} = \frac{\sum_\alpha g_{\alpha,t_\alpha} h_\alpha}{f^\mu} = \frac{f^{\mu-r}}{f^\mu} = \frac{1}{f^r}.$$

Hence, $\frac{1}{f^r} \in D_R \frac{1}{f^N}$, and the result follows. □

Remark IX.2.2. The conclusion of Lemma IX.2.1 is not necessarily true if f is not a homogeneous polynomial. Let m denote the homogeneous maximal ideal of R. If $f \in R$ is any polynomial such that R_m/fR_m is a regular local ring, then even if $D(R, K)\frac{1}{f^N} \neq R_f$, we have that $D(S, K)\frac{1}{f} = S_f$.

Remark IX.2.3. Let $b_f(s)$ denote the Bernstein-Sato polynomial of $f \in R$ over R (cf. Section II.4). If $N = \max\{j \in \mathbb{N} \mid b_f(-j) = 0\}$, then $D(R, K)\frac{1}{f^{N-1}} \neq R_f$ [Wal05, Lemma 1.3]. Therefore, if $f \in R$ is homogeneous, $\text{length}_{D(S,K)} H^1_{(f)}(S) \geq 2$ by Lemma IX.2.1.

Example IX.2.4. Let $R = K[X]$ be the polynomial ring over a field K in the entries of an $r \times r$ matrix X of indeterminates, and let m denote its homogeneous maximal ideal. Let Δ denote the principal ideal of R generated by the determinant of X. If K has characteristic zero, the Bernstein-Sato polynomial of the determinant of X over

R is $b_{\det(X)}(s) = (s+1)(s+2)\cdots(s+r)$, so by Remark IX.2.3, $\lambda_0^{r^2-1}(R_m/\Delta R_m) \geq 2$. In contrast, by Remark X.1.4, if K is instead a perfect field of characteristic $p > 0$, then $\lambda_0^{r^2-1}(R_m/\Delta R_m) = 1$. In particular, even when a specific Lyubeznik number is nonzero in both characteristic zero and characteristic $p > 0$, their values may differ.

Example IX.2.5. Now let R be the polynomial ring over a field K of characteristic zero in the entries of $X = [x_{ij}]$, an $r \times s$ matrix of indeterminates, where $r < s$. Let m denote its homogeneous maximal ideal, and let I_t be the ideal generated by the $t \times t$ minors of X, and let $I = I_r$ be the ideal generated by the maximal minors of X. By [Wit12, Theorem 1.1], $H_I^{r(s-r)+1}(R) \cong E_R(K)$, $0 \neq H_I^{i_t}(R) \hookrightarrow H_I^{i_t}(R)_{I_{t+1}} \cong E_R(R/I_{t+1})$ for $i_t = (r-t)(s-r)+1, 0 \leq t < r$, and all other $H_I^i(R) = 0$. Thus, $\lambda_0^{r^2-1}(R_m/IR_m) = \lambda_{m,0}^{0,r^2-1}(R_m/IR_m) (= \lambda_{0,r^2-1}(R_m/IR_m)) = 1$, and $\lambda_{m,0}^{0,i}(R_m/IR_m) = 0$ for every $i \neq r^2-1$.

Let $i_t = (r-t)(s-r)+1$, $t > 0$, and suppose that $\lambda_{m,0}^{1,rs-i_t}(R_m/IR_m) = 0$. Let C be the cokernel of the injection $H_I^{i_t}(R) \hookrightarrow E_R(R/I_{t+1})$, so the short exact sequence $0 \to H_I^{i_t}(R) \to E_R(R/I_{t+1}) \to C \to 0$ gives rise to the long exact sequence in local cohomology:

$$0 \to H_m^0 H_I^{i_t}(R) \to H_m^0(E_R(R/I_{t+1})) \to H_m^0(C) \to H_m^0 H_I^{i_t}(R) \to H_m^1(E_R(R/I_{t+1})) \to \dots$$

Since the I_{t+1} is the only associated prime of $E_R(R/I_{t+1})$ and of $H_I^{i_t}(R)$,

$$H_m^0 H_I^{i_t}(R) = H_m^0(E_R(R/I_{t+1})) = H_m^1(E_R(R/I_{t+1})) = 0,$$

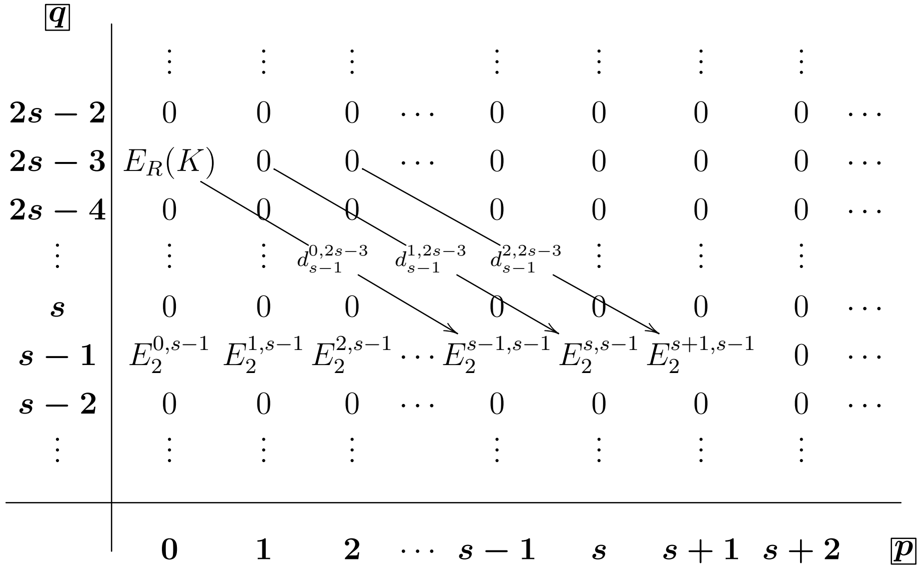
so $H_m^0(C) \cong H_m^1 H_I^{i_t}(R) = 0$.

If for some indeterminate $x_{\alpha\beta}$, the localization map $H_I^i(R) \to H_I^i(R)_{x_{\alpha\beta}}$ has a nonzero element u in the kernel, then $x_{\alpha\beta}^N \cdot u = 0$ for some N. But then, by symmetry, $x_{\alpha\beta}^N \cdot u = 0$ for *all* indeterminates $x_{\alpha\beta}$, forcing every element of $H_I^{i_t}(R)$ to be killed by a power of m, a contradiction. Similarly, the map $H_I^{i_t}(R)_{x_{11}} \to H_I^i(R)_{x_{11}\cdot x_{12}}$ is injective, and by induction, the composition of these localizations, $H_I^i(R) \to H_I^i(R)_{x_{11}x_{12}\cdot\ldots\cdot x_{rs}}$ will also be injective. In particular,

$$H_I^i(R)_{x_{\alpha\beta}} \hookrightarrow H_I^i(R)_{x_{11}\cdot x_{12}\cdot\ldots\cdot x_{rs}}, \text{ and } \bigcap_{\alpha,\beta} H_I^i(R)_{x_{\alpha\beta}} \hookrightarrow H_I^i(R)_{x_{11}\cdot x_{12}\cdot\ldots\cdot x_{rs}}.$$

Let M denote $\bigcap_{\alpha,\beta} H_I^i(R)_{x_{\alpha\beta}}$. Since $x_{\alpha\beta} \notin I_{t+1}$ and $H_I^i(R)_{I_{t+1}} \cong E_R(R/I_{t+1})$, M injects into $E_R(R/I_{t+1})$, and $M/H_I^i(R)$ injects into $E_R(R/I_{t+1})/H_I^i(R) = C$. Since every element of $M/H_I^i(R)$ is killed by a power of m, $M/H_I^i(R) = H_m^0(M/H_I^i(R)) \hookrightarrow$

Figure IX.2.6.1: $E_2^{p,q} = H_m^p H_I^q(R)$.



$H_m^0(C)) = 0$. Thus, $M = H_I^i(R)$.

Theorem IX.2.6. *Continuing with the notation above, if $r = 2$ and $s > 2$, then*

$$\lambda_{m,0}^{0,3}(R_m/IR_m) = \lambda_{m,0}^{s-1,s+1}(R_m/IR_m) = \lambda_{m,0}^{s+1,s+1}(R_m/IR_m) = 1,$$

and all other $\lambda_{m,0}^{i,j}(R_m/IR_m) = 0$. In particular, each $\lambda_{m,0}^{1,i}(R_m/IR_m) = 0$.

Proof. By [Wit12, Theorem 1.1], the only two nonzero local cohomology modules $H_I^i(R)$ are $H_I^{2s-3}(R) \cong E_R(K)$ and $H_I^{s-1}(R) \hookrightarrow E_R(R/I)$. Replace R by its localization at m, and consider the spectral sequence $E_2^{p,q} = H_m^p H_I^q(R) \overset{p}{\Longrightarrow} H_m^{p+q}(R) = E_\infty^{p,q}$ [Har77]. Now, $H_m^0 H_I^{2s-3}(R) \cong E_R(K)$ and $H_m^p H_I^{2s-3}(R) = 0$ for $p > 0$. In particular, $\lambda_{0,m}^{0,3}(R/I) = 1$. Also note that $\dim R/I = s+1$, since if a $2 \times s$ matrix has vanishing 2×2 minors, the second row is a multiple of the first row. Since $\operatorname{Ass}_R H_I^{s-1}(R) = \{I\}$, $H_m^p H_I^{s-1}(R) = 0$ for $p > s+1$. These observations are indicated in Figure IX.2.6.1.

As $H_m^{2s}(R) \cong E_R(K)$ is the only nonzero local cohomology module of R with support in m. The only possibly nonzero $E_2^{p,q} = H_m^p H_I^q(R)$ such that $p + q = 2s$ is $H_m^{s+1} H_I^{s-1}(R)$, and so, since the spectral sequence maps to and from $H_m^{s+1} H_I^{s-1}(R)$ must all be zero (since the terms from which they come or go are zero), we must have that $H_m^{s+1} H_I^{s-1}(R) \cong E_\infty^{s+1,s-1} = E_R(K)$, so that, as $\dim R - (s-1) = s+1$, $\lambda_{m,0}^{s+1,s+1}(R/IR) = 1$. Moreover, every other $E_\infty^{p,q}$ must vanish.

Since $E_{s-1}^{0,2s-3} \cong E_R(K)$, we see that the sole differential that is (possibly) nonzero is $d_{s-1}^{0,2s-3} : E_{s-1}^{0,2s-3} \cong E_R(K) \to E_{s-1}^{s-1,s-1}$. After taking cohomology with respect to the $d_{s-1}^{p,q}$ we must get zero at both the $(s-1, s-1)$ and $(0, 2s-3)$ spots, so $d_{s-1}^{0,2s-3}$ must be an isomorphism, and $H_m^{s-1} H_I^{s-1}(R) = E_{s-1}^{s-1,s-1} \cong E_R(K)$, so that, as $\dim R - (s-1) = s+1$, $\lambda_{m,0}^{s-1,s+1}(R/IR) = 1$. Since all other maps are the zero map, and after taking cohomology with respect to $d_{s-1}^{p,q}$ we must also get zero, all remaining local cohomology modules of the form $H_I^p H_m^q(R)$ must vanish (i.e., all except $p = 0$, $q = 3$, and $p = s-1$, $q = s-1$ and $p = s+1$, $q = s-1$), so that in these cases, $\lambda_{m,0}^{p,q}(R/I) = 0$. □

IX.3 Generalized Lyubeznik numbers of monomial ideals

In this section we characterize the generalized Lyubeznik numbers associated to monomial ideals. To do so, we make use of the categories of square-free and straight modules introduced by Yanagawa [Yan00, Yan01]; we begin with some definitions and notation he first introduced.

Notation IX.3.1. Let $S = K[x_1, \ldots, x_n]$, K a field, and consider the natural $\mathbb{N}^n$-grading on S. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define $\operatorname{Supp}(\alpha) = \{i \mid \alpha_i > 0\} \subseteq [n] = \{1, \ldots, n\}$. For a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\operatorname{Supp}(x^\alpha) := \operatorname{Supp}(\alpha)$. We say that x^α is *square-free* if, for every $i \in [n]$, α_i either vanishes or equals one. Let e_i denote the vector $(0, \ldots, 0, 1, 0 \ldots, 0) \in \mathbb{N}^n$, where "1" is in the i^{th} entry. If $F \subseteq [n]$, let P_F denote the prime ideal generated by $\{x_i \mid i \notin F\}$. If $F \subseteq [n]$, we will often use F instead of $\sum_{i \in F} e_i$; for instance, x^F denotes $\prod_{i \in F} x_i$.

Given a $\mathbb{Z}^n$-graded S-module M, and $\beta \in \mathbb{Z}$, $M(\beta)$ denotes the $\mathbb{N}^n$-graded S-module that has underlying S-module M, but with a shift in the grading: $M(\beta)_\alpha = M_{\alpha+\beta}$. Let $\omega_S = S(-1, \ldots, -1)$ denote the canonical module of S, and let ***Mon** denote the category of $\mathbb{Z}^n$-graded S-modules.

Definition IX.3.2 (Square-free monomial module)**.** An $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\beta \in \mathbb{N}^n} M_\beta$ is *square-free* if it is finitely generated, and the multiplication map $M_\alpha \xrightarrow{\cdot x_i} M_{\alpha+e_i}$ is bijective for all $\alpha \in \mathbb{N}$, and all $i \in \operatorname{Supp}(\alpha)$. The category of square-free S-modules is denoted **Sq**, a subcategory of ***Mon**.

If I is a square-free monomial ideal, then both I and S/I are square-free modules. Moreover, if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in ***Mon**, then M is a square-free module if and only if both M' and M'' are square-free modules. In addition, if M is a square-free module, then $\operatorname{Ext}_S^i(M, \omega_S)$ is a square-free module for every $i \in \mathbb{N}$ [Yan00]. Additionally, for any subset $G \subseteq F \subseteq [n]$, $S/P_F(-G)$ is a square-free module (where the grading of $P_F(-G)$ satisfies $[P_F(-G)]_\ell = [P_F]_{\ell-G}$).

Remark IX.3.3. An $\mathbb{N}^n$-graded square-free S-module M is a *simple square-free module* if it has no proper square-free non-trivial submodules. In fact, such a square-free module is simple if and only if it is isomorphic to $S/P_F(-F)$ for some $F \subseteq [n]$ [Yan00].

Proposition IX.3.4 ([Yan00, Proposition 2.5])**.** *An $\mathbb{N}^n$-graded S-module M is square-free if and only if there exists a filtration of $\mathbb{N}^n$-graded submodules $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_t = M$ such that, for each i $(0 \leq i \leq t-1)$, $\overline{M}_i = M_i/M_{i+1} \cong S/P_{F_i}(-F_i)$ for some $F_i \subseteq [n]$ (and so is, in particular, a simple square-free module).*

As a consequence of Proposition IX.3.4, every square-free module M has finite length in **Sq**. We now recall the following definition [Yan01].

Definition IX.3.5 (Straight module)**.** A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ is *straight* if $\dim(M_\beta) < \infty$ for all $\beta \in \mathbb{Z}^n$, and the multiplication map $M_\alpha \overset{\cdot x_i}{\to} M_{\alpha+e_i}$ is bijective for all $\alpha \in \mathbb{Z}^n$ and all $i \in \mathrm{Supp}(\alpha)$. The category of straight S-modules is denoted **Str**, a subcategory of ***Mon**.

Remark IX.3.6. If $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ is a straight module, then $\overline{M}$ denotes the $\mathbb{N}^n$-graded (square-free) submodule $\bigoplus_{\beta \in \mathbb{N}^n} M_\beta$. On the other hand, if M is a square-free module, we can define the *straight hull* of M, $\widetilde{M}$, as follows: For $\alpha \in \mathbb{N}^n$, let $\widetilde{M}_\alpha$ be a vector space isomorphic to $M_{\mathrm{Supp}(\alpha)}$, and let $\phi_\alpha : \widetilde{M}_\alpha \to M_{\mathrm{Supp}(\alpha)}$ denote such an isomorphism. Let $\beta = \alpha + e_i$ for some $i \in [n]$. If $\mathrm{Supp}(\alpha) = \mathrm{Supp}(\beta)$, we define $\widetilde{M}_\alpha \overset{\cdot x_i}{\to} \widetilde{M}_\beta$ by the composition $\widetilde{M}_\alpha \overset{\phi_\alpha}{\to} M_{\mathrm{Supp}(\alpha)} \overset{\phi_\beta^{-1}}{\to} \widetilde{M}_\beta$; otherwise, we define $\widetilde{M}_\alpha \overset{\cdot x_i}{\to} \widetilde{M}_\beta$ by the composition $\widetilde{M}_\alpha \overset{\phi_\alpha}{\to} M_{\mathrm{Supp}(\alpha)} \overset{x_i}{\to} M_{\mathrm{Supp}(\beta)} \overset{\phi_\beta^{-1}}{\to} \widetilde{M}_\beta$. Then $\widetilde{M}$ is straight, and its $\mathbb{N}^n$-graded part is isomorphic to M.

Proposition IX.3.7 ([Yan01, Proposition 2.7])**.** *Continuing with the notation above, the functor* **Str** $\to$ **Sq** *defined by $M \to \overline{M}$ is an equivalence of categories with inverse functor $N \to \widetilde{N}$.*

Remark IX.3.8. Let $L[F]$ denote the straight hull of $P_F(-F)$. By Proposition IX.3.7 (noting Remark IX.3.3), $L[F]$ is a simple straight module. We have that $L[F]_\alpha = k$ if $\mathrm{Supp}(\alpha) = F$, and is zero otherwise [Yan01]. Thus, $L[F] \cong H^\ell_{P_F}(\omega_S)$, where $\ell = n - |F|$.

Remark IX.3.9. Any straight module M may be given the structure of a $D(S, K)$-module. It suffices to define an action of $\frac{1}{t!}\frac{\partial^t}{\partial {x_i}^t}$, for every $1 \leq i \leq n$ and $t \geq 1$: Take $v \in M_\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$. If $1 \leq \alpha_i \leq t$, we define $\frac{1}{t!}\frac{\partial^t}{\partial {x_i}^t} v = 0$. Otherwise, there exist $w \in M_{\alpha - te_i}$ such that $x_i^t w = v$, and we define $\frac{1}{t!}\frac{\partial^t}{\partial {x_i}^t} v = \binom{\alpha_i}{t} w$ if $\alpha_i > 0$ and $\frac{1}{t!}\frac{\partial^t}{\partial {x_i}^t} v = (-1)^{-\alpha_i+1}\binom{-\alpha_i}{t} w$ if $\alpha_i < 0$. This observation extends in [Yan01, Remark 2.12] to any field. We note that giving this $D(S,K)$-structure gives an exact faithful functor from **Str** to the category of $D(S,K)$-modules.

Theorem IX.3.10. *Let K be a field, $S = K[x_1, \dots, x_n]$, and $\widehat{S} = K[[x_1, \dots, x_n]]$. Let $I_1, \dots, I_s \subseteq S$ be ideals generated by square-free monomials. Then*

$$\lambda^{i_s,\dots,i_1}_{I_s,\dots,I_1}(\widehat{S}) = \operatorname{length}_{\mathbf{Str}} H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(\omega_S) = \sum_{\alpha \in \{0,1\}^n} \dim_k \left[H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(\omega_S)\right]_{-\alpha}.$$

Moreover, if $\operatorname{char}(K) = 0$, *then* $\lambda^{i_1,\dots,i_s}_{I_1,\dots,I_s}(\widehat{S}) = e(H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(S))$, *where* $e(-)$ *denotes* $D(S,K)$*-module multiplicity* (see Definition II.4.5).

Proof. Let $M = H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(S)$, so that $\lambda^{i_1,\dots,i_s}_{I_1,\dots,I_s}(\widehat{S}) = \operatorname{length}_{D(\widehat{S},K)} M$. By applying [Yan01, Corollary 3.3] iteratively, we see that $H^{i_s}_{I_s} \cdots H^{i_2}_{I_2} H^{n-i_1}_{I_1}(\omega_S)$ is an straight module. By Propositions IX.3.4 and IX.3.7, there is a strict ascending filtration of $\mathbb{N}^n$-graded submodules $\widetilde{0 =} M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_t = M$ such that each quotient M_i/M_{i+1} is isomorphic to $\widetilde{P_{F_i}(-F_i)} \cong H^{n-|F_i|}_{P_{F_i}}(\omega_S)$, and is also a filtration of $D(S,K)$-modules by Remark IX.3.9. Moreover,

$$0 = M_0 \otimes_S \widehat{S} \subsetneq M_1 \otimes_S \widehat{S} \subsetneq \dots \subsetneq M_t \otimes_S \widehat{S} = M \otimes_S \widehat{S}$$

is a filtration of $D(\widehat{S},K)$-modules such that

$$\left(\widetilde{M_i} \otimes_S \widehat{S}\right) / \left(\widetilde{M}_{i-1} \otimes_S \widehat{S}\right) \cong \widetilde{P_{F_i}(-F_i)} \otimes_S \widehat{S} \cong H^{n-|F_i|}_{P_{F_i}}(\widehat{S}).$$

Since $H^{n-|F_i|}_{P_{F_i}}(\widehat{S})$ is a simple $D(\widehat{S},K)$-module for every $F \subseteq [n]$, $\operatorname{length}_{D(\widehat{S},K)} M \otimes_S \widehat{S} = t$ as well.

If K has characteristic zero, due to the filtration above and noting Remark II.4.6,

$$CC(M) = \sum_{i=1}^{t} CC\left(\widetilde{M}_i / \widetilde{M}_{i-1}\right) = \sum_{i=1}^{t} CC\left(H^{n-|F_i|}_{P_{F_i}}(S)\right),$$

where $CC(-)$ denotes the characteristic cycle (see Definition II.4.5). By [ÀM00, Corollary 3.3 and Remark 3.4], each $CC\left(H^{n-|F_i|}_{P_{F_i}}\right) = T^*_{\{x_i = 0 | x_i \in P_{F_i}\}} \operatorname{Spec}(S)$. As a result, each $e\left(H^{n-|F|}_{P_F}\right) = 1$ and so $e(M) = t$. Then $\lambda^{i_1,\dots,i_s}_{I_1,\dots,I_s}(\widehat{S}) = \operatorname{length}_{\mathbf{Sq}} \overline{M} = \operatorname{length}_{\mathbf{Str}} M = e(M)$. □

Remark IX.3.11. The Lyubeznik numbers with respect to monomial ideals may depend on the field, as shown in [ÀMV, Example 4.6].

Remark IX.3.12. For K a field of characteristic zero, let $S = K[x_1, \ldots, x_n]$, and take $I \subseteq S$ an ideal generated by monomials. Let $\widehat{S} = K[[x_1, \ldots, x_n]]$. Combining work of Álvarez Montaner in [ÀM00, Theorem 3.8 and Algorithm 1] with Theorem IX.3.10 provides an algorithm to compute $\lambda_0^i(\widehat{S}/I\widehat{S})$ in terms of $P_1, \ldots, P_N$, the minimal primes of I. A consequence of this algorithm is the following inequality:

$$\lambda_0^j(\widehat{S}/I\widehat{S}) \leq \sum_{1 \leq i_1 < \ldots < i_\ell < N} \delta^j_{i_1,\ldots,i_\ell},$$

where $\delta^j_{i_1,\ldots,i_\ell} = 1$ if $\operatorname{ht}(P_{i_1} + \ldots + P_{i_\ell}) = j + \ell - 1$, and equals zero otherwise.

Remark IX.3.13. By Corollary IX.3.12, there is a straightforward algorithm to compute the $\lambda_0^i(\widehat{S}/I\widehat{S})$ using the minimal primes of I.

Lemma IX.3.14. *Let $S = K[[x_1, \ldots, x_n]]$, K a field. For a monomial f with $|\operatorname{Supp}(f)| = j$, $\operatorname{length}_{D(S,K)} S_f = 2^j$.*

Proof. By IX.3.10, $\operatorname{length}_{D(S,K)} H^1_{(x_{i_1} \cdot \ldots \cdot x_{i_j})}(S) = 2^j - 1$. Since local cohomology is independent of radical, $H^1_f(S) = H^1_{(x_{i_1} \cdot \ldots \cdot x_{i_j})}(S) = 2^j - 1$. Due to the exact sequence $0 \to S \to S_f \to H^1_f(S) \to 0$ and the fact that S is a simple $D(S,K)$-module, we have that $\operatorname{length}_{D(S,K)} S_f = \operatorname{length}_{D(S,K)} S + \operatorname{length}_{D(S,K)} H^1_f(S) = 2^j$. □

Proposition IX.3.15. *Let K be a field, and let $S = K[[x_1, \ldots, x_n]]$, and let I be an ideal of S generated by square-free monomials $f_1, \ldots, f_\ell \in S$. Then*

$$\chi_\lambda(S/I) = (-1)^n \sum_{j=0}^{\ell} \sum_{1 \leq i_1 < \ldots < i_j \leq \ell} (-1)^j 2^{\deg \operatorname{lcm}(f_{i_1}, \ldots, f_{i_j})}.$$

Moreover, if S/I is also Cohen-Macaulay, the above equation equals $(-1)^d \lambda_0^d(S/I)$. If, further, $f_1, \ldots, f_\ell$ form a regular sequence, this equals $(-1)^{n-1} \prod_{i=1}^{\ell} (2^{\deg f_i} - 1)^\ell$.

Proof. Since $|\operatorname{Supp}(f_{i_1} \cdot \ldots \cdot f_{i_j})| = \deg \operatorname{lcm}(f_{i_1}, \ldots, f_{i_j})$, the first statement follows from Lemma IX.3.14 and Proposition IX.1.18.

If S/I is Cohen-Macaulay, then by [ÀM00, Proposition 3.1] (which is stated in characteristic zero, although the argument is characteristic independent), $H^j_I(S) = 0$ for all $j \neq \operatorname{ht} I = n - d$, and the statement follows. If the f_i also form a regular sequence, $\operatorname{lcm}(f_{i_1} \cdot \ldots \cdot f_{i_j}) = f_{i_1} \cdot \ldots \cdot f_{i_j}$ and $\deg(f_{i_1} \cdot \ldots \cdot f_{i_j}) = \sum_{r=1}^{j} \deg f_{i_r}$, and $\sum_{j=0}^{\ell} \sum_{1 \le i_1 < \ldots < i_j \le \ell} (-1)^j 2^{\left(\sum_{r=1}^{j} \deg f_{i_r}\right)} = \prod_{i=1}^{\ell}(1 - 2^{\deg f_i})^{\ell} = -\prod_{i=1}^{\ell}(2^{\deg f_i} - 1)^{\ell}$. □

IX.4 Lyubeznik characteristic of Stanley-Reisner rings

Definition IX.4.1 (Simplicial complex, faces/simplices, dimension of a face, i-face, facet)**.** A *simplicial complex* Δ on the vertex set $[n] = \{1, \ldots, n\}$ is a collection of subsets, called *faces* or *simplices*, that are closed under taking subsets. A face $\sigma \in \Delta$ of cardinality $|\sigma| = i + 1$ is said to have *dimension* i, and is called an *i-face* of Δ. The *dimension* of Δ, $\dim(\Delta)$, is the maximum of the dimensions of its faces (or $-\infty$ if $\Delta = \varnothing$). We denote the set of faces of dimension i of Δ by $F_i(\Delta)$. A face is a *facet* if it is not contained in any other face.

Remark IX.4.2. If Δ_1 and Δ_2 are simplicial complexes on the vertex set $[n]$, then $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$ are also simplicial complexes.

Definition IX.4.3 (Simple simplicial complex)**.** We say that a simplicial complex Δ on the vertex set $[n]$ is *simple* if it is equal to $\mathcal{P}(\sigma)$, the power set of a subset σ of $[n]$.

Remark IX.4.4. If $\sigma_1, \ldots, \sigma_\ell$ are the maximal facets of Δ, then $\Delta = \mathcal{P}(\sigma_1) \cup \ldots \cup \mathcal{P}(\sigma_\ell)$. In particular, a simplicial complex is determined by its facets.

Notation IX.4.5. If Δ is a simplicial complex on the vertex set $[n]$ and $\sigma \in \Delta$, then x^σ denotes $\prod_{i \in \sigma} x_i \in K[x_1, \ldots, x_n]$.

Definition IX.4.6 (Stanley-Reisner ideal of a simplicial complex)**.** The *Stanley-Reisner ideal of the simplicial complex* Δ is the square-free monomial ideal $I_\Delta = (x^\sigma \mid \sigma \notin \Delta)$ of $K[x_1, \ldots, x_n]$. The *Stanley-Reisner ring of* Δ is $K[x_1, \ldots, x_n]/I_\Delta$.

Theorem IX.4.7 ([MS05, Theorem 1.7])**.** *The correspondence $\Delta \mapsto I_\Delta$ defines a bijection from simplicial complexes on the vertex set $[n]$ to square-free monomial ideals of $K[x_1, \ldots, x_n]$. Furthermore, $I_\Delta = \bigcap_{\sigma \in \Delta} (x^{[n] \setminus \sigma})$.*

Proposition IX.4.8. *Under the correspondence in Theorem IX.4.7, $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$ and $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_1} \cap I_{\Delta_2}$ for all simplicial complexes Δ_1 and Δ_2.*

Proof. For the first statement, we see that

$$x^\sigma \in I_{\Delta_1 \cap \Delta_2} \Leftrightarrow \sigma \notin \Delta_1 \cap \Delta_2 \Leftrightarrow \sigma \notin \Delta_1 \text{ or } \sigma \notin \Delta_2$$

$$\Leftrightarrow x^\sigma \in I_{\Delta_1} \text{ or } x^\sigma \in I_{\Delta_1} \Leftrightarrow x^\sigma \in I_{\Delta_1} + I_{\Delta_2}.$$

The proof of the second statement is analogous. □

Example IX.4.9. Let $S = K[x_1, \ldots, x_6]$ and let I denote the monomial ideal of S generated by

$$x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6.$$

The simplicial complex associated to I corresponds to a minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$, and the projective algebraic set that I defines in $\mathbb{P}^5_K$ has been called *Reisner's variety* since he introduced it in [Rei76, Remark 3].

If $K = \mathbb{Q}$, then $\lambda_0^4(R) = 31$ and all other $\lambda_0^*(R)$ vanish

If $K = \mathbb{Z}/2\mathbb{Z}$, then $\lambda_0^4(R) = 32$, $\lambda_0^3(R) = 1$, and all other $\lambda_0^*(R) = 0$.

We notice that in the previos example we have $\chi_\lambda(R) = 31$ in both cases.

Theorem IX.4.10. *Take a simplicial complex Δ on the vertex set $[n]$. Let R be the Stanley-Reisner ring of Δ, and let m be its maximal homogeneous ideal. Then*

$$\chi_\lambda(R_m) = \sum_{i=-1}^{n} (-2)^{i+1} |F_i(\Delta)|.$$

Proof. Let $S = K[x_1, \ldots, x_n]$, and let η be its maximal homogeneous ideal. We proceed by induction on $d := \dim(\Delta)$. If $d = 0$, then $\Delta = \{\varnothing\}$. Then $I_\Delta = \eta$, and $R = K$, so that $\chi_\lambda(R_m) = 1 = (-2)^0 = \sum\limits_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta)|$.

Assume that the formula holds for all simplicial complexes of dimension less or equal to d. Take a simplicial complex Δ of dimension $d + 1$. Consider all its facets, $\sigma_1, \ldots, \sigma_\ell$. We now proceed by induction on ℓ. If $\ell = 1$, suppose that $\Delta_1 = \mathcal{P}(\sigma_1)$, where $\sigma_1 = \{i_1, \ldots, i_j\}$ and $\dim(\sigma_1) = j$. Then $I_{\Delta_1} = (x_i \mid i \notin \sigma_1)S$, $R \cong K[x_1, \ldots, x_{n-j}]$, and

$$\begin{aligned}\chi_\lambda(R_m) &= \operatorname{length}_{D(\widehat{S_\eta},K)} H^j_{I_{\Delta_1}}(\widehat{S_\eta}) = (-1)^j = (1-2)^j \\ &= \sum_{k=0}^{j} 1^{j-k}(-2)^k \binom{j}{k} = \sum_{k=-1}^{j-1} (-2)^{k+1} \binom{j}{k+1} = \sum_{k=-1}^{j-1} (-2)^{k+1} |F_k(\Delta)|.\end{aligned}$$

Assume that the formula is true for simplicial complexes of dimension $d+1$ with ℓ facets, and take a simplicial complex Δ of dimension $d+1$ with $\ell+1$ facets, $\sigma_1, \ldots, \sigma_\ell$. Let $\Delta_i = \mathcal{P}(\sigma_i)$ and $\Delta' = \Delta_1 \cup \ldots \cup \Delta_\ell$. Then $\Delta = \Delta' \cup \Delta_{\ell+1}$. We may assume, by renumbering, that $\dim(\Delta_\ell) = \dim(\Delta)$. Then $\dim(\Delta' \cap \Delta_\ell) < \dim(\Delta_\ell)$ by our choice of Δ_ℓ and as we chose the decomposition given by the maximal facets. Therefore $\chi_\lambda(R_m)$ equals

$$
\begin{aligned}
\chi_\lambda((S/I_{\Delta'\cup\Delta_\ell})_\eta) &= \chi_\lambda((S/I_{\Delta'} \cap I_{\Delta_\ell})_\eta) \text{ by IX.4.8} \\
&= \chi_\lambda((S/I_{\Delta'})_\eta) + \chi_\lambda((S/I_{\Delta_\ell})_\eta) - \chi_\lambda((S/(I_{\Delta'} + I_{\Delta_\ell}))_\eta) \text{ by IX.1.17} \\
&= \chi_\lambda((S/I_{\Delta'})_\eta) + \chi_\lambda((S/I_\Delta)_\ell)_\eta) - \chi_\lambda((S/(I_{\Delta'\cap\Delta_\ell}))_\eta) \text{ by. IX.4.8} \\
&= \sum_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta')| + \sum_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta_\ell)| - \sum_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta' \cap \Delta_\ell)| \\
&= \sum_{i=-1}^{n} (-2)^{i+1}(|F_i(\Delta')| + |F_i(\Delta_\ell)| - |F_i(\Delta' \cap \Delta_\ell)|) \\
&= \sum_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta' \cup \Delta_\ell)| = \sum_{i=-1}^{n} (-2)^{i+1}|F_i(\Delta)|.
\end{aligned}
$$

□

The above computation is related to work in [ÀMGLZA03].

Example IX.4.11. Let K be a field, $S = K[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring over K, and $m = (x_1, x_2, x_3, x_4, x_5)$ its maximal homogeneous ideal. Consider the ideal $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_2x_5)$ of S. Note that $R := S/I$ is the Stanley-Reisner ring of the simplicial complex is such that

$$|F_{-1}(\Delta' \cup \Delta_\ell)| = 1, |F_0(\Delta' \cup \Delta_\ell)| = 5, |F_1(\Delta' \cup \Delta_\ell)| = 5, \text{ and } |F_2(\Delta' \cup \Delta_\ell)| = 1,$$

Using Theorem IX.4.10, we get that $\chi_\lambda(R_m) = 1 \cdot 1 + (-2) \cdot 5 + 4 \cdot 5 + (-8) \cdot 1 = 3.$

Remark IX.4.12. In characteristic zero, Àlvarez Montaner has given formulas for $|F_k(\Delta)|$ in terms of the characteristic cycle multiplicities of $H^1_{I_\Delta}(K[x_1, \ldots, x_n])$ (cf. [ÀM00, Proposition 6.2])

Remark IX.4.13. Theorem IX.4.10 shows that the Lyubeznik characteristic of a Stanley-Reisner ring does not depend on its characteristic, although its Lyubeznik numbers do have such a dependence (cf. [ÀMV, Example 4.6]).

CHAPTER X

Lyubeznik numbers measure singularity

Results of Blickle [Bli04a] enable straightforward characterizations of F-regularity and F-rationality in terms of certain generalized Lyubeznik numbers.

If S is an F-finite regular local ring and $R = S/I$ is F-pure, Vassilev [Vas98] showed that there is a strictly ascending chain of ideals

$$I = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

such that $(\tau_i^{[p]} : \tau_i) \subset (\tau_{i+1}^{[p]} : \tau_{i+1})$ and τ_{i+1} is the pullback of the test ideal of S/τ_i.

Suppose that $R = S/fS$ is an F-pure hypersurface and that

$$0 \subset fS = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

is the flag of ideals previously introduced. The author and Pérez [NBP13] showed that $\ell \leq \lambda_0^{\dim(R)}(R; K')$ for every coefficient field K' of R.

Suppose that S is local, that $R = S/fS$ is F-pure, and that K is perfect. In this case, $\lambda_0^{\dim(R)}(R) = 1$ if and only if R is F-regular [Bli04a, NBW12a]. This fact and the previous theorem say that the Lyubeznik number, $\lambda_0^{\dim(R)}(R; K')$, measures how far R is from being F-regular. The main aim of this chapter is to generalize this property to all Gorestein F-pure rings (Theorem X.2.9). Moreover, we also give support that the generalized Lyubeznik numbers measure singularity for any F-pure rings by using $R\langle F\rangle$-modules (cf. Theorem X.3.1).

The results presented in this chapter are part of joint work with Hernández and Witt [HNBW13].

X.1 Relations with F-rationality and F-regularity

We recall Blickle's results [Bli04a, Theorem 4.9, Corollaries 4.10 and 4.16].

Theorem X.1.1 (Blickle)**.** *Let (S, m, K) be a regular local F-finite ring of characteristic $p > 0$. Let I be an ideal such that $R = S/I$ is a domain of dimension d and codimension c. Then $H^c_I(S)$ is a simple $D(S, \mathbb{Z})$-module if and only if $0^*_{H^d_m(R)}$ is F-nilpotent. As consequences,*

(1) *If R is F-rational, then $H^c_I(S)$ is a simple $D(S, \mathbb{Z})$-module. If R is F-injective, then R is F-rational if and only if $H^c_I(S)$ is a simple $D(S, \mathbb{Z})$-module.*

(2) *If $d = 1$, then $H^c_I(S)$ is a simple $D(S, \mathbb{Z})$-module if and only if R is unibranch.*

These results indicate that the generalized Lyubeznik numbers detect F-regularity and F-rationality, as we see in the following proposition.

Proposition X.1.2. *Let (R, m, K) be a complete local domain of characteristic $p > 0$ and of dimension d, such that K is F-finite. For any coefficient field K' of R, the following hold.*

(i) *If $\lambda^d_0(R; K') = 1$, then $0^*_{H^d_m(R)}$ is F-nilpotent.*

(ii) *If R is F-injective and $\lambda^d_0(R; K') = 1$, then R is F-rational.*

In addition, if K is perfect, then:

(iii) *$\lambda^d_0(R) = 1$ if and only if $0^*_{H^d_m(R)}$ is F-nilpotent.*

(iv) *If R is F-rational, then $\lambda^d_0(R) = 1$.*

(v) *If R is F-injective, then $\lambda^d_0(R) = 1$ if and only if R is F-rational.*

Moreover, if R is one-dimensional, we have that:

(vi) *If $\lambda^d_0(R; K') = 1$, then R is unibranch.*

(vii) *If K is perfect, then $\lambda^d_0(R) = 1$ if and only if R is unibranch.*

Proof. Take any surjective ring map $\pi : S \twoheadrightarrow R$, where $S = K[[x_1, \dots, x_n]]$ and $\pi(K) = K'$, and let $I = \mathrm{Ker}(\pi)$. Since $D(S, \mathbb{Z}) \subseteq D(S, K)$, $\mathrm{length}_{D(S,K)} H^{n-d}_I(S) = \lambda^d_0(R; K') = 1$ implies that $H^{n-d}_I(S)$ is a simple $D(S, \mathbb{Z})$-module. Then (i) and (ii) are consequences of the main statement and part (1) of Theorem X.1.1, respectively.

If K is perfect, $D(S,\mathbb{Z}) = D(S,K)$ by [Yek92], so $1 = \lambda_0^d(R)$ precisely when $H_I^{n-d}(S)$ is a simple $D(S,\mathbb{Z})$-module. Then (iii), (iv), and (v) are consequences of the main statement and part (1) of Theorem X.1.1. Similarly, (vi) and (vii) follow from Theorem X.1.1 (3). □

Corollary X.1.3. *Let (R, m, K) be a complete local Gorenstein domain of characteristic $p > 0$, of dimension d, and such that K is F-finite. The following hold:*

(i) *If R is F-pure and $\lambda_0^d(R) = 1$, then R is F-regular.*

(ii) *If R is F-pure and K is perfect, then R is F-regular if and only if $\lambda_0^d(R) = 1$.*

Proof. For a Gorenstein ring, F-rationality and F-regularity are equivalent [HH94a]; additionally, F-injectivity and F-purity are equivalent [Fed87, Lemma 3.3]. The result follows. □

Remark X.1.4. Let $R = K[X]$ be the polynomial ring over a perfect field K of characteristic $p > 0$ in the entries of an $r \times r$ matrix X of indeterminates. Let m denote its homogeneous maximal ideal, and let Δ denote the principal ideal of R generated by the determinant of X. Then R/Δ is F-rational [GS95, Theorem 9], so by Proposition X.1.2 (iv), $\lambda_0^d(R_m/\Delta R_m) = 1$.

Remark X.1.5. In general, the Lyubeznik number $\lambda_0^d(R)$ is bounded by below by the number of minimal primes of R that have dimension d. Let (R, m, K) be a complete local ring of dimension d. Take any surjective ring map $\pi : S \twoheadrightarrow R$, where $S = K[[x_1, \dots, x_n]]$ for some n. Let I denote the kernel of the surjection. Let $P_1, \dots, P_\ell$ be the minimal primes of I. By iteratively using the Mayer-Vietoris sequence, we find that $H_{P_1}^d(S) \oplus \dots \oplus H_{P_\ell}^d(S) \subseteq H_I^d(S)$. Therefore, $\lambda_0^d(R) \geq \ell$.

As a consequence, R is a domain if it is equimensional and $\lambda_0^d(R) = 1$. Thus, several results of Proposition X.1.2 can be obtained by assuming only that R is equidimensional.

Remark X.1.6. Let I be an ideal of an F-finite regular local ring S, and suppose that the quotient ring S/IS is F-pure. Let τ_1 denote the pullback of the test ideal of S/I to S, and inductively let τ_i denote the pullback of the test ideal of the ring S/τ_{i-1} to S. As demonstrated by Vassilev, the corresponding chain of ideals is of the form

$$I \subsetneq \tau_1 \subsetneq \tau_1 \subsetneq \dots \subsetneq \tau_\ell = S \tag{X.1.6.1}$$

for some $\ell \geq 1$, and each quotient S/τ_i is F-pure [Vas98]. Let $K' \subset \widehat{S/IS}$ be any coefficient field. The following result, which connects this filtration with the generalized Lyubeznik numbers, is due to the first author and Pérez [NBP13]: If $I = (f)$ is principal and ℓ is the length of the chain determined by the τ_i as in (X.1.6.1), then if $d = \dim(S/fS)$, $\lambda_0^d(S/fS; K') \geq \ell$.

By definition of the test ideals, we see that $\ell = 1$ if and only if the quotient S/fS is F-regular, and so the inequality above shows that the generalized Lyubeznik number $\lambda_0^d(S/fS; K')$ must be large whenever S/fS is "far" from being F-regular. This bound also shows that the hypersurface S/fS must be F-regular if $\lambda_0^d(S/fS; K') = 1$; Corollary X.1.3 provides a partial converse to this statement.

X.2 Lyubeznik numbers and test ideals

Discussion X.2.1. In this section we assume that R is a Gorenstein ring. We have that $A := S/I^{[p]}$ is also a Gorenstein ring. We also have a short exact sequence of S and A-modules

$$0 \to I/I^{[p]} \to A \to R \to 0$$

where the map $A \to R$ is the quotient morphism. We have an induced map in the local cohomology, $H^d_m(A) \to H^d_m(R)$. If we consider these modules over S and use local duality, we obtain a map:

$$\operatorname{Ext}^c_S(R, S) \to \operatorname{Ext}^c_S(A, S)$$

On the other hand, if we consider the local cohomology modules over A and use local duality for A, we obtain a map:

$$\operatorname{Hom}_A(R, A) \to \operatorname{Hom}_A(A, A).$$

Since $E_A(K) = \operatorname{Ann}_{E_S(K)} I^{[p]}$, we have that both maps are the same. We have that $R \cong \operatorname{Hom}_A(R, A) = \operatorname{Ext}^c_S(R, S)$ and that $A \cong \operatorname{Hom}_A(A, A) = \operatorname{Ext}^c_S(A, S)$ because both rings are Gorenstein. We fix an identification among the modules. We have that the map

$$R = \operatorname{Ext}^c_S(R, S) \to \operatorname{Ext}^c_S(A, S) = A$$

is defined by multiplication by an element $f \in S$. In fact, this element depends on the election of the identifications made before.

Definition X.2.2. We say that an element $f \in S$, as described in Discussion X.2.1 is a hypersurface reduction of I. We denote by $\mathcal{HR}(I)$ all the hypersurface reductions of I.

Remark X.2.3. Let $f \in \mathcal{HR}(I)$ and $g \in S$. Then $g \in \mathcal{HR}(I)$ if and only if there exists a unit $u \in R$ such that $f = u^{p-1}g \bmod I^{[p]}$.

Example X.2.4. If I is generated by a regular sequence, $g_1, \ldots, g_\ell$, then

$$g_1^{p-1} \cdots g_\ell^{p-1} \in \mathcal{HR}(I).$$

Example X.2.5. Suppose that $I \subset K[[x_1, \ldots, x_n]]$ is generated by square free monomials, $x^{\alpha_1}, \ldots, x^{\alpha_1}$, where $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,n}) \in \{0,1\}^n$ and $x^{\alpha_1} = x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$. Then

$$x_1^{(p-1)\mathrm{Max}\{\alpha_{i,1}\}} \cdots x_n^{(p-1)\mathrm{Max}\{\alpha_{i,n}\}} \in \mathcal{HR}(I).$$

Proposition X.2.6. *Every hypersurface reduction of I, $f \in \mathcal{HR}(I)$ satisfies the following properties:*

(a) $R \xrightarrow{f} A$ *is injective;*

(b) $I = (I^{[p]} : f)$;

(c) $f^2 \in I$, *and* $f \in I$ *if* $p \neq 2$;

(d) $\mathbf{fpt}(f) \leq \frac{1}{p-1}$.

Proof. (a) We have that $R \xrightarrow{f} A$ is equivalent to $\mathrm{Hom}_A(R, A) \to \mathrm{Hom}_A(A, A)$, which is injective.

(b) We have that $fI \subset I^{[p]}$ because that map $R \xrightarrow{f} A$ is well defined. Since it is injective, we have that $(I^{[p]} : f) \subset I$. Combining these two, we obtain that $I = (I^{[p]} : f)$.

(c) By the previous claim, we have that $IS_f = (I^{[p]}S_f : fS_f) = I^{[p]}S_f$. This is possible if and only if $IS_f = S_f$. Therefore $f \in \sqrt{I}$. Let $u \in \mathbb{N}$ be the minimum integer such that $f^u \in I$. Then, $f^{u+1} \in I^{[p]}$. Suppose that $u \neq 1$. Since S is regular, we have that $f^{p(u-1)} \notin I^{[p]}$. Thus, $p(u-1) \leq u$ so

$$p \leq \frac{u}{u-1} = 1 + \frac{1}{u-1},$$

which is possible only when $p = 2$ and $u = 2$.

(d) Since $f^2 \in I$, we have that $f^2 \cdot f^{1+p+\ldots p^{e-1}} \in I^{[p^e]}$ for every $e \in \mathbb{N}$. Then,

$$\mathbf{fpt}(f) \leq \frac{3+p+\ldots+p^{e-1}}{p^e}$$

By taking limits when $e \to \infty$, we obtain $\mathbf{fpt}(f) \leq \frac{1}{p-1}$.

□

Lemma X.2.7. *Let $f \in \mathcal{HR}(I)$ and $\lambda = \frac{1}{p-1}$. Then, R is F-pure if and only if $\mathbf{fpt}(f) = \lambda$. In particular, the locus in which R is not F-pure is $\mathcal{V}(\tau(f^{\lambda-\epsilon}) + I)$.*

Proof. We note that $R \xrightarrow{f} A$ is given by the map $\mathrm{Hom}_A(R, A) \to \mathrm{Hom}_A(A, A)$. The image of this map is the kernel of the induced map $\mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(I/I^{[p]}, A)$, which is given by all the elements $a \in A$ such that $aI \subset I^{[p]}$. Therefore, $fS + I^p = (I^{[p]} : I)$. We have that

$$\begin{aligned}\mathbf{fpt}(f) = \lambda &\Leftrightarrow f \notin m^{[p]} \text{ by Lemma X.2.6}\\ &\Leftrightarrow fS + I^{[p]} \not\subset m^{[p]}\\ &\Leftrightarrow (I^{[p]} : I) \not\subset m^{[p]}\\ &\Leftrightarrow R \text{ is } F\text{-pure by Fedder's Criterion.}\end{aligned}$$

Let $Q \subset S$ be prime ideal containing I. We notice that $(I^{[p]} : I)S_Q = (f + I^{[p]})S_Q$, and so, $f \in \mathcal{HR}(IS_Q)$. Therefore,

$$R_Q \text{ not } F-\text{pure} \Leftrightarrow f \in Q^{[p]}S_Q \Leftrightarrow \tau\left((fS_Q)^{\lambda-\epsilon}\right) \neq S_Q \Leftrightarrow \tau(f^{\lambda-\epsilon}) \subset Q.$$

□

Remark X.2.8. Since $\mathrm{Ext}^c_S(R, S) \to \mathrm{Ext}^c_S(A, S)$ is a root morphism for the local cohomology $H^c_I(S)$ and this is equivalent to $R \xrightarrow{f} A$, every F-submodule of $H^c_I(S)$ is given by an ideal $I \subset J \subset S$ such that $fJ \subset J^{[p]}$. Two ideals $J_1 \subset J_2$ generate the same F-submodule of $H^c_I(S)$ if and only if there is an $e \in \mathbb{N}$ such that $f^{1+p+\ldots p^{e-1}} J_2 \subset J_1^{[p^e]}$.

Theorem X.2.9. *Suppose that R is Gorenstein and F-pure. Let*

$$I = \tau_0 \subset \tau_1 \subset \ldots \subset \tau_\ell = R$$

be the flag of test ideals defined by Vassilev. Then, $\ell \leq \lambda^d_0(R; K')$ for every coefficient field K'.

Proof. Let $I = \tau_0 \subset \dots \tau_\ell = R$ denote Vassilev's flag of test ideals. Since

$$f \in (I^{[p]} : I) = (\tau_0^{[p]} : \tau_0) \subset (\tau_1^{[p]} : \tau_1) \subset \dots \subset (\tau_\ell^{[p]} : \tau_\ell),$$

we have that $f\tau_i \subset \tau_i^{[p]}$. It suffices to prove that τ_{i+1} generates a different F-submodule of $H^c_I(S)$ than τ_i.

Suppose not; then there exists an e such that $f^{1+p+\cdots+p^{e-1}}\tau_{i+1} \subset \tau_i^{[p^e]}$. Since R/τ_j is F-pure for every j, we have that both τ_i and τ_{i+1} are radical ideals. Therefore, we can choose a minimal prime Q of τ_i such that $(\tau_{i+1})_Q = S_Q$. Hence,

$$f^{1+p+\cdots+p^{e-1}}(\tau_{i+1})_Q =\subset (\tau_i)_Q^{[p^e]} \subset (QS_Q)^{[p^e]}.$$

Then the F-pure threshold of f is strictly smaller than $1/(p-1)$ and $\tau(f^{1/(p-1)-\epsilon}S_Q) \neq S_Q$, which is a contradiction because $\tau(f^{\frac{1}{p-1}-\epsilon}) = S$ by Lemma X.2.6 and this ideal commutes with localization. □

X.3 Lyubeznik Numbers and $R\langle F\rangle$-modules

Since $R = S/I$, we have that every $R\langle F\rangle$-module has a natural structure of $S\langle F\rangle$-module. In particular, $H^d_m(R)$ is an $S\langle F\rangle$-module. Smith [Smi97] proved that an F-pure Cohen-Macaulay ring R is F–rational if and only if $H^d_m(R)$ is a simple left $R\langle F\rangle$ module. We have that for Cohen-Macaulay rings, $\operatorname{length}_{R\langle F\rangle} H^d_m(R)$ gives a measure of how far R is from being F–rational. Using results of Lyubeznik on F-modules [Lyu97], of Blickle on intersection homology [Bli04a] and of Ma on $R\langle F\rangle$-modules [Ma12], we prove that the highest generalized Lyubeznik number $\lambda^d_0(R)$ is an upper bound for $\operatorname{length}_{R\langle F\rangle} H^d_m(R)$. This results holds for all F-finite rings even if they are not Cohen-Macaulay.

Theorem X.3.1. *Suppose that R is an F-pure ring. Then*

$$\operatorname{length}_{R\langle F\rangle} H^d_m(R) \leq \lambda^d_0(R; K')$$

for every coefficient field.

Proof. We have that

$$\operatorname{length}_{R\langle F\rangle} H^d_m(R) = \operatorname{length}_{F-\text{mod}} \mathcal{D}(H^d_m(R)).$$

by Lemma II.9.2, Theorem II.9.4, and II.9.5. Therefore, by Proposition II.9.6,

$$\operatorname{length}_{S\langle F\rangle} H^d_m(R) = \operatorname{length}_{F-\mathrm{mod}} \mathcal{D}(H^d_m(R))$$

$$= \operatorname{length}_{F-\mathrm{mod}} \mathcal{D}(H^c_I(S)) \leq \operatorname{length}_{D(S,K')} H^c_I(S) = \lambda^d_0(R; K').$$

□

CHAPTER XI

Lyubeznik numbers in mixed characteristic

Our aim in this chapter is to define a new family of invariants associated to *any* local ring whose residue field has prime characteristic. In particular, these new invariants are defined for local rings of mixed characteristic. These invariants are defined somewhat analogously to the Lyubeznik numbers. Moreover, we study properties of these *Lyubeznik numbers in mixed characteristic* and investigate when they agree with the (original) Lyubeznik numbers.

If S is a regular local ring of unramified mixed characteristic the Bass numbers of local cohomology modules $H^i_I(S)$ are finite (see Theorem IV.3.1 and [Lyu00b, NB12b]). Using the theory of p-bases, and explicit constructions used in the Cohen Structure Theorems, we prove that the Lyneznik numbers in mixed characteristic are well defined (see Theorem XI.1.6 and Definition XI.1.7)

Motivated by analogous properties of the Lyubeznik numbers in equal characteristic, we study properties of these invariants (cf. [Lyu93, Properties 4.4]). Some similar vanishing properties hold, as well as analogous computations for complete intersection rings (see Propositions XI.1.11 and XI.1.12). Moreover, the "highest" Lyubeznik number in mixed characteristic of a local ring for which these invariants are defined is a well-defined notion: if $d = \dim(R)$, then $\widetilde{\lambda}_{i,j}(R) = 0$ if either $i > d$ or $j > d$ (see Theorem XI.2.10 and Definition XI.2.11).

When R is a local ring of equal characteristic $p > 0$, both the Lyubeznik numbers and the Lyubeznik numbers in mixed characteristic of R are defined. We find that these invariants agree when R is Cohen-Macaulay, or if $\dim(R) \leq 2$ (see Corollary XI.3.4). However, we give a specific example for which $\widetilde{\lambda}_{i,j}(R) \neq \lambda_{i,j}(R)$ for some $i, j \in \mathbb{N}$, employing the work of Singh and Walther on Bockstein homomorphisms of local cohomology and a computation of Àlvarez Montaner and Vahidi [SW11, ÀMV] (see Remark XI.4.11 and Theorem XI.4.12).

The results presented in this chapter are part of joint work with Witt [NBW12b].

XI.1 Definition and properties

Lemma XI.1.1. *Let $S = R[[x]]$, where (R, m, K) is a Gorenstein local ring. Then for every ideal I of R, and all $i, j \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^i_S(K, H^{j+1}_{(I,x)S}(S)) = \dim_K \operatorname{Ext}^i_R(K, H^j_I(R)).$$

Proof. For G the functor defined in Chapter VIII, $G(H^j_I(R)) = H^{j+1}_{(I,x)}(S)$ by Lemma VIII.0.15. Since R is Gorenstein, Proposition XI.1.1 indicates that $\operatorname{Ext}^i_S(M, G(N)) = \operatorname{Ext}^i_R(M, N)$ for all R-modules M and N. Therefore,

$$\operatorname{Ext}^i_S(K, H^{j+1}_{(I,x)S}(S)) = \operatorname{Ext}^i_R(K, H^j_I(R)).$$

□

Corollary XI.1.2. *Let (R, m, K) be a Gorenstein local ring, and $S = R[[x_1, \ldots, x_n]]$. For every ideal I of R and all $i, j \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^i_S(K, H^{j+n}_{(I,x_1,\ldots,x_n)S}(S)) = \dim_K \operatorname{Ext}^i_R(K, H^j_I(R)).$$

Proof. Using Lemma XI.1.1, apply induction on n. □

Definition XI.1.3 (p-independent, p-base)**.** Let K be a field of characteristic $p > 0$. A finite set of elements $T_1, \ldots, T_\ell \in K - K^P$ is called *p-independent* if $[K^p[T_1, \ldots, T_\ell] : K^p] = p^n$. An inifinite set of elements in $K - K^p$ is *p-independent* if every finite subset is. A maximal p-independent subset of $K^p - K$ is called a *p-base for K.*

Remark XI.1.4. We recall some results related to the Cohen Structure Theorems that will be useful in proving that our new invariants are well defined. See [Coh46] for details.

For any field K of characteristic $p > 0$, there exists a complete Noetherian DVR $(V, \gamma V, K)$ with residue class field K. In fact, if $(V, \gamma V, K)$ and $(W, \gamma' W, K')$ are complete Noetherian DVRs of mixed characteristic $p > 0$ such that $K \cong K'$, then $V \cong W$ as well. Given an isomorphism $\varphi : K \to K'$, take a p-base $\Lambda \subseteq K$ for K, and let $\Lambda' \subseteq K'$ be the corresponding p-base for K' under φ. If $T \subseteq V$ is a lifting of Λ to V, and $T' \subseteq W$ is a lifting of Λ' to W, then the natural bijection $T \to T'$ extends uniquely to an isomorphism $V \to W$.

Suppose that (R, m, K) is a complete local ring of mixed characteristic $p > 0$. Then R contains a *coefficient ring* as a subring $V' \subseteq R$, i.e., $V' = V$ or $V' = V/\gamma^\ell V$

for some $\ell > 0$, where $(V, \gamma V, K)$ is a complete Noetherian DVR, and the induced map on residue fields $V/\gamma V \twoheadrightarrow R/m$ is an isomorphism. In fact, the Cohen Structure Theorems indicate that given a coefficient ring $V' \subseteq R$, there exists a surjection $\rho : V[[x_1, \ldots, x_n]] \twoheadrightarrow R$ such that $\rho(V) = V'$ (and we can take n to be the embedding dimension of R/pR). A key point in the proof of this fact is that for every lifting $T \subseteq R$ of a p-base $\Lambda \subseteq K$ of K to R, there is a unique coefficient ring $V' \subseteq R$ of R that contains T.

If (R, m, K) is a complete local ring of equal characteristic $p > 0$, then R is a homomorphic image of some $K[[x_1, \ldots, x_n]]$ by the Cohen Structure Theorems. Thus, if $(V, \gamma V, K)$ is a complete Noetherian DVR, the composition $V[[x_1, \ldots, x_n]] \twoheadrightarrow K[[x_1, \ldots, x_n]] \twoheadrightarrow R$ is surjective. Thus, any complete local ring (R, m, K) such that $\mathrm{char}(K) = p > 0$ is the homomorphic image of $V[[x_1, \ldots, x_n]]$, where V is a uniquely-determined (up to isomorphism) mixed characteristic complete Noetherian discrete valuation domain.

Lemma XI.1.5. *Let (R, m, K) be a complete local ring of mixed characteristic $p > 0$, and let $V', W' \subseteq R$ be coefficient rings of R. Let $(V, \nu V, K)$ and $(W, \gamma W, K)$ be complete Noetherian DVRs. Let $n = \dim_K(m/m^2)$. Then there exist surjective ring maps*

$$\rho_1 : S_1 := V[[x_1, \ldots, x_n]] \twoheadrightarrow R \quad \text{and} \quad \rho_2 : S_2 := W[[y_1, \ldots, y_n]] \twoheadrightarrow R$$

such that $\rho_1(V) = V'$ and $\rho_2(W) = W'$. Moreover, there is an isomorphism $\phi : S_1 \to S_2$ such that $\rho_1 = \rho_2 \circ \phi$.

Proof. Let Λ and Δ be a p-bases for K that are taken in the choice of V' and W'. There exist a map $\rho_1 : V \twoheadrightarrow V'$ given by choosing preimages $t_\lambda \in V$ of $\lambda \in \Lambda$ under the composition $V \twoheadrightarrow V' \twoheadrightarrow K$. Similarly, we have a map $\rho_2 : W \twoheadrightarrow W'$ given by $s_\delta \in W$, where $\delta \in \Delta$.

Pick elements $u_1, \ldots, u_n \in m$ such that the $\overline{u}_i$ form a basis for m/m^2, and extend ρ_1 to a map $S_1 \to R$ by $\rho_1(x_i) = u_i$. Similarly, extend ρ_2 to a map $S_2 \to R$ by $\rho_1(y_i) = u_i$.

By construction, ρ_1 and ρ_2 are surjective. Let $\sigma_\lambda \in S_2$ be elements such that $\rho_2(\sigma_\lambda) = \rho_1(t_\lambda)$ for every $\lambda \in \Lambda$. Then there exists a unique coefficient ring $\widetilde{V} \subseteq S_2$ such that $\sigma_\lambda \in \widetilde{V}$ for every $\lambda \in \Lambda$ (see Remark XI.1.4); moreover,

$t_\lambda \mapsto \sigma_\lambda$ defines an isomorphism $\phi : V \to \widetilde{V}$. Now, extend this map to $\phi : S_1 \to S_2$ by $\phi(x_i) = y_i$. The induced map $K \cong S_1/(\nu, x_1, \ldots, x_n) \to S_2/\phi(\nu, x_1, \ldots, x_n)S_2$ is well defined, $\mathrm{char}\left(S_2/\phi(\nu, x_1, \ldots, x_n)S_2\right) = p$, and γ must be in the image of ϕ. Thus,

ϕ is surjective. As S_1 and S_2 have the same dimension, ϕ is, in fact, an isomorphism. Since $\rho_2 \circ \phi(x_j) = \rho_1(y_j) = u_j$ and $\rho_2 \circ \phi(t_\lambda) = \rho_2(\sigma_\lambda) = \rho_1(t_\lambda)$ by construction, $\rho_1 = \rho_2 \circ \phi$. □

Theorem XI.1.6. *Let (R, m, K) be a local ring such that $\operatorname{char}(K) = p > 0$, admitting a surjection $\pi : S \twoheadrightarrow R$, where S is an n-dimensional unramified regular local ring of mixed characteristic. Let $I = \operatorname{Ker}(\pi)$, and take $i, j \in \mathbb{N}$. Then*

$$\dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S))$$

is finite and depends only on R, i, and j, but not on S, nor on π.

Proof. Each $\dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S))$ is finite (cf. Theorem IV.3.1 [Lyu00b, NB12b]), so it remains to prove that these numbers are well defined. As the Bass numbers with respect to the maximal ideal are not affected by completion, we may assume that the rings are complete.

Fix a coefficient ring W of R, and take $(V, \nu V, K)$ a complete Noetherian DVR such that $W = V$ or $W = V/\nu^\ell V$ for some $\ell > 0$. First, we take surjective ring maps $\pi : T \twoheadrightarrow R$ and $\pi' : T' \twoheadrightarrow R$, where $T = V[[x_1, \ldots, x_{n-1}]]$, $T' = V[[y_1, \ldots, y_{n'-1}]]$, $\pi(V) = W$, and $\pi|_V(r) = \pi'|_V(r)$ for every $r \in V$. Let m' denote the maximal ideal of T'.

Let $T'' = V[[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n'-1}]]$, and let $\pi'' : T'' \twoheadrightarrow R$ be the surjective ring map defined by $\pi''|_V(r) = \pi|_V(r) = \pi'|_V(r)$ for every $r \in V$, $\pi''(x_j) = \pi(x_j)$, and $\pi''(y_j) = \pi'(y_j)$. Let $I'' = \ker(\pi'')$, and let $\alpha : T \hookrightarrow T''$ denote the injection, so that $\pi'' \circ \alpha = \pi$. As π is surjective, there exist $f_1, \ldots, f_{n'-1} \in T$ such that $\pi''(y_j) = \pi(f_j)$ for $j \leq n' - 1$. Then $y_j - f_j \in \operatorname{Ker}(\pi'')$. Note that $\beta : T'' \twoheadrightarrow T$, defined by $\beta(x_j) = x_j$, $\beta(y_j) = f_j$, and $\beta|_V = \mathrm{id}_V$, is a splitting of α. Then

$$I'' = \operatorname{Im}(\alpha) \oplus \ker(\beta) = (I, y_1 - f_1, \ldots, y_{n'-1} - f_{n'-1})T''.$$

Since

$$\nu, x_1, \ldots, x_{n-1}, y_1 - f_1, \ldots, y_{n'-1} - f_{n'-1}$$

form a regular system of parameters for T'', Corollary XI.1.2 indicates that

$$\nu, z_1, \ldots, z_{n-1}, z_n - \alpha(f_1), \ldots, z_{n'+n-2} - \alpha(f_{n'-1})$$

form a regular system of parameters for T'', Corollary XI.1.2 indicates that

$$\dim_K \operatorname{Ext}^i_{T''}(K, H^{n+n'-j}_{I''}(T'')) = \dim_K \operatorname{Ext}^i_T(K, H^{n-j}_I(T)).$$

By an analogous argument,

$$\dim_K \operatorname{Ext}^i_{T''}(K, H^{n+n'-j}_{I''}(T'')) = \dim_K \operatorname{Ext}^i_{T'}(K, H^{n'-j}_{I'}(T')),$$

so

$$\text{(XI.1.6.1)} \qquad \dim_K \operatorname{Ext}^i_T(K, H^{n-j}_I(T)) = \dim_K \operatorname{Ext}^i_{T'}(K, H^{n'-j}_{I'}(T')).$$

Now we proceed to the general case. Take $\pi : S := V[[x_1, \ldots, x_{n-1}]] \twoheadrightarrow R$ and $\pi' : S' := W[[y_1, \ldots, y_{n'-1}]] \twoheadrightarrow R$, where V and W are complete Noetherian DVRs with residue field K, and $\pi|_V$ is a surjection of V onto V' and $\pi|_W$ is a surjection of W onto W', where V' and W' are coefficient rings of R.

Let $N = \dim_K(m/m^2)+1$. Let $S_1 = V[[x_1, \ldots, x_N]]$ and $S_2 = W[[y_1, \ldots, y_N]]$, and let $\rho_1 : S_1 \twoheadrightarrow R$, and $\rho_2 : S_2 \twoheadrightarrow R$, and $\phi : S_1 \to S_2$ be the maps ensured by Lemma XI.1.5; i.e., $\rho_1(V) = V'$ and $\rho_2(W) = W'$, and $\phi : S_1 \xrightarrow{\cong} S_2$; moreover, $\rho_1 = \rho_2 \circ \phi$.

By (XI.1.6.1), we have that

$$\dim_K \operatorname{Ext}^i_{S_2}(K, H^{N-j}_I(S_1)) = \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S)) \text{ and}$$
$$\dim_K \operatorname{Ext}^i_{S_2}(K, H^{N-j}_{\phi(I)}(S_2)) = \dim_K \operatorname{Ext}^i_{S'}(K, H^{n'-j}_{I'}(S')).$$

In addition, the isomorphism ϕ allows us to deduce that

$$\dim_K \operatorname{Ext}^i_S(K, H^{N-j}_I(S_1)) = \dim_K \operatorname{Ext}^i_{S_2}(K, H^{N-j}_{\phi(I)}(S_2)),$$

which concludes the proof.

□

Definition XI.1.7 (Lyubeznik numbers in mixed characteristic)**.** Let (R, m, K) be a local ring such that $\operatorname{char}(K) = p > 0$. By the Cohen Structure Theorems, the completion $\widehat{R}$ admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where S is an unramified regular local ring of mixed characteristic. Let $I = \operatorname{Ker}(\pi)$, $n = \dim(S)$, and $i, j \in \mathbb{N}$. Then the *Lyubeznik number in mixed characteristic of R with respect to i and j* is defined as

$$\widetilde{\lambda}_{i,j}(R) := \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S)).$$

Note that by Theorem XI.1.6, the $\widetilde{\lambda}_{i,j}(R)$ are well defined and depend only on R, i, and j.

Remark XI.1.8. In Definition XI.1.7, we need to take the completion of R for the Cohen Structure Theorems to ensure the existence of a surjection from an unramified regular local ring S of mixed characteristic, $\pi : S \twoheadrightarrow \widehat{R}$. If such a map exists without taking the completion, then $\widetilde{\lambda}_{i,j}(R) = \dim_K \operatorname{Ext}^i_{\widehat{S}}(K, H^{n-j}_{I\widehat{S}}(\widehat{S})) = \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S))$, where $I = \operatorname{Ker}(\pi)$.

Remark XI.1.9. Fix (R, m, K), a local ring of equal characteristic $p > 0$. There exists a surjection from an n-dimensional unramified regular local ring of mixed characteristic, $\pi : S \twoheadrightarrow \widehat{R}$; the induced map $\pi' : S/pS \twoheadrightarrow \widehat{R}$ is also surjective. If $I = \operatorname{Ker}(\pi)$ and $I' = \operatorname{Ker}(\pi')$, I is the preimage of I' under the canonical surjection $S \twoheadrightarrow S/pS$. In this case, both the Lyubeznik numbers, $\lambda_{i,j}(R) = \dim_K \operatorname{Ext}^i_{S/pS}(K, H^{n-j-1}_{I'}(S/pS))$, and the Lyubeznik numbers in mixed characteristic,

$$\widetilde{\lambda}_{i,j}(R) = \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S)),$$

are defined.

Remark XI.1.9 naturally incites the following question:

Question XI.1.10. Is $\widetilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R)$ whenever both are defined, i.e., for every local ring (R, m, K) of any equal characteristic $p > 0$ and all $i, j \in \mathbb{N}$?

In Corollary XI.3.4, we prove an affirmative answer to Question XI.1.10 when R is Cohen-Macaulay, or $\dim(R) \leq 2$. However, Remark XI.4.11 and Theorem XI.4.12 give an example of a Stanley-Reisner ring over $\mathbb{F}_2$ for which the answer is negative.

The Lyubeznik numbers in mixed characteristic satisfy similar vanishing properties to those of the original Lyubeznik numbers.

Proposition XI.1.11. *For (R, m, K) a local ring such that $\operatorname{char}(K) = p > 0$ and $d = \dim(R)$,*

(i) *$\widetilde{\lambda}_{i,j}(R) = 0$ if $j > d$ or $i > j + 1$, and*

(ii) *$\widetilde{\lambda}_{d,d}(R) \neq 0$.*

Proof. The completion of R, $\widehat{R}$, admits a surjection $\pi : S \twoheadrightarrow \widehat{R}$, where (S, η, K) is an unramified regular local ring of mixed characteristic and of dimension n. Let $I = \operatorname{Ker}(\pi)$.

For (i), the first statement holds since $H_I^{n-j}(S) = 0$ for $j > \dim(S/I) = \dim R = d$, and the second since $\operatorname{inj.dim}_S H_I^{n-j}(S) \leq \dim_S H_I^{n-j}(S) + 1 \leq j + 1$ [Zho98].

To prove (ii), first note that by an analogous argument to the proof of [Lyu93, Property 4.4(iii)], $H_\eta^d H_I^{n-d}(S) \neq 0$. We will prove that $\widetilde{\lambda}_{d,d}(R) \neq 0$ by contradicting this fact. Suppose that $\widetilde{\lambda}_{d,d}(R) = \operatorname{Ext}_S^d(K, H_I^{n-d}(S)) = 0$.

We claim that $\operatorname{Ext}_S^d(M, H_I^{n-d}(S)) = 0$ for every finite-length S-module M. We will prove this by induction on $h = \operatorname{length}_S(M)$. If $h = 1$, then $M = K$, and the statement holds by assumption. Suppose that the statement is true for all N with $\operatorname{length}_S N < h + 1$, and take M with $\operatorname{length}_S M = h + 1$. Then there exists a short exact sequence $0 \to K \to M \to M' \to 0$, where M' is an S-module of length h. The long exact sequence in Ext gives:

$$\cdots \to \operatorname{Ext}_S^d(M', H_I^{n-d}(S)) \to \operatorname{Ext}_S^d(M, H_I^{n-d}(S)) \to \operatorname{Ext}_S^d(K, H_I^{n-d}(S)) \to \cdots .$$

Now, $\operatorname{Ext}_S^d(K, H_I^{n-d}(S)) = \operatorname{Ext}_S^d(M', H_I^{n-d}(S)) = 0$ by the inductive hypothesis, so that $\operatorname{Ext}_S^d(M, H_I^{n-d+1}(S)) = 0$, and the claim follows.

This claim implies that $\operatorname{Ext}_S^d(S/\eta^\ell, H_I^{n-d}(S)) = 0$ for all $\ell \geq 1$. Then $H_\eta^d H_I^{n-d}(S) = \varinjlim_\ell \operatorname{Ext}_S^d(S/\eta^\ell, H_I^{n-d}(S)) = 0$, the sought contradiction. □

Proposition XI.1.12. *Let (V, pV, K) be an complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1, \ldots, x_n]]$. Let $f_1, \ldots, f_\ell \in S$ be a regular sequence. Then*

$$\widetilde{\lambda}_{i,j}\left(S/(f_1, \ldots, f_\ell)\right) = \begin{cases} 1 & i = j = n + 1 - \ell \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Our proof will be by induction on ℓ. If $\ell = 1$, since $\operatorname{Ext}_S^i(K, S_f) = 0$ for $i \geq 0$, the short exact sequence $0 \to S \to S_{f_1} \to H^1_{f_1 S}(S) \to 0$ indicates $\operatorname{Ext}_S^i(K, S) \cong \operatorname{Ext}_S^{i+1}(K, H^1_{f_1 S}(S))$.

Suppose that the formula holds for $\ell - 1$ and we will prove it for ℓ. From the exact sequence

$$0 \to H^{n-\ell-1}_{(f_1,\ldots,f_{\ell-1})S}(S) \to H^{n-\ell-1}_{(f_1,\ldots,f_{\ell-1})S}(S_{f_\ell}) \to H^{n-\ell}_{(f_1,\ldots,f_\ell)S}(S) \to 0,$$

we obtain that $\operatorname{Ext}_S^i(K, H^{n-\ell}_{(f_1,\ldots,f_\ell)S}(S)) = \operatorname{Ext}_S^{i+1}(K, H^{n-\ell-1}_{(f_1,\ldots,f_{\ell-1})S}(S))$ for every $i \geq 0$ because $\operatorname{Ext}_S^i(K, H^{n-\ell}_{(f_1,\ldots,f_\ell)S}(S_{f_{\ell+1}})) = 0$. □

XI.2 Existence of the highest Lyubeznik Number in Mixed Characteristic

Lemma XI.2.1. *Let* (V, pV, K) *be a complete DVR of unramified mixed characteristic* $p > 0$*, and let* $S = V[[x_1 \ldots, x_n]]$*. Then* $\operatorname{End}_{D(S,V)}(E_S(K)) = V$*.*

Proof. Let $\phi \in \operatorname{End}_{D(S,V)}(E_S(K)) \subseteq \operatorname{End}_S(E_S(K)) = S$; ϕ must correspond to multiplication by some $r \in S$. Thus, $\partial(rw) = r\partial(w)$ for every $w \in E_S(K)$ and $\partial \in D(S,V)$. We will prove that $r \in V$ by contradiction. If $r \notin V$, we may assume there exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$ such that $r = a + bx^\alpha + \sum_{\beta \in \mathbb{N}, \beta \geq_{\text{lex}} \alpha} c_\beta x^\beta$, where $a, b, c_\beta \in V$ and $b \neq 0$. Then for every $j \in \mathbb{N}$,

$$\begin{aligned} r\frac{(-1)^{\alpha_1 - 1}}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n - 1}}{\alpha_n!}\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\frac{1}{p^j x_1 \cdots x_n} &= \frac{r}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} \\ &= \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} + \frac{b}{p^j x_1 \cdots x_n} \end{aligned}$$

On the other hand, for every $j \in \mathbb{N}$,

$$\begin{aligned} &\frac{(-1)^{\alpha_1 - 1}}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n - 1}}{\alpha_n!}\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\frac{r}{p^j x_1 \cdots x_n} \\ =&\frac{(-1)^{\alpha_1 - 1}}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n - 1}}{\alpha_n!}\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\frac{a}{p^j x_1 \cdots x_n} = \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} \end{aligned}$$

Then $\frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n}} + \frac{b}{p^j x_1 \cdots x_n} = \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n}}$, so $b \in p^j V$ for every $j \in \mathbb{N}$. This means that $b = 0$, a contradiction. Thus, $r \in V$. Since every map given by multiplication by an element in V is already a map of $D(S,V)$-modules, we are done. $\square$

Proposition XI.2.2. *Let* (V, pV, K) *be a complete DVR of unramified mixed characteristic* $p > 0$*, and let* $S = V[[x_1 \ldots, x_n]]$*. Let* $N \subsetneq E_S(K)$ *be a proper* $D(S,V)$*-submodule. Then* $N = \operatorname{Ann}_{E_S(K)} p^\ell S$ *for some* $\ell \in \mathbb{N}$*.*

Proof. Let $v \in N$ be such that $v \in \operatorname{Ann}_{E_S(K)} p^\ell S$ but $v \notin \operatorname{Ann}_{E_S(K)} p^{\ell-1} S$. We claim that $D(S,V) \cdot v = \operatorname{Ann}_{E_S(K)} p^\ell S$ by induction on ℓ. If $\ell = 1$, $\operatorname{Ann}_{E_S(K)} pS = E_{S/pS}(K)$, a simple $D(S,K)$-module, and the claim holds. Now, we suppose the claim true for $\ell - 1$. Since $\operatorname{Ann}_{E_S(K)} p^\ell S / \operatorname{Ann}_{E_S(K)} p^{\ell-1} S = E_{S/pS}(K)$, there exists an operator $\partial \in D(S,V)$ such that $\partial v = 1/p^\ell x_1 \cdots x_n + w$ for an element $w \in \operatorname{Ann}_{E_S(K)} p^{\ell-1} S$. Then $p\partial v \in \operatorname{Ann}_{E_S(K)} p^{\ell-1} S \setminus \operatorname{Ann}_{E_S(K)} p^{\ell-2} S$. Thus, there exists an operator δ such that $p\delta\partial v = w$ by the induction hypothesis, so $1/p^\ell x_1 \cdots x_n = (\partial - p\delta\partial)v$. Therefore, $\operatorname{Ann}_{E_S(K)} p^\ell = D(S,V) \cdot 1/p^\ell x_1 \cdots x_n \subseteq D(S,V) \cdot v \subseteq \operatorname{Ann}_{E_S(K)} p^\ell$, proving our claim.

Since $N \neq E_S(K)$, $\ell = \inf\{j \in \mathbb{N} \mid 1/p^j x_1 \cdots x_n \in N\}$ is a natural number. Hence, $N = D(S, V) \cdot 1/p^\ell x_1 \cdots x_n = \operatorname{Ann}_{E_S(K)} p^\ell S$. $\square$

Lemma XI.2.3. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \dots, x_n]]$. Let $M \subseteq \bigoplus_{i=1}^{h} E_S(K)$ be a $D(S, V)$-submodule. Then $M \xrightarrow{\cdot p} M$ is surjective if and only if M is an injective S-module.*

Proof. Suppose that M is an injective S-module. Since $E_S(K) \xrightarrow{\cdot p} E_S(K)$ is surjective and $M = \bigoplus_{\ell} E_S(K)$, $M \xrightarrow{\cdot p} M$ is also surjective.

Now assume that $M \xrightarrow{\cdot p} M$ is surjective. We will show that M is injective by contradiction. Suppose that $M \neq E_S(M)$. As M is a $D(S, V)$-module supported only at the maximal ideal, $\operatorname{inj.dim}(M) \leq 1$ (see Theorem IV.4.3 and [NB12b, Zho98]). Let $0 \to M \to E_1 \xrightarrow{\phi} E_2 \to 0$ be a minimal injective resolution of M; in particular, $E_2 \neq 0$. Let $\mu_i = \operatorname{Ext}^i_S(K, M)$. Now, μ_1 is finite and less than or equal to h. Let $(-)^* = \operatorname{Hom}_S(-, E_S(K))$ be the Matlis duality functor. From the short exact sequence $0 \to E_2^* \xrightarrow{\phi^*} S^{\mu_1} \to M^* \to 0$, we obtain that E_2^* is a free module of finite rank less than or equal to μ_1, so, $E_2 = \bigoplus_{i=2}^{\mu_2} E_S(K)$. By Lemma XI.2.1, ϕ is given by a $\mu_1 \times \mu_2$-matrix with entries in V. Thus, $\phi^* : S^{\mu_2} \to S^{\mu_1}$ can be represented as a matrix by the transpose of a matrix that represents ϕ. We may consider ϕ^* as a map of free V-modules, $\phi^* : V^{\mu_2} \to V^{\mu_1}$. By the structure theorem for finitely-generated modules over a principal ideal domain, there are isomorphisms $\varphi_1 : V^{\mu_1} \to V^{\mu_1}$ and $\varphi_2 : V^{\mu_2} \to V^{\mu_2}$, such that $\varphi_1 \phi^* \varphi_2$ is a matrix whose entries are zero off the diagonal. That is, we have the following commutative diagram.

$$\begin{array}{ccc} V^{\mu_2} & \xrightarrow{\phi^*} & V^{\mu_1} \\ {\scriptstyle \varphi_2}\uparrow & & \downarrow{\scriptstyle \varphi_1} \\ V^{\mu_2} & \xrightarrow{\varphi_1 \phi^* \varphi_2} & V^{\mu_1} \end{array}$$

Let $a_1, \dots a_{\mu_1} \in V$ be the elements on the diagonal of $\varphi_1 \phi^* \varphi_2$, and let $v : V \to \mathbb{N}$ be the valuation. Since $E_S(M) = E_1 \to E_2$ is surjective, none of $a_1, \dots, a_{\mu_1}$ is zero. Since $E_2 \neq 0$ and the injective resolution $0 \to M \to E_1 \xrightarrow{\phi} E_2 \to 0$ is minimal, none of $a_1, \dots, a_{\mu_1}$ are units in V. Then $a_1, \dots, a_{\mu_1} \in pV \setminus \{0\}$. We extend φ_i as a isomorphism of S-modules, $\varphi_i : S^{\mu_i} \to S^{\mu_i}$. Then $\varphi_i^* : E_i \to E_i$ is an isomorphism of

$D(S,V)$-modules. We obtain the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & E_1 & \xrightarrow{\phi} & E_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \varphi_1^*} & & \uparrow{\scriptstyle \varphi_1^*} & & \downarrow{\scriptstyle \varphi_2^*} & & \\
0 & \longrightarrow & \operatorname{Ker}(\varphi_2^*\phi\varphi_1^*) & \longrightarrow & E_1 & \xrightarrow{\varphi_2^*\phi\varphi_1^*} & E_2 & \longrightarrow & 0
\end{array}$$

Therefore, $M \cong \operatorname{Ker}(\varphi_2^*\phi\varphi_1^*) = \bigoplus_{i=1}^{\mu_1-\mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \operatorname{Ann}_{E_S(K)} p^{v(a_i)}S$, a contradiction as

$$\bigoplus_{i=1}^{\mu_1-\mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \operatorname{Ann}_{E_S(K)} p^{v(a_i)}S \xrightarrow{\cdot p} \bigoplus_{i=1}^{\mu_1-\mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \operatorname{Ann}_{E_S(K)} p^{v(a_i)}S$$

is not surjective. □

Lemma XI.2.4. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let m denote the maximal ideal of S, and let $I \subseteq S$ be an ideal such that $\dim(S/I) = 1$. Then $H^0_m H^n_I(S) = 0$ and $H^1_m H^n_I(S) \cong E_R(K)$.*

Proof. Let $f \in m$ be an element not in any minimal prime of I, so $\sqrt{I + fS} = m$. We have the short exact sequence $0 \to H^n_I(S) \to H^n_I(S_f) \to H^{n+1}_{I+fS}(S) \cong E_S(K) \to 0$. Since $f \in m$ and $H^n_I(S_f) \cong H^n_I(S)_f$, $H^0_m H^n_I(S_f) = 0$, which implies that $H^0_m H^n_I(S) = 0$. Moreover, $H^1_m H^n_I(S) = E_S(K)$. □

Lemma XI.2.5. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let $I \subseteq S$ be an ideal of pure dimension 2. Then $H^n_I(S)$ is an injective S-module supported only at the maximal ideal.*

Proof. Let R denote S/pS. The short exact sequence $0 \to S \xrightarrow{\cdot p} S \to R \to 0$ induces the long exact sequence $\cdots \to H^n_I(S) \xrightarrow{\cdot p} H^n_I(S) \to H^n_I(R) \to 0$, where $H^n_I(R) = 0$ by the Hartshorne-Lichtenbaum Vanishing Theorem, as $\sqrt{I + pS} \neq m$. Thus, $H^n_I(S) \xrightarrow{\cdot p} H^n_I(S)$ is surjective. Now, $H^n_{IS_P}(S_P) = 0$ for every prime ideal P not containing I. If $I \subseteq P$ and $P \neq m$, then $\dim(S/P) = 1$ and $H^n_{IR_P}(R_P) = 0$ by the Hartshorne-Lichtenbaum Vanishing Theorem because I has pure dimension 2 and $\sqrt{IS_P} \neq PS_P$. Therefore, $H^n_I(R)$ is a $D(S,V)$-module supported only at the maximal ideal. Since $\dim_K \operatorname{Ext}^0_S(K, H^n_I(S))$ is finite, $H^n_I(S)$ is injective by Lemma XI.2.3. □

Lemma XI.2.6. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let m denote the maximal ideal of S, and let $I \subseteq S$*

be an ideal of pure dimension two. Then $H^0_m H^{n-1}_I(S) = H^1_m H^{n-1}_I(S) = 0$*. Moreover,* $H^n_I(S) \cong E_S(K)^{\bigoplus \alpha}$ *for some* $\alpha \in \mathbb{N}$*, and* $H^2_m H^{n-1}_I(S) \cong E_S(K)^{\bigoplus \alpha+1}$*. In particular,* $H^2_m H^{n-1}_I(S)$ *is an injective* S*-module.*

Proof. Let $f \in m$ be an element not in any minimal prime of I. Then $\sqrt{I + fS} \neq m$. Applying the Hartshorne-Lichtenbaum Vanishing Theorem, since $H^n_I(S)$ is supported at m by Lemma XI.2.5, we obtain the exact sequence

$$0 \to H^{n-1}_I(S) \to H^{n-1}_I(S_f) \to H^n_{I+fS}(S) \to H^n_I(S) \to 0.$$

Splitting the sequence into two short exact sequences, we obtain

$$0 \to H^{n-1}_I(S) \to H^{n-1}_I(S_f) \to M \to 0, \quad \text{and} \quad 0 \to M \to H^n_{I+fS}(S) \to H^n_I(S) \to 0.$$

These induce the following long exact sequences:

$$\begin{aligned} 0 \to H^0_m(H^{n-1}_I(S)) \to H^0_m(H^{n-1}_I(S_f)) \to H^0_m(M) \\ \to H^1_m(H^{n-1}_I(S)) \to H^1_m(H^{n-1}_I(S_f)) \to H^1_m(M) \\ \to H^2_m(H^{n-1}_I(S)) \to H^2_m(H^{n-1}_I(S_f)) \to H^2_m(M) \to 0, \end{aligned}$$

and

$$\begin{aligned} 0 \to H^0_m(M) \to H^0_m(H^n_{I+fS}(S)) \to H^0_m(H^n_I(S)) \\ \to H^1_m(M) \to H^1_m(H^n_{I+fS}(S)) \to H^1_m(H^n_I(S)) \\ \to H^2_m(M) \to H^2_m(H^n_{I+fS}(S)) \to H^2_m(H^n_I(S)) \to 0. \end{aligned}$$

Since all $H^j_m H^{n-1}_I(S_f) = 0$, we know that $H^0_m H^{n-1}_I(S) = H^2_m(M) = 0$. Since $\dim(S/(I + fS)) = 1$, $H^0_m H^n_{I+fS}(S) = H^2_m H^n_{I+fS}(S) = 0$ by Lemma XI.2.4, which implies both that $H^0_m(M) = H^1_m H^{n-1}_I(S) = 0$ and that $H^1_m(M) \cong H^2_m H^{n-1}_I(S)$. In addition, $H^1_m H^n_I(S) = H^2_m H^n_I(S) = 0$ by Lemma XI.2.5. Thus, we have a short exact sequence

$$0 \to H^0_m H^n_I(S) \to H^2_m H^{n-1}_I(S) \to H^1_m H^n_{I+fS}(S) \to 0.$$

By Lemma XI.2.5, $H^n_I(S)$ is an injective S-module supported only at m, and its Bass numbers are finite by Theorem IV.3.1 and [Lyu00b, NB12b], so $H^0_m H^n_I(S) = H^n_I(S) \cong E_R(K)^{\bigoplus \alpha}$ for some $\alpha \in \mathbb{N}$. Moreover, by Lemma XI.2.4, $H^1_m H^n_{I+fS}(S) \cong E_S(K)$.

Thus, we have the short exact sequence $0 \to E_S(K)^{\bigoplus \alpha} \to H^2_m H^{n-1}_I(S) \to E_S(K) \to 0$, which splits, so that $H^2_m H^{n-1}_I(S) \cong E_S(K)^{\bigoplus \alpha+1}$. □

Corollary XI.2.7. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \dots, x_n]]$. Let I be an ideal of S of pure dimension two. Then $H^j_Q H^i_I(S_Q)$ is injective for every prime ideal Q of S.*

Proof. This follows from Lemmas XI.2.4 and XI.2.6. □

Lemma XI.2.8. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \dots, x_n]]$. Let I be an ideal of S such that $\dim(S/I) = 2$, and let m denote its maximal ideal. Then $H^0_m H^{n-1}_I(S) = H^1_m H^{n-1}_I(S) = 0$ and $H^2_m H^{n-1}_I(S)$ is an injective S-module.*

Proof. Take J_1 and J_2, ideals of pure dimensions 1 and 2, respectively, such that $I = J_1 \cap J_2$. By the Mayer-Vietoris sequence of local cohomology, $H^{n-1}_I(S) = H^{n-1}_{J_2}(S)$. Thus, for all j, $H^j_m H^{n-1}_I(S) = H^j_m H^{n-1}_{J_2}(S)$, and the result follows by Lemma XI.2.6. □

Proposition XI.2.9. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, $S = V[[x_1 \dots, x_n]]$, and m the maximal ideal of S. For an ideal $I \subseteq S$ with $\dim(S/I) = d$, $H^d_m H^{n-d+1}_I(S)$ is an injective S-module and $H^j_m H^{n-d+1}_I(S) = 0$ for $j > d$.*

Proof. We proceed by induction on d. If $d = 0, 1$, or 2, we have the result by Lemmas XI.2.4 and XI.2.8. Suppose that $d \geq 3$ and the statement holds for $d - 1$. If $\operatorname{Ass}_S H^{n-d}_I(S) \neq \{m\}$, we pick an element $r \in m$ that is neither in any minimal prime of I, nor of $H^{n-d}_I(S)$, which is possible because $\operatorname{Ass}_S H^{n-d}_I(S)$ is finite (see Theorem IV.3.1 and [Lyu00b, NB12b]). On the other hand, $\operatorname{Ass}_S H^{n-d}_I(S) = \{m\}$, we pick an element $r \in m$ not in any minimal prime of I. We have that $H^d_m(H^{n-d+1}_I(S) = H^{d-1}_m H^{n-d+2}_{I+rS}(S)$ and $H^j_m H^{n-d+1}_I(S) = H^{j-1}_m H^{n-d+2}_{I+rS}(S) = 0$ for $j > d$ as in the proof of [Zha07, Proposition 2.1] because the conclusions of in [Zha07, Lemmas 2.3 and 2.4] hold in our case. Hence, the result follows by the induction hypothesis. □

Theorem XI.2.10. *Let (S, m, K) be either a regular local ring of unramified mixed characteristic, or a regular local ring containing a field. Let $n = \dim(S)$, and let I be an ideal of S such that $\dim(S/I) = d$. Then $\operatorname{inj.dim} H^{n-d}_I(S) = d$.*

Proof. We need to prove that $\operatorname{Ext}^j(R_Q/QR_Q, H^i_{IR_Q}(R_Q)) = 0$ for every prime ideal Q of R, all $i \in \mathbb{N}$, and all $j > d$. We may assume that Q is m because if $Q \subsetneq m$, then $\dim R_Q/IR_Q < d$ and $\operatorname{inj.dim}_{R_Q} H^i_{IR_Q}(R_Q) \leq \dim_{R_Q} H^i_{IR_Q}(R_Q) \leq d$ by [Zho98, Theorem 5.1].

We proceed by induction on n. If $n = 0$, S is a field and the result follows. Assume that the statement holds for all such S of dimension less than n.

Since the theorem is already true for regular local rings that contain a field (cf. [HS93, Lyu93, Lyu00c]), we will focus on the case where S is an unramified regular local ring of unramified mixed characteristic.

Let $E^* = (E^1 \to E^2 \to \ldots)$ be a minimal injective resolution for $H^{n-d+1}_I(S)$. By [Zho98, Theorem 5.1], $E^j = 0$ for $j > d+1$. For every prime ideal $Q \subseteq S$, S_Q is either an unramified regular local ring of mixed characteristic or a regular local ring containing a field. Moreover, $\dim(S_Q/IS_Q) \le d-1$ for every prime ideal $Q \subsetneq m$. Thus, $(E^d)_Q = (E^{d+1})_Q = 0$ by the inductive hypothesis. Hence, E^d and E^{d+1} are supported only at m.

Let $M = \operatorname{Im}(E^{d-1} \to E^d) = \operatorname{Ker}(E^d \to E^{d+1})$. It suffices prove that M is an injective S-module. The modules $H^j_m H^{n-d}_I(S)$ can be computed from the complex $H^0_m(E^*) = (H^0_m(E^1) \to H^0_m(E^2) \to H^0_m(E^3) \to \ldots)$. Let

$$B^j = \operatorname{Im}\left(H^0_m(E^{j-1}) \to H^0_m(E^j)\right) \text{ and } Z^j = \operatorname{Ker}\left(H^0_m(E^j) \to H^0_m(E^{j+1})\right).$$

Note that $Z^d = M$ since E_d and E_{d+1} are supported only at m. Since $\operatorname{inj.dim} Z^j \le 1$ and $\operatorname{inj.dim} H^j_m H^{n-d}_I(S) \le 1$ by the proof of [Zho98, Theorem 5.1] or by Theorem IV.4.3, as in the proof of [Zho98, Theorem 5.1], we obtain that B^j is injective from the following short exact sequences:

$$0 \to Z^j \to H^0_m(E^j) \to B^j \to 0, \quad \text{and} \quad 0 \to B^{j-1} \to Z^j \to H^j_m(H^{n-d}_I(S)) \to 0.$$

Since $H^d_m H^{n-d}_I(S)$ injective by Proposition XI.2.9, we know that $Z^d = M$ is injective due to the short exact sequence $0 \to B^{d-1} \to Z^d \to H^j_m H^{n-d}_I(S) \to 0$. Therefore, $E_{d+1} = 0$, so $\operatorname{inj.dim} H^{n-d}_I(S) = d$. □

Definition XI.2.11 (Highest Lyubeznik number in mixed characteristic)**.** For a local ring of dimension d, (R, m, K), such that $\operatorname{char}(K) = p > 0$, the *highest Lyubeznik number of R in mixed characteristic* is $\widetilde{\lambda}_{d,d}(R)$.

Note that the nomenclature "highest" is justified by Theorem XI.2.10. Moreover, we may also justify the following definition:

Definition XI.2.12 (Lyubeznik table in mixed characteristic)**.** For (R, m, K) a local ring such that $\operatorname{char}(K) = p > 0$ and $d = \dim(R)$, the *Lyubeznik table of R in mixed characteristic* is the $(d+1) \times (d+1)$ matrix $\widetilde{\Lambda}(R)$, where $\widetilde{\Lambda}(R)_{i,j} = \widetilde{\lambda}_{i,j}(R)$ for $0 \le i, j \le d$.

Remark XI.2.13. Recall that for a local ring R of dimension d containing a field, the *Lyubeznik table* of R is defined as the $(d+1) \times (d+1)$ matrix $\widetilde{\Lambda}(R)$ such that $\widetilde{\Lambda}(R)_{i,j} = \lambda_{i,j}(R)$ for $0 \leq i, j \leq R$. This matrix contains all nonzero Lyubeznik numbers, and is also upper triangular, since $\lambda_{i,j}(R) = 0$ if either $i > j$ or $j > d$ [Lyu93, Properties 4.4i, 4.4ii].

On the other hand, Proposition XI.1.11 and Theorem XI.2.10 imply that the Lyubeznik table in mixed characteristic contain all nonzero Lyubeznik numbers in mixed characteristic. However, Proposition XI.1.11 only implies that the Lyubeznik table in mixed characteristic is nonzero below the subdiagonal.

XI.3 Examples where the Lyubeznik numbers in equal characteristic and the Lyubeznik numbers in mixed characteristic are equal

Lemma XI.3.1. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let M be an S-module such that $\dim_K \operatorname{Ext}^i_S(K, M)$ is finite for all $i \in \mathbb{N}$. Suppose that $M \xrightarrow{\cdot p} M$ is surjective. Then for all $i \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^i_S(K, M) = \dim_K \operatorname{Ext}^i_{S/pS}(K, \operatorname{Ann}_M pS).$$

Proof. Let $R = S/pS$ and $N = \operatorname{Ann}_M(pS)$. The short exact sequence $0 \to N \to M \xrightarrow{\cdot p} M \to 0$ induces the long exact sequence

$$0 \to \operatorname{Ext}^0_S(K, N) \to \operatorname{Ext}^0_S(K, M) \xrightarrow{\cdot p} \operatorname{Ext}^0_S(K, M) \to \operatorname{Ext}^1_S(K, N) \to \cdots .$$

Since multiplication by p is zero on $\operatorname{Ext}^i_S(K, M)$, we have short exact sequences

$$0 \to \operatorname{Ext}^{i-1}_S(K, M) \to \operatorname{Ext}^i_S(K, N) \to \operatorname{Ext}^i_S(K, M) \to 0.$$

for all $i \in \mathbb{N}$, so that $\dim_K \operatorname{Ext}^i_S(K, N) = \dim_K \operatorname{Ext}^{i-1}_S(K, M) + \dim_K \operatorname{Ext}^i_S(K, M)$. We can compute $\operatorname{Ext}^i_S(K, N)$ using the Koszul complex, $\mathcal{K}$, with respect to the sequence $p, x_1, \ldots, x_n$ in S. On the other hand, we can compute $\operatorname{Ext}^i_R(K, N)$ using the Koszul complex, $\overline{\mathcal{K}}$, with respect to the sequence $\overline{x}_1, \ldots, \overline{x}_n$, in R. Now, $\mathcal{K}(N)$ is the direct sum of $\overline{\mathcal{K}}(N)$ and an indexing shift of the same complex by one. This means that

$$\dim_K \operatorname{Ext}^i_S(K, N) = \dim_K \operatorname{Ext}^{i-1}_R(K, N) + \dim_K \operatorname{Ext}^i_R(K, N), \text{ so}$$
$$\dim_K \operatorname{Ext}^{i-1}_S(K, M) + \dim_K \operatorname{Ext}^i_S(K, M) = \dim_K \operatorname{Ext}^{i-1}_R(K, N) + \dim_K \operatorname{Ext}^i_R(K, N).$$

Since $\dim_K \operatorname{Ext}_S^{-1}(K, M) = \dim_K \operatorname{Ext}_R^{-1}(K, N) = 0$, we have that $\dim_K \operatorname{Ext}_S^0(K, M) = \dim_K \operatorname{Ext}_R^0(K, N)$ as well. Inductively, $\dim_K \operatorname{Ext}_S^i(K, M) = \dim_K \operatorname{Ext}_R^i(K, N)$ for all $i \geq 0$. □

Corollary XI.3.2. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let I be an ideal of S such that S/I is a Cohen-Macaulay ring of characteristic p. Then for all $i, j \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^i_{S/pS}(K, H^{n-j}_{IS/pS}(S/pS)) = \dim_K \operatorname{Ext}^i_S(K, H^{n+1-j}_I(S)).$$

Proof. Let $R = S/pS$. The short exact sequence $0 \to S \xrightarrow{\cdot p} S \to R \to 0$ induces the short exact sequence $0 \to H^{n-d}_I(R) \to H^{n-d+1}_I(S) \xrightarrow{\cdot p} H^{n-d+1}_I(S) \to 0$ since $H^{n-d+1}_I(R) = 0$ by [PS73, Proposition 4.1]. Since $H^{n-d}_I(S) = 0$ $H^i_I(S) \xrightarrow{\cdot p} H^i_I(S)$ is injective for $i \neq n - d + 1$, and $H^{n-d}_I(S) = 0$. The result then follows from Lemma XI.3.1. □

Proposition XI.3.3. *Let (V, pV, K) be a complete DVR of unramified mixed characteristic $p > 0$, and let $S = V[[x_1 \ldots, x_n]]$. Let I be an ideal of S containing p, such that* $\dim(S/I) \leq 2$. *Then*

$$\dim_K \operatorname{Ext}^d_{S/pS}(K, H^{n-d}_{IS/pS}(S/pS)) = \dim_K \operatorname{Ext}^d_S(K, H^{n+1-d}_I(S)).$$

Proof. Let $R = S/pS$. Consider the following cases.

If $\dim(S/I) = 0$, $H^{n+1}_I(S) = E_S(K)$ and $H^n_I(S) = E_R(K)$. In this case, we have that $\dim_K \operatorname{Ext}^d_{S/pS}(K, H^{n-d}_{IS/pS}(S/pS)) = \dim_K \operatorname{Ext}^d_S(K, H^{n+1-d}_I(S)) = 1$.

If $\dim(S/I) = 1$, the short exact sequence $0 \to S \xrightarrow{\cdot p} S \to R \to 0$ induces a long exact sequence $0 \to H^{n-1}_I(R) \to H^n_I(S) \xrightarrow{\cdot p} H^n_I(S) \to 0$ by the Hartshorne-Lichtenbaum vanishing theorem. The proposition then follows from Lemma XI.3.1.

Suppose that $\dim(S/I) = 2$. First assume that I has pure dimension 2. Let α be the number of connected components of $\operatorname{Spec}(\widehat{A}) \setminus \{m\}$, where $A = \widehat{R/I}^{sh}$ is the strict Henselization of R/I. In fact, $\alpha = \dim_K \operatorname{Ext}^2_R(K, H^{n-2}_I(S/pS))$ (cf. [Wal01, Proposition 2.2]).

We prove the statement by induction on α. If $\alpha = 1$, the short exact sequence

$$0 \to S \xrightarrow{\cdot p} S \to R \to 0$$

induces the short exact sequence

$$0 \to H^{n-2}_I(R) \to H^{n-1}_I(S) \xrightarrow{\cdot p} H^{n-1}_I(S) \to 0,$$

since $H_I^{n-1}(R) = 0$ by [HL90, Theorem 2.9]. The proposition then follows from Lemma XI.3.1. If $\alpha > 1$, we pick ideals $J_1, \ldots, J_\alpha$ such that $I = J_1 \cap \ldots \cap J_\alpha$, and each J_k defines a connected component of $\operatorname{Spec}(\widehat{A}) \setminus \{m\}$. Let J denote $J_1 \cap \ldots \cap J_{\alpha-1}$. Using the Mayer-Vietoris sequence, we obtain an isomorphism $H_J^{n-1}(S) \oplus H_{J_\alpha}^{n-1}(S) \cong H_I^{n-1}(S)$ because $\sqrt{J + J_\alpha} = m$. Then

$$\dim_K \operatorname{Ext}_S^2(K, H_I^{n-1}(S)) = \dim_K \operatorname{Ext}_S^2(K, H_J^{n-1}(S)) + \dim_K \operatorname{Ext}_S^2(K, H_{J_\alpha}^{n-1}(S)) = \alpha.$$

By Lemma XI.2.6, [Lyu93, Lemma 1.4] and [Wal01, Proposition 2.2], the other numbers are determined by α.

For the general case such that $\dim(S/I) = 2$, let $P_1, \ldots, P_r$ be the minimal primes of dimension one of I, and let $Q_1, \ldots, Q_s$ be the minimal primes of dimension two of I. Let $J_1 = P_1 \cap \ldots \cap P_r$ and $J_2 = Q_1 \cap \ldots \cap Q_s$. We claim that $\operatorname{Ext}_S^j(K, H_I^{n-1}(S)) = \operatorname{Ext}_S^j(K, H_{J_2}^{n-1}(S))$. Let $f_1, \ldots, f_\ell \in J_2 \setminus I$ such that $I + (f_1, \ldots, f_\ell)S = J_2$. We proceed by induction on ℓ; first assume that $\ell = 1$. Since $H_I^{n-1}(S) = H_{J_2}^{n-1}(S)$, $H_I^{n-1}(S_{f_1}) = 0$. The long exact sequence

$$0 \to H_{I+f_1S}^{n-1}(S) \to H_I^{n-1}(S) \to H_I^{n-1}(S_{f_1}) \to H_{I+f_1S}^{n}(S) \to H_I^n(S) \to H_I^n(S_{f_1}) \to 0,$$

then indicates both that $H_{I+f_1S}^{n-1}(S) \cong H_I^{n-1}(S)$, and that $0 \to H_{I+f_1S}^n(S) \to H_I^n(S) \to H_I^n(S_f) \to 0$ is exact. Hence, $\operatorname{Ext}_S^j(K, H_I^{n-1}(S)) = \operatorname{Ext}_S^j(K, H_{I+f_1S}^{n-1}(S))$. Moreover, $I + f_1S \subseteq J_2$ is an ideal of dimension 2, whose minimal primes of dimension 2 are $P_1, \ldots, P_r$. If we assume that the claim is true for ℓ, the proof for $\ell + 1$ is analogous to the previous part. □

Corollary XI.3.4. *Let (R, m, K) be a local ring of characteristic $p > 0$. If R is a Cohen-Macaulay ring or if $\dim R \leq 2$, then for $i, j \in \mathbb{N}$, $\widetilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R)$.*

Proof. Since dimension, Cohen-Macaulayness, and both Lyubeznik numbers are preserved after completion, we can assume that R is complete. Then the result follows from Corollary XI.3.2 and Proposition XI.3.3. □

XI.4 An example for which the equal-characteristic and the Lyubeznik numbers in mixed characteristic differ

Remark XI.4.1. A certain minimal triangulation of the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ defines the Stanley-Reisner ideal of $K[x_1, \ldots, x_6]$ generated by the ten monomials

$$x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5,\ x_1x_4x_6,\ x_1x_5x_6,\ x_2x_3x_6,\ x_2x_4x_5,\ x_2x_5x_6,\ x_3x_4x_5,\ x_3x_4x_6.$$

The projective variety defined by this ideal is called *Reisner's variety* [Rei76, Remark 3].

Throughout this section, we will often refer to the following ring and ideal.

Notation XI.4.2. Let $R = \mathbb{Z}_{(2)}[x_1, \ldots, x_6]$. Moreover, let I denote the ideal of R generated by 2 and the ten monomial generators from Remark XI.4.1.

Remark XI.4.3. It is easily checked that for $I \subseteq R$ as in Notation XI.4.2, $\operatorname{depth}_I(R) = 4$. With $p = 2$, this means that the short exact sequence $0 \to R \to R_p \to R_p/R \to 0$ induces the long exact sequence

$$0 \to H^3_I(R_p/R) \to H^4_I(R) \to H^4_I(R_p) \to H^4_I(R_p/R) \to \cdots . \tag{XI.4.3.1}$$

Since $p = 2 \in I$, $H^i_I(R_p) = 0$ for all $i \in \mathbb{N}$, so $H^i_I(R_p/R) \cong H^{i+1}_I(R)$.

Remark XI.4.4. Given a polynomial ring A over $\mathbb{Z}$, an ideal $\mathfrak{a}$ of A, and a prime $p \in \mathbb{Z}$, the short exact sequence $0 \to A/pA \xrightarrow{\cdot p} A/p^2A \to A/pA \to 0$ induces the *Bockstein homomorphisms* $\partial_j : H^j_{\mathfrak{a}}(A/pA) \to H^{j+1}_{\mathfrak{a}}(A/pA)$ for each $j \in \mathbb{N}$, the connecting homomorphisms in the long exact sequence for local cohomology. For $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \ldots, x_6]$ generated by the ten monomials given in Remark XI.4.1, Singh and Walter showed that the Bockstein homomorphism ∂_3 is nonzero if and only if $p = 2$ [SW11, Example 5.10].

Proposition XI.4.5. *For $I \subseteq R$ from Notation XI.4.2 and $p = 2$, the map*

$$H^3_I(R_p/R) \xrightarrow{\cdot p} H^3_I(R_p/R)$$

is not surjective.

Proof. Since $\operatorname{depth}_I(R) = 4$ and $H^i_I(R_p/R) \cong H^{i+1}_I(R)$, $H^i_I(R_p/R) = 0$ for $i \leq 3$ by the long exact sequence in local cohomology (see Remark XI.4.3). For every $\ell \in \mathbb{N}$, the exact sequence $0 \to R/p^\ell R \to R_p/R \xrightarrow{\cdot p^\ell} R_p/R \to 0$ induces a long exact sequence

$$0 \to H^3_I(R/p^\ell R) \to H^3_I(R_p/R) \xrightarrow{\cdot p^\ell} H^3_I(R_p/R) \to H^4_I(R/p^\ell R) \to \cdots . \tag{XI.4.5.1}$$

In particular, $H^3_I(R/p^\ell R) \cong \operatorname{Ann}_{H^3_I(R_p/R)}(p^\ell R)$.

As the direct limit functor is exact, the limit of the direct system of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R/pR & \xrightarrow{\cdot p} & R/p^2R & \longrightarrow & R/pR & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \cdot p} & & \downarrow {\scriptstyle \cdot p} & & \\
0 & \longrightarrow & R/pR & \xrightarrow{\cdot p^2} & R/p^3R & \longrightarrow & R/p^2R & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \cdot p} & & \downarrow {\scriptstyle \cdot p} & & \\
0 & \longrightarrow & R/pR & \xrightarrow{\cdot p^3} & R/p^4R & \longrightarrow & R/p^3R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is the short exact sequence $0 \to R/pR \to R_p/R \xrightarrow{\cdot p} R_p/R \to 0$. Moreover, $H^j_I(R_p/R) = \varinjlim_\ell H^j_I(R/p^\ell R)$. We obtain the following isomorphism of sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^3_I(R/pR) & \xrightarrow{\cdot p} & H^3_I(R/p^2R) & \xrightarrow{\pi} & H^3_I(R/pR) & \xrightarrow{\partial_3} & \cdots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \operatorname{Ann}_{H^3_I(R_p/R)} pR & \longrightarrow & \operatorname{Ann}_{H^3_I(R_p/R)} p^2R & \xrightarrow{\cdot p} & \operatorname{Ann}_{H^3_I(R_p/R)} pR & \longrightarrow & \cdots
\end{array}$$

By Remark XI.4.4, ∂_3 is nonzero, so that π is not surjective, so the map

$$\operatorname{Ann}_{H^3_I(R_p/R)} p^2R \xrightarrow{\cdot p} \operatorname{Ann}_{H^3_I(R_p/R)} pR$$

is not either, and multiplication by p on $H^3_I(R_p/R)$ is not either. □

Remark XI.4.6. Let $A = \mathbb{F}_2[y_1, \ldots, y_5]$, and let $J = (y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1)$. Then $J = (y_2, y_3, y_5) \cap (y_1, y_3, y_4) \cap (y_1, y_2, y_4) \cap (y_1, y_3, y_5) \cap (y_2, y_4, y_5)$, and A/J is a graded Cohen-Macaulay ring of dimension 2, where the classes of $y_1 + y_2 + y_3$ and $y_1 + y_4 + y_5$ form a homogeneous system of parameters. Then $H^i_J(A) \neq 0$ if and only if $i = 3$ [PS73, Proposition 4.1]. (See [ÀMGLZA03, Proposition 3.1] for an analog in characteristic zero.)

Lemma XI.4.7. *For $I \subseteq R$ from Notation XI.4.2, $H^4_I(R/2R)$ is supported only at the maximal ideal $(2, x_1, \dots, x_6)$.*

Proof. Let $\overline{R} = R/2R \cong \mathbb{F}_2[x_1, \dots, x_6]$. As $I\overline{R}$ is a square-free monomial ideal, by [Yan00, Proposition 2.5] and [Yan01, Proposition 2.7], every prime in $\operatorname{Ass}_R H^4_I(\overline{R})$ is of the form $(2, x_{i_1}, \dots, x_{i_j})R$ for some $\{i_1, \dots i_j\} \subseteq \{1, \dots, 6\}$. Thus, it suffices to show that each $H^4_I(\overline{R})_{x_i} = 0$.

First consider $H^4_I(\overline{R})_{x_6}$; the other cases are analogous. For $A := \mathbb{F}_2[x_1, \dots, x_5]$ and $J := (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1) \subseteq A$, $H^4_J(A) = 0$ by Remark XI.4.6. Since $A[x_6]_{x_6} = \overline{R}_{x_6}$ is a flat extension, $H^4_J(\overline{R})_{x_6} = H^4_J(A) \otimes_A \overline{R}_{x_6} = 0$. As $J\overline{R}_{x_6} = I\overline{R}_{x_6}$, $H^4_I(\overline{R})_{x_6} = H^4_J(\overline{R})_{x_6} = 0$. □

Corollary XI.4.8. *Take $I \subseteq R$ from Notation XI.4.2, and let $S = \widehat{R}_m$, where m is the maximal ideal $(2, x_1, \dots, x_6)$ of R. Then for $p = 2$,*

$$\operatorname{Coker}\left(H^3_I(S_p/S) \xrightarrow{\cdot p} H^3_I(S_p/S)\right) \cong H^4_I(S/pS) \cong E_{S/pS}(\mathbb{F}_2).$$

Proof. Note that $S = \widehat{\mathbb{Z}}_{(2)}[[x_1, \dots, x_6]]$, $\widehat{\mathbb{Z}}_{(2)}$ the 2-adic integers, and that $S/pS \cong \mathbb{F}_2[[x_1, \dots, x_6]]$. By Lemma XI.4.7, $H^4_I(S/pS)$ is supported only at m, so $H^4_I(S/pS) = H^0_m H^4_I(S/pS)$, and thus is injective by [Lyu93, Corollary 3.6]; as its Bass numbers are finite [HS93], $H^4_I(S/pS) \cong E_{S/pS}(\mathbb{F}_2)^{\oplus \alpha}$ for some $\alpha \in \mathbb{N}$. By the calculation in [ÀMV, Example 4.8] (see Remark XI.4.11), $\dim_{\mathbb{F}_2} \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, H^4_I(S)) = \lambda_{0,2}(S/pS) = 1$, so that $\alpha = 1$ and $H^4_I(S/pS) \cong E_{S/2S}(\mathbb{F}_2)$.

Now, for $p = 2$, $\operatorname{Coker}\left(H^3_I(S_p/S) \xrightarrow{\cdot p} H^3_I(S_p/S)\right)$ injects into $H^4_I(S/pS) = E_{S/pS}(\mathbb{F}_2)$ by the long exact sequence (induced by $0 \to S/pS \to S_p/S \xrightarrow{\cdot p} S_p/S \to 0$)

$$0 \to H^3_I(S/pS) \to H^3_I(S_p/S) \xrightarrow{\cdot p} H^3_I(S_p/S) \to H^4_I(S/pS) \to \dots.$$

Thefore, this cokernel is a $D(S/pS, \mathbb{F}_2)$-submodule of $E_{S/pS}(\mathbb{F}_2)$, itself is a simple $D(S/pS, \mathbb{F}_2)$-module. Since it is nonzero by Proposition XI.4.5, we are done. □

Corollary XI.4.9. *There exists a regular local ring S of unramified mixed characteristic $p = 2$, and an ideal I of S containing p, so that the map*

$$H^4_I(S) \xrightarrow{\cdot p} H^4_I(S)$$

is not surjective.

Proof. Again, take $I \subseteq R$ from Notation XI.4.2, and let $S = \widehat{R}_m$, where $m =$

$(2, x_1, \ldots, x_6)R$. Then by Corollary XI.4.8, Coker $\left(H_I^4(S) \xrightarrow{\cdot p} H_I^4(S)\right) \cong E_{S/pS}(\mathbb{F}_2) \neq 0$. ☐

Proposition XI.4.10. *Take $I \subseteq R$ from Notation XI.4.2, and let $S = \widehat{R}_m$, where $m = (2, x_1, \ldots, x_6)$. Then $\widetilde{\lambda}_{i,j}(S/IS) = 1$ if $i = j = 3$, and vanishes otherwise.*

Proof. For brevity, let I denote IS, and let $p = 2$. In [Lyu84, Theorem 1, Example 1], it is shown that $H_I^5(S/pS) = 0$ (relying on the fact that $\mathrm{char}(S/pS) = 2$). By Corollary XI.4.8 and Remark XI.4.3, the short exact sequence $0 \xrightarrow{\cdot p} S \to S \to S/pS \to 0$ then gives rise to the long exact sequence

$$\ldots \to H_I^4(S) \xrightarrow{\cdot p} H_I^4(S) \to H_I^4(S/pS) \xrightarrow{0} H_I^5(S) \xrightarrow{\cdot p} H_I^5(S) \to 0 \to H_I^6(S) \xrightarrow{\cdot p} \ldots.$$

Thus, multiplication by p on $H_I^5(S)$ and $H_I^5(S)$ are injective maps, which implies that $H_I^5(S) = H_I^6(S) = 0$ since $p \in I$. Moreover, $H_I^i(S) = 0$ for $i \geq 6$ as well, so by again noting Corollary XI.4.8, $H_I^i(S) \neq 0$ if and only if $i = 4$.

This means that the spectral sequence $E_2^{p,q} = H_m^p H_I^q(S) \underset{p}{\Longrightarrow} H_m^{p+q}(S) = E_\infty^{p,q}$ converges at the second stage. Thus, $H_m^3 H_I^4(S) \cong H_m^7(S) \cong E_S(\mathbb{F}_2)$, and all other $H_m^p H_I^q(S)$ vanish. Since all $H_m^p H_I^q(S)$ are injective S-modules, [Lyu93, Lemma 1.4] indicates that the Bass number $\dim_{\mathbb{F}_2} \mathrm{Ext}_S^p(\mathbb{F}_2, H_I^q(S)) = \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, H_m^p H_I^q(S))$ for all $p, q \in \mathbb{N}$. Since $\dim(R) = 7$, we have that

$$\widetilde{\lambda}_{i,j}(R) = \dim_K \mathrm{Hom}_S(\mathbb{F}_2, H_m^i H_I^{7-j}(S)) = 1$$

if $i = j = 3$, and vanishes otherwise. ☐

Remark XI.4.11. Using work of Àlvarez Montaner and Vahidi, and of Singh and Walther, we finally may conclude that that the Lyubeznik numbers in mixed characteristic do not always agree the original Lyubeznik numbers. Take $I \subseteq R$ as defined in Notation XI.4.2. Let $S_1 = \mathbb{F}_2[[x_1, \ldots, x_6]]$, and $S_2 = \widehat{\mathbb{Z}}_{(2)}[[x_1, \ldots, x_6]]$.

$$\Lambda(S_1/IS_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \widetilde{\Lambda}(S_2/IS_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This indicates that if A is the completion of the Stanley Reisner ring of the ideal in Remark XI.4.1 with $K = \mathbb{F}_2$, then $\lambda_{0,2}(A) = \lambda_{2,3}(A) = 1$, while $\widetilde{\lambda}_{0,2}(A) = \widetilde{\lambda}_{2,3}(A) = 0$. In particular, this gives a negative answer to Question XI.1.10.

We note that the computation in Remark XI.4.11 is related to work in [ÀMGLZA03].

Theorem XI.4.12. *There exists a regular local ring (S, m, K) of unramified mixed characteristic $p = 2$, and an ideal I of S, such that for some $i, j \in \mathbb{N}$,*

$$\dim_K \operatorname{Ext}^j_S(K, H^i_I(S)) \neq \dim_K \operatorname{Ext}^j_{S/pS}(K, H^{i-1}_{IS/pS}(S/pS)).$$

Proof. Take $I \subseteq R$ from Notation XI.4.2, and let $S = \widehat{R}_m$, where $m = (2, x_1, \ldots, x_6)$. Then $\dim_K \operatorname{Ext}^0_R(K, H^5_I(S)) = 0 \neq 1 = \dim_K \operatorname{Ext}^0_{S/pS}(K, H^4_{IS/pS}(S/pS))$ by Proposition XI.4.10. □

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