

The Asymptotic Behavior of Generic Initial Systems

by

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This dissertation is dedicated to the memory of my grandmothers,
Donna Evelyn Nash Campbell (1924–1995)
and
Shirley Anne Plowman Mayes (1932–2006).

In the nine years that I spent with her, Grandma Campbell poured enough love and encouragement on me to last a lifetime. I will always remember her generosity, grace, and warmth.

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CHAPTER I

Introduction and Background for Non-Algebraists

This thesis is part of a powerful new research trend in algebra: the study of the asymptotic behavior of families of related objects. While mathematicians have traditionally worked to understand individual mathematical objects, recent work has demonstrated that infinite collections of algebraic objects often exhibit remarkable structures that are not seen when looking at the individual algebraic objects. Taking an asymptotic viewpoint in mathematics is akin to taking a big picture perspective of nature: the scene of an autumn forest with its multi-coloured trees is much more beautiful than you would expect by just looking at a single leaf.

The purpose of this chapter is to provide background information from the fields of computational commutative algebra and algebraic geometry necessary for understanding the main results of the thesis. Computational commutative algebra is concerned with studying the algebraic structure of certain collections of polynomials. One can produce a wide range of geometric objects by considering the zero-sets of polynomials; for example, the zero-set of the polynomial $y - x^2$ is a parabola while the zero-set of the polynomial $x^2 + y^2 - 8$ is a circle. The association between collections of polynomial equations and the geometry of their zero-sets provides a bridge between algebra and geometry; it is the subject of the field of algebraic geometry.

Our main asymptotic construction occurs in three steps which will be expanded upon throughout this chapter.

1. Begin with a geometric object, like a curve or a set of points, represented by a collection of polynomials called I .
2. Build an infinite collection of *generic initial ideals* from I : $\{\text{gin}(I^m)\}_m$.
3. Look at the asymptotics of the collection $\{\text{gin}(I^m)\}_m$ by studying its *limiting shape*

These three steps mirror the three main themes of this thesis: algebraic geometry, generic initial ideals, and asymptotic behavior. The presentation of material in Chapter I, then, will also follow these three themes.

Geometry (Section 1.2): We will begin by describing how sets of polynomials represent geometric objects in familiar spaces through their zero sets; such geometric objects are *affine algebraic sets*. As an example, we will look at the geometric objects associated to important sets of polynomials called *complete intersections* that will be the starting point for Chapters III and IV. We will then turn our attention to an alternative geometric space called the *projective plane* \mathbb{P}^2 and see how *homogeneous* polynomials cut out subsets of \mathbb{P}^2 . As an example, we will study sets of polynomials associated to the so-called *fat point* subsets of \mathbb{P}^2 ; these will be the starting point for our work in Chapters V through VII.

Generic Initial Ideals (Sections 1.3 and 1.4): It is often useful to associate a collection of polynomials to a related question of monomials since monomials are the easiest types of polynomials to manipulate. The *generic initial ideal* is a way of making such an association. In Section 1.3 we will study topics related to special collections of monomials called *monomial ideals*. In Section 1.4 we will define generic initial ideals and discuss two of the important properties that make studying them worthwhile: they are fixed under simple maps (that is, they are *independent of a choice of coordinates*) and they possess a combinatorial property called *Borel-fixedness*. Finally, we will prove that generic initial ideals actually exist.

Asymptotics (Section 1.5): We will describe how to associate a geometric shape, called a *limiting shape* to an infinite collection of generic initial ideals (or, more generally, to an infinite collection of monomial ideals). This limiting shape describes the asymptotic behavior of the entire collection of ideals; several of the main results in this thesis involve finding the limiting shape for various collections of generic initial ideals.

1.1 Preliminaries: Polynomials, Rings, and Ideals

We will begin by introducing the basic algebraic objects that form the basis of algebraic geometry and commutative algebra. The most fundamental of these objects are monomials in several variables familiar from high school math.

Definition 1.1.1. A **monomial** in the variables x_1, \dots, x_n is a product of the form

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where each α_i is a nonnegative integer: $\alpha_i \in \mathbb{N}$.¹

Example 1.1.1. $x_1x_2x_3 = x_2x_1x_3$, $x_1^8x_2^{77}$, and x_4^5 are monomials. $x_1 + x_2$, $6x_1^4$, and $x_1^{\frac{7}{8}}x_3^3$ are not monomials.

Sometimes we will write monomials in an abbreviated form, using *multi-index* notation. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ then

$$x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

While our definition of monomials only depends on the variables x_1, \dots, x_n that are chosen, polynomials depend on what *field* K we are working over. Roughly, a *field* is a set where the operations addition, subtraction, multiplication, and division can be defined.

Example 1.1.2. 1. The most familiar fields are: \mathbb{Q} , the set of all rational numbers; \mathbb{R} , the set of all real numbers; and \mathbb{C} , the set of all complex numbers.

2. \mathbb{N} , the set of all natural numbers, is *not* a field since the quotient of two natural numbers is not always a natural number. This means that division is not defined within \mathbb{N} .

Exercise 1. Prove that the set of numbers

$$\mathbb{Q}[i] := \{a + bi : a, b, \in \mathbb{Q}\}$$

contained in \mathbb{C} is a field by demonstrating that it is closed under addition, subtraction, and multiplication, and that any nonzero element $a + bi$ of $\mathbb{Q}[i]$ has an inverse $(a + bi)^{-1} \in \mathbb{Q}[i]$ such that $(a + bi) \cdot (a + bi)^{-1} = 1$.²

\mathbb{Q} , \mathbb{R} , \mathbb{C} , and $\mathbb{Q}[i]$ each contain all of the rational numbers \mathbb{Q} ; such fields are said to be of *characteristic zero*.³ While the main ideas of this thesis work for any field of characteristic zero, we will use the field \mathbb{C} throughout most of this chapter.

Our definition of a *polynomial* is the standard one in several variables: a polynomial is the sum of multiples of monomials.

¹The variables in this definition commute, so we may also write $x_2^{\alpha_2}x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} = x_1^{\alpha_1-1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}x_1$, and so on.

²To be a field, a set must also contain 1 and possess the associative and distributive properties; these properties are clear in this example. See Appendix A for solutions to the exercises in this chapter.

³More generally, the *characteristic* of a field is the smallest $p \in \mathbb{N}$ such that $p \cdot 1 = 0$, or is zero if no such p exists.

Definition 1.1.2. Let K be field, such as \mathbb{C} or \mathbb{R} . A **polynomial** f in the variables x_1, \dots, x_n with coefficients in K is of the form

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$$

where each c_α is an element of K , x^α is a monomial in the variables x_1, \dots, x_n , and all but a finite number of the c_α s are equal to zero.⁴

Example 1.1.3. $5x_1 + \frac{8}{3}x_1^2x_4$ is a polynomial in both $\mathbb{C}[x_1, \dots, x_4]$ and $\mathbb{R}[x_1, \dots, x_4]$. $ix_1^6x_3 + x_2^7$ is a polynomial in $\mathbb{C}[x_1, \dots, x_3]$ (and in $\mathbb{C}[x_1, \dots, x_4]$) but not in $\mathbb{R}[x_1, \dots, x_3]$.

Definition 1.1.3. The collection of all polynomials in the variables x_1, \dots, x_n with coefficients in K is denoted by

$$K[x_1, \dots, x_n]$$

and is called the **polynomial ring** in n variables with coefficients in K .

We add and multiply polynomials in the usual way. For example,

$$(x_1^2 + 5x_1x_2) + (x_1x_2 + x_2^{100}) = x_1^2 + 6x_1x_2 + x_2^{100}$$

and

$$\begin{aligned} (x_1^2 + 5x_1x_2) \cdot (x_1x_2 + x_2^{100}) &= x_1^2(x_1x_2 + x_2^{100}) + 5x_1x_2(x_1x_2 + x_2^{100}) \\ &= x_1^3x_2 + x_1^2x_2^{100} + 5x_1^2x_2^2 + 5x_1x_2^{101}. \end{aligned}$$

Sums and products of polynomials are always polynomials. Further, this addition and multiplication satisfies the commutative, associative, and distributive properties and interacts with the polynomials 0 and 1 in the way we would like. This tells us that $K[x_1, \dots, x_n]$ has the structure of a *commutative ring*.

Lemma 1.1.4. Let f, g , and h be polynomials in the variables x_1, \dots, x_n with coefficients in K (that is, $f, g, h \in K[x_1, \dots, x_n]$). Then

- $f + g, f \cdot g \in K[x_1, x_2, \dots, x_n]$;
- $(f + g) + h = f + (g + h)$ and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$;
- $0 + f = f + 0 = f$ and $1 \cdot f = f \cdot 1 = f$;

⁴We are using multi-index notation in this definition.

- $f \cdot (g + h) = f \cdot g + f \cdot h$ and $(f + g) \cdot h = f \cdot h + g \cdot h$; and
- $f + g = g + f$ and $f \cdot g = g \cdot f$.

We will be interested in studying special collections of polynomials in $R = K[x_1, \dots, x_n]$ that are closed under addition and that ‘absorb’ polynomials from R under multiplication; such collections are called *ideals*.

Definition 1.1.5. An **ideal** I of $R = K[x_1, \dots, x_n]$ is a subset of R with the following properties.

- For any polynomials f and g of I , $f + g$ is in I .
- For any polynomial f in I and an arbitrary polynomial $r \in R$, $rf \in I$. In particular, 0 is in every ideal.

To describe ideals in a reasonable way, we usually give a set of *generators*.

Definition 1.1.6. Let $R = K[x_1, \dots, x_n]$ and suppose that S is some subset of R (that is, S is a collection of polynomials). Then the **ideal of R generated by S** , denoted $\langle S \rangle$, is the smallest ideal containing S . When $S = \{f_1, \dots, f_r\}$, we may write $\langle S \rangle = \langle f_1, \dots, f_r \rangle$ or $\langle S \rangle = (f_1, \dots, f_r)$.

Exercise 2. Show that $\langle S \rangle$ is the collection of polynomials

$$\langle S \rangle = \left\{ \sum_{i=1}^s h_i f_i : h_i \in R, f_i \in S \right\}.$$

Example 1.1.4. Two different sets can generate the same ideal. For example, let $R = \mathbb{R}[x_1, x_2, x_3]$ and consider the sets

$$S_1 = \{x_1x_2, x_1x_2 + x_3^3, x_2x_3\}$$

and

$$S_2 = \{x_1x_2, x_3^3, x_2x_3\}.$$

To show that $\langle S_1 \rangle = \langle S_2 \rangle$, we must show that $x_1x_2 + x_3^3 \in \langle S_2 \rangle$ and $x_3^3 \in \langle S_1 \rangle$; we will then be able to build all of the polynomials in S_1 from S_2 and vice versa by Definition 1.1.6. Since $(x_1x_2 + x_3^3) - (x_1x_2) = x_3^3$ is in $\langle S_1 \rangle$ by definition and $(x_1x_2) + (x_3^3)$ is in $\langle S_2 \rangle$ by definition, $\langle S_1 \rangle = \langle S_2 \rangle$.

As suggested by the terminology, the ideal generated by a set of polynomials in R is in fact an ideal of R . Further, every polynomial ideal can be described by a finite generating set. This is the content of the following famous theorem of Hilbert (see Sections 1.4 and 2.5 of [CLO97] for proofs).

Theorem 1.1.7 (Hilbert’s Basis Theorem [Hil90]). *Let $R = K[x_1, \dots, x_n]$. For every ideal I of R , there is some finite set of polynomials $\{f_1, \dots, f_r\}$ of R such that $I = \langle f_1, \dots, f_r \rangle$.*

Proof. See Section 2.5 of [CLO97]. □

Just as we can add, multiply, and take powers of polynomials, we can add, multiply, and take powers of polynomial ideals.

Definition 1.1.8. Let I and J be ideals of $R = K[x_1, \dots, x_n]$ and let m be a non-negative integer.

- $I + J := \{f + g : f \in I, g \in J\}$
- $IJ := \langle f \cdot g : f \in I, g \in J \rangle$
- $I^m = \langle f_1 \cdot f_2 \cdots f_m : f_i \in I \rangle$

We will be working with a special class of ideals, called *homogeneous ideals*, throughout this thesis. To define them, we will use the notion of degree familiar from high school; the monomial $x_1^{18}x_2^7$, for example, has degree 25.

Definition 1.1.9. Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of $R = K[x_1, \dots, x_n]$. Then the **degree** of x^α is

$$\deg(x^\alpha) := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Exercise 3. Show that $R = K[x_1, \dots, x_n]$ contains exactly $\binom{n+d-1}{n-1} = \binom{n+d-1}{d}$ monomials of degree d .

Definition 1.1.10. If all of the monomials appearing in a polynomial $f \in R = \mathbb{C}[x_1, \dots, x_n]$ are of degree d , we say that f is **homogeneous of degree d** . An ideal $I \subseteq R$ that can be generated by homogeneous polynomials is called a **homogeneous ideal**.

1.2 Connecting Algebra and Geometry

In this section we will introduce the first theme of this thesis: geometry.

1.2.1 Affine Algebraic Sets

We will begin by discussing how to associate geometric objects – such as sets of points, curves, or surfaces – to algebraic objects, namely ideals.

As a starting point, consider the set of points (a_1, a_2) of \mathbb{R}^2 where the polynomial $x^2 - y \in \mathbb{R}[x, y]$ vanishes (that is, the set of points (a_1, a_2) of \mathbb{R}^2 where $a_1^2 - a_2 = 0$). From high school algebra, we know that this is an infinite set of points forming the parabola $y = x^2$ shown in Figure 1.1; this set is denoted $\mathbb{V}(x^2 - y)$.

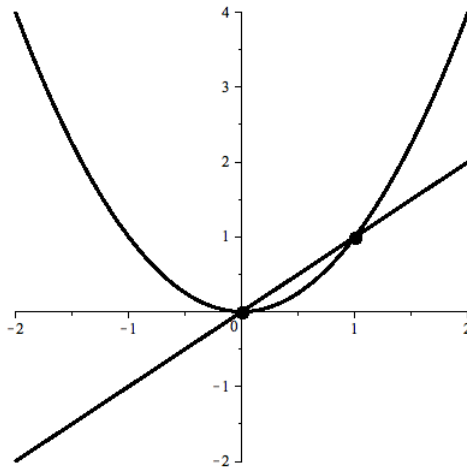


Figure 1.1: $\mathbb{V}(x^2 - y)$ and $\mathbb{V}(x - y)$.

Similarly, $\mathbb{V}(x - y)$ consists of the points of \mathbb{R}^2 where the polynomial $x - y$ vanishes. This is simply the line $y = x$ shown in Figure 1.1.

We may also consider vanishing sets of more than one polynomial. For example, $\mathbb{V}(x^2 - y, y - x)$ is the set of points of \mathbb{R}^2 where *both* $x^2 - y$ and $y - x$ vanish. It is easy to see that this is the set of two points $\{(0, 0), (1, 1)\}$, and that $\mathbb{V}(x^2 - y, x - y) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x - y)$ (these are the points in Figure 1.1 where the parabola and the line intersect).

On the other hand, $\mathbb{V}((x^2 - y)(y - x))$ consists of the points of \mathbb{R}^2 where $(x^2 - y)(y - x)$ vanishes. This is equal to the set of points where *either* $x^2 - y = 0$ or $x - y = 0$, and is given by the union of the parabola and line in Figure 1.1.

We may even consider the vanishing sets of an infinite number of polynomials. For example, consider the set $S = \{x, x^2, x^3, x^4, \dots\} \in \mathbb{R}[x]$. Then $\mathbb{V}(S)$ is the subset of points of $\mathbb{R}^1 = \mathbb{R}$ where x, x^2, x^3, \dots all vanish; thus, $\mathbb{V}(S) = \{0\}$. Note that this vanishing set is the same as the set where x vanishes: $\mathbb{V}(S) = \mathbb{V}(\{x\}) = \{0\}$. We will see that this is due to the fact that $\langle S \rangle = \langle x \rangle$: if two sets of polynomials generate the same ideal, then they have the same vanishing set.

We will let

$$f(a) = f(a_1, \dots, a_n)$$

where f is a polynomial of $K[x_1, \dots, x_n]$ and $a = (a_1, \dots, a_n)$ is a point of K^n . For example, if $f = x^2 - y \in \mathbb{R}[x, y]$ then $f(2, 1) = 2^2 - 1 = 3$.

Definition 1.2.1. Let S be a subset of $K[x_1, \dots, x_n]$. Then the **affine algebraic set** (or **affine variety**) defined by S is the set of points in K^n where all of the polynomials in S vanish:

$$\mathbb{V}(S) := \{a \in K^n : f(a) = 0 \text{ for all } f \in S\}.$$

If $S = \{f_1, \dots, f_s\}$ is finite, we will write $\mathbb{V}(f_1, \dots, f_s)$ in place of $\mathbb{V}(\{f_1, \dots, f_s\})$.

Lemma 1.2.2. *Let S and S' be sets of polynomials in $R = K[x_1, \dots, x_n]$.*

(i) *If $S \subseteq S'$ then $\mathbb{V}(S') \subseteq \mathbb{V}(S)$.*

(ii) *$\mathbb{V}(S) = \mathbb{V}(\langle S \rangle)$. This tells us that every affine algebraic set is the affine algebraic set of an ideal, so we can write any affine algebraic set as $\mathbb{V}(I)$ for some ideal I .*

(iii) *Every affine algebraic set is defined by a finite number of polynomials.*

(iv) *If the radical of a ideal I in R is*

$$\text{rad}(I) := \{f \in R : f^r \in I \text{ for some } r \in \mathbb{N}\}$$

then $\mathbb{V}(I) = \mathbb{V}(\text{rad}(I))$.

Proof. (i) is obvious from the definition. It also implies that $\mathbb{V}(\langle S \rangle) \subseteq \mathbb{V}(S)$. For the other inclusion in (ii), suppose that $a \in \mathbb{V}(S)$; we need to show that $f(a) = 0$ for all $f \in \langle S \rangle$. By Exercise 2, such an f can be written in the form $f = \sum_i b_i f_i$ for $f_i \in S$ and some polynomials b_i . Then $f(a) = \sum_i b_i(a) f_i(a) = \sum_i b_i(a) \cdot 0 = 0$. (iii) follows from (ii) and the fact that every ideal in a polynomial ring is finitely generated. The proof (iv) can be found in Section 4.2 of [CLO97]. \square

Example 1.2.1. 1. $\mathbb{V}(\{1\}) = \emptyset$.

2. Every point $a = (a_1, \dots, a_n)$ of K^n is the algebraic set

$$\mathbb{V}(x_1 - a_1, \dots, x_n - a_n).$$

3. *Hyperplanes* in K^n are of the form $\mathbb{V}(f)$ where $f \in K[x_1, \dots, x_n]$ is a homogeneous polynomial of degree one. For example, the hyperplanes of K^3 are simply planes cut out by polynomials of the form $f = ax + by + cz$ in $K[x, y, z]$.
4. Consider the ideal $I = \langle xy, xz \rangle \subseteq \mathbb{C}[x, y, z]$. Then $\mathbb{V}(I) \subseteq \mathbb{C}^3$ consists of the points where both xy and xz vanish; that is, points (x, y, z) where either x is 0 or where both y and z are 0. Such points lie in the two-dimensional plane

$$P := \{(0, y, z)\}$$

or on the one-dimensional line

$$L := \{(x, 0, 0)\}.$$

P and L are called *components* of $\mathbb{V}(I)$.

5. Let $I = \langle x^2 + y^2 - z^2 \rangle \subseteq \mathbb{C}[x, y, z]$. The points of $\mathbb{V}(I) \subseteq \mathbb{C}^3$ that are contained in \mathbb{R}^3 are shown in Figure 1.2. Mathematicians usually draw pictures of varieties contained in \mathbb{C}^n in \mathbb{R}^n because complex points are difficult to visualize. Notice that the region around the vertex $(0, 0, 0)$ of the cone looks different than the region around any other point; this point is said to be a *singular point* of $\mathbb{V}(I)$. We will revisit this example in the next subsection.
6. Consider the ideal $I = \langle x^2 + y^2 + 1 \rangle \subseteq \mathbb{R}[x, y]$. Then $\mathbb{V}(I) = \emptyset$ since there are no points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = -1$. On the other hand, if we consider $J = \langle x^2 + y^2 + 1 \rangle \in \mathbb{C}[x, y]$ $\mathbb{V}(J)$ is not empty: for example, it contains the point $(0, i)$ ($0^2 + i^2 + 1 = 0$).

The final example shows that the affine algebraic set associated to an ideal depends on whether we are considering the ideal as a subset of $\mathbb{R}[x_1, \dots, x_n]$ or as a subset of $\mathbb{C}[x_1, \dots, x_n]$. While the complex numbers \mathbb{C} are more difficult to picture than the real numbers \mathbb{R} , algebraic geometry is simpler over $\mathbb{C}[x_1, \dots, x_n]$. The reason for this is that *all* of the zeroes of a polynomial can be seen over \mathbb{C}^n . For polynomials in one variable, this is the familiar Fundamental Theorem of Algebra which says that a polynomial in $\mathbb{C}[x]$ factors completely into linear factors. An important consequence of the Fundamental Theorem is that a *radical* polynomial is completely determined by its roots, or zero set, up to a multiple.⁵

⁵A radical polynomial in $\mathbb{C}[x]$ is a polynomial with no repeated roots.

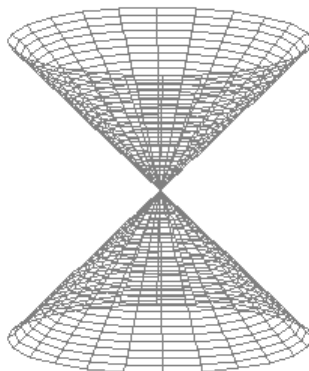


Figure 1.2: $\mathbb{V}(x^2 + y^2 - z^2)$.

The higher dimensional analogue of the Fundamental Theorem of Algebra is the Nullstellensatz of Hilbert. To write this down, we need some more notation.

Definition 1.2.3. Let V be a subset of \mathbb{C}^n . Then the set of polynomials that vanish at all of the points of V

$$I(V) := \{f \in \mathbb{C}[x_1, \dots, x_n] : f(a) = 0 \text{ for all } a \in V\}$$

is called the **ideal of V** .

The Nullstellensatz tells us that, over algebraically closed fields like \mathbb{C} , taking the ideal of an algebraic set and taking the algebraic set of an ideal are ‘opposite’ operations.

Theorem 1.2.4 (Strong Nullstellensatz, [Hil93]). *There is a one-to-one correspondence between algebraic sets of \mathbb{C}^n and radical ideals of $\mathbb{C}[x_1, \dots, x_n]$. More precisely,*

$$\begin{array}{ccc} \{ \text{algebraic sets of } \mathbb{C}^n \} & \iff & \{ \text{radical ideals of } \mathbb{C}[x_1, \dots, x_n] \} \\ V & \implies & I(V) \\ \mathbb{V}(I) & \impliedby & I \end{array}$$

Further,

$$I(\mathbb{V}(I)) = \text{rad}(I)$$

where $\text{rad}(I)$ is defined as in part (iv) of Lemma 1.2.2.⁶

⁶Part 5 of Example 1.2.1 demonstrates that this is not true when $R = \mathbb{R}[x, y]$.

Proof. See Section 4.2 of [CLO97]. □

The Nullstellensatz is the starting point for studying algebraic geometry. It allows us to establish ‘dictionary’ between algebra (radical ideals) and geometry (algebraic sets). Since the Nullstellensatz holds over the field \mathbb{C} , we will assume that our field is \mathbb{C} from this point on.

Exercise 4. Let I_j be ideals of $R = K[x_1, \dots, x_n]$. Show that

$$\bigcap_j \mathbb{V}(I_j) = \mathbb{V}\left(\bigcup_j I_j\right)$$

and

$$\bigcup_{j=1}^l \mathbb{V}(I_j) = \mathbb{V}\left(\prod_{i=1}^l I_j\right).$$

1.2.2 Complete Intersections

A *complete intersection* is a special kind of polynomial ideal that is generated by polynomials that are ‘unrelated’ in certain ways. For example, if a computer randomly generates polynomials f_1, f_2, \dots, f_r of $\mathbb{C}[x_1, \dots, x_n]$ where $r \leq n$, then the ideal $I = \langle f_1, \dots, f_r \rangle$ will almost certainly be a complete intersection.

In this section we will develop a more precise notion of a complete intersection; we start with the concept of *dimension* of a variety.

- As you would expect, \mathbb{C}^n has dimension n .
- Any line in \mathbb{C}^n has dimension 1. Bending a line doesn’t change its dimension, so the dimension of a curve is 1.
- Any plane in \mathbb{C}^n has dimension 2. Deforming a plane doesn’t change its dimension, so the dimension of a surface is 2.

Notice that if we look at a small section of a smooth curve (resp. of a surface) then it looks just like a small section of $\mathbb{C}^1 = \mathbb{C}$ (resp. \mathbb{C}^2). The same idea allows us to see the dimension of other algebraic sets.

Roughly, an affine algebraic set V with one component is of dimension n if almost all small sections, or neighborhoods, of V look like a small section of \mathbb{C}^n . If V has components V_1, \dots, V_k of different dimensions then the dimension of V is equal to the largest dimension of a V_i . The following gives a more precise definition of the dimension of an algebraic set; for alternative but equivalent definitions see, for

example, Section 9.1 of [CLO97], Chapter 8 of [Eis04], Section I.6 of [Sha88], and Chapter IV of [Per08].

Definition 1.2.5. Suppose that V is an affine algebraic set in \mathbb{C}^n . Intersect V with some random hyperplane H_1 ; we will denote this intersection $V_1 := V \cap H_1$ of \mathbb{C}^n . Then, choose another random hyperplane H_2 and intersect it with V_1 ; denote the result of this intersection is V_2 . Continuing this process, we will eventually get a V_d that is a finite set of points or the empty set.

If we repeat this process with different sets of random hyperplanes, the integer d for which V_d is a set of points or the empty set will almost always be the same.⁷ This d is equal to the **dimension** of V .

Example 1.2.2. Consider the hollow cone $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{C}^3$ in Figure 1.2. Then almost every small patch of the cone looks like a patch of the plane \mathbb{C}^2 ; the exception is a patch around the origin, which doesn't look like a piece of \mathbb{C}^n for any n . Thus, using the intuitive notion of dimension, V should have dimension 2.

We may use Definition 1.2.5 to see that V is 2 dimensional. The intersection of the cone and a random hyperplane is some conic section V_1 . The intersection of V_1 with different random hyperplane – V_2 – will either be one point, two points, or the empty set. Thus, the dimension of V is equal to 2 by the definition.

Notice that the cone V in the previous example was defined by a single polynomial and was a subset of \mathbb{C}^3 of dimension $3 - 1 = 2$; in fact, any affine variety of \mathbb{C}^n defined by a single polynomial will have dimension $n - 1$.

Theorem 1.2.6. *Let f be a nonconstant polynomial of $\mathbb{C}[x_1, \dots, x_n]$. Then $\mathbb{V}(f) \subseteq \mathbb{C}^n$ has dimension $n - 1$.*

Proof. See Theorem 2.3 of [Per08]. □

Complete intersections are a generalization of this idea.

Definition 1.2.7. Let I be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ such that $I = \langle f_1, \dots, f_r \rangle$ and I cannot be generated by fewer than r elements. Then I is a **complete intersection** (or an **r -complete intersection**) if the dimension of $\mathbb{V}(I)$ is equal to $n - r$. Further, if this condition holds and $\deg(f_i) = d_i$, we say that I is a **complete intersection of type** (d_1, \dots, d_r) .

⁷We will not prove this claim here; see Section I.6 of [Sha88].

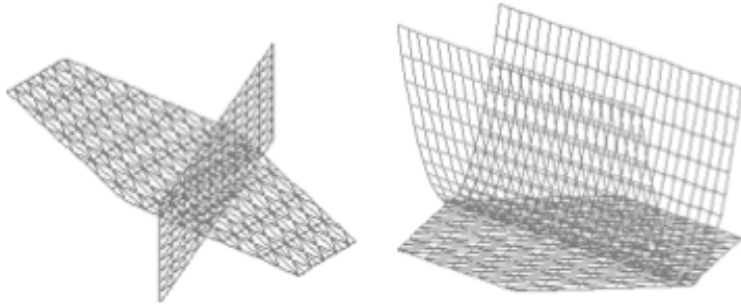


Figure 1.3: $\mathbb{V}(g_1)$ and $\mathbb{V}(g_2)$ from Example 1.2.3.

Intuitively, each polynomial in a complete intersection I adds a condition to the points on the $\mathbb{V}(I)$ and cuts the dimension of $\mathbb{V}(I)$ down by one. By Theorem 1.2.6, any ideal generated by a single polynomial is a complete intersection. The next example illustrates the how an ideal $I = \langle f, g \rangle$ generated by two polynomials can fail to be a complete intersection.

Example 1.2.3. Consider $I = (g_1, g_2) \subseteq \mathbb{C}[x, y, z]$ where $g_1 = x^2 + xy + xz - x - y - z$ and $g_2 = xy^2 - y^2 - xz + z$. We claim that $\mathbb{V}(I)$ contains a plane and that it has dimension 2. If I were a complete intersection, each of the polynomial conditions would cut the dimension of $\mathbb{V}(I)$ down by one, to make $\mathbb{V}(I)$ a one-dimensional subset of \mathbb{C}^3 . Thus, I is not a complete intersection.

The problem here is that both $g_1 = (x - 1)(x + y + z)$ and $g_2 = (x - 1)(y^2 - z)$ have a common factor of $(x - 1)$. Thus, both

$$\mathbb{V}(g_1) = \mathbb{V}(x - 1) \cup \mathbb{V}(x + y + z)$$

and

$$\mathbb{V}(g_2) = \mathbb{V}(x - 1) \cup \mathbb{V}(y^2 - z)$$

contain the plane given by the equation $x - 1 = 0$ (see Figure 1.3).⁸ Therefore, $\mathbb{V}(g_1, g_2) = \mathbb{V}(g_1) \cap \mathbb{V}(g_2)$ also contains this plane. The addition of the condition $g_2 = 0$ to $\mathbb{V}(g_1)$ does not cut down the dimension.

In fact, the only way that an ideal generated by two polynomials can fail to be a complete intersection is if the polynomials have a common factor.

⁸These images do not use the same orientation so that they can show all components.

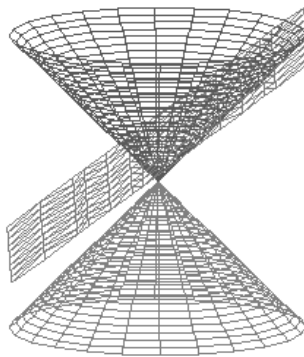


Figure 1.4: $\mathbb{V}(f_1)$ and $\mathbb{V}(f_2)$ from Example 1.2.4.

Example 1.2.4. Consider $I = \langle f_1, f_2 \rangle \subseteq \mathbb{C}[x, y, z]$ where $f_1 = x^2 + y^2 - z^2$ and $f_2 = 3x + 4z - 1$. Then, $\mathbb{V}(f_1)$, the cone from Example 1.2.2, has dimension 2. f_2 and f_1 do not share any common factors and $\mathbb{V}(f_2)$ is a plane that intersects the cone in a conic section; in particular, the conic section is an ellipse (see Figure 1.4). It follows that $\mathbb{V}(I) = \mathbb{V}(f_1) \cap \mathbb{V}(f_2)$ is a curve of dimension one, so I is a complete intersection.

1.2.3 The Zariski Topology

Zariski open subsets of \mathbb{C}^n will play an important role in defining generic initial ideals in Section 1.4. The main message of this subsection is that Zariski open subsets are very large.

Definition 1.2.8. A subset U of \mathbb{C}^n is **Zariski open** if it is the complement of an affine algebraic set; that is, if

$$U = \mathbb{C}^n \setminus \mathbb{V}(I)$$

for some ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$. Now, let X be a subset of \mathbb{C}^n . Then a subset $V \subseteq X$ is **Zariski open in X** if it is of the form $X \setminus (X \cap \mathbb{V}(I))$ for some ideal I .

Notice that proper affine algebraic sets are generally quite small compared to \mathbb{C}^n . For example, in \mathbb{C}^2 , the proper affine algebraic sets are points and curves; these miss nearly all of the points of \mathbb{C}^2 . This leads us to the key feature of Zariski open subsets: *nonempty Zariski open subsets are large – they fill nearly all of the space they are in.*

The following lemma makes this idea of nonempty Zariski open sets being large somewhat more precise.

Lemma 1.2.9. *Let X be a subset of \mathbb{C}^n and let U_1, \dots, U_r be nonempty Zariski open subsets of X . Then $U_1 \cap U_2 \cap \dots \cap U_r \neq \emptyset$.*

Proof. The intersection of any two Zariski open subsets $U_1 = X \setminus \mathbb{V}(I_1)$ and $U_2 = X \setminus \mathbb{V}(I_2)$ is Zariski open itself:

$$U_1 \cap U_2 = (X \setminus \mathbb{V}(I_1)) \cap (X \setminus \mathbb{V}(I_2)) = X \setminus (\mathbb{V}(I_1) \cup \mathbb{V}(I_2)) = X \setminus \mathbb{V}(I_1 I_2).$$

Thus, it suffices to prove the statement for $r = 2$. Suppose that $I_1 = \langle f_1, \dots, f_p \rangle$ and $I_2 = \langle g_1, \dots, g_q \rangle$. By definition, $\alpha \in U_1$ means that there is some f_i ($i = 1, \dots, p$) such that $f_i(\alpha) \neq 0$. Similarly, $\beta \in U_2$ means that there is some g_j ($j = 1, \dots, q$) such that $g_j(\beta) \neq 0$. Thus, $f_i g_j$ is not the 0 polynomial and, since \mathbb{C} is infinite, there is a $\gamma \in \mathbb{C}^n$ such that $f_i g_j(\gamma) = f_i(\gamma) g_j(\gamma) \neq 0$. Thus, $f_i(\gamma) \neq 0$ and $g_j(\gamma) \neq 0$ and $\gamma \in U_1 \cap U_2$ by definition. \square

1.2.4 The Projective Plane

So far, we have been discussing geometric subsets of

$$\mathbb{C}^n = \{(a_1, \dots, a_n) : a_i \in \mathbb{C}\}.$$

While these affine spaces are intuitive and easy to grasp, there are some inconsistencies that present problems when doing algebraic geometry. For example, two lines in \mathbb{C}^2 intersect in exactly one point *unless they are parallel*. This means that not all pairs of lines are the same; we have to treat pairs that happen to be parallel differently than other pairs. While this alone might not seem all that inconvenient, affine spaces also present many similar difficulties that prevent one from making clean, general statements. To overcome such issues, mathematicians often work in an expanded type of space, called *projective space*.

In this section, we will describe the complex projective plane \mathbb{P}^2 and subsets of it analogous to affine algebraic subsets of \mathbb{C}^n . Since giving a complete, intuitive description of projective space would take us too far afield, we will restrict ourselves to a very simple description of \mathbb{P}^2 . We encourage the reader encountering projective space for the first time to refer to [CLO97] and [SKKT00] for more complete descriptions.

Definition 1.2.10. The **complex projective plane**, denoted \mathbb{P}^2 , is the set of equivalence classes of $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ under the relation

$$(a_1, b_1, c_1) \sim (a_2, b_2, c_2) \iff \exists \lambda \in \mathbb{C} \setminus \{0\} \text{ such that } (a_1, b_1, c_1) = (\lambda a_2, \lambda b_2, \lambda c_2).$$

We write points of \mathbb{P}^2 with square brackets and colons instead of round braces and

commas to distinguish these from points in \mathbb{C}^3 . \mathbb{P}^2 then consists of points of the form

$$[a : b : c]$$

where a, b, c are not all 0 and $[a_1 : b_1 : c_1] = [a_2 : b_2 : c_2]$ if there is some $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$[a_1 : b_1 : c_1] = \lambda[a_2 : b_2 : c_2] = [\lambda a_2 : \lambda b_2 : \lambda c_2].^9$$

Next, we will define subsets of the projective plane \mathbb{P}^2 that are analogous to the affine algebraic subsets of \mathbb{C}^n . Like affine algebraic subsets, they should be defined by the vanishing of certain polynomials. Also, since each point $[a : b : c]$ in \mathbb{P}^2 has three coordinates, we will look at the vanishing sets of polynomials $f(x, y, z) \in \mathbb{C}[x, y, z]$ and ask: *for what points $[a : b : c]$ of \mathbb{P}^2 does $f(a, b, c) = 0$?* The next example shows that this question does not make sense for all polynomials of $\mathbb{C}[x, y, z]$.

Example 1.2.5. Consider the polynomial $f(x, y, z) = xy - 2z$. For what points $[a : b : c] \in \mathbb{P}^2$ does $f(a, b, c) = 0$?

Since $f(2, 1, 1) = 0$, we expect that the point $[2 : 1 : 1] \in \mathbb{P}^2$ should be in the vanishing set of f . However, in \mathbb{P}^2 ,

$$[2 : 1 : 1] = 2[2 : 1 : 1] = [4 : 2 : 2]$$

but $f(4, 2, 2) = 8 - 4 = 4 \neq 0$. That is, the vanishing set of $f(x, y, z)$ as the set of points $[a : b : c] \in \mathbb{P}^2$ where $f(a, b, c) = 0$ is not well-defined because it depends on which representation of a point we choose.

It turns out that the vanishing sets of homogeneous polynomials *are* well-defined in projective space (see Definition 1.1.10). Notice that the polynomial $f = xy - 2z$ that we considered in the previous example is not homogeneous since xy is of degree 2 while z is of degree 1.

Example 1.2.6. Consider the homogeneous degree 2 polynomial $g(x, y, z) = xy - 2z^2 \in \mathbb{C}[x, y, z]$ and note that $g(2, 1, 1) = 0$. We claim that g vanishes at the point $[2 : 1 : 1] \in \mathbb{P}^2$ no matter which representation of the point we choose. Let $[\lambda 2 : \lambda 1 : \lambda 1]$ be some other representation of $[2 : 1 : 1]$. Then

$$g(2\lambda, \lambda, \lambda) = (2\lambda)(\lambda) - 2(\lambda)^2 = \lambda^2(2 - 2) = 0$$

⁹Another way to describe \mathbb{P}^2 is in terms of lines in \mathbb{C}^3 so each point in \mathbb{P}^2 represents a line through the origin in \mathbb{C}^3 . The lines connecting $(a_1, a_2, a_3) \in \mathbb{C}^3$ and $(b_1, b_2, b_3) \in \mathbb{C}^3$ to the origin coincide exactly when there is a $\lambda \in \mathbb{C}$ such that $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$, or whenever $[a_1 : a_2 : a_3] = [b_1 : b_2 : b_3]$ in \mathbb{P}^2 .

as desired.

Proposition 1.2.11. *Let f be a homogeneous polynomial of $\mathbb{C}[x, y, z]$. If $f(a, b, c) = 0$ then $f(\lambda a, \lambda b, \lambda c) = 0$ as well. This means that if f vanishes at a point of \mathbb{P}^2 for one choice of coordinates, it vanishes for any choice of coordinates of the point. In particular,*

$$\mathbb{V}_{\mathbb{P}}(f) := \{p \in \mathbb{P}^2 : f(p) = 0\}$$

is a well-defined subset of \mathbb{P}^2 .

Proof. If f is homogenous of degree d then every term of f has the form $\gamma x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$ for a constant γ and integers $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = d$. Substituting $(x, y, z) = (\lambda a, \lambda b, \lambda c)$,

$$\gamma(\lambda a)^{\alpha_1}(\lambda b)^{\alpha_2}(\lambda c)^{\alpha_3} = \lambda^d \gamma(a^{\alpha_1} b^{\alpha_2} c^{\alpha_3}).$$

Thus, every term of $f(\lambda a, \lambda b, \lambda c)$ has a common factor of λ^d and

$$f(\lambda a, \lambda b, \lambda c) = \lambda^d f(a, b, c).$$

This means that $f(a, b, c) = 0$ if and only if $f(\lambda a, \lambda b, \lambda c) = 0$ for all $\lambda \neq 0$. □

Proposition 1.2.11 allows us to define the counterpart of affine algebraic subsets in the projective plane.

Definition 1.2.12. Let f_1, \dots, f_r be homogeneous polynomials of $\mathbb{C}[x, y, z]$ consisting of homogeneous polynomials. The **projective algebraic set** of \mathbb{P}^2 defined by f_1, \dots, f_r is

$$\mathbb{V}_{\mathbb{P}}(f_1, \dots, f_r) = \{[a : b : c] \in \mathbb{P}^2 : f_i(a, b, c) = 0 \text{ for all } i = 1, \dots, r\}.$$
¹⁰

As in the affine case, if I is the homogeneous ideal of $\mathbb{C}[x, y, z]$ generated by homogeneous polynomials f_1, \dots, f_r ,

$$\mathbb{V}_{\mathbb{P}}(I) = \mathbb{V}_{\mathbb{P}}(f_1, \dots, f_r).$$

Example 1.2.7. Recall that each point $(a_1, \dots, a_n) \in \mathbb{C}^n$ is an affine algebraic subset:

¹⁰When it is clear that we are working in \mathbb{P}^2 rather than in \mathbb{C}^3 , we may write $\mathbb{V}(I)$ instead of $\mathbb{V}_{\mathbb{P}}(I)$.

if $S = \{x_1 - a_1, \dots, x_n - a_n\}$, then

$$\mathbb{V}(S) = \{(a_1, \dots, a_n)\}.$$

Since none of the polynomials in S are homogeneous, S does not define a projective algebraic set. We would like to find a homogeneous ideal I such that $\mathbb{V}_{\mathbb{P}}(I)$ is the single point $[a : b : c]$. Note that we can always write

$$[a : b : c] = \frac{1}{a}[a : b : c] = [1 : b' : c']$$

where $b' = \frac{b}{a}$ and $c' = \frac{c}{a}$. Then

$$\mathbb{V}_{\mathbb{P}}(y - b'x, z - c'x) = \{[1 : b' : c']\} = \{[a : b : c]\}.$$

We will refer to the ideal $(y - b'x, z - c'x)$ as *the ideal of the point* $[a : b : c]$, because all homogeneous polynomials that vanish at the point $[a : b : c]$ are contained in this ideal. Note that

$$\mathbb{V}_{\mathbb{P}}(I_1) \cup \mathbb{V}_{\mathbb{P}}(I_2) \cup \dots \cup \mathbb{V}_{\mathbb{P}}(I_p) = \mathbb{V}_{\mathbb{P}}(I_1 I_2 \dots I_p)$$

as in the affine case. Thus, any finite set of points is a projective algebraic set. We often write a set of r points $p_1, \dots, p_r \in \mathbb{P}^2$ as $p_1 + \dots + p_r$.

Lines in \mathbb{P}^2 are defined in a way analogous to lines in \mathbb{C}^n .

Definition 1.2.13. A **line** in \mathbb{P}^2 (or a **projective line**) is a projective algebraic set defined by a homogeneous polynomial of degree 1. That is, it is of the form

$$\mathbb{V}_{\mathbb{P}}(Ax + By + Cz)$$

where at least one of the A, B, C is nonzero.

We can define ideals of subsets of \mathbb{P}^2 .

Definition 1.2.14. Let V be a subset of \mathbb{P}^2 . The **ideal of** V , denoted $I(V)$, is the ideal generated by all homogeneous polynomials of $\mathbb{C}[x, y, z]$ that vanish at every point of V . That is,

$$I(V) = \{f \in \mathbb{C}[x, y, z] : f(a, b, c) = 0 \text{ for all } [a : b : c] \in V\}.$$

In the next section we will be interested in ideals of points and sets of points in \mathbb{P}^2 . Notice that if p_1, \dots, p_r are points of \mathbb{P}^2

$$I(p_1 + p_2 + \dots + p_r) = I(p_1) \cap \dots \cap I(p_r)$$

since a polynomial vanishes at all of the points p_i if and only if it is in the vanishing set of each one. It turns out that this intersection has different properties depending on the relative positions of the points p_1, \dots, p_r . For example, if p_1, p_2, p_3 all lie on a single line L then the ideal $I := I(p_1 + p_2 + p_3)$ contains the polynomial corresponding to L whereas if they do not lie on a line then I does not contain any linear polynomials.

1.2.5 Fat Points of \mathbb{P}^2

In this section we will introduce a special type of geometric subset of \mathbb{P}^2 called *fat points* that will be the starting point for our work in Chapters V through VII. The study of fat points allows us to consider sets of points where points are repeated. For example, if p_1, \dots, p_r are points of \mathbb{P}^2 and m is some positive integer, the set of fat points

$$mp_1 + mp_2 + \dots + mp_r$$

can be thought of as the set

$$\underbrace{\{p_1, p_1, \dots, p_1\}}_m, \underbrace{\{p_2, p_2, \dots, p_2\}}_m, \dots, \underbrace{\{p_r, p_r, \dots, p_r\}}_m$$

where each point p_i is repeated m times. For example, $2p_1 + 3p_2$ can be thought of as $\{p_1, p_1, p_2, p_2, p_2\}$.

To really see what it means to ‘fatten’ points, we must turn to algebra. First, consider $R = \mathbb{C}[x]$. While both x and x^5 vanish at the point $x = 0$, intuitively it seems as though x^5 should vanish more times than x . One difference between x and x^5 is that the first derivative of x does not vanish at $x = 0$ while the first four derivatives of x^5 vanish at $x = 0$. The *order of vanishing* uses derivatives to quantify the number of times a polynomial vanishes at a point.

Definition 1.2.15. Suppose that f is a homogeneous polynomial of $R = \mathbb{C}[x, y, z]$ and $p = [a : b : c]$ is a point of \mathbb{P}^2 . The **order of vanishing** of f at p , denoted $\text{ord}_p(f)$, is the greatest t such that at least one of the t th partial derivatives of f vanishes at p , but none of the $(t + 1)$ st derivatives vanish at p .

This is used in the following definition.

Definition 1.2.16. Suppose that p, p_1, \dots, p_r are points of \mathbb{P}^2 and that $R = \mathbb{C}[x, y, z]$. The **ideal of the fat point** mp , denoted $I(mp)$, is the set of homogeneous polynomials of R that vanish at p to order at least m :

$$I(mp) := \{f \in R : f \text{ is homogeneous, } \text{ord}_p(f) \geq m\}.$$

The **ideal of fat points** $mp_1 + \dots + mp_r$, $I(mp_1 + \dots + mp_r)$ is the set of homogeneous polynomials of R that vanish to at least order m at every point p_1, \dots, p_r :

$$\begin{aligned} I(mp_1 + \dots + mp_r) &:= \{f \in R : f \text{ is homogeneous, } \text{ord}_{p_i}(f) \geq m \text{ for all } i\} \\ &= I(mp_1) \cap \dots \cap I(mp_r). \end{aligned}$$

Defining fat points formally requires something called *schemes*; the *fat point subscheme* $mp_1 + \dots + mp_r$ of \mathbb{P}^2 is the subscheme defined by the ideal $I(mp_1 + \dots + mp_r)$. Intuitively, you can just think of $mp_1 + \dots + mp_r$ as the set of points p_1, \dots, p_r where each point has been ‘fattened up’, or repeated m times. The following lemma tells us another way to describe the ideal of fat points $mp_1 + \dots + mp_r$ (see, for example, [Gli89]).

Lemma 1.2.17. *Let p_1, \dots, p_r be points of \mathbb{P}^2 and let $R = \mathbb{C}[x, y, z]$. If $I(p_i)$ is the ideal of the point p_i studied in Example 1.2.7, then*

$$I(mp_i) = I(p_i)^m.$$

Therefore,

$$\begin{aligned} I(mp_1 + mp_2 + \dots + mp_r) &= I(mp_1) \cap I(mp_2) \cap \dots \cap I(mp_r) \\ &= I(p_1)^m \cap I(p_2)^m \cap \dots \cap I(p_r)^m. \end{aligned}$$

We will give two alternative descriptions of the ideal of uniform fat points $mp_1 + \dots + mp_r$ in Proposition 1.2.20 that will be used in future chapters. First, we need two definitions from algebra.

Definition 1.2.18. If P is a *prime* ideal of a ring R then the m th **symbolic power** of P , $P^{(m)}$, is equal to

$$P^{(m)} := \{r \in R : sr \in P^m \text{ for some } s \in R \setminus P\}.$$

The *colon* of two ideals I and J of R is an operation on ideals analogous to

division; it gives us another ideal

$$(I : J) := \{r \in R : rJ \subseteq I\}.$$

Definition 1.2.19. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$. Then the **saturation** of I is

$$I^{\text{sat}} = \bigcup_{k \geq 0} (I : \mathfrak{m}^k).$$

A discussion of the following proposition and similar results can be found in [Gli89].

Proposition 1.2.20. Let p_1, \dots, p_r be r points of \mathbb{P}^2 and let m be some positive integer. Then

$$I(mp_1 + \dots + mp_r) = (I(p_1 + \dots + p_r))^{(m)} = ((I(p_1 + \dots + p_r))^m)^{\text{sat}}.$$

That is, the ideal of uniform fat points in \mathbb{P}^2 can be thought of as a symbolic power or as the saturation of an ordinary power.

1.3 Monomials and Monomial Ideals

We now turn our attention from algebraic geometry to computational algebra. In this section we will take our first steps towards defining generic initial ideals by studying monomials and ideals generated by monomials, called *monomial ideals*. Just as monomials are the simplest type of polynomials, monomial ideals are the simplest type of polynomial ideals.

We will begin by investigating some properties of monomial ideals that make them particularly nice to work with. Next, we will introduce *monomial orders*. In the final subsection, we will use monomial orders to describe how to ‘deform’ any polynomial ideal to a monomial ideal, called the *initial ideal*.

1.3.1 Monomial Ideals

From high school algebra, we know that monomials are easy to manipulate. Multiplying, dividing, and taking powers of monomials are very simple:

$$(x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}) \cdot (x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}) = (x_1^{i_1+j_1} x_2^{i_2+j_2} \dots x_n^{i_n+j_n})$$

$$\frac{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}} = (x_1^{i_1-j_1} x_2^{i_2-j_2} \cdots x_n^{i_n-j_n})$$

$$(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})^k = (x_1^{k \cdot i_1} x_2^{k \cdot i_2} \cdots x_n^{k \cdot i_n}).$$

Recall from Section 1.1 that we can also write a monomial $m = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in multi-index notation as $m = x^I$ where I is a vector $I := (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$. Since vectors are added, subtracted, and multiplied by constants entry-by-entry, the properties above can be rewritten:

$$x^I \cdot x^J = x^{I+J}, \frac{x^I}{x^J} = x^{I-J}, \text{ and } (x^I)^k = x^{k \cdot I}.$$

It is also easy to check if a monomial is divisible by another monomial: x^J is divisible by x^I if and only if $i_l \leq j_l$ for all $l = 1, \dots, n$. Further, a monomial x^I divides a polynomial $f(x_1, \dots, x_n) = \sum_{\alpha \in A} c_\alpha x^\alpha$ if and only if x^I divides each of the monomials x^α such that $c_\alpha \neq 0$. For example, it is easy to see that the monomial $x^2 y z$ divides $287x^3 y^2 z^5 + 8x^2 y^7 z$ but it does not divide $xy^2 z^8 + 5x^{811} y^{70} z^{45}$.

Checking the divisibility of two arbitrary polynomials is more complicated, requiring the use of a division algorithm. For example, it is not obvious that the polynomial $x^6 z + x^5 y z + x^4 z^2 + x^2 y^2 + x y^3 + x^2 y z + x y^2 z + x^3 + x^2 y + y^2 z + y z^2 + x z$ is divisible by $x^2 + y x + z$ with quotient $x^4 z + x + y z + y^2$.

Now we turn our attention to ideals generated by monomials.

Definition 1.3.1. An ideal $I \subseteq R = \mathbb{C}[x_1, \dots, x_n]$ is a **monomial ideal** if there is a set of monomials $\{u_1, u_2, \dots, u_s\}$ such that

$$I = \langle u_1, \dots, u_s \rangle.$$

In this case, the elements of I are exactly polynomials of the form $\sum_{i=1}^s f_i u_i$ where each f_i is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$.

Note that monomial ideals contain polynomials – for example, the sum of two monomials is not a monomial – and that they may be written with non-monomial generators (see Example 1.1.4). It turns out that working with ideals generated by monomials is just as easy as working with monomials themselves. For example, when I is a monomial ideal it is easy to see if a monomial lies inside of I .

Lemma 1.3.2. *Let I be a monomial ideal with monomial generators $\{u_1, \dots, u_s\}$. Then a monomial v is in I if and only if there is some $u_i \in \{u_1, u_2, \dots, u_s\}$ that divides v .*

Proof. If there is an u_l such that $u_l|v$, then v is a multiple of u_l so it lies in I by Exercise 2. Conversely, if $v \in I$ then $v = \sum_{i=1}^s f_i u_i$ for $f_i \in \mathbb{C}[x_1, \dots, x_n]$. When we expand the polynomial on the right hand side of the equation, each term is still divisible by a u_i . When we then simplify this polynomial, writing it as a sum of distinct monomials, each term must be divisible by some u_i , say u_l . However, the simplified polynomial is simply the monomial v ! Therefore, the one remaining term, v , is divisible by u_l . \square

We say that G is a *minimal set of generators* for a polynomial ideal I if $I = \langle G \rangle$ and if removing any polynomial from G results in a set of polynomials that no longer generates I . One can show that if two sets of polynomials G_1 and G_2 are minimal generating sets of an arbitrary ideal I , then G_1 and G_2 contain the same number of elements. The following result tells us that if I is a monomial ideal more is true: the minimal monomial generating set is unique.¹¹

Proposition 1.3.3. *A set of monomial generators $\{u_1, \dots, u_s\}$ of a monomial ideal I is minimal if and only if u_i does not divide u_j for all i, j where $i \neq j$. Further, every monomial ideal I has a unique minimal set of monomial generators.*

Proof. The first statement follows immediately from the Lemma 1.3.2: $G = \{u_1, \dots, u_s\}$ is not minimal $\iff G \setminus u_i$ generates I for some $i \iff u_i \in \langle G \setminus u_i \rangle \iff u_i$ divides some element of $G \setminus u_i$.

For the second statement, let $G_1 = \{u_1, \dots, u_s\}$ and $G_2 = \{v_1, \dots, v_s\}$ be two minimal sets of monomial generators for I . Since $u_i \in I = \langle G_2 \rangle$ and G_2 generates I , Lemma 1.3.2 implies that one of the v_j divides u_i ; in other words, there is some monomial w_1 such that $u_i = v_j w_1$. In the same way, there is a $u_k \in G_1$ and a monomial w_2 such that $v_j = u_k w_2$. Then

$$u_i = v_j w_1 = (u_k w_2) w_1 = u_k (w_1 w_2)$$

and u_k divides u_i . Since u_i is the only element of G_1 that divides u_i , we must have $k = i$, so $w_1 w_2 = 1$. Thus, $w_1 = 1$ and $u_i = v_j \in G_2$. Repeating this argument for each u_i , we see that $G_1 \subseteq G_2$ and by switching the roles of G_1 and G_2 we also have $G_2 \subseteq G_1$. Thus, $G_1 = G_2$. \square

Another nice feature of monomial ideals in $\mathbb{C}[x_1, \dots, x_n]$ is that they can be visualized as subsets of the lattice points \mathbb{N}^n . We will describe how this can be done when we construct the limiting shape in Section 1.5.

¹¹Remember that a *monomial* always has coefficient 1.

1.3.2 Monomial Orders

Our next goal is to find a way to deform an arbitrary polynomial ideal I of $R = \mathbb{C}[x_1, \dots, x_n]$ to a monomial ideal of R , denoted $\text{in}(I)$. To build such a monomial ideal, we will take one of the monomials appearing in each polynomial f of I and make it a generator of $\text{in}(I)$. The purpose of this section is to describe how to choose which monomial of f to single out by defining *monomial orders* – special ways to arrange the monomials of R so that any two monomials can be compared.

Definition 1.3.4. Let $\text{Mon}(R)$ denote the set of monomials of $R = \mathbb{C}[x_1, \dots, x_n]$. A **total order** on $\text{Mon}(R)$ is a relation \geq on $\text{Mon}(R)$ such that for all $u, v, w \in \text{Mon}(R)$:

1. u and v can always be compared: either $u \geq v$ or $v \geq u$;
2. $u \geq u$;
3. $u \geq v$ and $v \geq u$ implies that $u = v$; and
4. $u \geq v$ and $v \geq w$ implies that $u \geq w$.

If $u \geq v$ and $u \neq v$, we can also write $u > v$.¹²

A **monomial order** on $\text{Mon}(R)$ is a total order such that

1. $u > 1$ for all $u \in \text{Mon}(R) \setminus \{1\}$; and
2. if $u > v$ then $uw > vw$ for all $w \in \text{Mon}(R)$ (that is, the order respects multiplication).

Two of the most common monomial orders are the *graded lexicographic order* and the *graded reverse lexicographic order*.

Suppose that $R = K[x_1, x_2, x_3]$ is a polynomial ring in three variables, so that $\text{Mon}(R)$ is the set of monomials in x_1, x_2, x_3 . A monomial order on R should give us a way to compare *any* two monomials $x^I = x_1^{i_1} x_2^{i_2} x_3^{i_3}$ and $x^J = x_1^{j_1} x_2^{j_2} x_3^{j_3}$.

First, if x^I and x^J are of different degrees then we will say that the one with the greater degree should be greater in the monomial order. For example, $x_1 x_2 < x_1 x_2 x_3$ and $x_1^6 x_2^2 x_3^2 < x_1^8 x_2^8 x_3^{100}$.¹³

The lexicographic order and reverse lexicographic order break ties between monomials of the same degree in different ways. For both, we assume that $x_1 > x_2 > x_3$ is the order of the monomials of degree 1.

¹²The most familiar total order in mathematics is ‘counting’ order on \mathbb{N} : for $x, y \in \mathbb{N}$, $x \geq y$ exactly when x comes after y when counting up from 1.

¹³Recall that the *degree* of a monomial x^I is $i_1 + i_2 + i_3$, so that $x_1^6 x_2^2 x_3^2$ has degree 10 and $x_1^8 x_2^8 x_3^{100}$ has degree 116.

To illustrate how the lexicographic order breaks ties, suppose that we wanted to order the words `aardvark` and `abruptly` for a dictionary. We begin by comparing the first letter of the words; since both are the same – `a` – we need to move onto the next letter. The second letter of the word `aardvark` is an `a` while the second letter of the word `abruptly` is a `b`. Since `a` comes before `b` in the alphabet, `aardvark` should come before `abruptly` in the dictionary order; that is

$$\mathbf{a} > \mathbf{b} \Rightarrow \mathbf{aardvark} > \mathbf{abruptly}.$$

Notice that when ordering two words according to the dictionary order, we don't care about any letters past the first one where the words differ.

In the lexicographic order with $x_1 > x_2 > x_3$, we treat monomials of the same degree

$$x^I = x_1^{i_1} x_2^{i_2} x_3^{i_3} = \underbrace{x_1 x_1 \cdots x_1}_{i_1 \text{ repeats}} \underbrace{x_2 x_2 \cdots x_2}_{i_2 \text{ repeats}} \underbrace{x_3 x_3 \cdots x_3}_{i_3 \text{ repeats}}$$

and

$$x^J = x_1^{j_1} x_2^{j_2} x_3^{j_3} = \underbrace{x_1 x_1 \cdots x_1}_{j_1 \text{ repeats}} \underbrace{x_2 x_2 \cdots x_2}_{j_2 \text{ repeats}} \underbrace{x_3 x_3 \cdots x_3}_{j_3 \text{ repeats}}$$

as though they are words that we want to put in a dictionary.¹⁴ That is, we start by comparing the letters in the first position. If they are different, then the monomial that has the greater variable in the first position is automatically the greater monomial (recall that $x_1 > x_2 > x_3$). If the letters in the first position are the same, we compare the variables in the second position. Either they are different, in which case we say that the monomial with the greater variable in the second position is the greater monomial, or they are the same, in which case we move on to the third position. Continuing on in this way, we will eventually reach a position where there is not a tie.

Example 1.3.1. If $m_1 := x_1^2 x_3 = x_1 x_1 x_3$ and $m_2 := x_1 x_2^2 = x_1 x_2 x_2$ then $m_1 > m_2$ under the lexicographic order. In the first position, we have a tie: both monomials have an x_1 . In the second position m_1 has an x_1 and m_2 has an x_2 ; thus, since $x_1 > x_2$, $m_1 > m_2$. The monomials of $\text{Mon}(R)$ of degree 3 are ordered as follows when we have the lexicographic order with $x_1 > x_2 > x_3$:

$$x_1^3 > x_1^2 x_2 > x_1^2 x_3 > x_1 x_2^2 > x_1 x_2 x_3 > x_1 x_3^2 > x_2^3 > x_2^2 x_3 > x_2 x_3^2 > x_3^3.$$

¹⁴Note that we will always write the variables in this order, with all x_i s before x_{i+1} s.

Rather than breaking ties between monomials of the same degree by considering the first position first, the reverse lexicographic order breaks ties by first looking at the final position. In the lexicographic order, monomials are pulled up by having high variables at the beginning; in the reverse lexicographic order, monomials are pulled down by having low variables at the end.

In the reverse lexicographic order, we begin by comparing variables in the last position. If they are different, then the monomial with the lesser variable is automatically the lesser monomial. If they are the same, we compare the variables in the second to last position. Either these are different, in which case the monomial with the lesser variable in the second to last position is the lesser one, or they are the same, in which case we move on to the third to last position. Continuing on in this way, we will eventually reach a position where there is not a tie unless the two monomials are equal.

Example 1.3.2. Consider $m_1 = x_1^2 x_3 = x_1 x_1 x_3$ and $m_2 = x_1 x_2^2 = x_1 x_2 x_2$ under the reverse lexicographic order. In the final position m_1 has an x_3 while m_2 has an x_2 . The monomial m_1 is pulled down by having a lower variable in the final position, so we have $m_2 > m_1$. The monomials of $\text{Mon}(R)$ of degree 3 are ordered as follows when we have the reverse lexicographic order with $x_1 > x_2 > x_3$.

$$x_1^3 > x_1^2 x_2 > x_1 x_2^2 > x_2^3 > x_1^2 x_3 > x_1 x_2 x_3 > x_2^2 x_3 > x_1 x_3^2 > x_2 x_3^2 > x_3^3$$

The following definition makes these intuitive descriptions precise.

Definition 1.3.5. The **lexicographic order** $>_{\text{lex}}$ on $R = \mathbb{C}[x_1, \dots, x_n]$ with $x_1 > x_2 > \dots > x_n$ is defined by setting $x^I > x^J$ if either:

- $\deg(x^I) > \deg(x^J)$ (that is, $\sum_{l=1}^n i_l > \sum_{l=1}^n j_l$); or
- $\deg(x^I) = \deg(x^J)$ and the leftmost nonzero entry of the vector $I - J = (i_1 - j_1, i_2 - j_2, \dots, i_n - j_n)$ is positive.

The **reverse lexicographic order** $>_{\text{lex}}$ on R with $x_1 > x_2 > \dots > x_n$ is defined by setting $x^I > x^J$ if either:

- $\deg(x^I) > \deg(x^J)$; or
- $\deg(x^I) = \deg(x^J)$ and the rightmost nonzero entry of the vector $I - J = (i_1 - j_1, i_2 - j_2, \dots, i_n - j_n)$ is negative.

While the lexicographic order seems more intuitive, generic initial ideals – the main topic of this thesis – are generally nicer when we use the reverse lexicographic order. Therefore, our default monomial order is the reverse lexicographic order.

From now on, each polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ will also carry with it a monomial order on $\text{Mon}(R)$; we will say that R has the reverse lexicographic order (or the lexicographic order) with $x_1 > x_2 > \dots > x_n$.

1.3.3 Initial Ideals

The goal of this section is to describe how to deform an arbitrary polynomial ideal $I \subseteq R = \mathbb{C}[x_1, \dots, x_n]$ to a monomial ideal in a natural way. We will do this by choosing one special monomial appearing in each polynomial $f \in I$.

Definition 1.3.6. Suppose that $R = \mathbb{C}[x_1, \dots, x_n]$ is a polynomial ring with some fixed monomial order $>$. Let $f = \sum_{u \in \text{Mon}(R)} c_u u$ be a polynomial of R where $c_u \in \mathbb{C}$. The **initial monomial** (or **leading monomial**) of f , $\text{in}(f)$, is the largest monomial that appears in f with respect to $>$; that is, it is the largest u such that $c_u \neq 0$.

Example 1.3.3. Let $R = \mathbb{C}[x_1, x_2, x_3]$ be a polynomial ring in three variables with the reverse lexicographic order where $x_1 > x_2 > x_3$ and let $f = 2x_1^2x_3 + 4x_1x_2^2 + 6x_3^3 + 8x_2^2$. In the reverse lexicographic order with $x_1 > x_2 > x_3$, the monomials with nonzero coefficients in f are ordered:

$$x_1x_2^2 > x_1^2x_3 > x_3^3 > x_2^2$$

so

$$\text{in}(f) = x_1x_2^2.$$

In the lexicographic order with $x_1 > x_2 > x_3$,

$$x_1^2x_3 > x_1x_2^2 > x_3^3 > x_2^2$$

so if R instead has the lexicographic order with $x_1 > x_2 > x_3$

$$\text{in}(f) = x_1^2x_3.$$

Lemma 1.3.7. Let R be a polynomial ring with some fixed term order $>$. Suppose that f and g are elements of R and let u be a monomial in R . Then

$$(i) \text{ in}(uf) = u\text{in}(f);$$

(ii) $\text{in}(fg) = \text{in}(f)\text{in}(g)$;

(iii) $\text{in}(f + g) \leq \max\{\text{in}(f), \text{in}(g)\}$ with equality if $\text{in}(f) \neq \text{in}(g)$.

Proof. See Lemma 2.1.4 of [HH11]. □

The following definition gives one way to deform an arbitrary ideal to a monomial ideal.

Definition 1.3.8. Suppose that R is a polynomial ring with some fixed term order $>$ and that I is a nonzero ideal of R . Then the **initial ideal** of I with respect to $>$, $\text{in}(I)$, is the monomial ideal of R generated by $\{\text{in}(f) : 0 \neq f \in I\}$.

If the initial ideal truly represents a way of deforming an arbitrary ideal to a monomial ideal, it should deform a monomial ideal to itself. The following lemma tells us that this is in fact the case.

Lemma 1.3.9. *If I is a monomial ideal then $\text{in}(I) = I$.*

Proof. Suppose that $\{u_1, \dots, u_s\}$ is a set of monomial generators of I ; we may write

$$G(I) := \{u_1, \dots, u_s\}.$$

Since $\text{in}(u_i) = u_i$, $I = \langle G(I) \rangle \subseteq \text{in}(I)$. Now suppose that for $f \in I$, $w = \text{in}(f)$ is a generator of $\text{in}(I)$. As an element of I , $f = \sum_{i=1}^s f_i u_i$ for some $f_i \in R$. Thus, $\text{in}(f) = w u_j$ for some $j = 1, \dots, s$ and some monomial w , so $\text{in}(f) \in \langle G(I) \rangle$. Thus, $\text{in}(I) \subseteq \langle G(I) \rangle$, so $\text{in}(I) \subseteq \langle G(I) \rangle = I$. □

The following example demonstrates that when we have an arbitrary polynomial ideal, the initial ideal is *not* equal to the ideal generated by the initial monomials of a set of generators:

$$\text{in}(\langle f_1, \dots, f_r \rangle) \neq \langle \text{in}(f_1), \dots, \text{in}(f_r) \rangle.$$

Example 1.3.4. Let $R = \mathbb{C}[x_1, x_2, x_3]$ have the reverse lexicographic order and consider

$$I = \langle f_1, f_2, f_3 \rangle$$

where $f_1 = x_1^3 x_3 + 4x_1 x_2^2 x_3$, $f_2 = x_1^2 x_3 + 3x_1 x_2 x_3 + x_1 x_3^2$, and $f_3 = x_1^3 + 5x_1^2 x_2 + 2x_2 x_3^2$.

We might be tempted to write

$$\text{in}(I) = \langle \text{in}(f_1), \text{in}(f_2), \text{in}(f_3) \rangle = \langle x_1^3 x_3, x_1^2 x_3, x_1^3 \rangle = \langle x_1^2 x_3, x_1^3 \rangle.$$

However, it turns out that

$$\text{in}(I) = \langle x_1^3, x_1^2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_2^2x_3^3, x_1x_3^4, x_2x_3^5 \rangle.$$

Where did the extra elements of $\text{in}(I)$ come from? Notice that $\text{in}(I)$ must contain the initial terms of *all* elements of I , not just the initial terms of the generators. We can get other elements of I by adding polynomial multiples of the generators to each other; sometimes the initial monomials of generators will cancel, giving elements of I with different initial terms. For example,

$$\begin{aligned} f_1 + (3x_2 - x_1)f_2 &= (x_1^3x_3 + 4x_1x_2^2x_3) - x_1(x_1^2x_3 + 3x_1x_2x_3 + x_1x_3^2) \\ &\quad + 3x_2(x_1^2x_3 + 3x_1x_2x_3 + x_1x_3^2) \\ &= (4x_1x_2^2x_3 - 3x_1^2x_2x_3 - x_1^2x_3^2) + (3x_1^2x_2x_3 + 9x_1x_2^2x_3 + 3x_1x_2x_3^2) \\ &= 13x_1x_2^2x_3 - x_1^2x_3^2 + 3x_1x_2x_3^2 \in I. \end{aligned}$$

so $\text{in}(f_1 + (3x_2 - x_1)f_2) = x_1x_2^2x_3 \in \text{in}(I)$.

Generating sets of I whose initial monomials generate $\text{in}(I)$ are fundamental to the algorithms of computational commutative algebra; such generating sets are called Gröbner bases.

Definition 1.3.10 ([Buc65]). Suppose that R is a polynomial ring with some fixed term order $>$ and that I is a nonzero ideal of R . A finite set of polynomials $\{g_1, \dots, g_r\} \subseteq I$ is a **Gröbner basis** of I if

$$\text{in}(I) = \langle \text{in}(g_1), \dots, \text{in}(g_r) \rangle.$$

Example 1.3.5. Let $R = \mathbb{C}[x_1, x_2, x_3]$ have the reverse lexicographic order with $x_1 > x_2 > x_3$ and consider the ideal I from the previous example. It turns out that I is also generated by

$$\{f_1, f_2, \dots, f_7\}$$

where $f_1 = x_1^2x_3 + 3x_1x_2x_3 + x_1x_3^2$, $f_2 = x_1^3 + 5x_1^2x_2 + 2x_2x_3^2$, $f_3 = 49x_1x_2x_3^2 + 19x_1x_3^3 + 26x_2x_3^3$, $f_4 = 49x_1x_2^2x_3 - 5x_1x_3^3 - 12x_2x_3^3$, $f_5 = 73x_1x_3^4 - 158x_2x_3^4$, $f_6 = 73x_2^2x_3^3 - 9x_2x_3^4$, and $f_7 = x_2x_3^5$. Since

$$\text{in}(I) = \langle \text{in}(f_1), \dots, \text{in}(f_7) \rangle$$

$\{f_1, \dots, f_7\}$ is a Gröbner basis of I .

The following theorem says that Gröbner bases are in fact bases of ideals; see

Section 2.5 of [CLO97] or Theorem 2.1.8 of [HH11].

Theorem 1.3.11. *Given an ideal I in a polynomial ring R with some fixed term order:*

- (i) *a Gröbner basis for I always exists;*
- (ii) *any Gröbner basis for I generates I (that is, if $\{g_1, \dots, g_r\}$ is a Gröbner basis of I then $\langle g_1, \dots, g_r \rangle = I$).*

There is a simple algorithm, called *Buchberger's algorithm*, that computes the Gröbner basis of any ideal I in a polynomial ring R with any term order. This algorithm is the starting point for many of the most useful techniques from computational commutative algebra that allow us to answer questions such as the following.

- **The Ideal Membership Problem:** Given an ideal $I \subseteq R = \mathbb{C}[x_1, \dots, x_n]$ and an element $f \in R$, determine if $f \in I$.
- **Solving Polynomial Equations:** Given polynomials $f_1, \dots, f_r \in R$, find all of the $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that

$$f_1(a_1, \dots, a_n) = \dots = f_r(a_1, \dots, a_n).$$

In other words, find all the points in $\mathbb{V}(f_1, \dots, f_r)$.

Solving problems such as this is one of the main themes of [CLO97]; the reader is encouraged to look there for further information.

1.4 Generic Initial Ideals

In this section, we will define generic initial ideals, the main algebraic object studied in this thesis. The generic initial ideal is an alternative way to associate a monomial ideal to a homogeneous ideal. While they are more difficult to define and to compute than initial ideals, generic initial ideals work well with geometric questions and have a nice combinatorial property called *Borel-fixedness*.

We will discuss the general linear group of $n \times n$ invertible matrices and how it acts on $\mathbb{C}[x_1, \dots, x_n]$ in Subsections 1.4.2 and 1.4.1; this will prepare us for the definition of generic initial ideals in Subsection 1.4.3. The next two sections will be dedicated to illustrating what makes generic initial ideals important and nice to work with. Finally, we will prove that generic initial ideals always exist in Subsection 1.4.6.

1.4.1 $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}[x_1, \dots, x_n]$

Definition 1.4.1. Denote the set of all $n \times n$ matrices with entries in \mathbb{C} by $M_n(\mathbb{C})$. A matrix $g \in M_n(\mathbb{C})$ is **invertible** if its determinant is nonzero. The **general linear group** $\mathrm{GL}_n(\mathbb{C})$ is the set of invertible $n \times n$ matrices with entries in \mathbb{C} .

Suppose that $g = (g_{ij})$ is a matrix in $\mathrm{GL}_n(\mathbb{C})$ and that $R = \mathbb{C}[x_1, \dots, x_n]$. Then g sends a variable x_j to a linear combination of other variables:

$$g(x_j) := \sum_{i=1}^n g_{ij}x_i.$$

This *action* by g extends to all polynomials of R . In particular, g sends a polynomial $f(x_1, \dots, x_n)$ of R to

$$\begin{aligned} g(f(x_1, \dots, x_n)) &:= f(g(x_1), g(x_2), \dots, g(x_n)) \\ &= f\left(\sum_{i=1}^n g_{i1}x_i, \sum_{i=1}^n g_{i2}x_i, \dots, \sum_{i=1}^n g_{in}x_i\right). \end{aligned}$$

The action of g on R is often called a *coordinate change* on R , for reasons that we will explore in Section 1.4.4.

Example 1.4.1. Let $R = \mathbb{C}[x, y]$ and consider the matrix

$$g_1 := \begin{pmatrix} 3 & \pi \\ 100 & 7 \end{pmatrix}$$

Then g_1 maps

$$x \mapsto g_1(x) = 3x + 100y \text{ and } y \mapsto g_1(y) = \pi x + 7y.$$

Further, it maps the polynomial $f(x, y) = xy + x^2y^3$ to

$$\begin{aligned} g_1(f(x, y)) &= f(g_1(x), g_1(y)) \\ &= (3x + 100y)(\pi x + 7y) + (3x + 100y)^2(\pi x + 7y)^3. \end{aligned}$$

More generally, a matrix

$$g := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

of $\mathrm{GL}_2(\mathbb{C})$ with $g_{11}, g_{12}, g_{21}, g_{22} \in \mathbb{C}$ sends $f(x, y) = xy + x^2y^3$ to

$$\begin{aligned} g(f(x, y)) &= f(g(x), g(y)) \\ &= (g_{11}x + g_{21}y)(g_{12}x + g_{22}y) + (g_{11}x + g_{21}y)^2(g_{12}x + g_{22}y)^3. \end{aligned}$$

It is easy to check that this action extends even further to ideals of R , sending an ideal of R to another ideal of R . In particular, for an ideal $I = \langle f_1, \dots, f_r \rangle$ and an element $g \in \mathrm{GL}_n(\mathbb{C})$

$$g(I) := \langle g(f_1), \dots, g(f_r) \rangle.$$

Since the action sends each variable to a linear combination of other variables, it sends a homogeneous polynomial of degree d to another homogeneous polynomial of degree d , and thus a homogeneous ideal to another homogeneous ideal.

1.4.2 The Zariski open subsets of $\mathrm{GL}_n(\mathbb{C})$

The definition of the generic initial ideal requires describing *Zariski open subsets* of $\mathrm{GL}_n(\mathbb{C})$. To do this, we first note that the set of all $n \times n$ matrices $M_n(\mathbb{C})$ can be identified with the points of $\mathbb{C}^{n \times n}$:

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix} \in M_n(\mathbb{C}) \leftrightarrow (g_{11}, g_{12}, \dots, g_{21}, g_{22}, \dots, g_{nn}) \in \mathbb{C}^{n \times n}.$$

Recall that the expression for the determinant of a general $n \times n$ matrix $g = (g_{ij})$ can be written as a polynomial in the variables g_{ij} ; we will denote this polynomial $\det(g_{11}, \dots, g_{nn})$. Then, under the identification of $M_n(\mathbb{C})$ with $\mathbb{C}^{n \times n}$, $\mathrm{GL}_n(\mathbb{C})$ corresponds to the set of points in $\mathbb{C}^{n \times n}$ where $\det(g_{11}, \dots, g_{nn})$ does not vanish; this is exactly the Zariski open set

$$\mathbb{C}^{n \times n} \setminus \mathbb{V}(\det(g_{11}, \dots, g_{nn})).^{15}$$

It turns out that a subset U of $\mathrm{GL}_n(\mathbb{C})$ is Zariski open if and only if it is a Zariski open subset of $M_n(\mathbb{C})$, or equivalently of $\mathbb{C}^{n \times n}$. Recall that nonempty Zariski open subsets of $\mathrm{GL}_n(\mathbb{C})$ are very large in the following sense (see Lemma 1.2.9):

¹⁵Recall that Zariski open subsets of \mathbb{C}^m are of the form $\mathbb{C}^m \setminus \mathbb{V}(f_1, \dots, f_l)$ for some polynomials $f_i \in \mathbb{C}[x_1, \dots, x_m]$.

- the nonempty Zariski open subsets of $GL_n(\mathbb{C})$ are dense;
- the intersection of any finite number of nonempty Zariski open subsets of $GL_n(\mathbb{C})$ is nonempty; and
- for a fixed nonempty Zariski open subset U , a randomly chosen element of $GL_n(\mathbb{C})$ is likely inside of U .

1.4.3 Definition of Generic Initial Ideals

The following theorem is the key to defining the main subject of this thesis: generic initial ideals. It tells us that the generic initial ideal of a homogeneous ideal always exists. We will prove it in Section 1.4.6.

Theorem 1.4.2 ([Gal74]). *For any homogeneous ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$, there is a nonempty Zariski open subset $U \subseteq GL_n(\mathbb{C})$ such that the ideals $\text{in}(g(I))$ are equal for all $g \in U$.*

This theorem is quite remarkable. It says that if we take the initial ideal of $g(I)$ for a series of $g \in GL_n(\mathbb{C})$, we will almost always end up with the same monomial ideal. This is a special way to collapse a homogeneous polynomial ideal to a monomial ideal.

Definition 1.4.3. Let I be a homogeneous ideal of $\mathbb{C}[x_1, \dots, x_n]$ and let U be the Zariski open subset from Theorem 1.4.2. Then the **generic initial ideal** of I , denoted $\text{gin}(I)$, is the ideal $\text{in}(g(I))$ for any $g \in U$.

If we know the Zariski open subset U from Theorem 1.4.2, we can find the generic initial ideal of I in three steps.

1. Choose any matrix $g \in U$.
2. Send I to the ideal $g(I)$.
3. Take the initial ideal of $g(I)$ – this will be equal to $\text{gin}(I)$.

Generic initial ideals are not only remarkable for their existence, but also for special properties that they possess. In the next two sections we will see how they may help us to detect geometric properties related to the variety of an ideal and to compute invariants of an ideal.

Since Theorem 1.4.2 just tells us of the *existence* of the Zariski open subset needed to compute $\text{gin}(I)$ and not how to find it, a probabilistic method is usually used to compute specific generic initial ideals. The following example illustrates this method.

Example 1.4.2. Let $I = (x^3, xy, xz + z^2)$ be an ideal of $R = \mathbb{C}[x, y, z]$ with the reverse lexicographic order with $x > y > z$.¹⁶ Galligo's theorem tells us that

$$\text{gin}(I) = \text{in}(g(I))$$

for all g in some nonempty Zariski open subset U of $GL_3(\mathbb{C})$. While we do not know exactly what U is, the fact that it is Zariski open in $GL_3(\mathbb{C})$ means that almost all elements of $GL_3(\mathbb{C})$ will be inside of U . Therefore, if we choose some *random* $g \in GL_3(\mathbb{C})$, chances are that $\text{in}(g(I))$ will be equal to the generic initial ideal of I . If we choose a series of random g 's and compute $\text{in}(g(I))$ for each, the monomial ideal $\text{in}(g(I))$ that appears most often is likely to be equal to $\text{gin}(I)$.

Let g_1 and g_2 denote the following 3×3 matrices randomly generated by Macaulay 2:¹⁷

$$g_1 = \begin{pmatrix} 3/7 & 5/7 & 7/2 \\ 9/4 & 5/3 & 7/6 \\ 2 & 9/4 & 9 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 7/2 & 2/3 & 3/2 \\ 3/4 & 3 & 9/7 \\ 7/9 & 5/2 & 7/10 \end{pmatrix}.$$

Using Macaulay 2, we find that

$$\text{in}(g_1(I)) = (x^2, xy, y^3, y^2z, xz^3)$$

and

$$\text{in}(g_2(I)) = (x^2, xy, y^3, y^2z, xz^3).$$

On the other hand, if we choose

$$g_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\text{in}(g_3(I)) = (x^2, xy, xz^2, yz^2, z^4).$$

Thus, it is probable that

$$\text{gin}(I) = \text{in}(g_1(I)) = \text{in}(g_2(I)) = (x^2, xy, y^3, y^2z, xz^3).$$

We could be more certain of this choice if we repeated the calculation for more elements of $GL_3(\mathbb{C})$.

¹⁶You may think of $x = x_1$, $y = x_2$, and $z = x_3$.

¹⁷These matrices are actually generated over \mathbb{Q} rather than \mathbb{C} for the sake of simplicity.

Macaulay 2 does generic initial ideal computations in this way using the package `GenericInitialIdeal`. Its default program repeats the process of finding a random matrix, applying it to I , and taking the initial ideal six times, but the user can specify greater or fewer trials. Since this is only a probabilistic method, the generic initial ideals that Macaulay 2 generates are only correct $(100 - \epsilon)\%$ of the time.

1.4.4 Generic Initial Ideals of Points of \mathbb{P}^2

In this section, we will show how geometric properties of a set of points in \mathbb{P}^2 , such as the number of points, are reflected in the generic initial ideal of the corresponding ideal. The following example illustrates how generic initial ideals are ‘independent of a choice of coordinates.’

Example 1.4.3. Let $R = \mathbb{C}[x, y, z]$ have the reverse lexicographic order and consider the ideals $I_1 = (x - y, z)$ and $I_2 = (x - 2y, y - z)$ of R . A point $[a : b : c]$ is in $\mathbb{V}_{\mathbb{P}}(I_1)$ (that is, the equations $x - y$ and z vanish at the point) if and only if $a = b$ and $c = 0$; thus, $\mathbb{V}_{\mathbb{P}}(I_1) = [\lambda : \lambda : 0] = [1 : 1 : 0]$. Similarly, a point $[a : b : c]$ is in $\mathbb{V}_{\mathbb{P}}(I_2)$ if and only if $a = 2b$ and $b = c$; thus, $\mathbb{V}_{\mathbb{P}}(I_2) = [2\lambda : \lambda : \lambda] = [2 : 1 : 1]$. Using Macaulay 2,

$$\text{gin}(I_1) = \text{gin}(I_2) = (x, y).$$

The fact that $\text{gin}(I_1)$ is equal to $\text{gin}(I_2)$ is representative of the fact that I_1 and I_2 are related by a ‘change of coordinates.’ Intuitively, this means that the corresponding varieties $\mathbb{V}(I_1) = \{[1 : 1 : 0]\}$ and $\mathbb{V}(I_2) = \{[2 : 1 : 1]\}$ look the same: they are both single points that only differ because they are placed in different spots in \mathbb{P}^2 . Mathematically, this means that there is an invertible matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in \text{GL}_3(\mathbb{C})$$

such that $g(I_2) = I_1$. To see why this implies that $\text{gin}(I_1) = \text{gin}(I_2)$, let U be a nonempty Zariski open set of $\text{GL}_3(\mathbb{C})$ such that

$$\text{gin}(I_1) = \text{in}(hI_1)$$

for all $h \in U$. Then $Ug = \{h \cdot g : h \in U\}$ is a nonempty Zariski open subset of

$\mathrm{GL}_3(\mathbb{C})$ such that for all $h' = h \cdot g \in Ug$,

$$\mathrm{in}(h'I_2) = \mathrm{in}(h \cdot gI_2) = \mathrm{in}(hI_1) = \mathrm{gin}(I_1).$$

Thus, Ug is a nonempty Zariski open subset such that for all $h' \in Ug$, $\mathrm{in}(h'I_2)$ is constant. By definition, this means that $\mathrm{in}(h'I_2) = \mathrm{gin}(I_2)$ so $\mathrm{gin}(I_1) = \mathrm{gin}(I_2)$.

By the same argument as in the previous example, we have the following lemma; it says that generic initial ideals are independent of choice of coordinates.

Lemma 1.4.4. *Let I and I' be homogeneous ideals of $\mathbb{C}[x_1, \dots, x_n]$ and suppose that there is some $g \in \mathrm{GL}_n$ such that $g(I) = I'$ (that is, I and I' are the same up to a choice of coordinates). Then*

$$\mathrm{gin}(I) = \mathrm{gin}(I').$$

Example 1.4.4. Let p_1 and p_2 be as in Example 1.4.3. Then

$$\begin{aligned} I := I(\{p_1, p_2\}) = I(p_1) \cap I(p_2) &= (x - y, z) \cap (x - 2y, y - z) \\ &= (x - y - z, yz - z^2) \end{aligned}$$

where we use Macaulay 2 to find the intersection in the final line. One can check that

$$\mathrm{gin}(I) = (x, y^2).$$

Now consider the points $p_3 = [1 : 0 : 0]$ and $p_4 = [0 : 0 : 1]$. Then

$$\begin{aligned} I' := I(\{p_3, p_4\}) = I(p_3) \cap I(p_4) &= (y, z) \cap (y, x) \\ &= (y, xz). \end{aligned}$$

Intuitively, we expect that there is a change of coordinates sending I to I' since the corresponding sets of points in \mathbb{P}^2 look the ‘same’: the points $\{p_1, p_2\}$ can be sent to the points $\{p_3, p_4\}$ by translating, reflecting, and stretching. In fact, there is a change of coordinates

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C})$$

such that $g(I) = I'$. Thus,

$$\mathrm{gin}(I') = \mathrm{gin}(I) = (x, y^2)$$

by Lemma 1.4.4.

Example 1.4.5. Consider the set of points $S_1 = \{[1 : 0 : 0], [1 : 1 : 0], [2 : 1 : 0]\}$ and $S_2 := \{[1 : 0 : 0], [1 : 1 : 0], [2 : 0 : 1]\}$. Then

$$\begin{aligned} I_1 := I(S_1) &= I([1 : 0 : 0]) \cap I([1 : 1 : 0]) \cap I([2 : 1 : 0]) \\ &= (y, z) \cap (x - y, z) \cap (z, y - 2x) \\ &= (z, 2x^2y - 3xy^2 + y^3) \end{aligned}$$

and

$$\begin{aligned} I_2 := I(S_2) &= I([1 : 0 : 0]) \cap I([1 : 1 : 0]) \cap I([2 : 0 : 1]) \\ &= (y, z) \cap (x - y, z) \cap (y, x - 2z) \\ &= (yz, xz, xy - y^2). \end{aligned}$$

Note that the line $z = 0$ passes through all three points of S_1 (this is reflected by the fact that $z \in I_1$) while there is no line that passes through all three points of S_2 (this is reflected by the fact that I_2 does not contain any homogeneous polynomial of degree 1). Since reflections, stretches, and translations send lines to other lines, S_1 cannot be sent to S_2 by any series of these operations. Thus, we do not expect that there is a change of coordinates $g \in \text{GL}_3(\mathbb{C})$ such that $g(I_1) = I_2$. We can confirm that there is no such change of coordinates by computing generic initial ideals and seeing that

$$\text{gin}(I_1) = (x, y^3)$$

is not equal to

$$\text{gin}(I_2) = (x^2, xy, y^2).$$

If there was such a change of coordinates, $\text{gin}(I_1)$ would be equal to $\text{gin}(I_2)$ by Lemma 1.4.4.

We will see that generic initial ideals can reflect information about projective varieties beyond whether two varieties are the same up to a change of coordinates. To illustrate the type of information that we may obtain, we will continue to study the case of a set of points of \mathbb{P}^2 . In what follows, we will assume that $\Gamma = \{p_1, \dots, p_r\}$ where the p_i are points of \mathbb{P}^2 and that $I_\Gamma := I(\Gamma)$ is the corresponding ideal of \mathbb{P}^2 .

First note that the generic initial ideals in the examples above have generators that only contain variables x and y (that is, they are *generated in x and y*). The following

lemma tells us that this holds for all generic initial ideals of point arrangements (see Section 4 of [Gre98]).

Lemma 1.4.5 ([Gre98]). *Let $R = \mathbb{C}[x, y, z]$ with the reverse lexicographic order with $x > y > z$ and let $I_\Gamma \subseteq R$ be the ideal of r points as above.*

1. *The minimal set of generators of $\text{gin}(I_\Gamma)$ only contains the variables x and y .*
2. *There is some sequence of positive integers $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ such that*

$$\lambda_0 > \lambda_1 > \dots > \lambda_{k-1} > 0$$

and

$$\text{gin}(I) = \langle x^k, x^{k-1}y^{\lambda_{k-1}}, x^{k-2}y^{\lambda_{k-2}}, \dots, xy^{\lambda_1}, y^{\lambda_0} \rangle.$$

3. *The λ_i above are such that*

$$\lambda_0 + \lambda_1 + \dots + \lambda_{k-1} = r.$$

This lemma tells us that the generic initial ideal of I_Γ is completely determined by a sequence $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ of integers; we will call these numbers the **invariants** of Γ .

By part (c) of Lemma 1.4.5, the number of points that Γ contains is encoded by $\text{gin}(I_\Gamma)$; the following theorem of Ellia and Peskine says that $\text{gin}(I_\Gamma)$ can also detect whether some of the points in Γ lie on a curve.

Theorem 1.4.6 ([EP90]). *Let $\Gamma = \{p_1, \dots, p_r\} \subseteq \mathbb{P}^2$ and let $\lambda_0, \dots, \lambda_{k-1}$ be the invariants of $\text{gin}(I_\Gamma)$. If*

$$\lambda_i > \lambda_{i+1} + 2$$

for some $i < k - 1$ then Γ contains a subset of $\lambda_0 + \lambda_1 + \dots + \lambda_i$ points lying on a curve of degree $i + 1$.

Example 1.4.6. Suppose that I_Γ is the ideal of some point arrangement Γ and that

$$\text{gin}(I_\Gamma) = \langle x^2, xy, y^4 \rangle.$$

Then the invariants of I_Γ are $\lambda_0 = 4$ and $\lambda_1 = 1$. By the third part of Lemma 1.4.5, Γ contains

$$\lambda_0 + \lambda_1 = 4 + 1 = 5$$

points. Since $\lambda_0 = 4 > \lambda_1 + 2 = 3$, Theorem 1.4.6 tells us that there is a subset of $\lambda_0 = 4$ points in Γ lying on a curve of degree 1 (that is, a line). Therefore, Γ consists of 5 points, four of which lie on a line.

Example 1.4.7. The Ellia-Peskine theorem does not say that $\text{gin}(I_\Gamma)$ *always* reflects when points lie on a curve. For example, consider

$$\Gamma = \{[1 : 0 : 0], [0 : 1 : 0], [1 : 1 : 0], [0 : 0 : 1]\}.$$

The first three points in Γ lie on the line $z = 0$ while the fourth lies off of the line. If Theorem 1.4.6 detected the fact that three points were on a line then $\text{gin}(I_\Gamma)$ would have invariants λ_i such that $\lambda_0 > \lambda_1 + 2$ and $\lambda_0 = 3$. This would imply that $\lambda_1 \leq 0$, which contradicts the fact (from Lemma 1.4.5) that $\lambda_i > 0$ for all i . Using Macaulay 2 we see that $\text{gin}(I_\Gamma)$ is in fact equal to

$$\text{gin}(I_\Gamma) = (x^2, xy, y^3)$$

and the actual invariants are $\lambda_0 = 3$ and $\lambda_1 = 1$. The conditions of Theorem 1.4.6 do not hold for this ideal.

1.4.5 Generic Initial Ideals are Borel-fixed

In this section, we will see that generic initial ideals possess a very nice combinatorial property called Borel-fixedness that makes them even easier to work with than arbitrary monomial ideals. This property is the key to using the $\text{gin}(I)$ to uncover invariants of the original ideal I .

Definition 1.4.7. An $n \times n$ matrix $g = (g_{ij})$ is **upper triangular** if all of the entries below the diagonal are 0 (that is, $g_{ij} = 0$ whenever $i > j$). The subset of all invertible upper triangular matrices \mathcal{B} of $\text{GL}_n(\mathbb{C})$ is called the **Borel subgroup**. An ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is **Borel-fixed** if

$$g(I) = I$$

for *all* upper triangular matrices $g \in \mathcal{B}$.

Theorem 1.4.8 (Galligo's Theorem [Gal74]). *Let $R = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring and I be a homogeneous ideal of R . Then $\text{gin}(I)$ is Borel-fixed.*¹⁸

¹⁸This holds when \mathbb{C} is replaced with *any* field and R has any multiplicative monomial order.

At first glance, it seems as though verifying the Borel-fixed property for a given ideal would be quite complicated, as we have to check to see whether it is fixed by the action of an infinite number of matrices. The following well-known result gives an alternative – and much easier – way to check whether a monomial ideal is Borel-fixed.

Proposition 1.4.9. *Let I be a monomial ideal in $\mathbb{C}[x_1, \dots, x_n]$.¹⁹ Then the following statements are equivalent.*

1. I is Borel-fixed.
2. I is **strongly stable**: for any monomial $u \in I$ such that x_j divides u , $x_i \frac{u}{x_j} \in I$ for all $i < j$. In words, if we replace a variable in any monomial u in I with a lower-indexed variable, the new monomial will still be in I .

Proof. See Section 4.2 of [HH11]. □

The following lemma says that it is enough to check the strongly stable condition for any set of monomial generators of I . For a proof see Lemma 4.2.3 of [HH11].

Lemma 1.4.10. *Let I be a monomial ideal with a set of monomial generators $G(I)$. If $x_i \frac{u}{x_j} \in I$ for all monomials $u \in G(I)$ and all integers $i < j$ such that x_j divides u , then I is strongly stable, and thus is Borel-fixed.*

Example 1.4.8. In Example 1.4.2 we claimed that the generic initial ideal of $I = (x_1^3, x_1x_2, x_1x_3 + x_3^2) \subseteq \mathbb{C}[x_1, x_2, x_3]$ is $\text{gin}(I) = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^3)$. We may use Proposition 1.4.9 and Lemma 1.4.10 to verify that this generic initial ideal is Borel-fixed. In particular, we need to verify that when a lower-indexed variable is substituted for any higher-indexed variable in a monomial generator of $\text{gin}(I)$, we obtain another element of $\text{gin}(I)$.

- Since there are no higher-indexed variables in x_1^2 , there is nothing to check for this generator.
- When x_1 is substituted for x_2 in x_1x_2 , we get $x_1^2 \in I$.
- For x_2^3 , we need to check that if we substitute an x_1 for any or all of the x_2 's, we stay in I . In particular, $x_1x_2^2, x_1^2x_2, x_1^3 \in I$.

¹⁹This proposition also holds whenever I is a monomial ideal in a polynomial ring over a field of characteristic 0. The strongly stable condition can be reformulated when working over fields of characteristic $p > 0$ to get a similar statement; see Chapter 15 of [Eis04] for a complete description of the characteristic $p > 0$ case.

Completing similar substitutions in $x_2^2x_3$ and $x_1x_3^3$, the reader can verify that $\text{gin}(I)$ is Borel-fixed.

We saw in Section 1.3.3 that if I is a monomial ideal, $\text{in}(I) = I$. The following shows that this does not hold for generic initial ideals.

Example 1.4.9. Let $I = (x^3, xy) \subseteq \mathbb{C}[x, y]$. If $\text{gin}(I) = I$, then I would be Borel-fixed by Theorem 1.4.8. However, I is not Borel-fixed because it is not strongly stable: substituting x for y in $xy \in I$ gives $x^2 \notin I$. In fact, $\text{gin}(I) = (x^2, xy^2)$, which is Borel-fixed.

The following result tells us that $\text{gin}(I) = I$ exactly when I is a Borel-fixed monomial ideal. Its proof is not complicated (in fact, one direction is an immediate consequence of Theorem 1.4.8), but it requires some additional results that are not included here; for a proof, see Proposition 4.2.6 of [HH11].

Proposition 1.4.11 ([Con04]). *Let I be a monomial ideal of $R = \mathbb{C}[x_1, \dots, x_n]$. Then $\text{gin}(I) = I$ if and only if I is Borel-fixed.*

Example 1.4.10. $I = (x^3, x^2y, xy^2, z^3, x^2z^2) \subseteq \mathbb{C}[x, y, z]$ is a Borel-fixed ideal. Thus, $\text{gin}(I) = I$.

There are many numbers, or *invariants*, associated to an ideal I that can help us to understand I more deeply. Invariants such as the *regularity* $\text{reg}(I)$, *satiety* $\text{sat}(I)$, and *depth* $\text{depth}(R/I)$ of I are often difficult to compute. It turns out that knowing the generic initial ideal of I with respect to the reverse lexicographic order allows us to easily find these invariants. This is due to the following results of Bayer and Stillman ([BS87a]) and of Ahn and Migliore ([AM07]).

1. If J is a Borel-fixed ideal of $R = \mathbb{C}[x_1, \dots, x_n]$, $\text{reg}(J)$, $\text{sat}(J)$, and $\text{depth}(J)$ can be determined by simply looking at the generators of J . In particular,
 - $\text{reg}(J) =$ the degree of the largest generator of J
 - $\text{sat}(J) =$ the degree of the largest generator of J involving x_n
 - $\text{depth}(R/J) = n -$ largest variable contained in a minimal generator of J
2. When we use the reverse lexicographic order, for any homogeneous ideal I :
 - $\text{reg}(I) = \text{reg}(\text{gin}(I))$;
 - $\text{sat}(I) = \text{sat}(\text{gin}(I))$; and

- $\text{depth}(R/I) = \text{depth}(R/\text{gin}(I))$

We will discuss these invariants and the results above further in Subsection 2.2.2 of Chapter II.

Example 1.4.11. Let $I = (x_1^3, x_1x_2, x_1x_3 + x_3^2) \subseteq K[x, y, z]$. In Example 1.4.2, we saw that under the reverse lexicographic order

$$J := \text{gin}(I) = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^3).$$

Then:

1. the generator of J of with the largest degree is $x_1x_3^3$, which has degree 4;
2. the largest degree generator of J involving the variable z is $x_1x_3^3$, which has degree 4; and
3. the largest indexed variable, x_3 , appears in a generator of J .

Thus,

1. $\text{reg}(I) = \text{reg}(J) = 4$;
2. $\text{sat}(I) = \text{sat}(J) = 4$; and
3. $\text{depth}(R/I) = \text{reg}(R/J) = 3$.

1.4.6 Existence of Generic Initial Ideals

We will now prepare to prove Theorem 1.4.2 which claims that the Zariski open subset necessary for defining a generic initial ideal always exists. First, we define the d th graded component of an ideal.

Definition 1.4.12. Let $R = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring with the standard grading. Let R_d denote the homogeneous polynomials of degree d in R . If I is a homogeneous ideal of R , the **d th graded piece of I** is

$$I_d := I \cap R_d = \{f \in R : f \in R_d\}.$$

That is, I_d consists of polynomials of I that are sums of monomials of degree d .

Standard bases of graded parts of an ideal are analogous to Gröbner bases. Note that for each monomial $x^J \in \text{in}(I)$, there is at least one $f_J \in I$ with $\text{in}(f_J) = x^J$.

Definition 1.4.13. Let $\text{in}(f_J) = x^J$. A set of elements $\{f_J\}$ such that $\{x_J\}$ ranges over all monomials in $\text{in}(I)_d$ is called a **standard basis** for I_d .

For a proof of the following proposition, see Proposition 1.11 of [Gre98].

Proposition 1.4.14. *Let I be a homogeneous ideal. Then, for all d , a standard basis of I_d is actually a basis of I_d .*

Definition 1.4.15. Fix a homogeneous ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ and a positive integer d . Let $S = \{f_1, \dots, f_N\}$ be an ordered basis for I_d (that is, $\langle S \rangle = I_d$).²⁰ We will represent this basis as a matrix as follows. First, write the $\binom{n-1+d}{d}$ monomials of $\mathbb{C}[x_1, \dots, x_n]$ of degree d in decreasing order with respect to the term order, so that $m_{1,d}$ is the largest monomial of degree d and $m_{\binom{n-1+d}{d},d}$ is the smallest monomial of degree d . Then we can write each of the elements f_l in S as

$$f_l = \sum_{i=1}^{\binom{n-1+d}{d}} a_{l,i} m_{i,d}$$

where $a_{l,i} \in \mathbb{C}$. The matrix $M(I_{d,S})$ representing this basis is defined as

$$M(I_{d,S}) := (a_{l,i}).$$

In particular, each row of $M(I_{d,S})$ represents one element of the basis S while each column represents one of the monomials of degree d .

Example 1.4.12. Suppose that the monomials of $R = \mathbb{C}[x, y, z]$ are ordered according to the reverse lexicographic order with $x > y > z$. Then, with the notation from the previous definition, $m_{1,3} = x^3$, $m_{2,3} = x^2y$, $m_{3,3} = xy^2$, $m_{4,3} = y^3$, $m_{5,3} = x^2z$, $m_{6,3} = xyz$, $m_{7,3} = y^2z$, $m_{8,3} = xz^2$, $m_{9,3} = yz^2$, $m_{10,3} = z^3$.

Let $I = (x^3, xy, xz + z^2) \subseteq R$. Then using Macaulay 2,

$$S = \{x^3, xyz, x^2y, xy^2, xz^2 + z^3, x^2z + xz^2, xyz + yz^2\}$$

is a basis of I_3 and

²⁰The ‘ordered’ part here just means that we keep track of the position of the elements in S .

$$M(I_{3,S}) = \begin{pmatrix} x^3 & x^2y & xy^2 & y^3 & x^2z & xyz & y^2z & xz^2 & yz^2 & z^3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

For any homogeneous ideal I and basis S of I_d , the submatrix consisting of the first k columns of $M(I_{d,S})$ (that is, the columns corresponding to the highest k monomials of degree d) will be denoted

$$M_k(I_{d,S}).$$

Our interest in these submatrices comes from the following proposition.

Proposition 1.4.16. *Given a homogeneous ideal I and a positive integer d , the rank of $M_k(I_{d,S})$ is independent of the choice of basis S of I_d . Further, this rank is equal to the number of the highest k monomials of degree d contained in $\text{in}(I)_d$.*

Proof. Suppose that we have two different ordered bases for I_d : S_1 and S_2 . Then S_1 can be obtained from S_2 by multiplying elements of S_2 by constants, adding multiples of elements in S_2 to each other, and rearranging elements of S_2 . Since each of these operations on the basis elements corresponds to elementary row operations on the matrix $M(I_{d,S_2})$, the matrix $M(I_{d,S_1})$ can be obtained from $M(I_{d,S_2})$ by making a series of elementary row operations.

Recall from linear algebra that the rank of a matrix M is equal to its column rank, or the number of linearly independent columns of M . Row operations do not change the linear relations between the columns of a matrix M , so they do not change the column rank – or rank – of M or of any submatrix obtained by taking a subset of the columns of M . Thus, since $M(I_{d,S_1})$ and $M(I_{d,S_2})$ are related by row operations,

$$\text{rk}(M_k(I_{d,S_1})) = \text{rk}(M_k(I_{d,S_2}))$$

for any k and any two bases S_1 and S_2 .

To prove the second statement, we will fix $S = \{f_1, \dots, f_N\}$ to be a standard basis of I_d , ordered so that $\text{in}(f_1) > \text{in}(f_2) > \dots > \text{in}(f_N)$ and written so that the

coefficient of the leading term of each f_i is 1.²¹ With this choice, $M(I_{d,S})$ has the following characteristics.

- There are no nonzero rows.
- The first nonzero entry of a row is always strictly to the right of the first nonzero entry of the row above it. This is due to the way we have ordered the basis and since all $\text{in}(f_j)$ are distinct for f_j in a standard basis.
- The first nonzero entry of each row is a 1.

These three facts imply that the following hold for the submatrices $M_k(I_{d,S})$.

- Rows containing all zeroes are at the bottom of the matrix.
- The first nonzero entry of a row is always strictly to the right of the first nonzero entry of the row right above it.
- The first nonzero entry of each row is a 1.

These conditions are equivalent to saying that $M_k(I_{d,S})$ is in row-echelon form.

Recall that the rank of a matrix in row-echelon form is equal to the number of rows with leading nonzero entries. Also, having the leading 1 of the l th row of $M_k(I_{d,S})$ placed in the column corresponding to the j th highest monomial $m_{j,d}$ means exactly that

$$\text{in}(f_l) = m_{j,d}$$

because S is a standard basis of I_d . Then

$$\begin{aligned} \text{rk}(M_k(I_{d,S})) &= \text{number of rows of } M_k(I_{d,S}) \text{ with leading nonzero entries} \\ &= \text{number of the first } k \text{ monomials that are the initial terms} \\ &\quad \text{of some element } f_1, \dots, f_N \\ &= \text{number of the first } k \text{ monomials of degree } d \text{ that are in } \text{in}(I)_d \end{aligned}$$

where the last equality follows from the fact that we are working with a standard basis. □

Since rank of $M_k(I_{d,S})$ does not depend on the specific basis S chosen, we will just write $\text{rk}(M_k(I_d))$ without specifying a basis.

The following lemma will also be used in the proof of Theorem 1.4.2, but we will postpone its proof until after the main proof.

²¹Recall that S is a standard basis if $\text{in}(f_1), \dots, \text{in}(f_N)$ are exactly the monomials in $\text{in}(I)_d$.

Lemma 1.4.17. *Let I be a homogeneous ideal of $\mathbb{C}[x_1, \dots, x_n]$, let d and k be positive integers, and let $M_k(g(I)_d)$ be the matrix defined above. Then there is a nonempty Zariski open subset $U_{d,k} \subseteq GL_n(\mathbb{C})$ such that the rank of $M_k(g(I)_d)$ is constant for all $g \in U_{d,k}$.*

Proof of Theorem 1.4.2. Fix a natural number d . We claim that there is some nonempty Zariski open subset U_d of $GL_n(\mathbb{C})$ such that all of the ideals $\text{in}(g(I))_d$ are the same for $g \in U_d$.

To see this, note that for any k , Lemma 1.4.17 gives us a nonempty Zariski open subset $U_{d,k}$ such that

$$\text{rk}(M_k(g(I)_d)) = r_{d,k}$$

for all $g \in U_{d,k}$. By Proposition 1.4.16, this is equivalent to saying that $r_{d,k}$ of the highest k monomials of degree d are contained in $\text{in}(g(I))_d$ for all $g \in U_{d,k}$. Now define

$$U_d = \bigcap_{i=1}^{\binom{n-1+d}{d}} U_{d,i},$$

so exactly $r_{d,i}$ of the highest i monomials of degree d are contained in $\text{in}(g(I))_d$ for $i = 1, \dots, \binom{n-1+d}{d}$ and all $g \in U_d$.

It is easy to see that the sequence of numbers $\{r_{d,i}\}_i$ completely determines the monomials that are contained in $\text{in}(g(I))_d$ for all $g \in U_d$. For example, $r_{d,2} = 1$ means that only one of the two highest monomials of degree d , $m_{1,d}$ or $m_{2,d}$, is contained in each $\text{in}(g(I))_d$. To see which one, we only need to look at $r_{d,1}$: if $r_{d,1} = 1$, the highest monomial $m_{1,d}$ is in each $\text{in}(g(I))_d$, while if $r_{d,1} = 0$, $m_{1,d}$ is not in any $\text{in}(g(I))_d$, so $m_{2,d}$ is in each $\text{in}(g(I))_d$. Continuing inductively, we can use the sequence $\{r_{d,i}\}$ to find all of the monomials contained in $\text{in}(g(I))_d$ for $g \in U_d$.

Now, define $\text{gin}(I)$ to be the ideal such that

$$\text{gin}(I)_d = \text{in}(g(I))_d$$

for all $g \in U_1 \cap \dots \cap U_d$ and note that each $U_1 \cap \dots \cap U_d$ is Zariski open as the finite intersection of nonempty Zariski open sets. Since ideals of $\mathbb{C}[x_1, \dots, x_n]$ are finitely generated by Theorem 1.1.7, there is a generator of $\text{gin}(I)$ of highest degree; say it is of degree d_0 . Then

$$\begin{aligned} \text{gin}(I) &= \langle \text{gin}(I)_1, \text{gin}(I)_2, \dots, \text{gin}(I)_{d_0} \rangle \\ &= \langle \text{in}(g(I))_1, \text{in}(g(I))_2, \dots, \text{in}(g(I))_{d_0} \rangle \text{ for all } g \in U_1 \cap \dots \cap U_{d_0} \end{aligned}$$

$$= \text{in}(g(I)) \text{ for all } g \in U_1 \cap \cdots \cap U_{d_0}$$

where the final equality holds because $\text{in}(g(I))$ is generated in at most degree d_0 by assumption. Therefore, the ideals $\text{in}(g(I))$ are equal for all g in nonempty Zariski open subset $U_1 \cap \cdots \cap U_{d_0}$ of $\text{GL}_n(\mathbb{C})$, and the theorem holds. \square

It remains to prove Lemma 1.4.17.

Proof of Lemma 1.4.17. Recall that $\text{rk}(M) = r$ if and only if r is the largest number such that there is an $r \times r$ minor that does not vanish.²² Thus, we will be interested in studying the minors of the matrices $M_k(g(I)_d)$ as g varies.

Let $g = (g_{ij})$ be an arbitrary element of $\text{GL}_n(\mathbb{C})$. Note that the entries of $M_k(g(I)_d)$ are polynomials in the variables g_{ij} , so the minors of $M_k(g(I)_d)$ are also given by polynomials in the g_{ij} .

Consider the $r \times r$ minors of the matrix $M_k(g(I)_d)$ and suppose they are given by the polynomials $p_{r,1}(g_{ij}), \dots, p_{r,Q}(g_{ij})$. Two cases are possible.

1. All of the polynomials $p_{r,1}, \dots, p_{r,Q}$ are the 0 polynomial. This means for any change of coordinates $g = (g_{ij})$, all $r \times r$ minors of $M_k(g(I)_d)$ vanish.
2. At least one of the polynomials – say $p_{r,q}$ – is not the zero polynomial. Then for any $(g_{ij}) \in \text{GL}_n(\mathbb{C}) \setminus \mathbb{V}(p_{r,q})$

$$p_{r,q}(g_{ij}) \neq 0.$$

This means that there is an $r \times r$ minor of $M_k(g(I)_d)$ that does not vanish on a nonempty Zariski open subset of $\text{GL}_n(\mathbb{C})$; call this Zariski open set U_r .

Since $M_k(g(I)_d)$ has k columns, the minors of this matrix are of size $k \times k$ or less. Thus, we can find the greatest r such that the second condition holds; for such an r , set $U_{d,k} = U_r$. Then the rank of $M_k(g(I)_d)$ is equal to r for all $g \in U_{d,k}$ since the $r \times r$ minor given by $p_{r,q}$ does not vanish on $U_{d,k}$ but all larger minors do vanish. \square

1.5 Construction of the Limiting Shape

For each $m = 1, 2, 3, \dots$ let I_m be a monomial ideal. The goal of this section is to describe how to visualize what happens to an infinite collection of monomial ideals $\{I_m\}_m$ as m gets large. In particular, we will study the limiting behavior of *graded systems of monomial ideals*.

²²Recall that an $r \times r$ *minor* of a matrix M is the determinant of an $r \times r$ submatrix.

Definition 1.5.1 ([ELS01]). A collection $\{I_m\}$ of ideals of $R = \mathbb{C}[x_1, \dots, x_n]$ is a **graded system** if $I_j I_i \subseteq I_{i+j}$ for all i, j . If each of the ideals I_m is also a monomial ideal, such a collection is a **graded system of monomial ideals**.

Example 1.5.1. Let I be an ideal of $R = \mathbb{C}[x_1, \dots, x_n]$ and consider the infinite collection of ideals

$$\{I, I^2, I^3, I^4, \dots\} = \{I^j\}_{j=1}^{\infty} = \{I^j\}_j.$$

Notice that for every i, j ,

$$I^i \cdot I^j \subseteq I^{i+j};$$

this property makes $\{I^j\}_j$ a graded system of polynomial ideals.

The main examples of graded systems that we will consider are *generic initial systems* and *symbolic generic initial systems*.

Definition 1.5.2. Let I be a homogeneous ideal of $\mathbb{C}[x_1, \dots, x_n]$. The **generic initial system** of I is the collection of generic initial ideals of powers of I

$$\{\text{gin}(I^m)\}_m.$$

The **symbolic generic initial system** of I is the collection of generic initial ideals of symbolic powers of I :

$$\{\text{gin}(I^{(m)})\}_m.^{23}$$

Both of these collections of ideals are graded systems (this claim is proven in Chapters IV and V).

Let $\{I_m\}_m \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a graded system of monomial ideals. Then we can construct the *limiting shape* of $\{I_m\}_m$ in three steps detailed below.

1. Construct the *Newton polytope* of each ideal I_m .
2. Scale each of the Newton polytopes so they are nested.
3. Take the limit of the scaled polytopes as $m \rightarrow \infty$.

²³Recall from Definition 1.2.18 that the m th symbolic power of I is equal to

$$I^{(m)} := \{r \in R : sr \in I^m \text{ for some } s \in R \setminus I\}.$$

1.5.1 Construct the Newton polytope of each ideal

The *Newton polytope* of a monomial ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an infinite convex region of the first quadrant of \mathbb{R}^n . To define it, we first note that every monomial ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ can be thought of as a subset Γ_I of the lattice points \mathbb{N}^n where

$$(\lambda_1, \dots, \lambda_n) \in \Gamma_I \iff x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \in I.$$

Example 1.5.2. Let $I = \langle x^2, xy, y^4 \rangle$ be an ideal of $\mathbb{C}[x, y]$. Then the subset of lattice points corresponding to I , Γ_I , is shown in Figure 1.5. Note that all of the lattice points above and to the right of the region in the picture are also included in Γ_I .

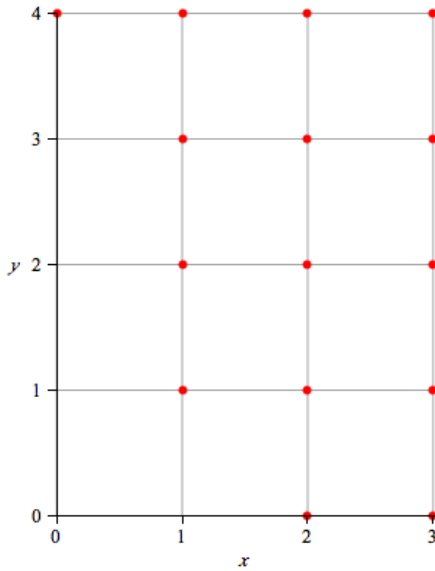


Figure 1.5: Γ_I when $I = \langle x^2, xy, y^4 \rangle$.

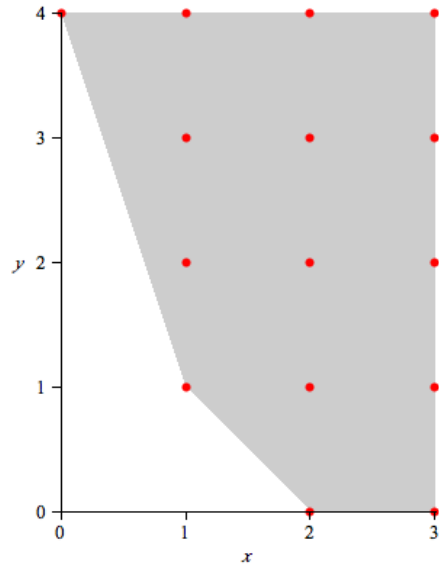


Figure 1.6: P_I when $I = \langle x^2, xy, y^4 \rangle$.

Definition 1.5.3. The **convex hull** of a subset Γ of \mathbb{R}^n is the smallest convex subset of \mathbb{R}^n containing Γ . The **Newton polytope** of a monomial ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is the convex hull of Γ_I when considered as a subset of \mathbb{R}^n . We will denote the Newton polytope of I by P_I .

Example 1.5.3. The shaded region in Figure 1.6 represents the Newton polytope P_I of $I = \langle x^2, xy, y^4 \rangle$ considered in the previous example. Note that the Newton polytope is an infinite region that continues above and to the right of the picture.

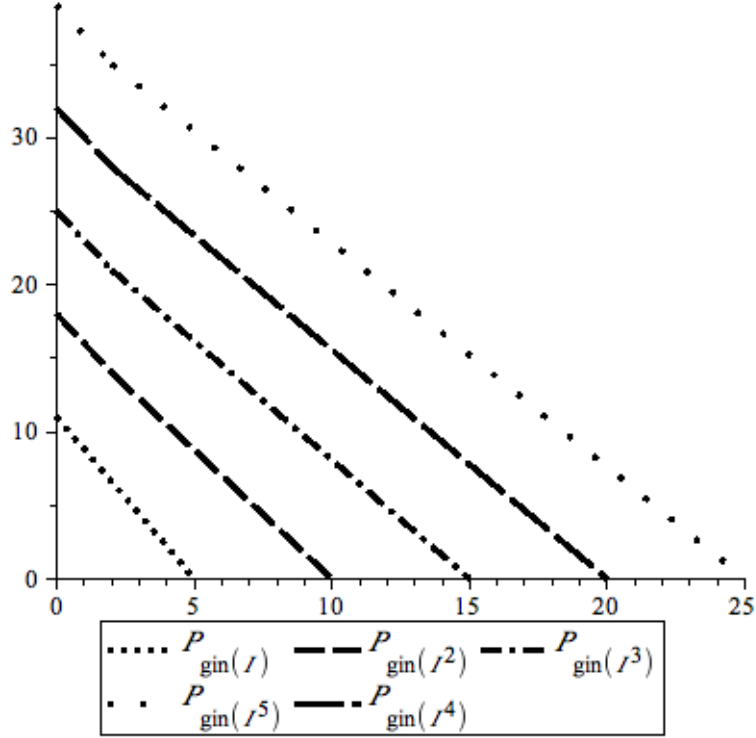


Figure 1.7: The boundaries of the Newton Polytopes $P_{\text{gin}(I^m)}$ for $m = 1, \dots, 5$ when $I = (x^5, y^7)$.

It turns out that the Newton polytope of a monomial ideal I is closely related to the *integral closure* of I . In particular, a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is contained in the integral closure of I if and only if $(\alpha_1, \dots, \alpha_n) \in P_I$ (see Section 1.4 of [SH06]).

When we have an entire system of monomial ideals $\{I_m\}_m$ of $\mathbb{C}[x_1, \dots, x_n]$, we can consider the entire collection of Newton polytopes of ideals in the system, $\{P_{I_m}\}_m$; by definition, each of these polytopes is a subset of \mathbb{R}^n .

Example 1.5.4. Let $I = (x^5, y^7)$ be an ideal of $\mathbb{C}[x, y]$; we will consider the generic initial system $\{\text{gin}(I^m)\}_m$. The Theorem 3.4.1 of Chapter III gives the generators of each of the generic initial ideals $\text{gin}(I^m)$ from which we can get the Newton polytopes of the ideals. Figure 1.7 shows the boundaries of the Newton polytopes $P_{\text{gin}(I^m)} \subseteq \mathbb{R}^2$ for $m = 1, \dots, 5$. For example, the Newton polytope of $\text{gin}(I^3)$ is the infinite region lying above the line composed of dashes and dots. Note that these Newton polytopes get smaller as m increases and that only the boundary of $P_{\text{gin}(I)}$ consists of a single line segment.

1.5.2 Scale each of the Newton polytopes so that they are nested

When we scale each of the Newton polytopes in $\{P_m\}_m$ by a factor according to its position in this set, we get another set of polytopes that are nested.

If Q is a subset of \mathbb{R}^n then, for a positive integer m , $\frac{1}{m}Q$ denotes the subset of \mathbb{R}^n defined by

$$(\alpha_1, \dots, \alpha_n) \in Q \iff \left(\frac{1}{m}\alpha_1, \dots, \frac{1}{m}\alpha_n \right) \in \frac{1}{m}Q.$$

Note that $\frac{1}{m}Q$ has the same shape as Q . We can then consider the set of scaled polytopes

$$\left\{ \frac{1}{m}P_m \right\}_m$$

where $\{P_m\}_m$ is the set of Newton polytopes from the first step. It turns out that these scaled polytopes are nested.

Proposition 1.5.4. *If $\{I_m\}_m$ is a graded system of monomial ideals in $R = \mathbb{C}[x_1, \dots, x_n]$ then the polytopes $\left\{ \frac{1}{m}P_m \right\}_m$ constructed above are nested:*

$$\frac{1}{p}P_{I_p} \subseteq \frac{1}{q}P_{I_q} \text{ whenever } p \text{ divides } q.$$

Proof. First note that if I and J are monomial ideals of $\mathbb{C}[x_1, \dots, x_n]$ then $\Gamma_I + \Gamma_J = \Gamma_{I+J}$; this essentially follows from the fact that multiplication of monomials corresponds to adding their exponents. Thus, if I, J , and L are monomial ideals of R such that $I \cdot J \subseteq L$,

$$\Gamma_{I \cdot J} = \Gamma_I + \Gamma_J \subseteq \Gamma_L$$

and

$$P_I + P_J \subseteq P_L.$$

Now, suppose that p divides q , so that there is an integer s such that $q = ps$. Since $\{I_m\}_m$ is a graded system of ideals, we have

$$\underbrace{I_p \cdot I_p \cdots I_p}_{s \text{ repeats}} \subseteq I_{ps} = I_q.$$

Thus,

$$\underbrace{P_{I_p} + \cdots + P_{I_p}}_{s \text{ repeats}} \subseteq P_{I_q}$$

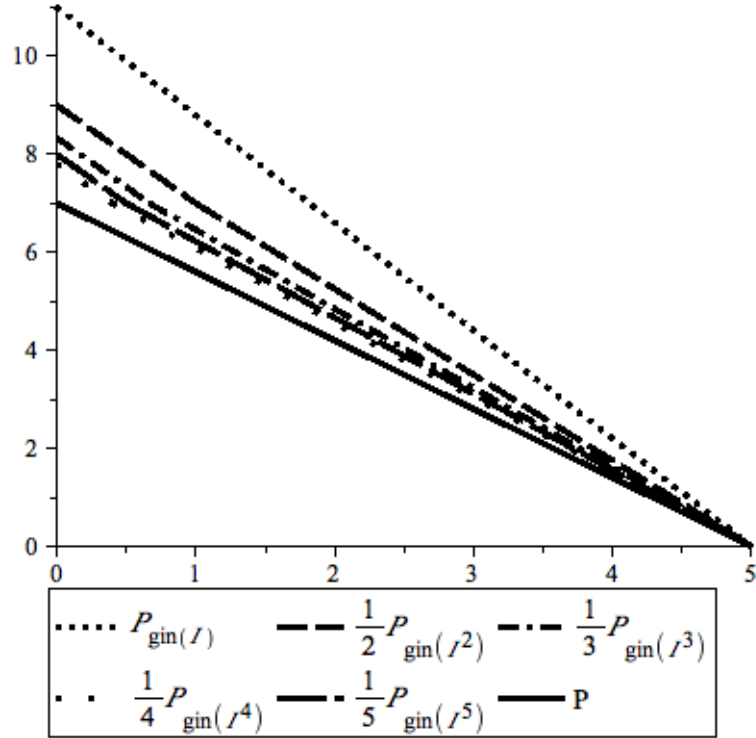


Figure 1.8: The boundaries of the scaled Newton Polytopes $\frac{1}{m}P_{\text{gin}(I^m)}$ for $m = 1, \dots, 5$ when $I = (x^5, y^7)$.

and, scaling each side by a factor of $\frac{1}{s}$,

$$P_{I_p} \subseteq \frac{1}{s}P_{I_q}.$$

If we then scale each side by a factor of $\frac{1}{p}$, we get the claim. \square

Example 1.5.5. Let $I = (x^5, y^7)$ be the ideal considered in Example 1.5.4. We will study the set of scaled polytopes

$$\left\{ \frac{1}{m}P_{\text{gin}(I^m)} \right\}_m.$$

The boundaries first five polytopes in this set are shown in Figure 1.8. They are obtained by scaling the boundaries of the polytopes from Figure 1.7.

1.5.3 Take the limit of the scaled polytopes

We can now define the geometric object that describes the limiting behavior of a graded system of monomial ideals.

Definition 1.5.5. The **limiting shape** P of a graded system of monomial ideals $\{I_m\}_m$ is defined to be

$$P := \overline{\bigcup_{m=1}^{\infty} \frac{1}{m} P_{I_m}}.$$

By Proposition 1.5.4, this is the limit of the scaled Newton polytopes $\{\frac{1}{m}P_m\}_m$.

Example 1.5.6. Let $I = (x^5, y^7)$ considered in the previous two examples. The main result of Chapter IV shows that the boundary of the limiting shape P of the generic initial system of I , $\{\text{gin}(I^m)\}_m$, is defined by the line segment through the points $(5, 0)$ and $(0, 7)$. This is the solid line segment shown in Figure 1.8.

1.6 The Main Question

The main question addressed in this thesis is the following.

Question 1.6.1. *Given a homogeneous ideal I , what can we say about the limiting shape of the generic initial system or the symbolic generic initial system of I ?*

We answer this question for two important classes of ideals: complete intersections and ideals corresponding to sets of points in \mathbb{P}^2 . The first section of the following chapter outlines some of these answers and details where they can be found in the main text.

CHAPTER II

Introduction

2.1 Motivation and Results

The study of the asymptotic behavior of families of related objects is an important research trend of the past twenty years. It is motivated by the philosophy that there is often a uniformity in the limit of a collection of algebraic objects that is not seen when studying individual members of the collection. Significant work along these lines includes: Huneke's uniform Artin-Rees lemma [Hun92]; Siu's work on the deformation-invariance of plurigena [Siu98]; Ein, Lazarsfeld, and Smith's introduction of graded systems and asymptotic multiplier ideals [ELS01]; and, most recently, Eisenbud and Schreyer's proof of the Boij-Söderberg conjectures [ES09].

This thesis applies asymptotics to computational commutative algebra and, in particular, to the study of generic initial ideals. As a coordinate-independent version of the initial ideal, generic initial ideals interact well with geometry. They are also Borel-fixed, which gives them a nice combinatorial structure. However, due to the fact that they are defined by an existence theorem rather than by an explicit construction, generic initial ideals are often very difficult to compute and can have complicated generating sets. We will show that much of this complexity can be overcome by taking an asymptotic approach.

Throughout, $R = K[x_1, \dots, x_n]$ is a polynomial ring over a field of characteristic 0 with the reverse lexicographic order where $x_1 > x_2 > \dots > x_n$ and I is a homogeneous ideal of R . We investigate infinite collections of generic initial ideals of the forms $\{\text{gin}(I^m)\}_m$ and $\{\text{gin}(I^{(m)})\}_m$, which we call *generic initial systems* and *symbolic generic initial systems*, respectively.¹ To describe the asymptotic behavior of graded systems of monomial ideals such as these, we use the notion of a *limiting shape*. If

¹Section 1.4 of Chapter I contains background on generic initial ideals.

$P_J \subseteq \mathbb{R}_{\geq 0}^n$ denotes the Newton polytope of a monomial ideal J , the limiting shape P of a graded system of monomial ideals \mathfrak{a}_\bullet is the limit of the scaled polytopes $\frac{1}{q}P_{\mathfrak{a}_q}$:

$$P = \overline{\bigcup_{q \in \mathbb{N}^*} \frac{1}{q}P_{\mathfrak{a}_q}}.$$

This concept arose from questions in algebraic geometry: the asymptotic multiplier ideals of a graded system of monomial ideals are completely determined by its limiting shape ([How01], [Mus02]; also see Subsection 2.2.3).

The main question that we address is this:

Question 2.1.1. *Given a homogeneous ideal I , what can we say about the limiting shape of the generic initial system or symbolic generic initial system of I ?*

We address this question for two classes of ideals: complete intersections and ideals corresponding to sets of points in \mathbb{P}^2 .

2.1.1 Complete Intersections

Chapters III and IV study the generic initial systems of complete intersections.² The following result of Chapter III demonstrates just how complicated the ideals $\text{gin}(I^m)$ of generic initial systems can be even when I is very simple.

Theorem 3.4.1. *Fix positive integers α , β , and m such that $\beta \geq \alpha$ and $m \geq 2$. If I is a complete intersection in $K[x_1, \dots, x_n]$ with minimal generators of degrees α and β then the reverse lexicographic generic initial ideal of I^m is*

$$\text{gin}(I^m) = (x_1^k, x_1^{k-1}x_2^{\lambda_{k-1}}, \dots, x_1x_2^{\lambda_1}, x_2^{\lambda_0})$$

where $k = m\alpha$ and $\{\lambda_i\}$ is the sequence of natural numbers arising from one of the algorithms given in Chapter III. The choice of algorithm depends on the relative sizes of α , β , and m .

Finding generators of $\text{gin}(I^m)$ when I is a 3-complete intersection is a very difficult and well-studied problem, even when $m = 1$. For example, results of Cimpoeaş ([Cim06]) and of Cho and Park ([CP08]) imply that demonstrating $\text{gin}(I)$ always depends only on the degrees of minimal generators of I is equivalent to showing

²When I is a complete intersection, $I^{(m)} = I^m$ for all m , so the generic initial system is equivalent to the symbolic generic initial system in this case.

that all 3-complete intersections are strongly Lefschetz; the latter question has been well-studied but remains open (see [MN11]).

The complexity of the individual ideals in generic initial system of a 2-complete intersection motivates the asymptotic approach of Chapter IV. The following theorem from the chapter answers Question 2.1.1 for any complete intersection I by completely characterizing the limiting shape of the generic initial system of I .

Theorem 4.1.1. *Let I be a complete intersection of type (d_1, \dots, d_r) in $K[x_1, \dots, x_n]$ where $d_1 \leq \dots \leq d_r$ and K is a field of characteristic 0.³ Then the limiting shape P of the reverse lexicographic generic initial system is defined by a single hyperplane. In particular, P consists of the points $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_{\geq 0}^n$ such that*

$$1 \leq \frac{\lambda_1}{d_1} + \dots + \frac{\lambda_r}{d_r}.$$

This result demonstrates that generic initial systems follow the philosophy guiding the study of asymptotics: the limiting behavior of generic initial systems of complete intersections is as nice as one could hope despite the fact that the individual ideals of the system are complicated. For example, the limiting shape depends only on the type of the complete intersection and not on the particular generators chosen; by contrast, it is not even known whether the reverse lexicographic generic initial ideals of 3-complete intersections depend on the type.

2.1.2 Ideals of Points

In Chapters V, VI, and VII we look at the symbolic generic initial systems of a second class of ideals: ideals corresponding to point arrangements in \mathbb{P}^2 . The motivation for understanding the asymptotics in this case is two-fold.

First, we would like to know the answer to Question 2.1.1 for an ideal of points in \mathbb{P}^2 is related to the geometry of the arrangement of the points. From the work of Ellia, Gruson, Peskine, and others, we know that $\text{gin}(I)$, when I is such an ideal, can reflect information about the arrangement, such as how many of the points lie on a curve of a certain degree ([EP90]; also see [Gre98]). Thus, it is reasonable to expect that the asymptotics of the generic initial system of I might also encode information about the point arrangement.

A second motivation for pursuing Question 2.1.1 for ideals of points comes from the study of fat points. Given a set of points of \mathbb{P}^2 with ideal $I \subseteq K[x, y, z]$, the ideals

³A complete intersection of type (d_1, \dots, d_r) is a complete intersection with minimal generators of degrees d_1, \dots, d_r .

$I^{(m)}$ are *uniform fat point ideals* (see Section 1.2.5 of Chapter I). Such fat point ideals have proven difficult to understand; there are still many open problems and unresolved conjectures related to finding the Hilbert function of $I^{(m)}$ and even the degree $\alpha(I^{(m)})$ of the smallest degree element of $I^{(m)}$ (for example, see [CHT11], [GH07], [GVT04], [GHM09], and [Har02]). It turns out that knowing the Hilbert function of $I^{(m)}$ is equivalent to knowing the minimal monomial generators of $\text{gin}(I^{(m)})$; thus, describing the limiting shape of $\{\text{gin}(I^{(m)})\}_m$ is equivalent to describing the Hilbert functions of the fat point ideals $I^{(m)}$ as m gets large. The answer to Question 2.1.1 may then be seen as a description of the asymptotics of a collection of fat point ideals; this injects the study of fat points into the asymptotic research trend.

If I is the ideal of a collection of points of \mathbb{P}^2 , the ideals $\text{gin}(I^{(m)})$ are generated in two variables. Thus, the limiting shape of $\{\text{gin}(I^{(m)})\}_m$ can be thought of as a convex region of $\mathbb{R}_{\geq 0}^2$. Chapters V, VI, and VII describe these regions for various arrangements of points.

The main result of Chapter V describes the limiting shape when the points of \mathbb{P}^2 are in general position, assuming that the *SHGH Conjecture* from the study of fat points holds when there are at least nine points (see Conjecture 5.3.1 or [Har02]).

Theorem 5.1.1. *Let $I \subseteq R = K[x, y, z]$ be the ideal of $r > 1$ distinct points p_1, \dots, p_r of \mathbb{P}^2 in general position and P be the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Then P can be characterized as follows.*

(a) *If $r \geq 9$ and the SHGH Conjecture holds for infinitely many m , then P has a boundary defined by the line through the points $(\sqrt{r}, 0)$ and $(0, \sqrt{r})$. See Figure 5.1.*

(b) *If $6 \leq r < 9$, then P has a boundary defined by the line through the points $(\gamma_1, 0)$ and $(0, \gamma_2)$ where:*

(i) $\gamma_1 = \frac{12}{5}$ and $\gamma_2 = \frac{5}{2}$ when $r = 6$;

(ii) $\gamma_1 = \frac{21}{8}$ and $\gamma_2 = \frac{8}{3}$ when $r = 7$; and

(iii) $\gamma_1 = \frac{48}{17}$ and $\gamma_2 = \frac{17}{6}$ when $r = 8$.

(c) *If $r = 4$ or $r = 5$, then P has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$. If $r = 2$ or $r = 3$, then P has a boundary defined by the line through the points $(\frac{r}{2}, 0)$ and $(0, 2)$.*

In Chapter VI we describe the limiting shape when the points of \mathbb{P}^2 lie on an irreducible conic.

Theorem 6.0.2. *Let $I \subseteq R = K[x, y, z]$ be the ideal of $r > 1$ distinct points p_1, \dots, p_r of \mathbb{P}^2 lying on an irreducible conic and let P be the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. If $r \geq 4$, then P has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$. If $r = 2$ or $r = 3$, then P has a boundary defined by the line through the points $(\frac{r}{2}, 0)$ and $(0, 2)$.*

In Chapter VII we describe the limiting shape for all arrangements of six points and make observations about how the geometry of a point arrangement is connected to features of the limiting shape.

Theorem 7.1.1. *Let $I \subseteq K[x, y, z]$ be the ideal corresponding to a set of six points in \mathbb{P}^2 . Then the limiting shape P of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$ is equal to the limiting shape P shown in Figures 7.1 and 7.2 corresponding to the configuration type of the six points.*

Finally, in Chapter VIII we outline some open questions related to generic initial systems and symbolic generic initial systems.

2.2 Background for Proofs

In Chapter I we reviewed background information necessary to understand the statements of our main results; the purpose of this section is to collect information for understanding the proofs of these results for readers who specialize in algebra. Since this thesis is a collection of manuscripts, each with a preliminary section of their own, we only includes background information that will be used frequently.

Throughout, K is a field of characteristic 0 and $R = K[x_1, \dots, x_n]$ is a polynomial ring with the standard grading and the reverse lexicographic with $x_1 > x_2 > \dots > x_n$.

2.2.1 Resolutions of Generic Initial Ideals

The main goal of this subsection is to demonstrate the following.

- The Betti numbers of Borel-fixed ideals are easy to find (Theorem 2.2.2).
- We can obtain information about the Betti numbers of an arbitrary homogeneous ideal from the Betti numbers of its generic initial ideal (Corollary 2.2.5).

Since generic initial ideals are Borel-fixed ([Gal74], Theorem 1.4.2 of Chapter I), this means that if we can compute $\text{gin}(I)$, we can obtain information about the Betti

numbers of I . This is a theme that is repeated throughout the study of generic initial ideals and is where much of their power comes from.

Recall that the *Betti numbers* of a graded ideal $I \subseteq R$ describe the free modules that appear in the the minimal free resolution of I . In particular, suppose that a free R -module of rank 1 whose generator has degree p is denoted by $R(-p)$ and that

$$\mathbf{F} : \cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots$$

is the minimal free resolution of I . Then each F_i can be written uniquely as

$$F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{\beta_{i,p}(I)}$$

since it is a free R -module; the Betti numbers of I are exactly these $\beta_{i,p}(I)$. Since $\beta_{i,j}(I) = 0$ whenever $j < i$, we often write the indices in this sequence as $\beta_{i,i+j}(I)$ (see Proposition I.12.3 of [Pee10]).

Recall from Chapter 0 that a monomial ideal $J \subseteq R$ is *Borel-fixed* if it is fixed by the action of the Borel subgroup of upper triangular matrices in $\mathrm{GL}_n(K)$. In characteristic 0, this is equivalent to J being *strongly stable*: for every monomial $m \in J$ and every variable x_i dividing m , $\frac{mx_j}{x_i} \in J$ for all $j \leq i$.

The following notation will be useful when describing properties of Borel-fixed ideals.

Definition 2.2.1. Let m be a monomial of $R = K[x_1, x_2, \dots, x_n]$. Then

$$\max(m) := \max\{i : x_i \text{ divides } m\}$$

and

$$\min(m) := \min\{i : x_i \text{ divides } m\}.$$

In 1990, Eliahou and Kervaire demonstrated that the minimal free resolution of any Borel-fixed ideal has a very nice structure ([EK90]). The following theorem describes the graded Betti numbers in such resolutions; for a nice descriptions of the maps needed to complete the resolution, see Section I.28 of [Pee10].⁴

Theorem 2.2.2 ([EK90]). *Let J be a Borel-fixed ideal of $R = K[x_1, \dots, x_n]$ with minimal monomial generators m_1, \dots, m_r . To describe the minimal free resolution of*

⁴When describing its maps, the Eliahou-Kervaire resolution is best described as a multi-graded resolution; see Section I.28 of [Pee10].

J we will consider positive integer sequences of the form

$$j_1, j_2, \dots, j_p \text{ where } 1 \leq j_1 < j_2 < \dots < j_p < \max(m). \quad (2.1)$$

Then the minimal free resolution of J is of the form

$$\mathbf{F} : 0 \rightarrow F_N \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow J \rightarrow 0$$

where:

- $F_0 = R$;
- $F_1 = \bigoplus R(-\deg(m_i))$;
- $F_2 = \bigoplus R(-\deg(m_i \cdot x_{j_1}))$;
- $F_3 = \bigoplus R(-\deg(m_i \cdot x_{j_1} \cdot x_{j_2}))$;
- in general, $F_{p+1} = \bigoplus R(-\deg(m_i \cdot x_{j_1} \cdot x_{j_2} \cdots x_{j_p}))$; and
- $N = \max\{\max(m_i)\}$.

Each of the sums above are over all generators m_i and sequences j_1, j_2, \dots, j_p such that $1 \leq j_1 < j_2 < \dots < j_p < \max(m_i)$.

Example 2.2.1. Let $I = (x_1, x_2^2, x_3^2)$. Then, using Macaulay 2 [GS],

$$\text{gin}(I) = (x_1, x_2^2, x_2x_3, x_3^3).^5$$

Since $\text{gin}(I)$ is Borel-fixed, its minimal free resolution is given by Theorem 2.2.2. The sequences of form (2.1) where m is a generator of $\text{gin}(I)$ are:

$$\begin{array}{ll} 1 \leq 1 < \max(x_2^2) & 1 \leq 2 < \max(x_3^3) \\ 1 \leq 1 < \max(x_2x_3) & 1 \leq 1 < 2 < \max(x_2x_3) \\ 1 \leq 2 < \max(x_2x_3) & 1 \leq 1 < 2 < \max(x_3^3) \\ 1 \leq 1 < \max(x_3^3). & \end{array}$$

⁵Since computer algebra systems use probabilistic methods to compute generic initial ideals, we can only be $(100 - \epsilon)\%$ certain of this computation. See the Theorem 3.4.1 of Chapter III for a proof that this is the correct answer.

Thus, the minimal free resolution is of the form

$$\mathbf{F} : 0 \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow \text{gin}(I) \rightarrow 0$$

where:

- $F_1 = R(-\deg(x_1)) \oplus R(-\deg(x_2^2)) \oplus R(-\deg(x_2x_3)) \oplus R(-\deg(x_3^3)) = R(-1) \oplus R^2(-2) \oplus R(-3)$;
- $F_2 = R(-\deg(x_2^2 \cdot x_1)) \oplus R(-\deg(x_2x_3 \cdot x_1)) \oplus R(-\deg(x_2x_3 \cdot x_2)) \oplus R(-\deg(x_3^3 \cdot x_1)) \oplus R(-\deg(x_3^3 \cdot x_2)) = R^3(-3) \oplus R^2(-4)$; and
- $F_3 = R(-\deg(x_2x_3 \cdot x_1 \cdot x_2)) \oplus R(-\deg(x_3^3 \cdot x_1 \cdot x_2)) = R(-4) \oplus R(-5)$.

The degeneration of an ideal I to its initial ideal $\text{in}(I)$ gives a relationship between the Betti numbers of I and $\text{in}(I)$. To describe this relationship, we will use the notion of *consecutive cancellations*.

Definition 2.2.3 ([Pee04]). A sequence $\{q_{i,j}\}$ is obtained from $\{p_{i,j}\}$ by a **consecutive cancellation** if there is a choice of s and r such that:

- $q_{s,r} = p_{s,r} - 1$;
- $q_{s+1,r} = p_{s+1,r} - 1$; and
- $q_{i,j} = p_{i,j}$ for all other i, j .

The proof of the following theorem uses a deformation from Gröbner basis theory; see Section I.22 of [Pee10].

Theorem 2.2.4 ([Pee04]). *Let I be a graded ideal. The sequence of graded Betti numbers of I can be obtained from the sequence of graded Betti numbers of $\text{in}(I)$ by consecutive cancellations.*

For a general change of coordinates g , the ideals I and gI have the same minimal free resolution. This fact gives the following corollary of the theorem.

Corollary 2.2.5. *Let I be a homogeneous ideal. Then the sequence of graded Betti numbers of I can be obtained from the sequence of graded Betti numbers of $\text{gin}(I)$ by consecutive cancellations.*

Example 2.2.2. Let $I = (x^4, y^5)$ be an ideal of $R = K[x, y]$. Then using Macaulay 2 ([GS]),

$$\text{gin}(I^2) = (x^8, x^7y^2, x^6y^4, x^5y^5, x^4y^7, x^3y^8, x^2y^{10}, xy^{11}, y^{13}).$$

Again using Macaulay 2 or Theorem 2.2.2, the minimal free resolution of $\text{gin}(I^2)$ is of the form

$$\begin{aligned} 0 &\rightarrow R(-10) \oplus R^2(-11) \oplus R^2(-12) \oplus R^2(-13) \oplus R(-14) \\ &\rightarrow R(-8) \oplus R(-9) \oplus R^2(-10) \oplus R^2(-11) \oplus R^2(-12) \oplus R(-13) \\ &\rightarrow R \rightarrow \text{gin}(I^2) \rightarrow 0. \end{aligned}$$

Further, the minimal free resolution of I^2 is of the form

$$0 \rightarrow R(-13) \oplus R(-14) \rightarrow R(-8) \oplus R(-9) \oplus R(-10) \rightarrow I^2 \rightarrow R \rightarrow 0.$$

Each consecutive cancellation in the sequence of Betti numbers of $\text{gin}(I^2)$ may be thought of as a cancellation of summands in two consecutive homological degrees with equal shifts. For example, to get the Betti numbers of I^2 from that of $\text{gin}(I^2)$, one $R(-10)$ summand, two $R(-11)$ summands, two $R(-12)$ summands, and one $R(-13)$ summand is cancelled from F_1 and F_2 . While all possible cancellations were made in this example, this is *not* always the case.

The degeneration of an ideal to its initial ideal, and thus to its generic initial ideal, does not change its Hilbert function. Macaulay used this result to classify all possible Hilbert functions.

Proposition 2.2.6 ([Mac27]). *Let I be a homogeneous ideal of $K[x_1, \dots, x_n]$ and suppose that $H_I(t) = \dim_K(I_t)$ is the Hilbert function of I . Then*

$$H_I(t) = H_{\text{gin}(I)}(t)$$

for all t .

Proof. The Hilbert function is invariant under invertible coordinate changes, so it suffices to show $H_I(t) = H_{\text{in}(I)}(t)$. Recall that a *standard basis* of I_d is a set of elements $\{f_1, \dots, f_l\} \subseteq I_d$ such that $\text{in}(f_1), \dots, \text{in}(f_l)$ is a basis for $\text{in}(I)_d$. To prove the theorem, we need to show that $\dim(I_d) = \dim(\text{in}(I)_d)$ for all d ; thus, it is sufficient to show that a standard basis f_1, \dots, f_l of I_d is actually a basis of I_d .

Let $C = \sum_{i=1}^l c_i f_i$ be a linear combination of the f_i . The initial terms $\text{in}(f_1), \dots,$

$\text{in}(f_i)$ are all distinct, so the highest $\text{in}(f_i)$ in C with $c_i \neq 0$ cannot cancel with another monomial in the sum. Thus, C is nonzero.

Suppose that the f_i do not span I_d . Let f be the element of I_d not in the span of the f_i that has the smallest initial monomial $\text{in}(f) = m$. If c is the coefficient of m in f then $f - cm$ is an element of I_d that is not in the span of the f_i but whose initial monomial is smaller than m . This contradicts our assumption. Thus, the f_i must span I_d and a standard basis of I_d is actually a basis. \square

2.2.2 Invariants of Generic Initial Ideals

We will see in this section that some invariants of an ideal may be easily seen from its generic initial ideal. The idea for each of these invariants is similar to what we saw for Hilbert functions and resolutions in the previous section.

1. If J is a Borel-fixed ideal, the invariant can be easily read off from the generators of J . This will give us the invariant of $\text{gin}(I)$ from the generators of $\text{gin}(I)$.
2. The invariants of I and $\text{gin}(I)$ are equal, so we can use the result of (1) to find the invariant of I .⁶

The first invariants that we will consider are dimension and depth.

Definition 2.2.7. Let I be an ideal of $R = K[x_1, \dots, x_n]$ and set $M = R/I$. The **depth** of R/I is the maximal length of a sequence of regular elements on M ; that is, it is the maximal length of a sequence r_1, \dots, r_l in R such that r_{i+1} is regular on $M/(r_1, \dots, r_i)M$ for all $i = 0, \dots, l-1$.

For a set of monomials S , denote

$$D(S) := \max\{\min(m) : m \in S\}$$

and

$$M(S) := \max\{\max(m) : m \in S\}$$

where $\min(m)$ and $\max(m)$ are as in Definition 2.2.1. If S is a set of minimal monomial generators of a monomial ideal J then $D(J) := D(S)$ and $M(J) := M(S)$.

The following lemma says that the dimension and depth are easy to read off from a Borel-fixed ideal.

⁶Recall that we have fixed the reverse lexicographic order: we will not always get this equality when using other monomial orders.

Lemma 2.2.8 ([AM07]). *Let J be a Borel-fixed monomial ideal of $R = K[x_1, \dots, x_n]$ with $n \geq 2$ and $\dim(R/J) > 0$. Then*

1. $\operatorname{codim}(R/J) = n - \dim(R/J) = D(J)$
2. $\operatorname{codepth}(R/J) = n - \operatorname{depth}(R/J) = M(J)$.

Sketch of Proof. For the first statement note that, by definition of $D(J)$, there is a minimal monomial generator m of J such that $x_{D(J)}$ divides m and is the smallest indexed variable dividing m . Since J is strongly stable, all higher-indexed variables in m can be replaced by $x_{D(J)}$ to get another element of J ; that is, some power of $x_{D(J)}$ is contained in J . This means that $x_{D(J)} \in \sqrt{J}$ so

$$\sqrt{J} = (x_1, x_2, \dots, x_{D(J)}).$$

Thus,

$$\dim(R/J) = \dim(R/\sqrt{J}) = n - D(J).$$

For a proof of the second statement, see Lemma 2.3 of [AM07]. □

Lemma 2.2.9. *Let I be a homogeneous ideal of $R = K[x_1, \dots, x_n]$ with the reverse lexicographic order. Then*

$$\dim(R/I) = \dim(R/\operatorname{gin}(I))$$

and

$$\operatorname{depth}(R/I) = \operatorname{depth}(R/\operatorname{gin}(I)).$$

Proof. The statement about dimension follows from the equality of the Hilbert functions in Proposition 2.2.6. For a proof of the depth relation, see Corollary 2.8 of [AM07]. □

Similar statements hold for the regularity and satiety of an ideal and its generic initial ideal.

Definition 2.2.10. Let I be an ideal of $R = K[x_1, \dots, x_n]$ with Betti numbers $\beta_{i,i+j}(I)$ and set $\mathfrak{m} = (x_1, \dots, x_n)$. The **regularity** of I is

$$\operatorname{reg}(I) = \max\{j : \beta_{i,i+j}(I) \neq 0 \text{ for some } i\}.$$

The **saturation** I^{sat} of I is

$$I^{\operatorname{sat}} = \bigcup_{k \geq 0} (I : \mathfrak{m}^k)$$

while the **satiety** of I is the smallest m for which $I_d = I_d^{\text{sat}}$ for all $d \geq m$.

The following results are due to Bayer and Stillman ([BS87a]). See Section 2 of [Gre98] for accessible proofs.

Theorem 2.2.11 ([BS87a]). *Let J be a Borel-fixed ideal in $R = K[x_1, \dots, x_n]$ and fix the reverse lexicographic order.*

1. $J^{\text{sat}} = \bigcup_{k=0}^{\infty} (J : x_n^k)$.
2. $\text{sat}(J) =$ the maximal degree of a generator of J involving the variable x_n .
3. $\text{reg}(J) =$ the maximal degree of a generator of J .

Lemma 2.2.12 ([BS87a]). *For any homogeneous ideal $I \subseteq R = K[x_1, \dots, x_n]$ with the reverse lexicographic order,*

$$\text{sat}(I) = \text{sat}(\text{gin}(I))$$

and

$$\text{reg}(I) = \text{reg}(\text{gin}(I)).$$

Example 2.2.3. Let $I = (x_1x_2 + x_3x_4, x_1x_2^2 + x_3^3, x_1x_4^3)$ be an ideal of $R = k[x_1, x_2, x_3, x_4]$ with the reverse lexicographic order. Then reverse lexicographic generic initial ideal of I is

$$\text{gin}(I) = (x_1^2, x_1x_2^2, x_2^4, x_2^3x_3, x_1x_2x_3^3, x_2^2x_3^3, x_1x_3^5, x_1x_3^4x_4).$$

We have the following.

- $\text{sat}(I) = \text{sat}(\text{gin}(I)) = 6$ since the highest degree generator of $\text{gin}(I)$ involving the variable x_4 , $x_1x_3^4x_4$, is of degree 6.
- $\text{reg}(I) = \text{reg}(\text{gin}(I)) = 6$ since the highest degree generator of $\text{gin}(I)$, $x_1x_3^4x_4$, is of degree 6.
- $\dim(R/I) = \dim(R/\text{gin}(I)) = 4 - 2 = 2$ since every generator involves an x_1 or x_2 .
- $\text{depth}(R/I) = \text{depth}(R/\text{gin}(I)) = 4 - 4 = 0$ since there is a generator of $\text{gin}(I)$ involving the variable x_4 .

2.2.3 The Limiting Shape: Volume and an Application

Recall that the *limiting shape* P of a graded system of monomial ideals $\{\mathfrak{a}_m\}_m$ in $K[x_1, \dots, x_n]$ is the limit

$$P = \overline{\bigcup_{q \in \mathbb{N}^*} P_{\mathfrak{a}_q}}$$

where P_J denotes the Newton polytope of J in \mathbb{R}^n .⁷

Since P is an infinite subset of the first quadrant $\mathbb{R}_{\geq 0}^n$ of \mathbb{R}^n , the complement of P in $\mathbb{R}_{\geq 0}^n$ is the region ‘under’ P . To make this notion precise, for a graded sequence of *zero-dimensional* monomial ideals $\{\mathfrak{a}_m\}_m$, let $Q_{\mathfrak{a}_m}$ be the closure of $\mathbb{R}_{\geq 0}^n \setminus P_{\mathfrak{a}_m}$ in \mathbb{R}^n . Then we may define the bounded region Q to be

$$Q := \bigcap_{q \in \mathbb{N}^*} \frac{1}{m} Q_{\mathfrak{a}_m};$$

this is essentially the complement of P in $\mathbb{R}_{\geq 0}^n$.

First, we will show how the geometric volume of the region Q is can be interpreted in terms of the algebraic volume of $\{\mathfrak{a}_m\}_m$.

Definition 2.2.13 ([ELS03]). Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_m$ be a graded system of zero-dimensional monomial ideals in $K[x_1, \dots, x_n]$. Then the volume of this system is defined by

$$\text{vol}(\mathfrak{a}_\bullet) := \limsup \frac{n! \cdot \text{length}(R/\mathfrak{a}_m)}{m^n}.$$

The following theorem of Mustařă relates this algebraic volume to the geometric volume of Q defined above.

Theorem 2.2.14 ([Mus02]). *Let \mathfrak{a}_\bullet be a graded system of zero-dimensional monomial ideals in $K[x_1, \dots, x_n]$. If Q is as defined above,*

$$\text{vol}(Q) = \text{vol}(\mathfrak{a}_\bullet).$$

The following well-known proposition is key to proving this theorem.

Proposition 2.2.15. *Let $J = (x^{v_1}, x^{v_2}, \dots, x^{v_r})$ be a zero-dimensional monomial ideal of $R = K[x_1, \dots, x_n]$, let P_J be the Newton polytope of J , and let Q_J denote the complement of P_J in $\mathbb{R}_{\geq 0}^n$. Then*

$$n! \text{vol}(Q) = e(J)$$

⁷A *graded system* is a collection of ideals such that the product of the i th and j th ideals is contained in the $(i + j)$ th ideal for all i, j ([ELS01]; also see Definition 1.5.1 of Chapter I).

where

$$e(J) := \lim_{m \rightarrow \infty} \frac{\lambda(R/J^m)}{m^n} n!$$

is the multiplicity of J .

Proof. The key step in this proof is to relate the length $\lambda(R/\overline{J^m})$ to the polytope P_J . Recall that for any monomial ideal I , the monomials in \overline{I} correspond to lattice points of \mathbb{N}^n lying inside of the Newton polytope P_I . Also, note that

$$\overline{J^m} = \overline{(x^{v_1 m}, x^{v_2 m}, \dots, x^{v_r m})}.$$

Thus,

$$\begin{aligned} & \lambda(R/\overline{J^m}) \\ &= \text{number of points in lattice } \mathbb{N}^n \text{ outside } P_{(x^{v_1 m}, \dots, x^{v_r m})} \\ &= \text{number of points in lattice } \mathbb{N}^n \text{ outside } mP_{(x^{v_1}, \dots, x^{v_r})} = mP_J \\ &= \text{number of points in lattice } \frac{1}{m}\mathbb{N}^n \text{ outside } P_J \\ &= \text{number of points in lattice } \frac{1}{m}\mathbb{N}^n \text{ inside } Q_J. \end{aligned}$$

When scaled by $\frac{1}{m^n}$, the last line is a Riemann sum that estimates $\text{vol}(Q)$ as m approaches infinity. Thus,

$$\lim_{m \rightarrow \infty} \frac{\lambda(R/\overline{J^m})}{m^n} = \text{vol}(Q)$$

so

$$\begin{aligned} n! \text{vol}(Q) &= n! \lim_{m \rightarrow \infty} \frac{\lambda(R/\overline{J^m})}{m^n} \\ &= n! \lim_{m \rightarrow \infty} \frac{\lambda(R/J^m)}{m^n} \\ &= e(J) \end{aligned}$$

□

Sketch of Proof of Theorem 2.2.14. By Lemma 2.13 of [Mus02], for every open neigh-

neighbourhood U of Q there is some m such that $\frac{1}{m}Q_{a_m} \subseteq U$. Thus,

$$\begin{aligned} \text{vol}(Q) &= \lim_{m \rightarrow \infty} \text{vol}\left(\frac{1}{m}Q_m\right) \\ &= \lim_{m \rightarrow \infty} \frac{\text{vol}(Q_{a_m})}{m^n} \\ &= \lim_{m \rightarrow \infty} \frac{e(a_m)}{n!} \end{aligned}$$

where the second line is a result of scaling each coordinate by a factor of $\frac{1}{m}$ and the third line follows from applying Proposition 2.2.15 to each monomial ideal a_m . By Theorem 1.7 of [Mus02], this limit is equal to $\text{vol}(a_\bullet)/n!$. Thus,

$$n!\text{vol}(Q) = \text{vol}(a_\bullet).$$

□

As we mentioned earlier, the notion of the limiting shape comes from algebraic geometry: it allows us to determine the asymptotic multiplier ideals of a graded system by Theorem 2.2.18 below. The survey article [BL04] provides an introduction to the theory of multiplier ideals for the reader who has not encountered these before. Roughly, for every ideal $a \subseteq K[x_1, \dots, x_n]$ and for each rational ‘weighting’ $c > 0$, there is a *multiplier ideal* $\mathcal{J}(a^c)$. If we have a graded system of ideals a_\bullet , we can attach multiplier ideals to each ideal in the system. To define asymptotic multiplier ideals, we consider multiplier ideals of the form $\mathcal{J}(a_p^{c/p})$.

Lemma 2.2.16 (Lemma 6.5 of [BL04]). *Let a_\bullet be a graded system of ideals in $K[x_1, \dots, x_n]$ and fix some rational $c > 0$. Then for $p \gg 0$, the multiplier ideals $\mathcal{J}(a_p^{c/p})$ coincide.*

Definition 2.2.17 ([ELS01]). Let a_\bullet be a graded system of ideals in $K[x_1, \dots, x_n]$. The **asymptotic multiplier ideal** of a_\bullet with exponent c is the ideal

$$\mathcal{J}(a_\bullet^c) := \mathcal{J}(a_p^{c/p})$$

for $p \gg 0$.

In 2001, Howald gave a simple way to compute the multiplier ideals of a monomial ideal ([How01]). Mustaŝă applied Howald’s result to the asymptotic setting soon after.

Theorem 2.2.18 ([Mus02]). *Let \mathfrak{a}_\bullet be a graded system of monomial ideals in $K[x_1, \dots, x_n]$, and let P be the limiting shape of P . Then $\mathcal{J}(c\mathfrak{a}_\bullet)$ is a monomial ideal. It contains the monomial x^v if and only if $v + (1, \dots, 1)$ is contained in the interior of $c \cdot P$.*

In Chapter IV we apply this theorem to the main result to obtain a formula asymptotic multiplier ideals of the generic initial system of a complete intersection. The theorem may be applied in the same way to find asymptotic multiplier ideals of the symbolic generic initial systems in Chapters V through VII.

CHAPTER III

The Generic Initial Ideals of Powers of a 2-Complete Intersection

3.1 Introduction

Consider the collection of ideals $\{\text{gin}(I^n)\}_n$ obtained by taking the generic initial ideals of powers of a fixed ideal I in a polynomial ring. Our study of such families of monomial ideals was initially motivated by the desire to understand their asymptotic behavior (see [May12d]). It soon became clear, however, that the individual ideals within such families are interesting in their own right. In this paper we compute the generators of the ideals $\text{gin}(I^n)$ with respect to the reverse lexicographic order where I is a 2-complete intersection and, in doing so, demonstrate relationships between such ideals.

Computing generic initial ideals is generally challenging because they are defined by an existence theorem rather than an explicit construction (see Galligo's Theorem, Theorem 3.2.1). As a result, there are few classes of ideals for which generic initial ideals have been explicitly computed (see [Gre98] for a survey, or [Cim06], [ACP07], [CP08], and [CR10] for more recent results).

The 2-complete intersections are amongst the ideals whose reverse lexicographic generic initial ideals are completely understood. In particular, if $I \subset K[x_1, \dots, x_m]$ is generated by a regular sequence of homogeneous polynomials of degrees α and β , with $\alpha \leq \beta$, then

$$\text{gin}(I) = (x_1^\alpha, x_1^{\alpha-1}x_2^{\lambda_0-2(\alpha-1)}, x_1^{\alpha-2}x_2^{\lambda_0-2(\alpha-2)}, \dots, x_1x_2^{\lambda_0-2}, x_2^{\lambda_0})$$

where $\lambda_0 = \beta + \alpha - 1$ (see Section 4 of [Gre98]). The generic initial ideals for larger complete intersections, however, have proven difficult to compute. For example,

Cimpoeaş [Cim06] has exhibited the minimal generators for the generic initial ideals of strongly Lefschetz 3-complete intersections; the structure of such generic initial ideals is relatively difficult to describe and depends on the relative degrees of the generators of the complete intersection.

In this paper we explicitly compute the generators of the reverse lexicographic generic initial ideals of powers of 2-complete intersections. In particular, we prove the following result.

Theorem 3.4.1. *Fix positive integers α , β , and n such that $\beta \geq \alpha$ and $n \geq 2$. If I is a type (α, β) complete intersection in $K[x_1, \dots, x_m]$, where K is a field of characteristic 0, then the reverse lexicographic generic initial ideal of I^n is*

$$\text{gin}(I^n) = (x_1^k, x_1^{k-1}x_2^{\lambda_{k-1}}, \dots, x_1x_2^{\lambda_1}, x_2^{\lambda_0})$$

where $k = n\alpha$ and $\{\lambda_i\}$ is the sequence of natural numbers arising from:

- Algorithm 1 if $\beta \geq 2\alpha - 1$;
- Algorithm 2 if $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha$;
- Algorithm 3 if $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \mid \alpha$, and $n \geq \frac{\alpha}{\beta - \alpha} + 1$;
- Algorithm 4 if $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \nmid \alpha$, and $n \geq \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$;
- Algorithm 5 if $\frac{3}{2}\alpha > \beta > \alpha$ and $2 \leq n < \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$; and
- Algorithm 6 if $\alpha = \beta$.

The algorithms referred to in this theorem are stated in Section 3.4. Although the particular choice of an algorithm in the theorem depends on n and on the relative sizes of α and β , all of the algorithms share common features. For example, they each compute the invariants λ_i one-by-one, starting with $\lambda_0 = n\beta + \alpha - 1$ and using the gaps $g_i = \lambda_{i-1} - \lambda_i$ to compute each successive invariant. The patterns amongst the invariants of the ideals $\text{gin}(I^n)$ are best seen by looking at the associated gap sequences of $\{g_i\}$, which consist entirely of the numbers 1, 2, and $\beta - 2\alpha + 2$.

This theorem adds powers of 2-complete intersections to the classes of ideals whose generic initial ideals can be explicitly computed. The complexity of this result even in this small case, however, gives further evidence that finding generators of the generic initial ideals of powers of larger complete intersections may be optimistic and provides motivation to instead study the asymptotic behavior of generic initial

systems $\{\text{gin}(I^n)\}_n$. One consequence of Theorem 3.4.1 is a different proof for the case of a 2-complete intersection of the main result of [May12d] on the asymptotic behavior of the generic initial system.

3.2 Preliminaries

In this section we will introduce some notation, definitions, and preliminary results related to generic initial ideals. Throughout, $R = K[x_1, \dots, x_m]$ is a polynomial ring over a field K of characteristic 0 with the standard grading and some fixed term order $>$ with $x_1 > x_2 > \dots > x_m$.

3.2.1 Generic Initial Ideals

An element $g = (g_{ij}) \in \text{GL}_m(K)$ acts on R and sends any homogeneous element $f(x_1, \dots, x_m)$ to the homogeneous element

$$f(g(x_1), \dots, g(x_m))$$

where $g(x_i) = \sum_{j=1}^m g_{ij}x_j$. If $g(I) = I$ for every upper triangular matrix g then we say that I is *Borel-fixed*. Borel-fixed ideals are *strongly stable* when K is of characteristic 0; that is, for every monomial f in the ideal such that x_i divides f , the monomials $\frac{x_j f}{x_i}$ for all $j < i$ are also in the ideal. This property makes such ideals particularly nice to work with.

To any homogeneous ideal I of R we can associate a Borel-fixed monomial ideal $\text{gin}_>(I)$ which can be thought of as a coordinate-independent version of the initial ideal.¹ Its existence is guaranteed by the following result known as Galligo's theorem (also see [Gre98, Theorem 1.27]).

Theorem 3.2.1 ([Gal74] and [BS87b]). *For any multiplicative monomial order $>$ on R and any homogeneous ideal $I \subset R$, there exists a Zariski open subset $U \subset \text{GL}_m$ such that $\text{In}_>(g(I))$ is constant and Borel-fixed for all $g \in U$.*

Definition 3.2.2. The **generic initial ideal** of I , denoted $\text{gin}_>(I)$, is defined to be $\text{In}_>(g(I))$ where $g \in U$ is as in Galligo's theorem.

The *reverse lexicographic order* $>$ is a total ordering on the monomials of R defined by:

¹For a polynomial $f = \sum a_i m_i$, $\text{in}_>(f)$ is the largest m_i with respect to $>$ such that a_i is nonzero. Further, for a polynomial ideal I , $\text{In}_>(I) = \{\text{in}(f) : f \in I\}$.

1. if $|I| = |J|$ then $x^I > x^J$ if there is a k such that $i_m = j_m$ for all $m > k$ and $i_k < j_k$; and
2. if $|I| > |J|$ then $x^I > x^J$.

For example, $x_1^2 x_3 < x_1 x_2^2$. From this point on, $\text{gin}(I) = \text{gin}_{>}(I)$ will denote the generic initial ideal with respect to the reverse lexicographic order.

3.2.2 The Hilbert Function and Notation

Recall that the Hilbert function $H_I(t)$ of a homogeneous ideal I is defined by $H_I(t) = \dim_K(I_t)$ where I_t denotes the t^{th} graded piece of I . The following theorem records two of the properties shared by $\text{gin}(I)$ and I . The first statement is a consequence of the fact that Hilbert functions are invariant under making changes of coordinates and taking initial ideals. The second statement is a result of Bayer and Stillman [BS87a]; for a simple proof see Corollary 2.8 of [AM07].

Theorem 3.2.3. *For any homogeneous ideal I in R :*

1. *the Hilbert functions of I and $\text{gin}(I)$ are equal; and*
2. *under the reverse lexicographic order, $\text{depth}(R/I) = \text{depth}(R/\text{gin}(I))$.*

Throughout this paper, $\binom{s}{t} = 0$ whenever $s \leq 0$ or $t > s$ so that $\binom{s}{t}$ is always non-negative. Under this assumption, the summation and recursive formulas for binomial coefficients hold:

$$\sum_{z=0}^L \binom{z}{p} = \binom{L+1}{p+1} \quad (3.1)$$

$$\binom{z}{p} = \binom{z-1}{p-1} + \binom{z-1}{p} \quad (3.2)$$

As a consequence of Equation 3.1 we have the following identities when $L_1, L_2 > 0$:

$$\sum_{j=L_1}^{L_2} \binom{z+j}{p} = \binom{z+L_2+1}{p+1} - \binom{z+L_1}{p+1} \quad (3.3)$$

$$\sum_{j=L_1}^{L_2} \binom{z-j}{p} = \binom{z-L_1+1}{p+1} - \binom{z-L_2}{p+1} \quad (3.4)$$

These are the only binomial coefficient identities that will be used in our later calculations.

Finally, since most of our work will only involve the first two variables x_1 and x_2 of $K[x_1, \dots, x_m]$, we will set $x_1 = x$ and $x_2 = y$.

3.3 Structure of Ideals in the Generic Initial System

A homogeneous ideal $I = (f_\alpha, f_\beta)$ is a *complete intersection of type* (α, β) if f_α, f_β is a regular sequence on R , $\deg(f_\alpha) = \alpha$, and $\deg(f_\beta) = \beta$. Since f_α and f_β are homogeneous, f_β, f_α is also a regular sequence; therefore, we may assume that $\alpha \leq \beta$. Throughout this section we assume that I is such a complete intersection.

3.3.1 Structure of $\text{gin}(I^n)$

The goal of this subsection is to describe the general structure of the reverse lexicographic generic initial ideals $\text{gin}(I^n)$ for a complete intersection I of type (α, β) . In particular, we will prove the following theorem.

Theorem 3.3.1. *Let I be a complete intersection of type (α, β) in $R = K[x_1, \dots, x_m]$ generated by the homogeneous polynomials f_α and f_β , and suppose that A_n is the set of minimal monomial generators of $\text{gin}(I^n)$. Then, setting $x = x_1$ and $y = x_2$,*

$$A_n = \{x^k, x^{k-1}y^{\lambda_{k-1}}, x^{k-2}y^{\lambda_{k-2}}, \dots, xy^{\lambda_1}, y^{\lambda_0}\}$$

where

- (i) $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{k-2} \geq \lambda_{k-1}$;
- (ii) $k = n\alpha$;
- (iii) $\lambda_0 = n\beta + \alpha - 1$; and
- (iv) $\lambda_{k-1} = \beta - \alpha + 1$.

We will refer to the λ_i as the *invariants* of $\text{gin}(I^n)$. This theorem will be proven in several parts. First, no matter how many variables the ambient ring R has, the minimal generators of these generic initial ideals will only involve the variables x_1 and x_2 .

Lemma 3.3.2. *Let I be a type (α, β) complete intersection in R and let A_n denote the set of minimal monomial generators of $\text{gin}(I^n)$. Then the elements of A_n are contained in $K[x_1, x_2]$. Furthermore, A_n contains a power of x_2 , say $x_2^{\lambda_0}$, and no element of A_n is of degree greater than λ_0 .*

This lemma is a consequence of the following result of Herzog and Srinivasan (see Lemma 3.1 of [HS98]) which relates the depth and dimension of a Borel-fixed monomial ideal to the variables appearing in its minimal generating set.

Proposition 3.3.3. *Let J be a Borel-fixed monomial ideal in R and define*

$$D(J) := \max\{t | x_t^j \in J \text{ for some positive integer } j\}$$

and

$$M(J) := \max\{t | x_t \text{ appears in some minimal generator of } J\}.$$

Then

1. $\dim(R/J) = m - D(J)$; and
2. $\text{depth}(R/J) = m - M(J)$.

Note that when I is a complete intersection of type (α, β) in R ,

$$\dim(R/I^n) = \text{depth}(R/I^n) = m - 2$$

for all $n \geq 1$. It then follows by Theorem 3.2.3 that the depth and dimension of $R/\text{gin}(I^n)$ are equal to $m - 2$ as well.

Proof of Lemma 3.3.2. By Proposition 3.3.3,

$$D(\text{gin}(I^n)) = m - \dim(R/\text{gin}(I^n)) = 2 = m - \text{depth}(R/\text{gin}(I^n)) = M(\text{gin}(I^n)).$$

This means that the minimal monomial generating set A_n of $\text{gin}(I^n)$ is contained in $S = K[x_1, x_2]$ and that A_n contains a power of x_2 , say $x_2^{\lambda_0}$. The fact that $\text{gin}(I^n)$ is strongly stable means that we can replace any number of x_2 variables in $x_2^{\lambda_0}$ with x_1 and still get an element of $\text{gin}(I^n)$. Therefore, any monomial $x^J \in S = K[x_1, x_2]$ of degree λ_0 is also contained in $\text{gin}(I^n)$. Now it is clear that the set of minimal monomial generators, $A_n \subset S$, cannot contain any element of degree greater than λ_0 . \square

Proof of Theorem 3.3.1 (i). By Lemma 3.3.2, $A_n \subset K[x, y]$ and $y^{\lambda_0} \in A_n$. Let m_k be a monomial of least degree k in $\text{gin}(I^n)$. Since $\text{gin}(I^n)$ is strongly stable, every variable appearing in m_k can be replaced by x and still stay inside of $\text{gin}(I^n)$; thus, $x^k \in \text{gin}(I^n)$ and, as a least degree element, x^k is also in A_n . Define λ_i by $\lambda_i =$

$\min\{t|x^i y^t \in \text{gin}(I^n)\}$ so that

$$A_n \subset \{x^k, x^{k-1}y^{\lambda_{k-1}}, x^{k-2}y^{\lambda_{k-2}}, \dots, xy^{\lambda_1}, y^{\lambda_0}\} \subset \text{gin}(I^n).$$

Since $x^i y^{\lambda_i}$ is in the strongly stable ideal $\text{gin}(I^n)$,

$$\frac{x(x^i y^{\lambda_i})}{y} = x^{i+1} y^{\lambda_i-1} \in \text{gin}(I^n)$$

for all $i = 1, \dots, k-2$. This condition holds if and only if $\lambda_i - 1 \geq \lambda_{i+1}$, or $\lambda_i \gneq \lambda_{i+1}$. Therefore, the λ_i s are strictly decreasing and $\lambda_{k-1} \geq 1$. Thus,

$$A_n = \{x^k, x^{k-1}y^{\lambda_{k-1}}, x^{k-2}y^{\lambda_{k-2}}, \dots, xy^{\lambda_1}, y^{\lambda_0}\}.$$

□

Proof of Theorem 3.3.1(ii). Note that, since $\alpha \leq \beta$, the homogeneous polynomial f_α^n is an element of I^n of the smallest degree. Under a general change of coordinates g , the smallest degree element of $g(I^n)$ is also of degree $n\alpha$ and its initial term is of degree $n\alpha$. Thus, the smallest degree element of $\text{in}(g(I^n)) = \text{gin}(I^n)$ has degree $n\alpha$ and, since $\text{gin}(I^n)$ is strongly stable, this is equal to the power of x in A_n . □

To determine the values of λ_0 and λ_{k-1} we will compare the *Betti numbers* of I^n and $\text{gin}(I^n)$ using ‘The Cancellation Principle’. Let

$$0 \rightarrow F_m \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow J \rightarrow 0$$

be the unique minimal free graded resolution of a homogeneous ideal J . The graded Betti numbers of J , $\beta_{i,j}(J)$, are defined by $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(J)}$. A *consecutive cancellation* takes a sequence $\{\beta_{i,j}\}$ to a new sequence by replacing $\beta_{i,j}$ by $\beta_{i,j} - 1$ and $\beta_{i+1,j}$ by $\beta_{i+1,j} + 1$. The ‘Cancellation Principle’ says that the graded Betti numbers $\beta_{i,j}(I^n)$ of I^n can be obtained from the graded Betti numbers $\beta_{i,j}(\text{gin}(I^n))$ of $\text{gin}(I^n)$ by making a series of consecutive cancellations (see Corollary 1.21 of [Gre98]).

In order to apply the Cancellation Principle to find λ_{k-1} and λ_0 , we need to know the Betti numbers of I^n and an ideal having the same form as $\text{gin}(I^n)$; this information is recorded in the following two propositions.

Proposition 3.3.4 ([GT05]). *Suppose that I is a complete intersection of type (α, β) .*

Then the minimal free resolution of I^n is of the form

$$0 \rightarrow \mathcal{H}_1 \rightarrow \mathcal{H}_0 \rightarrow I^n \rightarrow 0$$

where

$$\mathcal{H}_1 = \bigoplus_{p=1}^n R(-\alpha p - \beta(n+1-p))$$

and

$$\mathcal{H}_0 = \bigoplus_{p=0}^n R(-\alpha p - \beta(n-p)) = R(-\alpha n) \oplus \bigoplus_{p=0}^{n-1} R(-\alpha p - \beta(n-p)).$$

Proposition 3.3.5 (cf [EK90]). *The minimal free resolution of $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where $\lambda_0 > \lambda_1 > \dots > \lambda_{k-1}$ is of the form*

$$0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_0 \rightarrow J \rightarrow 0$$

where

$$\mathcal{G}_1 = \bigoplus_{i=0}^{k-1} R(-\lambda_i - i - 1)$$

and

$$\mathcal{G}_0 = \left(\bigoplus_{i=0}^{k-1} R(-\lambda_i - i) \right) \oplus R(-k).$$

Proof of Theorem 3.3.1(iii). Since the invariants λ_i are strictly decreasing, $\lambda_0 + 1 > \lambda_i + i \geq k$ for $i = 0, \dots, k-1$. Thus, if $\{\beta_{i,j}\}$ is the set of graded Betti numbers of $\text{gin}(I^n)$, $\beta_{1,\lambda_0+1} \geq 1$ and $\beta_{0,\lambda_0+1} = 0$ by Proposition 3.3.5. Therefore, no consecutive cancellation can replace β_{1,λ_0+1} and after any series of consecutive cancellations

$$\max\{t \mid \beta_{1,t} \geq 1\} = \lambda_0 + 1.$$

By Proposition 3.3.4, $\alpha + n\beta$ is the largest shift in \mathcal{H}_1 . Thus, by the Cancellation Principle, $\lambda_0 + 1 = \alpha + n\beta$, or

$$\lambda_0 = \alpha + n\beta - 1. \quad \square$$

Proof of Theorem 3.3.1(iv). Since the invariants λ_i are strictly decreasing and $\lambda_{k-1} \geq 1$, $k \leq \lambda_{k-1} + (k-1) < \lambda_i + i + 1$ for all $i = 0, \dots, k-1$. Thus, if $\{\beta_{i,j}\}$ is the set of graded Betti numbers of $\text{gin}(I^n)$, $\beta_{0,k} \geq 1$, $\beta_{0,\lambda_{k-1}+k-1} \geq 1$, $\beta_{1,k} = 0$, and $\beta_{1,\lambda_{k-1}+k-1} = 0$ by Proposition 3.3.5. Therefore, no consecutive cancellation can

replace $\beta_{0,k}$ or $\beta_{0,\lambda_{k-1}+k-1}$ and, for every t such that $t < k$ or $k < t < \lambda_{k-1} + k - 1$, $\beta_{0,t} = 0$ (note that it is possible to have $k = \lambda_{k-1} + k - 1$).

By Proposition 3.3.4, the two smallest shifts in \mathcal{H}_0 are $n\alpha$ and $\alpha(n-1) + \beta$. Thus, by the Cancellation Principle, $k = n\alpha$ (as we have seen in the proof of part (ii)) and $\lambda_{k-1} + k - 1 = \lambda_{n\alpha-1} + n\alpha - 1 = \alpha(n-1) + \beta$, or

$$\lambda_{n\alpha-1} = \beta - \alpha + 1. \quad \square$$

Note that we can write λ_0 and λ_{k-1} in terms of $l := \beta - \alpha$ and α as follows:

$$\lambda_0 = n(\alpha + l) + \alpha - 1 = (n+1)\alpha + nl - 1$$

$$\lambda_{k-1} = \lambda_{n\alpha-1} = \beta - \alpha + 1 = l + 1.$$

3.3.2 The Hilbert function of $\text{gin}(I^n)$

The following result tells us that the invariants of $\text{gin}(I^n)$ are completely determined by $H_{\text{gin}(I^n)}(t)$; this observation will be the key to computing these invariants.

Lemma 3.3.6. *Suppose that we have an ideal J of the form*

$$J = (x^k, x^{k-1}y^{\mu_{k-1}}, \dots, xy^{\mu_1}, y^{\mu_0})$$

where the μ_i s are strictly decreasing. If $H_J(t) = H_{I^n}(t)$ for a type (α, β) complete intersection ideal I then

$$\text{gin}(I^n) = J.$$

This lemma is an immediate consequence of the following well-known result. Although it is used in the literature (for example, it has the same content as Lemma 4.2 of [Gre98]), we record a complete proof here.

Lemma 3.3.7. *An ideal of the form*

$$J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$$

where $\lambda_0 > \lambda_1 > \dots > \lambda_{k-1}$ is uniquely determined by its Hilbert function.

Proof. The key observation here is that

$$\deg(x^i y^{\lambda_i}) = i + \lambda_i = (i-1) + (\lambda_i + 1) \leq (i-1) + \lambda_{i-1} = \deg(x^{i-1} y^{\lambda_{i-1}})$$

since $\lambda_i < \lambda_{i-1}$. Suppose that $H_J(t)$ is the Hilbert function of an ideal J as in the statement of the lemma.

First note that x^k is the smallest degree element of J so that $k = \min\{t | H_J(t) \neq 0\}$.

Consider the ideal $L_k = (x^k) \subset J$ and its Hilbert function $H_{L_k}(t)$. Set

$$S_k = \min\{t | H_J(t) \neq H_{L_k}(t)\}$$

so that the smallest degree monomial that is in J but not in L_k is of degree S_k . Since $\deg(x^i y^{\lambda_i}) \leq \deg(x^{i-1} y^{\lambda_{i-1}})$ for all i , we must have $\deg(x^{k-1} y^{\lambda_{k-1}}) = S_k$ and $\lambda_{k-1} = S_k - (k - 1)$.

The same argument works in general by induction. Suppose that we have determined the values of k and $\lambda_{k-1}, \dots, \lambda_T$ and let $L_T = (x^k, x^{k-1} y^{\lambda_{k-1}}, \dots, x^T y^{\lambda_T}) \subset J$. Set

$$S_T = \min\{t | H_J(t) \neq H_{L_T}(t)\}.$$

By the same argument as above, $\deg(x^{T-1} y^{\lambda_{T-1}}) = S_T$ and $\lambda_{T-1} = S_T - (T - 1)$. \square

Proof of Lemma 3.3.6. By Theorem 3.2.3,

$$H_{\text{gin}(I^n)}(t) = H_{I^n}(t) = H_J(t).$$

Since, J and $\text{gin}(I^n)$ are both of the form considered in Lemma 3.3.7, they are uniquely determined by their Hilbert functions and $J = \text{gin}(I^n)$. \square

To prove that the numbers $\{\lambda_i\}$ produced by the algorithms presented in Section 3.4 are indeed the invariants of $\text{gin}(I^n)$, we will compute the Hilbert function of the ideal

$$J = (x^k, x^{k-1} y^{\lambda_{k-1}}, \dots, x y^{\lambda_1}, y^{\lambda_0}).$$

By Lemma 3.3.6 it is then sufficient to show that $H_J(t)$ is equal to $H_{I^n}(t)$. We will now record expressions for the Hilbert functions of I^n and J that will be used to carry out this procedure.

Proposition 3.3.8. *If I is the ideal of a type (α, β) complete intersection in $K[x_1, \dots, x_m]$ then*

$$H_{I^n}(t) = \sum_{j=1}^n \left[\binom{t - \alpha(n-j) - \beta j + (m-1)}{(m-1)} - \binom{t - \alpha j - \beta(n+1-j) + (m-1)}{(m-1)} \right]$$

$$+ \binom{t - n\alpha + (m-1)}{(m-1)}.$$

Setting $l := \beta - \alpha$,

$$\begin{aligned} H_{I^n}(t) &= \sum_{j=1}^n \left[\binom{t - \alpha n - jl + (m-1)}{(m-1)} - \binom{t - \alpha(n+1) - lj + (m-1)}{(m-1)} \right] \\ &\quad + \binom{t - n\alpha + (m-1)}{(m-1)}. \end{aligned}$$

Proof. By Proposition 3.3.4,

$$\begin{aligned} H_{I^n}(t) &= \dim_K [R(-\alpha n)]_t + \sum_{p=0}^{n-1} \dim_K [R(-\alpha p - \beta(n-p))]_t \\ &\quad - \sum_{p=1}^n \dim_K [R(-\alpha p - \beta(n+1-p))]_t \\ &= \binom{t - n\alpha + (m-1)}{(m-1)} + \sum_{p=0}^{n-1} \binom{t - \alpha p - \beta(n-p) + (m-1)}{(m-1)} \\ &\quad - \sum_{p=1}^n \binom{t - \alpha p - \beta(n+1-p) + (m-1)}{(m-1)} \\ &= \sum_{j=1}^n \left[\binom{t - \alpha(n-j) - \beta j + (m-1)}{(m-1)} \right. \\ &\quad \left. - \binom{t - \alpha j - \beta(n+1-j) + (m-1)}{(m-1)} \right] + \binom{t - n\alpha + (m-1)}{(m-1)} \end{aligned}$$

If $l = \beta - \alpha$, the *sum* in the above expression is

$$\begin{aligned} &\sum_{j=1}^n \left[\binom{t - \alpha n - j(\beta - \alpha) + (m-1)}{(m-1)} - \binom{t - \beta(n+1) - j(\alpha - \beta) + (m-1)}{(m-1)} \right] \\ &= \sum_{j=1}^n \left[\binom{t - \alpha n - jl + (m-1)}{(m-1)} - \binom{t - (\alpha + l)(n+1) + jl + (m-1)}{(m-1)} \right] \\ &= \sum_{j=1}^n \left[\binom{t - \alpha n - jl + (m-1)}{(m-1)} - \binom{t - \alpha(n+1) - l(n+1-j) + (m-1)}{(m-1)} \right] \\ &= \sum_{j=1}^n \left[\binom{t - \alpha n - jl + (m-1)}{(m-1)} - \binom{t - \alpha(n+1) - lj + (m-1)}{(m-1)} \right]. \end{aligned}$$

Note that the last equality follows by changing the indexing. □

Proposition 3.3.9. *Suppose that we have an ideal J of the form*

$$J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$$

where $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{k-1}$. Then

$$H_J(t) = \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-2)}{(m-2)} + \binom{t - k + (m-1)}{(m-1)}.$$

Proof. From Proposition 3.3.5,

$$\begin{aligned} H_J(t) &= \dim_K[R(-k)] + \sum_{i=0}^{k-1} \dim_K[R(-\lambda_i - i)] - \sum_{i=0}^{k-1} \dim_K[R(-\lambda_i - i - 1)] \\ &= \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-1)}{(m-1)} - \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-2)}{(m-1)} + \binom{t - k + (m-1)}{(m-1)} \\ &= \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-2)}{(m-2)} + \binom{t - k + (m-1)}{(m-1)}. \end{aligned}$$

The last equality follows from the recursive formula for binomial coefficients (see Equation 3.2). □

3.4 Main Theorem and the Proposed Invariants

In the previous section we determined the general structure of $\text{gin}(I^n)$ where I is a 2-complete intersection of type (α, β) and showed that it was defined by a strictly decreasing sequence of invariants $\{\lambda_i\}$. We also found expressions for λ_0 and λ_{k-1} in terms of n , α , and β (see Theorem 3.3.1). In this section we propose algorithms to determine the remaining invariants, and thus the minimal generators, of $\text{gin}(I^n)$. Throughout $l := \beta - \alpha$.

Theorem 3.4.1. *Fix positive integers α , β , and n such that $\beta \geq \alpha$ and $n \geq 2$. Compute the sequence of invariants $\{\lambda_i\}$ using:*

- *Algorithm 1 if $\beta \geq 2\alpha - 1$;*

- Algorithm 2 if $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha$;
- Algorithm 3 if $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \mid \alpha$, and $n \geq \frac{\alpha}{\beta - \alpha} + 1$;
- Algorithm 4 if $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \nmid \alpha$, and $n \geq \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$;
- Algorithm 5 if $\frac{3}{2}\alpha > \beta > \alpha$ and $2 \leq n < \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$; and
- Algorithm 6 if $\alpha = \beta$.

If I is a type (α, β) complete intersection in R then the reverse lexicographic generic initial ideal of I^n is

$$\text{gin}(I^n) = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$$

where $k = n\alpha$.

Each of the algorithms, and thus the invariants λ_i that they produce and the resulting gaps $g_i := \lambda_{i-1} - \lambda_i$, can be divided into three consecutive phases which we refer to as the Build, the Pattern, and the Reverse Build. As the names of the phases suggest, the gap sequences $\{g_i\}$ arising from the Reverse Build and the Build are almost mirror images of each other while the gap sequences arising from the Pattern consists of a number of repeats of the same sub-sequence called a Pattern Block.

3.4.1 Algorithms Producing the Proposed Invariants

Given three positive integers n , α , and β where $n \geq 2$ and $\beta \geq \alpha$, the following algorithms produce a sequence of positive integers $\lambda_0, \dots, \lambda_{k-1}$ which Theorem 3.4.1 claims are the invariants of $\text{gin}(I^n)$ for a type (α, β) complete intersection I .

For examples and illustrations of the outputs of these algorithms, see Appendix B.

Algorithm 1 Determine $\{\lambda_i\}$ for $\beta \geq 2\alpha - 1$, $n \geq 1$

```

i = 1
 $\lambda_0 = n\beta + \alpha - 1$ 
h = 1
while h ≤ n - 1 do
  BlockFar(i,  $\lambda_{i-1}$ ,  $\alpha$ ,  $\beta$ )
  h = h + 1
end while
PartialBlockFar(i,  $\lambda_{i-1}$ ,  $\alpha$ )

```

Sub-routines for Algorithm 1

BlockFar($i, \lambda_{i-1}, \alpha, \beta$)

$t = 1$

while $t \leq \alpha - 1$ **do**

$\lambda_i = \lambda_{i-1} - 2$

$t = t + 1$

$i = i + 1$

end while

$\lambda_i = \lambda_{i-1} - (\beta - 2\alpha + 2)$

$i = i + 1$

RETURN

PartialBlockFar(i, λ_{i-1}, α)

$h = 1$

while $h \leq \alpha - 1$ **do**

$\lambda_i = \lambda_{i-1} - 2$

$i = i + 1$

$h = h + 1$

end while

RETURN

The subroutines not called by Algorithm 1 appear after the statements of the remaining algorithms.

Algorithm 2 Determine $\{\lambda_i\}$ for $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha$, $n \geq 2$

$l = \beta - \alpha$

$r = 2\alpha - \beta$

$\lambda_0 = n\beta + \alpha - 1$

$i = 1$

Build($0, i, \lambda_{i-1}$)

if $n \geq 3$ **then**

$h = 1$

while $h \leq n - 2$ **do**

BlockMid($i, \lambda_{i-1}, r, \alpha$)

$h = h + 1$

end while

end if

if $n \geq 2$ **then**

PartialBlockMid(i, λ_{i-1}, r)

end if

ReverseBuild($0, i, \lambda_{i-1}, l$)

Algorithm 3 Determine $\{\lambda_i\}$ for $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \mid \alpha$, $n \geq \frac{\alpha}{\beta - \alpha} + 1$

```

 $l = \beta - \alpha$ 
 $c = \lceil \frac{\alpha}{l} \rceil = \frac{\alpha}{l}$ 
 $d = \alpha \bmod l = 0$ 
 $\lambda_0 = n\beta + \alpha - 1$ 
 $i = 1$ 
Build( $c - 2, i, \lambda_{i-1}$ )
 $h = 1$ 
while  $h \leq nl - \alpha + l$  do
    BlockClose( $i, \lambda_{i-1}, c, d, l, \alpha$ )
     $h = h + 1$ 
end while
ReverseBuildPartial( $c - 2, i, \lambda_{i-1}, l$ )
ReverseBuild( $c - 3, i, \lambda_{i-1}, l$ )

```

Algorithm 4 Determine $\{\lambda_i\}$ for $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \nmid \alpha$, $n \geq \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$

```

 $l = \beta - \alpha$ 
 $c = \lceil \frac{\alpha}{l} \rceil$ 
 $d = \alpha \bmod l$ 
 $\lambda_0 = n\beta + \alpha - 1$ 
 $i = 1$ 
Build( $c - 2, i, \lambda_{i-1}$ )
 $h = 1$ 
while  $h \leq n - c$  do
    BlockClose( $i, \lambda_{i-1}, c, d, l, \alpha$ )
     $h = h + 1$ 
end while
PartialBlockClose( $i, \lambda_{i-1}, c, d$ )
ReverseBuildPartial( $c - 2, i, \lambda_{i-1}, l$ )
ReverseBuild( $c - 3, i, \lambda_{i-1}, l$ )

```

Algorithm 5 Determine $\{\lambda_i\}$ for $\frac{3}{2}\alpha > \beta > \alpha$, $2 \leq n < \lceil \frac{\alpha}{l} \rceil + 1$

```

 $l = \beta - \alpha$ 
 $\lambda_0 = n\beta + \alpha - 1$ 
 $i = 1$ 
Build( $n - 2, i, \lambda_{i-1}$ )
 $h = 1$ 
while  $h \leq \beta - nl$  do
  onestwo( $n - 1, i, \lambda_{i-1}$ )
   $h = h + 1$ 
end while
ReverseBuildPartial( $n - 2, i, \lambda_{i-1}, l$ )
if  $n \geq 3$  then
  ReverseBuild( $n - 3, i, \lambda_{i-1}, l$ )
end if

```

Algorithm 6 Determine $\{\lambda_i\}$ for $\alpha = \beta$, $n \geq 1$

```

 $i = 1$ 
 $\lambda_0 = (n + 1)\alpha - 1$ 
 $h = 1$ 
while  $h \leq \alpha - 1$  do
  onestwo( $n - 1, i, \lambda_{i-1}$ )
   $h = h + 1$ 
end while
PartialBlockEqual( $i, \lambda_{i-1}, n$ )

```

Sub-routines for Remaining Algorithms

onestwo(x, i, λ_{i-1})

$t = 1$

while $t \leq x$ **do**

$\lambda_i = \lambda_{i-1} - 1$

$t = t + 1$

$i = i + 1$

end while

$\lambda_i = \lambda_{i-1} - 2$

$i = i + 1$

RETURN

revonestwo(x, i, λ_{i-1})

$\lambda_i = \lambda_{i-1} - 2$

$i = i + 1$

$t = 1$

while $t \leq x$ **do**

$\lambda_i = \lambda_{i-1} - 1$

$t = t + 1$

$i = i + 1$

end while

RETURN

Build($limq, i, \lambda_{i-1}$)

$q = 0$

while $q \leq limq$ **do**

$j = 1$

while $j \leq l$ **do**

onestwo(q, i, λ_{i-1})

$j = j + 1$

end while

$q = q + 1$

end while

RETURN

ReverseBuild($limq, i, \lambda_{i-1}, l$)

$q = limq$

while $q \geq 0$ **do**

$j = 1$

while $j \leq l$ **do**

revonestwo(q, i, λ_{i-1})

$j = j + 1$

end while

$q = q - 1$

end while

RETURN

ReverseBuildPartial($limq, i, \lambda_{i-1}, l$)

$j = 1$

while $j \leq limq$ **do**

$\lambda_i = \lambda_{i-1} - 1$

$j = j + 1$

end while

$j = 2$

while $j \leq l$ **do**

revonestwo($limq, i, \lambda_{i-1}$)

$j = j + 1$

end while

RETURN

BlockMid($i, \lambda_{i-1}, r, \alpha$)

$t = 1$

while $t \leq 2r - 1$ **do**

if t is odd **then**

$\lambda_i = \lambda_{i-1} - 1$

else $\{t$ is even $\}$

$\lambda_i = \lambda_{i-1} - 2$

end if

$t = t + 1$

$i = i + 1$

end while

$t = 1$

while $t \leq \alpha - (2r - 1) = \beta - 3\alpha + 1$

do

$\lambda_i = \lambda_{i-1} - 2$

$t = t + 1$

$i = i + 1$

end while

RETURN

PartialBlockMid(i, λ_{i-1}, r)

$t = 1$

while $t \leq 2r - 1$ **do**

if t is odd **then**

$\lambda_i = \lambda_{i-1} - 1$

else $\{t$ is even $\}$

$\lambda_i = \lambda_{i-1} - 2$

end if

$t = t + 1$

$i = i + 1$

end while

RETURN

BlockClose($i, \lambda_{i-1}, c, d, l, \alpha$)

if $l \mid \alpha$ **then**

onestwo($c - 1, i, \lambda_{i-1}$)

else

$j = 1$

while $j \leq d$ **do**

onestwo($c - 1, i, \lambda_{i-1}$)

$j = j + 1$

end while

while $j \leq l$ **do**

onestwo($c - 2, i, \lambda_{i-1}$)

$j = j + 1$

end while

end if

RETURN

PartialBlockClose(i, λ_{i-1}, c, d)

$j = 1$

while $j \leq d$ **do**

onestwo($c - 1, i, \lambda_{i-1}$)

$j = j + 1$

end while

RETURN

PartialBlockEqual(i, λ_{i-1}, n)

$h = 1$

while $h \leq n - 1$ **do**

$\lambda_i = \lambda_{i-1} - 1$

$i = i + 1$

$h = h + 1$

end while

RETURN

3.4.2 Description of the Algorithms

We will call the λ_i s produced by these algorithms the *proposed invariants* of $\text{gin}(I^n)$. Each of the algorithms can be divided into the following three stages:

1. the Build (absent in the cases where $\beta \geq 2\alpha - 1$ and $\alpha = \beta$);

2. the Pattern (consists of full or partial repetitions of a Pattern Block²); and
3. the Reverse Build (also absent in the cases where $\beta \geq 2\alpha - 1$ and $\alpha = \beta$).

It will be convenient to divide the proposed invariants produced by an algorithm into the same three categories; for example, a λ_i produced by the Build stage of an algorithm will be said to be part of the Build.

Each of the algorithms begins with defining $\lambda_0 = n\beta + \alpha - 1$ (see Theorem 3.3.1). The other invariants are obtained one-by-one by subtracting 1, 2, or $\beta - 2\alpha + 2$ from the previous invariant in the sequence. Patterns in the sequences $\{\lambda_0, \dots, \lambda_{k-1}\}$ emerge by looking at the sequences of gaps between the λ_i s; thus, we set g_i to be equal to the number subtracted from λ_{i-1} to obtain λ_i , or $g_i := \lambda_{i-1} - \lambda_i$. The sequence $\{g_i\}$ will be called the *gap sequence* corresponding to the sequence of proposed invariants; note that this sequence will consist entirely of the numbers 1, 2, and $\beta - 2\alpha + 2$.

Observe the following:

- Since all of the numbers g_i are greater than 0, the sequences $\{\lambda_i\}$ produced by the algorithms are strictly decreasing.
- The gap sequence of the Build written backwards generally gives the gap sequence of the Reverse Build. The exception to this is in the algorithms corresponding to the cases where $\frac{3}{2}\alpha > \beta > \alpha$. In these cases, everything but the final gap of the Build is reflected in the Reverse Build; this is why it is necessary to include the **ReverseBuildPartial** subroutine.
- The gap sequences of the Build, the Reverse Build, and the Pattern Blocks, are independent of n except in Algorithm 5 corresponding to the case where $\frac{3}{2}\alpha > \beta > \alpha$, $2 \leq n < \lceil \frac{\alpha}{l} \rceil + 1$ and in Algorithm 6 corresponding to the case where $\alpha = \beta$. The only part of the other algorithms that changes as n increases is the number of times that the Pattern Block is repeated.
- The last λ_i produced by the algorithms is $\lambda_{k-1} = \beta - \alpha + 1$. Note that, by Theorem 3.3.1, this condition must be satisfied for the algorithms to produce the invariants of $\text{gin}(I^n)$. We can check that this condition holds by showing

²As their names suggest, **BlockFar**, **BlockMid**, and **BlockClose** are Pattern Blocks. In the cases where $\alpha = \beta$ and $\frac{3}{2} > \beta > \alpha$, $2 \leq n < \lceil \frac{\alpha}{l} \rceil + 1$, the Pattern Block is simply the **onestwo** subroutine.

that

$$\begin{aligned}
\sum_{i=1}^{k-1} g_i &= \lambda_0 - \lambda_{k-1} \\
&= (n\beta + \alpha - 1) - (\beta - \alpha + 1) \\
&= (n-1)\beta + 2\alpha - 2.
\end{aligned}$$

- The conditions on α and β ensure that the algorithms make sense. For example, in Algorithm 3 **ReverseBuild**($c-3, \dots$) is well-defined because when $\frac{3}{2}\alpha > \beta > \alpha$,

$$c = \frac{\alpha}{l} > \frac{\alpha}{3\alpha/2 - \alpha} = 2.$$

We encourage the reader to consult Appendix B to get a better feel for the sequences $\{g_i\}$ and $\{\lambda_i\}$ produced by these algorithms. It contains examples of the outputs for fixed α , β , and n and illustrations that indicate what happens for a general n in most cases.

3.5 Proof of Theorem 3.4.1

In this section we will prove Theorem 3.4.1. The proof is divided into six parts, one for each of the cases and algorithms referred to in the theorem. The proof in each case will involve the following steps:

1. write non-recursive formulas for the proposed invariants λ_i produced by the algorithm;
2. compute $H_J(t)$ where

$$J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$$

and the invariants λ_i are as above; and

3. rewrite $H_{I^n}(t)$ in an appropriate form, sometimes using the assumptions on α and β for the particular case, and simplify the expression to show that it is equal to $H_J(t)$. By Lemma 3.3.6 this will prove that $J = \text{gin}(I^n)$ so that the invariants produced by the algorithm are the invariants of $\text{gin}(I^n)$.

Since the required calculations are routine and long, details are left to the Appendices C and D. In particular, Appendix C contains the derivations of the non-

recursive formulas for the λ_i from the algorithms while Appendix D contains details of the $H_{J^n}(t)$ calculation and simplifications of the partial sums $\sum_i \binom{t-\lambda_i-i+m-2}{m-2}$ that appear in the expression for $H_J(t)$ from Section 3.3.2.

For convenience, we will divide the formulas and long calculations into parts according to whether they involve invariants and indexing from the Build, Pattern, or Reverse Build of the algorithm as described in Section 3.4.2.

As before, $l = \beta - \alpha$ and $g_i = \lambda_{i-1} - \lambda_i$.

3.5.1 The Case $\beta \geq 2\alpha - 1$, $n \geq 1$

3.5.1.1 Formulas for the proposed invariants

First we write a closed form expression for the λ_v produced by Algorithm 1. For details on how this formula follows from the algorithm, see Section C.1 of Appendix C.

If $v = j\alpha + s$ where $s = 0, \dots, \alpha - 1$ and $j = 0, \dots, n - 1$,

$$\lambda_v = (n - j)\beta + \alpha - 1 - 2s.$$

3.5.1.2 The Hilbert function of J

Suppose that $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_v are given by the formula in Section 3.5.1.1. Then, by Proposition 3.3.9, the Hilbert function for $H_J(t)$ is

$$\begin{aligned} H_J(t) &= \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-2)}{(m-2)} + \binom{t - k + (m-1)}{(m-1)} \\ &= \sum_{j=0}^{n-1} \sum_{s=0}^{\alpha-1} \binom{t - [(n-j)\beta + \alpha - 1 - 2s] - [j\alpha + s] + (m-2)}{(m-2)} \\ &\quad + \binom{t - n\alpha + (m-1)}{(m-1)} \\ &= \sum_{j=0}^{n-1} \sum_{s=0}^{\alpha-1} \binom{t - (n-j)\beta - \alpha - j\alpha + s + m - 1}{m-2} + \binom{t - n\alpha + m - 1}{m-1} \\ &= \sum_{j=0}^{n-1} \left[\binom{t - (n-j)\beta - \alpha - j\alpha + \alpha + m - 1}{m-1} \right] \end{aligned}$$

$$\begin{aligned}
& - \binom{t - (n-j)\beta - \alpha - j\alpha + m - 1}{m-1} \Big] + \binom{t - n\alpha + m - 1}{m-1} \\
= & \sum_{p=1}^n \binom{t - p\beta - (n-p)\alpha + m - 1}{m-1} \\
& - \sum_{q=1}^n \binom{t - (n-q+1)\beta - q\alpha + m - 1}{m-1} + \binom{t - n\alpha + m - 1}{m-1}.
\end{aligned}$$

3.5.1.3 The Hilbert function of I^n

By Proposition 3.3.8,

$$\begin{aligned}
H_{I^n}(t) = & \sum_{j=1}^n \left(\binom{t - \alpha(n-j) - \beta j + (m-1)}{(m-1)} - \binom{t - \alpha j - \beta(n+1-j) + (m-1)}{(m-1)} \right) \\
& + \binom{t - n\alpha + (m-1)}{(m-1)}
\end{aligned}$$

Since $H_J(t) = H_{I^n}(t)$, Lemma 3.3.6 implies that $J = \text{gin}(I^n)$ and thus that the numbers produced by Algorithm 1 are the invariants of $\text{gin}(I^n)$ when $\beta \geq 2\alpha - 1$ and $n \geq 1$.

3.5.2 The Case $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha, n \geq 2$

Throughout this section, we will set $r := 2\alpha - \beta > 0$.

3.5.2.1 Formulas for the proposed invariants

First we will write closed-form expressions for the numbers λ_i produced by Algorithm 2. Details about how these formulas are obtained from the algorithm can be found in Section C.2 of Appendix C. To match the work in the appendix, we distinguish the formulas for invariants produced by the Build, the Reverse Build, and the Pattern phases of the Algorithm 2.

Formula for λ_i in the Build. For $v = 0, \dots, l$,

$$\lambda_v = \alpha(n+1) + l \cdot n - 1 - 2v.$$

Formulas for λ_i in the Reverse Build. For $v = k - i = n\alpha - i$ where $i = 1, \dots, l+1$,

$$\lambda_v = l + 1 + 2(i - 1) = l + 2i - 1.$$

Formulas for λ_i in the Pattern.

- ❷ For $v = l + j\alpha + y$ where $j = 0, \dots, (n - 3)$ and $y = 2r - 1, \dots, \alpha - 1$,

$$\lambda_v = \lambda_0 - [2l + (2\alpha - r)j + 2y - r] = \lambda_0 - [2l + (\alpha + l)j + 2y - (\alpha - l)].$$

- ❸ For $v = l + j\alpha$ where $j = 1, \dots, n - 2$,

$$\lambda_v = \lambda_0 - [j(2\alpha - r) + 2l] = \lambda_0 - [j(l + \alpha) + 2l].^3$$

- ❹ For $v = l + j\alpha + 2p$ where $j = 0, \dots, n - 2$ and $p = 1, \dots, r - 1 = \alpha - l - 1$,

$$\lambda_v = \lambda_0 - [2l + (\alpha + l)j + 2p + p].$$

- ❺ For $v = l + j\alpha + 2p - 1$ where $j = 0, \dots, (n - 2)$ and $p = 1, \dots, r - 1$,

$$\lambda_v = \lambda_0 - [2l + (\alpha + l)j + 2p - 2 + p]$$

3.5.2.2 The Hilbert function of J

Suppose that $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_i are the invariants produced by Algorithm 2 and are given by the formulas in Section 3.5.1.1. By Proposition 3.3.9

$$H_J(t) = \sum_{i=0}^{k-1} \binom{t - \lambda_i - i + (m-2)}{(m-2)} + \binom{t - k + (m-1)}{(m-1)}.$$

Set

$$X_j := t - n\alpha - jl$$

and

$$Y_j := t - (n + 1)\alpha - jl.$$

In Section D.2 of Appendix D, we simplify the partial sums $\sum_i \binom{t - \lambda_i - i + m - 2}{m - 2}$ as i ranges over the Build, Reverse Build, and Pattern. Adding these together with $\binom{t - n\alpha + m - 1}{m - 1}$ we get the following expression for $H_J(t)$ when $n \geq 3$.

³Note that when $n = 2$, the ranges for j in ❷ and ❸ are empty. This reflects the fact that Algorithm 2 only includes the Partial Pattern Block when $n = 2$.

$$\begin{aligned}
H_J(t) &= \sum_{j=2}^n \binom{X_j + m-1}{m-1} + \sum_{j=2}^n \binom{X_j + m-2}{m-1} - \sum_{j=3}^n \binom{X_j + m-2}{m-1} - \sum_{j=1}^{n-1} \binom{Y_j + m}{m-1} \\
&\quad - \sum_{j=1}^{n-1} \binom{Y_j + m-1}{m-1} + \sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-1} + \sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-2} + \binom{Y_{n-1} + m}{m-1} \\
&\quad - \binom{Y_n + m-1}{m-1} + \binom{X_1 + m-1}{m-1} - \binom{X_2 + m-2}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\
&= \sum_{j=2}^n \binom{X_j + m-1}{m-1} + \binom{X_2 + m-2}{m-1} + \left(\sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-2} - \sum_{j=1}^{n-1} \binom{Y_j + m}{m-1} \right) \\
&\quad - \binom{Y_{n-1} + m-1}{m-1} + \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m-1}{m-1} + \binom{X_1 + m-1}{m-1} - \\
&\quad \binom{X_2 + m-2}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\
&= \sum_{j=2}^n \binom{X_j + m-1}{m-1} - \sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-1} - \binom{Y_{n-1} + m}{m-1} - \binom{Y_{n-1} + m-1}{m-1} \\
&\quad + \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m-1}{m-1} + \binom{X_1 + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\
&= \sum_{j=1}^n \binom{X_j + m-1}{m-1} - \sum_{j=1}^n \binom{Y_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1}.
\end{aligned}$$

When $n = 2$,

$$\begin{aligned}
H_J(t) &= \binom{X_2 + m-1}{m-1} - \binom{Y_1 + m}{m-1} + \binom{X_2 + m-2}{m-1} - \binom{Y_1 + m-1}{m-1} \\
&\quad + \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m-1}{m-1} + \binom{X_1 + m-1}{m-1} - \binom{X_2 + m-2}{m-1} \\
&= \binom{X_2 + m-1}{m-1} - \binom{Y_1 + m-1}{m-1} - \binom{Y_2 + m-1}{m-1} + \binom{X_1 + m-1}{m-1}.
\end{aligned}$$

3.5.2.3 The Hilbert function of I^n

By Proposition 3.3.8,

$$H_{I^n}(t) = \sum_{j=1}^n \left(\binom{t - \alpha n - jl + (m-1)}{(m-1)} - \binom{t - \alpha(n+1) - lj + (m-1)}{(m-1)} \right)$$

$$+ \binom{t - n\alpha + m - 1}{m - 1}.$$

Thus, $H_{I^n}(t) = H_J(t)$ so, by Lemma 3.3.6, $J = \text{gin}(I^n)$. Therefore, Algorithm 2 produces the invariants of $\text{gin}(I^n)$ when $\beta \geq 2\alpha - 1$ and $n \geq 1$.

3.5.3 The Case $\frac{3}{2}\alpha > \beta > \alpha$, $l|\alpha$, $n \geq \frac{\alpha}{l} + 1$

Throughout this section, set $c := \alpha/l$; this is an integer by assumption.

3.5.3.1 Formulas for the proposed invariants

As in previous cases, in this section we write closed form expressions for the numbers λ_i produced by Algorithm 3. See Section C.3 of Appendix C for details on how the formulas stated here were obtained. To be consistent with the work there, the formulas coming from each of the three phases of the algorithm (the Build, Reverse Build, and Pattern) are recorded separately.

Formulas for λ_i the Build.

- ❶ For $v = 0, \dots, l$,

$$\lambda_v = \lambda_0 - 2v.$$

- ❷ For $v = l(1 + \dots + q) + (q + 1)j$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_0 - [(2 + \dots + (q + 1))l + (q + 1)j + j]$$

- ❸ For $v = l(1 + \dots + q) + (q + 1)j - x$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,
 $x = 1, \dots, q$,

$$\lambda_v = \lambda_0 - [(2 + \dots + (q + 1))l + (q + 1)j - x + j - 1]$$

Formulas for λ_i in the Reverse Build.

- ❶ For $v = (k - 1) - j$ where $j = 0, \dots, l$,

$$\lambda_v = \lambda_{k-1} + 2j.$$

- ❷ For $v = (k - 1) - (l(1 + \dots + q) + (q + 1)j)$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_{k-1} + [(2 + \dots + (q + 1))l + (q + 1)j + j].$$

However, when $q = c - 2$ and $j = l$, λ_v is in the Pattern, not in the Reverse Build.

- ③ For $v = (k - 1) - (l(1 + \dots + q) + (q + 1)j - x)$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$, $x = 1, \dots, q$,

$$\lambda_v = \lambda_{k-1} + [(2 + \dots + (q + 1))l + (q + 1)j - x + j - 1].$$

However, when $q = c - 2$, $j = l$, and $x = 1$, λ_v is in the Pattern, not the Reverse Build.

Formulas for λ_i in the Pattern.

Let

$$E := l(1 + \dots + (c - 1))$$

and

$$B := l(2 + \dots + c).$$

- ① For $v = E + jc + i$ where $j = 0, \dots, ln - \alpha + l - 1$, $i = 1, \dots, c - 1$,

$$\lambda_v = \lambda_0 - B - [j(c + 1) + i]$$

- ② For $v = E + jc$ where $j = 1, \dots, ln - \alpha + l - 1$,

$$\lambda_v = \lambda_0 - B - [j(c + 1)]$$

3.5.3.2 The Hilbert function of J

As before we set

$$X_j = t - n\alpha - jl$$

and

$$Y_j = t - (n + 1)\alpha - jl.$$

Note that $X_c = Y_0$ and $Y_{n-c+1} = X_{n+1}$:

$$X_c = t - n\alpha - cl = t - (n + 1)\alpha = Y_0$$

$$Y_{n-c+1} = t - (n + 1)\alpha - nl + lc - l = t - n\alpha - l(n + 1) = X_{n+1}.$$

We will apply these relations in the calculations below.

Let $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_i are the numbers produced by Algorithm 3 and are given by the formulas in Section 3.5.3.1. To compute $H_J(t)$ we use the formula from Proposition 3.3.9.

Section D.3 of Appendix D contains expressions for the partial sums

$$\sum_i \binom{t - \lambda_i - i + m - 2}{m - 2}$$

as i ranges over the Build, the Reverse Build, and the Pattern. Adding these together with $\binom{t - n\alpha + m - 1}{m - 1}$, we obtain the following expression for $H_J(t)$.

$$\begin{aligned} H_J(t) &= \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m-1}{m-1} + \binom{Y_{n-c+1} + m}{m-1} - \binom{Y_{n-1} + m}{m-1} \\ &\quad + (c-2) \binom{Y_{n-c+1} + m-1}{m-1} - \sum_{j=n-c+2}^{n-1} \binom{Y_j + m-1}{m-1} + \binom{X_1 + m-1}{m-1} \\ &\quad - \binom{X_2 + m-2}{m-1} + \binom{X_2 + m-2}{m-1} - \binom{X_c + m-2}{m-1} + \sum_{j=2}^{c-1} \binom{X_j + m-1}{m-1} \\ &\quad - (c-2) \binom{X_c + m-1}{m-1} + (c-1) \left[\binom{Y_0 + m-1}{m-1} - \binom{X_{n+1} + m-1}{m-1} \right] \\ &\quad + \binom{Y_0 + m}{m-1} - \binom{X_{n+1} + m}{m-1} - \binom{Y_0 + m-2}{m-2} - \binom{Y_0 + m-1}{m-2} \\ &\quad + \binom{t - n\alpha + m-1}{m-1} \\ &= - \sum_{j=n-c+2}^n \binom{Y_j + m-1}{m-1} + \binom{Y_{n-c+1} + m}{m-1} - \binom{X_{n+1} + m}{m-1} \\ &\quad + (c-2) \binom{Y_{n-c+1} + m-1}{m-1} - (c-1) \binom{X_{n+1} + m-1}{m-1} + \binom{Y_0 + m-1}{m-1} \\ &\quad + (c-1) \binom{Y_0 + m-1}{m-1} - \binom{Y_0 + m-2}{m-2} - (c-2) \binom{Y_0 + m-1}{m-1} \\ &\quad - \binom{X_c + m-2}{m-1} + \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\ &= - \sum_{j=n-c+2}^n \binom{Y_j + m-1}{m-1} - \binom{Y_{n-c+1} + m-1}{m-1} + c \binom{Y_0 + m-1}{m-1} \\ &\quad - (c-2) \binom{Y_0 + m-1}{m-1} - \binom{Y_0 + m-2}{m-2} - \binom{Y_0 + m-2}{m-1} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\
= & - \sum_{j=n-c+1}^n \binom{Y_j + m-1}{m-1} + 2 \binom{Y_0 + m-1}{m-1} - \binom{Y_0 + m-1}{m-1} \\
& + \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1} \\
= & - \sum_{j=n-c+1}^n \binom{Y_j + m-1}{m-1} + \sum_{j=1}^c \binom{X_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1}
\end{aligned}$$

3.5.3.3 The Hilbert function of I^n

We may take advantage of the assumptions that $\frac{3}{2}\alpha > \beta > \alpha$ and $l|\alpha$ to rewrite the expression of $H_{I^n}(t)$ from Proposition 3.3.8 as follows:

$$H_{I^n}(t) = \sum_{j=1}^c \binom{X_j + m-1}{m-1} - \sum_{j=n-c+1}^n \binom{Y_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1}.$$

For the details of this calculation see Section D.3 of Appendix D.

Since $H_{I^n}(t) = H_J(t)$, Lemma 3.3.6 implies that $\text{gin}(I^n) = J$ and that the numbers produced by Algorithm 3 are the invariants of $\text{gin}(I^n)$ when $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha)|\alpha$, and $n \geq \frac{\alpha}{\beta - \alpha} + 1$.

3.5.4 The Case $\frac{3}{2}\alpha > \beta > \alpha$, $l \nmid \alpha$, $n \geq \lceil \frac{\alpha}{l} \rceil + 1$

Throughout this section, set $c := \lceil \frac{\alpha}{l} \rceil$ and $d := \alpha \bmod l$. Then

$$d = \alpha - l(c-1) = \alpha - lc + l$$

so

$$lc = \alpha + l - d.$$

This relation will be used during the calculations in this section.

3.5.4.1 Formulas for the proposed invariants

In this section we present closed form expressions for the λ_i produced by Algorithm 4. As before, formulas for the λ_i produced by the Build, the Reverse Build, and the

Pattern are recorded separately. Details of how these formulas were obtained from Algorithm 4 can be found in Section C.4 of Appendix C.

Formulas for λ_i in the Build and Reverse Build.

The formulas for λ_i produced by the Build and the Reverse Build are exactly the same as those in the Build and Reverse Build in the previous case; see Section 3.5.3.1 for the formulas.

Formulas for λ_i in the Pattern.

Set $E = l(1 + 2 + \cdots + (c - 1))$ and $B = l(2 + 3 + \cdots + c)$.

- ❶ For $v = E + p\alpha + jc + i$ where $p = 0, \dots, n - c$, $j = 0, \dots, d - 1$, and $i = 1, \dots, c - 1$,

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + j(c + 1) + i]$$

- ❷ For $v = E + p\alpha + jc$ where $p = 0, \dots, n - c$, and $j = 1, \dots, d$,

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + j(c + 1)]$$

- ❸ For $v = E + p\alpha + dc + j(c - 1) + i$ where $p = 0, \dots, n - c - 1$, $j = 0, \dots, l - d - 1$, and $i = 1, \dots, c - 2$

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + d(c + 1) + jc + i]$$

- ❹ For $v = E + p\alpha + dc + j(c - 1)$ where $p = 0, \dots, n - c - 1$, and $j = 1, \dots, l - d$,

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + d(c + 1) + jc]$$

3.5.4.2 The Hilbert function of J

Set $X_j := t - n\alpha - lj$ and $Y_j := t - (n + 1)\alpha - lj$ as before; we also set

$$Z_j := t - (n + 1)\alpha - lj + d.$$

Note that

$$Y_0 = X_c = t - n\alpha - cl = t - n\alpha - \alpha - l + d = t - (n + 1)\alpha - l + d = Z_1;$$

we will apply this identity to get the third equality below.

Let $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_i are the numbers produced by Algorithm 4. We use the closed form expressions for the λ_i given above in the formula for $H_J(t)$ from Proposition 3.3.9. Section D.4 of Appendix D contains expressions for the partial sums $\sum_i \binom{t-\lambda_i-i+m-2}{m-2}$ as i ranges over the Build, the Reverse Build, and the Pattern. Adding these together with $\binom{t+n\alpha+m-1}{m-1}$, we obtain the following expression for $H_J(t)$.

$$\begin{aligned}
H_J(t) &= \binom{X_1 + m-1}{m-1} - \binom{X_2 + m-2}{m-1} + \binom{X_2 + m-2}{m-1} - \binom{X_c + m-2}{m-1} \\
&+ \sum_{j=2}^{c-1} \binom{X_j + m-1}{m-1} - (c-2) \binom{X_c + m-1}{m-1} + \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m-1}{m-1} \\
&+ \binom{Y_{n-c+1} + m}{m-1} - \binom{Y_{n-1} + m}{m-1} + (c-2) \binom{Y_{n-c+1} + m-1}{m-1} \\
&- \sum_{j=n-c+2}^{n-1} \binom{Y_j + m-1}{m-1} + (c-1) \sum_{j=1}^{n-c+1} \binom{Z_j + m-1}{m-1} + \sum_{j=1}^{n-c+1} \binom{Z_j + m}{m-1} \\
&- (c-2) \sum_{j=2}^{n-c+1} \binom{Z_j + m-1}{m-1} - \sum_{j=2}^{n-c+1} \binom{Z_j + m}{m-1} \\
&- (c-1) \sum_{j=1}^{n-c+1} \binom{Y_j + m-1}{m-1} - \sum_{j=1}^{n-c+1} \binom{Y_j + m}{m-1} + (c-2) \sum_{j=1}^{n-c} \binom{Y_j + m-1}{m-1} \\
&+ \sum_{j=1}^{n-c} \binom{Y_j + m}{m-1} s - \binom{Z_1 + m-2}{m-2} - \binom{Z_1 + m-1}{m-2} \\
&+ \binom{t - n\alpha + m-1}{m-1} \\
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \binom{X_c + m-2}{m-1} - (c-2) \binom{X_c + m-1}{m-1} \\
&- \sum_{j=n-c+2}^n \binom{Y_j + m-1}{m-1} + \binom{Y_{n-c+1} + m}{m-1} + (c-2) \binom{Y_{n-c+1} + m-1}{m-1} \\
&- \sum_{j=1}^{n-c} \binom{Y_j + m-1}{m-1} - (c-1) \binom{Y_{n-c+1} + m-1}{m-1} - \binom{Y_{n-c+1} + m}{m-1} \\
&+ \sum_{j=2}^{n-c+1} \binom{Z_j + m-1}{m-1} + (c-1) \binom{Z_1 + m-1}{m-1} + \binom{Z_1 + m}{m-1} - \binom{Z_1 + m-2}{m-2} \\
&- \binom{Z_1 + m-1}{m-2} + \binom{t - n\alpha + m-1}{m-1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \sum_{j=1}^n \binom{Y_j + m-1}{m-1} + \sum_{j=2}^{n-c+1} \binom{Z_j + m-1}{m-1} \\
&\quad + \binom{Z_1 + m-1}{m-1} - \binom{Z_1 + m-2}{m-1} + \binom{Z_1 + m}{m-1} - \binom{Z_1 + m-2}{m-2} \\
&\quad - \binom{Z_1 + m-1}{m-2} + \binom{t - n\alpha + m-1}{m-1} \\
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \sum_{j=1}^n \binom{Y_j + m-1}{m-1} + \sum_{j=1}^{n-c+1} \binom{Z_j + m-1}{m-1} \\
&\quad + \binom{t - n\alpha + m-1}{m-1}
\end{aligned}$$

3.5.4.3 The Hilbert function of I^n

Using the assumption that $\frac{3}{2}\alpha > \beta > \alpha$ we may rewrite the Hilbert function of I^n from Proposition 3.3.8 in terms of c , d , l , and n . As before, we set $X_j = t - n\alpha - lj$, $Y_j = t - (n+1)\alpha - lj$, and $Z_j := t - (n+1)\alpha - lj + d$. Then

$$H_{I^n}(t) = \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} + \sum_{i=1}^{n-c+1} \binom{Z_i + m-1}{m-1} - \sum_{j=1}^n \binom{Y_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1}.$$

For details of this calculation see Section D.4 of Appendix D.

Since $H_{I^n}(t) = H_J(t)$, we conclude by Lemma 3.3.6 that $J = \text{gin}(I^n)$. That is, the λ_i produced by Algorithm 4 are the invariants of $\text{gin}(I^n)$ when $\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \nmid \alpha$, and $n \geq \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$.

3.5.5 The Case $\frac{3}{2}\alpha > \beta > \alpha$, $n < \lceil \frac{\alpha}{l} \rceil + 1$

3.5.5.1 Formulas for the proposed invariants

In this section we present closed form expressions for the λ_i produced by Algorithm 5. As before, formulas for the λ_i arising from the Build, the Reverse Build, and the Pattern phases of the algorithm are recorded separately. Details of the derivations of these formulas can be found in Section C.5 of Appendix C.

Formulas for λ_i the Build.

- For $v = 0, \dots, l$,

$$\lambda_v = \lambda_0 - 2v.$$

- ② For $v = l(1 + \cdots + q) + (q + 1)j$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_0 - [(2 + \cdots + (q + 1))l + (q + 1)j + j]$$

- ③ For $v = l(1 + \cdots + q) + (q + 1)j - x$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,
 $x = 1, \dots, q$,

$$\lambda_v = \lambda_0 - [(2 + \cdots + (q + 1))l + (q + 1)j - x + j - 1]$$

Formulas for λ_i in the Reverse Build.

- ① For $v = (k - 1) - j$ where $j = 0, \dots, l$,

$$\lambda_v = \lambda_{k-1} + 2j.$$

- ② For $v = (k - 1) - (l(1 + \cdots + q) + (q + 1)j)$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_{k-1} + [(2 + \cdots + (q + 1))l + (q + 1)j + j].$$

However, when $q = n - 2$ and $j = l$, λ_v is in the Pattern, not in the Reverse Build.

- ③ For $v = (k - 1) - (l(1 + \cdots + q) + (q + 1)j - x)$ where $q = 1, \dots, (n - 2)$,
 $j = 1, \dots, l$, $x = 1, \dots, q$,

$$\lambda_v = \lambda_{k-1} + [(2 + \cdots + (q + 1))l + (q + 1)j - x + j - 1].$$

However, when $q = n - 2$, $j = l$, and $x = 1$, λ_v is in the Pattern, not the Reverse Build.

Formulas for λ_i in the Pattern.

Let

$$E := l(1 + \cdots + (n - 1))$$

and

$$B := l(2 + \cdots + n).$$

- ① For $v = E + jn + i$ where $j = 0, \dots, \beta - nl - 1$, $i = 1, \dots, n - 1$,

$$\lambda_v = \lambda_0 - B - [j(n + 1) + i].$$

② For $v = E + jn$ where $j = 1, \dots, \beta - nl - 1$,

$$\lambda_v = \lambda_0 - B - [j(n+1)].$$

3.5.5.2 The Hilbert function of J

As before, we set $X_j = t - n\alpha - lj$ and $Y_j = t - (n+1)\alpha - lj$.

Let $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_i are given by the formulas above and are the invariants produced by Algorithm 5. In this section we will compute the Hilbert function $H_J(t)$ using the expression from Proposition 3.3.9. Section D.5 of Appendix D contains simplifications of the partial sums $\sum_i \binom{t-\lambda_i-i+m-2}{m-2}$ as i ranges over the Build, the Reverse Build, and the Pattern. Adding these together with $\binom{t-n\alpha+m-1}{m-1}$ we have:

$$\begin{aligned} H_J(t) &= \binom{Y_{n-1}+m}{m-1} - \binom{Y_n+m-1}{m-1} + \binom{Y_1+m}{m-1} - \binom{Y_{n-1}+m}{m-1} \\ &\quad + (n-2) \binom{Y_1+m-1}{m-1} - \sum_{j=2}^{n-1} \binom{Y_j+m-1}{m-1} + (n-1) \binom{X_n+m-1}{m-1} \\ &\quad - (n-1) \binom{Y_1+m-1}{m-1} + \binom{X_n+m-1}{m-1} - \binom{Y_1+m}{m-1} + \binom{X_1+m-1}{m-1} \\ &\quad - \binom{X_2+m-2}{m-1} + \binom{X_2+m-2}{m-1} - \binom{X_n+m-2}{m-1} - \binom{X_n+m-2}{m-2} \\ &\quad + \sum_{j=2}^{n-1} \binom{X_j+m-1}{m-1} - (n-2) \binom{X_n+m-1}{m-1} - \binom{X_n+m-1}{m-2} \\ &\quad + \binom{t-n\alpha+m-1}{m-1} \\ &= - \sum_{j=2}^n \binom{Y_j+m-1}{m-1} + \binom{Y_1+m}{m-1} - \binom{Y_1+m-1}{m-1} - \binom{Y_1+m}{m-1} \\ &\quad + \sum_{j=1}^{n-1} \binom{X_j+m-1}{m-1} + \binom{X_n+m-1}{m-1} + \binom{X_n+m-1}{m-1} - \binom{X_n+m-2}{m-1} \\ &\quad - \binom{X_n+m-2}{m-2} + \binom{t-n\alpha+m-1}{m-1} \\ &= - \sum_{j=1}^n \binom{Y_j+m-1}{m-1} + \sum_{j=1}^{n-1} \binom{X_j+m-1}{m-1} + 2 \binom{X_n+m-1}{m-1} \\ &\quad - \binom{X_n+m-1}{m-1} + \binom{t-n\alpha+m-1}{m-1} \end{aligned}$$

$$= \sum_{j=1}^n \binom{X_j + m - 1}{m - 1} - \sum_{j=1}^n \binom{Y_j + m - 1}{m - 1} + \binom{t - n\alpha + m - 1}{m - 1}.$$

3.5.5.3 The Hilbert function of I^n

By Proposition 3.3.8, the Hilbert function of I^n is equal to

$$H_{I^n}(t) = \sum_{j=1}^n \left(\binom{t - X_j + (m-1)}{(m-1)} - \binom{t - Y_j + (m-1)}{(m-1)} \right) + \binom{t - n\alpha + m - 1}{m - 1}.$$

Since $H_J(t) = H_{I^n}(t)$, we conclude by Lemma 3.3.6 that $J = \text{gin}(I^n)$. That is, the λ_i produced by Algorithm 5 are the invariants of $\text{gin}(I^n)$ when $\frac{3}{2}\alpha > \beta > \alpha$ and $2 \leq n < \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$.

3.5.6 The Case $\alpha = \beta$, $n \geq 1$.

3.5.6.1 Formulas for the proposed invariants

In this section we write a closed-form expression for the invariants produced by Algorithm 6. See Section C.6 of Appendix C for details of how this follows from the algorithm. For $v = qn + j$ where $q = 0, \dots, \alpha - 1$ and $j = 0, \dots, n - 1$,

$$\lambda_v = (n + 1)\alpha - 1 - q(n + 1) - j.$$

3.5.6.2 The Hilbert function of J

Consider $J = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0})$ where the λ_i are given by the formula from Section 3.5.6.1 and are the invariants produced by Algorithm 6. By Proposition 3.3.9, the Hilbert function of J is

$$\begin{aligned} H_J(t) &= \sum_{i=0}^{n\alpha-1} \binom{t - \lambda_{qn+j} - (qn + j) + (m-2)}{(m-2)} + \binom{t - n\alpha + (m-1)}{(m-1)} \\ &= \sum_{q=0}^{\alpha-1} \sum_{j=0}^{n-1} \binom{t - (n+1)\alpha - 1 - q + m - 2}{m - 2} + \binom{t - n\alpha + (m-1)}{(m-1)} \\ &= n \left[\binom{t - (n+1)\alpha + m - 1 + \alpha}{m - 1} - \binom{t - (n+1)\alpha + m - 1}{m - 1} \right] \end{aligned}$$

$$\begin{aligned}
& + \binom{t - n\alpha + (m-1)}{(m-1)} \\
= & n \left[\binom{t - n\alpha + m - 1}{m - 1} - \binom{t - (n+1)\alpha + m - 1}{m - 1} \right] + \binom{t - n\alpha + (m-1)}{(m-1)}.
\end{aligned}$$

3.5.6.3 Rewriting the Hilbert function of I^n

Under the assumption that $\alpha = \beta$, the Hilbert function of I^n from Proposition 3.3.8 is

$$\begin{aligned}
H_{I^n}(t) &= \sum_{j=1}^n \left[\binom{t - \alpha(n-j) - \alpha j + m - 1}{m - 1} - \binom{t - \alpha j - \alpha n - \alpha + j\alpha + m - 1}{m - 1} \right] \\
& \quad + \binom{t - n\alpha + (m-1)}{(m-1)} \\
&= n \left[\binom{t - n\alpha + m - 1}{m - 1} - \binom{t - \alpha(n+1) + m - 1}{m - 1} \right] + \binom{t - n\alpha + (m-1)}{(m-1)}.
\end{aligned}$$

Since $H_J(t) = H_{I^n}(t)$, Lemma 3.3.6 implies that $J = \text{gin}(I^n)$. Thus, the λ_i produced by Algorithm 6 are the invariants of $\text{gin}(I^n)$ when $\alpha = \beta$.

CHAPTER IV

The Limiting Shape of the Generic Initial System of a Complete Intersection

4.1 Introduction

The asymptotic behavior of algebraic objects has been a fruitful research trend of the past twenty years, motivated by the philosophy that there is often a uniformity achieved in the limit that is hidden when studying individual objects. Significant work along these lines includes: Huneke's uniform Artin-Rees lemma [Hun92]; Siu's work on the deformation-invariance of plurigena [Siu98]; Ein, Lazarsfeld, and Smith's introduction of graded systems and asymptotic multiplier ideals [ELS01]; and, most recently, Eisenbud and Schreyer's proof of the Boij-Söderberg conjectures [ES09]. We study the asymptotic behavior of generic initial ideals - a rich research topic in their own right - using techniques similar to those in [ELS01].

Consider a homogeneous ideal I in a polynomial ring $R = K[x_1, \dots, x_m]$ with the standard grading and some fixed term order. The generic initial ideal of I , $\text{gin}(I)$, is a coordinate-independent version of the initial ideal; as a monomial ideal, there is a Newton polytope $P_{\text{gin}(I)}$ of \mathbb{R}^m associated to it. In this paper we introduce the *generic initial system* $\{\text{gin}(I^n)\}_n$ of I and define the *limiting shape* of this system to be the limit of the polytopes $\frac{1}{n}P_{\text{gin}(I^n)}$. We study the case where I is a complete intersection with minimal generators of degrees d_1, d_2, \dots, d_r . Our main result is the following theorem describing the limiting shape of the reverse lexicographic generic initial system of such an ideal.

Theorem 4.1.1. *Let I be a complete intersection of type (d_1, \dots, d_r) in $K[x_1, \dots, x_r]$ where $d_1 \leq \dots \leq d_r$ and K is a field of characteristic 0. The limiting shape of the reverse lexicographic generic initial system $\{\text{gin}(I^n)\}_n$ is the closure*

of the complement in $\mathbb{R}_{\geq 0}^r$ of the r -simplex with vertices at the points

$$(0, 0, \dots, 0), (d_1, 0, \dots, 0), (0, d_2, 0, \dots, 0), \dots, (0, \dots, 0, d_r).$$

This result easily extends to the case where I is an r -complete intersection in $K[x_1, \dots, x_m]$ and shows that the asymptotic behavior of the generic initial system of a complete intersection is as nice as one could hope. The simplicity of the limiting shape contrasts sharply with the structure of the individual ideals $\text{gin}(I^n)$.

First, it is not clear that the ideals $\text{gin}(I^n)$ depend only on the degrees of the generators of I , even when $n = 1$. Consider the case where $R = k[x_1, x_2, x_3]$ and I is a triple complete intersection with minimal generators of degrees d_1, d_2 and d_3 . Cimpoeaş [Cim06] has computed the generators of $\text{gin}(I)$ under the additional condition that I be strongly Lefschetz; in this case $\text{gin}(I)$ is almost reverse lexicographic and depends only on d_1, d_2 , and d_3 . Cho and Park [CP08] prove that an ideal J in $R = k[x_1, x_2, x_3]$ is strongly Lefschetz if and only if $\text{gin}(J)$ is almost reverse lexicographic. Therefore, in this case, showing that $\text{gin}(I)$ depends only on the degrees of the generators of I is equivalent to showing that all triple complete intersections are strongly Lefschetz; the latter question has been well-studied and seems difficult to prove in general (see [MN11]).

Second, the generators of the ideals $\text{gin}(I^n)$ are complicated in even the simplest cases and they depend heavily on the relative magnitudes of d_1, \dots, d_r . For explicit descriptions of the generators of $\text{gin}(I^n)$ see [May12c] for the case where $r = 2$ and [Cim06] for the case where $n = 1, r = 3$, and I is strongly Lefschetz.

The generic initial system of a complete intersection, then, follows the philosophy guiding the study of asymptotic objects: the ideals $\text{gin}(I^n)$ are complex but uniformity is gained in the limit. We anticipate that this will hold for other generic initial systems; further research is required to determine conditions on the limiting shapes of such systems.

4.2 Preliminaries

In this section we will introduce some notation, definitions, and preliminary results related to generic initial ideals and systems of ideals. Throughout $R = K[x_1, \dots, x_m]$ is a polynomial ring over a field K of characteristic 0 with the standard grading and some fixed term order $>$ with $x_1 > \dots > x_m$. A monomial $x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}$ of R may also be written in multi-index notation as x^J where $J = (j_1, \dots, j_m)$.

4.2.1 Generic Initial Ideals

An element $g = (g_{ij}) \in \mathrm{GL}_m(K)$ acts on R and sends any homogeneous element $f(x_1, \dots, x_n)$ to the homogeneous element

$$f(g(x_1), \dots, g(x_n))$$

where $g(x_i) = \sum_{j=1}^m g_{ij}x_j$. If $g(I) = I$ for every upper triangular matrix g then we say that I is *Borel-fixed*. Borel-fixed ideals are *strongly stable* when K is of characteristic 0; that is, for every monomial m in the ideal such that x_i divides m , the monomials $\frac{x_j^m}{x_i}$ for all $j < i$ are also in the ideal. This property makes such ideals particularly nice to work with.

To any homogeneous ideal I of R we can associate a Borel-fixed monomial ideal $\mathrm{gin}_>(I)$ which can be thought of as a coordinate-independent version of the initial ideal. Its existence is guaranteed by Galligo's theorem (also see [Gre98, Theorem 1.27]).

Theorem (Galligo's theorem [Gal74]). *For any multiplicative monomial order $>$ on R and any homogeneous ideal $I \subset R$, there exists a Zariski open subset $U \subset \mathrm{GL}_m(K)$ such that $\mathrm{In}_>(g(I))$ is constant and Borel-fixed for all $g \in U$.*

Definition 4.2.1. The **generic initial ideal of I** , denoted $\mathrm{gin}_>(I)$, is defined to be $\mathrm{In}_>(g(I))$ where $g \in U$ is as in Galligo's theorem.

The *reverse lexicographic order* $>$ is a total ordering on the monomials of R defined by

1. if $|I| = |J|$ then $x^I > x^J$ if there is a k such that $i_m = j_m$ for all $m > k$ and $i_k < j_k$; and
2. if $|I| > |J|$ then $x^I > x^J$.

For example, $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$. From this point on, $\mathrm{gin}(I) = \mathrm{gin}_>(I)$ will denote the generic initial ideal with respect to the reverse lexicographic order.

The following theorem records two of the properties shared by $\mathrm{gin}(I)$ and I . The first statement is a consequence of the fact that Hilbert functions are invariant under making changes of coordinates and taking initial ideals. The second statement is a result of Bayer and Stillman; for a simple proof see Corollary 2.8 of [AM07].

Theorem 4.2.2. *For any homogeneous ideal I in R :*

1. the Hilbert functions of I and $\text{gin}(I)$ are equal; and
2. $\text{depth}(R/I) = \text{depth}(R/\text{gin}(I))$.

4.2.2 The Generic Initial System

The focus of this paper is on understanding the behavior of the generic initial ideals of the powers of a fixed ideal.

Definition 4.2.3. The **generic initial system** of a homogeneous ideal I is the collection of ideals J_\bullet such that $J_i = \text{gin}(I^i)$.

Definition 4.2.4 ([ELS01]). A **graded system of ideals** is a collection of ideals $J_\bullet = \{J_i\}_{i=1}^\infty$ such that

$$J_i \cdot J_j \subseteq J_{i+j} \quad \text{for all } i, j \geq 1.$$

Lemma 4.2.5. *The generic initial system is a graded system of monomial ideals.*

Proof. By definition, $\text{gin}(I^i)$ is a monomial ideal. We need to show that for all $i, j \geq 1$, $\text{gin}(I^i) \cdot \text{gin}(I^j) \subseteq \text{gin}(I^{i+j})$. For any $l \geq 1$, let U_l be the Zariski open subset of GL_m such that $\text{gin}(I^l) = \text{In}(g \cdot (I^l))$ for all g in U_l . Since U_i, U_j , and U_{i+j} are Zariski open they have a nonempty intersection; fix some $g \in U_i \cap U_j \cap U_{i+j}$. Given monomials $f' \in \text{gin}(I^i) = \text{In}(g(I^i))$ and $h' \in \text{gin}(I^j) = \text{In}(g(I^j))$, suppose that $f' = \text{In}(g(f))$ and $h' = \text{In}(g(h))$ for $f \in I^i$ and $h \in I^j$. Now

$$f' \cdot h' = \text{In}(g(f))\text{In}(g(h)) = \text{In}(g(f) \cdot g(h)) = \text{In}(g(f \cdot h)) \in \text{In}(g(I^{i+j}))$$

since $f \cdot h \in I^{i+j}$. Thus $f' \cdot h' \in \text{gin}(I^{i+j})$ as desired. \square

4.2.3 Volume and multiplier ideals

In this subsection we will discuss the geometric interpretations of the volume and of the asymptotic multiplier ideal associated to a graded system of monomial ideals.

Definition 4.2.6 ([ELS03]). Let \mathbf{a}_\bullet be a graded system of zero-dimensional ideals in $R = K[x_1, \dots, x_m]$. The **volume of \mathbf{a}_\bullet** is

$$\text{vol}(\mathbf{a}_\bullet) := \limsup_{n \rightarrow \infty} \frac{m! \cdot \text{length}(R/\mathbf{a}_n)}{n^m}.$$

Let J be a monomial ideal of R . We may regard J as a subset Λ of the lattice points \mathbb{N}^m consisting of the points λ such that $x^\lambda \in J$. The *Newton polytope* P_J of J is the convex hull of Λ regarded as a subset of \mathbb{R}^m . Scaling the polytope P_J by a factor of r gives another polytope which we will denote rP_J .

If \mathfrak{a}_\bullet is a graded system of monomial ideals in R then the polytopes of $\{\frac{1}{q}P_{\mathfrak{a}_q}\}_q$ are nested: $\frac{1}{p}P_{I_p} \subseteq \frac{1}{q}P_{I_q}$ whenever p divides q .¹ The *limiting shape* P of \mathfrak{a}_\bullet is the limit of the polytopes in this set:

$$P = \bigcup_{q \in \mathbb{N}^*} \frac{1}{q}P_{\mathfrak{a}_q}.$$

Under the additional assumption that the ideals of \mathfrak{a}_\bullet are zero-dimensional, the closure of each set $\mathbb{R}_{\geq 0}^m \setminus P_{\mathfrak{a}_q}$ in \mathbb{R}^m is compact. This closure is denoted by Q_q and we let

$$Q = \bigcap_{q \in \mathbb{N}^*} \frac{1}{q}Q_q.$$

Note that finding the volume of Q , $\text{vol}(Q)$, is the same as finding the volume underneath of the limiting shape P . It turns out that this geometric volume is closely tied to the algebraic volume of \mathfrak{a}_\bullet .

Proposition 4.2.7 ([Mus02]). *If \mathfrak{a}_\bullet is a graded system of zero-dimensional monomial ideals in $R = K[x_1, \dots, x_m]$ and Q is as defined above,*

$$\text{vol}(\mathfrak{a}_\bullet) = m! \text{vol}(Q).$$

Proof. This is an immediate consequence of Theorem 1.7 and Lemma 2.13 of [Mus02]. □

Although multiplier ideals are an important tool in algebraic geometry, they are usually difficult to compute explicitly. Since we will only be concerned with calculating multiplier ideals of monomial ideals we will give a definition only for this special case. See [BL04] for an introduction to multiplier ideals or [Laz04] for a more general treatment.

Definition 4.2.8 ([How01]). Let $J \subset R$ be a monomial ideal and let P_J be its Newton polytope. The **multiplier ideal of J with coefficient c** , denoted $\mathcal{J}(c \cdot J)$, is the monomial ideal

$$\mathcal{J}(c \cdot J) := \{x^\lambda : \lambda + \mathbf{1} \in \text{Int}(cP_J) \cap \mathbb{N}^m\}$$

¹This follows from the fact that multiplication of monomial ideals corresponds to adding Newton polytopes: if $q = ps$ then $\mathfrak{a}_p^s \subseteq \mathfrak{a}_{ps} = \mathfrak{a}_q$ implies that $sP_{\mathfrak{a}_p} \subseteq P_{\mathfrak{a}_q}$ so, scaling by $\frac{1}{q}$, $\frac{1}{p}P_{\mathfrak{a}_p} \subseteq \frac{1}{q}P_{\mathfrak{a}_q}$ (also see [Mus02]).

where $\text{Int}(cP_J)$ denotes the interior of the polytope cP_J .

The asymptotic definition requires the following lemma.

Lemma 4.2.9 (Lemma 1.3 of [ELS01]). *Let J_\bullet be a graded system of ideals and fix a rational number $c > 0$. Then for $p \gg 0$ the multiplier ideals $\mathcal{J}(\frac{c}{p} \cdot J_p)$ coincide.*

Definition 4.2.10. Let $J_\bullet = \{J_k\}_{k \in \mathbb{N}}$ be a graded system of ideals on R . Given $c > 0$ the **asymptotic multiplier ideal of J_\bullet with coefficient c** is

$$\mathcal{J}(c \cdot J_\bullet) = \mathcal{J}\left(\frac{c}{p} \cdot J_p\right)$$

for any sufficiently large p (guaranteed by Lemma 4.2.9). When J_\bullet is a graded system of monomial ideals with limiting shape P ,

$$\mathcal{J}(c \cdot J_\bullet) = \{x^\lambda : \lambda + \mathbf{1} \in \text{Int}\left(\frac{c}{p}P_{J_p}\right) \cap \mathbb{N}^m\}$$

for $p \gg 0$.

Therefore, the volume and the asymptotic multiplier ideal of a graded system of monomial ideals are entirely determined by the limiting shape P of the system.

4.3 The Generic Initial System of a Complete Intersection

A homogeneous ideal $I = (f_1, \dots, f_r)$ is a *complete intersection of type (d_1, \dots, d_r)* if f_1, \dots, f_r is a regular sequence on R and $\deg(f_i) = d_i$. Since the f_i are homogeneous, any permutation of f_1, \dots, f_r is still a regular sequence; therefore, we may assume that $d_1 \leq \dots \leq d_r$.

The goal of this section is to describe the reverse lexicographic generic initial ideals $\text{gin}(I^n)$ for such a complete intersection. The following result tells us that no matter how many variables the ambient ring R has the minimal generators of these generic initial ideals only involve the first r variables.

Lemma 4.3.1. *Let I be a type (d_1, \dots, d_r) complete intersection in $R = K[x_1, \dots, x_m]$ (so that $m \geq r$) and let A_n denote the set of minimal generators of $\text{gin}(I^n)$. Then the monomials of A_n are contained in $K[x_1, \dots, x_r]$ and one of the largest degree elements of A_n is of the form $x_r^{p_r(n)}$ for some $p_r(n) \geq 1$.*

This lemma is a consequence of the following result of Herzog and Srinivasan (see Lemma 3.1 of [HS98]) which relates the depth and dimension of a Borel-fixed monomial ideal to the variables appearing in its minimal generating set.

Proposition 4.3.2. *Let J be a Borel-fixed monomial ideal in R and define*

$$D(J) := \max\{t | x_t^j \in J \text{ for some positive integer } j\}$$

and

$$M(J) := \max\{t | x_t \text{ appears in some minimal generator of } J\}.$$

Then

1. $\dim(R/J) = m - D(J)$; and
2. $\text{depth}(R/J) = m - M(J)$.

Note that when I is a complete intersection of type (d_1, \dots, d_r) in R ,

$$\dim(R/I^n) = \text{depth}(R/I^n) = m - r$$

for all $n \geq 1$. It then follows by Theorem 4.2.2 that the depth and dimension of $R/\text{gin}(I^n)$ are equal to $m - r$ as well.

Proof of Lemma 4.3.1. By Proposition 4.3.2,

$$D(\text{gin}(I^n)) = m - \dim(R/\text{gin}(I^n)) = r = m - \text{depth}(R/\text{gin}(I^n)) = M(\text{gin}(I^n))$$

This means that the minimal generating set A_n of $\text{gin}(I^n)$ is contained in $S = K[x_1, \dots, x_r]$ and that A_n contains a power of x_r , say $x_r^{p_r(n)}$. The fact that $\text{gin}(I^n)$ is strongly stable means that we can replace each x_r in $x_r^{p_r(n)}$ with any variable x_1, \dots, x_r and still get an element of $\text{gin}(I^n)$. Therefore, any monomial $x^J \in S$ of degree $p_r(n)$ is also contained in $\text{gin}(I^n)$. Now it is clear that $A_n \subset S$ cannot contain any element of degree greater than $p_r(n)$. \square

Given the fact that the generators of $\text{gin}(I^n)$ only involve the first r variables it is natural to wonder whether we can restrict our attention to the case where $R = K[x_1, \dots, x_m]$ (that is, $r = m$). The following result tells us that the answer is essentially ‘yes’.

Proposition 4.3.3. *Given a type (d_1, \dots, d_r) complete intersection I in $R = K[x_1, \dots, x_m]$, so that $m \geq r$, there exists a type (d_1, \dots, d_r) complete intersection L of $S = K[x_1, \dots, x_r]$ such that the minimal generators of $\text{gin}(L^n)$ are the same as the minimal generators of $\text{gin}(I^n)$ for all $n \geq 1$.*

Proof. We will proceed by induction on m . The statement is trivial in the case where $m = r$. Assume that it holds for all $m \leq M$ such that $m \geq r$ and let I be a complete intersection of type (d_1, \dots, d_r) in $K[x_1, \dots, x_{M+1}]$. If $h = a_1x_1 + \dots + a_{M+1}x_{M+1}$ is a general linear form in $K[x_1, \dots, x_{M+1}]$, consider the ring isomorphism

$$\phi : \frac{K[x_1, \dots, x_{M+1}]}{(h)} \rightarrow K[x_1, \dots, x_M]$$

given by sending x_i to x_i for $i = 1, \dots, M$ and x_{M+1} to $-\sum_{i=1}^M \frac{a_i}{a_{M+1}}x_i$. If J is an ideal of $K[x_1, \dots, x_{M+1}]$, let $J_H = \phi((J+(h))/(h))$. Since h is a general linear form, the ideal I_H is a complete intersection of type (d_1, \dots, d_r) in $K[x_1, \dots, x_M]$.

If

$$\psi : K[x_1, \dots, x_{M+1}] \rightarrow K[x_1, \dots, x_M]$$

is the map given by sending x_i to x_i for $i = 1, \dots, M$ and x_{M+1} to 0, we have the following well-known relation between the generic initial ideal of any ideal J of $K[x_1, \dots, x_{M+1}]$ and the generic initial ideal of J_H (see [BS87a] and Corollary 2.5 of [Gre98]):

$$\text{gin}(J_H) = \psi(\text{gin}(J)).$$

By Lemma 3.1, no minimal generator of $\text{gin}(I^n)$ involves the variable x_{M+1} and so

$$\text{gin}((I_H)^n) = \text{gin}((I^n)_H) = \psi(\text{gin}(I^n))$$

has the same generators as $\text{gin}(I^n)$. By the inductive assumption, there exists a type (d_1, \dots, d_r) complete intersection $L \subset K[x_1, \dots, x_r]$ such that $\text{gin}((I_H)^n)$ - and thus $\text{gin}(I^n)$ - has the same minimal generators as $\text{gin}(L^n)$ for all $n \geq 1$. Therefore, the statement holds for all $m \geq r$. \square

Fix an r -complete intersection I of type (d_1, \dots, d_r) . Let $p_i(n)$ denote the smallest power of x_i contained inside of $\text{gin}(I^n)$ so that $(x_1^{p_1(n)}, x_2^{p_2(n)}, \dots, x_r^{p_r(n)})$ is the largest ideal generated by variable powers that is contained inside of $\text{gin}(I^n)$.

Lemma 4.3.4. $p_1(n) = nd_1$ for all $n \geq 1$.

Proof. Since $\text{gin}(I^n)$ is strongly stable there is no element of $\text{gin}(I^n)$ of degree smaller than $p_1(n)$: if there was such a monomial x^J we could replace each x_2, \dots, x_m in x^J with x_1 and get a power of x_1 smaller than $p_1(n)$ contained in $\text{gin}(I^n)$. If f_1 is the generator of I of degree d_1 , then the smallest degree elements of I^n , and thus of $\text{gin}(I^n)$, are of degree nd_1 . Therefore, $p_1(n) = nd_1$. \square

Lemma 4.3.5. $p_1(n) \leq p_2(n) \leq \cdots \leq p_r(n)$ for all $n \geq 1$.

Proof. Since $\text{gin}(I^n)$ is strongly stable we can replace each x_i in $x_i^{p_i(n)} \in \text{gin}(I^n)$ with x_{i-1} and stay inside of $\text{gin}(I^n)$: $x_{i-1}^{p_i(n)} \in \text{gin}(I^n)$. $x_{i-1}^{p_{i-1}(n)}$ is a minimal generator of $\text{gin}(I^n)$ so $p_{i-1}(n) \leq p_i(n)$ for each $i = 2, \dots, r$. \square

To determine the value of $p_r(n)$ we will compare the *Betti numbers* of I^n and $\text{gin}(I^n)$. Let

$$0 \rightarrow F_m \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow J \rightarrow 0$$

be the unique minimal free resolution of an ideal J . The graded Betti numbers of J , $\beta_{i,j}(J)$, are defined by $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(J)}$. A *consecutive cancellation* takes a sequence $\{\beta_{i,j}\}$ to a new sequence by replacing $\beta_{i,j}$ by $\beta_{i,j} - 1$ and $\beta_{i+1,j}$ by $\beta_{i+1,j} - 1$. The ‘Cancellation Principle’ says that the graded Betti numbers $\beta_{i,j}(I^n)$ of I^n can be obtained from the graded Betti numbers of $\beta_{i,j}(\text{gin}(I^n))$ by making a series of consecutive cancellations (see Corollary 1.21 of [Gre98]).

Lemma 4.3.6. $p_r(n) = d_1 + \cdots + d_{r-1} + nd_r - r + 1$ for all $n \geq 1$.

Proof. Let $\text{gin}(I^n)$ have minimal generators x^{J_1}, \dots, x^{J_N} . By a theorem of Eliahou and Kevaire (see [EK90] or Theorem 1.31 of [Gre98]), the initial module of p th syzygies of $\text{gin}(I^n)$ is minimally generated by $x_{i_p} \otimes x_{i_{p-1}} \otimes \cdots \otimes x_{i_1} \otimes x^{J_j}$ where $1 \leq j \leq N$, $i_p < i_{p-1} < \cdots < i_1$, and i_1 is less than the index of the largest variable appearing in J_j .

Since $x_r^{p_r(n)}$ is a generator of $\text{gin}(I^n)$, $x_1 \otimes x_2 \otimes \cdots \otimes x_{r-1} \otimes x_r^{p_r(n)}$ is in the initial module of $(r-1)$ st syzygies and $\beta_{r-1, p_r(n)+(r-1)}(\text{gin}(I^n)) \neq 0$. Further, since $\text{gin}(I^n)$ is Borel-fixed, one of the largest degree elements in its minimal generating set is $x_r^{p_r(n)}$. Thus, by Eliahou-Kervaire, the largest possible degree of a minimal generator of the initial module of $(r-2)$ nd syzygies is $p_r(n) + (r-2)$; this means that $\beta_{r-2, p_r(n)+(r-1)}(\text{gin}(I^n)) = 0$. Therefore, no consecutive cancellation can occur between $\beta_{r-2, p_r(n)+(r-1)}(\text{gin}(I^n))$ and $\beta_{r-1, p_r(n)+(r-1)}(\text{gin}(I^n))$ so, by the Cancellation Principle, $\beta_{r-1, p_r(n)+(r-1)}(I^n) \geq 1$ and $\beta_{r-1, s}(I^n) = 0$ for all $s > p_r(n) + (r-1)$.

By the main result of [GT05], $\beta_{r-1, d_1+\cdots+d_{r-1}+nd_r}(I^n) \geq 1$ and $\beta_{r-1, s}(I^n) = 0$ for all $s > d_1 + \cdots + d_{r-1} + nd_r$. Therefore,

$$d_1 + \cdots + d_{r-1} + nd_r = p_r(n) + r - 1$$

and $p_r(n) = d_1 + d_2 + \cdots + d_{r-1} + nd_r - r + 1$ as claimed. \square

4.4 The Limiting Shape of the Generic Initial System of a Complete Intersection

In this section we prove Theorem 4.1.1 describing the asymptotic behavior of the reverse lexicographic generic initial system of a complete intersection. Throughout I will be a complete intersection of type (d_1, \dots, d_r) and $p_i(n)$ will be the minimal power of x_i contained in $\text{gin}(I^n)$ as described in the previous section. The following result bounding the growth of the $p_i(n)$ will be used in the proof of the main theorem.

Lemma 4.4.1. *Suppose that I is a type (d_1, \dots, d_r) complete intersection in $S = K[x_1, \dots, x_r]$. For all $i \leq r$,*

$$\lim_{n \rightarrow \infty} \frac{p_i(n)}{n} \leq d_i.$$

Proof. First note that this limit exists. By Lemma 1.4 of [Mus02], if $\{\alpha_n\}_{n \in \mathbb{N}^*}$ is a sequence of positive real numbers such that $\alpha_{n_1+n_2} \leq \alpha_{n_1} + \alpha_{n_2}$ for all $n_1, n_2 \in \mathbb{N}^*$ then $\lim_{n \rightarrow \infty} \frac{\alpha_n}{n}$ exists. Since the generic initial system is a graded system of ideals,

$$\text{gin}(I^{n_1}) \cdot \text{gin}(I^{n_2}) \subseteq \text{gin}(I^{n_1+n_2})$$

for all $n_1, n_2 \in \mathbb{N}^*$. In particular,

$$x_i^{p_i(n_1)+p_i(n_2)} = x_i^{p_i(n_1)} \cdot x_i^{p_i(n_2)} \in \text{gin}(I^{n_1+n_2})$$

so $p_i(n_1 + n_2) \leq p_i(n_1) + p_i(n_2)$. Thus, the limit exists.

We will proceed by induction on r . If $r = 2$, Lemma 4.3.4 gives

$$\lim_{n \rightarrow \infty} \frac{p_1(n)}{n} = \lim_{n \rightarrow \infty} \frac{nd_1}{n} = d_1$$

and Lemma 4.3.6 gives

$$\lim_{n \rightarrow \infty} \frac{p_2(n)}{n} = \lim_{n \rightarrow \infty} \frac{d_1 + nd_2 - 1}{n} = d_2$$

so the result holds in this case.

Assume that the statement holds for all $r \leq R$ and that I is a complete intersection of type (d_1, \dots, d_{R+1}) in $K[x_1, \dots, x_{R+1}]$ generated by homogeneous polynomials f_1, \dots, f_{R+1} where $\deg(f_i) = d_i$. Consider the ideal $J \subseteq K[x_1, \dots, x_{R+1}]$ generated by f_1, \dots, f_R and note that $J \subseteq I$ implies $\text{gin}(J^n) \subseteq \text{gin}(I^n)$. If $p'_i(n)$ denotes the minimal power of x_i contained in $\text{gin}(J^n)$, then $p'_i(n) \geq p_i(n)$ for all $i = 1, \dots, R$.

By Proposition 4.3.3 there exists an ideal $L \subset R[x_1, \dots, x_R]$ such that the minimal generators of $\text{gin}(L^n)$ are the same as the minimal generators of $\text{gin}(J^n)$ for all $n \geq 1$; in particular, the minimal power of x_i contained in $\text{gin}(L^n)$ is equal to the minimal power $p'_i(n)$ contained in $\text{gin}(J^n)$. The statement of the lemma holds for L by the inductive assumption so

$$\lim_{n \rightarrow \infty} \frac{p_i(n)}{n} \leq \lim_{n \rightarrow \infty} \frac{p'_i(n)}{n} \leq d_i$$

for all $i \leq R$. For $i = R + 1$ we have

$$\lim_{n \rightarrow \infty} \frac{p_{R+1}(n)}{n} = \lim_{n \rightarrow \infty} \frac{d_1 + \dots + d_R + nd_{R+1} - (R + 1) + 1}{n} = d_{R+1}$$

by Lemma 4.3.6. Therefore, the claim holds for $r = R + 1$ and thus for all r . \square

Suppose that T is an r -simplex in \mathbb{R}^r with vertices at the origin and at $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$. Recall that the volume of T is equal to

$$\text{vol}(T) = \frac{\det(A)}{r!}$$

where A is a matrix with columns $\vec{v}_1, \dots, \vec{v}_r$ arranged so that A has a positive determinant (see, for example, Chapter 5 of [Lax96]).

We are now prepared to prove that the limiting shape of the generic initial system \mathbf{a}_\bullet of a type (d_1, \dots, d_r) complete intersection in $K[x_1, \dots, x_r]$ is the closure of the complement in $\mathbb{R}_{\geq 0}^r$ of the r -simplex with vertices at the points

$$(0, 0, \dots, 0), (d_1, 0, \dots, 0), (0, d_2, 0, \dots, 0), \dots, (0, \dots, 0, d_r).$$

As in Section 4.2.3, the limiting shape is denoted by P and the closure of its complement in $\mathbb{R}_{\geq 0}^r$ is denoted by Q .

Proof of Theorem 4.1.1. First note that, since $x_i^{p_i(n)} \in \text{gin}(I^n)$ for $i = 1, \dots, r$, there is a vertex of the limiting shape lying on each coordinate axis of \mathbb{R}^r . The vertex on the x_i axis has nonzero coordinate $\lim_{n \rightarrow \infty} \frac{p_i(n)}{n}$ which is at most d_i by Lemma 4.4.1. Under these conditions, the maximum possible volume beneath the convex limiting shape P is attained if and only if its boundary is defined by the coordinate planes and the hyperplane through the points $(d_1, 0, \dots, 0), (0, d_2, 0, \dots, 0), \dots, (0, \dots, 0, d_r)$. In

this case, Q is the r -simplex described in the theorem and

$$\text{vol}(Q) = \frac{d_1 \cdot d_2 \cdots d_r}{r!}.$$

To show that the maximum volume is attained and thus that the limiting shape is as claimed we will compute the algebraic volume of the graded system. It follows from Exercise 12.3 of [Eis04] and the fact that $\text{length}(I^n/I^{n+1}) = \text{length}(R/I^{n+1}) - \text{length}(R/I^n)$ that $\text{length}(R/I^n) = \binom{n+r-1}{r} d_1 \cdots d_r$. Since the lengths of R/I^n and $R/\text{gin}(I^n)$ are equal, the volume of the generic initial system \mathbf{a}_\bullet of I is

$$\begin{aligned} \text{vol}(\mathbf{a}_\bullet) &= \limsup_{n \rightarrow \infty} \frac{r! \cdot \text{length}(R/\text{gin}(I^n))}{n^r} \\ &= \limsup_{n \rightarrow \infty} r! \cdot \binom{n+r-1}{r} \frac{d_1 \cdots d_r}{n^r} \\ &= \limsup_{n \rightarrow \infty} r! \cdot \frac{(n+r-1)(n+r-2) \cdots (n)}{r!} \frac{d_1 \cdots d_r}{n^r} \\ &= d_1 \cdots d_r \end{aligned}$$

By Proposition 4.2.7, $\text{vol}(\mathbf{a}_\bullet) = \text{vol}(Q)r!$ so the maximum possible geometric volume is attained. \square

By Proposition 4.3.3, if I is a type (d_1, \dots, d_r) complete intersection in any polynomial ring $K[x_1, \dots, x_m]$, there exists a complete intersection $L \subset K[x_1, \dots, x_r]$ of the same type such that the minimal generators of $\text{gin}(L^n)$ are the same as the minimal generators of $\text{gin}(I^n)$. Thus, the above theorem gives the limiting shape P of *any* complete intersection. In particular, P consists of the points $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}_{\geq 0}^m$ such that

$$1 \leq \frac{\lambda_1}{d_1} + \cdots + \frac{\lambda_r}{d_r}.$$

Note that there are no conditions on the last $m - r$ coordinates.

From Subsection 4.2.3, knowing the limiting shape of a graded system is enough to compute its asymptotic multiplier ideal.

Corollary 4.4.2. *Let I be a complete intersection of type (d_1, \dots, d_r) in $K[x_1, \dots, x_m]$ and let \mathbf{a}_\bullet denote the generic initial system of I ; that is, $\mathbf{a}_n = \text{gin}(I^n)$. Then the asymptotic multiplier ideal of \mathbf{a}_\bullet is*

$$\mathcal{J}(\mathbf{a}_\bullet) = \{x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m} \mid (c_1, \dots, c_m) \in \mathbb{N}^m \text{ and } \frac{(c_1+1)}{d_1} + \cdots + \frac{(c_r+1)}{d_r} > 1\}$$

Proof. By the discussion above, the hyperplane $1 = \frac{x_1}{d_1} + \cdots + \frac{x_r}{d_r}$ bounds the limiting shape of the generic initial system so, using Definition 4.2.10, $\mathcal{J}(\mathbf{a}_\bullet)$ is as claimed. \square

CHAPTER V

The Asymptotic Behavior of Symbolic Generic Initial Systems of Points in General Position

5.1 Introduction

Generic initial ideals can be viewed as a coordinate-independent version of initial ideals, which carry much of the same information as the initial ideal with the added benefit of preserving, and even revealing, certain geometric information. Given an ideal $I \subseteq K[x, y, z]$ of distinct points p_1, \dots, p_r in \mathbb{P}^2 , the reverse lexicographic generic initial ideal of I , $\text{gin}(I)$, can detect if a subset of the points lies on a curve of a certain degree (see [EP90] or Theorem 4.4 of [Gre98]). If we instead consider the ideal $I^{(m)}$ of the fat point subscheme $Z_m = mp_1 + \dots + mp_r \subseteq \mathbb{P}^2$, one might ask what $\text{gin}(I^{(m)})$ says about Z_m ; this question motivated the work in this paper.

Despite being simple to describe, ideals $I^{(m)}$ of fat point subschemes $Z_m = m(p_1 + \dots + p_r)$ have proven difficult to understand. For example, there are still many open problems and unresolved conjectures related to finding the Hilbert function of $I^{(m)}$ and even the degree $\alpha(I^{(m)})$ of the smallest degree element of $I^{(m)}$. Many of the challenges in understanding the individual ideals $I^{(m)}$ can be overcome by changing one's focus to studying the general behavior of the entire family of ideals $\{I^{(m)}\}_m$. For instance, more can be said about the Seshadri constant

$$\epsilon(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{rm}$$

than the invariants $\alpha(I^{(m)})$ of each ideal (see [BH10] and [Har02] for further background on these constants). Thus, we will explore the asymptotic behavior of the entire symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$ as a first step to understanding the generic initial ideals of fat point subschemes.

To describe limiting behavior, we define the *limiting shape* P of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}$ of the ideal $I \subseteq K[x, y, z]$ corresponding to an arrangement of points in \mathbb{P}^2 to be the limit $\lim_{m \rightarrow \infty} \frac{1}{m} P_{\text{gin}(I^{(m)})}$, where $P_{\text{gin}(I^{(m)})}$ denotes the Newton polytope of $\text{gin}(I^{(m)})$. We will see that each of the ideals $\text{gin}(I^{(m)})$ is generated in the variables x and y , so that $P_{\text{gin}(I^{(m)})}$, and thus P , can be thought of as a subset of \mathbb{R}^2 . One reason for studying the limiting shape of a system of monomial ideals is that it completely determines the asymptotic multiplier ideals of the system (see [How01] and [May12d]).

When the point arrangement has an ideal I that is a complete intersection of type (α, β) with $\alpha \leq \beta$, a special case of the main result of [May12d] shows that the limiting shape of the symbolic generic initial system has a boundary defined by the line through the points $(\alpha, 0)$ and $(0, \beta)$. The main result of this paper is the following theorem describing the limiting shape of the symbolic generic initial system of an ideal of r distinct points of \mathbb{P}^2 in general position, assuming that the SHGH Conjecture 5.3.1 holds for the case where $r \geq 9$.

Theorem 5.1.1. *Let $I \subseteq R = K[x, y, z]$ be the ideal of $r > 1$ distinct points p_1, \dots, p_r of \mathbb{P}^2 in general position and P be the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Then P can be characterized as follows.*

- (a) *If $r \geq 9$ and the SHGH Conjecture holds for infinitely many m , then P has a boundary defined by the line through the points $(\sqrt{r}, 0)$ and $(0, \sqrt{r})$. See Figure 5.1.*
- (b) *If $6 \leq r < 9$, then P has a boundary defined by the line through the points $(\gamma_1, 0)$ and $(0, \gamma_2)$ where:*
 - (i) $\gamma_1 = \frac{12}{5}$ and $\gamma_2 = \frac{5}{2}$ when $r = 6$;
 - (ii) $\gamma_1 = \frac{21}{8}$ and $\gamma_2 = \frac{8}{3}$ when $r = 7$; and
 - (iii) $\gamma_1 = \frac{48}{17}$ and $\gamma_2 = \frac{17}{6}$ when $r = 8$.
- (c) *If $r = 4$ or $r = 5$, then P has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$. If $r = 2$ or $r = 3$, then P has a boundary defined by the line through the points $(\frac{r}{2}, 0)$ and $(0, 2)$.*

Precisely what information is carried by the limiting shape of the symbolic generic initial system of other point arrangements is still uncertain. While one can prove that the x -intercept of the boundary of P is equal to $r\epsilon(I)$ (see Section 5.2), that the y -intercept reflects the asymptotic behavior of the regularity of the ideals $I^{(m)}$ (see

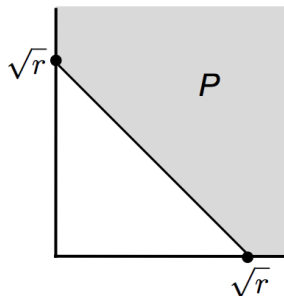


Figure 5.1: The limiting shape P of $\{\text{gin}(I^{(m)})\}_m$ where I is the ideal of $r \geq 9$ points in general position, assuming that the SHGH Conjecture holds for infinitely many m .

[May12d]), and that the volume under P is equal to $\frac{r}{2}$ (Proposition 5.2.14), there is likely additional geometric information encoded within P . Two important questions concern the form of P : is P always a polytope, and what does it mean for the boundary of P to be defined by a certain number of line segments?

Following background information in Section 5.2, the three parts of Theorem 5.1.1 are proven in Sections 5.3, 5.4, and 5.5. The final section contains an example demonstrating that there are point arrangements for which the boundary of the limiting shape of the symbolic generic initial system is not defined by a single line segment.

5.2 Preliminaries

In this section we will introduce some notation, definitions, and preliminary results related to fat points in \mathbb{P}^2 , generic initial ideals, and systems of ideals. Unless stated otherwise, $R = K[x, y, z]$ is the polynomial ring in three variables over a field K of characteristic 0 with the standard grading and some fixed term order $>$ with $x > y > z$.

5.2.1 Fat Points in \mathbb{P}^2

Definition 5.2.1. Let p_1, \dots, p_r be distinct points of \mathbb{P}^2 , I_j be the ideal of $K[\mathbb{P}^2] = R$ consisting of all forms vanishing at the point p_j , and $I = I_1 \cap \dots \cap I_r$ be the ideal of the points p_1, \dots, p_r . A **fat point subscheme** $Z = m_1 p_1 + \dots + m_r p_r$, where the m_i are nonnegative integers, is the subscheme of \mathbb{P}^2 defined by the ideal $I_Z = I_1^{m_1} \cap \dots \cap I_r^{m_r}$ consisting of forms that vanish at the points p_i to multiplicity at least m_i . When

$m_i = m$ for all i , we say that Z is **uniform**; in this case, I_Z is equal to the m^{th} symbolic power of I , $I^{(m)}$.

The following lemma relates the symbolic and ordinary powers of I in the case we are interested in (see, for example, Lemma 1.3 of [AV03]).

Lemma 5.2.2. *If I is the ideal of distinct points in \mathbb{P}^2 ,*

$$(I^m)^{\text{sat}} = I^{(m)},$$

where $J^{\text{sat}} = \bigcup_{k \geq 0} (J : \mathfrak{m}^k)$ denotes the saturation of J .

In this paper we will be interested in studying the ideals of uniform fat point subschemes $Z = mp_1 + \cdots + mp_r$ such that the points p_1, \dots, p_r are in *general position*.

Definition 5.2.3. A collection of points in \mathbb{P}^2 is in **general position** if, for each $d \in \mathbb{N}$, no subset of cardinality $\binom{d+2}{2}$ lies on any curve of degree d .

5.2.2 Generic Initial Ideals

An element $g = (g_{ij}) \in \text{GL}_n(K)$ acts on $R = K[x_1, \dots, x_n]$ and sends any homogeneous element $f(x_1, \dots, x_n)$ to the homogeneous element

$$f(g(x_1), \dots, g(x_n))$$

where $g(x_i) = \sum_{j=1}^n g_{ij}x_j$. If $g(I) = I$ for every upper triangular matrix g then we say that I is *Borel-fixed*. Borel-fixed ideals are *strongly stable* when K is of characteristic 0; that is, for every monomial m in the ideal such that x_i divides m , the monomials $\frac{x_j m}{x_i}$ are also in the ideal for all $j < i$. This property makes such ideals particularly nice to work with.

To any homogeneous ideal I of R we can associate a Borel-fixed monomial ideal $\text{gin}_{>}(I)$ which can be thought of as a coordinate-independent version of the initial ideal. Its existence is guaranteed by Galligo's theorem (also see [Gre98, Theorem 1.27]).

Theorem 5.2.4 ([Gal74] and [BS87b]). *For any multiplicative monomial order $>$ on R and any homogeneous ideal $I \subset R$, there exists a Zariski open subset $U \subset \text{GL}_n$ such that $\text{In}_{>}(g(I))$ is constant and Borel-fixed for all $g \in U$.*

Definition 5.2.5. The **generic initial ideal** of I , denoted $\text{gin}_{>}(I)$, is defined to be $\text{In}_{>}(g(I))$ where $g \in U$ is as in Galligo's theorem.

The *reverse lexicographic order* $>$ is a total ordering on the monomials of R defined by:

1. if $|I| = |J|$ then $x^I > x^J$ if there is a k such that $i_m = j_m$ for all $m > k$ and $i_k < j_k$; and
2. if $|I| > |J|$ then $x^I > x^J$.

For example, $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$. From this point on, $\text{gin}(I) = \text{gin}_{>}(I)$ will denote the generic initial ideal with respect to the reverse lexicographic order.

Recall that the Hilbert function $H_I(t)$ of I is defined by $H_I(t) = \dim(I_t)$. The following theorem is a consequence of the fact that Hilbert functions are invariant under making changes of coordinates and taking initial ideals; we will use it frequently and freely throughout this paper.

Theorem 5.2.6. *For any homogeneous ideal I in R , the Hilbert functions of I and $\text{gin}(I)$ are equal.*

In this paper we will be studying the set of reverse lexicographic generic initial ideals of symbolic powers of a fixed ideal I , $\{\text{gin}(I^{(m)})\}_m$. One reason for our interest in these ideals is the following proposition which tells us that we can get information about the ideals $\text{gin}(I^m)$ from the ideals $\text{gin}(I^{(m)})$.

Proposition 5.2.7 (Proposition 2.21 of [Gre98]). *Fix the reverse lexicographic order on $K[x_1, \dots, x_n]$ with $x_1 > x_2 > \dots > x_n$ and let $\mathfrak{m} = (x_1, \dots, x_n)$. Then, if $I^{\text{sat}} = \bigcup_{k \geq 0} (I : \mathfrak{m}^k)$ denotes the saturation of I ,*

$$\text{gin}(I^{\text{sat}}) = \bigcup_{k \geq 0} (\text{gin}(I) : \mathfrak{m}^k) = (\text{gin}(I))^{\text{sat}}.$$

In particular, when I is the ideal of distinct points in \mathbb{P}^2 ,

$$\text{gin}(I^{(m)}) = \bigcup_{k \geq 0} (\text{gin}(I^m) : \mathfrak{m}^k) = (\text{gin}(I^m))^{\text{sat}}.$$

for all $m \geq 1$ by Lemma 5.2.2.

The following result due to Bayer and Stillman ([BS87a]).

Proposition 5.2.8 (Theorem 2.21 of [Gre98]). *Fix the reverse lexicographic order on $K[x_1, \dots, x_n]$ with $x_1 > x_2 > \dots > x_n$. An ideal I of R is saturated if and only if no minimal generator of $\text{gin}(I)$ involves the variable x_n . In particular, when $I \subset K[x, y, z]$ is the (saturated) ideal of a set of distinct points of \mathbb{P}^2 , no minimal generator of $\text{gin}(I^{(m)})$ involves the variable z .*

Corollary 5.2.9. *Suppose that $I \subset K[x, y, z]$ is the ideal of a set of distinct points of \mathbb{P}^2 . Then the minimal generators of $\text{gin}(I^{(m)})$ under the reverse lexicographic order are of the form*

$$\{x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_{\alpha(m)-1}(m)}, \dots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\}$$

where $\lambda_0(m) > \lambda_1(m) > \dots > \lambda_{\alpha(m)-1}(m) \geq 1$.

Proof. By a result of Herzog and Srinivasan relating the dimension of a Borel-fixed monomial ideal J to the variable powers that it contains, $\text{gin}(I^{(m)})$ contains a power of y (see Lemma 3.1 of [HS98]). Now the result is immediate from Proposition 5.2.8 and the fact that $\text{gin}(I^{(m)})$ is a Borel-fixed ideal. \square

5.2.3 Graded Systems

In this subsection we introduce some tools for studying certain collections of monomial ideals.

Definition 5.2.10 ([ELS01]). A **graded system of ideals** is a collection of ideals $J_\bullet = \{J_i\}_{i=1}^\infty$ such that

$$J_i \cdot J_j \subseteq J_{i+j} \quad \text{for all } i, j \geq 1.$$

Definition 5.2.11. The **generic initial system** of a homogeneous ideal I is the collection of ideals J_\bullet such that $J_i = \text{gin}(I^i)$. The **symbolic generic initial system** of a homogeneous ideal I is the collection of ideals J_\bullet such that $J_i = \text{gin}(I^{(i)})$.

Lemma 5.2.12. *The symbolic generic initial system is a graded system of ideals.*

Proof. By definition, $\text{gin}(I^{(i)})$ is a monomial ideal. We need to show that for all $i, j \geq 1$, $\text{gin}(I^{(i)}) \cdot \text{gin}(I^{(j)}) \subseteq \text{gin}(I^{(i+j)})$. For any $l \geq 1$, let U_l be the Zariski open subset of GL_n such that $\text{gin}(I^{(l)}) = \text{In}(g \cdot (I^{(l)}))$ for all g in U_l . Since U_i , U_j , and U_{i+j} are Zariski open they have a nonempty intersection; fix some $g \in U_i \cap U_j \cap U_{i+j}$.

Given monomials $f' \in \text{gin}(I^{(i)}) = \text{In}(g(I^{(i)}))$ and $h' \in \text{gin}(I^{(j)}) = \text{In}(g(I^{(j)}))$, suppose that $f' = \text{In}(g(f))$ and $h' = \text{In}(g(h))$ for $f \in I^{(i)}$ and $h \in I^{(j)}$. Now

$$f' \cdot h' = \text{In}(g(f))\text{In}(g(h)) = \text{In}(g(f) \cdot g(h)) = \text{In}(g(f \cdot h)) \in \text{In}(g(I^{(i+j)}))$$

since $f \cdot h \in I^{(i+j)}$.¹ Thus $f' \cdot h' \in \text{gin}(I^{(i+j)})$ as desired. \square

The same proof with $I^{(i)}$ replaced by I^i shows that the generic initial system is also a graded system of ideals.

Definition 5.2.13 ([ELS03]). Let \mathbf{a}_\bullet be a graded system of zero-dimensional ideals in $R = K[x_1, \dots, x_n]$. The **volume** of \mathbf{a}_\bullet is

$$\text{vol}(\mathbf{a}_\bullet) := \limsup_{m \rightarrow \infty} \frac{n! \cdot \text{length}(R/\mathbf{a}_m)}{m^n}.$$

Let J be a monomial ideal of R . We may associate to J a subset Λ of \mathbb{N}^n consisting of the points λ such that $x^\lambda \in J$. The *Newton polytope* P_J of J is the convex hull of Λ regarded as a subset of \mathbb{R}^n . Scaling the polytope P_J by a factor of r gives another polytope which we will denote rP_J .

If \mathbf{a}_\bullet is a graded system of monomial ideals in R , the polytopes of $\{\frac{1}{q}P_{\mathbf{a}_q}\}_q$ are nested: $\frac{1}{p}P_{I_p} \subseteq \frac{1}{q}P_{I_q}$ whenever p divides q . The *limiting shape* P of \mathbf{a}_\bullet is the limit of the polytopes in this set:

$$P = \bigcup_{q \in \mathbb{N}^*} \frac{1}{q}P_{\mathbf{a}_q}.$$

Under the additional assumption that the ideals of \mathbf{a}_\bullet are zero-dimensional, the closure of each set $\mathbb{R}_{\geq 0}^n \setminus P_{\mathbf{a}_q}$ in \mathbb{R}^n is compact. This closure is denoted by Q_q and we let

$$Q = \bigcap_{q \in \mathbb{N}^*} \frac{1}{q}Q_q.$$

Proposition 5.2.14 ([Mus02]). *If \mathbf{a}_\bullet is a graded system of zero-dimensional monomial ideals in $R = K[x_1, \dots, x_n]$ and Q is as defined above,*

$$\text{vol}(\mathbf{a}_\bullet) = n! \text{vol}(Q).$$

Proof. This is an immediate consequence of Theorem 1.7 and Lemma 2.13 of [Mus02]. \square

¹This holds since the set of symbolic powers of a fixed ideal is itself a graded system: $I^{(i)} \cdot I^{(j)} \subseteq I^{(i+j)}$.

We now turn our attention to using the concept of the limiting shape to study the asymptotic behavior of the system of ideals $\{\text{gin}(I^{(m)})\}_m$ where I is an ideal of r distinct points in \mathbb{P}^2 . By Corollary 5.2.9, the ideals $\text{gin}(I^{(m)})$ for such an I are generated in the variables x and y and contain a power of both x and y . Therefore, we can think of the ideals $\text{gin}(I^{(m)})$ as zero-dimensional in $K[x, y]$ and consider a two dimensional limiting shape P of the symbolic generic initial system.

Lemma 5.2.15. *Suppose that I is the ideal of r distinct points p_1, p_2, \dots, p_r in \mathbb{P}^2 and $J_m = \text{gin}(I^{(m)}) \subseteq K[x, y]$. If P is the limiting shape of J_\bullet and $Q \subseteq \mathbb{R}^2$ is as above,*

$$\text{vol}(Q) = \frac{r}{2}.$$

Proof. Let $h = ax + by + cz$ be a general linear form in $K[x, y, z]$. To reduce our calculations to $K[x, y]$, consider the ring isomorphism

$$\phi : \frac{K[x, y, z]}{(h)} \rightarrow K[x, y]$$

given by sending x to x , y to y , and z to $-\frac{a}{c}x - \frac{b}{c}y$. If $I_i \subseteq K[x, y, z]$ is the ideal of the point p_i in \mathbb{P}^2 then $\phi(\bar{I}_i) \cong (x, y)^m$. Further, $\phi(\overline{I^{(m)}}) = \phi(\bar{I}_1^m \cap \dots \cap \bar{I}_r^m)$ and $\text{length}\left(\frac{K[x, y]}{(x, y)^m}\right) = \binom{m+1}{2}$ so

$$\begin{aligned} \text{length}\left(\frac{K[x, y]}{\phi(\overline{I^{(m)}})}\right) &= \text{length}\left(\frac{K[x, y]}{I_1^m} \times \dots \times \frac{K[x, y]}{I_r^m}\right) \\ &= \text{length}\left(\frac{K[x, y]}{(x, y)^m} \times \dots \times \frac{K[x, y]}{(x, y)^m}\right) \\ &= r(1 + \dots + m). \end{aligned}$$

The fact that $\text{gin}(I^{(m)})$ is generated in x and y (Proposition 5.2.8) together with a well-known relation between the generic initial ideals of J and $\phi(\bar{J})$ (see Corollary 2.5 of [Gre98]) imply that $\text{gin}(I^{(m)})$ and $\text{gin}(\phi(\overline{I^{(m)}}))$ have the same generators. Thus, thinking of $\text{gin}(I^{(m)})$ as being contained in $K[x, y]$,

$$\begin{aligned} \text{length}\left(\frac{K[x, y]}{\text{gin}(I^{(m)})}\right) &= \text{length}\left(\frac{K[x, y]}{\text{gin}(\phi(\overline{I^{(m)}}))}\right) \\ &= \text{length}\left(\frac{K[x, y]}{\phi(\overline{I^{(m)}})}\right) \\ &= r(1 + \dots + m) = r\left(\frac{m^2 + m}{2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned}
\text{vol}(Q) &= \lim_{m \rightarrow \infty} \frac{\text{length}(K[x, y]/\text{gin}(I^{(m)}))}{m^2} \\
&= \lim_{m \rightarrow \infty} \frac{(m^2 + m)r}{2m^2} \\
&= \frac{r}{2}.
\end{aligned}$$

□

If I is the ideal of distinct points in \mathbb{P}^2 , the minimal generating set of each ideal $\text{gin}(I^{(m)})$ contains a power of x and a power of y , say $x^{\alpha(m)}$ and $y^{\zeta(m)}$ by Corollary 5.2.9. It is clear that $\lim_{m \rightarrow \infty} \frac{\alpha(m)}{m}$ and $\lim_{m \rightarrow \infty} \frac{\zeta(m)}{m}$ are the x - and y -intercepts of the limiting shape P of $\{\text{gin}(I^{(m)})\}_m$.

Corollary 5.2.16. *Let $I \subseteq K[x, y, z]$ be the ideal of r distinct points in \mathbb{P}^2 and P be the limiting shape of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Suppose that the x -intercept γ_1 and the y -intercept γ_2 of the boundary of P are such that $\gamma_1 \cdot \gamma_2 = r$. Then the limiting shape P has a boundary defined by the line passing through $(\gamma_1, 0)$ and $(0, \gamma_2)$.*

Proof. The smallest possible limiting shape P satisfying the given conditions is the one defined by the line segment through $(\gamma_1, 0)$ and $(0, \gamma_2)$ since P is convex by definition. This extreme case is the only one in which the maximum volume under P is achieved, in which case $\text{vol}(Q) = \frac{\gamma_1 \gamma_2}{2}$. Under the assumptions stated, $\gamma_1 \cdot \gamma_2 = r$ so, by the previous lemma, the maximum volume must be attained and P is as claimed. □

5.3 The Symbolic Generic Initial System of Greater than 8 Uniform Points in General Position

Throughout this section, $I \subseteq R[x, y, z]$ will denote the ideal of $r \geq 9$ points p_1, \dots, p_r of \mathbb{P}^2 in general position. We will frequently use the fact that the Hilbert function of an ideal and its generic initial ideal are equal (see Theorem 5.2.6).

Computing the Hilbert functions of ideals of fat points in \mathbb{P}^2 can be very difficult. However, the following conjecture of Segre, Harbourne, Gimigliano, and Hirschowitz proposes that when Z is the ideal of $r \geq 9$ uniform fat points in general position, $H_{I_Z}(t)$ has a very simple form. See [HC12] for a statement similar to what follows and [Har02] for more general versions of the conjecture.

Conjecture 5.3.1 (SHGH Conjecture). *Let $R = K[x, y, z]$ and I be the ideal of $r \geq 9$ generic points $p_i \in \mathbb{P}^2$. Then, if $I^{(m)}$ is the ideal of the uniform fat point subscheme $Z = m(p_1 + \cdots + p_r)$,*

$$H_{I^{(m)}}(t) = \max \left\{ \binom{t+2}{2} - r \binom{m+1}{2}, 0 \right\}.$$

The SHGH Conjecture is known to hold for certain special cases. For example, it holds for infinitely many m when r is a square by [HR04], and for all m when r is a square not divisible by a prime bigger than 5 by [Eva99].

The main goal of this section is to prove the first part of Theorem 5.1.1.

Theorem 5.1.1(a). *Fix $r \geq 9$ points of \mathbb{P}^2 in general position and suppose that the SHGH Conjecture 5.3.1 holds for infinitely many m . Let I be the ideal of r general points in \mathbb{P}^2 and P be the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Then the boundary of P is defined by the line through the points $(\sqrt{r}, 0)$ and $(0, \sqrt{r})$.*

The proof of this statement is contained in Section 5.3.2. In preparation for this proof, we compute the minimal generators of the generic initial ideals $\text{gin}(I^{(m)})$ in Section 5.3.1 under the assumption that the SHGH Conjecture holds.

5.3.1 Structure of $\text{gin}(I^{(m)})$

The following lemma records the degree of the smallest degree element of $I^{(m)}$.

Lemma 5.3.2. *Let I be the ideal of p_1, \dots, p_r points of \mathbb{P}^2 in general position where $r \geq 9$ and suppose that $\alpha(m)$ is the least integer t such that $H_{I^{(m)}}(t) > 0$. Then, if the SHGH Conjecture holds for $Z = m(p_1 + \cdots + p_r)$,*

$$\alpha(m) = \left\lfloor -\frac{1}{2} + \sqrt{\frac{1}{4} + rm^2 + rm} \right\rfloor.$$

Proof. By the SHGH Conjecture, $\alpha(m)$ is the smallest integer t such that $\binom{t+2}{2} - r \binom{m+1}{2} > 0$.

$$\begin{aligned} & \frac{(t+2)(t+1)}{2} - r \frac{(m+1)m}{2} > 0 \\ \Leftrightarrow & t^2 + 3t + 2 - rm^2 - rm > 0 \end{aligned}$$

If this is an equality, the positive root is

$$t = -\frac{3}{2} + \frac{1}{2}\sqrt{1 + 4rm^2 + 4rm}.$$

Then the least integer that will make the expression positive is

$$\alpha(m) = \left\lceil -\frac{3}{2} + \frac{1}{2}\sqrt{1 + 4rm^2 + 4rm} + 1 \right\rceil.$$

□

If the SHGH Conjecture holds, the structure of the generic initial ideals $\text{gin}(I^{(m)})$ is very simple.

Proposition 5.3.3. *Let I be the ideal of $r \geq 9$ points of \mathbb{P}^2 in general position, fix a non-negative integer m , and suppose that the SHGH Conjecture holds for $I^{(m)}$. Set $\alpha = \alpha(m)$ and $\eta := H_{I^{(m)}}(\alpha) = \binom{\alpha+2}{2} - r \binom{m+1}{2}$ so that $\eta \leq \alpha + 1$. Then*

$$\text{gin}(I^{(m)}) = (x^\alpha, x^{\alpha-1}y, \dots, x^{\alpha-\eta+1}y^{\eta-1}, x^{\alpha-\eta}y^{\eta+1}, x^{\alpha-\eta-1}y^{\eta+2}, \dots, xy^\alpha, y^{\alpha+1})$$

when $\eta < \alpha + 1$ and

$$\text{gin}(I^{(m)}) = (x^\alpha, x^{\alpha-1}y, \dots, xy^{\alpha-1}, y^\alpha)$$

when $\eta = \alpha + 1$.

Proof. Since there is no element of $\text{gin}(I^{(m)})$ of degree smaller than $\alpha(m)$, all monomials of degree $\alpha(m)$ in $\text{gin}(I^{(m)})$ must be generators and thus contain only the variables x and y by Proposition 5.2.8. There are at most $\alpha + 1$ monomials of degree α in the variables x and y so $\eta := H_{\text{gin}(I^{(m)})}(\alpha) \leq \alpha + 1$.

If $\eta = \alpha + 1$ then all $\alpha + 1$ monomials of degree α in the variables x and y are minimal generators of $\text{gin}(I^{(m)})$. By Corollary 5.2.9, $\text{gin}(I^{(m)})$ has exactly $\alpha + 1$ minimal generators. Thus, all minimal generators of $\text{gin}(I^{(m)})$ are of degree α and are the ones given.

Now suppose that $\eta < \alpha + 1$. The η monomials of $\text{gin}(I^{(m)})$ of degree α must be minimal generators. In fact, since generic initial ideals are Borel-fixed, these must be the largest η monomials in x and y of degree α with respect to the reverse lexicographic order:

$$[\text{gin}(I^{(m)})]_\alpha = \{x^\alpha, x^{\alpha-1}y, \dots, x^{\alpha-\eta+1}y^{\eta-1}\}.$$

There are exactly η elements of $\text{gin}(I^{(m)})$ of degree $\alpha + 1$ involving the variable z , obtained by multiplying each of the η generators of $[\text{gin}(I^{(m)})]_\alpha$ by z . By the SHGH Conjecture 5.3.1,

$$\begin{aligned} H_{I^{(m)}}(\alpha + 1) - \eta &= \left[\binom{\alpha + 1 + 2}{2} - r \binom{m + 1}{2} \right] - \left[\binom{\alpha + 2}{2} - r \binom{m + 1}{2} \right] \\ &= \binom{\alpha + 2 + 1}{2} - \binom{\alpha + 2}{2} \\ &= \binom{\alpha + 2}{1} = \alpha + 2 \end{aligned}$$

and there are $\alpha + 2$ monomials in $\text{gin}(I^{(m)})$ of degree $\alpha + 1$ containing only the variables x and y . Since there are exactly $\alpha + 2$ monomials of degree $\alpha + 1$ in x and y , $\text{gin}(I^{(m)})$ contains all of them. Thus, the remaining generators of $\text{gin}(I^{(m)})$ are of degree $\alpha + 1$; they are

$$x^{\alpha-\eta}y^{\eta+1}, x^{\alpha-\eta-1}y^{\eta+2}, \dots, xy^\alpha, y^{\alpha+1}$$

by Corollary 5.2.9. □

5.3.2 Proof of Theorem 5.1.1 (a)

Proof of Theorem 5.1.1 (a). By Proposition 5.3.3, $x^{\alpha(m)}$ and $y^{\alpha(m)+1}$ or $y^{\alpha(m)}$ are the smallest variable powers contained in $\text{gin}(I^{(m)})$ for all m such that the SHGH Conjecture holds. Thus, the x -intercept of the boundary of P is

$$\lim_{m \rightarrow \infty} \frac{\alpha(m)}{m}$$

while the y -intercept of the boundary of P is

$$\lim_{m \rightarrow \infty} \frac{\alpha(m) + 1}{m} = \lim_{m \rightarrow \infty} \frac{\alpha(m)}{m}$$

where we take the limits over the infinite subset such that the SHGH Conjecture holds.

By Lemma 5.3.2,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\alpha(m)}{m} &= \lim_{m \rightarrow \infty} \frac{\lfloor -\frac{1}{2} + \sqrt{\frac{1}{4} + rm^2 + rm} \rfloor}{m} \\ &= \sqrt{r} \end{aligned}$$

so the x and y intercepts of the limiting shape P are both equal to \sqrt{r} . Since $\sqrt{r} \cdot \sqrt{r} = r$, Corollary 5.2.16 tells us that the boundary of P is defined by the line through the x - and the y -intercepts as claimed. \square

5.4 The Symbolic Generic Initial System of 6, 7, and 8 Uniform Fat Points in General Position

As before, $I \subseteq R[x, y, z]$ will denote the ideal of points p_1, \dots, p_r of \mathbb{P}^2 in general position. The goal of this section is to prove the second part of Theorem 5.1.1.

Theorem 5.1.1 (b). *Suppose that $I \subseteq K[x, y, z]$ is the ideal of $r = 6, 7$, or 8 points of \mathbb{P}^2 in general position and that $P \subseteq \mathbb{R}^2$ is the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Then the boundary of P is defined by the line segment through the points $(\gamma_1, 0)$ and $(0, \gamma_2)$ where*

- (a) $\gamma_1 = \frac{12}{5}$ and $\gamma_2 = \frac{5}{2}$ when $r = 6$;
- (b) $\gamma_1 = \frac{21}{8}$ and $\gamma_2 = \frac{8}{3}$ when $r = 7$; and
- (c) $\gamma_1 = \frac{48}{17}$ and $\gamma_2 = \frac{17}{6}$ when $r = 8$.

The proof of this result relies on knowing certain values of the Hilbert functions $H_{I^{(m)}}(t)$ of the ideals $I^{(m)}$ where I is the ideal of 6, 7, or 8 general points. Techniques for computing $H_{I^{(m)}}(t)$ in these cases are not new (for example, see [Nag60]), but they can be complicated. Thus, we take time in Section 5.4.1 to review a modern technique for finding the Hilbert functions, and then apply these results to the proof of Theorem 5.1.1(b) in Section 5.4.2.

5.4.1 Background on Surfaces

The method we use to compute $H_{I^{(m)}}(t)$ follows the work of Fichett, Harbourne, and Holay in [FHH01].

Suppose that $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up of distinct points p_1, \dots, p_r of \mathbb{P}^2 . Let $E_i = \pi^{-1}(p_i)$ for $i = 1, \dots, r$ and L be the total transform in X of a line not passing through any of the points p_1, \dots, p_r . The classes of these divisors form a basis of $\text{Cl}(X)$; for convenience, we will write e_i for the class $[E_i]$ of E_i and e_0 for the class $[L]$. Further, the intersection product in $\text{Cl}(X)$ is defined by $e_i^2 = -1$ for $i = 1, \dots, r$; $e_0^2 = 1$; and $e_i \cdot e_j = 0$ for all $i \neq j$.

Let $Z_m = m(p_1 + \cdots + p_r)$ be a uniform fat point subscheme with sheaf of ideals \mathcal{I}_{Z_m} ; set

$$F_d = dE_0 - m(E_1 + E_2 + \cdots + E_r)$$

and $\mathcal{F}_d = \mathcal{O}_X(F_d)$. Then $\pi_*(\mathcal{F}_d) = \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(d)$ so

$$\dim((I_{Z_m})_d) = h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(d)) = h^0(X, \mathcal{F}_d)$$

for all d . In particular, if $I \subseteq K[x, y, z]$ is the ideal of the points p_1, \dots, p_r in \mathbb{P}^2 ,

$$H_{I^{(m)}}(d) = h^0(X, \mathcal{F}_d)$$

and so we can study the Hilbert function of the symbolic powers $I^{(m)}$ by working with divisors on the surface X . For convenience, we will often write $h^0(X, F) = h^0(X, \mathcal{O}_X(F))$.

Recall that if $[F]$ not the class of an effective divisor then $h^0(X, F) = 0$. On the other hand, if F is effective, then we will see that we can compute $h^0(X, F)$ by computing $h^0(X, H)$ for some *numerically effective* divisor H .

Definition 5.4.1. A divisor H is **numerically effective** if $[F] \cdot [H] \geq 0$ for every effective divisor F , where $[F] \cdot [H]$ denotes the intersection multiplicity. The cone of classes of numerically effective divisors in $\text{Cl}(X)$ is denoted by $\text{NEF}(X)$.

Lemma 5.4.2. *Suppose that X is the blow-up of \mathbb{P}^2 at $r \leq 8$ points in general position and that $F \in \text{NEF}(X)$. Then F is effective and*

$$h^0(X, F) = ([F]^2 - [F] \cdot [K_X])/2 + 1$$

where $K_X = -3e_0 + e_1 + \cdots + e_r$.

Proof. This is a consequence of Riemann-Roch and the fact that $h^1(X, F) = 0$ for any numerically effective divisor F . See Theorem 8 of [Har96] or Section 1 of [FHH01] for a discussion. \square

Corollary 5.4.3. *Let $F_t = tL - m(E_1 + E_2 + \cdots + E_r)$. If F_t is numerically effective then*

$$h^0(X, F_t) = \binom{t+2}{2} - r \binom{m+1}{2}.$$

A divisor class $[C]$ on X is said to be *exceptional* if it is the class of an exceptional

divisor C on X (that is, a smooth curve isomorphic to \mathbb{P}^1 such that $[C]^2 = -1$).² The following result of Fichett, Harbourne, and Holay [FHH01] tells us how to detect if a divisor is numerically effective if we know the exceptional curves.

Lemma 5.4.4 (Lemma 4(b) of [FHH01]). *Suppose that X is the surface obtained by blowing up $2 \leq r \leq 8$ points of \mathbb{P}^2 . Then F is numerically effective if the intersection multiplicity of $[F]$ with all exceptional classes is greater than or equal to 0.*

Another result from [FHH01] tells us what the exceptional curves of X are in the cases that we are interested in.

Lemma 5.4.5 (Lemma 3(a) of [FHH01]). *Let C be a curve on the blow-up X of \mathbb{P}^2 at 8 points in general position. Then, with the notation above, the exceptional classes are the following, up to permutation of indices $1, 2, \dots, 8$:*

- $h_1 = e_8$
- $h_5 = 4e_0 - 2e_1 - 2e_2 - 3e_3 - e_4 - \dots - e_8$
- $h_2 = e_0 - e_1 - e_2$
- $h_6 = 5e_0 - 2e_1 - \dots - 2e_6 - e_7 - e_8$
- $h_3 = 2e_0 - e_1 - \dots - e_5$
- $h_7 = 6e_0 - 3e_1 - 2e_2 - \dots - 2e_8$.
- $h_4 = 3e_0 - 2e_1 - e_2 - \dots - e_7$

When X is the blow-up of \mathbb{P}^2 at $n \leq 8$ points, the exceptional classes of X are the ones listed above with $8 - n$ of the e_i ($i = 1, \dots, 8$) set to 0.

It turns out that knowing how to compute $h^0(X, H)$ for a numerically effective divisor H will actually allow us to compute $h^0(X, F)$ for *any* divisor F . In particular, for any divisor F , there exists a divisor H such that $h^0(X, F) = h^0(X, H)$ and either:

- (a) H is numerically effective so

$$h^0(X, F) = h^0(X, H) = (H^2 - H \cdot K_X)/2 + 1$$

by Lemma 5.4.2; or

- (b) There is a numerically effective divisor G such that $[G] \cdot [H] < 0$ so $[H]$ is not the class of an effective divisor and $h^0(X, F) = h^0(X, H) = 0$.

The following result will be used in Procedure 5.4.7 to find such an H .

²Note that if $[C]$ is an exceptional class, there is a unique effective divisor in this class, typically called the *exceptional curve*

Lemma 5.4.6. *Suppose that $[C]$ is an exceptional class such that $[F] \cdot [C] < 0$. Then $h^0(X, F) = h^0(X, F - C)$.*

Proof. Note that it suffices to prove this statement for the case where C is a smooth curve isomorphic to \mathbb{P}^1 : if $[C']$ is an exceptional class then there exists a smooth curve C isomorphic to \mathbb{P}^1 such that $[C'] = [C]$ so $[C'] \cdot [F] = [C] \cdot [F]$ and $h^0(X, F - C') = h^0(X, F - C)$. Note that we have an exact sequence

$$\mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C(F) \cong \mathcal{O}_{\mathbb{P}^1}([F] \cdot [C]) \rightarrow 0$$

induced by tensoring the exact sequence

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

with $\mathcal{O}_X(F)$. Then, from the long exact sequence of cohomology, $h^0(X, \mathcal{O}_X(F - C)) = h^0(X, \mathcal{O}_X(F))$ since $h^0(X, \mathcal{O}_{\mathbb{P}^1}([F] \cdot [C])) = 0$ ($[F] \cdot [C] < 0$). \square

The method for finding such the H described above is as follows.

Procedure 5.4.7. *Given a divisor F we can find a divisor H with $h^0(X, F) = h^0(X, H)$ satisfying either condition (a) or (b) above as follows.*

1. *Reduce to the case where $[F] \cdot e_i \geq 0$ for all $i = 1, \dots, n$: if $[F] \cdot e_i < 0$ for some i , $h^0(X, F) = h^0(X, F - ([F] \cdot e_i)E_i)$, so we can replace F with $F - ([F] \cdot e_i)E_i$.*
2. *Since L is numerically effective, if $[F] \cdot e_0 < 0$ then $[F]$ is not the class of an effective divisor and we can take $H = F$ (case (b)).*
3. *If $[F] \cdot [C] \geq 0$ for every exceptional class $[C]$ then, by Lemma 5.4.4, F is numerically effective, so we can take $H = F$ (case (a)).*
4. *If $[F] \cdot [C] < 0$ for some exceptional class $[C]$ then $h^0(X, F) = h^0(X, F - C)$ by Lemma 5.4.6. Then replace F with $F - C$ and repeat from Step 2.*

There are only a finite number of exceptional classes to check by Lemma 5.4.5 so it is possible to complete Step 3. Further, it is easy to see with Lemma 5.4.5 that $F \cdot e_0 > [F - C] \cdot e_0$ when $[C]$ is an exceptional curve, so the condition in Step 2 will be satisfied after at most $[F] \cdot e_0 + 1$ repetitions. Thus, this process will eventually terminate.³

³The decomposition $F = H + (F - H)$ has been referred to as a *Zariski decomposition* in some of the literature on fat points (for example, in [FHH01]), but we avoid this terminology here because it is not consistent with definitions elsewhere (for example, in [Laz04]).

5.4.2 Proof of Theorem 5.1.1(b)

The proof of each part of Theorem 5.1.1(b) follows the same five steps outlined below. In Step 4, we will use the following lemma.

Lemma 5.4.8. *Let I be the ideal of r points in \mathbb{P}^2 . The number of monomials in $\text{gin}(I^{(m)}) \subseteq K[x, y, z]$ of degree t involving the variable z is equal to $H_I(t-1)$.*

Proof. Since, by Proposition 5.2.8, $\text{gin}(I^{(m)})$ is generated in the variables x and y , the only elements of $\text{gin}(I^{(m)})_t$ that involve z have to arise by multiplying monomials of $\text{gin}(I^{(m)})_{t-1}$ by z . Since multiplying each of the $H_{I^{(m)}}(t-1)$ monomials in $\text{gin}(I^{(m)})_{t-1}$ by z gives distinct monomials, the result follows. \square

As in Section 5.4.1, $Z_m = m(p_1 + \cdots + p_r)$ is a uniform fat point subscheme supported at r distinct general points p_1, \dots, p_r and I is the ideal of p_1, \dots, p_r so that $I^{(m)} = I_{Z_m}$. Recall that if $F_t = tL - m(E_1 + \cdots + E_r)$ then

$$H_{I_{Z_m}}(t) = h^0(X, \mathcal{F}_t);$$

we also write $H_{I_{Z_m}}(t) = H_Z(t)$. Finally,

$$\alpha(m) := \min\{t : H_{I_{Z_m}}(t) \neq 0\}.$$

Step 1: Find the smallest N such that $F_t = tE_0 - m(E_1 + \cdots + E_r)$ is numerically effective for all $t \geq N$. To do this we will find the smallest N such that $[F_t] \cdot [C] \geq 0$ for all $t \geq N$ (see Lemma 5.4.4). By Corollary 5.4.3,

$$h^0(X, \mathcal{F}_t) = \binom{t+2}{2} - r \binom{m+1}{2}$$

for all $t \geq N$.

Step 2: Use some optimal numerically effective divisor D to find M such that $[F_t] \cdot [D] < 0$ for all $t < M$. By the definition of a numerically effective divisor, this will show that $[F_t]$ for $t < M$ is not the class of an effective divisor, and thus that $H_{I_{Z_m}}(t) = h^0(X, \mathcal{F}_t) = 0$ for all $t < M$.

Step 3: Show that $h^0(X, \mathcal{F}_M) \neq 0$ where M is as in Step 2. To do this, we will use Procedure 5.4.7 to find a numerically effective H such that $h^0(X, F_M) = h^0(X, H_M)$. Together with Step 2, this will show that $\alpha(m) = M$, and x^M is the smallest power of x in $\text{gin}(I^{(m)})$.

Step 4: By Lemma 5.4.8, the number of monomials of degree t in the $\text{gin}(I_{Z_m})$ involving only the variables x and y is equal to $H_Z(t) - H_Z(t - 1)$. Using this, show that the number of monomials in $\text{gin}(I_{Z_m})$ of degree $N + 1$ in x and y is exactly $N + 2$ (here N is as in Step 1). This implies that all monomials in x and y of degree $N + 1$ are in the $\text{gin}(I_{Z_m})$.

Use Lemma 5.4.8 again to show that the number of monomials in $\text{gin}(I_{Z_m})$ of degree N involving only x and y is strictly less than $N + 1$, so not all monomials of degree N in x and y are in $\text{gin}(I^{(m)})$. Since the ideals of the symbolic generic initial system are generated in x and y (Proposition 5.2.8), this will imply that y^{N+1} is the smallest power of y in $\text{gin}(I^{(m)})$.

Step 5: The smallest power of y in $\text{gin}(I^{(m)})$ is $N + 1$ by Step 4 and the smallest power of x in $\text{gin}(I^{(m)})$ is $\alpha(m) = M$ by Step 3. Thus, the intercepts of the limiting shape P of the symbolic generic initial system of I are $(0, \lim_{m \rightarrow \infty} \frac{N+1}{m})$ and $(\lim_{m \rightarrow \infty} \frac{M}{m}, 0)$. Since

$$\left(\lim_{m \rightarrow \infty} \frac{N+1}{m} \right) \cdot \left(\lim_{m \rightarrow \infty} \frac{M}{m} \right) = r,$$

Corollary 5.2.16 implies that the limiting shape P is as claimed in Theorem 5.1.1(b).

5.4.2.1 6 General Points

Throughout this section I is the ideal of 6 points p_1, \dots, p_6 of \mathbb{P}^2 in general position and $Z_m = m(p_1 + \dots + p_6)$ so $I_{Z_m} = I^{(m)}$. The exceptional classes of the blow-up X of \mathbb{P}^2 at p_1, \dots, p_6 are those in Lemma 5.4.5 with two of the e_i set to 0.

Step 1: To find an N such that $F_t = tL - m(E_1 + \dots + E_6)$ is numerically effective for all $t \geq N$ we will use the permutation of the exceptional curves from Lemma 5.4.5 that is most likely to make $h_i \cdot [F_t]$ negative.

$$\begin{aligned} F_t \cdot h_2 = t - 2m \geq 0 & \iff t \geq 2m \\ F_t \cdot h_3 = 2t - 5m \geq 0 & \iff t \geq \frac{5}{2}m \\ F_t \cdot h_4 = 3t - 2m - 5m \geq 0 & \iff t \geq \frac{7}{3}m \\ F_t \cdot h_5 = 4t - 3 \cdot 2m - 3m \geq 0 & \iff t \geq \frac{9}{4}m \\ F_t \cdot h_6 = 5t - 2 \cdot 6m \geq 0 & \iff t \geq \frac{12}{5}m \end{aligned}$$

$$F_t \cdot h_7 = 6t - 3m - 2 \cdot 5m \geq 0 \iff t \geq \frac{13}{6}m$$

The strongest condition on t is $t \geq \frac{5}{2}m$. Thus, $N = \frac{5}{2}m$ and F_t is numerically effective for all $t \geq \frac{5}{2}m$. Thus,

$$H_{I^{(m)}}(t) = h^0(X, \mathcal{F}_t) = \binom{t+2}{2} - 6 \binom{m+1}{2}$$

for all $t \geq \frac{5}{2}m$.

Step 2: Now we want to find an optimal numerically effective divisor D such that $[F_t] \cdot [D] < 0$ for small t . By the calculations in Step 1,

$$D = 5L - 2(E_1 + \cdots + E_6)$$

is numerically effective ($D = F_5$ when $m = 2$).

If $[F_t]$ is the class of an effective divisor then $[D] \cdot [F_t] \geq 0$. Thus, if $[D] \cdot [F_t] < 0$ then $[F_t]$ is not effective. Note that

$$[D] \cdot [F_t] = 5t - 2 \cdot 6m < 0 \iff t < \frac{12m}{5}.$$

Thus, $[F_t]$ is not the class of an effective divisor when $t < \frac{12m}{5}$ so $h^0(X, \mathcal{F}_t) = 0$ for $t < \frac{12}{5}m$. We set $M = \frac{12}{5}m$.

Step 3: Starting with this step we will make a divisibility assumption on m . Suppose that m is divisible by both 5 and 2, so

$$m = 10m'$$

for some integer m' . The goal of this step is to show that $F_{\frac{12}{5}m} = F_{24m'}$ is in the class of an effective divisor; to do this we follow Procedure 5.4.7. One can check that the only exceptional class that has a negative intersection multiplicity with $[F_{24m'}]$ is $h_3 = [2L - E_1 - \cdots - E_5]$:

$$[F_{24m'}] \cdot h_3 = 2 \cdot 24m' - 5 \cdot 10m' = -2m'.$$

At this point it will be useful to distinguish between the permutations of h_3 ; we will denote $[2L - E_1 - \cdots - \widehat{E}_i - \cdots - E_6]$ by h_{3_i} .

Lemma 5.4.9. *Let $F_t = tL - m(E_1 + \cdots + E_6)$. If $[F_t] \cdot h_{3_1} < 0$ then $([F_t] - h_{3_1} - \cdots - h_{3_i}) \cdot h_{3_{i+1}} < 0$ for $i = 1, \dots, 5$.*

Proof. Suppose that $[F_t] \cdot h_{3_1} < 0$, or equivalently, that $t < \frac{5}{2}m$. Then

$$[F_t] - h_{3_1} - \cdots - h_{3_i} = (t - 2i)e_0 - (m - (i - 1))(e_1 + \cdots + e_i) - (m - i)(e_{i+1} + \cdots + e_6)$$

so

$$([F_t] - h_{3_1} - \cdots - h_{3_i}) \cdot h_{3_{i+1}} = 2(t - 2i) - (m - i + 1)(i) - (m - i)(6 - i - 1) = 2t - 5m$$

which is less than 0 since $t < \frac{5}{2}m$. □

We will denote the sum $h_{3_1} + \cdots + h_{3_5}$ by $[Y]$ and call it a *cycle*. Note that

$$[Y] = (2 \cdot 6)e_0 - 5(e_1 + \cdots + e_6).$$

By Lemma 5.4.9, if $[F_t] \cdot h_3 < 0$ for one permutation then we subtract an entire cycle from $[F_t]$ when following Procedure 5.4.7.

When following Procedure 5.4.7, we subtract off $2m'$ full cycles from $[F_{24m'}]$;

$$\begin{aligned} [F_{24m'}] - 2m'[Y] &= (24m' - 12 \cdot 2m')e_0 - (10m' - 5 \cdot 2m')(e_1 + \cdots + e_6) \\ &= 0e_0 - 0(e_1 + \cdots + e_6) \end{aligned}$$

so $H_{24m'} = 0$. Therefore, $H_{I^{(m)}}(24m') = h^0(X, \mathcal{F}_{24m'}) = h^0(X, 0) = 1$ and $\alpha(m) = \frac{12m}{5}$ when m is divisible by 10.

Step 4: Again, in this section we will assume that 10 divides m and we write $m = 10m'$ for some integer m' . Then $N = \frac{5}{2}m = 25m'$. By Lemma 5.4.8, there are $H_Z(N + 1) - H_Z(N)$ monomials in $\text{gin}(I^{(m)})$ that involve only x and y . Since F_N and F_{N+1} are numerically effective by Step 1,

$$\begin{aligned} H_Z(N + 1) - H_Z(N) &= h^0(X, \mathcal{F}_{N+1}) - h^0(X, \mathcal{F}_N) \\ &= \binom{N + 2}{1} = N + 2. \end{aligned}$$

Thus, $\text{gin}(I_{Z_m})_{N+1}$ contains all monomials of degree $N + 1$ in the variables x and y .

Now we need to determine $H_Z(N) - H_Z(N - 1)$ and show that it is less than $N + 1$ (that is, $\text{gin}(I_{Z_m})_N$ does not contain all monomials in x and y of degree N). Consider

$$F_{N-1} = (25m' - 1)L - 10m'(E_1 + \cdots + E_6).$$

Then, following Procedure 5.4.7, we can subtract exactly 2 cycles $[Y]$ from $[F_{N-1}]$ to obtain $[H_{N-1}]$. We get

$$[H_{N-1}] = (25m' - 1 - 24)e_0 - (10m' - 10)(e_1 + \cdots + e_6)$$

so, by Corollary 5.4.3,

$$\begin{aligned} h^0(X, \mathcal{F}_{N-1}) &= h^0(X, \mathcal{H}_{N-1}) \\ &= \frac{25}{2}m'^2 - \frac{35}{2}m' + 6. \end{aligned}$$

By Step 1, F_N is numerically effective so, again by Corollary 5.4.3,

$$h^0(X, \mathcal{F}_N) = \frac{25}{2}m'^2 + \frac{15}{2}m' + 1$$

Thus,

$$H_{I_{Z_m}}(N) - H_{I_{Z_m}}(N-1) = 25m' - 5 < N + 1 = 25m' + 1$$

and not all monomials in x and y of degree N are contained in $\text{gin}(I^{(m)})$. Therefore, the largest degree generator of $\text{gin}(I_{Z_m})$ is of degree $N + 1 = \frac{5}{2}m + 1$ when m is divisible by 10.

Step 5: By Step 4, the highest degree generator of $\text{gin}(I^{(m)})$ is of degree $\frac{5}{2}m + 1$ when m is divisible by 10. By Step 3, the smallest degree element of $\text{gin}(I^{(m)})$ is of degree $\alpha(m) = \frac{12m}{5}$ when m is divisible by 10. Thus, the intercepts of the limiting shape of the symbolic generic initial system of I are $(0, \frac{5}{2})$ and $(\frac{12}{5}, 0)$. Since

$$\frac{12}{5} \cdot \frac{5}{2} = 6,$$

Corollary 5.2.16 tells us that the boundary of the limiting shape is defined by the line through the intercepts and is as claimed in Theorem 5.1.1(b).

5.4.2.2 7 General Points

Throughout this section I is the ideal of 7 points p_1, \dots, p_7 of \mathbb{P}^2 in general position and $Z = m(p_1 + \cdots + p_7)$ so $I_{Z_m} = I^{(m)}$. The exceptional classes of the blow-up X of \mathbb{P}^2 at p_1, \dots, p_7 are those in Lemma 5.4.5 with one of the e_i set to 0.

Step 1: To find an N such that $F_t = tL - m(E_1 + \cdots + E_7)$ is numerically effective for all $t \geq N$ we will use the permutation of the exceptional curves from Lemma 5.4.5

that is most likely to make $h_i \cdot [F_t]$ negative. Similar to the case of six points, the strongest condition on t from $h_i \cdot [F_t] \geq 0$ is $t \geq \frac{8}{3}m$. Thus, $N = \frac{8}{3}m$ and F_t is numerically effective for all $t \geq N$. Further,

$$H_{I^{(m)}}(t) = h^0(X, F_t) = \binom{t+2}{2} - 7 \binom{m+1}{2}$$

for all $t \geq \frac{8}{3}m$.

Step 2: Now we want to find an optimal numerically effective divisor D . By the calculations in Step 1,

$$D = 8L - 3(E_1 + \cdots + E_7)$$

is numerically effective ($D = F_8$ when $m = 3$).

If $[F_t]$ is the class of an effective divisor then $[D] \cdot [F_t] \geq 0$. We want to know when $[D] \cdot [F_t]$ is strictly less than 0 because this will imply that $[F_t]$ is not the class of an effective divisor. Note that

$$[D] \cdot [F_t] = 8t - 3 \cdot 7m < 0 \iff t < \frac{21m}{8}.$$

Thus, $h^0(X, \mathcal{F}_t) = 0$ for $t < \frac{21}{8}m$. We set $M = \frac{21}{8}m$.

Our next goal is to show that this is an optimal value. That is, if $\frac{21}{8}m$ is an integer, then $h^0(X, \mathcal{F}_{\frac{21}{8}m}) \neq 0$.

Step 3: Starting with this step we will make a divisibility assumption on m and suppose that m is divisible by both 8 and 3, so

$$m = 24m'$$

for some integer m' . The goal of this step is to show that $F_{\frac{21}{8}m} = F_{21 \cdot 3m'}$ is in the class of an effective divisor; to do this we will follow Procedure 5.4.7. One can check that the only exceptional class which has a negative intersection multiplicity with $[F_{63m'}]$ is $h_4 = [3L - 2E_1 - E_2 - \cdots - E_7]$:

$$[F_{21 \cdot 3m'}] \cdot h_4 = 3 \cdot 63m' - 2 \cdot 24m' - 6 \cdot 24m' = -3m'.$$

At this point it will be useful to distinguish between the permutations of h_4 ; we will denote $[3L - 2E_i - E_1 - \cdots - \widehat{E_i} - \cdots - E_7]$ by h_{4_i} .

Lemma 5.4.10. *Let $F_t = tL - m(E_1 + \cdots + E_7)$. If $[F_t] \cdot h_{4_1} < 0$ then $([F_t] - h_{4_1} - \cdots - h_{4_i}) \cdot h_{4_{i+1}} < 0$ for $i = 1, \dots, 6$.*

We will denote the sum of all seven permutations of h_4 by $[Y]$ and call it one *cycle*. Note that

$$[Y] = 21e_0 - 8(e_1 + \cdots + e_7).$$

It is easy to see that we can subtract off $3m'$ full cycles from $[F_{21 \cdot 3m'}]$ to obtain $[H_{21 \cdot 3m'}]$. We have

$$\begin{aligned} [F_{21 \cdot 3m'}] - 3m'[Y] &= (21 \cdot 3m' - 21 \cdot 3m')e_0 - (24m' - 8 \cdot 3m')(e_1 + \cdots + e_7) \\ &= 0L - 0(e_1 + \cdots + e_7). \end{aligned}$$

so $H_{21 \cdot 3m'} = 0$ and $h_{I_{Z_m}}(21 \cdot 3m') = h^0(X, F_{21 \cdot 3m'}) = h^0(X, 0) = 1$ and $\alpha(m) = 21 \cdot 3m' = \frac{21m}{8}$ when m is divisible by $8 \cdot 3$.

Step 4: Again, in this section we will assume that 24 divides m , so we write $m = 24m'$ for some integer m' . Then $N = \frac{8}{3}m = 8^2m'$. We now want to show that the number of monomials in the variables x and y in $\text{gin}(I^{(m)})_{N+1}$ is equal to $N + 2$ and that the number of monomials of degree N in x and y in $\text{gin}(I^{(m)})_N$ is less than $N + 1$. This will prove that the highest degree generator occurs in degree $N + 1$.

By Lemma 5.4.8, there are $H_Z(N + 1) - H_Z(N)$ monomials in $\text{gin}(I^{(m)})$ that involve only x and y . Then, since F_N and F_{N+1} are numerically effective by Step 1,

$$H_Z(N + 1) - H_Z(N) = \binom{N + 2}{1} = N + 2.$$

Thus, $\text{gin}(I_{Z_m})_{N+1}$ contains all monomials of degrees $N + 1$ in the variables x and y .

Now we need to determine $H_Z(N) - H_Z(N - 1)$ and show that it is less than $N + 1$ (that is, $\text{gin}(I_{Z_m})_N$ does not contain all monomials in x and y of degree N).

Consider

$$F_{N-1} = (8^2m' - 1)L - 24m'(E_1 + \cdots + E_7).$$

Recall from Step 3 that one cycle is equal to $[Y] = 3 \cdot 7e_0 - 8(e_1 + \cdots + e_7)$. By Procedure 5.4.7,

$$[H_{N-1}] = (8^2m' - 1 - 63)e_0 - (24m' - 24)(e_1 + \cdots + e_7)$$

so, by Corollary 5.4.3,

$$h^0(X, \mathcal{F}_{N-1}) = h^0(X, \mathcal{H}_{N-1}) = 32m'^2 - 52m' + 21.$$

From Step 1 we know that F_N is numerically effective and

$$h^0(X, \mathcal{F}_N) = 32m'^2 + 12m' + 1$$

Thus,

$$H_{I_{Z_m}}(N) - H_{I_{Z_m}}(N - 1) = 64m' - 20 < N + 1 = 64m' + 1$$

and not all monomials in x and y of degree N are contained in $\text{gin}(I^{(m)})$. Therefore, the largest degree generator of $\text{gin}(I_{Z_m})$ is of degree $N + 1 = \frac{8}{3}m + 1$ when m is divisible by 24.

Step 5: By Step 4, the highest degree generator of $\text{gin}(I^{(m)})$ is of degree $\frac{8}{3}m + 1$ when m is divisible by 24. By Step 3, the smallest degree element of $\text{gin}(I^{(m)})$ is of degree $\alpha(m) = \frac{21m}{8}$ when m is divisible by 24. Thus, the intercepts of the limiting shape of the symbolic generic initial system of I are $(0, \frac{8}{3})$ and $(\frac{21}{8}, 0)$. Since

$$\frac{8}{3} \cdot \frac{21}{8} = 7,$$

Corollary 5.2.16 tells us that the limiting shape is defined by the line through the intercepts and is as claimed in Theorem 5.4.

5.4.2.3 8 General Points

Throughout this section I is the ideal of 8 points p_1, \dots, p_8 of \mathbb{P}^2 in general position and $Z_m = m(p_1 + \dots + p_8)$ so $I_{Z_m} = I^{(m)}$. The exceptional classes of the blow-up X of \mathbb{P}^2 at p_1, \dots, p_8 are those in Lemma 5.4.5.

Step 1: To find an N such that $F_t = tL - m(E_1 + \dots + E_8)$ is numerically effective for all $t \geq N$ we compute $[C] \cdot [F_t]$ for all exceptional classes $[C]$ of X . Similar to the case of six points, we can check that the strongest condition on t in resulting from $[C] \cdot [F_t] \geq 0$ is $t \geq \frac{17}{6}m$. Thus, $N = \frac{17}{6}m$ and F_t is numerically effective for all $t \geq N$. Further,

$$h^0(X, \mathcal{F}_t) = \binom{t+2}{2} - 8 \binom{m+1}{2}$$

for all $t \geq \frac{17}{6}m$.

Step 2: Now we want to find an optimal numerically effective divisor D . By the calculations in Step 1,

$$D = 17L - 6(E_1 + \dots + E_8)$$

is numerically effective ($D = F_{17}$ when $m = 6$).

If $[F_t]$ is the class of an effective divisor then $[D] \cdot [F_t] \geq 0$. We want to know when $[D] \cdot [F_t]$ is strictly less than 0 which will imply that $[F_t]$ is not the class of an effective divisor. Note that

$$[D] \cdot [F_t] = 17t - 6 \cdot 8m < 0 \iff t < \frac{48m}{17}.$$

Thus, $h^0(X, \mathcal{F}_t) = 0$ for $t < \frac{48}{17}m$. We set $M = \frac{48}{17}m$.

Our next goal is to show that this is an optimal value. That is, if $\frac{48}{17}m$ is an integer, then $h^0(X, \mathcal{F}_{\frac{48}{17}m}) \neq 0$.

Step 3: Starting with this step we will make a divisibility assumption on m and suppose that m is divisible by both 17 and 6, so

$$m = 17 \cdot 6m'$$

for some integer m' . The goal of this step is to show that $F_{\frac{48}{17}m} = F_{48 \cdot 6m'}$ is in the class of an effective divisor. To do this we will follow Procedure 5.4.7 to find $H_{48 \cdot 6m'}$. One can check that the only exceptional class that has a negative intersection multiplicity with $[\mathcal{F}_{48 \cdot 6m'}]$ is $h_7 = [6L - 3E_1 - 2E_2 - \dots - 2E_8]$:

$$[F_{48 \cdot 6m'}] \cdot h_7 = 6 \cdot 48 \cdot 6m' - 3 \cdot 17 \cdot 6m' - 2 \cdot 7 \cdot 17 \cdot 6m' = -6m'.$$

It will be useful to distinguish between the permutations of h_7 . We will denote $[6L - 3E_i - 2E_1 - \dots - \widehat{2E_i} - \dots - 2E_7]$ by h_{7_i} .

Lemma 5.4.11. *Let $F_t = tL - m(E_1 + \dots + E_8)$. If $[F_t] \cdot h_{7_1} < 0$ then $([F_t] - h_{7_1} - \dots - h_{7_i}) \cdot h_{7_{i+1}} < 0$ for $i = 1, \dots, 7$.*

We will denote the sum of all eight permutations of h_7 by $[Y]$ and call it a *cycle*. Note that

$$Y = 48e_0 - 17(e_1 + \dots + e_8).$$

Following Procedure 5.4.7, we subtract off $6m'$ full cycles from $[F_{48 \cdot 6m'}]$ to get $H_{48 \cdot 6m'}$. Then

$$\begin{aligned} [F_{48 \cdot 6m'}] - 6m'[Y] &= (48 \cdot 6m' - 48 \cdot 6m')e_0 - (17 \cdot 6m' - 17 \cdot 6m')(e_1 + \dots + e_8) \\ &= 0e_0 - 0(e_1 + \dots + e_7). \end{aligned}$$

Therefore, $h^0(X, \mathcal{F}_{48 \cdot 6m'}) = h^0(X, 0) = 1$ and $\alpha(m) = 48 \cdot 6m' = \frac{48m}{17}$ in this case.

Step 4: Again, in this section we will assume that 6 and 17 divide m , so $m = 17 \cdot 6m'$ for some integer m' and $N = \frac{17}{6}m = 17^2m'$. We now want to show that the number of monomials in only x and y in $\text{gin}(I^{(m)})_{N+1}$ is $N + 2$ and that the number of monomials of degree N in the variables x and y in $\text{gin}(I^{(m)})$ is less than $N + 1$. This will show that the highest degree generator occurs in degree $N + 1$.

By Lemma 5.4.8, there are $H_Z(N + 1) - H_Z(N)$ monomials in $\text{gin}(I^{(m)})$ that involve only x and y . Then

$$H_Z(N + 1) - H_Z(N) = \binom{N + 2}{1} = N + 2$$

Thus, $\text{gin}(I_{Z_m})_{N+1}$ contains all monomials of degrees $N + 1$ in the variables x and y .

Now we need to determine $H_Z(N) - H_Z(N - 1)$ and show that it is less than $N + 1$ (that is, $\text{gin}(I_{Z_m})_N$ does not contain all monomials in x and y of degree N).

Consider

$$F_{N-1} = (17^2m' - 1)L - 17 \cdot 6m'(E_1 + \cdots + E_8).$$

Recall that one cycle is equal to $[Y] = 48e_0 - 17(e_1 + \cdots + e_7)$. It is easy to see that exactly 6 cycles can be subtracted off of $[F_{N-1}]$ to obtain $[H_{N-1}]$. We have

$$[H_{N-1}] = (17^2m' - 1 - 6 \cdot 48)e_0 - (17 \cdot 6m' - 17 \cdot 6)(e_1 + \cdots + e_8)$$

so, by Corollary 5.4.3,

$$h^0(X, \mathcal{F}_t) = \frac{289}{2}m'^2 - \frac{527}{2}m' + 120$$

From Step 1 we know that F_N is numerically effective so

$$h^0(X, \mathcal{F}_N) = \frac{289}{2}m'^2 + \frac{51}{2}m' + 1$$

Thus,

$$H_Z(N) - H_Z(N - 1) = 238m' - 119 < N + 1 = 289m' + 1$$

and not all monomials in x and y of degree N are contained in $\text{gin}(I^{(m)})$. Therefore, the largest degree generator of $\text{gin}(I_{Z_m})$ is of degree $N + 1 = \frac{17}{6}m + 1$ when m is divisible by $17 \cdot 6$.

Step 5: By Step 4, the highest degree generator of $\text{gin}(I^{(m)})$ is of degree $\frac{17}{6}m + 1$ when m is divisible by $17 \cdot 6$. By Step 3, the smallest degree element of $\text{gin}(I^{(m)})$ is of

degree $\alpha(m) = \frac{48m}{17}$ when m is divisible by $17 \cdot 6$. Thus, the intercepts of the limiting shape of the symbolic generic initial system of I are $(0, \frac{17}{6})$ and $(\frac{48}{17}, 0)$. Since

$$\frac{17}{6} \cdot \frac{48}{17} = 8,$$

Corollary 5.2.16 tells us that the limiting shape is defined by the line through the intercepts and is as claimed in Theorem 5.4.

5.5 The Symbolic Generic Initial System of 5 or Fewer Uniform Fat Points in General Position

In this section we prove part (c) of the main theorem.

Theorem 5.1.1 (c). *Suppose that $1 < r \leq 5$ and I is the ideal of r points in general position. Then the limiting shape of the symbolic generic initial system $\{gin(I^{(m)})\}$ has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$ when $\frac{r}{2} \geq 2$ and $(\frac{r}{2}, 0)$ and $(0, 2)$ when $\frac{r}{2} < 2$.*

This is an immediate consequence of the following result of [May12e] since five or fewer points of \mathbb{P}^2 in general position lie on an irreducible conic.

Theorem 5.5.1. *Suppose that I is the ideal of r points in \mathbb{P}^2 lying on an irreducible conic. Then the limiting shape of the symbolic generic initial system $gin(I^{(m)})$ has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$ when $\frac{r}{2} \geq 2$ and $(\frac{r}{2}, 0)$ and $(0, 2)$ when $\frac{r}{2} < 2$.*

5.6 Final Example

The results presented here and in [May12d] may lead the reader to believe that the limiting shape of any symbolic generic initial system is defined by a single hyperplane. The following example shows that this does not hold even for ideals of points in \mathbb{P}^2 .

Example 5.6.1. Suppose that I is the ideal of the $l + 1 \geq 4$ points p_1, \dots, p_l, p_{l+1} of \mathbb{P}^2 where p_1, \dots, p_l lie on a line and p_{l+1} lies off of the line.

Proposition 5.6.1. *Let I be the ideal of $l + 1$ distinct points of \mathbb{P}^2 where l of the points lie on a line and suppose that $l(l - 1)$ divides m . Then the highest degree generator of $gin(I^{(m)})$ is of degree lm and the lowest degree generator of $gin(I^{(m)})$ is of degree $2m - \frac{m}{l}$.*

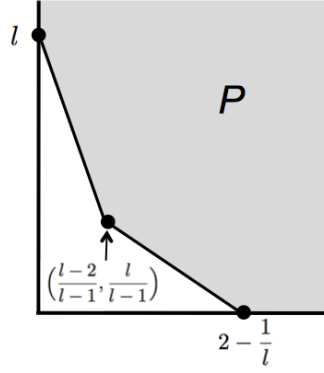


Figure 5.2: The limiting shape P of the symbolic generic initial system of the ideal of l points on a line and one point off.

Idea of Proof. The proof of this proposition is similar to the work contained in Section 5.4 with the following considerations. In this case, the blow-up $\pi : X \rightarrow \mathbb{P}^2$ of p_1, \dots, p_{l+1} has exceptional curves with classes $[L - E_1 - E_2 - \dots - E_l]$ and $[L - E_i - E_{l+1}]$ for $i = 1, \dots, l$ where $E_j = \pi^{-1}(p_j)$ and L is the total transform of a general line in \mathbb{P}^2 (note that the exceptional curves are the total transforms of lines through the points \mathbb{P}^2 ; see [Har98]). \square

If P is the limiting shape of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$, then Proposition 5.6.1 implies that the boundary of P has y -intercept

$$\lim_{m \rightarrow \infty} \frac{lm}{m} = l$$

and x -intercept

$$\lim_{m \rightarrow \infty} \frac{2m - \frac{m}{l}}{m} = 2 - \frac{1}{l}.$$

If the boundary of P was defined by the line through these intercepts, the volume under of P would be

$$\text{vol}(Q) = \frac{(l)(2 - \frac{1}{l})}{2} = l - \frac{1}{2}.$$

However, by Lemma 5.2.15, the volume under of P must be $\frac{l+1}{2}$ which is strictly smaller than $l - \frac{1}{2}$ ($l \geq 3$). Thus, P is not defined by the line through the intercepts. In fact, one can prove that the limiting shape P is the one shown in Figure 5.2.

CHAPTER VI

The Symbolic Generic Initial System of Points on an Irreducible Conic

The general research trend looking at the asymptotic behavior of collections of algebraic objects is motivated by the idea that there is often a structure revealed in the limit that is difficult to see when studying individual objects (see, for example, [ELS01], [Siu98], [Hun92], and [ES09]). The asymptotic behavior of a collection of monomial ideals \mathfrak{a}_\bullet such that $\mathfrak{a}_i \cdot \mathfrak{a}_j \subseteq \mathfrak{a}_{i+j}$ (a *graded system of monomial ideals*) can be described by its *limiting shape* P . If $P_{\mathfrak{a}_i}$ denotes the Newton polytope of \mathfrak{a}_i , then the limiting shape P is defined to be the limit $\lim_{m \rightarrow \infty} \frac{1}{m} P_{\mathfrak{a}_m}$ ([May12c]). In addition to giving a simple geometric interpretation of the limiting behavior, P completely determines the asymptotic multiplier ideals of \mathfrak{a}_\bullet (see [How01]).

Generic initial ideals have a nice combinatorial structure, but are often difficult to compute and usually have complicated sets of generators (see [Gre98] for a survey or [Cim06] and [May12a] for examples). This motivates a series of work describing the limiting shape of generic initial systems, $\{\text{gin}(I^m)\}_m$, and of symbolic generic initial systems, $\{\text{gin}(I^{(m)})\}_m$ ([May12c], [May12c], [May12a], [May12b]). The goal of this paper is to describe the limiting shape of the symbolic generic initial system of the ideal of r points in \mathbb{P}^2 lying on an irreducible conic.

We will see that when I is the ideal of points in \mathbb{P}^2 , each of the polytopes $P_{\text{gin}(I^{(m)})}$, and thus P itself, can be thought of as a subset of \mathbb{R}^2 . The following theorem describes P in the case we are interested in.

Theorem 6.0.2. *Let $I \subseteq R = K[x, y, z]$ be the ideal of $r > 1$ distinct points p_1, \dots, p_r of \mathbb{P}^2 lying on an irreducible conic and let $P \subseteq \mathbb{R}_{\geq 0}^2$ be the limiting shape of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. If $r \geq 4$, then P has a boundary defined by the line through the points $(2, 0)$ and $(0, \frac{r}{2})$ (see Figure 6.1). If*

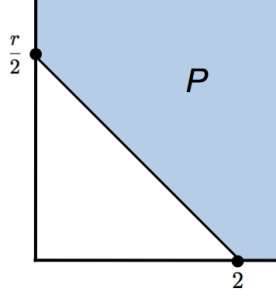


Figure 6.1: The limiting shape P of $\{\text{gin}(I^{(m)})\}_m$ where I is the ideal of $r \geq 4$ points lying on an irreducible conic.

$r = 2$ or $r = 3$, then P has a boundary defined by the line through the points $(\frac{r}{2}, 0)$ and $(0, 2)$.

The proof of this theorem is an application of ideas that have been described elsewhere. Rather than repeating arguments here, we refer the reader elsewhere for details where necessary.

The following result describes the structure of the individual ideals $\text{gin}(I^{(m)})$ that make up the generic initial system.

Theorem 6.0.3. *Suppose that $I \subseteq K[x, y, z]$ be the ideal of a set of distinct points of \mathbb{P}^2 . Then the minimal generators of $\text{gin}(I^{(m)})$ are*

$$\{x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_{\alpha(m)-1}(m)}, \dots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\}$$

for some positive integers $\lambda_0(m), \dots, \lambda_{\alpha(m)-1}(m)$ such that $\lambda_0(m) > \lambda_1(m) > \dots > \lambda_{\alpha(m)-1}(m)$. Further, if the minimal free resolution of $I^{(m)}$ is of the form

$$0 \rightarrow F_1 = \bigoplus_{i=1}^{\psi} R(-u_i) \rightarrow F_1 = \bigoplus_{i=1}^{\mu} R(-d_i) \rightarrow I^{(m)} \rightarrow 0$$

with $U(m) = \max\{u_i\}$ and $D(m) = \min\{d_i\}$, then

$$\alpha(m) = D(m)$$

and

$$\lambda_0(m) = U(m) - 1.$$

Proof. The first part of the theorem is Corollary 2.9 of [May12a] and follows from results in [BS87a] and [HS98]. The second statement follows the a result of Hilbert-

Burch, which says that, with the notation in the theorem, the minimal free resolution of $\text{gin}(I^{(m)})$ is of the form

$$0 \rightarrow G_1 \rightarrow G_0 \rightarrow \text{gin}(I^{(m)}) \rightarrow 0$$

where $G_1 = \bigoplus_{i=0}^{\alpha(m)} R(-\lambda_i(m) - i - 1)$ and $G_0 = [\bigoplus_{i=0}^{\alpha(m)} R(-\lambda_i(m) - i)] \oplus R(-\alpha(m))$ (Corollary 4.15 of [Gre98]). A *consecutive cancellation* takes a sequence $\{\beta_{i,j}\}$ to a new sequence by replacing $\beta_{i,j}$ by $\beta_{i,j} - 1$ and $\beta_{i+1,j}$ by $\beta_{i+1,j} - 1$. The ‘Cancellation Principle’ says that the graded Betti numbers of J can be obtained by the graded Betti numbers of $\text{gin}(J)$ by making a series of consecutive cancellations (see Corollary 1.21 of [Gre98]). Since $\lambda_0(m) + 1 > \lambda_i(m) + i$ for all i , $\beta_{1,\lambda_0+1} \geq 1$ does not change with any such consecutive cancellation; thus, $R(-\lambda_0(m) - 1)$ is the summand of F_1 with the largest shift. Likewise, $\alpha(m) < \lambda_i(m) + i + 1$ for all i so $\beta_{0,\alpha(m)} \geq 1$ does not change with a consecutive cancellation; thus, $R(-\alpha(m))$ is the summand of F_0 with the smallest shift. \square

In the case where m is even and I is the ideal of $r \geq 3$ points lying on an irreducible conic in \mathbb{P}^2 , we will see in Proposition 6.0.6 that we can write down the entire minimal free resolution of $I^{(m)}$. This will give us $D(m)$ and $U(m)$ when m is even so that we can find the powers of x and y , $x^{\alpha(m)} = x^{D(m)}$ and $y^{\lambda_0(m)} = y^{U(m)-1}$, that appear in a minimal generating set of $\text{gin}(I^{(m)})$. In particular, Proposition 6.0.6 implies the following.

Lemma 6.0.4. *Suppose that I is the ideal of $r \geq 3$ points in \mathbb{P}^2 lying on an irreducible conic and use the notation of the previous theorem.*

- (a) *If $r \geq 4$ is even, $D(m) = 2m$ and $U(m) = \frac{rm}{2} + 2$.*
- (b) *If $r > 4$ is odd and m is even, then $D(m) = 2m$ and $U(m) = \frac{rm}{2} + 2$.*
- (c) *If $r = 3$ and m is even, then $D(m) = \frac{3m}{2}$ and $U(m) = 2m + 1$.*

By Lemma 6.0.3, each of the generic initial ideals $\text{gin}(I^{(m)})$ is generated in the variables of x and y , so we can think of each Newton polytope $P_{\text{gin}(I^{(m)})}$, and thus the limiting shape P itself, as a subset of \mathbb{R}^2 .

The following result is the key for proving the main theorem: it describes when the limiting shape P of the symbolic generic initial system in \mathbb{P}^2 is defined by a single boundary line. The proof is contained in [May12a].

Proposition 6.0.5 (Corollary 2.16 of [May12a]). *Let $I \subseteq K[x, y, z]$ be the ideal of r distinct points in \mathbb{P}^2 and let P be the limiting shape of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Suppose that the x -intercept γ_1 and the y -intercept γ_2 of the boundary of P are such that $\gamma_1 \cdot \gamma_2 = r$. Then the limiting shape P has a boundary defined by the line passing through $(\gamma_1, 0)$ and $(0, \gamma_2)$.*

With these results in mind, we can now prove the main theorem.

Proof of Theorem 6.0.2. Suppose first that $r \geq 4$ and that m is even if r is odd. By Theorem 6.0.3 and Lemma 6.0.4, $x^{D(m)} = x^{2m}$ and $y^{U(m)-1} = y^{\frac{rm}{2}+1}$ are the smallest powers of x and y contained in $\text{gin}(I^{(m)})$. This means that the intercepts of the boundary of $P_{\text{gin}(I^{(m)})}$ are $(2m, 0)$ and $(0, \frac{rm}{2} + 1)$. Thus, the intercepts of the boundary of the limiting shape P of the entire symbolic generic initial system are $(\lim_{m \rightarrow \infty} \frac{2m}{m}, 0) = (2, 0)$ and $(0, \lim_{m \rightarrow \infty} \frac{rm/2}{m} + 1) = (0, \frac{r}{2})$.¹ By Proposition 6.0.5, the fact that $\frac{r}{2} \cdot 2 = r$ implies that the limiting shape P is as claimed.

Now suppose that $r = 3$ and that m is even. By the same argument as above, the intercepts of the boundary of the limiting shape P are $(\lim_{m \rightarrow \infty} \frac{3m/2}{m}, 0) = (\frac{3}{2}, 0)$ and $(0, \lim_{m \rightarrow \infty} \frac{2m}{m}) = (0, 2)$. Since $\frac{3}{2} \cdot 2 = 3$, the limiting shape is as claimed.

The case where $r = 2$ follows from the main theorem of [May12c] since I is a type (1,2) complete intersection in this case. \square

It remains to prove Lemma 6.0.4 which follows immediately from the next proposition. In particular, we will write the minimal free resolutions of the ideals $I^{(m)}$ when I is the ideal of $r \geq 3$ points on an irreducible conic and m is even if r is odd.

Proposition 6.0.6. *Let I be the ideal of $r \geq 3$ points of \mathbb{P}^2 lying on an irreducible conic and suppose that the minimal free resolution of $I^{(m)}$ is of the form*

$$0 \rightarrow G_1 \rightarrow G_0 \rightarrow I^{(m)} \rightarrow 0.$$

(a) *If r is even,*

$$G_0 = \bigoplus_{j=0}^m R(-2(m-j) - \frac{rj}{2})$$

and

$$G_1 = \bigoplus_{j=1}^m R(-2(m-j) - \frac{rj}{2} - 2).$$

¹In the case where r is odd we take the limits over even m .

(b) If $r \geq 5$ is odd and m is even,

$$G_0 = \left[\bigoplus_{j=0}^{m/2} R(-2m - j(r-4)) \right] \oplus \left[\bigoplus_{j=0}^{m/2-1} R^2\left(-2m - j(r-4) - \frac{r-1}{2} + 1\right) \right]$$

and

$$G_1 = \left[\bigoplus_{j=1}^{m/2} R(-2m - j(r-4) - 2) \right] \oplus \left[\bigoplus_{j=0}^{m/2-1} R^2\left(-2m - j(r-4) - \frac{r-1}{2}\right) \right].$$

(c) If $r = 3$ and m is even,

$$G_0 = R\left(-\frac{3m}{2}\right) \oplus \left[\bigoplus_{j=0}^{m/2-1} R^3\left(-\frac{3m}{2} - j - 1\right) \right]$$

and

$$G_1 = \bigoplus_{j=0}^{m/2-1} R^3\left(-\frac{3m}{2} - j - 2\right).$$

To prove this proposition we will follow the results of Catalisano described in [Cat91] that can be used to compute the minimal free resolution of *any* fat point ideal

$$I_{(m_1, \dots, m_r)} = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \dots \cap I_{p_r}^{m_r}$$

as long as the points p_1, \dots, p_r lie on an irreducible conic. The following is a specialization of Catalisano's work to the case where $r \geq 4$ and $m_i = m$ for all i (that is, when I is the ideal of a uniform fat point subscheme).

Proposition 6.0.7 ([Cat91]). *Let I be the ideal of $r \geq 4$ points of \mathbb{P}^2 lying on an irreducible conic. Suppose that the minimal free resolution of $I^{(t-1)}$ is of the form*

$$0 \rightarrow F'_1 \rightarrow F'_0 \rightarrow I^{(t-1)} \rightarrow 0$$

where $F'_1 = \bigoplus_{i=1}^{\mu-1} R(-u_i)$ and $F'_0 = \bigoplus_{i=1}^{\mu} R(-d_i)$ and that the minimal free resolution of $I^{(t)}$ is of the form

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow I^{(t)} \rightarrow 0.$$

If rt is even

$$(1) \quad F_1 = \left[\bigoplus_{i=1}^{\mu-1} R(-u_i - 2) \right] \oplus R\left(-\frac{rt}{2} - 2\right) \text{ and} \\ F_0 = \left[\bigoplus_{i=1}^{\mu} R(-d_i - 2) \right] \oplus R\left(-\frac{rt}{2}\right)$$

while if rt is odd

$$(2) \quad F_1 = [\oplus_{i=1}^{\mu-1} R(-u_i - 2)] \oplus R^2(-\frac{rt+1}{2} - 1) \text{ and} \\ F_0 = [\oplus_{i=1}^{\mu} R(-d_i - 2)] \oplus R^2(-\frac{rt+1}{2}).$$

Therefore, one can apply this result m times to find the minimal free resolution of $I^{(m)}$. That is, first find the minimal free resolution of $I^{(1)} = I$ from $0 \rightarrow 0 \rightarrow R(0) \rightarrow I^{(0)} \rightarrow 0$, then find the minimal free resolution of $I^{(2)}$ from that of I , and so on.

Sketch of Proof of Proposition 6.0.6. We will give an idea of how to find the minimal free resolutions of the ideals $I^{(m)}$ using the algorithm in Proposition 6.0.7.

If we are in case (a) where r is even, rt is even for all t , so to find the resolution of $I^{(t)}$ from the minimal free resolution of $I^{(t-1)}$ we follow the first case of Proposition 6.0.7. In particular, to find the resolution of $I^{(m)}$ from the resolution of $I^{(0)}$, we apply part (1) exactly m times for $t = 1, \dots, m$.

If we are in case (b) where r is odd, rt is odd for odd t and rt is even for even t . Thus, we need to apply both cases of Proposition 6.0.7 to find the resolution of $I^{(m)}$. To obtain the resolution

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow I^{(t)} \rightarrow 0$$

of $I^{(t)}$ from the resolution

$$0 \rightarrow F_1'' \rightarrow F_0'' \rightarrow I^{(t-2)} \rightarrow 0$$

of $I^{(t-2)}$ when t is even, one needs to:

- shift each summand of F_0 and F_1 by -4 ;
- add $R^2(-\frac{r(t-1)+1}{2} - 3)$ and $R(-\frac{rt}{2} - 2)$ to F_1'' ; and
- add $R^2(-\frac{r(t-1)+1}{2} - 3)$ and $R(-\frac{rt}{2})$ to F_0'' .

If m is even, we can follow this procedure $\frac{m}{2}$ times with $t = 2, 4, 6, \dots, m$ to find the resolution of $I^{(m)}$ from that of $I^{(0)}$.

For case (c) when $r = 3$ and m is even, one needs to use other results of [Cat91] beyond those stated in Proposition 6.0.7. The general idea is the same as above: use a sequence of fat point schemes $Z_0 = m(p_1 + p_2 + p_3), Z_1, Z_2, \dots, Z_H = 0(p_1 + p_2 + p_3)$ and find the minimal free resolution of $I_{Z_{H-1}}$ from that of I_{Z_H} , then find the minimal free resolution of $I_{Z_{H-2}}$ from that of $I_{Z_{H-1}}$, and so on, until we can find the minimal free resolution of $I^{(m)} = I_{Z_0}$ from that of I_{Z_1} . However, when $r = 3$ not all of

the Z_i will be uniform fat point subschemes. In particular, subsequences of the form $Z_l = t(p_1+p_2+p_3)$, $Z_{l+1} = (t-1)p_1+(t-1)p_2+tp_3$, $Z_{l+2} = (t-1)p_1+(t-2)p_2+(t-1)p_3$, $Z_{l+3} = (t-2)(p_1+p_2+p_3)$ come together to form the sequence Z_0, \dots, Z_H . Refer to [Cat91] for more details. \square

CHAPTER VII

The Asymptotics of Symbolic Generic Initial Systems of Six Points in \mathbb{P}^2

7.1 Introduction

Given a set of six points of \mathbb{P}^{n-1} with ideal $I \subseteq k[\mathbb{P}^{n-1}]$, we may consider the ideal $I^{(m)}$ generated by the polynomials that vanish to at least order m at each of the points. Such ideals are called *uniform fat point ideals* and, although they are easy to describe, they have proven difficult to understand. There are still many open problems and unresolved conjectures related to finding the Hilbert function of $I^{(m)}$ and even the degree $\alpha(I^{(m)})$ of the smallest degree element of $I^{(m)}$ (for example, see [CHT11], [GH07], [GVT04], [GHM09], and [Har02]).

In this paper we will study a *limiting shape* that describes the behavior of the Hilbert functions of the set of fat point ideals $\{I^{(m)}\}_m$ as m approaches infinity. Studying asymptotic behavior has been an important research trend of the past twenty years; while individual algebraic objects may be complicated, the limit of a collection of such objects is often quite nice (see, for example, [Hun92], [Siu98],[ELS01], and [ES09]). Research on fat point ideals has shown that certain challenges in understanding these ideals can be overcome by studying the entire collection $\{I^{(m)}\}_m$. For instance, more can be said about the Seshadri constant

$$\epsilon(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{rm}$$

than the invariants $\alpha(I^{(m)})$ of each ideal (see [BH10] and [Har02]).

To describe the limiting behavior of the Hilbert functions of fat point ideals, we will study the *symbolic generic initial system*, $\{\text{gin}(I^{(m)})\}_m$, obtained by taking the reverse lexicographic generic initial ideals of fat point ideals. When $I \subseteq K[x, y, z]$ is

an ideal of points of \mathbb{P}^2 , knowing the Hilbert function of $I^{(m)}$ is equivalent to knowing the generators of $\text{gin}(I^{(m)})$; thus, describing the limiting behavior of the symbolic generic initial system of I is equivalent to describing that of the Hilbert functions of the fat point ideals $I^{(m)}$ as m gets large.

We define the *limiting shape* P of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$ of the ideal I to be the limit $\lim_{m \rightarrow \infty} \frac{1}{m} P_{\text{gin}(I^{(m)})}$, where $P_{\text{gin}(I^{(m)})}$ denotes the Newton polytope of $\text{gin}(I^{(m)})$. When $I \subseteq K[x, y, z]$ corresponds to an arrangement of points in \mathbb{P}^2 , each of the ideals $\text{gin}(I^{(m)})$ is generated in the variables x and y , so $P_{\text{gin}(I^{(m)})}$, and thus P , can be thought of as a subset of \mathbb{R}^2 .

The main result of this paper is the following theorem describing the limiting shape of the symbolic generic initial system of an ideal corresponding to any collection of 6 points in \mathbb{P}^2 . The concept of *configuration type* mentioned is intuitive; for example, $\{p_1, \dots, p_6\}$ are of configuration type B pictured in Figure 7.1 when there is one line through three of the points but no lines through any other three points and no conics through all six points (see Definition 7.2.3).

Theorem 7.1.1. *Let $I \subseteq K[x, y, z]$ be the ideal corresponding to a set of six points in \mathbb{P}^2 . Then the limiting shape P of the reverse lexicographic symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$ is equal to the limiting shape P shown in Figures 7.1 and 7.2 corresponding to the configuration type of the six points.*

This theorem will be proved in Section 7.4; Sections 7.2 and 7.3 contain background information necessary for the proof.

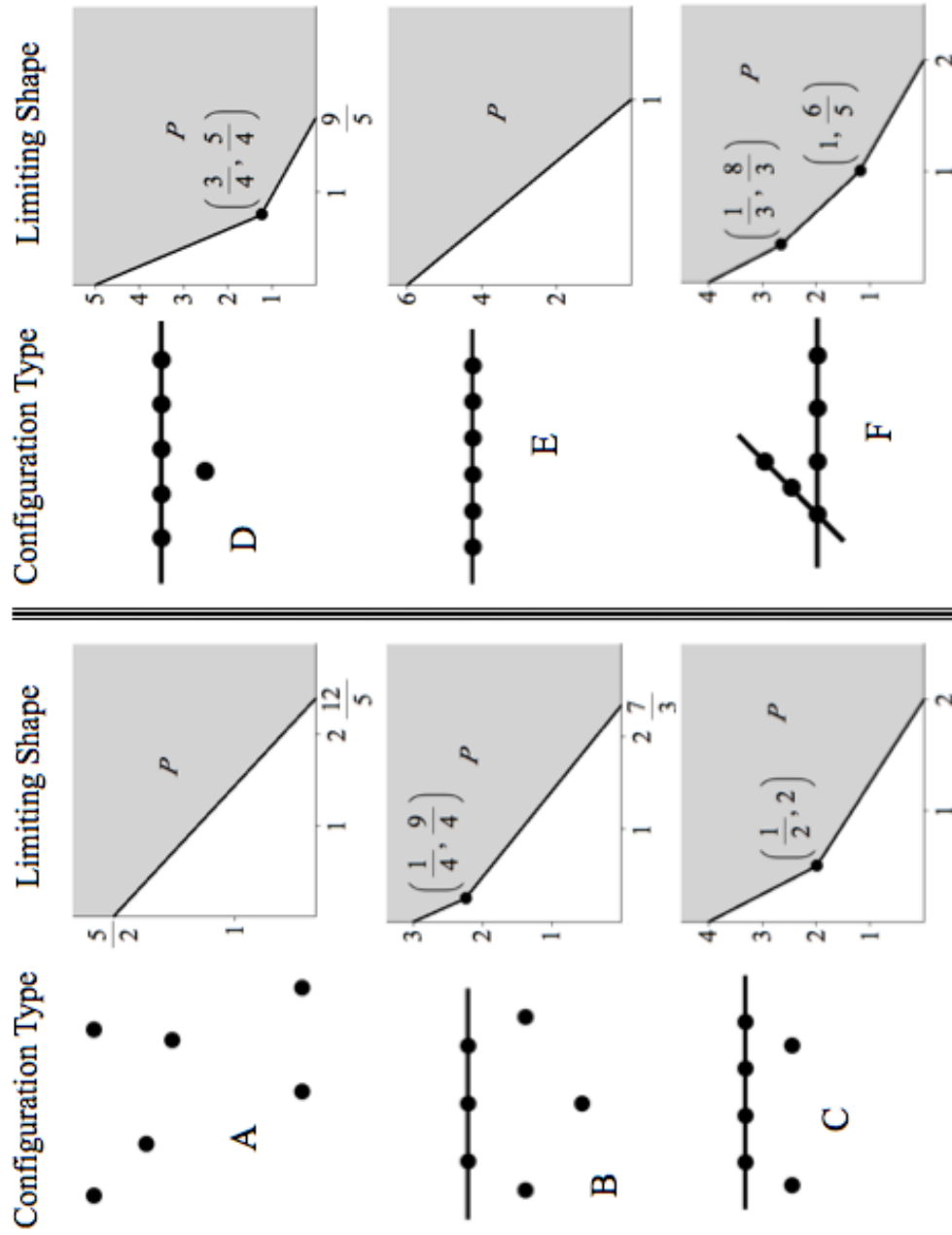


Figure 7.1: The limiting shape P of the generic initial systems $\{\text{gin}(I^{(m)})\}_m$ when I is the ideal corresponding to points $\{p_1, \dots, p_6\}$ in configuration types A through F.

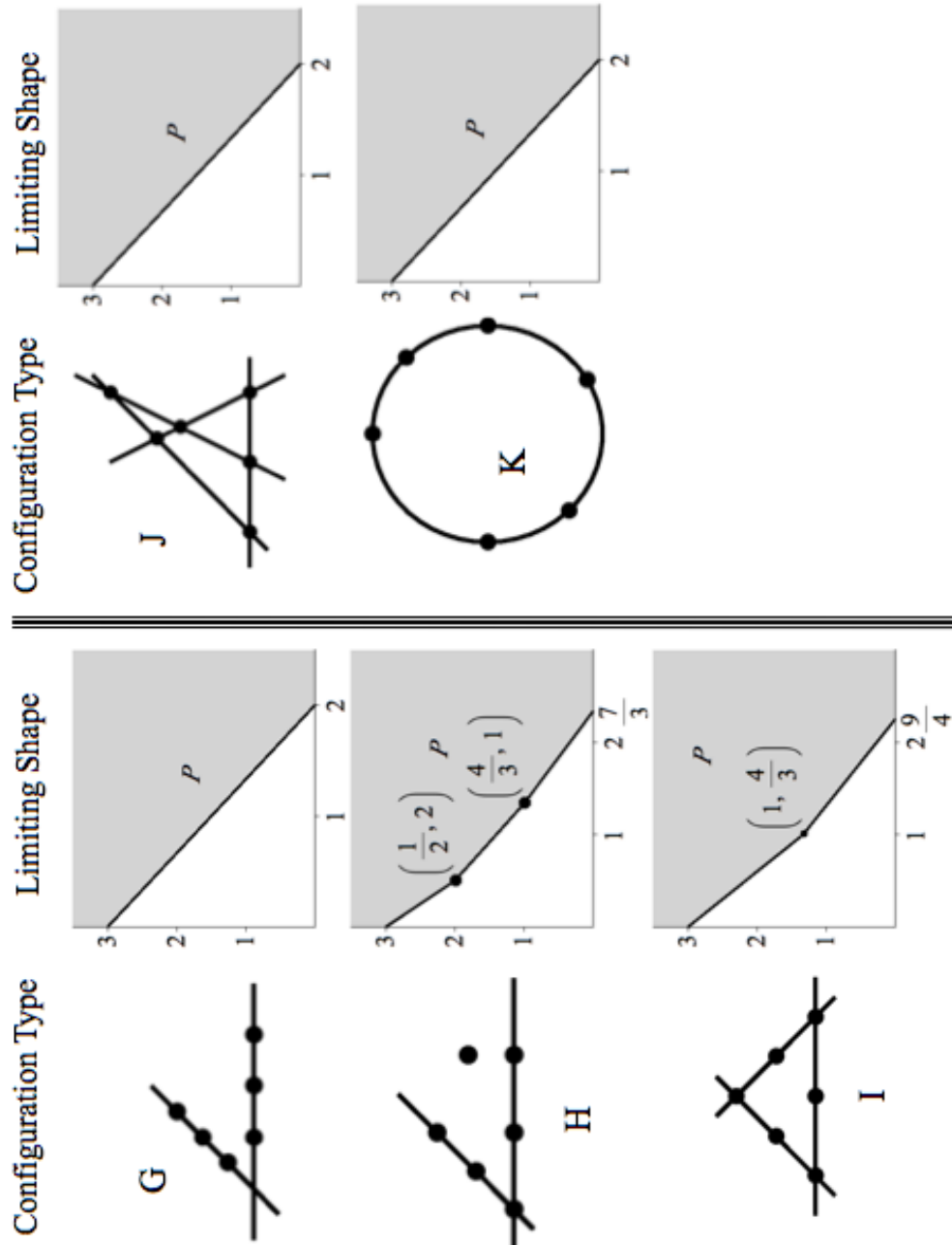


Figure 7.2: The limiting shape P of the generic initial systems $\{\text{gin}(I^{(m)})\}_m$ when I is the ideal corresponding to points $\{p_1, \dots, p_6\}$ in configuration types G through K.

7.2 Background

In this section we will introduce notation, definitions, and results related to fat points in \mathbb{P}^2 , generic initial ideals, and systems of ideals. Unless stated otherwise, $R = K[x, y, z]$ is the polynomial ring in three variables over a field K of characteristic 0 with the standard grading and the reverse lexicographic order $>$ with $x > y > z$.

7.2.1 Fat Points in \mathbb{P}^2

Definition 7.2.1. Let p_1, \dots, p_r be distinct points of \mathbb{P}^2 , I_j be the ideal of $K[\mathbb{P}^2] = R$ consisting of all forms vanishing at the point p_j , and $I = I_1 \cap \dots \cap I_r$ be the ideal of the points p_1, \dots, p_r . A **fat point subscheme** $Z = m_1 p_1 + \dots + m_r p_r$, where the m_i are nonnegative integers, is the subscheme of \mathbb{P}^2 defined by the ideal $I_Z = I_1^{m_1} \cap \dots \cap I_r^{m_r}$ consisting of forms that vanish at the points p_i with multiplicity at least m_i . When $m_i = m$ for all i , we say that Z is **uniform**; in this case, I_Z is equal to the m^{th} symbolic power of I , $I^{(m)}$.

The following lemma relates the symbolic and ordinary powers of I in the case we are interested in (see, for example, Lemma 1.3 of [AV03]).

Lemma 7.2.2. *If I is the ideal of distinct points in \mathbb{P}^2 ,*

$$(I^m)^{\text{sat}} = I^{(m)},$$

where $J^{\text{sat}} = \bigcup_{k \geq 0} (J : \mathfrak{m}^k)$ denotes the saturation of J .

The precise definition of a *configuration type* mentioned in the statement of Theorem 7.1.1 is as follows.

Definition 7.2.3 ([GH07]). Two sets of points $\{p_1, \dots, p_r\}$ and $\{p'_1, \dots, p'_r\}$ of \mathbb{P}^2 have the same **configuration type** if for all sequences of positive integers m_1, \dots, m_r the ideals of the fat point subschemes $Z = m_1 p_1 + \dots + m_r p_r$ and $Z' = m_1 p'_1 + \dots + m_r p'_r$ have the same Hilbert function, possibly after reordering.

Proposition 7.2.4 ([GH07]). *The configuration types for six distinct points in \mathbb{P}^2 are exactly the configurations A through K shown in Figures 7.1 and 7.2.*

7.2.2 Generic Initial Ideals

An element $g = (g_{ij}) \in \text{GL}_n(K)$ acts on $R = K[x_1, \dots, x_n]$ and sends any homogeneous element $f(x_1, \dots, x_n)$ to the homogeneous element

$$f(g(x_1), \dots, g(x_n))$$

where $g(x_i) = \sum_{j=1}^n g_{ij}x_j$. If $g(I) = I$ for every upper triangular matrix g then we say that I is *Borel-fixed*. Borel-fixed ideals are *strongly stable* when K is of characteristic 0; that is, for every monomial m in the ideal such that x_i divides m , the monomials $\frac{x_j m}{x_i}$ are also in the ideal for all $j < i$. This property makes such ideals particularly nice to work with.

To any homogeneous ideal I of R we can associate a Borel-fixed monomial ideal $\text{gin}_>(I)$ which can be thought of as a coordinate-independent version of the initial ideal. Its existence is guaranteed by Galligo's theorem (also see Theorem 1.27 of [Gre98]).

Theorem 7.2.5 ([Gal74] and [BS87b]). *For any multiplicative monomial order $>$ on R and any homogeneous ideal $I \subset R$, there exists a Zariski open subset $U \subset \text{GL}_n$ such that $\text{In}_>(g(I))$ is constant and Borel-fixed for all $g \in U$.*

Definition 7.2.6. The **generic initial ideal of I** , denoted $\text{gin}_>(I)$, is defined to be $\text{In}_>(g(I))$ where $g \in U$ is as in Galligo's theorem.

The *reverse lexicographic order* $>$ is a total ordering on the monomials of R defined by:

1. if $|I| = |J|$ then $x^I > x^J$ if there is a k such that $i_m = j_m$ for all $m > k$ and $i_k < j_k$; and
2. if $|I| > |J|$ then $x^I > x^J$.

For example, $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$. From this point on, $\text{gin}(I) = \text{gin}_>(I)$ will denote the generic initial ideal with respect to the reverse lexicographic order.

Recall that the Hilbert function $H_I(t)$ of I is defined by $H_I(t) = \dim(I_t)$. The following result is a consequence of the fact that Hilbert functions are invariant under making changes of coordinates and taking initial ideals ([Gre98]).

Proposition 7.2.7. *For any homogeneous ideal I in R , the Hilbert functions of I and $\text{gin}(I)$ are equal.*

We now describe the structure of the ideals $\text{gin}(I^{(m)})$ where I is an ideal corresponding to points in \mathbb{P}^2 . The proof of this result is contained in [May12a] and follows from results of Bayer and Stillman ([BS87a]) and of Herzog and Srinivasan ([HS98])

Proposition 7.2.8 (Corollary 12.9 of [May12a]). *Suppose $I \subseteq K[x, y, z]$ is the ideal of distinct points in \mathbb{P}^2 . Then the minimal generators of $\text{gin}(I^{(m)})$ are*

$$\{x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_{\alpha(m)-1}}, \dots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\}$$

for $\lambda_0(m), \dots, \lambda_{\alpha(m)-1}$ such that $\lambda_0(m) > \lambda_1(m) > \dots > \lambda_{\alpha(m)-1}(m) \geq 1$.

Since Borel-fixed ideals generated in two variables are determined by their Hilbert functions (see, for example, Lemma 3.7 of [May12d]), we have the following corollary of Propositions 7.2.7 and 7.2.8.

Corollary 7.2.9. *If I and I' are ideals corresponding to two point arrangements of the same configuration type, $\text{gin}(I^{(m)}) = \text{gin}(I'^{(m)})$ for all m .*

Actually finding the Hilbert functions of fat point ideals is not easy and is a significant area of research. (for example, see [CHT11], [GH07], [GVT04], [GHM09], and [Har02]) When I is the ideal of less than 9 points, however, techniques exist for computing these Hilbert functions. In Section 7.3 we will outline the method used in this paper, following [GH07]. Other techniques, such as those in [CHT11], can also be used for some of the point arrangements A through K.

7.2.3 Graded Systems

In this subsection we introduce the limiting shape of a graded system of monomial ideals.

Definition 7.2.10 ([ELS01]). A **graded system of ideals** is a collection of ideals $J_\bullet = \{J_i\}_{i=1}^\infty$ such that

$$J_i \cdot J_j \subseteq J_{i+j} \quad \text{for all } i, j \geq 1.$$

Definition 7.2.11. The **generic initial system** of a homogeneous ideal I is the collection of ideals J_\bullet such that $J_i = \text{gin}(I^i)$. The **symbolic generic initial system** of a homogeneous ideal I is the collection J_\bullet such that $J_i = \text{gin}(I^{(i)})$.

The following lemma justifies calling these collections ‘systems’; see Lemma 2.5 of [May12d] and Lemma 2.2 of [May12a] for proofs.

Lemma 7.2.12. *Generic initial systems and symbolic generic initial systems are graded system of ideals.*

Let J be a monomial ideal of $R = K[x_1, \dots, x_n]$. We may associate to J a subset Λ of \mathbb{N}^n consisting of the points λ such that $x^\lambda \in J$. The *Newton polytope* P_J of J is the convex hull of Λ regarded as a subset of \mathbb{R}^n . Scaling the polytope P_J by a factor of r gives another polytope that we will denote rP_J .

If \mathfrak{a}_\bullet is a graded system of monomial ideals in R , the polytopes of $\{\frac{1}{q}P_{\mathfrak{a}_q}\}_q$ are nested: $\frac{1}{p}P_{I_p} \subseteq \frac{1}{q}P_{I_q}$ whenever p divides q . The *limiting shape* P of \mathfrak{a}_\bullet is the limit of the polytopes in this set:

$$P = \bigcup_{q \in \mathbb{N}^*} \frac{1}{q}P_{\mathfrak{a}_q}.$$

When I is the ideal of points in \mathbb{P}^2 $\text{gin}(I^{(m)})$ is generated in the variables x and y by Proposition 7.2.8, so we can think of each $P_{\text{gin}(I^{(m)})}$, and thus P , as a subset of \mathbb{R}^2 .

7.3 Technique for Computing the Hilbert Function

Here we summarize the method that is used to compute $H_{I^{(m)}}(t)$ in this paper. It follows the work of Guardo and Harbourne in [GH07]; details can be found there.

Suppose that $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up of distinct points p_1, \dots, p_r of \mathbb{P}^2 . Let $E_i = \pi^{-1}(p_i)$ for $i = 1, \dots, r$ and L be the total transform in X of a line not passing through any of the points p_1, \dots, p_r . The classes of these divisors form a basis of $\text{Cl}(X)$; for convenience, we will write e_i in place of $[E_i]$ and e_0 in place of $[L]$. Further, the intersection product in $\text{Cl}(X)$ is defined by $e_i^2 = -1$ for $i = 1, \dots, r$; $e_0^2 = 1$; and $e_i \cdot e_j = 0$ for all $i \neq j$.

Let $Z = m(p_1 + \dots + p_r)$ be a uniform fat point subscheme with sheaf of ideals \mathcal{I}_Z ; set

$$F_d = dE_0 - m(E_1 + E_2 + \dots + E_r)$$

and $\mathcal{F}_d = \mathcal{O}_X(F_d)$.

The following lemma relates divisors on X to the Hilbert function of $I^{(m)}$.

Lemma 7.3.1. *If $F_d = dE_0 - m(E_1 + \dots + E_r)$ then $h^0(X, \mathcal{F}_d) = H_{I^{(m)}}(d)$.*

Proof. Since $\pi_*(\mathcal{F}_d) = \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(d)$,

$$H_{I^{(m)}}(d) = \dim((I_Z)_d) = h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(d)) = h^0(X, \mathcal{F}_d)$$

for all d . □

For convenience, we will sometimes write $h^0(X, F) = h^0(X, \mathcal{O}_X(F))$. Recall that if $[F]$ not the class of an effective divisor then $h^0(X, F) = 0$. On the other hand, if F is effective, then we will see that we can compute $h^0(X, F)$ by finding $h^0(X, H)$ for some *numerically effective* divisor H .

Definition 7.3.2. A divisor H is **numerically effective** if $[F] \cdot [H] \geq 0$ for every effective divisor F , where $[F] \cdot [H]$ denotes the intersection multiplicity. The cone of classes of numerically effective divisors in $\text{Cl}(X)$ is denoted by $\text{NEF}(X)$.

Lemma 7.3.3. *Suppose that X is the blow-up of \mathbb{P}^2 at $r \leq 8$ points in general position and that $F \in \text{NEF}(X)$. Then F is effective and*

$$h^0(X, F) = ([F]^2 - [F] \cdot [K_X])/2 + 1$$

where $K_X = -3E_0 + E_1 + \cdots + E_r$.

Proof. This is a consequence of Riemann-Roch and the fact that $h^1(X, F) = 0$ for any numerically effective divisor F . See Lemma 2.1b of [GH07] for a discussion. \square

The set of classes of effective, reduced, and irreducible curves of negative intersection is

$$\text{NEG}(X) := \{[C] \in \text{Cl}(X) : [C]^2 < 0, C \text{ is effective, reduced, and irreducible}\}.$$

The set of classes in $\text{NEG}(X)$ with self intersection less than -1 is

$$\text{neg}(X) := \{[C] \in \text{NEG}(X) : [C]^2 < -1\}.$$

The following result of Guardo and Harbourne allows us to easily identify divisor classes belonging to $\text{NEG}(X)$. In the lemma, the *curves defining the configuration type* are lines that pass through any three points or conics that pass through any six points. For example, the divisors defining the configuration type shown in Figure 7.3 are $E_0 - E_1 - E_2 - E_3$ and $E_0 - E_1 - E_4 - E_5$.

Lemma 7.3.4 (Lemma 2.1d of [GH07]). *The elements of $\text{neg}(X)$ are the classes of divisors that correspond to the curves defining the configuration types. Further,*

$$\text{NEG}(X) = \text{neg}(X) \cup \{[C] \in \mathcal{B} \cup \mathcal{L} \cup \mathcal{Q} : [C]^2 = -1, [C] \cdot [D] \geq 0 \text{ for all } D \in \text{neg}(X)\}$$

where $\mathcal{B} = \{e_i : i > 0\}$, $\mathcal{L} = \{e_0 - e_{i_1} - \cdots - e_{i_r} : r \geq 2, 0 < i_1 < \cdots < i_r \leq 6\}$, and $\mathcal{Q} = \{2e_0 - e_{i_1} - \cdots - e_{i_r} : r \geq 5, 0 < i_1 < \cdots < i_r \leq 6\}$.

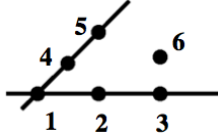


Figure 7.3: Points p_1, \dots, p_6 of configuration type H.

The following result will be used in Procedure 7.3.6; see Section 2 of [GH07].

Lemma 7.3.5. *Suppose that $[C] \in \text{NEG}(X)$ is such that $[F] \cdot [C] < 0$. Then $h^0(X, F) = h^0(X, F - C)$.*

Knowing how to compute $h^0(X, H)$ for a numerically effective divisor H will allow us to compute $h^0(X, F)$ for *any* divisor F . In particular, given a divisor F , there exists a divisor H such that $h^0(X, F) = h^0(X, H)$ and either:

- (a) H is numerically effective so

$$h^0(X, F) = h^0(X, H) = (H^2 - H \cdot K_X)/2 + 1$$

by Lemma 7.3.3; or

- (b) there is a numerically effective divisor G such that $[G] \cdot [H] < 0$ so $[H]$ is not the class of an effective divisor and $h^0(X, F) = h^0(X, H) = 0$.

The method for finding such an H is as follows.

Procedure 7.3.6 (Remark 2.4 of [GH07]). *Given a divisor F we can find a divisor H with $h^0(X, F) = h^0(X, H)$ satisfying either condition (a) or (b) above as follows.*

1. *Reduce to the case where $[F] \cdot e_i \geq 0$ for all $i = 1, \dots, n$: if $[F] \cdot e_i < 0$ for some i , $h^0(X, F) = h^0(X, F - ([F] \cdot e_i)E_i)$, so we can replace F with $F - ([F] \cdot e_i)E_i$.*
2. *Since L is numerically effective, if $[F] \cdot e_0 < 0$ then $[F]$ is not the class of an effective divisor and we can take $H = F$ (case (b)).*
3. *If $[F] \cdot [C] \geq 0$ for every $[C] \in \text{NEG}(X)$ then, by Lemma 7.3.4, F is numerically effective, so we can take $H = F$ (case (a)).*
4. *If $[F] \cdot [C] < 0$ for some $[C] \in \text{NEG}(X)$ then $h^0(X, F) = h^0(X, F - C)$ by Lemma 7.3.5. Then replace F with $F - C$ and repeat from Step 2.*

There are only a finite number of elements in $\text{NEG}(X)$ to check by Lemma 7.3.4 so it is possible to complete Step 3. Further, $[F] \cdot e_0 > [F - C] \cdot e_0$ when $[C] \in \text{NEG}(X)$, so the condition in Step 2 will be satisfied after at most $[F] \cdot e_0 + 1$ repetitions. Thus, the process will terminate.

Taking these results together we can compute the Hilbert function of $I^{(m)}$ as follows.

1. Compute $\text{NEG}(X)$ from $\text{neg}(X)$ using Lemma 7.3.4.
2. Find H_t corresponding to F_t using Procedure 7.3.6 for all t .
3. Compute $H_{I^{(m)}}(t) = h^0(X, F_t) = h^0(X, H_t)$ with Lemma 7.3.3.

7.4 Proof of the Main Theorem

In this section, we will outline the proof of Theorem 7.1.1. Recall that ideals of points with the same configuration type have the same symbolic generic initial system by Corollary 7.2.9 so the statement of the theorem makes sense. Further, Proposition 7.2.4 ensures that the theorem includes all possible sets of six points.

If I is the ideal of a set of six points having configuration type E , G , or K , the theorem follows from the main result of [May12d]. Likewise, if I is the ideal of six points of configuration type A , the theorem follows from the main result of [May12a].

For the remaining cases we can find the limiting shape of $\{\text{gin}(I^{(m)})\}_m$ by following the five steps below. First, we record a lemma that will be used in Step 2.

Lemma 7.4.1. *Let J be a monomial ideal of $K[x, y, z]$ generated in the variables x and y . Then the number of elements of J of degree t only involving the variables x and y is equal to $H_J(t) - H_J(t - 1)$. The number of minimal generators of J in degree t is equal to $H_J(t) - H_J(t - 2) - 1$.*

Proof. The first statement follows from the fact that there are exactly $H_J(t - 1)$ monomials of J of degree t involving the variable z . The number of generators in degree t is equal to the number of monomials of J in the variables x and y of degree t minus the number of monomials of J that arise from multiplying the elements of degree $t - 1$ in x and y by the variables x and y . Using this, the last statement follows from the first. \square

Step 1: Find the Hilbert function of $I^{(m)}$ for infinitely many m by using the method outlined in Section 7.3.

Step 2: Find the number of minimal generators of $\text{gin}(I^{(m)})$ of each degree t for infinitely many m . We can use Lemma 7.4.1 for this computation because $\text{gin}(I^{(m)})$ is an ideal generated in the variables x and y (Proposition 7.2.8) and we know the Hilbert function of $\text{gin}(I^{(m)})$ from Proposition 7.2.7 and Step 1.

Step 3: Write down the generators of $\text{gin}(I^{(m)})$ for infinitely many m . Note that this follows from Step 2 since

$$\text{gin}(I^{(m)}) = (x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_{\alpha(m)-1}}, \dots, xy^{\lambda_1(m)}, y^{\lambda_0(m)})$$

where $\lambda_0(m) > \dots > \lambda_{k-1}(m) \geq 1$ by Proposition 7.2.8.

Step 4: Compute the Newton polytope $P_{\text{gin}(I^{(m)})}$ of each $\text{gin}(I^{(m)})$ for infinitely many m . Recall that the boundary of these polytopes is determined by the convex hull of the points $(i, \lambda_i(m))$ and $(\alpha(m), 0)$.

Step 5: Find the limiting shape of the symbolic generic initial system of I . To do this it suffices to take the limit

$$P = \bigcup_{m \in \mathbb{N}^*} \frac{1}{m} P_{a_m}$$

over an infinite subset of \mathbb{N}^* .

All of the remaining calculations are similar but long so, for the sake of space, we will only record the proof here for configuration H.

7.4.1 Proof of main theorem for configuration H

Let I be the ideal of points p_1, \dots, p_6 of configuration type H, ordered as in Figure 7.3.

Step 1.

First we will follow the method outlined in Section 3 to find $H_{I^{(m)}}$ for infinitely many m . We will use the notation from Section 7.3 and will often denote the divisor $a_0E_0 - (a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 + a_5E_5 + a_6E_6)$ by $(a_0; a_1, a_2, a_3, a_4, a_5, a_6)$. Also, if F_1 and F_2 are divisors, $F_1 \cdot F_2$ denotes $[F_1] \cdot [F_2]$, the intersection multiplicity of their classes.

First we need to determine $\text{NEG}(X)$. Note that the configuration type H is defined by a line through points 1, 2, and 3 and another line through points 1, 4, and 5. Thus, $\text{neg}(X)$ consists of the classes of $A_1 := E_0 - E_1 - E_2 - E_3$ and $A_2 := E_0 - E_1 - E_4 - E_5$. The other elements of $\text{NEG}(X)$ are exactly those $[C] \in \mathcal{B} \cup \mathcal{L} \cup \mathcal{Q}$ such that $[C]^2 = -1$

and $[C] \cdot [D] \geq 0$ for all $[D] \in \text{neg}(X)$ by Lemma 7.3.4. Using this, one can check that $\text{NEG}(X)$ consists of the classes of the divisors

$$\begin{aligned} A_1 &:= E_0 - E_1 - E_2 - E_3, A_2 := E_0 - E_1 - E_4 - E_5, \\ B &:= E_0 - E_1 - E_6, \\ C_i &:= E_0 - E_i - E_6 \text{ for } i = 2, 3, 4, 5, \\ D_{ij} &:= E_0 - E_i - E_j \text{ for } i = 2, 3 \text{ and } j = 4, 5, \\ Q &:= 2E_0 - E_2 - E_3 - E_4 - E_5 - E_6. \end{aligned}$$

Next, we will follow Procedure 7.3.6 for each F_t once we fix m divisible by 12. The procedure produces a divisor H_t that is either numerically effective or is in the class of an effective divisor such that

$$H_{I(m)}(t) = h^0(X, F_t) = h^0(X, H_t).$$

First, we will make some observations about which elements of $\text{NEG}(X)$ may be subtracted during the procedure.

Suppose that J is a divisor of the form $J := (a; b, c, c, c, c, d)$. We will show that if the procedure allows us to subtract one A_i (respectively, one C_i or one D_{ij}) from J , we can subtract them all consecutively. This is equivalent to showing that if the intersection multiplicity of J with A_1 is negative then the intersection multiplicity of $J - A_1$ with A_2 is also negative; parallel statements hold for the C_i and D_{ij} .

A₁ :

$$\begin{aligned} J \cdot A_1 &= a - b - 2c \\ (J - A_1) \cdot A_2 &= (a - 1; b - 1, c - 1, c - 1, c, c, d) \cdot A_2 \\ &= a - 1 - (b - 1) - 2c = a - b - 2c \end{aligned}$$

C₁ :

$$\begin{aligned} J \cdot C_2 &= a - c - d \\ (J - C_2) \cdot C_3 &= (a - 1; b, c - 1, c, c, c, d - 1) \cdot C_3 \\ &= (a - 1) - c - (d - 1) = a - c - d \\ (J - C_2 - C_3) \cdot C_4 &= (a - 2; b, c - 1, c - 1, c, c, d - 2) \cdot C_4 \end{aligned}$$

$$\begin{aligned}
&= (a - 2) - c - (d - 2) = a - c - d \\
(J - C_2 - C_3 - C_4) \cdot C_5 &= (a - 3; b; c - 1, c - 1, c - 1, c, d - 3) \cdot C_5 \\
&= (a - 3) - c - (d - 3) = a - c - d
\end{aligned}$$

\mathbf{D}_{ij} :

$$\begin{aligned}
J \cdot D_{24} &= a - 2c \\
(J - D_{24}) \cdot D_{25} &= (a - 1; b, c - 1, c, c - 1, c, d) \cdot D_{25} \\
&= (a - 1) - (c - 1) - c = a - 2c \\
(J - D_{24} - D_{25}) \cdot D_{34} &= (a - 2; b, c - 2, c, c - 1, c - 1, d) \cdot D_{34} \\
&= (a - 2) - c - (c - 1) = a - 2c - 1 \\
(J - D_{24} - D_{25} - D_{34}) \cdot D_{35} &= (a - 3; b, c - 2, c - 1, c - 2, c - 1, d) \cdot D_{35} \\
&= (a - 3) - 2(c - 1) = a - 2c - 1
\end{aligned}$$

Define

$$A := A_1 + A_2, \quad C := C_2 + C_3 + C_4 + C_5, \quad D := D_{24} + D_{25} + D_{34} + D_{35}.$$

The calculations above show that if $J \cdot A_1 < 0$ (if $J \cdot C_2 < 0$, $J \cdot D_{24} < 0$, respectively) then the procedure will allow us to subtract one entire copy of A (C , D). If we begin with a divisor of the form $J = (a; b, c, c, c, c, d)$ then $J - A$, $J - B$, $J - C$, $J - D$, and $J - Q$ have the same form. These facts taken together mean that that H_t is obtained from F_t - a divisor with the same form as J - by subtracting off copies of A , B , C , D , and Q .

In Procedure 7.3.6, the requirement for being able to subtract an element of $\text{NEG}(X)$ from J is that the intersection of that element with J is strictly negative. Thus, it is of interest how the intersection multiplicities with elements of $\text{NEG}(X)$ *change* as other elements of $\text{NEG}(X)$ are subtracted from a divisor of the form $(a; b, c, c, c, c, d)$.

If $J = (a; b, c, c, c, c, d)$ as above, we have the following.

	value of G				
	A_i	B	C_i	D_{ij}	Q
$(J - A) \cdot G - J \cdot G$	2	0	-1	0	0
$(J - B) \cdot G - J \cdot G$	0	1	0	-1	-1
$(J - C) \cdot G - J \cdot G$	-2	0	1	-2	0
$(J - D) \cdot G - J \cdot G$	0	-4	-2	0	0
$(J - Q) \cdot G - J \cdot G$	0	-1	0	0	1

We now use this set-up to obtain H_t from F_t by successively subtracting elements of $\text{NEG}(X)$ that have negative intersection with the remaining divisor. First note that

$$F_t \cdot A_i = t - 3m < 0 \iff t < 3m,$$

$$F_t \cdot B = F_t \cdot C_i = F_t \cdot D_{ij} = t - 2m < 0 \iff t < 2m,$$

and

$$F_t \cdot Q = 2t - 5m < 0 \iff t < \frac{5m}{2}.$$

Therefore, $[F_t] = [H_t]$ (that is, F_t is numerically effective) if and only if $t \geq 3m$. In this case, $h^0(X, F_t) = \frac{1}{2}t^2 - 3m^2 + \frac{3}{2}t - 3m + 1$ by Lemma 7.3.3.

We will assume from this point on that $12|m$.

Now suppose that $3m > t \geq \frac{5m}{2}$. In this case, $[A_i] \cdot [F_t] < 0$, but $[C] \cdot [F_t] \geq 0$ for all other $[C] \in \text{NEG}(X)$; thus, Procedure 7.3.6 allows us to subtract A_i - and thus A - but no other divisors initially. How many copies can we subtract? From the table, we see that the intersection multiplicity of the remaining divisor with A_i increases by 2 each time we subtract a copy of A_i . We can keep subtracting copies of A as long as the intersection multiplicity with A_i is strictly negative; thus, we can subtract exactly

$$\left\lceil -\frac{F_t \cdot A_i}{2} \right\rceil = \left\lceil \frac{3m - t}{2} \right\rceil$$

copies of A . The only other intersection multiplicity that changes through the process subtracting A s is with the C_i , which decreases by one for each copy of A subtracted.

Thus,

$$\left(F_t - \left\lceil \frac{3m - t}{2} \right\rceil A \right) \cdot C_i = t - 2m - \left\lceil \frac{3m - t}{2} \right\rceil$$

and this is never negative when $t \geq \frac{5m}{2}$ (t must be at most $\frac{7m}{3}$ for this expression to be negative). Thus, the intersection multiplicity of $F_t - \left\lceil \frac{3m - t}{2} \right\rceil A$ with all $[C] \in \text{NEG}(X)$

is nonnegative, so

$$H_t = \left(t - 2 \left\lfloor \frac{3m-t}{2} \right\rfloor; m - 2 \left\lfloor \frac{3m-t}{2} \right\rfloor, m - \left\lfloor \frac{3m-t}{2} \right\rfloor, \dots, m \right).$$

When t is even,

$$H_t = \left(2t - 3m; t - 2m, \frac{t-m}{2}, \dots, m \right)$$

and $h^0(X, F_t) = t^2 - 3tm - \frac{3}{2}m^2 + \frac{3}{2}t - 3m + 1$ while when t is odd

$$H_t = \left(2t - 3m - 1; t - 2m - 1, \frac{t-m-1}{2}, \dots, m \right)$$

and $h^0(X, F_t) = t^2 - 3tm - \frac{3}{2}m^2 + \frac{3}{2}t - 3m + \frac{1}{2}$.

Now suppose that $\frac{5m}{2} > t \geq \frac{7m}{3}$. In this case, Procedure 7.3.6 allows us to subtract copies of Q because $F_t \cdot Q < 0$. From the table, for each copy of Q subtracted, the intersection multiplicity increases by 1; since we can keep subtracting copies of Q as long as the intersection multiplicity with the remaining divisor is negative, we can subtract exactly $-F_t \cdot Q = 5m - 2t$ copies. We may also subtract $\left\lfloor \frac{3m-t}{2} \right\rfloor A$ by the same argument as in the previous case, since subtracting copies of A doesn't change the intersection multiplicity with Q and vice versa.

Through the process of subtracting A s and Q s the intersection multiplicities with C_i and B have changed; in particular,

$$(F_t - \left\lfloor \frac{3m-t}{2} \right\rfloor A - (5m-2t)Q) \cdot C_i = t - 2m - \left\lfloor \frac{3m-t}{2} \right\rfloor$$

and

$$(F_t - \left\lfloor \frac{3m-t}{2} \right\rfloor A - (5m-2t)Q) \cdot B = t - 2m - (5m-2t) = 3t - 7m.$$

These are both nonnegative, as $t \geq \frac{7m}{3}$, so the intersection multiplicity of the remaining divisor with all elements of $\text{NEG}(X)$ is nonnegative and Procedure 7.3.6 terminates.¹ Therefore, when t is even

$$H_t = (6t - 13m; t - 2m, \frac{5t - 11m}{2}, \dots, 2t - 4m)$$

¹ $\frac{7m}{3}$ is always an integer under the divisibility assumption so we don't have to worry about t being the smallest odd integer less than $\frac{7m+1}{3}$.

and $h^0(X, F_t) = 3t^2 - 13tm + 14m^2 + \frac{5}{2}t - \frac{11}{2}m + 1$, while when t is odd

$$H_t = (6t - 13m - 1; t - 2m - 1, \frac{5t - 11m - 1}{2}, \dots, 2t - 4m)$$

and $h^0(X, F_t) = 3t^2 - 13tm + 14m^2 + \frac{5}{2}t - \frac{11}{2}m + \frac{1}{2}$.

Now suppose that $t = \frac{7m}{3} - 1$. By the same arguments as above, we can subtract $\lceil \frac{3m-t}{2} \rceil = \frac{2m}{6}$ copies of A and $5m - 2t = \frac{m}{3} + 2$ copies of Q when following Procedure 7.3.6. Then

$$F_{\frac{7m}{3}-1} - \frac{2m}{6}A - \left(\frac{m}{3} + 2\right)Q = \left(m - 7; \frac{m}{3} - 2, \frac{m}{3} - 3, \dots, \frac{2m}{3} - 2\right)$$

has intersection multiplicity 1 with A_i and -2 with C_i . At this point, Procedure 7.3.6 allows us to do the following.

- Subtract one copy of C . Now the intersection multiplicity with A_i is -1 and the intersection multiplicity with C_i is -1 .
- Subtract one copy of A . Now the intersection multiplicity with A is 1 and the intersection multiplicity with C_i is -2 .

It is clear that we can repeat this process as many times as we wish when we follow the procedure; eventually, we will end up with a divisor that has a negative E_0 coefficient. We have that $H_{I^{(m)}}(t) = h^0(X, F_t) = 0$ when $t = \frac{7m}{3} - 1$ and thus $H_{I^{(m)}}(t) = 0$ for all $t < \frac{7m}{3}$.

Step 2.

Assume that $12|m$.

Now we will turn our attention to the generic initial ideals of $I^{(m)}$. We compute the number of generators of $\text{gin}(I^{(m)})$ in each degree using Lemma 7.4.1 and the Hilbert function values from Step 1. We have the following.

Value of t	Number of generators of degree t
$t < \frac{7m}{3}$	0
$t = \frac{7m}{3}$	$\frac{1}{3}m + 1$
$t = \frac{7m}{3} + 1$	$\frac{2}{3}m + 3$
$\frac{5m}{2} > t \geq \frac{7m}{3} + 2, t$ even	6
$\frac{5m}{2} > t \geq \frac{7m}{3} + 2, t$ odd	4
$t = \frac{5m}{2}$	6
$t = \frac{5m}{2} + 1$	1
$3m > t \geq \frac{5m}{2} + 2, t$ even	2
$3m > t \geq \frac{5m}{2} + 2, t$ odd	0
$t = 3m$	2
$t > 3m$	0

Step 3.

Assume once again that $12|m$.

Note that there are

$$\frac{\frac{5m}{2} - \frac{7m}{3} - 2}{2} = \frac{\frac{m}{6} - 2}{2} = \frac{m}{12} - 1$$

even (or odd) integers t such that $\frac{5m}{2} > t \geq \frac{7m}{3} + 2$, and

$$\frac{3m - \frac{5m}{2} - 2}{2} = \frac{m}{4} - 1$$

even (or odd) integers t such that $3m > t \geq \frac{5m}{2} + 2$.

Using the results of Step 2, we can find strictly decreasing λ_i such that

$$\text{gin}(I^{(m)}) = (x^k, x^{k-1}y^{\lambda_{k-1}}, \dots, xy^{\lambda_1}, y^{\lambda_0}).$$

Since the smallest degree generator is of degree $\frac{7m}{3}$, $k = \frac{7m}{3}$.

The values of λ_i that we obtain are shown in the following table.

degree	$\frac{7m}{3}$					$\frac{7m}{3} + 1$					$\frac{7m}{3} + 2$
i	$\frac{7m}{3}$	$\frac{7m}{3} - 1$	\dots	$2m$	$2m - 1$	$2m - 2$	\dots	$\frac{4m}{3} - 3$	$\frac{4m}{3} - 4$	\dots	$\frac{4m}{3} - 9$
λ_i	0	1	\dots	$\frac{m}{3}$	$\frac{m}{3} + 2$	$\frac{m}{3} + 3$	\dots	$m + 4$	$m + 6$	\dots	$m + 11$
$\frac{7m}{3} + 3$				\dots	$\frac{5m}{2}$						
$\frac{4m}{3} - 10$	\dots	$\frac{4m}{3} - 13$	\dots	$\frac{4m}{3} - 4 - 10(\frac{m}{12} - 1) = \frac{m}{2} + 6$	$\frac{m}{2} + 5$	\dots	$\frac{m}{2} + 1$				
$m + 13$	\dots	$m + 16$	\dots	$m + 6 + 12(\frac{m}{12} - 1) = 2m - 6$	$2m - 5$	\dots	$2m - 1$				
$\frac{5m}{2} + 1$	$\frac{5m}{2} + 2$	$\frac{5m}{2} + 4$		\dots	$3m$						
$\frac{m}{2}$	$\frac{m}{2} - 1$	$\frac{m}{2} - 2$	$\frac{m}{2} - 3$	$\frac{m}{2} - 4$	\dots	$\frac{m}{2} - 1 - 2(\frac{m}{4} - 1) = 1$	0				
$2m + 1$	$2m + 3$	$2m + 4$	$2m + 7$	$2m + 8$	\dots	$2m + 3 + 4(\frac{m}{4} - 1) = 3m - 1$	$3m$				

Step 4.

Assume that $12|m$.

The Newton polytope of $\text{gin}(I^{(m)})$ is the convex hull of the ideal when thought of as a subset of \mathbb{R}^2 . In particular, its boundary is determined by the points (i, λ_i) recorded in the table from Step 3. Plotting these points, one can see that the boundary of $P_{\text{gin}(I^{(m)})}$ is defined by the line segments through the points $(0, 3m)$, $(\frac{m}{2} - 2, 2m + 4)$, $(\frac{m}{2} + 1, 2m - 1)$, $(\frac{4m}{3} - 9, m + 11)$, $(\frac{4m}{3} - 3, m + 4)$, $(2m, \frac{m}{3})$, and $(\frac{7m}{3}, 0)$.

Step 5.

Scaling $P_{\text{gin}(I^{(m)})}$ from the previous step by $\frac{1}{m}$ and taking the limit as m approaches infinity, the limiting shape of the symbolic generic initial system is defined by the line segments through the following points.

$$(0, 3) = \lim_{m \rightarrow \infty} \left(0, \frac{3m}{m}\right)$$

$$\left(\frac{1}{2}, 2\right) = \lim_{m \rightarrow \infty} \left(\frac{m/2 - 2}{m}, \frac{2m + 4}{m}\right) = \lim_{m \rightarrow \infty} \left(\frac{m/2 + 1}{m}, \frac{2m - 1}{m}\right)$$

$$\left(\frac{4}{3}, 1\right) = \lim_{m \rightarrow \infty} \left(\frac{4m/3 - 9}{m}, \frac{m + 11}{m}\right) = \lim_{m \rightarrow \infty} \left(\frac{4m/3 - 3}{m}, \frac{m + 4}{m}\right)$$

$$\left(2, \frac{1}{3}\right) = \lim_{m \rightarrow \infty} \left(\frac{2m}{m}, \frac{m/3}{m}\right)$$

$$\left(\frac{7}{3}, 0\right) = \lim_{m \rightarrow \infty} \left(\frac{7m/3}{m}, 0\right)$$

Note that $(2, \frac{1}{3})$ lies on the line segment connecting $(\frac{4}{3}, 1)$ with $(\frac{7}{3}, 0)$ so it is not a vertex of the boundary of the limiting shape.

CHAPTER VIII

Further Questions

8.1 Restrictions on the Limiting Shape

This thesis considered limiting shapes of specific generic initial systems. We would like to be able to make general statements about these limiting shapes; one question is the following.

Question 8.1.1. *What infinite convex regions of $\mathbb{R}_{\geq 0}^n$ can be the limiting shape of a generic initial system?*

For example, all of the limiting shapes of generic initial systems that we have studied have boundaries defined by a finite number of line segments, planes, or higher-dimensional hyperplanes. Is this true for the limiting shapes *every* generic initial system?

Question 8.1.2. *Do the limiting shape of all generic initial systems have boundaries defined by hyperplanes? Is there a generic initial system whose limiting shape has a ‘curved’ boundary?*

Michael von Korff showed that any convex region of $\mathbb{R}_{\geq 0}^n$ satisfying some obvious necessary conditions can be the limiting shape of a graded system of monomial ideals ([von10]). However, the proof of this result involved constructing graded systems in somewhat unnatural ways. Since generic initial systems are quite special, it is reasonable to expect that all of their limiting shapes could be defined by hyperplanes. For example, if an ideal $I \subseteq K[x, y]$ had a generic initial system with a limiting shape in $\mathbb{R}_{\geq 0}^2$ whose boundary was curved, the Hilbert functions of the powers I^m would display more and more distinct ‘phases’ as m increases. The Hilbert functions of powers of a fixed ideal may be too closely related for this to happen.

8.2 Ideal Properties Reflected by the Limiting Behavior of Generic Initial Systems

When studying the symbolic generic initial systems of ideals of points in \mathbb{P}^2 , we asked how information about the arrangement of points is reflected or encoded in the limiting shape. We could ask the following more general question.

Question 8.2.1. *What properties of I are reflected in the limiting shape of the generic initial system or symbolic generic initial system of I ?*

More specifically, we could consider properties having to do with the form of the limiting shape.

Question 8.2.2. *Can we classify those ideals that have a generic initial system with limiting shape defined by a single hyperplane? Two hyperplanes?*

Complete intersections and the ideals of points in \mathbb{P}^2 in general position, for example, both have symbolic generic initial systems with limiting shapes defined by a single hyperplane. This is a commonality that is only seeing in the asymptotics of the system and not in the individual ideals. At this stage it is not clear which shared property dictates this simple limiting shape.

8.3 The GIS of Ideals of Points in \mathbb{P}^2

There are many open questions related to the generic initial systems of specific classes of ideals, including one that was studied in this thesis: ideals corresponding to points in \mathbb{P}^2 . In this subsection, we will assume that $I \subseteq K[x, y, z]$ is the ideal of points p_1, \dots, p_r in \mathbb{P}^2 .

Recall that, for such an ideal,

$$I^{(m)} = (I^m)^{\text{sat}}$$

and so

$$\text{gin}(I^{(m)}) = \text{gin}((I^m)^{\text{sat}}) = \bigcup_{k \geq 0} (\text{gin}(I^m) : z^k)$$

by Theorem 2.2.11 of Chapter II. This means that we can determine the ideals of the symbolic generic initial system of I from the ideals of the generic initial system of I . Further, $\text{gin}(I^m)$ has generators in the variables x, y , and z , while $\text{gin}(I^{(m)})$ can

be generated in just the variables x and y . This is one of the reasons that we began to study *symbolic* generic initial systems of points in \mathbb{P}^2 .

The relationship between the ideals of these two systems can be visualized by using *triangle diagrams* of Borel-fixed ideals. The circles in triangle diagram of a Borel-fixed ideal I represent monomials in x and y and are empty, filled in, or contain a number based on what monomials are in I .

Example 8.3.1. Let I be the ideal of 3 points in general position. Then the triangle diagram in Figure 8.1 represents the ideal

$$\text{gin}(I^2) = (x^4, x^3y, x^2y^2, xy^3, y^4, x^3z).$$

The one circle in the top row of the diagram represents the monomial x^0y^0 ; it is empty since there is no l such that $x^0y^0z^l \in \text{gin}(I^2)$. The two circles in the top row represent the monomials x and y , respectively; they are also empty since there is no l such that $xz^l \in \text{gin}(I^2)$ or $yz^l \in \text{gin}(I^2)$. Continuing in this way, the first circle in the fourth row represents the monomial x^3 ; it contains the number 1 since 1 is the least l such that $x^3z^l \in \text{gin}(I^2)$. All of the circles in the fifth row and below are solid since all monomials of degree 4 and greater in the variables x and y are contained in I . The operation that gives us the generic initial ideal of the saturation of I^2

$$\text{gin}(I^{(2)}) = (x^3, x^2y^2, xy^3, y^4)$$

in terms of $\text{gin}(I^2)$ corresponds to ‘filling in’ circles in the triangle diagram that have numbers in them (see Figure 8.2).

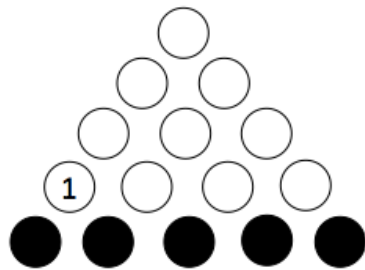


Figure 8.1: Triangle diagram of $\text{gin}(I^2)$.

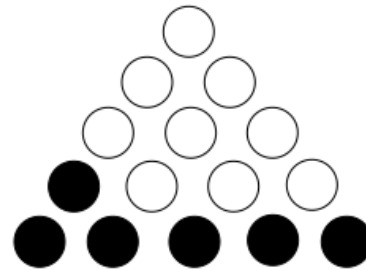


Figure 8.2: Triangle diagram of $\text{gin}(I^{(2)})$.

We would also like to work the other way, from the symbolic generic initial system to the generic initial system.

Question 8.3.1. *What is the limiting shape of the generic initial system of I ? What features of the arrangement of points p_1, \dots, p_r are reflected?*

While the asymptotic behavior of the generic initial system does not have the advantage of reflecting the behavior of fat points, this question is relevant to our original motivating questions and to the questions raised in Sections 8.1 and 8.2.

APPENDICES

APPENDIX A

Solutions to Chapter I Exercises

Solution to Exercise 1

Let $a + bi$ and $c + di$ be two elements of $\mathbb{Q}[i]$ where $a, b, c, d \in \mathbb{Q}$. Since the sum, difference, and product of two rational numbers is also a rational number, we have

$$(a + bi) + (c + di) = (a + c) + (b + d)i \in \mathbb{Q}[i],$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i \in \mathbb{Q}[i],$$

and

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i \in \mathbb{Q}[i]$$

so $\mathbb{Q}[i]$ is closed under addition, subtraction, and multiplication. Let

$$(a + bi)^{-1} := \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

$(a + bi)^{-1}$ is an element of $\mathbb{Q}[i]$ since the quotient of rational numbers is also rational. Further, $(a + bi)^{-1}$ is actually the inverse of $a + bi$ since

$$\begin{aligned} (a + bi) \cdot (a + bi)^{-1} &= \left(a \frac{a}{a^2 + b^2} + b \frac{b}{a^2 + b^2} \right) + \left(\frac{a}{a^2 + b^2} b - a \frac{b}{a^2 + b^2} \right) \\ &= 1 + 0i = 1. \end{aligned}$$

Solution to Exercise 2

By definition, $\langle S \rangle$ is the smallest ideal containing S . Since an ideal absorbs polynomials under multiplication, all products of the form hf where $h \in R, f \in S$ are

in $\langle S \rangle$. Further, the fact that an ideal is closed under addition means that all sums $\sum_{i=1}^s h_i f_i : h_i \in R, f_i \in S$ are in $\langle S \rangle$ as well. Therefore,

$$S \subseteq \left\{ \sum_{i=1}^s h_i f_i : h_i \in R, f_i \in S \right\} \subseteq \langle S \rangle.$$

To complete the exercise we need to show that $\{\sum_{i=1}^s h_i f_i : h_i \in R, f_i \in S\}$ is itself an ideal. It is clear that this set is closed under addition. If h is an arbitrary polynomial of R ,

$$h \cdot \left(\sum_{i=1}^s h_i f_i \right) = \sum_{i=1}^s (h \cdot h_i) f_i$$

so this set absorbs polynomials in R .

Solution to Exercise 3

How many monomials of $R = K[x_1, \dots, x_n]$ are of degree d ? Since a monomial of degree d can be written as the product of d (not necessarily distinct) variables, we can rephrase this question as:

How many ways can we fill d ‘stars’

$$\underbrace{***\cdots*}_d$$

with the variables x_1, x_2, \dots, x_n where all of the x_1 s come before the x_2 s, all of the x_2 s come before the x_3 s, etc.?

A way to fill-in the stars with variables is uniquely determined by the positions where we switch from one variable to another; we can represent these transition points by ‘bars’ between the stars. For example, if $d = 4$ and $n = 5$,

$$| * || * * | *$$

represents the monomial $x_2 x_4 x_4 x_5$ and

$$* | * | * * ||$$

represents the monomial $x_1 x_2 x_3 x_3$. For a monomial of degree d , we have d stars and $n - 1$ bars representing transitions between the variables. Thus, we can rephrase the previous question as

In how many ways can we place $n - 1$ bars in between d stars?

The string of stars and bars has length $n - 1 + d$ of which $n - 1$ are bars. Also, the particular string of stars and bars is determined by the placement of the $n - 1$ bars.

There are

$$\binom{n + d - 1}{n - 1} = \binom{n + d - 1}{d}$$

different ways to place the bars and, returning to the original question, $\binom{n+d-1}{d}$ monomials of R of degree d .

Solution to Exercise 4

$\bigcap_j \mathbb{V}(I_j) = \mathbb{V}(\bigcup_j I_j)$: A point p of K^n is in $\bigcap_j \mathbb{V}(I_j)$ if and only if it is in every $\mathbb{V}(I_j)$. This holds if and only if all the polynomials of every I_j vanish at p ; this is exactly the condition for p to belong to $\mathbb{V}(\bigcup_j I_j)$ on the left hand side.

$\bigcup_{j=1}^l \mathbb{V}(I_j) = \mathbb{V}(\prod_{i=1}^l I_i)$: It is enough to prove this statement when $l = 2$. Thus, let $V = \mathbb{V}(f_1, \dots, f_s)$ and $W = \mathbb{V}(g_1, \dots, g_t)$. We want to show that

$$V \cup W = \mathbb{V}(\langle f_1, \dots, f_s \rangle \cdot \langle g_1, \dots, g_t \rangle);$$

note that the right hand side is equal to $\mathbb{V}(\langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle)$. We will denote this variety by $\mathbb{V}(f_i g_j)$.

If the point p of K^n is contained in V , then all of the f_i s vanish at p ; thus, all of the polynomials $f_i g_j$ vanish at p as well so $V \subseteq \mathbb{V}(f_i g_j)$. Similarly, $W \subseteq \mathbb{V}(f_i g_j)$ so

$$V \cup W \subseteq \mathbb{V}(f_i g_j).$$

Conversely, suppose that a point p is contained in $\mathbb{V}(f_i g_j)$. If p lies in V then we are done. If p does not lie in V then there is some f_{i_0} such that $f_{i_0}(p) \neq 0$. Since $f_{i_0} g_j$ vanishes at p for all j then $g_j(p) = 0$ for all j . This means that $p \in W$. Thus, $\mathbb{V}(f_i g_j) \subseteq V \cup W$.

APPENDIX B

Examples for Chapter III

This appendix contains examples of the sequence of numbers $\{\lambda_i\}$ produced by each algorithm in Section 3.4 of Chapter III. We also record the sequence of gaps $\{g_i\}$ where $g_i = \lambda_{i-1} - \lambda_i$. It is in these gap sequences that the patterns can be most clearly seen and the spacing used when listing the gap sequences is meant to highlight these patterns. The figures following the example further illustrate the patterns.

B.1 $\beta \geq 2\alpha - 1, n \geq 1.$

- Let $(\alpha, \beta) = (4, 12), n = 3$. In this case $\beta - 2\alpha + 2 = 12 - 2(4) + 2 = 6$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 39, 37, 35, 33, 27, 25, 23, 21, 15, 13, 11, 9 = \lambda_{k-1} = \lambda_{11}$$

The sequence of gaps $\{g_i\}$ is

$$2, 2, 2, 6, \quad 2, 2, 2, 6, \quad 2, 2, 2$$

See Figure B.1 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (4, 12)$.

- Let $(\alpha, \beta) = (4, 9), n = 4$. In this case $\beta - 2\alpha + 2 = 9 - 2(4) + 2 = 3$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 39, 37, 35, 33, 30, 28, 26, 24, 21, 19, 17, 15, 12, 10, 8, 6 = \lambda_{k-1} = \lambda_{15}$$

The sequence of gaps $\{g_i\}$ is

$$2, 2, 2, 3, \quad 2, 2, 2, 3, \quad 2, 2, 2, 3, \quad 2, 2, 2$$

B.2 $2\alpha > \beta > \frac{3}{2}\alpha, n \geq 2.$

- Let $(\alpha, \beta) = (6, 10), n = 5.$

In this case $r = 2(6) - 10 = 2$ and $l = 10 - 6 = 4.$ We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 55, 53, 51, 49, 47, 46, 44, 43, 41, 39, 37, 36, 34, 33, 31,$$

$$29, 27, 26, 24, 23, 21, 19, 17, 16, 14, 13, 11, 9, 7, 5 = \lambda_{k-1} = \lambda_{29}.$$

The sequence of gaps $\{g_i\}$ is

$$2, 2, 2, 2, \quad 1, 2, 1, \quad 2, 2, 2, \quad 1, 2, 1, 2, 2, 2, \quad 1, 2, 1, 2, 2, 2, \quad 1, 2, 1, \quad 2, 2, 2, 2$$

See Figure B.2 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (6, 10).$

- Let $(\alpha, \beta) = (7, 12), n = 4.$

In this case $r = 2(7) - 12 = 2$ and $l = 12 - 7 = 5.$ We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 54, 52, 50, 48, 46, 44, 43, 41, 40, 38, 36, 34, 32, 31, 29, 28, 26, 24, 22, 20, 19,$$

$$17, 16, 14, 12, 10, 8, 6 = \lambda_{k-1} = \lambda_{27}.$$

The sequence of gaps $\{g_i\}$ is

$$2, 2, 2, 2, 2, \quad 1, 2, \quad 1, 2, 2, 2, 2, \quad 1, 2, \quad 1, 2, 2, 2, 2, \quad 1, 2, \quad 1, \quad 2, 2, 2, 2, 2$$

B.3 $\frac{3}{2}\alpha > \beta > \alpha, l|\alpha, n \geq \frac{\alpha}{l} + 1.$

- Let $(\alpha, \beta) = (12, 15)$, $n = 5$.

In this case $l = 15 - 12 = 3$ and $c = 12/3 = 4$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\begin{aligned} \lambda_0 = & 86, 84, 82, 80, 79, 77, 76, 74, 73, 71, 70, 69, 67, 66, 65, 63, 62, 61, 59, 58, 57, \\ & 56, 54, 53, 52, 51, 49, 48, 47, 46, 44, 43, 42, 41, 39, 38, 37, 36, 34, 33, 32, \\ & 31, 29, 28, 27, 25, 24, 23, 21, 20, 19, 17, 16, 14, 13, 11, 10, 8, 6, 4 = \lambda_{k-1} = \lambda_{59}. \end{aligned}$$

The sequence of gaps between the λ_i s is

$$\begin{aligned} & 2, 2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 1, 2, \\ & 1, 1, 1, 2, \quad 1, 1, 1, 2, \quad 1, 1, 1, 2, \quad 1, 1, 1, 2, \quad 1, 1, 1, 2, \quad 1, 1, \quad 2, 1, 1, \\ & 2, 1, 1, \quad 2, 1, \quad 2, 1, \quad 2, 1, \quad 2, 2, 2 \end{aligned}$$

See Figure B.3 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (12, 15)$.

- Let $(\alpha, \beta) = (9, 12)$, $n = 4$.

In this case $l = 12 - 9 = 3$ and $c = 9/3 = 3$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\begin{aligned} \lambda_0 = & 56, 54, 52, 50, 49, 47, 46, 44, 43, 41, 40, 39, 37, 36, 35, 33, 32, 31, 29, \\ & 28, 27, 25, 24, 23, 21, 20, 19, 17, 16, 14, 13, 11, 10, 8, 6, 4 = \lambda_{k-1} = \lambda_{35}. \end{aligned}$$

The sequence of gaps $\{g_i\}$ is

$$\begin{aligned} & 2, 2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, \\ & 2, \quad 1, \quad 2, 1, \quad 2, 1, \quad 2, 2, 2 \end{aligned}$$

B.4 $\frac{3}{2}\alpha > \beta > \alpha$, $l \nmid \alpha$, $n \geq \lceil \alpha/l \rceil + 1$.

- Let $(\alpha, \beta) = (10, 14)$, $n = 4$.

In this case $l = 14 - 10 = 4$, $c = \lceil 10/4 \rceil = 3$, and $d = 2$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 65, 63, 31, 59, 57, 56, 54, 53, 51, 50, 48, 47, 45, 44, 43, 41, 40, 39, 37,$$

$$36, 34, 33, 31, 30, 29, 27, 26, 25, 23, 22, 20, 19, 17, 16, 14, 13, 11, 9, 7, 5 = \lambda_{k-1} = \lambda_{39}.$$

The sequence of gaps $\{g_i\}$ in this example is

$$2, 2, 2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2,$$

$$1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, \quad 2, 1, \quad 2, 1, \quad 2, 1, \quad 2, 2, 2, 2$$

See Figure B.4 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (10, 14)$.

- Let $(\alpha, \beta) = (7, 9)$, $n = 6$.

In this case $l = 12 - 9 = 3$, $c = \lceil 9/2 \rceil = 5$, and $d = 2$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 60, 58, 56, 55, 53, 52, 50, 49, 48, 46, 45, 44, 42, 41, 40, 39, 37, 36, 35, 33, 32, 31,$$

$$30, 28, 27, 26, 24, 23, 22, 21, 19, 18, 17, 15, 14, 13, 11, 10, 8, 7, 5, 3 = \lambda_{k-1} = \lambda_{41}.$$

The sequence of gaps $\{g_i\}$ in this example is

$$2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 1, 2, \quad 1, 1, 2, \quad 1, 1, 1, 2,$$

$$1, 1, 2, \quad 1, 1, 1, \quad 2, \quad 1, 1, \quad 2, 1, 1, \quad 2, 1, \quad 2, 1, \quad 2, 2$$

B.5 $\frac{3}{2}\alpha > \beta > \alpha$, $n < \lceil \alpha/l \rceil + 1$.

- Let $(\alpha, \beta) = (6, 8)$, $n = 3$.

In this case $l = 8 - 6 = 2$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 29, 27, 25, 24, 22, 21, 19, 18, 17, 15, 14, 13, 11, 10, 8, 7, 5, 3 = \lambda_{k-1} = \lambda_{23}.$$

The sequence of gaps $\{g_i\}$ in this example is

$$2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 1, 2, \quad 1, 1, 2, \quad 1, \quad 2, 1, \quad 2, 2$$

See Figure B.5 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (6, 8)$ and $n = 3$.

- Let $(\alpha, \beta) = (7, 10)$, $n = 2$.

In this case $l = 10 - 7 = 3$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 26, 24, 22, 20, 19, 17, 16, 14, 13, 11, 10, 8, 6, 4 = \lambda_{k-1} = \lambda_{13}.$$

The sequence of gaps $\{g_i\}$ in this example is

$$2, 2, 2, \quad 1, 2, \quad 1, 2, \quad 1, 2, \quad 1, \quad 2, \quad 2, 2$$

B.6 $\alpha = \beta$, $n \geq 1$.

- Let $(\alpha, \beta) = (3, 3)$, $n = 5$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 17, 16, 15, 14, 13, 11, 10, 9, 8, 7, 5, 4, 3, 2, 1 = \lambda_{k-1} = \lambda_{15}.$$

The sequence of gaps $\{g_i\}$ is

$$1, 1, 1, 1, 2, \quad 1, 1, 1, 1, 2, \quad 1, 1, 1, 1.$$

See Figure B.6 for an illustration of the patterns found in the gap sequence when $(\alpha, \beta) = (3, 3)$ and $n = 5$.

- Let $(\alpha, \beta) = (4, 4)$, $n = 2$. We have the following sequence $\{\lambda_i\}$ of invariants:

$$\lambda_0 = 11, 10, 8, 7, 5, 4, 2, 1 = \lambda_{k-1} = \lambda_7.$$

The sequence of gaps $\{g_i\}$ is

$$1, 2, \quad 1, 2, \quad 1, 2, \quad 1$$

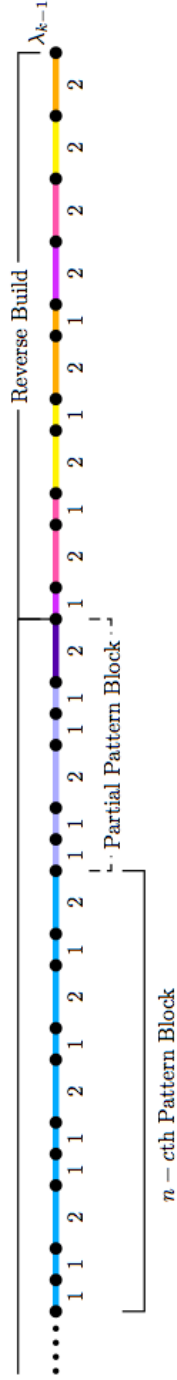
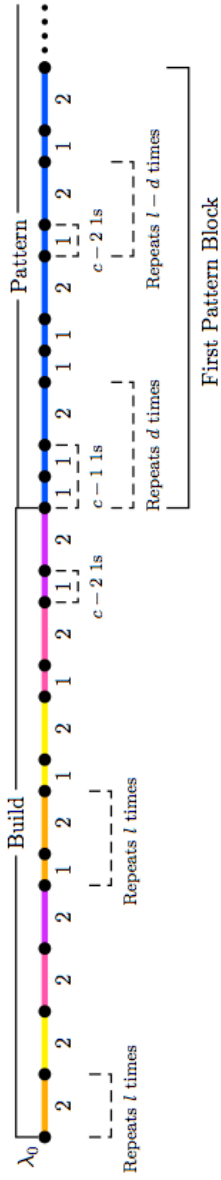


Figure B.4: Example of the output of Algorithm 4 ($\frac{3}{2}\alpha > \beta > \alpha$, $(\beta - \alpha) \nmid \alpha$, and $n \geq \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$) for case $\alpha = 10$ and $\beta = 14$. Note that $l = 4$, $c = 3$, and $d = \alpha \bmod l = 2$.

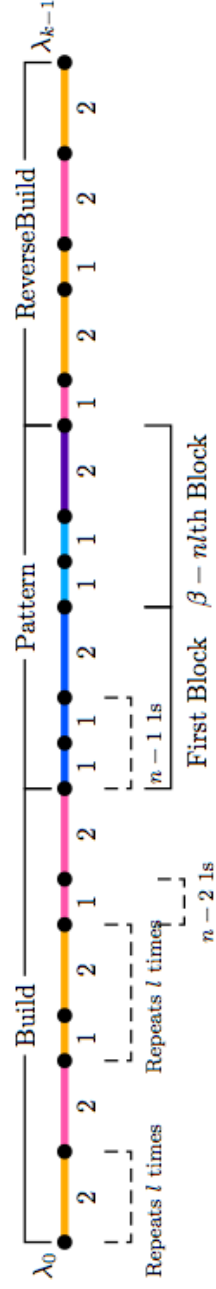


Figure B.5: Example of the output of Algorithm 5 ($\frac{3}{2}\alpha > \beta > \alpha$ and $2 \leq n < \lceil \frac{\alpha}{\beta - \alpha} \rceil + 1$) for case $\alpha = 6$, $\beta = 8$, and $n = 3$. Note that $l = 2$.

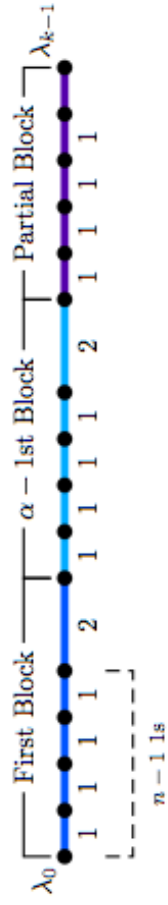


Figure B.6: Example of the output of Algorithm 6 ($\alpha = \beta$) for case $\alpha = \beta = 3$ and $n = 5$.

APPENDIX C

Formulas for Invariants of Chapter III

In this appendix we fill in details of how the formulas for the proposed invariants follow from the algorithms in Section 3.4 of Chapter III. Recall from 3.4 that each of the algorithms may be divided into three phases: the Build, the Reverse Build, and the Pattern. The sequence of gaps $g_i = \lambda_{i-1} - \lambda_i$ arising from the Reverse Build in a particular case is the reverse of the sequence of gaps arising from the Build. Therefore, we can use closed form expressions for invariants arising from the Build to write expressions for invariants arising from the Reverse Build. Further, the Pattern consists of repeats of some Pattern Blocks so it will be convenient to write indices within the Pattern in a way that reflects both the number of Pattern Blocks which have passed and the position in the current Pattern Block. Referring to the examples and pictures in Appendix B may help to clarify the work in this subappendix.

C.1 $\beta \geq 2\alpha - 1, n \geq 1$

Note that each repeat of the Pattern Block $\mathbf{BlockFar}(i, \lambda_{i-1}, \alpha, \beta)$ produces α invariants. Then it makes sense to write each index v in the unique form $v = j\alpha + s$ where $0 \leq s < \alpha$ so that j denotes the number of repeats of $\mathbf{BlockFar}(i, \lambda_{i-1}, \alpha, \beta)$ that have preceded v and s denotes the position of v in $\mathbf{BlockFar}(i, \lambda_{i-1}, \alpha, \beta)$. Then

$$\begin{aligned}\lambda_v &= \lambda_0 - (\beta - 2\alpha + 2)j - 2(\alpha - 1)j - 2s \\ &= (n - j)\beta + \alpha - 1 - 2s\end{aligned}$$

where $s = 0, \alpha - 1$ and $j = 0, \dots, n - 1$.

C.2 $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha$

Throughout this subappendix, $r := 2\alpha - \beta$.

Formula for λ_i in the Build. For $v = 0, \dots, l$, $\sum_{i=1}^v g_i = 2v$ so

$$\lambda_v = \lambda_0 - 2v = \alpha(n+1) + l \cdot n - 1 - 2v.$$

Formulas for λ_i in the Reverse Build

We can confirm that Algorithm 2 produces $\lambda_{k-1} = \beta - \alpha + 1$ by computing the sum of the gaps g_i as follows:

$$\begin{aligned} \lambda_0 - \lambda_{k-1} &= \sum_{i=1}^k g_i \\ &= 2l + [(m-1) \cdot 2 + m + 2(\alpha - (2m-1))] + 2(m-1) + m + 2l \\ &= \beta(n-1) + 2\alpha - 2 \\ &= \lambda_0 - (\beta - \alpha + 1) \end{aligned}$$

Thus, for $v = k - l - 1, \dots, k - 1$,

$$\lambda_v = \lambda_{k-1} + \sum_{i=v+1}^{k-1} g_i = \lambda_{k-1} + 2(k - v - 1)$$

or, for $i = 1, \dots, l + 1$,

$$\lambda_{k-i} = (l+1) + 2(k - (k-i) - 1) = l+1 + 2(i-1) = l+2i-1.$$

Formulas for λ_i in the Pattern.

Note that the total number of invariants produced by one complete repeat of **BlockMid**($i, \lambda_{i-1}, r, \alpha$) is $2(r-1) + \alpha - (2r-1) + 1 = \alpha$. Further, $\sum g_i$ where the sum is over one repeat of **BlockMid**($i, \lambda_{i-1}, r, \alpha$) is

$$\alpha + \# \text{ of 2s in difference sequence} = \alpha + [r-1 + \alpha - (2r-1)] = 2\alpha - r.$$

Note that the indices $l+1, l+2, \dots, n\alpha - l - 2$ are in the Pattern (repeats of **BlockMid**($i, \lambda_{i-1}, r, \alpha$)); we will divide these indices as follows to reflect different parts of the Pattern.

	indices i	gaps g_i
	$l + 1, \dots, l + 2r - 2$	$\rightarrow 12 \cdots 12$
	$l + 2r - 1, \dots, l + \alpha - 1$	$\rightarrow 12 \cdots 2$
	$l + \alpha$	$\rightarrow 2$
	\vdots	\vdots
	$l + (n - 3)\alpha + 1, \dots, l + (n - 3)\alpha + 2r - 2$	$\rightarrow 12 \cdots 12$
	$l + (n - 3)\alpha + 2r - 1, \dots, l + (n - 2)\alpha - 1$	$\rightarrow 12 \cdots 2$
	$l + (n - 2)\alpha$	$\rightarrow 2$
	$l + (n - 2)\alpha + 1, \dots, l + (n - 2)\alpha + 2(r - 1) = n\alpha - l - 2$	$\rightarrow 12 \cdots 12$

In the following, j represents how many full repeats of **BlockMid** have come before v and y represents the position in the current repeat of **BlockMid**.

- ❶ $v = l + j\alpha + y$ where $j = 0, \dots, (n - 2)$ and $y = 1, \dots, 2r - 2$. Then

$$\lambda_v = \lambda_0 - \left[2l + (2\alpha - m)j + y + \left\lfloor \frac{y}{2} \right\rfloor \right].$$

- ❷ $v = l + j\alpha + y$ where $j = 0, \dots, (n - 3)$ and $y = 2r - 1, \dots, \alpha - 1$. Then

$$\lambda_0 - [2l + (2\alpha - r)j + 2y - r]$$

- ❸ $v = l + j\alpha$ where $j = 1, \dots, n - 2$. Then

$$\lambda_v = \lambda_0 - [j(2\alpha - r) + 2l]$$

To get rid of the floor function in the first expression, we break this part into two. For $p = 1, \dots, r - 1$, when $y = 2p$, $\lfloor y/2 \rfloor = p$ and when $y = 2p - 1$, $\lfloor y/2 \rfloor = p - 1$. We now have the following expressions *replacing* ❶:

- ❹ $v = l + j\alpha + 2p$ where $j = 0, \dots, n - 2$ and $p = 1, \dots, r - 1 = \alpha - l - 1$. Then

$$\lambda_v = \lambda_0 - [2l + (\alpha + l)j + 2p + p].$$

⑤ $v = l + j\alpha + 2p - 1$ where $j = 0, \dots, (n - 2)$ and $p = 1, \dots, r - 1$. Then

$$\lambda_v = \lambda_0 - [2l + (\alpha + l)j + 2p - 2 + p]$$

Thus, the expressions denoted ②, ③, ④, and ⑤ taken together give the formulas for all of the invariants produced by the Pattern phase of Algorithm 2.

C.3 $\frac{3}{2}\alpha > \beta > \alpha, l|\alpha, n \geq \frac{\alpha}{l} + 1$

Throughout this subappendix $c := \frac{\alpha}{l}$, which is an integer by assumption.

Formulas for λ_i in the Build.

Note that the Build in Algorithm 3 spans the indices $1, 2, \dots, l(1 + \dots + (c - 1))$. We divide these indices to reflect different parts of the Build as follows.

indices i	gaps g_i
$1, \dots, l$	$\rightarrow 2 \cdots 2$
$l + 1, \dots, 2l + l$	$\rightarrow 12 \cdots 12$
$2l + l + 1, \dots, 3l + 2l + l$	$\rightarrow 112 \cdots 112$
\vdots	\vdots
$l(1 + \dots + (c - 2)) + 1, \dots, l(1 + \dots + (c - 1))$	$\rightarrow 1 \cdots 12 \cdots 1 \cdots 12$

This suggests that we write an index v between $l + 1$ and $l(1 + \dots + (c - 1))$ as $v = l(1 + \dots + q) + p$ where $q = 1, \dots, (c - 2)$ and $p = 1, \dots, (q + 1)l$. We have the following cases:

1. If $p = (q + 1)j$ for $j = 1, \dots, l$ then

$$\sum_{i=1}^v g_i = (2 + \dots + (q + 1))l + (q + 1)j + \left\lfloor \frac{(q + 1)j}{q + 1} \right\rfloor = (2 + \dots + (q + 1))l + (q + 1)j + j.$$

2. If $p = (q + 1)j - x$ for $j = 1, \dots, l$ and $x = 1, \dots, q$ then

$$\begin{aligned} \sum_{i=1}^v g_i &= (2 + \dots + (q + 1))l + (q + 1)j - x + \left\lfloor \frac{(q + 1)j - x}{q + 1} \right\rfloor \\ &= (2 + \dots + (q + 1))l + (q + 2)j - x - 1. \end{aligned}$$

Since $\lambda_v = \lambda_0 - \sum_{i=1}^v g_i$,

❶ For $v = 0, \dots, l$,

$$\lambda_v = \lambda_0 - 2v.$$

❷ For $v = l(1 + \dots + q) + (q + 1)j$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_0 - [(2 + \dots + (q + 1))l + (q + 1)j + j].$$

❸ For $v = l(1 + \dots + q) + (q + 1)j - x$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,
 $x = 1, \dots, q$,

$$\lambda_v = \lambda_0 - [(2 + \dots + (q + 1))l + (q + 1)j - x + j - 1].$$

Formula for λ_i in the Reverse Build.

We claim that the value of λ_{k-1} produced by Algorithm 3 is equal to $l + 1$ as we would expect by Theorem 3.3.1. This can be checked as follows:

$$\begin{aligned} \lambda_0 - \lambda_{k-1} &= \sum_{i=1}^{k-1} g_i \\ &= l(2 + \dots + c) + [(c - 1) + 2][nl - \alpha + l] + l(2 + \dots + c) - 2 \\ &= \alpha - l + \alpha n + nl - 2 \\ &= \beta(n - 1) + 2\alpha - 2 = \lambda_0 - (l + 1) \end{aligned}$$

The sequence of gaps in the Reverse Build is almost the opposite of the sequence of gaps in the Build; the exception is that the final part of the Build which does not appear in the Reverse Build.¹ Thus, we can use the formulas for $\sum g_i$ derived above to find formulas for λ_v where v is in the Build. In particular,

$$\lambda_v = \lambda_{k-1-T} = \lambda_{k-1} + \sum_{i=0}^T g_i$$

where v ranges over the index set of the Build. We take $v = \lambda_{k-1-T}$ where $T = 0, \dots, l(1 + \dots + (c - 2)) + (c - 1)l - 1$.² The final formulas are listed in subappendix 3.5.3.1 of the main text.

¹This is why the **ReverseBuildPartial** subroutine is necessary

²Note that for the Build we considered indices $t = 0, \dots, l(1 + \dots + (c - 2)) + (c - 1)l$.

Formulas for λ_i in the Pattern.

We introduce some additional notation. Let E be the final index of the Build, $E := l(1 + \dots + (c - 1))$ and let $B := \sum_{i=1}^E g_i$ so $B = l(2 + \dots + c)$. Then $v = E + 1, E + 1, \dots, E + (ln - \alpha + l)c$ are the indices that are part of the Pattern. Since each repeat of the **BlockClose**($i, \lambda_{i-1}, c, d, l, \alpha$) subroutine produces c invariants, we divide these indices as follows to reflect different parts of the Pattern.

indices i	gaps g_i
$E + 1, \dots, E + c - 1$	$\rightarrow 1 \dots 1$
$E + c$	$\rightarrow 2$
$E + c + 1, \dots, E + c + (c - 1)$	$\rightarrow 1 \dots 1$
$E + 2c$	$\rightarrow 2$
\vdots	\vdots
$E + (ln - \alpha + l - 1)c + 1, \dots, (ln - \alpha + l - 1)c + (c - 1)$	$\rightarrow 1 \dots 1$
$E + (ln - \alpha + l)c$	$\rightarrow 2$

This suggests the following formulas.

- ❶ For $v = E + jc + i$ where $j = 0, \dots, ln - \alpha + l - 1, i = 1, \dots, c - 1$,

$$\lambda_v = \lambda_0 - B - [j(c + 1) + i]$$

- ❷ For $v = E + jc$ where $j = 1, \dots, ln - \alpha + l - 1$,

$$\lambda_v = \lambda_0 - B - [j(c + 1)]$$

C.4 $\frac{3}{2}\alpha > \beta > \alpha, l \nmid \alpha, n \geq \lceil \frac{\alpha}{l} \rceil + 1$

Throughout this subappendix, set $c := \lceil \frac{\alpha}{l} \rceil$ and $d := \alpha \bmod l$.

Formulas for λ_i in the Build and Reverse Build.

The Build and the Reverse Build are identical to the $\frac{3}{2}\alpha > \beta > \alpha, l \mid \alpha, n \geq \frac{\alpha}{l}$

case.³ Further, the λ_{k-1} value produced by Algorithm 4 is equal to $l + 1$ since

$$\begin{aligned}\lambda_0 - \lambda_{k-1} &= \sum_{i=0}^{k-1} g_i \\ &= 2l(2 + \dots + c) + (n - c)[d(c + 1) + (l - d)c] - 2 + d(c + 1) \\ &= \beta(n - 1) + 2\alpha - 2.\end{aligned}$$

Therefore, the formulas for λ_i in the Build and the Reverse Build are the same as those in subappendix C.3.

Formulas for λ_i in the Pattern.

As before, we introduce some notation to denote where the Pattern begins. Let E be the index where the Build ends so that

$$E := l(1 + \dots + (c - 1)).$$

Let $B = \sum_{i=1}^E g_i$, so that

$$B := l(2 + \dots + c).$$

Consider the first Pattern Block; we divide the indexing as follows.

indices i	gaps g_i
$E + 1, \dots, E + c - 1$	$\rightarrow 1 \cdots 1$
$E + c$	$\rightarrow 2$
$E + c + 1, \dots, E + 2c - 1$	$\rightarrow 1 \cdots 1$
$E + 2c$	$\rightarrow 2$
\vdots	\vdots
$E + (d - 1)c + 1, \dots, E + (d - 1)c + (c - 1)$	$\rightarrow 1 \cdots 1$
$E + dc$	$\rightarrow 2$
$E + dc + 1, \dots, E + dc + (c - 2)$	$\rightarrow 1 \cdots 1$
$E + dc + (c - 1)$	$\rightarrow 2$
$E + dc + (c - 1) + 1, \dots, E + dc + (c - 1) + (c - 2)$	$\rightarrow 1 \cdots 1$
$E + dc + 2(c - 1)$	$\rightarrow 2$
\vdots	\vdots

³This is because the **Build**, **ReverseBuild**, and **ReverseBuildPartial** with the same inputs are found in both Algorithm 3 and Algorithm 4.

$$E + dc + (l - d)(c - 1) = E + \alpha \quad \rightarrow \quad 2$$

Note that each Pattern Block has total length α so we divide the remainder of the indexing in the same way. We then the following formulas. **❶** and **❷** describe v such that g_v are the 1s and 2s respectively from $1 \cdots 12$ with $(c - 1)$ 1s. **❸** and **❹** describe v such that g_v are the 1s and 2s from the $1 \cdots 12$ with $(c - 2)$ 1s. Note that p signifies the number of full patterns that have preceded the one that v belongs to.

- ❶** For $v = E + p\alpha + jc + i$ where $p = 0, \dots, n - c, j = 0, \dots, d - 1, i = 1, \dots, c - 1,$

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + j(c + 1) + i]$$

- ❷** For $v = E + p\alpha + jc$ where $p = 0, \dots, n - c, j = 1, \dots, d,$

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + j(c + 1)]$$

- ❸** For $v = E + p\alpha + dc + j(c - 1) + i$ where $p = 0, \dots, n - c - 1, j = 0, \dots, l - d - 1, i = 1, \dots, c - 2$

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + d(c + 1) + jc + i]$$

- ❹** For $v = E + p\alpha + dc + j(c - 1)$ where $p = 0, \dots, n - c - 1, j = 1, \dots, l - d$

$$\lambda_v = \lambda_0 - B - [p(l + \alpha) + d(c + 1) + jc]$$

C.5 $\frac{3}{2}\alpha > \beta > \alpha, 2 \leq n < \lceil \frac{\alpha}{7} \rceil + 1$

Algorithm 5 for this case is similar to Algorithm 3 for the case where $\frac{3}{2}\alpha > \beta > \alpha, l|\alpha, n \gg 0$ so we will refer to the results in subappendix 3.5.3 and only mention where replacements are made

Formulas for λ_i the Build. The Build is the same as in the $\frac{3}{2}\alpha > \beta > \alpha, l|\alpha, n \gg 0$ case except with c replaced by n .

- ❶** For $v = 0, \dots, l,$

$$\lambda_v = \lambda_0 - 2v.$$

- ② For $v = l(1 + \cdots + q) + (q + 1)j$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_0 - [(2 + \cdots + (q + 1))l + (q + 1)j + j]$$

- ③ For $v = l(1 + \cdots + q) + (q + 1)j - x$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,
 $x = 1, \dots, q$,

$$\lambda_v = \lambda_0 - [(2 + \cdots + (q + 1))l + (q + 1)j - x + j - 1]$$

Formulas for λ_i in the Reverse Build. The Build is the same as in the $\frac{3}{2}\alpha > \beta > \alpha$, $n \gg 0$ case except with c replaced by n . It is easy to check that $\sum_{i=1}^{k-1} g_i = \lambda_0 - (l + 1)$ so these formulas are valid when we set $\lambda_{k-1} = l + 1$.

- ① For $v = (k - 1) - j$ where $j = 0, \dots, l$,

$$\lambda_v = \lambda_{k-1} + 2j.$$

- ② For $v = (k - 1) - (l(1 + \cdots + q) + (q + 1)j)$ where $q = 1, \dots, (n - 2)$, $j = 1, \dots, l$,

$$\lambda_v = \lambda_{k-1} + [(2 + \cdots + (q + 1))l + (q + 1)j + j].$$

However, when $q = n - 2$ and $j = l$, λ_v is in the Pattern, not in the Reverse Build.

- ③ For $v = (k - 1) - (l(1 + \cdots + q) + (q + 1)j - x)$ where $q = 1, \dots, (n - 2)$,
 $j = 1, \dots, l$, $x = 1, \dots, q$,

$$\lambda_v = \lambda_{k-1} + [(2 + \cdots + (q + 1))l + (q + 1)j - x + j - 1].$$

However, when $q = n - 2$, $j = l$, and $x = 1$, λ_v is in the Pattern, not the Reverse Build.

Formulas for λ_i in the Pattern.

Let E be the index where the Build ends; that is

$$E := l(1 + \cdots + (n - 1)).$$

Let $B = \sum_{i=1}^E g_i$ so

$$B := l(2 + \cdots + n).$$

The Pattern is similar to the case where $\frac{3}{2}\alpha > \beta > \alpha$, $l|\alpha$, and $n \gg 0$ except that c is replaced by n within each Pattern Block and the Pattern repeats $\beta - nl$ times instead of $nl - \alpha + l$ times. Thus, we can make appropriate substitutions in the formulas for the invariants in the Pattern from Section C.3 to determine the formulas here.

- ❶ For $v = E + jn + i$ where $j = 0, \dots, \beta - nl - 1$, $i = 1, \dots, n - 1$,

$$\lambda_v = \lambda_0 - B - [j(n+1) + i].$$

- ❷ For $v = E + jn$ where $j = 1, \dots, \beta - nl - 1$,

$$\lambda_v = \lambda_0 - B - [j(n+1)].$$

C.6 $\alpha = \beta$, $n \geq 1$

The only phase of Algorithm 6 is the Pattern. Note that the Pattern Block subroutine **onestwo** $(n - 1, i, \lambda_{i-1})$ produces n invariants. Thus, it is convenient to write $v = qn + j$ where $0 \leq j \leq n - 1$. Since $\lambda_{qn} - \lambda_{(q+1)n} = n + 1$ and $\lambda_{qn+j} - \lambda_{qn+j+1} = 1$ for $j \neq n - 1$, we have

$$\begin{aligned} \lambda_v &= \lambda_0 - q(n+1) - j \\ &= (n+1)\alpha - 1 - q(n+1) - j. \end{aligned}$$

for $q = 0, \dots, \alpha - 1$ and $j = 0, \dots, n - 1$.

APPENDIX D

Calculation Details for Chapter III

This appendix contains the details of the routine calculations found in Section 3.5. In particular, we rewrite $H_{J^n}(t)$ from Proposition 3.3.8 according to the relations between α and β and we simplify partial sums of the form $\sum_i \binom{t-\lambda_i-i+m-2}{m-2}$ where i ranges over subsets of the invariants. These simplified expressions are added together in subappendix 3.5 to give

$$H_J(t) = \sum_{i=0}^{n\alpha-1} \binom{t-\lambda_i-i+m-2}{m-2} + \binom{t-n\alpha+m-1}{m-1}$$

as in Proposition 3.3.9. The numbering of subsets of the Build, Reverse Build, and Pattern matches that in Appendix C and in the main text. The only combinatorial identities used are equations (1) through (4) from Section 3.2.

D.1 $\beta \geq 2\alpha - 1, n \geq 1$

All details are contained in the main text.

D.2 $2\alpha - 1 > \beta \geq \frac{3}{2}\alpha$

We begin by computing $\lambda_v + v$ and then simplify the sums of the associated binomial coefficients. Set $X_j = t - n\alpha - jl$ and $Y_j = t - (n+1)\alpha - jl$.

Partial sums from the invariants in the Build.

For $v = 0, \dots, l$,

$$\lambda_v + v = \lambda_0 - 2v + v = (n+1)\alpha + nl - 1 - v.$$

Then

$$\begin{aligned}
& \sum_{v=0}^l \binom{t - [(n+1)\alpha + nl - 1 - v] + m - 2}{m-2} \\
&= \sum_{v=0}^l \binom{t - (n+1)\alpha - nl + m - 1 + v}{m-2} \\
&= \binom{t - (n+1)\alpha - nl + m - 1 + l + 1}{m-1} - \binom{t - (n+1)\alpha - nl + m - 1}{m-1} \\
&= \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m - 1}{m-1}.
\end{aligned}$$

Partial sums from the invariants in the Reverse Build.

For $i = 1, \dots, l + 1$,

$$\lambda_{k-i} + k - i = (l + 2i - 1) + (k - i) = l + i + n\alpha - 1.$$

Then

$$\begin{aligned}
& \sum_{i=1}^{l+1} \binom{t - [l + n\alpha + i - 1] + m - 2}{m-2} \\
&= \sum_{i=1}^{l+1} \binom{t - l - n\alpha + m - 1 - i}{m-2} \\
&= \binom{t - l - n\alpha + m - 1}{m-1} - \binom{t - l - n\alpha + m - 1 - (l + 1)}{m-1} \\
&= \binom{X_1 + m - 1}{m-1} - \binom{X_2 + m - 2}{m-1}.
\end{aligned}$$

Partial sums from the invariants in the Pattern.

② For $v = l + j\alpha + y$ where $j = 0, \dots, (n - 3)$ and $y = 2r - 1, \dots, \alpha - 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - [2l + (\alpha + l)j + 2y - (\alpha - l)] + l + j\alpha + y \\
&= \lambda_0 - l - jl - y + (\alpha - l) \\
&= (n + 2)\alpha + l(n - j - 2) - 1 - y.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{j=0}^{n-3} \sum_{y=2r-1}^{\alpha-1} \binom{t - [(n+2)\alpha + l(n-j-2) - 1 - y] + m-2}{m-2} \\
&= \sum_{j=0}^{n-3} \sum_{y=2(\alpha-l)-1}^{\alpha-1} \binom{t - (n+2)\alpha - l(n-j-2) + m-1 + y}{m-2} \\
&= \sum_{j=0}^{n-3} \left[\binom{t - (n+2)\alpha - l(n-j-2) + m-1 + \alpha}{m-1} \right. \\
&\quad \left. - \binom{t - (n+2)\alpha - l(n-j-2) + m-1 + 2(\alpha-l) - 1}{3} \right] \\
&= \sum_{j=0}^{n-3} \left[\binom{t - (n+1)\alpha - l(n-j-2) + m-1}{m-1} - \binom{t - n\alpha - l(n-j) + m-2}{m-1} \right] \\
&= \sum_{p=3}^n \left[\binom{t - (n+1)\alpha - l(p-2) + m-1}{m-1} - \binom{t - n\alpha - lp + m-2}{m-1} \right] \\
&= \sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-1} - \sum_{j=3}^n \binom{X_j + m-2}{m-1}
\end{aligned}$$

③ For $v = l + j\alpha$ where $j = 1, \dots, n-2$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - [j(\alpha + l) + 2l] + l + j\alpha \\
&= \lambda_0 - jl - l \\
&= (n+1)\alpha + l(n-j-1) - 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{j=1}^{n-2} \binom{t - [(n+1)\alpha + l(n-j-1) - 1] + m-2}{m-2} \\
&= \sum_{j=1}^{n-2} \binom{t - (n+1)\alpha - l(n-j-1) + m-1}{m-2} \\
&= \sum_{p=1}^{n-2} \binom{t - (n+1)\alpha - lp + m-1}{m-2} \\
&= \sum_{j=1}^{n-2} \binom{Y_j + m-1}{m-2}
\end{aligned}$$

④ For $v = l + j\alpha + 2p$ where $j = 0, \dots, n-2$ and $p = 1, \dots, r-1 = \alpha - l - 1$,

$$\begin{aligned}\lambda_v + v &= \lambda_0 - [2l + (\alpha + l)j + 3p] + l + j\alpha + 2p \\ &= \lambda_0 - l - lj - p \\ &= (n+1)\alpha + (n-j-1)l - 1 - p.\end{aligned}$$

Then

$$\begin{aligned}& \sum_{j=0}^{n-2} \sum_{p=1}^{\alpha-l-1} \binom{t - [(n+1)\alpha + (n-j-1)l - 1 - p] + m-2}{m-2} \\ &= \sum_{j=0}^{n-2} \left[\binom{t - (n+1)\alpha - (n-j-1)l + m-1 + \alpha - l}{m-1} \right. \\ & \quad \left. - \binom{t - (n+1)\alpha - (n-j-1)l + m-1 + 1}{m-1} \right] \\ &= \sum_{j=0}^{n-2} \left[\binom{t - n\alpha - (n-j)l + m-1}{m-1} - \binom{t - (n+1)\alpha - (n-j-1)l + m}{m-1} \right] \\ &= \sum_{p=2}^n \binom{X_p + m-1}{m-1} - \sum_{p'=1}^{n-1} \binom{Y_{p'} + m}{m-1}\end{aligned}$$

⑤ For $v = l + j\alpha + 2p - 1$ where $j = 0, \dots, (n-2)$ and $p = 1, \dots, r-1$,

$$\begin{aligned}\lambda_v + v &= \lambda_0 - [2l + (\alpha + l)j + 3p - 2] + l + j\alpha + 2p - 1 \\ &= \lambda_0 - l - lj - p + 1 \\ &= (n+1)\alpha + l(n-j-1) - p.\end{aligned}$$

Then

$$\begin{aligned}& \sum_{j=0}^{n-2} \sum_{p=1}^{\alpha-l-1} \binom{t - [(n+1)\alpha + l(n-j-1) - p] + m-2}{m-2} \\ &= \sum_{j=0}^{n-2} \sum_{p=1}^{\alpha-l-1} \binom{t - (n+1)\alpha - l(n-j-1) + m-2 + p}{m-2} \\ &= \sum_{j=0}^{n-2} \left[\binom{t - (n+1)\alpha - l(n-j-1) + m-2 + \alpha - l}{m-1} \right. \\ & \quad \left. - \binom{t - (n+1)\alpha - l(n-j-1) + m-2 + 1}{m-1} \right]\end{aligned}$$

$$\begin{aligned}
&= \sum_{p=2}^n \left[\binom{t - n\alpha - lp + m - 2}{m-1} - \binom{t - (n+1)\alpha - l(p-1) + m - 1}{m-1} \right] \\
&= \sum_{j=2}^n \binom{X_j + m - 2}{m-1} - \sum_{j=1}^{n-1} \binom{Y_j + m - 1}{m-1}
\end{aligned}$$

Adding these partial sums together with $\binom{t - n\alpha + m - 1}{m-1}$ gives the Hilbert function sum in subappendix 3.5.2.2.

D.3 $\frac{3}{2}\alpha > \beta > \alpha$, $l|\alpha$, $n \geq \frac{\alpha}{l} + 1$

Throughout this subappendix, $c := \frac{\alpha}{l}$; this is an integer by assumption.

Partial sums from invariants in the Build.

Recall that $\lambda_0 = (n+1)\alpha + ln - 1$ and set $X_j = t - n\alpha - jl$ and $Y_j = t - (n+1)\alpha - jl$ as above.

❶

For $v = 0, \dots, l$,

$$\lambda_v + v = \lambda_0 - 2v + v = \lambda_0 - v = (n+1)\alpha + nl - v - 1.$$

Then

$$\begin{aligned}
&\sum_{j=0}^l \binom{t - ((n+1)\alpha + nl - j - 1) + m - 2}{m-2} \\
&= \sum_{j=0}^l \binom{t - (n+1)\alpha - nl + (m-1) + j}{m-2} \\
&= \left[\binom{t - (n+1)\alpha - nl + m - 1 + l + 1}{m-1} - \binom{t - (n+1)\alpha - nl + m - 1}{m-1} \right] \\
&= \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m - 1}{m-1}
\end{aligned}$$

②

For $v = l(1 + \cdots + q) + (q + 1)j$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,

$$\begin{aligned}\lambda_v + v &= \lambda_0 - (2 + \cdots + (q + 1))l - (q + 1)j - j + l(1 + \cdots + q) + (q + 1)j \\ &= (n + 1)\alpha + l(n - q) - j - 1.\end{aligned}$$

Then

$$\begin{aligned}& \sum_{q=1}^{c-2} \sum_{j=1}^l \binom{t - (n + 1)\alpha - l(n - q) + j + 1 + m - 2}{m - 2} \\ &= \sum_{q=1}^{c-2} \left[\binom{t - (n + 1)\alpha - l(n - q) + m - 1 + l + 1}{m - 1} \right. \\ & \quad \left. - \binom{t - (n + 1)\alpha - l(n - q) + m}{m - 1} \right] \\ &= \sum_{j=n-c+1}^{n-2} \binom{Y_j + m}{m - 1} - \sum_{j=n-c+2}^{n-1} \binom{Y_j + m}{m - 1} \\ &= \binom{Y_{n-c+1} + m}{m - 1} - \binom{Y_{n-1} + m}{m - 1}\end{aligned}$$

③

For $v = l(1 + \cdots + q) + (q + 1)j - x$ where $q = 1, \dots, (c - 2)$, $j = 1, \dots, l$,
 $x = 1, \dots, q$,

$$\begin{aligned}\lambda_v + v &= \lambda_0 - (2 + \cdots + (q + 1))l - (q + 1)j + x - j + 1 + l(1 + \cdots + q) \\ & \quad + (q + 1)j - x \\ &= (n + 1)\alpha + l(n - q) - j.\end{aligned}$$

Then

$$\begin{aligned}& \sum_{q=1}^{c-2} q \left[\sum_{j=1}^l \binom{t - (n + 1)\alpha - l(n - q) + j + m - 2}{m - 2} \right] \\ &= \sum_{q=1}^{c-2} q \left[\binom{t - (n + 1)\alpha - l(n - q) + m - 2 + l + 1}{m - 1} \right. \\ & \quad \left. - \binom{t - (n + 1)\alpha - l(n - q) + m - 2 + 1}{m - 1} \right]\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=n-c+1}^{n-2} (n-j-1) \binom{t-(n+1)\alpha-lj+m-1}{m-1} \\
&\quad - \sum_{j'=n-c+2}^{n-1} (n-j') \binom{t-(n+1)\alpha-lj'+m-1}{m-1} \\
&= (n-(n-c+1)-1) \binom{Y_{n-c+1}+m-1}{m-1} - (n-(n-1)-1) \binom{Y_{n-1}+m-1}{m-1} \\
&\quad - \sum_{j=n-c+2}^{n-1} \binom{Y_j+m-1}{m-1} \\
&= (c-2) \binom{Y_{n-c+1}+m-1}{m-1} - \sum_{j=n-c+2}^{n-1} \binom{Y_j+m-1}{m-1}
\end{aligned}$$

Partial sums from invariants in the Reverse Build.

Recall that $\lambda_{k-1} = l + 1$.

❶

For $v = (k-1) - j$ where $j = 0, \dots, l$,

$$\lambda_v + v = \lambda_{k-1} + 2j + (k-1) - j = l + n\alpha + j.$$

Then

$$\begin{aligned}
&\sum_{j=0}^l \binom{t-n\alpha-l-j+m-2}{m-2} \\
&= \binom{t-n\alpha-l+m-2+1}{m-1} - \binom{t-n\alpha-2l+m-2}{m-1} \\
&= \binom{X_1+m-1}{m-1} - \binom{X_2+m-2}{m-1}
\end{aligned}$$

❷

For $v = (k-1) - (l(1+\dots+m) + (q+1)j)$ where $q = 1, \dots, (c-2)$, $j = 1, \dots, l$, with the exception of $q = c-2$ and $j = l$,

$$\begin{aligned}
\lambda_v + v &= \lambda_{k-1} + (2 + \dots + (q+1))l + (q+1)j + j + (k-1) \\
&\quad - (l(1+\dots+q) + (q+1)j) \\
&= (q+1)l + n\alpha + j.
\end{aligned}$$

Note that λ_v where $q = c-2$ and $j = l$ is not in the Reverse Build. Then the sum

$\sum_v \binom{t-v-\lambda_v+m-2}{m-2}$ where v ranges over this part of the Reverse Build is:

$$\begin{aligned}
& \sum_{q=1}^{c-2} \sum_{j=1}^l \binom{t-n\alpha-(q+1)l-j+m-2}{m-2} - \binom{t-(n+1)\alpha+m-2}{m-2} \\
&= \sum_{q=1}^{c-2} \left[\binom{t-n\alpha-(q+1)l+m-2}{m-1} - \binom{t-n\alpha-(q+1)l+m-2-l}{m-1} \right] \\
&\quad - \binom{Y_0+m-2}{m-2} \\
&= \sum_{j=2}^{c-1} \binom{X_j+m-2}{m-1} - \sum_{j'=3}^c \binom{X_{j'}+m-2}{m-1} - \binom{Y_0+m-2}{m-2} \\
&= \binom{X_2+m-2}{m-1} - \binom{X_c+m-2}{m-1} \\
&\quad - \binom{Y_0+m-2}{m-2}
\end{aligned}$$

③

For $v = (k-1) - (l(1+\dots+q) + (q+1)j - x)$ where $q = 1, \dots, (c-2)$, $j = 1, \dots, l$, $x = 1, \dots, q$, with the exception of $q = c-2$, $j = l$, and $x = 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_{k-1} + (2 + \dots + (q+1))l - x + j - 1 + (k-1) - l(1 + \dots + q) + x \\
&= n\alpha + (q+1)l + j - 1.
\end{aligned}$$

Note that λ_v where $q = c-2$, $j = l$, and $x = 1$ is not in the Reverse Build. Then the sum $\sum_v \binom{t-v-\lambda_v+m-2}{m-2}$ where v ranges over this part of the Reverse Build is:

$$\begin{aligned}
& \sum_{q=1}^{c-2} q \left[\sum_{j=1}^l \binom{t-n\alpha-(q+1)l-j-1+m-2}{m-2} \right] - \binom{t-(n+1)\alpha+m-1}{m-2} \\
&\quad - \binom{Y_0+m-1}{m-2} \\
&= \sum_{q=1}^{c-2} q \left[\binom{t-n\alpha-(q+1)l+m-1}{m-1} - \binom{t-n\alpha-(q+1)l+m-1-l}{m-1} \right] \\
&\quad - \binom{Y_0+m-1}{m-2} \\
&= \sum_{j=2}^{c-1} (j-1) \binom{X_j+m-1}{m-1} - \sum_{j'=3}^c (j'-2) \binom{X_{j'}+m-1}{m-1} -
\end{aligned}$$

$$\begin{aligned}
& \binom{Y_0 + m - 1}{m - 2} \\
= & \sum_{j=2}^{c-1} (j-2) \binom{X_j + m - 1}{m-1} + \sum_{j=2}^{c-1} \binom{X_j + m - 1}{m-1} - \sum_{j=3}^c (j-2) \binom{X_j + m - 1}{m-1} \\
& - \binom{Y_0 + m - 1}{m - 2} \\
= & \sum_{j=2}^{c-1} \binom{X_j + m - 1}{m-1} + 0 - (c-2) \binom{X_c + m - 1}{m-1} - \binom{Y_0 + m - 1}{m - 2}
\end{aligned}$$

Partial Sums from invariants in the Pattern.

Note that $E - B = l(1 - c)$.

❶ For $v = E + jc + i$ where $j = 0, \dots, ln - \alpha + l - 1$, $i = 1, \dots, c - 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - B - jc - j - i + E + jc + i \\
&= n\alpha + (n + 1)l - 1 - j.
\end{aligned}$$

Then

$$\begin{aligned}
& (c-1) \sum_{j=0}^{l(n+1)-\alpha-1} \binom{t - n\alpha - l(n+1) + j + 1 + m - 2}{m-2} \\
= & (c-1) \left[\binom{t - n\alpha - l(n+1) + m - 1 + l(n+1) - \alpha - 1 + 1}{m-1} \right. \\
& \left. - \binom{t - n\alpha - l(n+1) + m - 1}{m-1} \right] \\
= & (c-1) \left[\binom{Y_0 + m - 1}{m-1} - \binom{X_{n+1} + m - 1}{m-1} \right]
\end{aligned}$$

❷ For $v = E + jc$ where $j = 1, \dots, ln - \alpha + l - 1$

$$\lambda_v + v = \lambda_0 - B - jc - j + E + jc = n\alpha + (n + 1)l - j - 1.$$

Then

$$\sum_{j=1}^{l(n+1)-\alpha} \binom{t - n\alpha - (n+1)l + j + 1 + m - 2}{m-2}$$

$$\begin{aligned}
&= \left[\binom{t - n\alpha - (n+1)l + m-1 + l(n+1) - \alpha + 1}{m-1} \right. \\
&\quad \left. - \binom{t - n\alpha - (n+1)l + m-1 + 1}{m-1} \right] \\
&= \binom{Y_0 + m}{m-1} - \binom{X_{n+1} + m}{m-1}
\end{aligned}$$

Rewriting the Hilbert function of I^n .

The Hilbert function of I^n is

$$\begin{aligned}
H_{I^n}(t) &= \sum_{j=1}^n \left(\binom{t - \alpha n - jl + m-1}{m-1} - \binom{t - \alpha(n+1) - lj + m-1}{m-1} \right) \\
&\quad + \binom{t - n\alpha + m-1}{m-1}
\end{aligned}$$

Now we will simplify this expression further by taking advantage of the assumption that α is divisible by l . We can rewrite the indexing set $j = 1, \dots, n$ as follows:

$$\begin{aligned}
&1, 2, \dots, c-1 \\
&c, c+1, \dots, 2c-1 \\
&\vdots \\
&(\lfloor n/c \rfloor - 1)c, \dots, \lfloor n/c \rfloor \cdot c - 1 \\
&\lfloor n/c \rfloor \cdot c, \dots, n = \lfloor n/c \rfloor \cdot c + (n - \lfloor n/c \rfloor \cdot c)
\end{aligned}$$

Using the fact that $c \cdot l = \alpha$, reindexing, and simplifying, the *sum* in the above Hilbert Function becomes:

$$\begin{aligned}
&\sum_{j=1}^{c-1} \left(\binom{t - \alpha n - jl + m-1}{m-1} - \binom{t - \alpha(n+1) - lj + m-1}{m-1} \right) + \\
&\sum_{p=1}^{\lfloor n/c \rfloor - 1} \sum_{j=0}^{c-1} \left(\binom{t - \alpha n - (pc + j)l + m-1}{m-1} - \binom{t - \alpha(n+1) - l(pc + j) + m-1}{m-1} \right) \\
&+ \sum_{j=0}^{n - \lfloor n/c \rfloor \cdot c} \binom{t - \alpha n - (\lfloor n/c \rfloor c + j)l + m-1}{m-1}
\end{aligned}$$

$$\begin{aligned}
& - \binom{t - \alpha(n+1) - (\lfloor n/c \rfloor c + j)l + m-1}{m-1} \Big) \\
= & \sum_{j=1}^{c-1} \left(\binom{t - \alpha n - jl + m-1}{m-1} - \binom{t - \alpha(n+1) - lj + m-1}{m-1} \right) \\
& + \sum_{j=0}^{c-1} \left(\sum_{p=1}^{\lfloor n/c \rfloor - 1} \binom{t - \alpha(n+p) - lj + m-1}{m-1} - \sum_{p'=2}^{\lfloor n/c \rfloor} \binom{t - \alpha(n+p') - lj + m-1}{m-1} \right) + \\
& \sum_{j=0}^{n - \lfloor n/c \rfloor \cdot c} \left(\binom{t - \alpha(n + \lfloor n/c \rfloor) - jl + m-1}{m-1} \right. \\
& \left. - \binom{t - \alpha(n+1 + \lfloor n/c \rfloor) - jl + m-1}{m-1} \right) \\
= & \sum_{j=1}^{c-1} \left(\binom{t - \alpha n - jl + m-1}{m-1} - \binom{t - \alpha(n+1) - lj + m-1}{m-1} \right) + \\
& \sum_{j=0}^{c-1} \binom{t - \alpha(n+1) - lj + m-1}{m-1} - \sum_{j=0}^{c-1} \binom{t - \alpha(n + \lfloor n/c \rfloor) - lj + m-1}{m-1} \\
& \sum_{j=0}^{n - \lfloor n/c \rfloor \cdot c} \left(\binom{t - \alpha(n + \lfloor n/c \rfloor) - jl + m-1}{m-1} \right. \\
& \left. - \binom{t - \alpha(n+1 + \lfloor n/c \rfloor) - jl + m-1}{m-1} \right) \\
= & \sum_{j=1}^{c-1} \binom{t - \alpha n - lj + m-1}{m-1} + \binom{t - \alpha(n+1) + m-1}{m-1} - \\
& \sum_{j=n - \lfloor n/c \rfloor c + 1}^{c-1} \binom{t - \alpha(n + \lfloor n/c \rfloor) - lj + m-1}{m-1} \\
& - \sum_{j=0}^{n - \lfloor n/c \rfloor c} \binom{t - \alpha(n+1 + \lfloor n/c \rfloor) - lj + m-1}{m-1} \\
= & \sum_{j=1}^{c-1} \binom{t - \alpha n - lj + m-1}{m-1} + \binom{t - \alpha(n+1) + m-1}{m-1} \\
& - \sum_{p=n+1}^{c \lfloor n/c \rfloor + c - 1} \binom{t - \alpha n - lp + m-1}{m-1} - \sum_{p=c \lfloor n/c \rfloor}^n \binom{t - \alpha(n+1) - lp + m-1}{m-1}
\end{aligned}$$

Letting $X_j = t - \alpha n - lj$ and $Y_j = t - (n+1)\alpha - lj$,

$$\begin{aligned}
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \sum_{j'=n-c+1}^{c\lceil n/c \rceil - 1} \binom{t - n\alpha - l(j' + c) + m-1}{m-1} \\
&\quad - \sum_{j=c\lceil n/c \rceil}^n \binom{Y_j + m-1}{m-1} + \binom{Y_0 + m-1}{m-1} \\
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \sum_{j=n-c+1}^{c\lceil n/c \rceil - 1} \binom{t - n\alpha - \alpha - lj + m-1}{m-1} - \\
&\quad \sum_{j=c\lceil n/c \rceil}^n \binom{Y_j + m-1}{m-1} + \binom{Y_0 + m-1}{m-1} \\
&= \sum_{j=1}^{c-1} \binom{X_j + m-1}{m-1} - \sum_{j=n-c+1}^n \binom{Y_j + m-1}{m-1} + \binom{Y_0 + m-1}{m-1}
\end{aligned}$$

Also note that $X_c = Y_0$ so that, when $l|\alpha$,

$$H_I^n(t) = \sum_{j=1}^c \binom{X_j + m-1}{m-1} - \sum_{j=n-c+1}^n \binom{Y_j + m-1}{m-1} + \binom{t - n\alpha + m-1}{m-1}$$

D.4 $\frac{3}{2}\alpha > \beta > \alpha$, $l \nmid \alpha$, $n \geq \lceil \frac{\alpha}{l} \rceil + 1$

As before, $c := \lceil \frac{\alpha}{l} \rceil$ and $d := \alpha \bmod l$.

Partial sums appearing in $H_J(t)$.

We now simplify the partial sums $\sum_v \binom{t - \lambda_v - v + m - 2}{m-2}$ where v ranges over parts of the Build, Reverse Build, and Pattern. We set $X_j = t - n\alpha - lj$, $Y_j = t - (n+1)\alpha - lj$, and $Z_j = t - (n+1)\alpha - lj + d$.

Partial sums from invariants in the Build and the Reverse Build.

These are the same as in the case $\frac{3}{2}\alpha > \beta > \alpha$, $l|\alpha$; see Section D.3 for details.

Partial sums from invariants in the Pattern

Recall that $E - B = l(1 - c)$.

❶ For $v = E + p\alpha + jc + i$ where $p = 0, \dots, n - c$, $j = 0, \dots, d - 1$, $i = 1, \dots, c - 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - B - pl - p\alpha - jc - j - i + E + p\alpha + jc + i \\
&= (n+1)\alpha + nl - 1 + l(1 - c) - pl - j
\end{aligned}$$

$$= (n+1)\alpha + l(n-c-p+1) - j - 1.$$

Then

$$\begin{aligned} & (c-1) \left[\sum_{p=0}^{n-c} \sum_{j=0}^{d-1} \binom{t - (n+1)\alpha - l(n-c-p+1) + j + 1 + m-2}{m-2} \right] \\ = & (c-1) \left[\sum_{p=0}^{n-c} \left(\binom{t - (n+1)\alpha - l(n-c-p+1) + d + m-1 + d}{m-1} \right. \right. \\ & \left. \left. - \binom{t - (n+1)\alpha - l(n-c-p+1) + m-1}{m-1} \right) \right] \\ = & (c-1) \left[\sum_{j=1}^{n-c+1} \left(\binom{Z_j + m-1}{m-1} - \binom{Y_j + m-1}{m-1} \right) \right]. \end{aligned}$$

② For $v = E + p\alpha + jc$ where $p = 0, \dots, n-c$, $j = 1, \dots, d$,

$$\begin{aligned} \lambda_v + v &= \lambda_0 - B - pl - p\alpha - jc - j + E + p\alpha + jc \\ &= (n+1)\alpha + nl - 1 + l(1-c) - pl - j \\ &= (n+1)\alpha + l(n-c-p+1) - j - 1. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{p=0}^{n-c} \sum_{j=1}^d \binom{t - (n+1)\alpha - l(n-c-p+1) + j + 1 + m-2}{m-2} \\ = & \sum_{p=0}^{n-c} \left[\binom{t - (n+1)\alpha - l(n-c-p+1) + d + 1 + m-1}{m-1} \right. \\ & \left. - \binom{t - (n+1)\alpha - l(n-c-p+1) + 1 + m-1}{m-1} \right] \\ = & \sum_{j=1}^{n-c+1} \left[\binom{Z_j + m}{m-1} - \binom{Y_j + m}{m-1} \right]. \end{aligned}$$

③ For $v = E + p\alpha + dc + j(c-1) + i$ where $p = 0, \dots, n-c-1$, $j = 0, \dots, l-d-1$, $i = 1, \dots, c-2$,

$$\lambda_v + v = \lambda_0 - B - pl - p\alpha - dc - d - jc - i + E + p\alpha + dc + jc - j + i$$

$$\begin{aligned}
&= (n+1)\alpha + nl - 1 + l(1-c) - pl - d - j \\
&= (n+1)\alpha + l(n-c-p+1) - j - d - 1.
\end{aligned}$$

Then

$$\begin{aligned}
&(c-2) \left[\sum_{p=0}^{n-c-1} \sum_{j=0}^{l-d-1} \binom{t - (n+1)\alpha - l(n-c-p+1) + j + d + 1 + m-2}{m-2} \right] \\
&= (c-2) \left[\sum_{p=0}^{n-c-1} \left(\binom{t - (n+1)\alpha - l(n-c-p+1) + d + l - d + m-1}{m-1} \right) \right. \\
&\quad \left. - \binom{t - (n+1)\alpha - l(n-c-p+1) + d + m-1}{m-1} \right] \\
&= (c-2) \sum_{j=1}^{n-c} \binom{Y_j + m-1}{m-1} - (c-2) \sum_{j'=2}^{n-c+1} \binom{Z_{j'} + m-1}{m-1}.
\end{aligned}$$

④ For $v = E + p\alpha + dc + j(c-1)$ where $p = 0, \dots, n-c-1, j = 1, \dots, l-d,$

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - B - pl - p\alpha - dc - d - jc + E + p\alpha + dc + jc - j \\
&= (n+1)\alpha + nl - 1 + l(1-c) - pl - d - j \\
&= (n+1)\alpha + l(n-c-p+1) - d - j - 1.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{p=0}^{n-c-1} \sum_{j=1}^{l-d} \binom{t - (n+1)\alpha - l(n-c-p+1) + j + d + 1 + m-2}{m-2} \\
&= \sum_{p=0}^{n-c-1} \left[\binom{t - (n+1)\alpha - l(n-c-p) + d + l - d + 1 + m-1}{m-1} \right. \\
&\quad \left. - \binom{t - (n+1)\alpha - l(n-c-p+1) + d + m}{m-1} \right] \\
&= \sum_{j=1}^{n-c} \binom{Y_j + m}{m-1} - \sum_{j'=2}^{n-c+1} \binom{Z_{j'} + m}{m-1}
\end{aligned}$$

Rewriting the Hilbert function.

The expression for the Hilbert function of I^n in terms of α and l found in Section

3.3.2 is

$$H_{I^n}(t) = \sum_{j=1}^n \left(\binom{t - \alpha n - jl + m - 1}{m - 1} - \binom{t - \alpha(n + 1) - lj + m - 1}{m - 1} \right) + \binom{t - n\alpha + m - 1}{m - 1}.$$

Using the fact that $lc = \alpha + l - d$ the summation in this expression is

$$\begin{aligned} & \sum_{j=1}^{c-1} \binom{t - n\alpha - lj + m - 1}{m - 1} + \sum_{j=c}^n \binom{t - n\alpha - lj + m - 1}{m - 1} \\ & - \sum_{j=1}^n \binom{t - (n + 1)\alpha - lj + m - 1}{m - 1} \\ = & \sum_{j=1}^{c-1} \binom{t - n\alpha - lj + m - 1}{m - 1} + \sum_{i=0}^{n-c} \binom{t - n\alpha - l(c + i) + m - 1}{m - 1} \\ & - \sum_{j=1}^n \binom{t - (n + 1)\alpha - lj + m - 1}{m - 1} \\ = & \sum_{j=1}^{c-1} \binom{t - n\alpha - lj + m - 1}{m - 1} + \sum_{i=0}^{n-c} \binom{t - n\alpha - \alpha - l + d - li + m - 1}{m - 1} \\ & - \sum_{j=1}^n \binom{t - (n + 1)\alpha - lj + m - 1}{m - 1} \\ = & \sum_{j=1}^{c-1} \binom{t - n\alpha - lj + m - 1}{m - 1} + \sum_{j'=1}^{n-c+1} \binom{t - (n + 1)\alpha - lj' + d + m - 1}{m - 1} \\ & - \sum_{j=1}^n \binom{t - (n + 1)\alpha - lj + m - 1}{m - 1} \end{aligned}$$

This leads to the formula for $H_{I^n}(t)$ found in Section 3.5.4.3.

D.5 $\frac{3}{2}\alpha > \beta > \alpha$, $2 \leq n < \frac{\alpha}{l} + 1$

Partial sums appearing in $H_J(t)$.

We now simplify the partial sums $\sum_v \binom{t - \lambda_v - v + m - 2}{m - 2}$ where v ranges over parts of the Build, Reverse Build, and Pattern. As above, we set $X_j = t - n\alpha - lj$, $Y_j = t - (n + 1)\alpha - lj$, and $Z_j = t - (n + 1)\alpha - lj + d$.

Partial sums from invariants in the Build.

❶ For $v = 0, \dots, l$,

$$\lambda_0 - 2v + v = \lambda_0 - v.$$

Then

$$\begin{aligned} & \sum_{j=0}^l \binom{t - (n+1)\alpha - nl + 1 + j + m - 2}{m-2} \\ &= \binom{t - (n+1)\alpha - (n-1)l + m - 1 + 1}{m-1} - \binom{t - (n+1)\alpha - nl + m - 1}{m-1} \\ &= \binom{Y_{n-1} + m}{m-1} - \binom{Y_n + m - 1}{m-1} \end{aligned}$$

❷ For $v = l(1 + \dots + q) + (q+1)j$ where $q = 1, \dots, n-2$ and $j = 1, \dots, l$,

$$\begin{aligned} \lambda_v + v &= \lambda_0 - (2 + \dots + (q+1))l - (q+1)j - j + l(1 + \dots + q) + (q+1)j \\ &= \lambda_0 - ql - j. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{q=1}^{n-2} \sum_{j=1}^l \binom{t - (n+1)\alpha - nl + 1 + ql + j + m - 2}{m-2} \\ &= \sum_{q=1}^{n-2} \left[\binom{t - (n+1)\alpha - (n-q)l + m - 1 + l + 1}{m-1} \right. \\ & \quad \left. - \binom{t - (n+1)\alpha - (n-q)l + m - 1 + 1}{m-1} \right] \\ &= \sum_{j=1}^{n-2} \binom{Y_j + m}{m-1} - \sum_{j'=2}^{n-1} \binom{Y_{j'} + m}{m-1} \\ &= \binom{Y_1 + m}{m-1} - \binom{Y_{n-1} + m}{m-1}. \end{aligned}$$

❸ For $v = l(1 + \dots + q) + (q+1)j - x$ where $q = 1, \dots, n-2$, $j = 1, \dots, l$, and $x = 1, \dots, q$,

$$\lambda_v + v = \lambda_0 - (2 + \dots + (q+1))l - (q+1)j + x - j + 1 +$$

$$\begin{aligned}
& l(1 + \cdots + q) + (q + 1)j - x \\
&= \lambda_0 - (q + 1)l_i - j + 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{q=1}^{n-2} q \left[\sum_{j=1}^l \binom{t - (n+1)\alpha - nl + ql + j + m - 2}{m-2} \right] \\
&= \sum_{q=1}^{n-2} q \left[\binom{t - (n+1)\alpha - l(n-q-1) + m - 1}{m-1} \right. \\
&\quad \left. - \binom{t - (n+1)\alpha - l(n-q) + m - 1}{m-1} \right] \\
&= \sum_{j=1}^{n-2} (n-j-1) \binom{Y_j + m - 1}{m-1} - \sum_{j'=2}^n (n-j') \binom{Y_{j'} + m - 1}{m-1} \\
&= \sum_{j=1}^{n-2} (n-j-1) \binom{Y_j + m - 1}{m-1} - \sum_{j=2}^{n-1} (n-j-1) \binom{Y_j + m - 1}{m-1} \\
&\quad - \sum_{j=2}^{n-1} \binom{Y_j + m - 1}{m-1} \\
&= (n-2) \binom{Y_1 + m - 1}{m-1} - \sum_{j=2}^{n-1} \binom{Y_j + m - 1}{m-1}.
\end{aligned}$$

Partial sums from invariants in the Reverse Build.

❶ For $v = (k-1) - j$ where $j = 0, \dots, l$,

$$\lambda_v + v = (k-1) - j + \lambda_{k-1} + 2j = l + k + j.$$

Then

$$\begin{aligned}
\sum_{j=0}^l \binom{t - n\alpha - l - j + m - 2}{m-2} &= \left[\binom{t - n\alpha - l + m - 2 + 1}{m-1} \right. \\
&\quad \left. - \binom{t - n\alpha - l + m - 2 - l}{m-1} \right] \\
&= \binom{X_1 + m - 1}{m-1} - \binom{X_2 + m - 2}{m-1}.
\end{aligned}$$

② For $v = (k-1) - (l(1+\dots+q) + (q+1)j)$ where $q = 1, \dots, n-2$ and $j = 0, \dots, l$

$$\begin{aligned}
\lambda_v + v &= (k-1) - l(1+\dots+q) - j(q+1) + \lambda_{k-1} \\
&\quad + (2+\dots+(q+1))l + (q+1)j + j \\
&= n\alpha - 1 - l + l + 1 + (q+1)l + j \\
&= n\alpha + (q+1)l + j.
\end{aligned}$$

Note that λ_v such that $q = n-2$ and $j = l$ is not in the Reverse Build.

$$\begin{aligned}
&\sum_{q=1}^{n-2} \sum_{j=1}^l \binom{t - n\alpha - (q+1)l - j + m - 2}{m-2} - \binom{t - n\alpha - (n-1)l - l + m - 2}{m-2} \\
&= \sum_{q=1}^{n-2} \left[\binom{t - n\alpha - (q+1)l + m - 2}{m-1} - \binom{t - n\alpha - (q+1)l + m - 2 - l}{m-1} \right] \\
&\quad - \binom{X_n + m - 2}{m-2} \\
&= \sum_{j=2}^{n-1} \binom{X_j + m - 2}{m-1} - \sum_{j'=3}^n \binom{X_{j'} + m - 2}{m-1} \\
&\quad - \binom{X_n + m - 2}{m-2} \\
&= \binom{X_2 + m - 2}{m-2} - \binom{X_n + m - 2}{m-1} - \binom{X_n + m - 2}{m-2}
\end{aligned}$$

③ For $v = (k-1) - (l(1+\dots+q) + (q+1)j - x)$ where $q = 1, \dots, n-2$, $j = 1, \dots, l$, and $x = 1, \dots, q$,

$$\begin{aligned}
\lambda_v + v &= n\alpha - l(1+\dots+q) - (q+1)j + x + \lambda_{k-1} \\
&\quad + (2+\dots+(q+1))l + (q+1)j - x + j - 1 \\
&= n\alpha + ql + l + 1 + j - 1 \\
&= n\alpha + l(q+1) + j - 1.
\end{aligned}$$

Note that λ_v such that $q = n-2$, $j = l$, and $x = 1$ is not in the Reverse Build.

$$\begin{aligned}
& \sum_{q=1}^{n-2} q \left[\sum_{j=1}^l \binom{t - n\alpha - l(q+1) - j + 1 + m - 2}{m-2} \right] \\
& - \binom{t - n\alpha - l(n-1) - l + 1 + m - 2}{m-2} \\
= & \sum_{q=1}^{n-2} q \left[\binom{t - n\alpha - l(q+1) + m - 2}{m-1} - \binom{t - n\alpha - l(q+1) + m - 1 - l}{m-1} \right] \\
& - \binom{X_n + m - 1}{m-2} \\
= & \sum_{j=2}^{n-1} (j-1) \binom{X_j + m - 1}{m-1} - \sum_{j'=3}^n (j'-2) \binom{X_{j'} + m - 1}{m-1} \\
& - \binom{X_n + m - 1}{m-2} \\
= & \sum_{j=2}^{n-1} \binom{X_j + m - 1}{m-1} + \sum_{j=2}^{n-1} (j-2) \binom{X_j + m - 1}{m-1} - \sum_{j=3}^n (j-2) \binom{X_j + m - 1}{m-1} \\
& - \binom{X_n + m - 1}{m-2} \\
= & \sum_{j=2}^{n-1} \binom{X_j + m - 1}{m-1} - (n-2) \binom{X_n + m - 1}{m-1} - \binom{X_n + m - 1}{m-2}
\end{aligned}$$

Partial sums from invariants in the Pattern.

Note that $E - B = l - ln$.

❶ For $v = E = jn + i$ where $j = 0, \dots, \beta - nl - 1$ and $i = 1, \dots, n - 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - B - jn - j - i + E + jn + i \\
&= \lambda_0 + l - ln - j.
\end{aligned}$$

Then

$$\begin{aligned}
& (n-1) \sum_{j=0}^{\beta-nl-1} \binom{t - (n+1)\alpha - nl + 1 - l + ln + j + m - 2}{m-2} \\
= & (n-1) \sum_{j=0}^{\beta-nl-1} \binom{t - (n+1)\alpha - l + j + m - 1}{m-2}
\end{aligned}$$

$$\begin{aligned}
&= (n-1) \left[\binom{t - (n+1)\alpha - l + m - 1 + \beta - nl}{m-1} - \binom{t - (n+1)\alpha - l + m - 1}{m-1} \right] \\
&= (n-1) \left[\binom{t - n\alpha - nl + m - 1}{m-1} - \binom{Y_1 + m - 1}{m-1} \right] \\
&= (n-1) \left[\binom{X_n + m - 1}{m-1} - \binom{Y_1 + m - 1}{m-1} \right].
\end{aligned}$$

② For $v = E + jn$ where $j = 1, \dots, \beta - nl - 1$,

$$\begin{aligned}
\lambda_v + v &= \lambda_0 - B - jn - j + e + jn \\
&= (n+1)\alpha + nl - 1 + l - ln - j \\
&= (n+1)\alpha + l - 1.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{j=1}^{\beta - nl - 1} \binom{t - (n+1)\alpha - l + 1 + j + m - 2}{m-2} \\
&= \left[\binom{t - (n+1)\alpha - l + m - 1 + \beta - nl}{m-1} - \binom{t - (n+1)\alpha - l + m - 1 + 1}{m-1} \right] \\
&= \binom{X_n + m - 1}{m-1} - \binom{Y_1 + m}{m-1}.
\end{aligned}$$

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