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Correcting detection errors in quantum state engineering through data processing

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Abstract. Verifying correlations within a quantum state is an important task in quantum state engineering, while realistic detectors with imperfect efficiency usually introduce errors into the detection results. With these detection errors, genuine multi-partite entanglement can be invisible. In this paper, we propose a simple method for correcting arbitrary uncorrelated detection errors. The method is based on the processing of data from repetitive experiments and can correct detection errors of any magnitude as long as the error magnitude is calibrated. The method is illustrated with its application in the detection of multi-partite entanglement from quantum state engineering.

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1. Introduction

The prospect of quantum computation and quantum communication [1] strongly stimulates interest in the engineering of various non-classical states with multi-partite entanglement as a resource for quantum information. Due to the volatile and elusive nature of entanglement, verifying it alone is not a trivial problem. Many approaches to this problem have been proposed (for a review see [5]). Those detection schemes, e.g. entanglement witnesses, generally involve an inequality of certain observables, the violation of which indicates entanglement. However, there is an issue with these schemes unconsidered previously, associated with the imperfect efficiency of the detectors used in real experiments, which causes detection errors and potentially false estimation of the state.

In this paper, we propose a simple method for correcting the detection error caused by the detectors, which is in practice a significant obstacle to the observation of multi-partite entanglement in quantum state engineering [2]. We show here that this type of error can be corrected at any magnitude as long as the error magnitude has been calibrated (for instance, through prior test experiments). The detection error distorts the experimental data by a transformation that depends on the magnitudes of various error possibilities. When the relevant error magnitudes have been calibrated for the detectors by the prior test experiments, the form of the distortion transformation induced by the detection error is known, and then we can find a way of inverting this transformation to reconstruct the original signal. In this way, we can use imperfect detectors to simulate perfect detectors as long as their imperfection has been calibrated. The proposed method is straightforward for experimental implementation as it is based on data processing and does not increase the complexity of the setup. To correct the detection error, we only need to repeat the same experiments by some additional rounds to have small statistical error for the inverse transformation. To illustrate its applications, the method is used to significantly improve the detection of multi-qubit entanglement and spin squeezing. In many cases, the signal of multi-partite entanglement only becomes visible after the proposed correction of the detection error, in particular when the number of qubits is large.

2. Mathematical formulation

Any measurement in quantum information can be reduced to a population measurement in a certain basis (including possibly several complementary bases). If we want to measure properties associated with a state ρ (generally mixed) of n qubits, for each chosen measurement basis there are 2^n possible measurement outcomes. By carrying out measurements we determine the probability f_i associated with each outcome i ($i = 1, 2, \ldots, 2^n$). For instance, if we repeat the same experiment N times and get the ith outcome N_i times, we estimate the probability f_i by $f_i = N_i/N$ and its standard deviation (the error bar) by $\Delta f_i = \sqrt{f_i(1-f_i)/N}$ using the binomial distribution. If the detectors are perfect, the measured probabilities f_i just give the distribution $g_i \equiv \langle i|\,\rho\,|i\rangle$ of the state ρ in the measurement basis $\{|i\rangle\}$. In reality, however, the detectors always have errors, which distort the distribution g_i, potentially making the measured distribution f_i significantly different from g_i. The purpose of this paper is to show how to reconstruct the real distribution g_i from the measured distorted probabilities f_i.

2.1. The individual qubit addressing case

We first consider the case where the measurements have individual addressing, and each qubit is measured by an independent detector. For detection on a qubit, the most general error model is characterized by a 2×2 matrix

$$D = \begin{bmatrix} 1-p_0 & p_1 \\ p_0 & 1-p_1 \end{bmatrix}, \tag{1}$$

where p_0 (p_1) denotes respectively the error probability that the detector gives outcome 1 (0) for the input signal of 0 (1). For simplicity of notation, we assume that the error matrix D has the same form for the detection of each qubit (it is straightforward to generalize the formalism to the case when the error rates p_0 and p_1 in the D matrix are qubit-dependent). Furthermore, we assume that p_0 and p_1 have been well calibrated by a prior test experiment. For instance, we may input a known state to the detector and can calibrate p_0 and p_1 easily from the measurement data.

For n qubits, the error model for the detection is then characterized by a $2^n \times 2^n$ matrix $M = [M_{ji}]$, with the element M_{ji} corresponding to the probability of recording the outcome j with the input signal i. Assume that the detection error rates on different qubits are independent of each other and the binary string i has n_0 zeros and $n_1 = n - n_0$ ones. If we need α flips from 0 to 1 and β flips from 1 to 0 to change the string from i to j, the matrix element M_{ji} is given by

$$M_{ji} = (1-p_0)^{n_0-\alpha}\,(1-p_0)^{n_1-\beta}\,p_0^{\alpha}\,p_1^{\beta}. \tag{2}$$

The measured probabilities f_j are connected with the real distribution g_i through the distortion transformation $f_j = \sum_{i=1}^{2^n} M_{ji} g_i$. To reconstruct the real signal g_i from the measured distribution f_j, in principle we only need to invert the matrix M. However, as M is a huge $2^n \times 2^n$ matrix, it is not clear how to invert this matrix (it is even a question whether the inverse exists).

Our key observation is that the matrix M, with the elements given by equation (2), has a simple tensor product structure. It is straightforward to show by mathematical induction that

$$M = \bigotimes_{k=1}^{n} D_k, \tag{3}$$

where all the D_k are identical and given by D in equation (1). Therefore, the inverse can be easily done in an analytic form with

$$M^{-1} = \bigotimes_{k=1}^{n} D_k^{-1} = \bigotimes_{k=1}^{n} \begin{bmatrix} 1-p_0' & p_1' \\ p_0' & 1-p_1' \end{bmatrix}_i, \tag{4}$$

where the parameters p_0' and p_1' are given by

$$\begin{aligned} p_0' &= p_0/(p_0+p_1-1), \\ p_1' &= p_1/(p_0+p_1-1). \end{aligned} \tag{5}$$

Note that with the substitution into equation (5), M^{-1} and M have the same form except that p_0' and p_1' can no longer be interpreted as error rates since in general they are not in the range $[0, 1]$. The formula also shows that the inverse transformation M^{-1} always exists except for the special case with $p_0 + p_1 = 1$.

2.2. The collective measurement case

In some experimental systems, we do not have the ability to resolve individual qubits. Instead, we make collective measurements on n qubits by detecting how many qubits (denoted by j, $j = 0, 1, \ldots, n$) are in the state $|1\rangle$ in a chosen detection basis (this is equivalent to the measurement of the collective spin operator along a certain direction). In this case, the detection only has $n+1$ outcomes for an n-qubit system. For collective measurements on n qubits, the detection error matrix is represented by an $(n+1) \times (n+1)$ matrix $L = [L_{ij}]$. The matrix element L_{ij} corresponds to the probability of registering outcome i when j qubits are in the $|1\rangle$ state. If the detection error matrix for an individual qubit is still given by D in equation (1), we can directly calculate L_{ij} from D: from signal j to i if n_{10} qubits flip from 0 to 1 and n_{01} qubits flip from 1 to 0, with the constraints $0 \leqslant n_{01} \leqslant j$, $0 \leqslant n_{10} \leqslant n-j$ and $n_{01} - n_{10} = j - i$; L_{ij} is given by

$$\begin{aligned} L_{ij} &= \sum_{\substack{0\leqslant n_{01}\leqslant j,\\ 0\leqslant n_{10}\leqslant n-j,\\ kn_{01}-n_{10}=j-i}} B(j, p_1, n_{01}) B(n-j, p_0, n_{10}) \\ &= \sum_{q=\max\{0,i+j-n\}}^{\min\{i,j\}} B(j, 1-p_1, q) B(n-j, p_0, i-q), \end{aligned} \tag{6}$$

where $B(n, p, k) \equiv \binom{n}{k} p^k (1-p)^{n-k}$ and we have let $q = i - n_{10}$, and hence q satisfies the constraint $\max\{0, i+j-n\} \leqslant q \leqslant \min\{i, j\}$. As the dimension of the L matrix depends linearly on the qubit number n, it is typically not difficult to numerically calculate its inverse matrix L^{-1} if n is not very large. As will be shown in the appendix, there is also a simple analytic formula for $L^{-1} = [L_{ij}^{-1}]$: if we denote the dependence of L_{ij} in equation (6) on p_0, p_1 as $L_{ij} = L_{ij}(p_0, p_1)$, we have

$$L_{ij}^{-1} = L_{ij}\left(p_0', p_1'\right), \tag{7}$$

where p_0', p_1' are given by the simple substitution into equation (5). With the inverse matrix L^{-1}, the real signal g_i can be similarly reconstructed from the measured data f_j as $g_i = \sum_j L_{ij}^{-1} f_j$.

The above formulation can be extended straightforwardly to qudit (d-dimensional) systems where the individual detection error matrix D in equation (1) is replaced by a $d \times d$ matrix.

For the independent detection of n-qudits, the overall error matrix M still has the tensor-product structure as shown by equation (3), which allows for an easy calculation of M^{-1} from D^{-1}.

2.3. Cost of error correction

With the inverse error matrix M^{-1}, it is straightforward to reconstruct the true distribution g_i from the measured data f_i. The price we need to pay is that compared with $\Delta f_i = \sqrt{f_i(1-f_i)/N}$, there is an increase of the standard deviation (error bar) Δg_i in our estimate of g_i by the formula $g_i = \sum_{j=1}^{2^n} M_{ij}^{-1} f_j$. With some tedious but straightforward calculation, we find that

$$\Delta g_i = \sqrt{\left[\sum_j (M_{ij}^{-1})^2 f_j - g_i^2\right] \Big/ N}. \tag{8}$$

As $M^{-1} = \bigotimes_{k=1}^{n} D_k^{-1}$ and D_k^{-1} has matrix element $1 - p_0' \approx \mathrm{e}^{p} > 1$ (when $p_0 \sim p_1 \sim p \ll 1$), M^{-1} has matrix element $\sim \mathrm{e}^{np}$, which leads to an exponential increase of the error bar Δg_i with the qubit number n. To maintain the same error bar Δg_i, the number of repetitions N of the experiment eventually needs to increase exponentially with n. For practical applications, this exponential increase of N by the factor e^{np} is typically not a problem for two reasons. Firstly, as the detection error rate p is usually at a few per cent level, the exponential factor e^{np} remains moderate even for hundreds of qubits. Secondly, this exponential increase only applies when we need to measure each element of the distribution g_i. In most of the quantum information applications, we only need to measure certain operators which are expressed as tensor products of a constant number of Pauli operators for different qubits. In this case, N does not have the exponential increase, as we show now.

Suppose we need to measure an operator $\hat{O}$, which is expressed as $\hat{O} = \otimes_{k=1}^{n} \sigma_k^{\mu_k}$, where $\sigma_k^{\mu_k}$ is a component of the Pauli matrices when $\mu_k = 1, 2, 3$ or the identity operator when $\mu_k = 0$. The number of the Pauli matrices n_p in the tensor product expansion of $\hat{O}$ is called the support of $\hat{O}$. To measure the operator $\hat{O}$, we choose the measurement basis to be the eigenbasis of $\sigma_k^{\mu_k}$ for the kth qubit. In this measurement basis, $\hat{O}$ is diagonal with the matrix element $\hat{O} = \otimes_{k=1}^{n} \mathrm{diag}\left(\sigma_k^{\mu_k}\right)$, where $\mathrm{diag}\left(\sigma_k^{\mu_k}\right) = [1, 1]$ for $\mu_k = 0$ and $\mathrm{diag}\left(\sigma_k^{\mu_k}\right) = [1, -1]$ for $\mu_k = 1, 2, 3$. Under the distribution g_i, the expectation value of $\hat{O}$ is given by $\langle \hat{O} \rangle = \sum_i \hat{O}_i g_i = \sum_i \hat{O}_i \sum_j M_{ij}^{-1} f_j = \sum_j \left(\sum_i \hat{O}_i M_{ij}^{-1}\right) f_j \equiv \sum_j \hat{O}_j^{\mathrm{c}} f_j$, where $\hat{O}_i$ denotes the diagonal matrix element of $\hat{O}$. Therefore, by defining a corrected operator $\hat{O}^{\mathrm{c}}$, we can get the true expectation value $\langle \hat{O} \rangle$ directly from the experimental data f_j. Using the relation $M^{-1} = \bigotimes_{k=1}^{n} D_k^{-1}$, $\hat{O}^{\mathrm{c}}$ is expressed as $\hat{O}^{\mathrm{c}} = \bigotimes_{k=1}^{n} \left[\mathrm{diag}(\sigma_k^{\mu_k}) D_k^{-1}\right]$. For $\mu_k = 1, 2, 3$,

$$\begin{aligned} \mathrm{diag}(\sigma_k^{\mu_k}) D_k^{-1} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 - p_0' & p_1' \\ p_0' & 1 - p_1' \end{bmatrix} \\ &= \begin{bmatrix} (1 - 2p_0') & -(1 - 2p_1') \end{bmatrix}, \end{aligned} \tag{9}$$

and for $\mu_k = 0$, $\mathrm{diag}(\sigma_k^{\mu_k}) D_k^{-1} = [1, 1]$. For simplicity of notation, we take $p_0 = p_1 = p$. In this case, $\mathrm{diag}\ (\sigma_k^{\mu_k}) D_k^{-1} = (1 - 2p)^{-1} \mathrm{diag}(\sigma_k^{\mu_k})$ for $\mu_k = 1, 2, 3$, and the corrected operator $\hat{O}^{\mathrm{c}}$ is related with the original operator $\hat{O}$ by a simple scaling transformation $\hat{O}^{\mathrm{c}} = (1 - 2p)^{-n_p} \hat{O}$.

The scaling transformation is independent of the qubit number n, so the error bar of $\langle \hat{O} \rangle$ does not have an exponential increase with n when the operator $\hat{O}$ has a constant support n_p.

The scaling transformation also applies to collective operators, but some caution needs to be taken for the calculation of their variance. For instance, if we take the collective spin operator $J_z \equiv \sum_{k=1}^{n} \sigma_k^z/2$, it is easy to see that $J_z^c = (1-2p)^{-1} J_z$, as each of the terms of J_z has support $n_p = 1$. However, as $J_z^2 \equiv n/4 + \sum_{k \neq l} \sigma_k^z \sigma_l^z/4$ which has non-uniform support for its superposition terms, one finds that $\left(J_z^2\right)^c = n/4 + (1-2p)^{-2}\left(J_z^2 - n/4\right) = (1-2p)^{-2}\left[J_z^2 - np\,(1-p)\right]$. With this transformation, we can correct the distortion to the spin squeezing parameter by the detection error. Assume that the mean value of $\langle \mathbf{J} \rangle$ is along the x-direction with $\langle \mathbf{J} \rangle = \langle J_x \rangle$ and the squeezing is along the z-direction. The squeezing parameter is given by $\xi = \sqrt{n\left\langle J_z^2 \right\rangle / \langle J_x \rangle^2}$ [3]. Using the transformation for $\left(J_z^2\right)^c$ and J_x^c, we find that

$$\xi^c = \sqrt{n\left\langle \left(J_z^2\right)^c \right\rangle / \left\langle J_x^c \right\rangle^2} = \sqrt{\xi^2 - \xi_d^2}, \tag{10}$$

where $\xi_d^2 = n^2 p(1-p)(1-2p)^{-2} \langle J_x \rangle^{-2}$ is the contribution to ξ^2 by the detection noise. After the correction of the detection error, ξ^c decreases significantly compared with ξ, in particular when the qubit number n is large, and thus can be used to verify a much bigger entanglement depth using the criterion in [4]. From equation (10), we find that the variation $\Delta\xi^c/\Delta\xi = \xi/\xi^c$. As typically $\xi \gg \xi^c$, the error bar for ξ^c after correction of the detection error significantly increases, and we need to correspondingly increase the rounds of the experiment N to reduce the statistical error.

3. Application example

To illustrate the application of the error correction method here, as an example, we apply it to the detection of genuine multi-partite entanglement in graph states. For a graph state $|G_n\rangle$ of n qubits associated with a q-colorable graph G, the genuine n-party entanglement can be detected with the following witness operator [5]:

$$W_{G_n} = 3\mathbb{I} - 2\left[\sum_{l=1}^{q}\left(\prod_{k \in Q_l}(S_k + \mathbb{I})/2\right)\right], \tag{11}$$

where Q_l denotes the set of qubits with the lth color ($l = 1, 2, \ldots, q$), $\mathbb{I}$ is the identity operator and S_k is the stabilizer operator for the kth qubit (which is a tensor product of the Pauli operators σ_k^x for the kth qubit and $\sigma_{k'}^z$ for all it neighbors k' in the graph G). A state ρ has genuine n-qubit entanglement if $\mathrm{tr}\left(\rho W_{G_n}\right) = \left\langle W_{G_n} \right\rangle < 0$. For an ideal graph state, all its stabilizer operators S_k have expectation values $\langle S_k \rangle = 1$. With detection error, the value of $\langle S_k \rangle$ gets significantly degraded. As an example, figure 1 shows the values of all $\langle S_k \rangle$ for two particular two-colorable graph states: a ten-qubit GHZ state (GHZ_{10}) and a linear cluster state (LC_{10}). We assume 3% detection error with $p_0 = p_1 = p = 0.03$ for each qubit. With a known magnitude p, the detection error can be easily corrected by a scaling transformation $S_k^c = (1-2p)^{-n_{pk}} S_k$, where n_{pk} is the support of the corresponding stabilizer operator S_k. Figure 1 shows that after error correction, $\left\langle S_k^c \right\rangle$ is almost unity. Its error bar increases slightly after the correction, but is still small. To show the influence on the entanglement detection, we assume that the experimentally prepared graph state ρ_{ex} corresponds to the ideal target state ρ_{id} distorted by small depolarization noise independently acting on each qubit, so $\rho_{ex} = \widehat{\$}(\rho_{id})$, where the noise super-operator

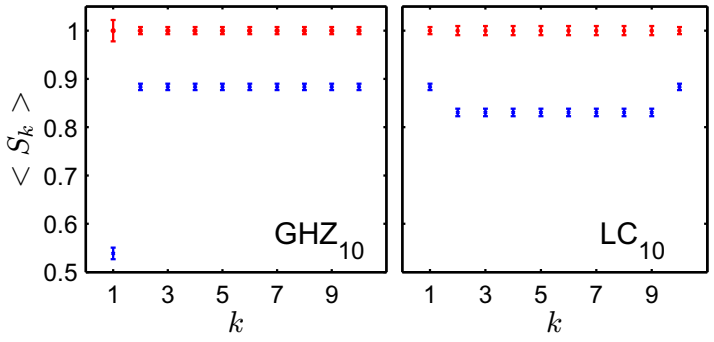

Figure 1. Values of stabilizers before (lower points) and after (upper points) correction of the detection error (with the error rate $p = 0.03$) for the ten-qubit GHZ state (GHZ_{10}) and the linear cluster state (LC_{10}). Error bars account for the statistical error by assuming $N = 5000$ independent measurements in each detection setting.

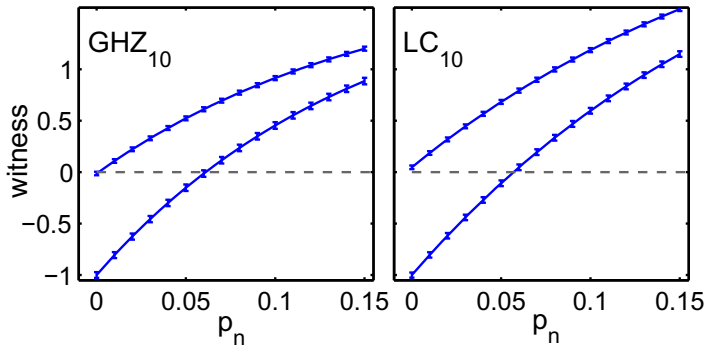

Figure 2. The entanglement witness $\langle W_{G_n} \rangle$ under different state preparation errors p_n for GHZ (GHZ_{10}) and cluster (LC_{10}) states before (upper lines) and after (lower lines) correction of the detection error (error rate $p = 0.03$). The error bars are obtained by assuming $N = 5000$ rounds of measurements in each detection setting.

$\widehat{\$} = \bigotimes_{k=1}^{n} \widehat{\$}_k$ and $\widehat{\$}_k(\rho_{\mathrm{id}}) = (1 - 3p_n/4)\rho_{\mathrm{id}} + p_n/4 \sum_{\mu=x,y,z} \sigma_k^{(\mu)} \rho_{\mathrm{id}} \sigma_k^{(\mu)}$ [6]. In figure 2, we show the witness $\langle W_{G_n} \rangle$ as a function of the preparation error rate p_n, both before and after correction of the detection error (with an error rate $p = 3\%$). For both GHZ_{10} and LC_{10} states, without correction of the detection error, we cannot detect any n-qubit entanglement even for a perfectly prepared state with $p_n = 0$. After correction of the detection error, we can confirm genuine n-qubit entanglement as long as the preparation error $p_n \lesssim 5\%$. So, correction of the detection error significantly improves the experimental performance, and the improvement gets more dramatic when the qubit number increases.

4. Sensitivity on detector calibration

We briefly comment on the sensitivity of our error correction method to calibration of the detection error. In this method, the error magnitude p (or magnitudes p_i, $i = 0, 1, \ldots,$ for

the general cases) is assumed to be known. If we have a relative error e in the calibration of the magnitude p, i.e. $\delta p/p \sim e$, the scaling transformation on the detected operator $\hat{O}$ leads to a relative error in the observed quantity $\delta\langle\hat{O}\rangle/\langle\hat{O}\rangle \sim 2n_p\delta p\,(1-2p)^{-1} \sim 2n_p pe$. As long as $2n_p p \lesssim 1$, which is typically the case as $p \ll 1$, the relative error is actually reduced and the method here can tolerate some uncertainty in the calibration of the error magnitude p.

5. Conclusion

In summary, we have shown a method for correcting any detection error through simple processing of the experimental data. The method applies to measurements in general many-particle settings, with or without separate addressing. Moreover, the method does not require change of the experimental setup and works under arbitrary magnitudes of the detection noise, as long as the error magnitude has been calibrated. The cost of this method is moderate as it only requires repetition of the same experiment by some additional rounds to gain enough statistics and thus the method can readily apply to many experimental settings used in quantum state engineering.

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Appendix. Proof of equation (7)

We can relate the L matrix to the M matrix defined in equation (4). Denote the space of n-bit binary strings with i bits of 1 as S_i, and S_i has dimension $\binom{n}{i}$. The matrix element $M_{\sigma\rho}$ represents the probability of recording an n-bit binary string ρ as σ, and L_{ij} is the probability of recording a signal $\rho \in S_j$ as any string in the S_i space. As a collective measurement does not distinguish the binary strings in the same space S_i, L_{ij} is related to $M_{\sigma\rho}$ by $L_{ij} = \sum_{\sigma\in S_i} M_{\sigma\rho}$. The probability L_{ij} is apparently independent of the exact form of ρ as long as ρ belongs to the space S_j, so we can pick up any $\rho \in S_j$ in $L_{ij} = \sum_{\sigma\in S_i} M_{\sigma\rho}$ without alteration of the result of summation. From equation (4), we know $M^{-1}_{\mu\nu} = M_{\mu\nu}\left(p'_0, p'_1\right)$. Let us define

$$N_{jk} \equiv \sum_{\mu\in S_j} M^{-1}_{\mu\nu} = \sum_{\mu\in S_j} M_{\mu\nu}\left(p'_0, p'_1\right) = L_{jk}\left(p'_0, p'_1\right), \tag{A.1}$$

where ν is an arbitrary element in S_k. Now we show that N gives the inverse of the matrix L:

$$\begin{aligned}\sum_j L_{ij}N_{jk} &= \sum_j \sum_{\sigma\in S_i,\mu\in S_j} M_{\sigma\rho}M^{-1}_{\mu\nu} \\ &= \sum_{\sigma\in S_i}\sum_j\sum_{\mu\in S_j} M_{\sigma\mu}M^{-1}_{\mu\nu} = \sum_{\sigma\in S_i}\delta_{\sigma\nu} = \delta_{ik}.\end{aligned}$$

In the second line, we have changed the subscript ρ in $M_{\sigma\rho}$ to μ as both ρ, μ belong to S_j. This proves equation (7) in the text. □

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