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Isothermal, mass-limited rarefactions in planar and spherical geometry

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Solutions are developed for isothermal, mass-limited rarefactions in which the temperature is held constant by continued heating. The case of spatially constant temperature and fixed total energy is also revisited. These solutions are useful in the context of experiments in which plasmas are continuously heated for long periods (up to tens of ns) by present-day lasers or hohlraums.

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The expansion of plasmas is of fundamental importance in a wide range of circumstances, for example, including laser-heated systems¹ and supernova explosions.² This lends value to simple models of such expansions, from the very basic, self-similar, adiabatic and isothermal rarefactions of semi-infinite slabs often discussed in books^{3,4} to other simple models discussed in the literature. In particular, see Qi and Krishnan⁵ and references therein. Much of this work^{6,7} was motivated by the exploding laser targets used for x-ray lasers and for laser-plasma-interaction studies,⁸ and there has been some⁹ but relatively little work since 1990 on single-fluid expansions. There is more recent work on two-fluid expansions,^{10,11} in which some electrons eventually pull ahead of the ions, motivated by particle acceleration from surfaces driven by high-irradiance lasers. Here, however, we are concerned with the behavior away from the ion front and thus with single-fluid models.

An aspect of the prior work is the absence of models for the expansion of a constant-temperature plasma of finite mass, save for the approximate result given by London and Rosen.⁶ Yet this case is of some relevance, both for a period of time in exploding-foil targets and for the expansion of plasma of finite mass whose temperature is sustained in some way, such as by a hohlraum or by a radiative shock. Electron heat conduction is quite rapid in densities below solid density, so that constant-temperature models are often sensible approximations. The present brief communication is intended to address this finite mass, constant-temperature regime. We refer below to the paper⁶ by London and Rosen as LR, to the paper⁵ by Qi and Krishnan as QK, and to the paper⁷ by Hunter and London as HL.

Before beginning a calculation, it is worthwhile to recall some of the relevant physical context. The single fluid treatment is justified as follows. In the expanding plasma, the electron fluid motion is continuously in steady state relative to that of the ion fluid, establishing an electric field E of magnitude given by $en_e E = -\nabla p_e$, where the electron charge, number density, and pressure are e , n_e , and p_e , respectively. This electric field in turn accelerates the ions with the effect that the net source in the equation for the momentum density is $-\nabla p$, with p being the total pressure. The

importance of the cases with spatially constant temperature considered here is that these are realistic because for typical conditions electron heat conduction is fast. One can find the timescale for electron heat conduction from

$$\rho C_v \frac{\partial T_e}{\partial t} \Big|_{hc} = -\nabla \cdot \kappa \nabla T_e, \quad (1)$$

where the mass density, electron temperature, specific heat at constant volume, and heat conduction coefficient are ρ , T_e , C_v , and κ , respectively. For the fully ionized plasmas typically of interest here $C_v = (1+Z)k_B/(Am_p)$, where the average charge, average atomic number, proton mass, and Boltzmann constant are Z , A , m_p , and k_B , respectively, and

$$\kappa = \frac{128}{3\pi} \frac{n_e k_B T_e}{m_e \nu_{ei}} k_B, \quad (2)$$

in which the electron mass is m_e and the electron-ion collision rate is $\nu_{ei} = Z\omega_{pe}^4 \ln \Lambda / (6\sqrt{2}\pi^{3/2} n_e (k_B T_e / m_e)^{3/2})$, which in s^{-1} is $3 \times 10^{-6} \ln \Lambda n_e Z T_e^{-3/2}$ for n_e in cm^{-3} and T_e in eV. (Note $k_B = 1.6 \times 10^{-12}$ ergs/eV, ω_{pe} is the electron plasma frequency, and $\ln \Lambda$ is the Coulomb logarithm.) From this one can find the ratio of the temperature equilibration timescale, t_{equil} , to the expansion timescale, t_{exp} . The equilibration timescale is defined in typical fashion from Eq. (1) as

$$t_{equil} = \frac{\rho C_v L^2}{\kappa}, \quad (3)$$

while t_{exp} , the experiment duration, is the period of time during which the expansion has continued. This typically would be the time interval in an experiment from the initial deposition of energy to the time of observation. Here L is the spatial scale of the expanding system at any t_{exp} , for which a basic estimate would be t_{exp} times the sound speed. The ratio t_{equil}/t_{exp} , shown in Fig. 1, is proportional to t_{exp} , rather than to $1/t_{exp}$, because of the factor L^2 in t_{equil} . Typical laser blowoff plasmas are below $10^{21} cm^{-3}$ in electron density. One can see that heat conduction will be rapid for most cases of interest.

We commence the derivation with the usual equations for the conservation of mass and momentum, being

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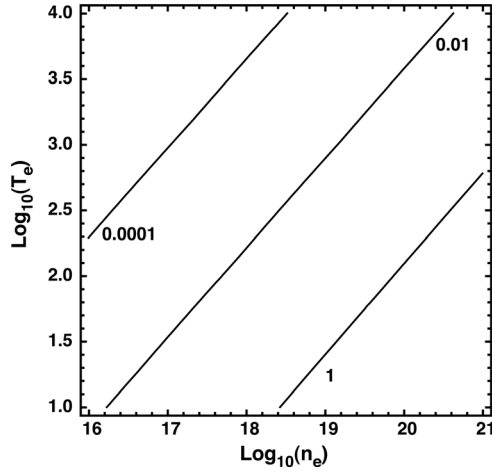


FIG. 1. This is the ratio of the electron temperature equilibration time to the experiment duration t_{exp} , evaluated for $t_{exp} = 10$ ns and for a fully ionized Be plasma. When this quantity is small, the plasma will remain isothermal.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{and} \quad (4)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (5)$$

respectively, where ρ is mass density, \mathbf{u} is velocity, and p is pressure. We seek here symmetric planar or spherical expansions, so \mathbf{u} is radial and these equations become

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left(\frac{\partial u}{\partial r} + \frac{su}{r} \right) \text{ and} \quad (6)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (7)$$

where s is zero for planar expansions and 2 for spherical ones.

For the assumed free, homogeneous expansion $u \propto r$ and we use the ultimately convenient form that

$$u = \frac{1}{L} \frac{dL}{dt} r, \quad (8)$$

where the scale length L is a function of t only. We also use an ideal-gas equation of state with gas constant R and temperature T and assume R and T to be constant in space as is justified above. Then Eq. (7) becomes

$$\frac{rL''}{L} + \frac{RT}{\rho} \frac{\partial \rho}{\partial r} = 0, \quad (9)$$

where the prime indicates a derivative in time. This equation has spatial solution

$$\rho = \hat{\rho} \exp \left[-\frac{r^2 L''}{2RTL} \right], \quad (10)$$

where $\hat{\rho}$ is the time-dependent density at $r=0$. This maximum density is related to other quantities by conservation of total mass. For the planar case with areal mass density σ one finds

$$\hat{\rho}_{pl} = \sigma \sqrt{\frac{2}{\pi}} \sqrt{\frac{L''}{LRT}}, \quad (11)$$

while for the spherical case with total mass M one finds

$$\hat{\rho}_{sph} = \frac{M}{(2\pi)^{3/2}} \left(\frac{L''}{LRT} \right)^{3/2}. \quad (12)$$

We find the implications of the mass equation by substituting Eqs. (8) and (11) or (12) and into Eq. (6) with the appropriate value for s . In both cases, one finds that the solution is $RT = LL''$. As a result the profiles become

$$\hat{\rho}_{pl} = \sqrt{\frac{2}{\pi}} \frac{\sigma}{L} \exp \left[-\frac{r^2}{2L^2} \right] \quad \text{and} \quad (13)$$

$$\hat{\rho}_{sph} = \frac{M}{(2\pi)^{3/2}} \frac{1}{L^3} \exp \left[-\frac{r^2}{2L^2} \right]. \quad (14)$$

These density profiles are well known but the unmet challenge is posed by the seemingly simple equation $RT = LL''$. We consider the case when RT is held constant for all time by some heat source. This turns out to have a solution, which can be evaluated numerically. The solution to

$$\frac{\partial}{\partial t} \left[L \frac{\partial^2 L}{\partial t^2} \right] = 0 \quad (15)$$

can be written

$$L = \frac{L_o}{1.88} \exp \left[-\left(\text{Erf}^{-1} \left[i \frac{2}{\sqrt{\pi}} \frac{t}{t_o} \right] \right)^2 \right], \quad (16)$$

where as we will see the initial state has $L = L_o$ at $t = t_o$ and the origin of the factor 1.88 is described below. To convert this to a more tractable form, one must further discuss the error function and its inverse. One has

$$iz = \text{Erf}^{-1}[\text{Erf}(iz)] = \text{Erf}^{-1} \left[i \frac{2}{\sqrt{\pi}} e^{z^2} D(z) \right], \quad (17)$$

where

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy, \quad \text{so} \quad (18)$$

$$\text{Erf}(iz) = i \frac{2}{\sqrt{\pi}} \int_0^z e^{y^2} dy = i \frac{2}{\sqrt{\pi}} e^{z^2} D(z), \quad (19)$$

where $D(z)$ is the Dawson function,

$$D(z) = e^{-z^2} \int_0^z e^{y^2} dy. \quad (20)$$

With this definition of z we have

$$L = \frac{L_o e^{z^2}}{1.88}, \quad (21)$$

where z is a solution of

$$\int_0^z e^{t^2} dt = \frac{t}{t_o}. \quad (22)$$

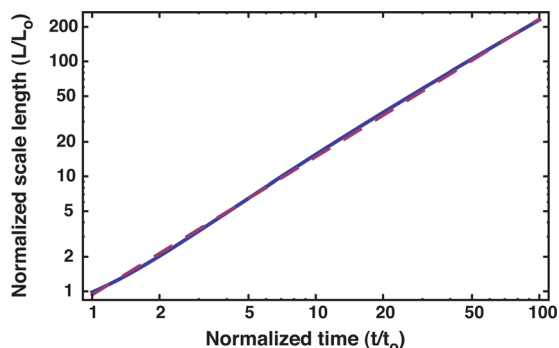


FIG. 2. (Color online) The ratio L/L_o from Eq. (21) is shown against t/t_o , along with the fit $0.95(t/t_o)^{1.2}$.

The factor 1.88 is, to three-digit accuracy, the solution of Eq. (22) for $t = t_o$, assuring that $L = L_o$ for $t = t_o$. Figure 2 compares the solution for L/L_o from Eqs. (21) and (22) with the fit

$$L/L_o = 0.95(t/t_o)^{1.2}, \quad (23)$$

which is accurate to better than 7% over the range $1 \leq t/t_o \leq 100$. Thus, the time dependent profiles are given by Eqs. (13) and (14) with L from Eq. (23) or a numerical evaluation of Eq. (22), and one needs to know a value of L at some initial time. One can also note that LR report a solution of Eq. (15), valid in the (very late time) limit that $\ln[t/t_o] \gg 1$, which can be written as

$$L/L_o = \sqrt{2}(t/t_o) \sqrt{\ln[\sqrt{2}(t/t_o)]}. \quad (24)$$

To provide concrete examples, one may consider two possible experiments using large plasmas at the National Ignition Facility¹² (NIF). The goal of such experiments might be to produce collisionless and/or magnetized flows relevant to collisionless shocks or magnetization of blast waves, and indeed some experiments similar to these have been selected for NIF. In the design of such experiments, one is likely to focus on ion density rather than mass density, in order to evaluate relevant quantities such as ion skin depth or collision lengths, so Fig. 3 shows ion density. Using 1 MJ of laser energy at a wavelength of $0.35 \mu\text{m}$, NIF can irradiate a 5 mm diameter target with an irradiance of 10^{14} W/cm^2 for 50 ns, or a larger area for a shorter time. For the case of a C target this will sustain an electron temperature near 2 keV, so that the sound speed is $330 \mu\text{m/ns}$. The mass ablation rate will be about $2.6 \mu\text{m/ns}$. In experiments seeking to drive collisionless shocks, for example, one might use a $5\text{-}\mu\text{m}$ -thick target having $\sigma = 0.0012 \text{ g/cm}^2$ with $L_o = 660 \mu\text{m}$ at $t_o = 2 \text{ ns}$, and observe

the system at 10 ns when it is still planar. Fig. 3(a) shows this case. In experiments seeking to produce spherical blast waves one might use a $26\text{-}\mu\text{m}$ -thick target of 5 mm diameter, having $M = 0.0012 \text{ g}$, with $L_o = 6.6 \text{ mm}$ at $t_o = 20 \text{ ns}$, and observe the spherically expanding system at 50 ns. Fig. 3(b) shows this case. In both cases shown, the right hand edge of the figure corresponds to a velocity of about 1000 km/s .

The other relevant case is one in which heat deposition stops at some time but heat conduction remains strong enough to keep T constant in space. This case has been treated in various limits by LR, QK, and HL. (One may note that all three previous papers incorrectly state that the expansion is adiabatic after the heating stops, which is not the case because there is heat conduction. QK also do not mention that their expansion is mass limited, although this is the case in their solutions.) HL, in their Eqs. (20)–(24), include solutions for the time dependence of the scale length and for the relation of temperature and scale length. These solutions require that one know the scale length L_o , the temperature T_o , and $\partial L/\partial t$ at time t_o , along with the total mass M .

For the aid of readers who may at some point seek to connect these papers, and to consider whether their assumptions are sufficiently general for some given application, some discussion of the relevant equation of state properties may be useful. One writes the pressure as $p = \rho RT$ and the specific internal energy as $\epsilon = C_v T$, where T varies in time but not space, and we take the gas constant R to in fact be constant (a simplification). Identifying this as an “isothermal” case, one might be tempted to take the polytropic index γ to be 1. If this were the case, then the results in the previous papers would be incorrect. However, in this case, there is not only one γ . It is the “acoustic γ ” whose value is one.⁴ The internal energy per particle remains $(n/2)k_B T$, where n is the number of degrees of freedom, which is 3 for constant R . The corresponding “shock γ ” is $\gamma_s = 1 + 2/n = 5/3$, and the specific heat is $C_v = \partial(3RT/2)/\partial T|_v = 3R/2$. These results with the above profiles enable us to reduce the equation for conservation of energy,

$$\frac{\partial}{\partial t} \left(\rho \epsilon + \frac{\rho u^2}{2} \right) + \nabla \cdot (\rho u (\epsilon + u^2/2) + pu) = 0, \quad (25)$$

to

$$TL' + (3/2)LT' = 0, \quad (26)$$

for the planar case, in agreement with Eq. (7) of LR (Ref. 6) for no heating and to

$$TL' + (1/2)LT' = 0, \quad (27)$$

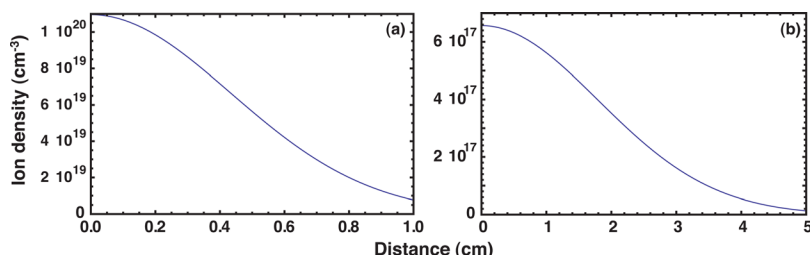


FIG. 3. (Color online) The ion density is shown for two isothermal cases corresponding to potential NIF experiments described in the text. (a) planar and (b) spherical.

for the spherical case. These are consistent with the general result shown in HL.

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