Equidistribution of Preimages in Nonarchimedean Dynamics

by

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Preface

Ergodic theoretic methods play a central role in studying the dynamics of holomorphic endomorphisms $f\colon X \to X$ of complex projective varieties. At the heart of these methods are equidistribution results, which allow one to construct dynamically interesting f-invariant probability measures on X. The most important of these is the *equidistribution of preimages theorem*, stated below in what is currently its most general form [51].

Theorem. *Let X be an irreducible complex projective variety, and let $f\colon X \to X$ be a polarized dynamical system of algebraic degree $d \geq 2$. Then there is an f-invariant probability measure μ_f on X and a proper Zariski closed subset $\mathscr{E}_f \subsetneq X$ such that the iterated preimages $f^{-n}(x)$ of any point $x \in X \smallsetminus \mathscr{E}_f$ equidistribute to μ_f as $n \to \infty$.*

The terminology used in the statement of the theorem will be precisely defined and explained in the main body of this dissertation, but for now suffice it to say that the theorem constructs a canonical invariant probability measure μ_f for a very interesting class of holomorphic dynamical systems, including all dynamically interesting endomorphisms of projective space $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$. This theorem opens the door to doing meaningful ergodic theory for these systems. An in depth background discussion of the theorem, including its history and attributions, will be given Chapter 1.

Given its importance to complex dynamics, it is natural to wonder if some version of the equidistribution of preimages theorem holds for endomorphisms $f\colon X \to X$ of projective varieties over fields other than the complex numbers. The main question we will address in this thesis is whether an analogue holds over *nonarchimedean fields.*

Main Question. *If K is a complete, algebraically closed nonarchimedean field and $f\colon X \to X$ is a polarized endomorphism of a projective variety over K, do the iterated preimages of most points $x \in X$ equidistribute to some probability measure μ_f on X?*

As stated, we will see that this question is not well-posed, due to the fact that the topological structure of nonarchimedean fields is not conducive to measure theory and analytic geometry. In order to make the question well-posed, one must replace X with its *Berkovich analytification* X^{an}. The space X^{an} is a natural compactification of X with good topological and geometric properties—good enough, in fact, to have a robust measure theory and analytic geometry. We will give a fairly detailed background discussion of nonarchimedean fields and Berkovich analytifications in Chapter 4. In the language of Berkovich analytifications, we can rephrase our question into a well-posed conjecture.

Conjecture. *Let K be a complete, algebraically closed nonarchimedean field, and suppose $f\colon X \to X$ is a polarized dynamical system of algebraic degree $d \geq 2$. Let $\mathscr{E}_f \subset X^{\mathrm{an}}$ be the union of all analytifications $E^{\mathrm{an}} \subset X^{\mathrm{an}}$ of proper closed subvarieties $E \subsetneq X$ satisfying $f^{-1}(E) = E$. Then there is an invariant probability measure μ_f on X such that the iterated preimages $f^{-n}(x)$ of all points $x \in X^{\mathrm{an}} \smallsetminus \mathscr{E}_f$ equidistribute to μ_f.*

It should be noted that in general $\mathscr{E}_f$ will not itself be the analytification of a proper closed subvariety of X, but nonetheless $X^{\mathrm{an}} \smallsetminus \mathscr{E}_f$ is always a large subset of X^{an}, containing at least all points $x \in X^{\mathrm{an}}$ corresponding to admissible norms on the function field of X.

One can actually make this conjecture more precise in a couple of ways. First, there is a natural candidate for the measure μ_f: to any such dynamical system f, Chambert-Loir has constructed an f-invariant probability measure [34], which should agree with μ_f. Second, one should be able to say precisely what happens to the preimages of points $x \in \mathscr{E}_f$. If $x \in \mathscr{E}_f$, then there exists some minimal proper closed set $F \subsetneq X$ such that $f^{-m}(F) = F$ for some $m \geq 1$ and $x \in F^{\mathrm{an}}$. In this case, the iterated f^m-preimages of x should equidistribute to the Chambert-Loir measure associated to the dynamical system $f^m\colon Y^{\mathrm{an}} \to Y^{\mathrm{an}}$.

The conjecture is known to hold when f is an endomorphism $\mathbf{P}^1_K \to \mathbf{P}^1_K$ by work of Favre and Rivera-Letelier [67], but until the work of this thesis, nothing was known about the validity of the conjecture in the higher dimensional setting. Here we will prove the conjecture holds when f is a sufficiently generic map of *good reduction*. More precisely, our main result is the following, stated, for simplicity, in the most interesting special case when f is an endomorphism of $\mathbf{P}^r_K$.

Theorem A. *Let K be a complete, algebraically closed nonarchimedean field, with residue field k. Let $f\colon \mathbf{P}^r_K \to \mathbf{P}^r_K$ be an endomorphism of algebraic degree $d \geq 2$ that has good reduction, and let $\tilde{f}\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ be the reduction of f. Suppose that $\mathrm{char}(k) \nmid d$. Then:*

1. *There is a (unique) maximal proper Zariski closed subset $\mathscr{E} \subsetneq \mathbf{P}^r_k$ such that $\tilde{f}^{-1}(\mathscr{E}) = \mathscr{E}$.*

2. *For every point $x \in \mathbf{P}^{r,\mathrm{an}}_K$ whose reduction does not lie in $\mathscr{E}$, the iterated preimages of x equidistribute to the Dirac probability measure supported at the Gauss point of $\mathbf{P}^{r,\mathrm{an}}_K$.*

In particular, if $\mathscr{E} = \varnothing$, then the conjecture holds for f.

For generically chosen f, the set $\mathscr{E}$ will in fact be empty, so Theorem A gives the full conjecture for a typical map of good reduction. Moreover, in the special case when K is a *trivially valued field*, every map f has good reduction, so in this case Theorem A gives the full conjecture in any dimension for almost every f.

The strategy for proving Theorem A is simple to describe. By definition, f having good reduction means that there is a *reduction map* $\mathsf{red}\colon \mathbf{P}_K^{r,\mathrm{an}} \to \mathbf{P}_k^r$ which makes the following diagram commute:

$$\begin{tikzcd} \mathbf{P}_K^{r,\mathrm{an}} \arrow[r, "f"] \arrow[d, "\mathsf{red}"'] & \mathbf{P}_K^{r,\mathrm{an}} \arrow[d, "\mathsf{red}"] \\ \mathbf{P}_k^r \arrow[r, "\tilde{f}"] & \mathbf{P}_k^r \end{tikzcd}$$

This allows us to *approximate* the dynamics of f on $\mathbf{P}_K^{r,\mathrm{an}}$ by the dynamics of $\tilde{f}$ on $\mathbf{P}_k^r$. Theorem A will then be proved in two broad steps. First, we will prove a version of Theorem A for the reduced map $\tilde{f}$, namely Theorem B below. Second, we lift Theorem B via the reduction map red to deduce Theorem A.

Theorem B. *Let k be any algebraically closed field, and let $\tilde{f}\colon \mathbf{P}_k^r \to \mathbf{P}_k^r$ be an endomorphism of algebraic degree $d \geq 2$. Suppose* $\mathrm{char}(k) \nmid d$. *Then:*

1. *There is a (unique) maximal proper Zariski closed subset $\mathscr{E} \subsetneq \mathbf{P}_k^r$ such that $\tilde{f}^{-1}(\mathscr{E}) = \mathscr{E}$.*

2. *If $x \in \mathbf{P}_k^r$ is any scheme theoretic point not lying in $\mathscr{E}$, then the iterated preimages of x equidistribute to the Dirac probability measure at the generic point of $\mathbf{P}_k^r$.*

On the face of it, statement 2 of Theorem B is ambiguous at best. This is because the space $\mathbf{P}_k^r$ has only an algebraic (scheme theoretic) structure, not very conducive to statements about measures and equidistribution. It is possible to make sense of it, however, by developing a good theory of measures on (Noetherian) schemes, see Appendix A. Granting this, the proof of Theorem B will still take a considerable amount of work, done in Chapters 2 and 3. Lifting Theorem B to Theorem A is a little easier, and will be done in Chapter 5, where we will also prove a strengthening of Theorem A in the case when K is a trivially valued field.

The main body of work detailed in this dissertation has been published in the author's paper [73], except for Appendix A, which summarizes the paper [74]. While they will not be used or discussed here, the author's papers [76, 75] are also very related to the topic at hand.

Contents

List of Figures

Chapter 1

Equidistribution in complex dynamics

The results and methods of this dissertation, as is the case generally in nonarchimedean dynamics, have their origins and motivations in the field of complex dynamics. This chapter is dedicated to providing a brief survey of the complex dynamical theory that underpins this work, specifically the theory pertaining to *equidistribution of preimages*. Much has been written on this and related topics, particularly in the setting of complex dynamics in several variables, where the subject is an important and active area of research. The reader is especially encouraged to peruse the surveys [101, 80], which treat equidistribution problems in great detail. Our focus will be more narrow and tailored to aspects of the theory that will be relevant in later chapters.

Our discussion starts in dimension one, where we will motivate, state, and sketch a proof of the complex equidistribution of preimages theorem. We only sketch a proof, since most of the details of the argument will not be used in later chapters. On the other hand, during the course of our sketch we will derive an *equidistribution condition*, a generalization of which is central to the proof of the main results in this work. Starting in §1.3, we move on to the higher dimensional setting, where we will settle for surveying the main results, with no attempt to prove any of them.

1.1. Equidistribution of preimages in dimension one

We begin our discussion of equidistribution of preimages in (complex) dimension one, where the theory is at its cleanest and the results are readily visualizable. Specifically, we consider the dynamical systems given by endomorphisms $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ of the complex projective line. These systems have been the subject of a century of research starting with the pioneering work of Fatou and Julia, leading to a very well developed elementary theory [31, 94], as well as a number of very active current research directions. We will start by recalling some of the basic aspects of the elementary theory that are relevant for the present topic.

After identifying the complex projective line $\mathbf{P}^1_{\mathbf{C}}$ with the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, recall that any nonconstant endomorphism $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ can be written as a rational function $f = P/Q \in \mathbf{C}(z)$. The *degree* d of f is by definition the maximum of the degrees of the polynomials P and Q, where it is assumed that P and Q have no common factor. The degree d can also be characterized topologically as the degree of f as a ramified covering map from $\mathbf{P}^1_{\mathbf{C}} \cong \mathbf{S}^2$ to itself; thus every point $w \in \mathbf{P}^1_{\mathbf{C}}$ has d preimages under f when counted according to multiplicity. As covering degrees are multiplicative, w has d^n preimages under f^n for all $n \geq 1$, when counted according to multiplicity. We will always assume our dynamical systems have degree $d \geq 2$, as this is the dynamically interesting case.

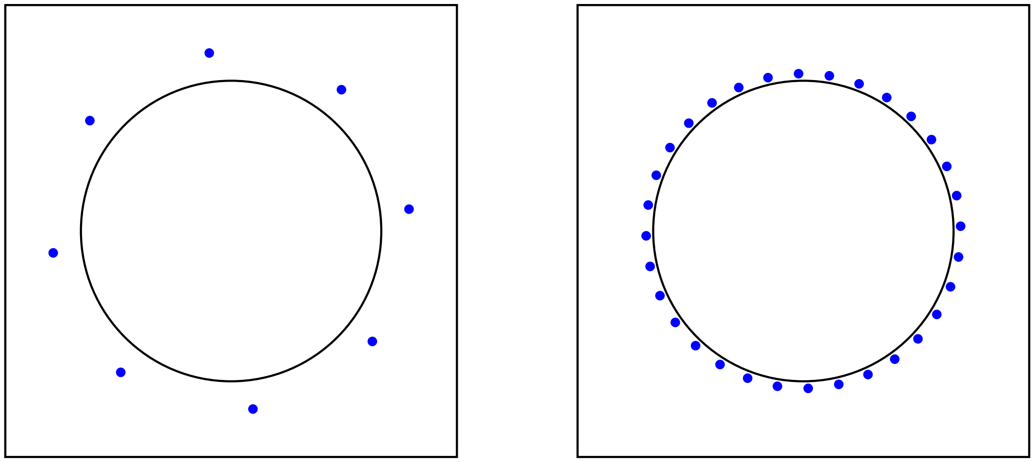
Given a dynamical system $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ of degree $d \geq 2$, the *Fatou set* $\mathcal{F}_f$ of f is defined to be the set of points $z \in \mathbf{P}^1_{\mathbf{C}}$ which have a neighborhood U on which the family of iterates $\{f^n|_U\colon U \to \mathbf{P}^1_{\mathbf{C}}\}_{n\geq 1}$ is equicontinuous. Rephrased dynamically, a point z lies in the Fatou set if the orbit of every point z' close to z remains close to the orbit of z for all time. Thus the Fatou set is the largest open set on which the dynamics of f is *non-chaotic*. The complement of the Fatou set, called the *Julia set* $\mathcal{J}_f$, is therefore the set of points at which the dynamics of f is chaotic. The Fatou and Julia sets are both totally invariant, that is, $f^{-1}(\mathcal{F}_f) = \mathcal{F}_f$ and $f^{-1}(\mathcal{J}_f) = \mathcal{J}_f$. While the Fatou set of f may be empty, the Julia set of f never is. When studying the Fatou and Julia sets of f, essential use is made of Montel's theorem, which says the following.

Montel's Theorem. *Let $U \subseteq \mathbf{P}^1_{\mathbf{C}}$ be an open set, and $\{f_\alpha\}$ a family of holomorphic functions $f_\alpha\colon U \to \mathbf{P}^1_{\mathbf{C}}$. If there exist three distinct points of $\mathbf{P}^1_{\mathbf{C}}$ that do not lie in the image of any f_α, then $\{f_\alpha\}$ is an equicontinuous family on U.*

Montel's Theorem, combined with definition of the Julia set of f, immediately implies the next corollary.

Corollary 1.1.1. *Suppose $z \in \mathcal{J}_f$. Then for any neighborhood U of z, the union $\bigcup_{n\geq 0} f^n(U)$ contains all but at most two points of $\mathbf{P}^1_{\mathbf{C}}$.*

With just a little more work, Corollary 1.1.1 can be strengthened to say the following: there is a set $\mathscr{E}_f \subseteq \mathbf{P}^1_{\mathbf{C}}$ consisting of at most two points, depending only on f and not on the point $z \in \mathcal{J}_f$, such that $\bigcup_{n\geq 0} f^n(U) \supseteq \mathbf{P}^1_{\mathbf{C}} \setminus \mathscr{E}_f$ for all neighborhoods U of z. This set $\mathscr{E}_f$ is called the *exceptional set* of f, and is characterized by being the largest finite subset of $\mathbf{P}^1_{\mathbf{C}}$ which is totally invariant for f. For generically chosen dynamical systems f of degree $d \geq 2$, the exceptional set $\mathscr{E}_f$ will be empty. This strengthening of Corollary 1.1.1 has the following interesting corollary.



(a) $f^{-3}(w)$ (b) $f^{-5}(w)$

Figure 1.1: Iterated preimages of a point $w \in \mathbf{C}$ for the polynomial $f(z) = z^2$.

Corollary 1.1.2. *Suppose $w \in \mathbf{P}^1_{\mathbf{C}}$ is a point that does not lie in the exceptional set $\mathscr{E}_f$. Then the set of iterated preimages $\bigcup_{n\geq 0} f^{-n}(w)$ of w clusters at every point of $\mathcal{J}_f$.*

The equidistribution of preimages theorem, which we will state soon, is in essence a quantitative strengthening of Corollary 1.1.2. Intuitively, it says that the iterated preimages of every non-exceptional point w cluster along the Julia set in a *uniform way*, and, remarkably, in a way that is independent of the choice of starting point w.

Before stating the equidistribution of preimages theorem, let us illustrate it in the simplest possible case, namely for the dynamical system given by the polynomial map $f(z) = z^2$. For this map, the Julia set $\mathcal{J}_f$ is the unit circle $\mathbf{S}^1 \subseteq \mathbf{C}$, and the exceptional set $\mathscr{E}_f$ consists of the two points $0, \infty \in \hat{\mathbf{C}}$. For any point $w \in \hat{\mathbf{C}}$, the nth preimages $f^{-n}(w)$ are exactly the 2^n-th roots of w. Assuming that $w \notin \mathscr{E}_f$, there are exactly 2^n such roots, and they are spaced out evenly along the circle of radius $|w|^{1/2^n}$ centered at the origin, as depicted in Figure 1.1. We then see that, as we let $n \to \infty$, the preimages $f^{-n}(w)$ converge to the Julia set $\mathcal{J}_f = \mathbf{S}^1$ in a very regular manner. We make this regularity precise in the following way. For each $n \geq 1$, let μ_n be the probability measure on $\mathbf{P}^1_{\mathbf{C}}$ that gives each point of $f^{-n}(w)$ equal weight 2^{-n}. Then the measures μ_n converge weakly to the Lebesgue probability measure Λ on the unit circle. We say that Λ gives the asymptotic distribution of the preimages of w, or that *the preimages of w equidistribute to* Λ. Note that w was an arbitrary non-exceptional point, so this equidistribution holds for all such w. Moreover, the limiting distribution Λ is *independent of w*. This brings us to the statement of the equidistribution of preimages theorem, which says this behavior is completely general.

Equidistribution of Preimages Theorem 1. *Let $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ be an endomorphism of degree $d \geq 2$. Then there is a probability measure μ_f on $\mathbf{P}^1_{\mathbf{C}}$ such that the preimages of all non-exceptional points $w \in \mathbf{P}^1_{\mathbf{C}}$ equidistribute to μ_f. Moreover, the support of the measure μ_f is exactly the Julia set of f.*

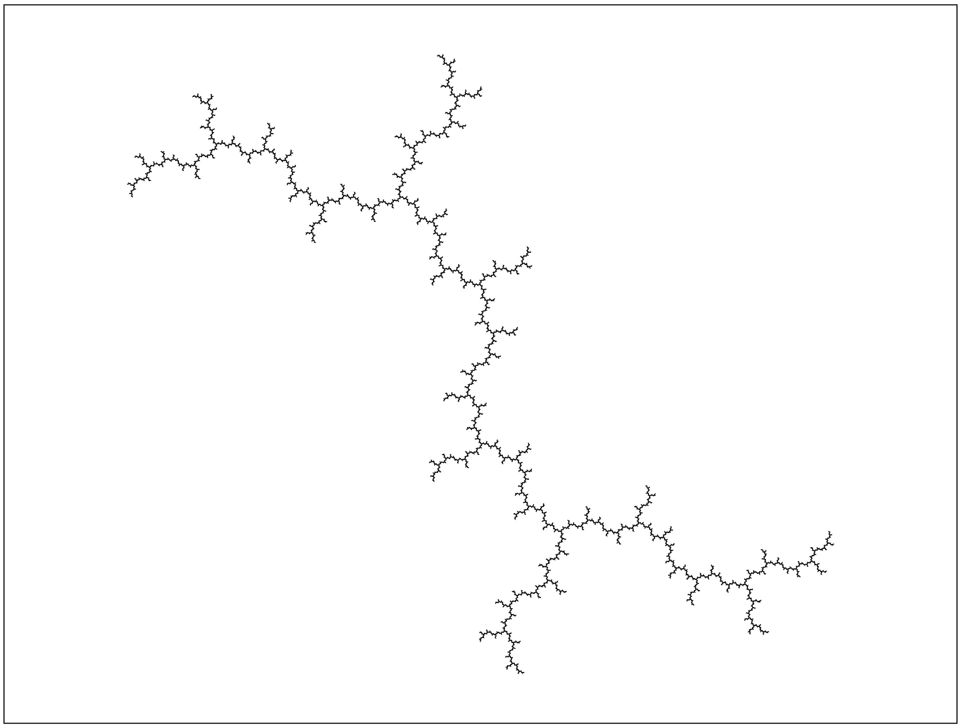
To be clear, this theorem is saying that for any non-exceptional point w, the sequence of probability measures μ_n that weights each element of $f^{-n}(w)$ according to their multiplicity as an f^n-preimage of w converges weakly to μ_f as $n \to \infty$.

Definition 1.1.3. The measure μ_f is sometimes referred to as the Green measure, the equilibrium measure, or the measure of maximal entropy for f. We will simply call it the *canonical measure* of f, since this will be more suitable when we are in the nonarchimedean setting.

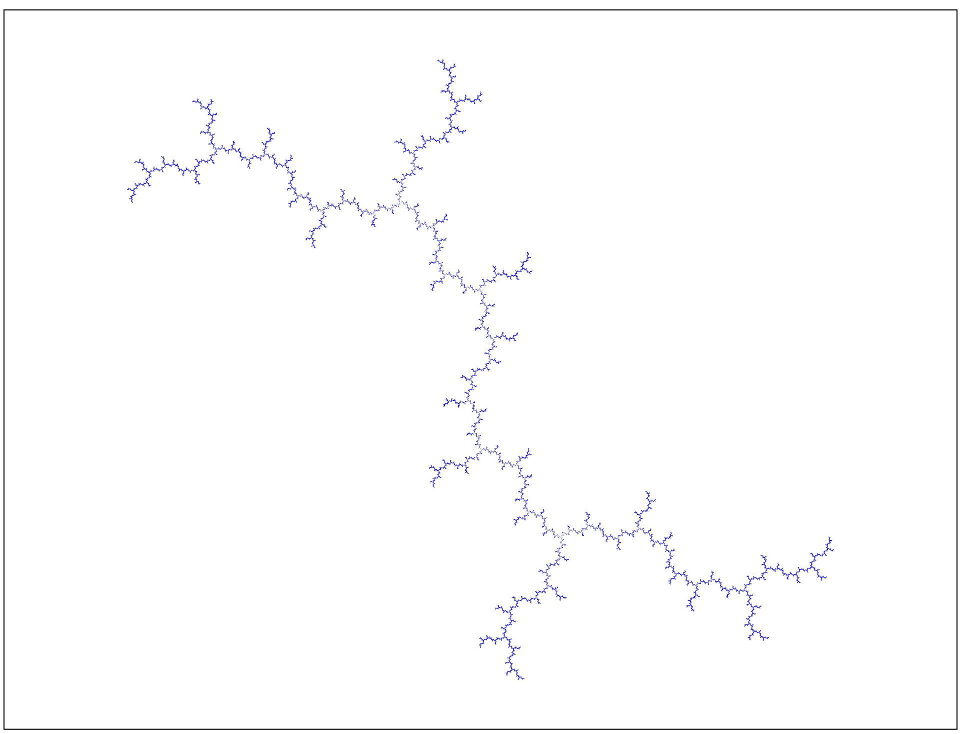
On the next several pages, we have included four figures in the spirit of Figure 1.1 meant to illustrate the equidistribution of preimages theorem for more general quadratic polynomials than $f(z) = z^2$. In each figure we plot (a) the Julia set of a quadratic polynomial $f(z)$ in the complex plane and (b) the 20th preimages $f^{-20}(w)$ of the arbitrarily chosen starting point $w = 1 + i$. As in Figure 1.1, in each figure the Julia set is colored black, while the preimage points are colored blue. In Figures 1.2 and 1.3 we see that the preimages $f^{-20}(1+i)$ appear to give a very good approximation of the entire Julia set $\mathcal{J}_f$, while in Figures 1.4 and 1.5 there are parts of $\mathcal{J}_f$ which do not appear in the picture of the preimages $f^{-20}(1+i)$. This suggests that the mass of the canonical measures μ_f for the maps in Figures 1.2 and 1.3 will be fairly evenly spread out along all parts of the Julia sets $\mathcal{J}_f$, while the mass of the canonical measures μ_f for the maps in Figures 1.4 and 1.5 should be more concentrated along the "outermost" parts of $\mathcal{J}_f$.

1.2. A potential theoretic approach to equidistribution

In the literature, there are currently two different broad approaches to proving the equidistribution of preimages theorem; one approach uses techniques from potential theory, and the other makes use of more geometric arguments. Historically, the potential theoretic approach came first. Hans Brolin, who first observed the equidistribution of preimages phenomenon in his Ph.D. thesis [30, Ch. III], proved the theorem for polynomials $f\colon \mathbf{C} \to \mathbf{C}$ using potential theory. It was not clear how to generalize his methods to arbitrary endomorphisms f, leading to the development twenty years later of the geometric approach by Lyubich [92] and independently by Freire-Lopes-Mañé [70]. The potential theoretic approach of Brolin was extended to arbitrary f ten years later by Fornæss-Sibony [69] and Hubbard-Papadopol [84].

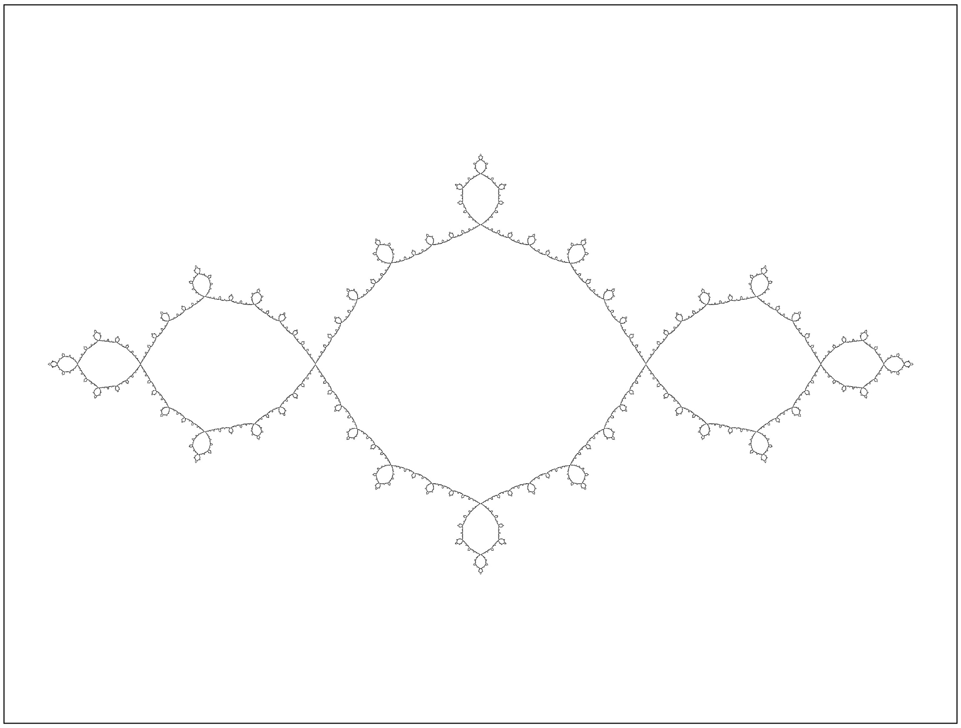


(a) The Julia set of $f(z) = z^2 + i$.

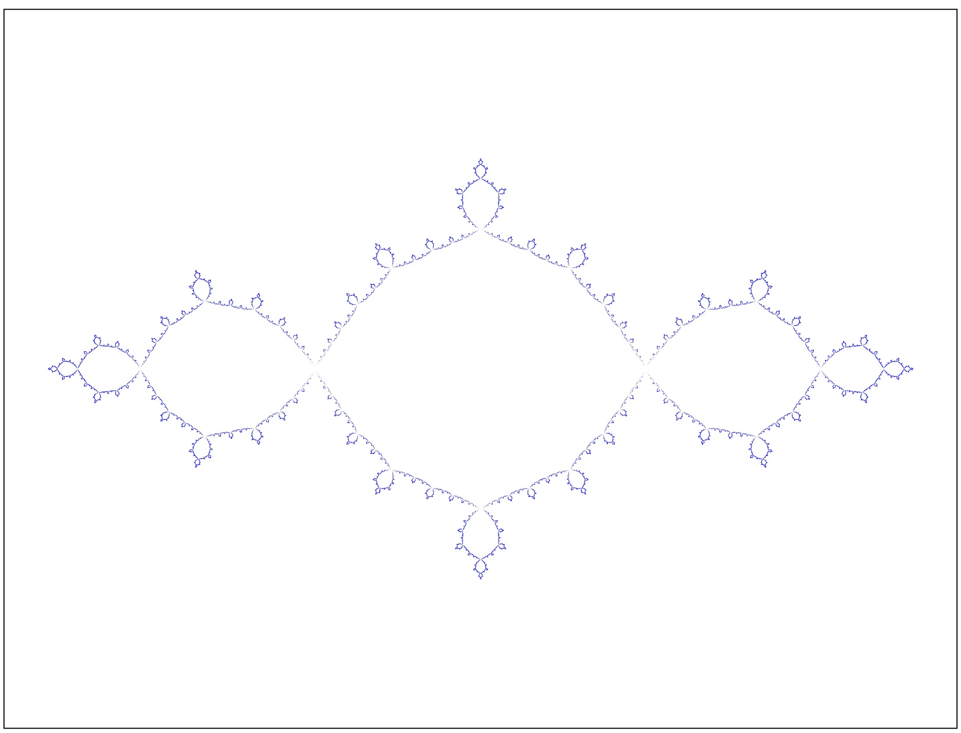


(b) The preimage points $f^{-20}(1 + i)$.

Figure 1.2: Equidistribution of preimages for $f(z) = z^2 + i$.

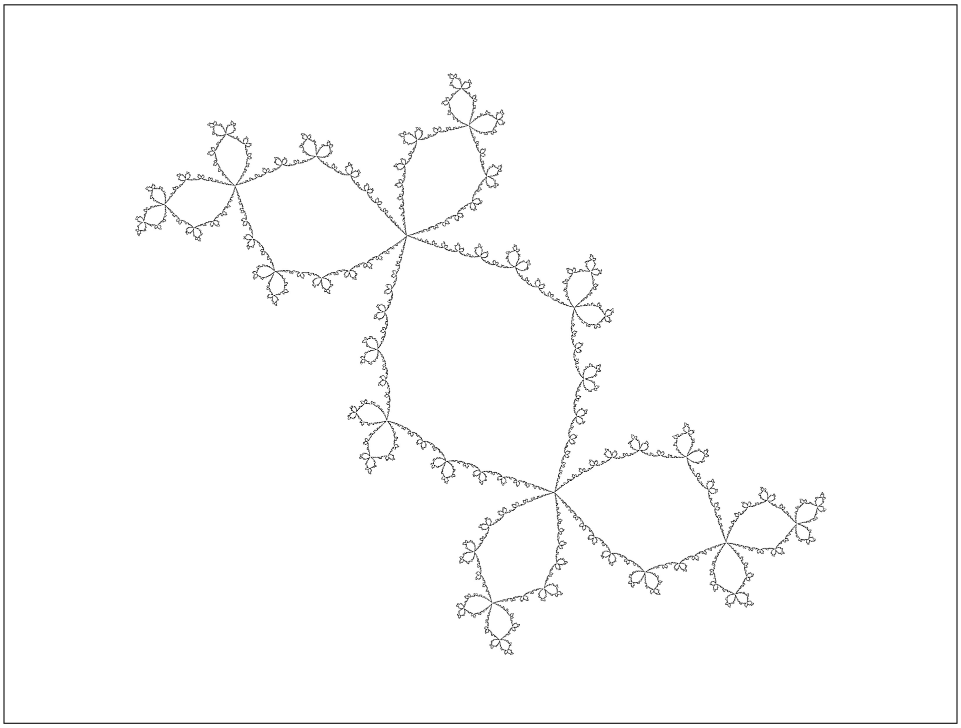


(a) The Julia set of $f(z) = z^2 - 1$.

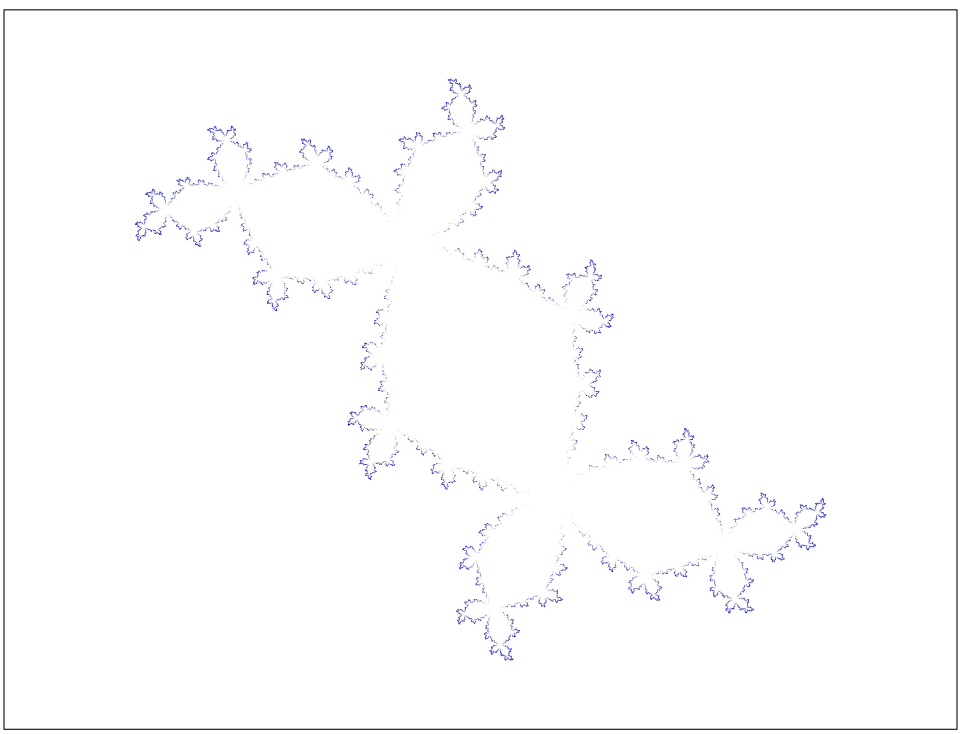


(b) The preimage points $f^{-20}(1 + i)$.

Figure 1.3: Equidistribution of preimages for $f(z) = z^2 - 1$.

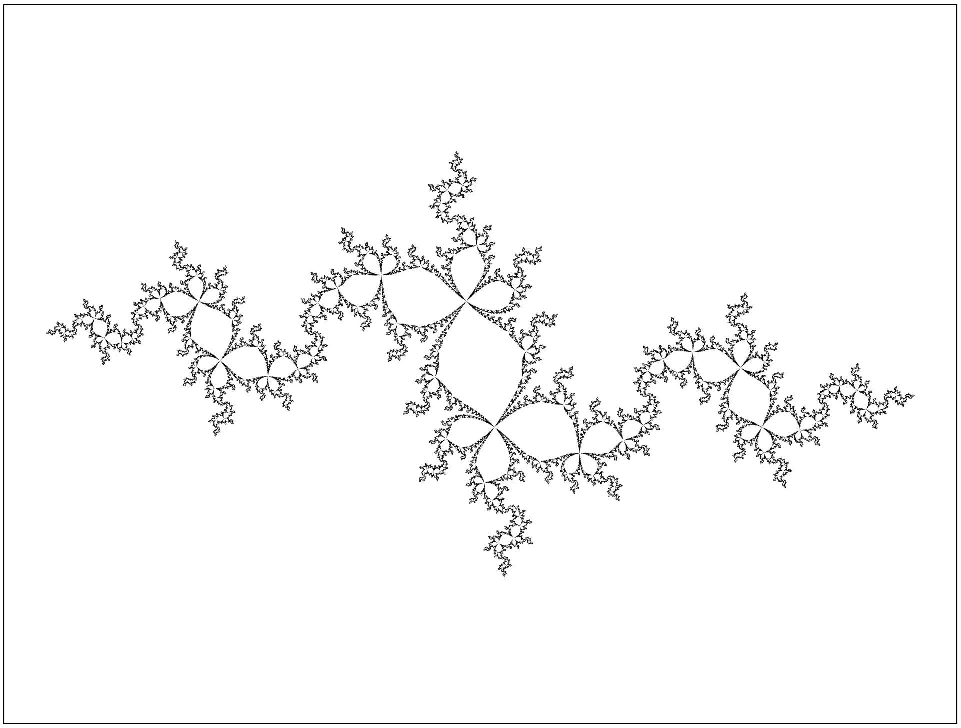


(a) The Julia set of $f(z) = z^2 - .122 + .745i$.

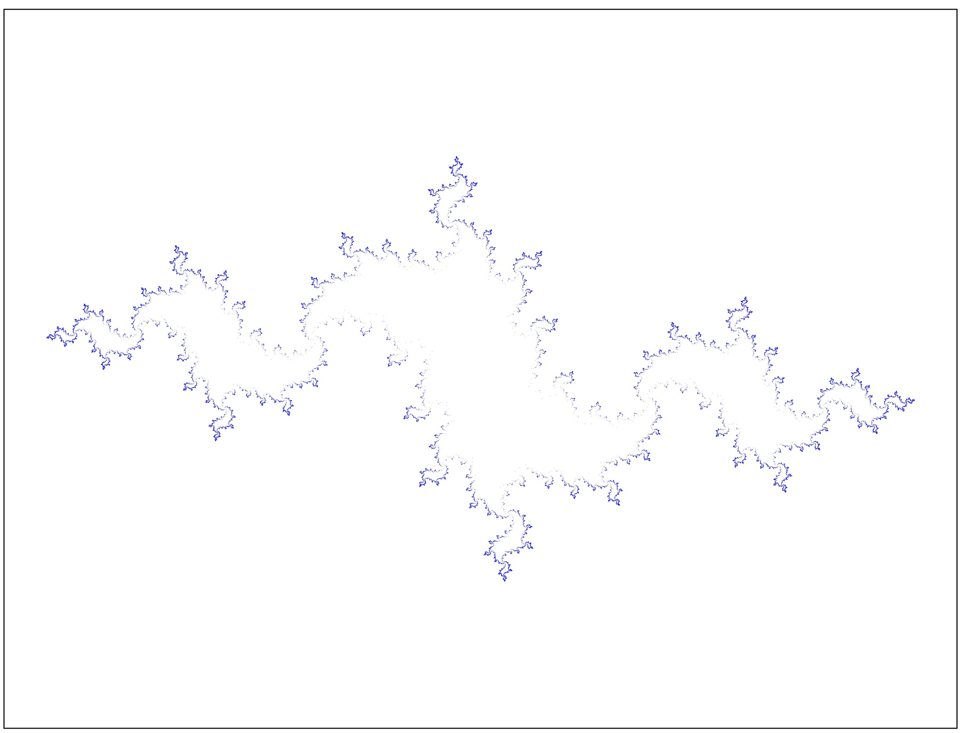


(b) The preimage points $f^{-20}(1+i)$.

Figure 1.4: Equidistribution of preimages for $f(z) = z^2 - .122 + .745i$.



(a) The Julia set of $f(z) = z^2 - 1 + .26i$.



(b) The preimage points $f^{-20}(1 + i)$.

Figure 1.5: Equidistribution of preimages for $f(z) = z^2 - 1 + .26i$.

Both approaches have been successfully generalized to complex dynamics in higher dimensions, as we will discuss later.

In this section, we will briefly outline a potential theoretic proof the equidistribution of preimages theorem for polynomial maps f. It will not be a completely rigorous argument, since our objective in doing so is *not* simply to see a proof, but rather to derive an alternate characterization of points $w \in \mathbf{C}$ for which equidistribution fails. We will see that in order for equidistribution of preimages to fail for w, a certain sequence of multiplicities associated to w must grow rapidly. This, in turn, is equivalent to w lying in the exceptional set $\mathscr{E}_f$. The proof given below essentially follows the proof in [80, Ch. 1], but is somewhat simpler due to our assumption that f is a polynomial.

Fix a polynomial map $f\colon \mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$. Recall that the *filled Julia set* of f is the compact set K_f of points $z \in \mathbf{C}$ whose f-orbit is bounded. The Julia set $\mathscr{J}_f$ of f is exactly the topological boundary of ∂K_f of K_f. One defines the *rate of escape* function of f to be the function $g_f\colon \mathbf{C} \to \mathbf{R}$ given by

$$g_f(z) := \lim_{n\to\infty} d^{-n} \log^+ |f^n(z)|.$$

The value $g_f(z)$ is an asymptotic measure of how quickly the sequence $|f^n(z)|$ grows. It is not difficult to show that this limit always exists, and that the function g_f has the following properties:

1. $g_f(z) \geq 0$ for all $z \in \mathbf{C}$, with $g_f(z) = 0$ if and only if $z \in K_f$.
2. g_f is subharmonic, and is in fact harmonic away from the Julia set $\mathscr{J}_f$.
3. $g_f(z) = \log |z| + C + o(1)$ as $z \to \infty$, where C is some constant.

In other words, g_f is exactly the Green's function with pole at ∞ of the domain $\mathbf{P}^1_{\mathbf{C}} \smallsetminus K_f$. One therefore obtains a probability measure supported within $\mathscr{J}_f$ by taking the normalized Laplacian Δg_f of g_f (in the sense of distributions), called the *equilibrium measure* of K_f. The canonical measure μ_f of f is defined to be this measure, $\mu_f := \Delta g_f$. While this construction of μ_f is valid only for polynomial maps f, a similar procedure can be used to define μ_f for general endomorphisms f.

Now fix any point $w \in \mathbf{C}$, and define μ_n to be the probability measure that weights each element of $f^{-n}(w)$ according to its multiplicity as an f^n-preimage of w. To show that μ_n converges weakly to μ as $n \to \infty$ it suffices to find potentials g_n of μ_n that converge to g_f in $L^1_{\mathrm{loc}}(\mathbf{C})$ as $n \to \infty$. We will prove this for the potentials g_n given by

$$g_n(z) = d^{-n} \log |f^n(z) - w|.$$

Observe that if we decompose g_n as the sum

$$g_n(z) = d^{-n} \log^+ |f^n(z) - w| + d^{-n} \log^- |f^n(z) - w|,$$

then the $\log^+$ terms clearly converge to g_f as $n \to \infty$, and we are left to show that the $\log^-$ terms converge to 0. That is, one has the equidistribution $\mu_n \to \mu$ if and only if the functions $u_n(z) := d^{-n} \log^- |f^n(z) - w|$ converge to 0 in $L^1_{\text{loc}}(\mathbf{C})$ as $n \to \infty$. This, in turn, can be shown to be equivalent to having the Lebesgue measure $|\Omega_n(\varepsilon)|$ of the open sets

$$\Omega_n(\varepsilon) := \{z \in \mathbf{C} : u_n(z) < -\varepsilon\}$$

converge to 0 as $n \to \infty$ for every fixed $\varepsilon > 0$.

Choose a ball B centered at the origin that is both large enough to contain the ball $\Omega_0(\varepsilon)$ and to satisfy $f^{-1}(B) \subseteq B$. Then one has $\Omega_n(\varepsilon) = f^{-(n-m)}(\Omega_m(\varepsilon d^{n-m})) \subseteq B$ for all integers $n \geq m \geq 0$. Using these relations and some elementary estimates that are omitted here (see [80, Theorem 1.7]), it is possible to show that for some $C > 0$,

$$|\Omega_m(\varepsilon d^{n-m})| \geq \exp\left\{-\frac{C}{|\Omega_n(\varepsilon)|} d^{n-m}\right\}. \tag{1.1}$$

On the other hand, we can bound $|\Omega_m(\varepsilon d^{n-m})|$ above in the following way. Let $v_{-m}(w)$ denote the maximum multiplicity of the roots of the polynomial $f^m(z) - w$, and let $\alpha_m = d^m / v_{-m}(w)$. Observe that the function $\exp(-\alpha_m u_m)$ is integrable on B. Indeed, the only discontinuities of $\exp(-\alpha_m u_m)$ are at the preimages $\zeta \in f^{-m}(w)$, and in a small neighborhood of ζ, one has

$$\exp(-\alpha_m u_m(z)) = \left(\frac{1}{|f^n(z) - w|}\right)^{1/v_{-m}(w)},$$

which is integrable. Thus $\exp(-\alpha_m u_m)$ is integrable on B, and we use Chebyshev's inequality to conclude that

$$|\Omega_m(\varepsilon d^{n-m})| \leq D \exp(-\varepsilon \alpha_m d^{n-m}) \quad \text{for some constant } D > 0. \tag{1.2}$$

Combining Equation 1.1 with Equation 1.2 and then taking a logarithm yields

$$-\frac{C}{|\Omega_n(\varepsilon)|} d^{n-m} \leq \log D - \varepsilon \alpha_m d^{n-m}.$$

In the limit as $n \to \infty$, this simplifies to

$$\limsup_{n \to \infty} |\Omega_n(\varepsilon)| \leq \frac{C}{\varepsilon \alpha_m} = \frac{C}{\varepsilon} \frac{v_{-m}(w)}{d^m}.$$

Recall that one has the equidistribution $\mu_n \to \mu_f$ if and only if the left hand side of this inequality vanishes for all $\varepsilon > 0$. Thus we have derived a condition for equidistribution:

Equidistribution Condition. *If $v_{-m}(w) = o(d^m)$ as $m \to \infty$, then the iterated preimages of w equidistribute to μ_f.*

In this one-dimensional setting, it is not hard to show that $v_{-m}(w) = o(d^m)$ if and only if $w \notin \mathscr{E}_f$, so this essentially completes the proof of the equidistribution of preimages theorem, at least for polynomial maps.

The main idea from this argument that we will use in subsequent chapters is that whether or not the preimages of a point w equidistribute to μ_f can be determined by measuring the growth rate of the multiplicities $v_{-m}(w)$ as $m \to \infty$. This idea will take considerable work to develop in the setting we will be working in, namely, for dynamical systems in arbitrary dimension over arbitrary algebraically closed fields. In Chapter 2 we will develop the relevant theory of multiplicities in higher dimensions, and in Chapter 3 we will derive an analogous condition for equidistribution of preimages based on the growth rate of an analogous sequence v_{-m} of multiplicities.

1.3. Equidistribution of preimages in higher dimensions

Complex dynamics in several variables began as an active area of research in the late 1980s and early 1990s, and it did not take long to realize that equidistribution of preimages should admit an analogue in the higher dimensional setting. When Fornæss-Sibony and Hubbard-Papadopol generalized Brolin's potential theoretic construction of the canonical measure μ_f to arbitrary endomorphisms f of the projective line [69, 84] using pluripotential theoretic methods, they observed that their construction could easily be carried out in any dimension. However, perhaps surprisingly, they found that the appropriate analogue of the canonical measure μ_f in higher dimensions is not a measure at all, but a *current* T_f.

Theorem 1.3.1 ([69, 84]). *Suppose $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ is an endomorphism of algebraic degree $d \geq 2$, that is, $f = [F_0 : \cdots : F_r]$ where the F_i are homogeneous polynomials of degree d with no nontrivial common zeros. Then there is a positive closed $(1,1)$-current T_f on $\mathbf{P}^r_{\mathbf{C}}$ with the property for any smooth positive closed $(1,1)$-form α on $\mathbf{P}^r_{\mathbf{C}}$ cohomologous to the Fubini-Study form ω, the iterated pullbacks $d^{-n} f^{n*}\alpha$ converge weakly to T_f as $n \to \infty$.*

We will not discuss either currents or positivity here; for a reference, see for instance [39, Chapters 1,3]. The original proof of this theorem involved lifting f to an endomorphism $F\colon \mathbf{A}^{r+1}_{\mathbf{C}} \to \mathbf{A}^{r+1}_{\mathbf{C}}$ and using pluripotential theoretic techniques there. Using language that has developed over the last twenty years, however, this is unnecessary, and a proof can be given in a few lines.

Proof of Theorem 1.3.1. Since α and ω are cohomologous, the dd^c-lemma of Hodge theory implies that there is a smooth function θ on $\mathbf{P}^r_{\mathbf{C}}$ such that $\alpha = \omega + dd^c\theta$ (the operator dd^c is the pluripotential theoretic analogue of the Laplacian operator in dimension one). Similarly, there is a smooth function φ such that $d^{-1}f^*\omega = \omega + dd^c\varphi$. By a simple induction,

$$d^{-n}f^{n*}\alpha = \omega + dd^c\left\{\varphi + \frac{\varphi\circ f}{d} + \cdots + \frac{\varphi\circ f^{n-1}}{d^{n-1}} + \frac{\theta\circ f^n}{d^n}\right\}.$$

As $n\to\infty$, the expression in braces converges uniformly to $g_f = \sum_{n=0}^{\infty} d^{-n}\varphi\circ f^n$. Thus the pullbacks $d^{-n}f^{n*}\alpha$ converge weakly to $T_f := \omega + dd^c g_f$ as $n\to\infty$. Noting that g_f is defined only in terms of φ, we conclude that T_f is independent of α. $\square$

Note that the only thing we used about α in this proof is that it is of the form $\alpha = \omega + dd^c\theta$ where θ is *bounded*. Thus the normalized pullbacks of singular positive closed $(1,1)$-currents α cohomologous to ω converge to T_f as well, so long as $\alpha = \omega + dd^c\theta$ for bounded θ. This then begs the question: what happens when α is even more singular, i.e., when $\alpha = \omega + dd^c\theta$ for unbounded θ?

The mathematics used to date in addressing this question is much more challenging and subtle than that of either Theorem 1.3.1 or the one dimensional equidistribution of preimages theorem. Most of the early work on this question was done by Fornæss-Sibony, as they were doing their pioneering work on complex dynamics in several variables [68, 69], but it has since been the subject of a large volume of work by many researchers [99, 62, 79, 49, 50, 51, 47, 53, 52, 54, 96, 105], see also the very related works [8, 9, 10, 42, 28, 48, 46, 6, 43, 45, 44, 65]. Interesting as this question is, it is not the subject of the present work, so for our purposes it is enough to simply state the following theorem as an answer to the question and note that the question is by no means settled.

Equidistribution of Preimages Theorem 2. *Let $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ be an endomorphism of algebraic degree $d \geq 2$. Then, assuming that f is sufficiently general, the normalized pullbacks $d^{-n}f^{n*}S$ of any positive closed $(1,1)$-current S cohomologous to the Fubini-Study form ω converge weakly to T_f as $n\to\infty$.*

The current T_f is usually called the *Green current* of f. One consequence of our proof of Theorem 1.3.1 is that $T_f = \omega + dd^c g_f$ for a *continuous* function g_f. The continuity of g_f allows us to take the r-fold wedge product $\mu_f := T_f\wedge\cdots\wedge T_f$ of T_f, as defined by Bedford-Taylor [11], to get a probability measure μ_f on $\mathbf{P}^r_{\mathbf{C}}$.

Definition 1.3.2. The measure $\mu_f = T_f\wedge\cdots\wedge T_f$, sometimes called the Green measure or equilibrium measure of f, will be called the *canonical measure* here.

As an essentially direct consequence of Theorem 1.3.1, the normalized pullbacks by f of any *smooth* probability measure μ on $\mathbf{P}^r_{\mathbf{C}}$ converge weakly to μ_f, giving a measure-theoretic form of equidistribution. However, when μ is a singular probability measure, or more specifically, when $\mu = \delta_w$ is the Dirac mass at a point $w \in \mathbf{P}^r_{\mathbf{C}}$, the theorem does not apply. Thus it remained unclear for a number of years exactly when the iterated preimages of a point$w \in \mathbf{P}^r_{\mathbf{C}}$ equidistributes to the canonical measure μ_f. After initial work of Fornæss-Sibony [69], the question was finally settled ten years ago by Briend-Duval and Dinh-Sibony.

Equidistribution of Preimages Theorem 3 ([28, 48])**.** *Let $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ be an endomorphism of algebraic degree $d \geq 2$. Then there exists a proper Zariski closed subset $\mathscr{E}_f \subsetneq \mathbf{P}^r_{\mathbf{C}}$ such that if $w \notin \mathscr{E}_f$, the iterated preimages of w equidistribute to μ_f.*

The set $\mathscr{E}_f$ is called the *exceptional set* of f, in analogy with the one dimensional case. It can be characterized as the largest proper Zariski closed subset $E \subsetneq \mathbf{P}^r_{\mathbf{C}}$ that is totally invariant for f, i.e., for which $f^{-1}(E) = E$. For sufficiently general f, the exceptional set $\mathscr{E}_f$ will be empty. See Chapter 3 for a more in depth discussion of the exceptional set.

It is this version of the equidistribution of preimages theorem that is the subject of the present work. We will be proving an analogue of the theorem for dynamical systems f on varieties over nonarchimedean fields. The original proof of this theorem by Briend-Duval eschews the pluripotential theoretic methods discussed to this point, and instead adapts the geometric approach of Lyubich [92] and Freire-Lopes-Mañé [70] for the one dimensional case to higher dimensions. While we won't discuss these methods here, it is worth pointing out that the exceptional set $\mathscr{E}_f$ is constructed as the set of points where a sequence of multiplicities associated to f grow quickly, similar to the equidistribution condition derived in the previous section. Finally, we note that this version of the equidistribution of preimages theorem has been extended to all polarized complex dynamical systems by Dinh-Sibony [51]; see Chapter 3 for a discussion of these systems.

1.4. Ergodic theory of the canonical measure

Fix an endomorphism $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ of algebraic degree $d \geq 2$, let T_f be its Green current, and let $\mu_f = T_f \wedge \cdots \wedge T_f$ be the canonical measure of f. By its very construction, T_f is *totally invariant*, that is, $d^{-1} f^* T_f = T_f$. It follows immediately that the canonical measure μ_f is also totally invariant, $d^{-r} f^* \mu_f = \mu_f$. This is an incredibly strong dynamical property for a measure to possess, and it in particular implies that μ_f is invariant, $f_* \mu_f = \mu_f$. This opens up the door to studying the ergodic properties of f with respect to μ_f. We will not say much about this ergodic theory, but it is worth stating some of the major results.

Theorem 1.4.1. *Consider the measure preserving transformation* $f\colon (\mathbf{P}^r_{\mathbf{C}}, \mu_f) \to (\mathbf{P}^r_{\mathbf{C}}, \mu_f)$. *Then:*

1. *The topological entropy of* f *is exactly* $h_{\text{top}}(f) = r \log d$. *Here the inequality* $\geq$ *is due to Yomdin [111] and the inequality* $\leq$ *is due to Gromov [77].*

2. *The canonical measure* μ_f *is the unique* f*-invariant probability measure on* $\mathbf{P}^r_{\mathbf{C}}$ *whose metric entropy is maximal,* $h_{\mu_f} = r \log d$ *[28]. Intuitively, the canonical measure is the invariant measure which best reflects the dynamics of* f.

3. *The map* f *is ergodic, and even mixing, with respect to* μ_f *[69].*

4. *All Lyapunov exponents of* f *with respect to* μ_f *are* $\geq \frac{1}{2}\log d$, *and thus in particular are positive [27].*

These rather strong ergodic statements represent some of the major successes in complex dynamics in several variables.

1.5. Arithmetic equidistribution I

To close out the chapter, we begin a brief discussion of another (related) equidistribution result, namely Yuan's arithmetic equidistribution theorem [112]. This deep result, which generalizes the earlier works [104, 3, 66, 34] (see also [78, 58]), has, since its publication, been used to great effect in dynamics [113, 1, 72, 2, 61]. Yuan's theorem concerns the equidistribution of *points of small height* for certain height functions on projective varieties defined over $\overline{\mathbf{Q}}$. The theorem has two parts: an *archimedean* (complex) part, and a *nonarchimedean* (p-adic) part. In this section we will only touch on the archimedean part, leaving the nonarchimedean part for §4.6. We will also, for simplicity, only state Yuan's theorem in a relevant special case, that is, for the *canonical height* associated to an endomorphism $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ defined over $\overline{\mathbf{Q}}$.

Before being able to define the canonical height associated to f, we must first discuss the (logarithmic) *Weil height* on projective space $\mathbf{P}^r$. For a detailed introductory treatment of both the Weil height and canonical heights, we refer to [102, Ch. 3]. The Weil height h on $\mathbf{P}^r$ is a function $h\colon \mathbf{P}^r(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$ which, intuitively, is meant to measure the arithmetic complexity or size of the $\overline{\mathbf{Q}}$-points of projective space. In this section we will only define h on the $\mathbf{Q}$-points of $\mathbf{P}^r$, postponing the definition of h on general $\overline{\mathbf{Q}}$-points until §4.6. Observe that every $\mathbf{Q}$-point $x \in \mathbf{P}^r(\mathbf{Q})$ can be represented uniquely in homogeneous coordinates as $x = [x_0 : \cdots : x_r]$, where the x_i are integers with no common prime factors. The (logarithmic) Weil height of x is defined to be $h(x) := \log \max_i |x_i|$.

One of the most important properties of the Weil height h is the *Northcott property*: if $B > 0$ is a real constant and $D \geq 1$ is an integer, then the set of points $x \in \mathbf{P}^r(\overline{\mathbf{Q}})$ of degree at most D and Weil height at most B is a finite set. This is very easy to verify on $\mathbf{Q}$-points, since if $x \in \mathbf{P}^r(\mathbf{Q})$ has $h(x) \leq B$, then x must be one of the finitely many points of the form $x = [x_0 : \cdots : x_r]$ where each x_i is an integer with $|x_i| \leq e^B$. The Weil height is also Galois invariant, in that $h(\sigma x) = h(x)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

In the study of heights on projective varieties over $\overline{\mathbf{Q}}$, it is typical not to distinguish functions which differ by a bounded function. For instance, if h' is a function of the form $h' = h + O(1)$, then h' gives roughly the same measure of the arithmetic complexity of points in $\mathbf{P}^r(\overline{\mathbf{Q}})$ as h, and also satisfies the Northcott property; we might wish to call h' a *height function* on $\mathbf{P}^r$ equivalent to h. From this point of view, there is no canonical choice of height function on $\mathbf{P}^r$ equivalent to h. However, if we fix an endomorphism $f\colon \mathbf{P}^r \to \mathbf{P}^r$ defined over $\overline{\mathbf{Q}}$ of algebraic degree $d \geq 2$, there is a unique height function $\hat{h}_f$ on $\mathbf{P}^r$, called the *canonical height* for f, which is of the form $\hat{h}_f = h + O(1)$ and is compatible with the dynamics of f in the sense that $\hat{h}_f \circ f = d \cdot \hat{h}_f$. The canonical height $\hat{h}_f$ is constructed, like most invariant dynamical objects, via a limiting procedure:

$$\hat{h}_f(x) := \lim_{n\to\infty} d^{-n} h(f^n(x)).$$

While we will not take the time to prove this limit always exists or that $\hat{h}_f = h + O(1)$, both of these facts can be proved without much trouble from the estimate $h \circ f = d \cdot h + O(1)$, which in turn is a consequence of Hilbert's Nullstellensatz, see [102, §3.2].

Yuan's arithmetic equidistribution theorem (Archimedean part)**.** *Let $f\colon \mathbf{P}^r \to \mathbf{P}^r$ be an endomorphism of algebraic degree $d \geq 2$ defined over a number field K. Fix an embedding $\overline{K} \subset \mathbf{C}$, and let μ_f be the canonical measure of the induced endomorphism $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$. Let A_n be a sequence of finite subsets of $\mathbf{P}^r(\overline{K})$ for which:*

1. *Each A_n is $\mathrm{Gal}(\overline{K}/K)$-invariant.*
2. *The sequence A_n is Zariski generic in the sense that for all closed subvarieties $D \subsetneq \mathbf{P}^r_{\mathbf{C}}$ defined over K, one has $A_n \cap D = \varnothing$ for sufficiently large n.*
3. *The sequence $\max_{x\in A_n} \hat{h}_f(x)$ tends to 0 as $n \to \infty$.*

Then the A_n equidistribute to μ_f, that is, the probability measures $\mu_n = (\#A_n)^{-1} \sum_{x\in A_n} \delta_x$ converge weakly to μ_f as $n \to \infty$.

To see how this relates to the equidistribution of preimages theorem, let us try to apply the theorem to the sequence $A_n = f^{-n}(x)$, where $x \in \mathbf{P}^r(K)$ is a K-point of projective space. Since f is defined over K, condition 1 is satisfied for these A_n. Condition 3 is also satisfied, since for any $y \in A_n$, the compatibility of $\hat{h}_f$ with the dynamics of f implies that $\hat{h}_f(y) = d^{-n}\hat{h}_f(x)$. In general condition 2 will not be satisfied, but if x is a point for which it is satisfied, then Yuan's theorem will imply that the preimages of x equidistribute to μ_f (note, this is not an immediate implication, since Yuan's theorem does not count the preimages with multiplicities, but it does not take much work deal with this wrinkle).

In dimension $r = 1$, condition 2 is much easier to handle, since proper closed subvarieties of $\mathbf{P}^1_{\mathbf{C}}$ are just points. It is not hard to see that $A_n = f^{-n}(x)$ satisfies condition 2 if and only if x is not periodic for f. Moreover, even when x is periodic, so long as it is not exceptional for f we can still use Yuan's theorem to conclude that the preimages of x equidistribute to μ_f by simply excluding elements in the periodic cycle of x from the sets A_n. However, in dimension $r \geq 2$, it is not as clear when A_n is generic. One of the main results in this dissertation, Theorem B, which we will prove in §3.4, implies that if $x \in \mathbf{P}^r_{\mathbf{C}}$ does not lie in $\mathscr{E}_f$, then the preimage sets $A_n = f^{-n}(x)$ are *asymptotically generic*, in the sense that for any closed subvariety $D \subsetneq \mathbf{P}^r_{\mathbf{C}}$, the number of f^n-preimages of x in D, counted with multiplicity, is $= o(d^{rn})$.

Though not directly related to equidistribution of preimages, it is worth mentioning another application of Yuan's theorem, namely to the problem of *equidistribution of periodic points*. It is a straightforward consequence of the Northcott property that $\hat{h}_f(x) = 0$ if and only if x is preperiodic for f. Let $A_n \subset \mathbf{P}^r(\overline{K})$ be the set of f-periodic points with minimal period n. It is again clear that A_n satisfies conditions 1 and 3 of Yuan's theorem. Moreover, in dimension $r = 1$, it is also clear that A_n satisfies condition 2, proving that the n-periodic points of f equidistribute to μ_f. Of course, this is true for all $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ regardless of whether or not f is defined over a number field, as was proved by Lyubich [92], see also the higher dimensional results [7, 27, 55, 90].

At the moment, we have given no indication for why the canonical measure μ_f should appear in a theorem about heights. We will explore this connection more in §4.6. Finally, we should reiterate that the full Yuan arithmetic equidistribution theorem [112] is far more general than the version stated above, in that it holds for more general height functions than just the canonical height, and on general projective varieties as well.

Chapter 2

Multiplicities associated to finite morphisms

In the previous chapter, we derived a condition for when the iterated preimages of a point $x \in \mathbf{C}$ under a polynomial $f\colon \mathbf{C} \to \mathbf{C}$ equidistribute the canonical measure μ_f associated to f. This condition was phrased in terms the asymptotic growth rate of multiplicities of the preimages of x. Estimating this growth rate is at the heart of every proof of the complex equidistribution of preimages theorem, even in the higher dimensional setting, so it unsurprising that we should need to do the same in the nonarchimedean setting.

In complex dynamics, it is typical to use analytic methods to define and work with these multiplicities. Unfortunately, these analytic formulations are not available over more general fields. For this reason, we devote the chapter to algebraically developing the relevant theory of multiplicities for (finite and flat) morphisms between varieties over any algebraically closed field. The results in this chapter are not dynamical; we apply them in the dynamical setting starting in Chapter 3.

2.1. Setup and notation

The setup for the entirety of this chapter is as follows. Fix an algebraically closed field k, and let $f\colon X \to Y$ be a finite surjective morphism between two irreducible varieties over k. In later chapters we will apply the work done here in the dynamical setting, namely, when f is a (polarized) morphism $X \to X$, but in this chapter there will be no dynamics. We will always view X and Y as schemes over k. In particular, when we refer to a point of X or Y, we are allowing the possibility that the point be non-closed.

We will make a further technical assumption on the morphism f, namely that it be *flat*. This assumption is used in an essential way in this chapter (see Theorem 2.3.4), but may be unnecessary in the (more restrictive) dynamical setting. In general, the interaction between flatness and dynamics is not well understood; the recent work [93] has taken some initial steps towards its elucidation. At the very least, however, the assumption of flatness simplifies the exposition, and in cases of particular interest is automatic. For instance, finite surjective morphisms $f\colon X \to Y$ between *smooth* varieties are necessarily flat, see [57, Corollary 18.17] and [82, Exercise III.9.3].

Throughout this chapter (and thesis) we use the following notation. The structure sheaf of X and Y will be denoted $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Given a (not necessarily closed) point $x \in X$, we write $\mathfrak{m}_x$ for the maximal ideal of the local ring $\mathcal{O}_{X,x}$, and $\kappa(x)$ for the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$. Recall that the *degree* of f is the degree of the field extension $k(X)/f^*k(Y)$, where $k(X)$ and $k(Y)$ are the function fields of X and Y. This degree will be written $[X :_f Y]$. Similarly, $[X :_f Y]_s$ and $[X :_f Y]_i$ will denote the separable and purely inseparable factors of this degree.

2.2. Multiplicities at a point

Suppose $f\colon X \to Y$ is a finite surjective flat morphism between irreducible varieties over an algebraically closed field k. To any point $x \in X$, we now attach two positive integers $m_f(x)$ and $v_f(x)$, which we call the *multiplicity* and *generic multiplicity* of f at x, respectively. These multiplicities are the main object of study in this chapter, and are of central importance in understanding equidistribution of preimages. The multiplicity function $m_f\colon X \to \mathbf{N}$ will be used to define the pullback of measures on varieties, and the generic multiplicity function $v_f\colon X \to \mathbf{N}$ is one of the main tools used in detecting totally invariant sets.

Definition 2.2.1. Let $x \in X$ and $y = f(x) \in Y$. The multiplicity of f at x is the integer

$$m_f(x) := \dim_{\kappa(y)}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}),$$

where here $\mathcal{O}_{X,x}$ is viewed as an $\mathcal{O}_{Y,y}$-module via f. Let $E = \overline{\{x\}} \subseteq X$ and $F = \overline{\{y\}} \subseteq Y$ be the subvarieties associated to x and y. Then the *generic multiplicity* of f at x is the integer

$$v_f(x) := [E :_f F]_i \times \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}).$$

It will sometimes be convenient to write $m_f(E)$ and $v_f(E)$ in place of $m_f(x)$ and $v_f(x)$.

Before exploring the properties of these multiplicities in detail, it is worth seeing some explicit example computations.

Example 2.2.2. Let us start with a one-dimensional example, namely the case when f is a non-constant polynomial map $f\colon \mathbf{A}^1_k \to \mathbf{A}^1_k$. Suppose $x \in \mathbf{A}^1_k$ is a *closed* point, and $y = f(x)$. Then x is a root of the polynomial $f - y$ of some order $e \geq 1$, and one easily computes

$$\mathcal{O}_{\mathbf{A}^1_k,x}/\mathfrak{m}_y\mathcal{O}_{\mathbf{A}^1_k,x} = k[t]_{(t-x)}/(f(t) - y) \cong k[t]/(t-x)^e \cong k^e.$$

Thus $m_f(x) = e$. Moreover, since the function fields of the points x and y are both just k, the inseparable degree $[x :_f y]_i$ is simply 1, so that $v_f(x) = e$ as well.

Now suppose that x is the generic point of $\mathbf{A}^1_k$, and hence $f(x) = x$. The local ring $\mathcal{O}_{\mathbf{A}^1_k,x}$ is precisely the function field $k(t)$ of $\mathbf{A}^1_k$, and $\mathfrak{m}_x = 0$. By unraveling definitions, we then see that $m_f(x)$ is exactly the degree of the field extension $k(t)/f^*k(t)$, which is the degree of the polynomial f. Similarly, $v_f(x) = [\mathbf{A}^1_k :_f \mathbf{A}^1_k]_i$ is the inseparable degree of f.

Example 2.2.3. We now move on to higher dimensional examples. Assume k does not have characteristic 2, and let $f\colon \mathbf{A}^2_k \to \mathbf{A}^2_k$ be the polynomial map $f(x,y) := (x^2 - y^2, x^2)$. The preimage $f^{-1}(L_x)$ of the line $L_x := \{x = 0\}$ is the union of the two lines $L_\pm := \{x = \pm y\}$. One now computes

$$\mathcal{O}_{\mathbf{A}^2_k,L_+}/\mathfrak{m}_{L_x}\mathcal{O}_{\mathbf{A}^2_k,L_+} = k[x,y]_{(x-y)}/(x^2-y^2) \cong k[x,y]_{(x-y)}/(x-y) = k(L_+).$$

Therefore $m_f(L_+)$ is the degree of the field extension $k(L_+)/f^*k(L_x)$, i.e., $m_f(L_+) = 2$. On the other hand, the length of $k(L_+)$ is 1, and the field extension $k(L_+)/f^*k(L_x)$ is separable, so $v_f(L_+) = 1$. Of course, nearly identical computations hold for L_-, so $m_f(L_-) = 2$ and $v_f(L_-) = 1$ as well.

The preimage $f^{-1}(L_y)$ of the line $L_y := \{y = 0\}$ has one component, namely the line L_x. One then sees that

$$\mathcal{O}_{\mathbf{A}^2_k,L_x}/\mathfrak{m}_{L_y}\mathcal{O}_{\mathbf{A}^2_k,L_x} = k[x,y]_{(x)}/(x^2) \cong k(L_x)^2,$$

which has length $v_f(L_x) = 2$. Moreover, the dimension $m_f(L_x)$ of this ring as a vector space over $k(L_y)$ is twice the dimension of $k(L_x)$ over $k(L_y)$, that is, $m_f(L_x) = 4$.

Example 2.2.4. In our final example, we assume k has characteristic $p > 0$. Let $f\colon \mathbf{P}^2_k \to \mathbf{P}^2_k$ be the endomorphism $f[x:y:z] := [x^{p-1}(z+x) : y^{p-1}(z+y) : z^p]$. The line $L_z := \{z = 0\}$ is totally invariant for f, that is, $f^{-1}(L_z) = L_z$. Moreover, the restriction $f|_{L_z} : L_z \to L_z$ is the Frobenius morphism $[x:y] \mapsto [x^p : y^p]$, so $[L_z :_f L_z]_i = p$. In the standard affine chart with coordinates $\tilde{y} = y/x$ and $\tilde{z} = z/x$, the map f is given by

$$f(\tilde{y},\tilde{z}) = \left(\frac{\tilde{y}^p + \tilde{z}\tilde{y}^{p-1}}{\tilde{z}+1}, \frac{\tilde{z}^p}{\tilde{z}+1}\right),$$

and hence

$$\mathcal{O}_{\mathbf{P}^2_k,L_z}/f^*\mathfrak{m}_{L_z} = k[\tilde{y},\tilde{z}]_{(\tilde{z})}/(\tilde{z}^p/[\tilde{z}+1]) \cong k[\tilde{y},\tilde{z}]_{(\tilde{z})}/(\tilde{z}^p) \cong k(L_z)^p.$$

The length of this ring is p, giving that $v_f(L_z) = p^2$. The dimension of this ring as a vector space over $f^*k(L_z)$ is p times the dimension of $k(L_z)$ over $f^*k(L_z)$, that is, $m_f(L_z) = p^2$.

2.3. Properties of the multiplicities

We now outline in some, but not complete, detail the algebraic properties satisfied by these two multiplicities that will be relevant for us later on.

Lemma 2.3.1. *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be Noetherian local rings, with B a finite flat A-module. Let $\mathfrak{a}$ be any $\mathfrak{m}$-primary ideal of A. Then the following identities hold:*

1. $\operatorname{length}_B(B/\mathfrak{a}B) = \operatorname{length}_A(A/\mathfrak{a}) \times \operatorname{length}_B(B/\mathfrak{m}B)$.

2. $\operatorname{length}_A(B/\mathfrak{a}B) = \operatorname{length}_B(B/\mathfrak{a}B) \times [B/\mathfrak{n} : A/\mathfrak{m}]$.

Proof. (1) Let $A/\mathfrak{a} = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_N = 0$ be a composition series of $A/\mathfrak{a}$. Since A is local, the successive quotients I_i/I_{i+1} are each isomorphic to $A/\mathfrak{m}$. Because B is flat over A, one obtains a filtration $B/\mathfrak{a}B = B \otimes_A I_0 \supseteq B \otimes_A I_1 \supseteq \cdots \supseteq B \otimes_A I_N = 0$ of $B/\mathfrak{a}B$, whose successive quotients are $(B \otimes_A I_i)/(B \otimes_A I_{i+1}) \cong B \otimes_A (I_i/I_{i+1}) \cong B \otimes_A A/\mathfrak{m} \cong B/\mathfrak{m}B$. Thus $\operatorname{length}_B(B/\mathfrak{a}B) = N \times \operatorname{length}_B(B/\mathfrak{m}B)$, as desired.

(2) Now fix a composition series $B/\mathfrak{a}B = J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_M = 0$ of $B/\mathfrak{a}B$ as a B-module. Since B is local, the quotients J_i/J_{i+1} are all isomorphic to $B/\mathfrak{n}$. Thus

$$\operatorname{length}_A(B/\mathfrak{a}B) = M \times \operatorname{length}_A(B/\mathfrak{n}) = M \times [B/\mathfrak{n} : A/\mathfrak{m}],$$

as desired. □

This lemma allows us to relate the two multiplicities m_f and v_f, and to show how they behave under compositions.

Proposition 2.3.2. *The multiplicity functions $m_f \colon X \to \mathbf{N}$ and $v_f \colon X \to \mathbf{N}$ are related as follows. Let $x \in X$ and $y = f(x) \in Y$. Suppose $E = \overline{\{x\}}$ and $F = \overline{\{y\}}$ are the subvarieties corresponding to x and y. Then one has the identity*

$$m_f(x) = v_f(x) \times [E :_f F]_s.$$

In particular, if x is a closed point, then $m_f(x) = v_f(x)$.

Proof. Applying Lemma 2.3.1(2) to the case where $A = \kappa(y)$, $\mathfrak{a} = 0$, and $B = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ yields $\dim_{\kappa(y)}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) = \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) \times [\kappa(x) : \kappa(y)]$, which is exactly the desired identity $m_f(x) = v_f(x) \times [E :_f F]_s$. If x and y are closed points, then $\kappa(x) = \kappa(y) = k$, since k is algebraically closed. Thus $[E :_f F] = 1$ in this case, giving $m_f(x) = v_f(x)$. □

Proposition 2.3.3. *Suppose that $g\colon Y \to Z$ is another finite surjective flat morphism between irreducible varieties. Let $x \in X$ and $y = f(x)$. Then the multiplicity m_f and generic multiplicity v_f are multiplicative in the sense that*

$$m_{g\circ f}(x) = m_f(x)m_g(y) \quad \text{and} \quad v_{g\circ f}(x) = v_f(x)v_g(y).$$

Proof. By Proposition 2.3.2, it suffices to show that the generic multiplicity is multiplicative. Moreover, since degrees of inseparability for fields extensions are multiplicative, it suffices to show that

$$\operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_{g(y)}\mathcal{O}_{X,x}) = \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) \times \operatorname{length}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/\mathfrak{m}_{g(y)}\mathcal{O}_{Y,y}).$$

This is exactly Lemma 2.3.1(1). □

The next two theorems give a geometric interpretation of m_f and v_f. In Theorem 2.3.4, we prove the fact that, for our finite flat morphism $f\colon X \to Y$, every point $y \in Y$ has the same number of preimages in X, when counted according to the appropriate multiplicity, which is exactly m_f. As we will see in Theorem 2.3.5, the generic multiplicity $v_f(x)$ is exactly the multiplicity $m_f(z)$ of general closed points z specializing x. In other words, if x is the generic point of a subvariety $E \subseteq X$, then $v_f(x)$ is the multiplicity $m_f(z)$ of a generically chosen closed point $z \in E$. This explains the name *generic multiplicity* for the function v_f.

Theorem 2.3.4. *Every point $y \in Y$ as exactly $[X :_f Y]$ preimages in X, counted according to the multiplicity m_f. That is, $[X :_f Y] = \sum_{f(x)=y} m_f(x)$.*

Proof. Since f is finite and flat, $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of some rank $r < \infty$. The fiber of $f_*\mathcal{O}_X$ at a point $y \in Y$ is

$$(f_*\mathcal{O}_X)_y/\mathfrak{m}_y(f_*\mathcal{O}_X)_y \cong \bigoplus_{f(x)=y} \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

Comparing the $\kappa(y)$-dimension of both sides of this isomorphism, we see $r = \sum_{f(x)=y} m_f(x)$. In the special case where y is the generic point of Y, this identity yields $r = [X :_f Y]$. □

Theorem 2.3.5 (Lejeune-Jalabert and Teissier)**.** *There is a coherent sheaf $\mathcal{F}$ on X whose fiber dimensions are given by v_f. As a consequence, the function $v_f\colon X \to \mathbf{N}$ is Zariski upper semicontinuous.*

Sketch. The sheaf $\mathcal{F}$ is a *relative jet sheaf*, constructed as follows. Let $Z = X \times_f X$, and let $\mathcal{I}$ denote the ideal sheaf of the diagonal $\Delta \subseteq Z$. Write $\pi\colon Z \to X$ for the projection onto the first coordinate. We then set $\mathcal{F} = \pi_*(\mathcal{O}_Z/\mathcal{I}^n)$, where n is a large enough integer;

indeed, any $n \geq [X :_f Y]$ will suffice. The fiber dimension of $\mathcal{F}$ at any point $x \in X$ is computed in [91, Proposition 4.7] to be exactly $v_f(x)$. In the case when x is a closed point, a very concrete computation is also given in [12, Lemma 2.1]. The upper semicontinuity statement is then a consequence of the fact that the fiber dimensions of any coherent sheaf on X are Zariski upper semicontinuous, which follows from Nakayama's Lemma, see [82, Example III.12.7.2]. $\square$

The previous theorem allows us to relate the quantity $v_f(x)$ to the multiplicities $m_f(z)$ of general closed points z specializing x, but we have still not related the multiplicity $m_f(x)$ to the multiplicities $m_f(z)$ of points z specializing x. This we will do in Proposition 2.3.7 before closing out the section.

Lemma 2.3.6. *There is a nonempty Zariski open subset $U \subseteq Y$ such that any closed point $y \in U$ has exactly $[X :_f Y]_s$ preimages in X.*

Proof. Without loss of generality, we may assume that X and Y are both affine and smooth, with coordinate rings $k[Y] \subseteq k[X]$. Let L be the unique intermediate field $k(Y) \subseteq L \subseteq k(X)$ such that L is separable over $k(Y)$ and $k(X)$ is purely inseparable over L. Let A be the integral closure of $k[Y]$ in L. Then A is the coordinate ring of some irreducible affine variety by [57, Corollary 13.13], and the inclusions $k[Y] \subseteq A \subseteq k[X]$ induce morphisms

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[rr, bend left, "f"] & Z \arrow[r, "h"'] & Y. \end{tikzcd}$$

We first note that g must be injective, or, in terms of rings, every maximal ideal $\mathfrak{m}$ of A has exactly one maximal ideal $\mathfrak{n}$ of $k[X]$ lying over it. Indeed, suppose that $\mathfrak{n}, \mathfrak{n}'$ were two maximal ideals of $k[X]$ lying over $\mathfrak{m}$. Let $g \in \mathfrak{n}$. Since $k(X)$ is purely inseparable over L, the minimal polynomial of g over L is of the form $t^{p^m} - a$, where $a \in \mathfrak{m}$, where of course $p = \operatorname{char}(k) > 0$. Thus $b^{p^m} = a \in \mathfrak{m} \subseteq \mathfrak{n}'$, so that $b \in \mathfrak{n}'$. It follows that $\mathfrak{n} = \mathfrak{n}'$.

As g is injective, it suffices to prove the theorem for h. In other words, we may without loss of generality assume $[X :_f Y]_i = 1$. In this case, the lemma is proved in [100, §II.6.3]. $\square$

Proposition 2.3.7. *Let $E \subseteq X$ be an irreducible closed subvariety with generic point z, and let $F = f(E)$. Then there is a nonempty Zariski open subset U of F such that for all $y \in U$,*

$$m_f(z) = \sum_{x \in f^{-1}(y) \cap E} m_f(x).$$

Proof. Applying Theorem 2.3.5 and Lemma 2.3.6, there is a nonempty Zariski open subset U of F with the following two properties:

1. If $w \in U$ is a closed point, then w has exactly $[E :_f F]_s$ preimages in E.

2. If $y \in U$ is a (not necessarily closed) point, then $v_f(z) = v_f(x)$ for all $x \in f^{-1}(y) \cap E$.

Suppose that $y \in U$ is any point, and let $x_1, \dots, x_r$ be the preimages of y that lie in E. Set $W = \overline{\{y\}}$ and $V_i = \overline{\{x_i\}}$ for each i. By another application of Lemma 2.3.6, we can find some closed point $w \in W \cap U$ such that $f^{-1}(w) \cap E \subseteq V_1 \cup \cdots \cup V_r$, no preimage of w lies in two distinct V_i, and the number of preimages of w in each V_i is exactly $[V_i :_f W]_s$. But then

$$\begin{aligned}\sum_{i=1}^{r} m_f(x_i) &= \sum_{i=1}^{r} v_f(x_i)[V_i :_f W]_s = v_f(z) \sum_{i=1}^{r} [V_i :_f W]_s = v_f(z) \times \#(f^{-1}(w) \cap E) \\ &= v_f(z)[E :_f F]_s = m_f(z).\end{aligned}$$

This completes the proof. ☐

2.4. Measures on classical varieties

In order to generalize the complex equidistribution of preimages theorem to endomorphisms of varieties defined over general algebraically closed fields k, we need to be able to talk about measures and weak convergence of measures on these varieties. In the complex setting, the natural measures to consider are Radon measures for the analytic topology, but over arbitrary k there is no analytic topology. Instead, we use the only topology we have available to us, namely the Zariski topology. In this section we will discuss Borel measures on varieties in the Zariski topology, what it means for such measures to converge weakly, and define a pullback operation on measures. A full, self-contained development of these topics would take us pretty far afield from our current focus, however, so we will only outline the relevant results here, and refer to Appendix A and the author's article [74] for details.

We will stay in the same setup as the rest of this chapter, namely, we fix a finite flat surjective morphism $f : X \to Y$ between irreducible varieties over k. It should be noted that in this section it is *absolutely essential* to view X and Y as schemes, allowing for non-closed points, as otherwise Theorems 2.4.1 and 2.4.2 below can fail.

We denote by $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ the real vector spaces of all finite signed Borel measures on X and Y with respect to their Zariski topologies. We let $SC(X)$ denote the real vector space of all *semicontinuous functions* on X, by which we mean functions $g : X \to \mathbf{R}$ of the form $g = h_1 - h_2$, where the $h_i : X \to \mathbf{R}$ are bounded upper semicontinuous functions.

Similarly, $SC(Y)$ will denote the space of semicontinuous functions on Y. We equip both $SC(X)$ and $SC(Y)$ with the supremum norm, making them normed linear spaces.

Theorem 2.4.1 ([74] or §A.2)**.** *We have the following characterization of measures on X.*

1. *Any $\mu \in \mathcal{M}(X)$ can be written uniquely as an absolutely convergent sum $\mu = \sum_{x\in X} c_x\delta_x$, where $c_x \in \mathbf{R}$ for each x, and δ_x denotes the Dirac probability measure at x.*

2. *Integration induces a duality $\mathcal{M}(X) \cong SC(X)^*$, analogous to the duality between Radon measures and continuous functions on a compact Hausdorff space.*

Here $SC(X)^*$ is the continuous dual space of $SC(X)$.

The space $SC(X)^*$ has two useful topologies, the *strong topology* induced by dual norm on $SC(X)^*$, and the *weak topology*, which is simply the topology of pointwise convergence. The isomorphism $\mathcal{M}(X) \cong SC(X)^*$ given in Theorem 2.4.1 allows us to push these topologies forward to $\mathcal{M}(X)$. The weak topology on $\mathcal{M}(X)$ has a particularly simple interpretation in terms of measures: a sequence μ_n of measures converges weakly to a measure μ if and only if $\mu_n(E) \to \mu(E)$ as $n \to \infty$ for all closed sets $E \subseteq X$.

Theorem 2.4.2 ([74])**.** *The collection of Borel probability measures on X is both compact and sequentially compact in the weak topology.*

The first part of Theorem 2.4.2 (compactness) is an immediate corollary of the Banach-Alaoglu theorem, which is itself a nontrivial consequence of the axiom of choice. The second part of the theorem (sequential compactness) does not follow in any obvious way from compactness, and also requires a technical axiom of choice argument.

Now that we have a language of measures on X and Y, we want to define a pullback operation on measures, similar to that defined over the complex numbers. This is the content of the final proposition of the chapter.

Proposition 2.4.3. *There is a unique linear operator $f^*\colon \mathcal{M}(Y) \to \mathcal{M}(X)$ which satisfies the following two conditions:*

1. *f^* is continuous in both the weak and strong topologies.*

2. *If $y \in Y$, then $f^*\delta_y = \sum_{f(x)=y} m_f(x)\delta_x$.*

For any $\mu \in \mathcal{M}(Y)$, one has $f_*f^*\mu = [X :_f Y]\mu$, where f_* denotes the ordinary push-forward operation on measures. If μ is positive and has total mass R, then $f^*\mu$ is again positive, and has total mass $[X :_f Y]R$.

Proof. First, assume such an operator f^* does exist. Given a $\mu \in \mathcal{M}(Y)$, we can decompose μ uniquely as $\mu = \sum_{y \in Y} c_y \delta_y$ by Theorem 2.4.1. Let $y_1, y_2, \ldots$ be an enumeration of the points $y \in Y$ such that $c_y \neq 0$; there must be a countable number, as otherwise the sum $\sum c_y \delta_y$ would not converge. The measures μ_N defined by $\mu_N = \sum_{i=1}^N c_{y_i} \delta_{y_i}$ converge strongly to μ as $N \to \infty$, so by conditions 1. and 2. we must have

$$f^*\mu = \lim_{N\to\infty} f^*\mu_N = \lim_{N\to\infty} \sum_{i=1}^N c_{y_i} f^*\delta_{y_i} = \lim_{N\to\infty} \sum_{i=1}^N \sum_{f(x)=y_i} c_{y_i} m_f(x)\delta_x = \sum_{x\in X} c_{f(x)} m_f(x)\delta_x.$$

This derivation shows that f^* is uniquely determined. Moreover, combining this identity with Theorem 2.4.1 yields the remaining statements of the proposition. It then only remains to show the existence of f^*. We could try to *define* f^* using the above identity, but it turns out not to be clear how to show that f^* is continuous. For this reason, we will use a different approach.

To prove existence, we will exploit the duality $\mathcal{M} \cong SC^*$ from Theorem 2.4.1, and define f^* to be the adjoint of a certain linear operator $f_*\colon \overline{SC}(X) \to \overline{SC}(Y)$, where $\overline{SC}$ denotes the Banach space closure of SC, i.e., the space of all functions which are uniform limits of semicontinuous functions. The operator f_* is defined as follows:

$$(f_*\varphi)(y) := \sum_{f(x)=y} m_f(x)\varphi(x).$$

Of course, we must check that f_* does actually map $\overline{SC}(X)$ into $\overline{SC}(Y)$. The continuity of f_*, however, is clear, since Theorem 2.3.4 gives that $\|f_*\varphi\| \leq [X :_f Y]\|\varphi\|$. To prove that f_* maps $\overline{SC}(X)$ into $\overline{SC}(Y)$, it will suffice to prove the following statement: if $E \subseteq X$ is closed and χ_E is the characteristic function of E, then $f_*\chi_E \in SC(Y)$. This is because the vector subspace of $SC(X)$ spanned by all such characteristic functions χ_E is dense in $SC(X)$, and hence also in $\overline{SC}(X)$, see Lemma A.2.5 or [74, Lemma 3.4].

Let T denote the set of all closed sets $E \subseteq X$ for which $f_*\chi_E \notin SC(Y)$. Assume for contradiction that T is not empty, and let $E \in T$ be a minimal element under inclusion. If E were reducible, say with $E = E_1 \cup E_2$ a nontrivial decomposition, then

$$f_*\chi_E = f_*\chi_{E_1} + f_*\chi_{E_2} - f_*\chi_{E_1\cap E_2}$$

lies in $SC(Y)$ by the minimality of E, a contradiction. Therefore E must be irreducible. As is easily seen, the function $f_*\chi_E$ is supported on the closed set $F = f(E)$. Furthermore, by Proposition 2.3.7 there is a nonempty open subset $U \subseteq F$ such that $f_*\chi_E \equiv m_f(E)$ on U. If $V = F \setminus U$ and $W = f^{-1}(V) \cap E$, this says that $f_*(\chi_E - \chi_W) = m_f(E)\chi_U \in SC(Y)$. By the minimality of E, we know that $f_*\chi_W \in SC(Y)$, and therefore

$$f_*\chi_E = f_*(\chi_E - \chi_W) + f_*\chi_W \in SC(Y),$$

a contradiction. We conclude that T must be empty, and, consequently, f_* is a continuous linear map $\overline{SC}(X) \to \overline{SC}(Y)$.

We now define $f^* \colon \mathcal{M}(Y) \to \mathcal{M}(X)$ to be the adjoint of f_*. The continuity of f^* in both the weak and strong topologies is then immediate, and we only need to check that condition 2. holds. Let $y \in Y$ be any point, and let $E \subseteq X$ be closed. Then

$$(f^*\delta_y)(E) = \int f_*\chi_E \, d\delta_y = (f_*\chi_E)(y) = \sum_{x \in f^{-1}(y) \cap E} m_f(x) = \sum_{f(x)=y} m_f(x)\delta_x(E).$$

This shows that $f^*\delta_y$ agrees with the measure $\sum_{f(x)=y} \delta_x$ when evaluated on closed sets. It turns out that this is enough to guarantee $f^*\delta_y$ and $\sum_{f(x)=y} m_f(x)\delta_x$ are in fact the same measure, see Lemma A.2.1 or [74, Lemma 2.7]. $\square$

Chapter 3

Equidistribution of preimages on classical varieties

In Chapter 2 we developed a language of measures on varieties over an algebraically closed field k and a theory of multiplicities for certain morphisms between these varieties. This puts us in a position to state and prove an analogue of the complex equidistribution of preimages theorem over more general fields. While we will prove a general equidistribution theorem for *polarized dynamical systems*, in the simplest case our main result is the following.

Theorem 3.0.1. *Let $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ be an endomorphism of algebraic degree $d \geq 2$ satisfying a mild separability hypothesis. Then there is a proper, totally invariant Zariski closed subset $\mathscr{E}_f \subsetneq \mathbf{P}^r_k$ such that the preimages of any point $x \notin \mathscr{E}_f$ equidistribute to the Dirac mass at the generic point of $\mathbf{P}^r_k$.*

The basic outline of this chapter is as follows. In §3.1 we define and discuss the class of polarized endomorphisms of projective varieties, and describe some of their useful intersection theoretic properties. In §3.2 we show how the generic multiplicity function v_f defined in Chapter 2 can be used to detect totally invariant behavior in our dynamical systems. This will allow us to construct the exceptional set $\mathscr{E}_f$, a nontrivial procedure. In §3.3 we generalize a well-known theorem of Fornæss-Sibony [69] from complex dynamics by showing that sufficiently generic endomorphisms f of projective space have $\mathscr{E}_f = \varnothing$. Finally, in §3.4 we prove several versions of our equidistribution of preimages theorem.

Throughout this chapter, k denotes a fixed algebraically closed field, and X denotes an irreducible projective variety over k. As always, X will be viewed as a scheme. We continue to use the notations of §2.1.

3.1. Polarized dynamical systems

Informally, an endomorphism $f\colon X \to X$ of a projective variety is *polarized* if it is induced by an endomorphism of a projective space $\mathbf{P}^r_k$, in a sense we will make precise shortly. Many of the dynamical systems studied in complex and nonarchimedean dynamics are polarized, since the polarization hypothesis makes available certain tools that otherwise one would not have. For the current problem, the polarization hypothesis allows us to make use of the very simple nature of intersection theory of $\mathbf{P}^r_k$. Specifically, we need it to prove the intersection theoretic results Propositions 3.1.3 and 3.1.4 below.

Definition 3.1.1. Let X be an irreducible projective variety over k, and $f\colon X \to X$ an endomorphism of X. A polarization of f is an ample line bundle L on X such that $f^*L \cong L^d$ for some integer $d \geq 1$. If a polarization L of f is specified, we will say that f is a *polarized dynamical system*, and write $f\colon (X, L) \to (X, L)$ to signify this. The integer d will be called the *algebraic degree* of f, which should not be confused with the degree $[X :_f X]$ of f. Not every endomorphism f admits a polarization.

This definition has the benefit of being intrinsic, if rather abstract. The following theorem of Fakhruddin [59] clarifies the definition, making it very concrete.

Theorem 3.1.2 (Fakhruddin)**.** *Suppose $f\colon (X, L) \to (X, L)$ is a polarized dynamical system of algebraic degree d. Then there is an embedding $X \subseteq \mathbf{P}^r_k$ into a projective space and an endomorphism $F\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ with $F^*\mathcal{O}(1) = \mathcal{O}(d)$ such that $F(X) = X$ and $F|_X = f$.*

Simply put, polarized endomorphisms of algebraic degree d are exactly the restrictions of endomorphisms of $\mathbf{P}^r_k$ of algebraic degree d to invariant subvarieties. An endomorphism $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ is of algebraic degree d if and only if it can be written as $f = [F_0 : \cdots : F_r]$, where the F_i are homogeneous polynomials of degree d with no nontrivial common zeros. As an easy consequence, all polarized dynamical systems are finite.

To prove the next two propositions we will need to use some basic intersection theory. For an overview of ample line bundles and intersection theory, we refer to [38]. Given an ample line bundle L on X and an irreducible dimension q subvariety $E \subseteq X$, the *L-degree* of E, denoted $\deg_L E$, is the intersection number $\deg_L E = (E \cdot L \cdot \ldots \cdot L)$, where here there are q factors of L. If $s_1, \ldots, s_q$ are general enough divisors representing L, then $\deg_L E$ is exactly the number of points in the intersection $E \cap \mathrm{Div}(s_1) \cap \cdots \cap \mathrm{Div}(s_q)$, counted with multiplicity. The next proposition gives the relation between the algebraic degree of a polarized dynamical system $f\colon (X, L) \to (X, L)$ and the degree $[X :_f X]$ of f.

Proposition 3.1.3. *Suppose $f\colon (X, L) \to (X, L)$ is a polarized dynamical system of algebraic degree d. Let $E \subseteq X$ be an irreducible subvariety of dimension q such that $f^n(E) = E$ for some $n \geq 1$. Then $[E :_{f^n} E] = d^{nq}$. In particular, $[X :_f X] = d^{\dim X}$.*

Proof. The projection formula [38, p.73, Remark (f)] gives that $[E :_{f^n} E] \deg_L E = \deg_{f^{n*}L} E$. The polarization hypothesis tells us that $f^{n*}L = L^{d^n}$, and thus

$$[E :_{f^n} E] \deg_L E = \deg_{f^{n*}L} E = \deg_{L^{d^n}} E = d^{nq} \deg_L E.$$

Thus $[E :_{f^n} E] = d^{nq}$, as desired. $\square$

The next proposition gives a bound on the exponential growth rate of the (separable) degree of f along subvarieties of a fixed subvariety $W \subseteq X$, in terms of the dimension of W.

Proposition 3.1.4. *Suppose $f: (X, L) \to (X, L)$ is a polarized dynamical system of algebraic degree d. Let $W \subseteq X$ be an irreducible subvariety of dimension q, and let F be an irreducible subvariety of $f^n(W)$. Let $E_1, \ldots, E_m$ be the components of $f^{-n}(F)$ contained in W. Then there is a $C > 0$ depending only on L and q such that*

$$\sum_{i=1}^{m} [E_i :_{f^n} F]_s \leq C d^{nq}.$$

Proof. Replacing L by a power L^s, we may assume with no loss of generality that L is very ample. We first prove the inequality in the case when $F = x$ is a closed point of $f^n(W)$. Let $s_1, \ldots, s_q$ be sections of L such that $f^n(W) \cap \mathrm{Div}(s_1) \cap \cdots \cap \mathrm{Div}(s_q)$ is finite and contains x. Then one has

$$\# f^{-1}(x) \cap W \leq \# W \cap f^{n*}\mathrm{Div}(s_1) \cap \cdots \cap f^{n*}\mathrm{Div}(s_q) \leq \deg_{f^{n*}L} W = d^{nq} \deg_L W.$$

We may therefore take $C = \deg_L W$. To prove the general case, we use Lemma 2.3.6 to find a nonempty open subset U of F with the following property: if x is a closed point of U, then every element of $f^{-n}(x)$ lies in exactly one E_i, and moreover $\# f^{-n}(x) \cap E_i = [E_i :_{f^n} F]_s$. But then if $x \in U$ is a closed point, $\sum_i [E_i :_{f^n} F]_s = \# f^{-n}(x) \cap W \leq d^{nq} \deg_L W$ by what we proved for closed points. We therefore take $C = \deg_L W$ again. $\square$

3.2. Detecting total invariance

Let us now fix a polarized dynamical system $f: (X, L) \to (X, L)$ of algebraic degree $d \geq 2$. We will also assume that f is flat, so that we can apply all the results of Chapter 2. In this section, we will use the generic multiplicity function v_f to detect totally invariant subvarieties of X. Recall that a subset $A \subseteq X$ is *totally invariant* if $f^{-1}(A) = A$. This condition is strictly stronger than ordinary invariance $f(A) = A$. We will say that an irreducible closed set $E \subseteq X$ is part of a *totally invariant cycle* for f if E is totally invariant for some iterate f^n of f. In this case $F := E \cup f(E) \cup \cdots \cup f^{n-1}(E)$ is totally invariant for f, and f permutes the irreducible components of F cyclically.

In the complex setting, it has long been recognized that totally invariant subvarieties of X are those subvarieties on which the generic multiplicity functions v_{f^n} grow as quickly as possible. There are essentially two different, though very related, techniques for showing that high growth rates for v_{f^n} imply the existence of totally invariant subvarieties. The first shows that if the multiplicities v_f are high along the *forward orbit* of a point $x \in X$, then x must belong to an exceptional subvariety. This is essentially the technique used by Briend-Duval [28, 29] in their original proof of the complex equidistribution of preimages theorem. A more recent technique is to characterize points in totally invariant subvarieties as those on which the multiplicities v_f are high along *reverse orbits*. In dimension > 1, this technique was developed by Dinh [47]. This is the approach we will take here. Following Dinh, we make the following definition.

Definition 3.2.1. For each point $y \in X$ and each $n \geq 1$, define

$$v_{-n}(y) := \max_{f^n(x)=y} v_{f^n}(x) \quad \text{and} \quad v_-(y) := \lim_{n\to\infty} [v_{-n}(y)]^{1/n}.$$

The function $v_- \colon X \to \mathbf{N}$ will be called the *reverse asymptotic multiplicity* function for f, since it gives an upper bound on the asymptotic growth rate of the function v_{f^n} along reverse orbits. As in Chapter 2, it will sometimes be convenient to write $v_-(E)$ in place of $v_-(y)$ when $E = \overline{\{y\}}$.

We know from Theorem 2.3.5 that each of the generic multiplicity functions v_{f^n} are Zariski upper semicontinuous. Rather surprisingly, so is the reverse asymptotic multiplicity function v_-.

Theorem 3.2.2 (Theorem A.3.5, see also [47, 74])**.** *For each point $y \in X$, the limit $v_-(y)$ exists. Moreover, $v_- \colon X \to \mathbf{R}$ is Zariski upper semicontinuous.*

In order to proceed any further, we will need to impose one additional technical assumption on our dynamical system f to rule out complications resulting from inseparability in f when working over fields of positive characteristic.

Assumption 3.2.3. We assume that whenever $E \subseteq X$ is an irreducible subvariety which is periodic for f, say with period n, one has $[E :_{f^n} E]_i = 1$.

Proposition 3.2.4. *If* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) = p \nmid d$*, then Assumption 3.2.3 is automatically satisfied.*

Proof. The proposition is clear when $\mathrm{char}(k) = 0$, so assume $\mathrm{char}(k) = p > 0$ and $p \nmid d$. By Proposition 3.1.3, $[E :_{f^n} E] = d^{n \dim(E)}$. Since $p \nmid d$, it follows that the field extension $k(E)/f^{n*}k(E)$ must be separable. □

Under this assumption, we are now in a position to show how the reverse asymptotic multiplicity function v_- picks out the totally invariant subvarieties of X.

Theorem 3.2.5. *Suppose $f\colon (X, L) \to (X, L)$ is a polarized dynamical system of algebraic degree $d \geq 2$ which is flat and satisfies Assumption 3.2.3. Let E be an irreducible subvariety of codimension q in X. Then $v_-(E) \leq d^q$, with equality if and only if E is part of a totally invariant cycle for f.*

Proof. For any n-periodic irreducible subvariety $F \subseteq X$, denote by $v_+(F)$ the quantity $v_+(F) := v_{f^n}(F)^{1/n}$. Note, this is independent of the choice of period n by Proposition 2.3.3. Using Theorem A.3.5, one has the identity $v_-(E) = \max v_+(F)$, where the maximum is taken over all periodic irreducible subvarieties $F \subseteq X$ which contain E. Fix a periodic subvariety F for which the maximum is attained, and suppose it has period n. We then derive that

$$v_-(E) = v_+(F) = v_{f^n}(F)^{1/n} = \left(\frac{m_{f^n}(F)}{[F :_{f^n} F]_s}\right)^{1/n} \leq \left(\frac{[X :_{f^n} X]}{[F :_{f^n} F]_s}\right)^{1/n}, \tag{3.1}$$

where the third equality comes from Proposition 2.3.2 and the inequality is a consequence of Theorem 2.3.4. Moreover, the inequality is strict unless $m_{f^n}(F) = [X :_{f^n} X]$, which by Theorem 2.3.4 happens if and only if F is totally invariant for f^n. Using Assumption 3.2.3, we have $[F :_{f^n} F]_s = [F :_{f^n} F]$. If we then apply Proposition 3.1.3, we see $[F :_{f^n} F] = d^{n \dim F}$ and $[X :_{f^n} X] = d^{n \dim X}$. Thus Equation 3.1 becomes

$$v_-(E) = v_+(F) \leq d^{\operatorname{codim} F},$$

with equality if and only if F is totally invariant for f^n. Since $E \subseteq F$ and both are irreducible, $q \geq \operatorname{codim} F$ with equality if and only if $E = F$. We conclude $v_-(E) \leq d^q$, with equality if and only if $E = F$ is totally invariant for f^n. ☐

Corollary 3.2.6. *There are only finitely many irreducible subvarieties $E \subseteq X$ that are part of a totally invariant cycle for f.*

Proof. It is enough to prove there are only finitely many such E of a fixed codimension q. Indeed, by Theorem 3.2.5, the codimension q irreducible subvarieties of X that are part of a totally invariant cycle are precisely the codimension q components of the Zariski closed set $\{x \in X : v_-(x) \geq d^q\}$. Note we are heavily using Theorem 3.2.2 here. ☐

Definition 3.2.7. The *exceptional set* of f is the set $\mathscr{E}_f \subseteq X$ which is the union of all totally invariant proper closed subsets $E \subsetneq X$. Under the hypotheses of Theorem 3.2.5, Corollary 3.2.6 says precisely that this union is finite, so that $\mathscr{E}_f$ is itself a totally invariant proper closed subset of X. It is thus the maximal such subset.

It should be noted that Assumption 3.2.3 cannot be removed from Theorem 3.2.5. Indeed, for the Frobenius map $f\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ over $k = \overline{\mathbf{F}}_p$ defined by $f[x:y] = [x^p : y^p]$, one easily checks that *every* point of $\mathbf{P}^1_k$ is part of a totally invariant cycle for f. In particular, there is no maximal proper Zariski closed subset of $\mathbf{P}^1_k$ which is totally invariant. This shows that the techniques developed in the complex setting for detecting total invariance can fail in characteristic p in the presence of inseparable behavior.

3.3. Generic dynamical systems have empty exceptional set

In complex dynamics, the geometry of the exceptional set $\mathscr{E}_f$ of a polarized dynamical system is rather mysterious. In dimension 1, the exceptional set is completely understood: if $f\colon \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ is a rational map of degree $d \geq 2$, then $\mathscr{E}_f$ consists of at most 2 points, and we have an explicit characterization of when f has nonempty exceptional set. This picture generalizes to fields k of positive characteristic as well, though in this setting it is possible that the exceptional set be infinite when f is conjugate to an iterate of the Frobenius map, as we discussed at the end of §3.2.

The situation in dimension 2 is still fairly well understood. If $f\colon \mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^2_{\mathbf{C}}$ is an endomorphism of algebraic degree $d \geq 2$, then the irreducible components of the exceptional set $\mathscr{E}_f$ are always linear subspaces of $\mathbf{P}^2_{\mathbf{C}}$, and $\mathscr{E}_f$ can contain up to 3 lines [81]. Once we move to dimensions greater than 2, however, very little has been said. It is conjectured that for any endomorphism $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of algebraic degree $d \geq 2$, the exceptional set $\mathscr{E}_f$ will be a union of linear subspaces. Some partial results towards this conjecture have been proved. For instance, any totally invariant hypersurface of $\mathbf{P}^r_k$ that is *smooth* must be a hyperplane [33, 5]. In dimension $r = 3$, recent work of De-Qi Zhang [114] has shown that totally invariant hypersurfaces of $\mathbf{P}^3_k$ are either hyperplanes or one of 4 singular cubics. It should also be noted that a published proof of the general conjecture [26] is considered to be incomplete, though not necessarily incorrect.

One of the central difficulties in studying the exceptional set is that it is difficult to write very many dynamical systems f with nonempty exceptional set. Having totally invariant subvarieties is apparently a very restrictive property. This qualitative statement was first made quantitative by Fornæss-Sibony [69], who proved that suitably "generic" endomorphisms $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of a fixed algebraic degree d have $\mathscr{E}_f = \varnothing$. This section is devoted to generalizing this result to dynamical systems over arbitrary algebraically closed fields k. Instead of modeling our approach on the proof in [69], we give a proof in the spirit of [51, Theorem 1.3], which in turn builds off a construction of Ueda [110]. What we prove here must certainly be well-known, but a convenient reference is lacking.

We first must make precise what is meant by "generic" dynamical systems. For the rest of the section, fix once and for all an algebraically closed field k, a dimension $r \geq 1$, and an integer $d \geq 2$. We will consider only endomorphisms $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of algebraic degree d. Any such f is of the form $f = [F_0 : \cdots : F_r]$, where the F_i are homogeneous polynomials of degree d with no nontrivial common zeros. This allows us to parameterize all such maps f by the coefficients of the polynomials F_i. Of course, the coefficients of the F_i are only determined up to a scalar factor. Thus endomorphisms $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of algebraic degree d are parameterized by a subset $\mathcal{H}_d \subseteq \mathbf{P}^N_k$ of a large enough projective space, where here N is one less than the total number of coefficients of the F_i.

Observe that the set $\mathcal{H}_d$ consists of exactly those coefficients for which the F_i have no nontrivial common zeros. That is, $\mathcal{H}_d$ is the complement of the hypersurface in $\mathbf{P}^N_k$ defined by the resultant $\operatorname{Res}(F_0, \dots, F_r)$ of the polynomials F_i. In particular, $\mathcal{H}_d$ naturally has the structure of an irreducible smooth affine variety of dimension N. Somewhat abusively, we will consider our maps f as being (closed) *points* of $\mathcal{H}_d$. We will prove that $\mathscr{E}_f = \varnothing$ for all f lying outside of a proper Zariski closed set of $\mathcal{H}_d$; this is what we mean by "generic" f having empty exceptional set. For more information about the parameter space $\mathcal{H}_d$ and related topics, see the recent book [103].

Proposition 3.3.1. *Let $v\colon \mathcal{H}_d \times \mathbf{P}^r_k \to \mathbf{R}$ be the map, defined on closed points, that is given by $v(f, x) = v_f(x)$. Then v is Zariski upper semicontinuous.*

Proof. Let $X \subseteq \mathcal{H}_d \times \mathbf{P}^r_k \times \mathbf{P}^r_k$ be the subvariety $X = \{(f, x, y) : f(x) = f(y)\}$, and let $\mathcal{I}$ denote the ideal sheaf of $\Delta = \{(f, x, y) : x = y\}$ in X. We will denote by $\mathcal{F}$ the sheaf $\mathcal{O}_X/\mathcal{I}^N$, where N is any integer $\geq d^r$. Let $\pi\colon X \to \mathcal{H}_d \times \mathbf{P}^r_k$ denote the projection onto the first two coordinates. For any fixed $f \in \mathcal{H}_d$, one obtains embeddings $i_f\colon \mathbf{P}^r_k \times_f \mathbf{P}^r_k \to X$ and $j_f\colon \mathbf{P}^r_k \to \mathcal{H}_d \times \mathbf{P}^r_k$, namely $i_f(x, y) = (f, x, y)$ and $j_f(x) = (f, x)$. If $\eta\colon \mathbf{P}^r_k \times_f \mathbf{P}^r_k \to \mathbf{P}^r_k$ is the projection onto the first coordinate, then $\pi i_f = j_f \eta$. We saw in the proof of Theorem 2.3.5 that $v_f(x)$ is exactly the fiber dimension of $\eta_* j_f^* \mathcal{F}$ at $x \in \mathbf{P}^r_k$. Therefore $v_f(x)$ is also the fiber dimension of $i_f^* \pi_* \mathcal{F}$ at x, which is exactly the fiber dimension of $\pi_* \mathcal{F}$ at $i_f(x) = (f, x)$. We have shown that the fiber dimension of $\pi_* \mathcal{F}$ at (f, x) is $v_f(x)$ for any $f \in \mathcal{H}_d$ and $x \in \mathbf{P}^r_k$. Since $\pi_* \mathcal{F}$ is a coherent sheaf on $\mathcal{H}_d \times \mathbf{P}^r_k$, its fiber dimensions are Zariski upper semicontinuous. $\square$

Corollary 3.3.2. *For any $a \in \mathbf{R}$, the set of $f \in \mathcal{H}_d$ such that $v_f(x) < a$ for all $x \in \mathbf{P}^r_k$ is Zariski open.*

Proof. The set of $f \in \mathcal{H}_d$ for which there exists point $x \in \mathbf{P}^r_k$ with $v_f(x) \geq a$ is the image under the projection map $\pi\colon \mathcal{H}_d \times \mathbf{P}^r_k \to \mathcal{H}_d$ of the closed set $\{(f, x) : v(f, x) \geq a\}$. The corollary then follows from the fact that π is closed, see [100, Theorem I.5.3]. $\square$

Proposition 3.3.3. *Let $f \in \mathcal{H}_d$, and suppose there is an integer $N \geq 1$ such that $v_{f^N}(x) < d^N$ for all points $x \in \mathbf{P}^r_k$. Then $\mathscr{E}_f = \varnothing$.*

Proof. Replacing f by an iterate if necessary, we may assume that $N = 1$ and that all irreducible components of $\mathscr{E}$ are totally invariant. If E is such a component, then one would have that $v_-(E) = v_f(E) = d^{\operatorname{codim}(E)}[E :_f E]_i \geq d$. Using the upper semicontinuity of v_f, this implies that $v_f(x) \geq d$ for every closed point $x \in E$, a contradiction of our assumption that $v_f(x) < d$ for all $x \in \mathbf{P}^r_k$. Thus we must have $\mathscr{E}_f = \varnothing$. $\square$

Combining Corollary 3.3.2 with Proposition 3.3.3, it follows that if we can find *just one* $f \in \mathcal{H}_d$ such that $v_{f^N}(x) < d^N$ for some $N \geq 1$ and all $x \in \mathbf{P}^r_k$, then there will be a whole nonempty Zariski open subset of $\mathcal{H}_d$ of such maps, and each will have empty exceptional set. We have therefore reduced the problem to constructing a single map f. We will first do this in dimension $r = 1$, and then we will lift this to the $r > 1$ case via the idea of Ueda previously mentioned.

Theorem 3.3.4. *There is an endomorphism $h\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ of degree d and a constant $B > 0$ such that $v_{h^n}(x) \leq B$ for all $n \geq 1$ and all $x \in \mathbf{P}^1_k$.*

Proof. Suppose first that $d \neq p^m$, where $p = \operatorname{char}(k) > 0$. Then there is an $a \in k^\times$ such that $(a+1)^d = 1$. Define $h\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ by $h(z) = (z+a)^d/z^d$. This map h satisfies the conditions of the theorem, since h has two critical points of order $d-1$, namely 0 and $-a$, and both are strictly preperiodic. It follows that $v_{h^n}(z) \leq d^2$ for all $n \geq 1$ and all $z \in \mathbf{P}^1_k$. Next, assume that $\operatorname{char}(k) = p > 0$ and $d = p^m$ for some $m \geq 1$. In this case, a similar argument holds for the map $h(z) = (z+1)^d/z^{d-1}$. This h again has two critical points 0 and -1, which have orders $d-2$ and $d-1$, respectively. Both are strictly preperiodic, so $v_{h^n}(z) \leq d(d-1)$ for all $n \geq 1$ and $z \in \mathbf{P}^1_k$. $\square$

In order to lift this theorem to dimensions $r > 1$, we will use the geometric observation that $\mathbf{P}^r_k$ can be obtained as the quotient of the r-fold product $\mathbf{P}^1_k \times \cdots \times \mathbf{P}^1_k$ by the natural action of the symmetric group S_r, given by permutation of coordinates. Moreover, in this case the quotient morphism $\pi\colon \mathbf{P}^1_k \times \cdots \times \mathbf{P}^1_k \to \mathbf{P}^r_k$ is rather explicit: the jth homogeneous coordinate of $\pi([x_1 : y_1], \ldots, [x_r : y_r])$ is $(-1)^j y_1^{r-j} \cdots y_r^{r-j} \Sigma_j(x_1/y_1, \ldots, x_r/y_r)$ for $0 \leq j \leq r$, where Σ_j is the elementary symmetric polynomial in r variables of degree j. We see that π is a finite morphism of degree $r!$. Moreover, if $h\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ is an endomorphism of degree d, then the product endomorphism $H = h \times \cdots \times h$ of $\mathbf{P}^1_k \times \cdots \times \mathbf{P}^1_k$ descends via π to an endomorphism $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of algebraic degree d.

Theorem 3.3.5. *There is an endomorphism $f\colon \mathbf{P}^r_k \to \mathbf{P}^r_k$ of algebraic degree d and a constant $C > 0$ such that $v_{f^n}(x) \le C$ for all $n \ge 1$ and all $x \in \mathbf{P}^r_k$.*

Proof. Let h and B be as given in Theorem 3.3.4. Let $H = h \times \cdots \times h$ be the r-fold product of h, and let f be the quotient of H under the action of S_r. It is straightforward to check that for any point $z = (z_1, \ldots, z_r) \in \mathbf{P}^1_k \times \cdots \times \mathbf{P}^1_k$, one has $v_H(z) = v_h(z_1) \cdots v_h(z_r)$. Let $x \in \mathbf{P}^r_k$, and let $z = (z_1, \ldots, z_r) \in \pi^{-1}(x)$ be any lift of x. Then $v_{f^n \circ \pi}(z) = v_\pi(z) v_{f^n}(x)$ by Proposition 2.3.3. On the other hand, $f^n \circ \pi = \pi \circ H^n$, giving

$$v_{f^n}(x) = \frac{v_{f^n}(x) v_\pi(z)}{v_\pi(z)} = \frac{v_{H^n}(z) v_\pi(H^n(z))}{v_\pi(z)}.$$

Since $v_{H^n}(z) = v_{h^n}(z_1) \cdots v_{h^n}(z_r) \le B^r$ and $v_\pi(H^n(z)) \le r!$ by Theorem 2.3.4, we conclude that $v_{f^n(x)} \le B^r r!$ for all $n \ge 1$ and all $x \in \mathbf{P}^r_k$. □

3.4. The equidistribution theorem

We are now ready to state and prove one of the central results in this thesis, the equidistribution of preimages theorem for classical varieties. While much of the work done to this point has been a (nontrivial) generalizing of complex methods to arbitrary fields, the proof of the equidistribution theorem below is unlike any in the complex setting, since our setup is quite different.

Theorem 3.4.1 (Equidistribution for classical varieties)**.** *Let k be an algebraically closed field, and let X be an irreducible projective variety of dimension r over k. Suppose that $f\colon (X, L) \to (X, L)$ is a flat polarized endomorphism of X with algebraic degree $d \ge 2$ which satisfies Assumption 3.2.3. Let $x \in X$ be any (not necessarily closed) point of X, and let $V \subseteq X$ be the smallest totally invariant closed set containing x. Assume that V is irreducible with generic point y. Then the sequence of probability measures $d^{-rn} f^{n*}\delta_x$ on X converges weakly to δ_y as $n \to \infty$.*

As a special case of this theorem, if $x \notin \mathscr{E}_f$ then $V = X$ and y is the generic point of X, so the preimages of x equidistribute to the Dirac mass at the generic point of X. This is exactly what was stated in Theorem 3.0.1. Theorem 3.4.1 is stronger than Theorem 3.0.1, since it says what the preimages of x equidistribute to even when $x \in \mathscr{E}_f$, at least under the hypothesis that V is irreducible. We will consider the case when V is reducible in Corollary 3.4.3.

Proof of Theorem 3.4.1. For simplicity, let μ_n denote the measure $d^{-rn} f^{n*}\delta_x$ for each $n \ge 1$. Recall from Theorem 2.4.2 that the space of Borel probability measures on X is sequentially compact in the weak topology. It therefore suffices to prove that any weakly convergent subsequence μ_{n_i} of the μ_n converges to δ_y. We therefore fix a weakly convergent subsequence μ_{n_i}, converging to some measure μ. Since X is Noetherian, there is a minimal closed set W such that $\mu(W) > 0$. If W were reducible, say $W = W_1 \cup W_2$, then by the minimality of W we would have $\mu(W) \leq \mu(W_1) \cup \mu(W_2) = 0$, a contradiction. Therefore W is irreducible. One easily sees that $\mu_n(V) = 1$ for all n, and hence in the limit $\mu(V) = 1$. In particular, $\mu(W \cap V) = \mu(W) > 0$, so the minimality of W implies that $W \subseteq V$.

To prove the theorem, it will suffice to show that W is part of a totally invariant cycle for f. Indeed, if we can do this, then the minimality of V implies $W = V$. But then $\mu(V) = 1$ and $\mu(Z) = 0$ for all closed sets $Z \subsetneq V$, implying that $\mu = \delta_y$, as desired. We will prove that W is part of a totally invariant cycle for f by contradiction. Suppose that W is *not* part of a totally invariant cycle for f. By Theorem 3.2.5, one then has $v_-(W) < d^q$, where q is the codimension of W in X. We need the following lemma to proceed.

Lemma 3.4.2. *There is an integer $I \geq 0$ and a preimage $z \in f^{-n_I}(x)$ such that*

1. *$z \in W$ and $v_-(z) < d^q$.*

2. *$\limsup_{i\to\infty} d^{-r(n_i-n_I)}[f^{(n_i-n_I)*}\delta_z](W) > 0$.*

Proof. Recall from Theorem 3.2.2 that the reverse asymptotic multiplicity function v_- is Zariski upper semicontinuous. Since $v_-(W) < d^q$, it follows that there is a nonempty open subset U of W such that $v_- < d^q$ on U. Again by the minimality of W, one has $\mu(W) = \mu(U) = \lim_{i\to\infty} \mu_{n_i}(U)$. We will prove the lemma by contradiction, so suppose no such z and I exist. To simplify notation, set

$$R(z, I) := \limsup_{i\to\infty} d^{-r(n_i-n_I)}[f^{(n_i-n_I)*}\delta_z](U)$$

whenever $I \geq 0$ is an integer at $z \in f^{-n_I}(x)$. Note that $R(z, I) \leq 1$, and by our contradiction assumption $R(z, I) = 0$ whenever $z \in U$.

Claim: If $I \geq 0$ and $z \in f^{-n_I}(x)$ are such that $R(z, I) \geq c > 0$, then there is an integer $J > I$ and a preimage $z' \in f^{-(n_J-n_I)}(z)$ such that $R(z, I) \leq (1 - c/2)R(z', J)$. To prove the claim, let $J > I$ be any integer large enough that $d^{-r(n_J-n_I)}[f^{(n_J-n_I)*}\delta_z](U) \geq c/2$. Suppose that $z_1, \ldots, z_s$ are the element of $f^{-(n_J-n_I)}(z) \cap U$, and that $z_{s+1}, \ldots, z_t$ are the elements of $f^{-(n_J-n_I)}(z)$ lying outside U. Then

$$R(z, I) \leq \sum_{i=1}^{t} \frac{m_{f^{n_J-n_I}}(z_i)}{d^{r(n_J-n_I)}} R(z_i, J) = \sum_{i=s+1}^{t} \frac{m_{f^{n_J-n_I}}(z_i)}{d^{r(n_J-n_I)}} R(z_i, J),$$

where the last equality is because $R(z_i, J) = 0$ for all $i \leq s$. We then have the easy upper bound

$$\begin{aligned} R(z, I) &\leq \max\{R(z_{s+1}, J), \dots, R(z_t, J)\} \sum_{i=s+1}^{t} \frac{m_{f^{n_J - n_I}}(z_i)}{d^{r(n_J - n_I)}} \\ &= \max\{R(z_{s+1}, J), \dots, R(z_t, J)\} d^{-r(n_J - n_I)} [f^{(n_J - n_I)*} \delta_z](X \smallsetminus U). \end{aligned}$$

It then follows from our choice of J that

$$R(z, I) \leq (1 - c/2) \max\{R(z_{s+1}, J), \dots, R(z_t, J)\},$$

which proves the claim.

Let $c = \mu(W)$. By definition, $R(x, 0) = c$, so the claim yields an integer $I_1 > 0$ and a preimage $z_1 \in f^{-n_I}(x)$ such that $\mu(W) = c = R(x, 0) \leq (1 - c/2) R(z_1, I_1)$. In particular,

$$R(z_1, I_1) \geq \frac{c}{1 - c/2} > c.$$

We can thus apply the claim again to find an integer $I_2 > I_1$ and a $z_2 \in f^{(-n_{I_2} - n_{I_1})}(z_1)$ such that $R(z_1, I_1) \leq (1 - c/2) R(z_2, I_2)$. This gives $\mu(W) = R(x, 0) \leq (1 - c/2)^2 R(z_2, I_2)$. Continuing in this fashion, we construct sequences I_j and z_j such that

$$\mu(W) \leq (1 - c/2)^j R(z_j, I_j) \leq (1 - c/2)^j \to 0.$$

This contradicts the fact that $\mu(W) > 0$, and completes the proof of the lemma. $\square$

We now continue with the proof of Theorem 3.4.1. Let I and z be as in the statement of Lemma 3.4.2. Let $\Delta \in \mathbf{R}$ be such that $v_-(z) < \Delta < d^q$. Passing to a subsequence if necessary, we may assume that the limit

$$c := \lim_{i \to \infty} d^{-r(n_i - n_I)} [f^{(n_i - n_I)*} \delta_z](W) \tag{3.2}$$

exists and is positive. For each $i \geq I$, let $z_1^i, \dots, z_{s_i}^i$ denote the elements of $f^{-(n_i - n_I)}(z)$ which lie in W. Then, using Proposition 2.3.2, the right hand side of Equation 3.2 is

$$\lim_{i \to \infty} d^{-r(n_i - n_I)} \sum_{j=1}^{s_i} m_{f^{n_i - n_I}}(z_j^i) = \lim_{i \to \infty} d^{-r(n_i - n_I)} \sum_{j=1}^{s_i} v_{f^{n_i - n_I}}(z_j^i) [E_j^i :_{f^{n_i - n_I}} E]_s,$$

where $E_j^i = \overline{\{s_j^i\}}$ and $E = \overline{\{z\}}$. Since $v_-(z) < \Delta$, we have $v_{f^{n_i - n_I}}(z_j^i) \leq \Delta^{n_i - n_I}$ for every j whenever i is sufficiently large. Using Proposition 3.1.4,

$$\sum_{j=1}^{s_i} [E_j^i :_{f^{n_i - n_I}} E]_s \leq C d^{(n_i - N_I) \dim(W)}$$

for some $C > 0$ independent of i. Combining these inequalities, we see that

$$c \leq \limsup_{i\to\infty} d^{-r(n_i-n_I)}\Delta^{n_i-N_I}Cd^{(n_i-n_I)\dim(W)} = C\limsup_{i\to\infty}(d^{-q}\Delta)^{n_i-n_I} = 0,$$

where here the last equality results from the fact that $\Delta < d^q$. We have then reached a contradiction, since $c > 0$. We conclude W is totally invariant, completing the proof. $\square$

From this theorem, we are able to deduce a couple of easy variants.

Corollary 3.4.3. *Let f be as in Theorem 3.4.1. Let $x \in X$ be any point, and let V be the smallest totally invariant closed subset of X containing x. Let $V = V_0 \cup \cdots \cup V_{s-1}$ be the irreducible decomposition of V, and let y_i be the generic point of V_i for each i. Then, after relabeling the V_i if necessary, one has for each $i = 0, \ldots, s-1$ that $d^{-r(i+sn)}f^{(i+sn)*}\delta_x \to \delta_{y_i}$ weakly as $n \to \infty$.*

Proof. Without loss of generality, we may assume $x \in V_0$ and that $f(V_i) = V_{i-1}$, the indices taken modulo s. Note, in particular, that $d^{-r}f^*\delta_{y_i} = \delta_{y_{i+1}}$. The set V_0 is totally invariant for the iterate f^s, and is in fact the minimal f^s-totally invariant closed set containing x. Thus by Theorem 3.4.1, $d^{-rsn}f^{sn*}\delta_x \to \delta_{y_0}$ weakly as $n \to \infty$. By Proposition 2.4.3, the pullback operator f^* on measures is weakly continuous. Thus for any $i = 0, \ldots, s-1$ we see that

$$d^{-r(i+sn)}f^{(i+sn)*}\delta_x = d^{-ri}f^{i*}[d^{-rsn}f^{sn*}\delta_x] \to d^{-ri}f^{i*}\delta_{y_0} = \delta_{y_i},$$

as desired. $\square$

Corollary 3.4.4. *Let f be as in Theorem 3.4.1. Let μ be a Borel probability measure on X that gives no mass to the exceptional set $\mathscr{E}_f$ of f. Then $d^{-rn}f^{n*}\mu \to \delta_y$ weakly as $n \to \infty$, where y is the generic point of X.*

Proof. From Theorem 2.4.1, μ can be written as a convergent sum $\mu = \sum_{x\in X} c_x\delta_x$. The condition that μ gives no mass to $\mathscr{E}_f$ says exactly that if $c_x \neq 0$, then $x \notin \mathscr{E}_f$. For any such x, Theorem 3.4.1 says that the preimages of x equidistribute to δ_y. Let $x_1, x_2, \ldots$ be an enumeration of those $x \in X$ with $c_x \neq 0$. For each integer $N \geq 1$, let $\mu_N = \sum_{i=1}^N c_{x_i}\delta_{x_i}$ and $\nu_N = \sum_{i>N} c_{x_i}\delta_{x_i}$. Let $\varepsilon > 0$ be given, and choose N large enough that $\nu_N(X) < \varepsilon$. For any closed set $E \subseteq X$, one then has

$$|d^{-rn}(f^{n*}\mu)(E) - \delta_y(E)| \leq |d^{-rn}(f^{n*}\mu_N)(E) - \delta_y(E)| + \varepsilon$$

for every $n \geq 1$. When n is sufficiently large, however, Theorem 3.4.1 tells us that

$$|d^{-rn}(f^{n*}\mu_N)(E) - \mu_N(X)\delta_y(E)| \leq \varepsilon.$$

Combining this with the previous inequality, we see that

$$|d^{-rn}(f^{n*}\mu)(E) - \delta_y(E)| \leq (1 - \mu_N(X))\delta_y(E) + 2\varepsilon \leq 3\varepsilon.$$

Therefore $d^{-rn}f^{n*}\mu \to \delta_y$ as $n \to \infty$. $\square$

Chapter 4

Berkovich analytic spaces

Starting in this chapter, we move into the nonarchimedean setting, with the eventual goal of proving an equidistribution of preimages theorem for dynamical systems on varieties over nonarchimedean fields. However, as we will discuss shortly, the topology of nonarchimedean fields prevents even the statement of the complex equidistribution of preimages theorem from making sense in this setting, to say nothing about the proof. To overcome these difficulties, it has become standard in nonarchimedean dynamics to work not on ordinary varieties, but instead on *Berkovich analytic varieties.* The primary goal of this chapter is to give a very quick overview of nonarchimedean geometry, with particular emphasis placed on Berkovich analytic geometry, that is suited to our needs.

4.1. Nonarchimedean fields

Before beginning our discussion of Berkovich analytic spaces, let us first briefly review some of the relevant elementary theory of nonarchimedean fields. Recall that an *absolute value* on a field K is a function $|\cdot|\colon K \to \mathbf{R}$ that is positive definite, multiplicative, and satisfies the triangle inequality $|a+b| \leq |a| + |b|$. Any such absolute value induces a metric on K in the standard way, making K both a topological and an algebraic object. A field K equipped with an absolute value is a *valued field.* In this thesis, we will only be concerned with valued fields K that are both algebraically closed and complete as a metric space, as we view these as analogues of the field complex numbers. Some insight into their structure is given by the following theorem of Ostrowski [32, Theorem 1.1].

Theorem 4.1.1 (Ostrowski)**.** *Let K be valued field that is algebraically closed and complete as a metric space. Then exactly one of the following holds.*

1. *$K \cong \mathbf{C}$ in the sense that there is a field isomorphism between K and $\mathbf{C}$ that is also a homeomorphism.*

2. *The absolute value on K satisfies the strong triangle inequality* $|a+b| \leq \max\{|a|, |b|\}$.

In the latter case, the absolute value on K is said to be nonarchimedean.

Therefore, when trying to generalize the complex equidistribution of preimages theorem to dynamical systems defined over other valued fields, we are justified in considering only the *nonarchimedean fields*, that is, valued fields whose absolute value is nonarchimedean.

Example 4.1.2. Common examples of algebraically closed, complete nonarchimedean fields are the following.

1. The field $\mathbf{C}_p$ of *p-adic complex numbers*, where $p \in \mathbf{N}$ is a prime number. This field is constructed similarly to the usual construction of $\mathbf{C}$ from $\mathbf{Q}$, just using the p-adic absolute value instead of the usual Euclidean absolute value. The p-adic absolute value $|\cdot|_p$ on $\mathbf{Q}$ is the nonarchimedean absolute value defined by setting $|a/b|_p := p^{-s}$, where $s = \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$. The completion of $\mathbf{Q}$ with respect to $|\cdot|_p$ is the nonarchimedean field $\mathbf{Q}_p$ of *p-adic numbers.* The field $\mathbf{C}_p$ is the completion of the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ with respect to the (unique) extension of $|\cdot|_p$ to $\overline{\mathbf{Q}}_p$.

2. The field $\mathbf{L}_k$ of *generalized Puiseux series* over an algebraically closed field k, constructed as follows. First, equip the Laurent series field $k((t))$ with the nonarchimedean absolute value defined by $|f(t)| := \exp(-\mathrm{ord}_t(f))$. With respect to this absolute value, $k((t))$ is complete, but not algebraically closed. The absolute value on $k((t))$ extends uniquely to the algebraic closure $\overline{k((t))}$, which is exactly the field of Puiseux series in t over k. Completing $\overline{k((t))}$ with respect to this absolute value yields the field $\mathbf{L}_k$ of generalized Puiseux series over k.

3. All *trivially valued* algebraically closed fields K. Given any algebraically closed field, the trivial absolute value $|\cdot|$ on K defined by $|a| = 1$ for all $a \in K^\times$ and $|0| = 0$ is a nonarchimedean absolute value which respect to which K is obviously complete.

Remark 4.1.3. While at first glance it may seem of little use to study trivially valued fields, we will see that from the standpoint of Berkovich analytic geometry, varieties defined over trivially valued fields actually have very nontrivial geometry. These spaces have appeared in numerous applications, see for instance [64, 65, 17, 88, 76, 75]

Let K be a complete, algebraically closed nonarchimedean field. It follows from the strong triangle inequality that the "closed" unit ball $\{a \in K : |a| \leq 1\}$ is a subring of K; we will call it the *ring of integers* of K and denote it by K°. It is a local ring, whose maximal ideal $\mathfrak{m}_K$ is the "open" unit ball $\mathfrak{m}_K = \{a \in K : |a| < 1\}$ in K. The quotient $k = K^\circ/\mathfrak{m}_K$ is called the *residue field* of K; it is also algebraically closed. The characteristics of K and k may differ. For instance, the residue field of $\mathbf{C}_p$, which has characteristic 0, is the field $\overline{\mathbf{F}}_p$. However, if K has characteristic $p > 0$, then so does k. It is easy to check that the residue field of the generalized Puiseux series field $\mathbf{L}_k$ over an algebraically closed field k is exactly k, whereas the residue field of a trivially valued field K is K itself.

Every coset in the quotient $k = K/\mathfrak{m}_K$ is by definition a translate of $\mathfrak{m}_K$, and thus is itself an "open" ball of radius 1 contained in K°. From this simple observation, one sees that K has very different topological properties than the field of complex numbers:

1. K° is covered by the pairwise disjoint family k of open balls, of which $\mathfrak{m}_K$ is a member. Thus K° is open and $\mathfrak{m}_K$ is closed. More generally, all balls of positive radius in K are both open and closed.

2. K is totally disconnected. Indeed, the connected component of a point $a \in K$ must be contained within any open and closed subset containing a, and hence must be contained in all balls of positive radius around a.

3. Since k is algebraically closed, it is in particular infinite, so K° is covered by an infinite collection of disjoint nonempty open sets. This proves that the "closed" unit ball K° is not compact.

4. A coset $z \in k$ is equal to $a + \mathfrak{m}_K$, where a is *any* representative of the coset. Thus every point within the ball z is a "center" of z. This holds, more generally, for all balls in K. It follows from this that any two balls in K are either disjoint, or one is contained in the other.

These topological properties of course carry over to varieties defined over K. Thus, for instance, any variety over K is totally disconnected and, assuming K is nontrivially valued, not locally compact.

4.2. Obstructions to equidistribution

Fix an algebraically closed, complete nonarchimedean field K, and let $f\colon \mathbf{P}^r_K \to \mathbf{P}^r_K$ be an endomorphism of algebraic degree $d \geq 2$. Naively, one might hope that a direct analogue of the complex equidistribution of preimages theorem holds for f:

Naive Hope. *There is a probability measure μ_f on $\mathbf{P}^r_K$ such that the iterated preimages of every nonexceptional point $x \in \mathbf{P}^r_K$ equidistribute to μ_f.*

There are several serious problems with this hope, however. For instance, equidistribution of preimages is formulated in terms of weak convergence of measures: the probability measures μ_n supported on $f^{-n}(x)$ which weight each preimage according to their multiplicity is supposed to converge *weakly* to μ_f. Weak convergence of measures is a notion that is only valid for Radon measures, which in turn only exist on locally compact Hausdorff spaces. As we saw in the previous section, $\mathbf{P}^r_K$ will not be locally compact, so the notion of weak convergence of measures is simply not available on $\mathbf{P}^r_K$, or indeed on any other variety over K of positive dimension.

An even more serious problem is the following. In the complex setting, the support of the canonical measure μ_f is contained in the Julia set $\mathcal{J}_f$ of f, so if a nonarchimedean analogue of the equidistribution theorem were to hold, one would expect that the support of μ_f also be contained within the Julia set of f (which is well-defined in the nonarchimedean setting [89]). However, unlike the complex setting, there exist dynamical systems f with *empty* Julia set. For example, it is easy to see that the iterates of the polynomial endomorphism $f(z) = z^2$ of $\mathbf{P}^1_K$ are everywhere equicontinuous on $\mathbf{P}^1_K$, so $\mathcal{J}_f = \varnothing$. More generally, any f that has *good reduction* has empty Julia set [89]. See §5.2 for a definition of good reduction.

Perhaps the most serious problem of all, however, is that there is no straightforward way to carry over many of the tools of complex analytic geometry to the nonarchimedean setting. As an example, there is no useful notion of analytic continuation of functions, or at least not naively, because nonarchimedean varieties are totally disconnected. Consider, for instance, a function $\varphi\colon K \to K$ that is $\equiv 0$ on some open ball B around the origin and $\equiv 1$ outside of B. This φ is an entire and analytic (since it is locally constant), vanishes in a whole neighborhood of the origin, and yet does φ does not vanish everywhere. Such phenomenon make a good theory of analytic geometry over K difficult to develop.

4.3. Rigid analytic geometry

In order to approach equidistribution of preimages over nonarchimedean fields, we will need to overcome each of these difficulties. The key to doing this will be to work not on the space $\mathbf{P}^r_K$ itself, but on its *Berkovich analytification*. We will outline Berkovich's approach to nonarchimedean analytic geometry in §4.4, but before doing so, it is worth briefly describing the first successful development of a nonarchimedean analytic geometry, so-called *rigid analytic geometry*, initiated by Tate in the 1960s [106]. The main reference for the subject is the tome [19], see also [71, 37].

The definition of *rigid analytic spaces* in Tate's theory parallels the definition of general varieties in classical algebraic geometry. One general schema for defining varieties is to first define the affine space $\mathbf{A}_K^r$, then define affine varieties within $\mathbf{A}_K^r$, and then finally to define general varieties as geometric objects that are locally isomorphic to affine varieties. In Tate's theory, one first defines the *unit polydisk* $\mathbf{D}_K^r$, then defines *affinoid subsets* of $\mathbf{D}_K^r$, and then finally defines rigid analytic spaces as geometric objects that are locally isomorphic (in a suitable sense) to these affinoids.

The unit polydisk $\mathbf{D}_K^r$ is, as a set, simply the maximal ideal spectrum of the *Tate algebra* T_r, that is, the ring of formal power series $\sum_{\alpha\in\mathbf{N}^r} c_\alpha X^\alpha$ in r-variables with coefficients $c_\alpha \in K$ such that $|c_\alpha| \to 0$ as $|\alpha| \to \infty$. When K is algebraically closed (as we always assume it is), there is a correspondence between maximal ideals of T_r and points $a = (a_1, \ldots, a_r) \in K^r$ with $|a_i| \leq 1$ for each i.

An *affinoid subset* of $\mathbf{D}_K^r$ is a subset which is the vanishing set of some ideal of T_r. These are the local models of rigid analytic spaces. However, before being able to glue affinoids together into rigid analytic spaces, one must first develop a reasonable sheaf theory on affinoids. However, because of the topological difficulties previously mentioned, this step is actually rather complicated. The idea is to *not* use the analytic topology on affinoids, but rather to endow them with a Grothendieck topology with respect to which a good sheaf theory can be developed. This Grothendieck topology is based in least in part on the use of *affinoid domains*, which are the rigid analytic geometry analogues of affine open subsets in algebraic geometry. We won't go into any more detail on the matter here.

While Tate's approach goes a long way towards solving the problem of doing meaningful analytic geometry over nonarchimedean fields, it does *not* resolve the other obstructions to equidistribution of preimages outlined in the previous section. There is still no theory of Radon measures and weak convergence on rigid analytic spaces, nor does it give a fix for the problem of some dynamical systems having empty Julia set. Thus for our purposes this formulation of analytic geometry is inadequate.

4.4. Berkovich analytic geometry

In the 1980s, Vladimir Berkovich introduced an alternative formulation of nonarchimedean analytic geometry which does resolve the aforementioned obstructions to equidistribution. We dedicate this section to describing in some detail the Berkovich analytification of varieties over nonarchimedean fields K. The main references for this subject are [13, 14], but see also [4, 63, 86] for a more detailed discussion of the structure of Berkovich analytic spaces in low dimensions.

Fix a field K that is algebraically closed and complete with respect to a (possibly trivial) nonarchimedean absolute value. We begin by defining the Berkovich analytification of affine varieties over K. Let X be an affine variety over K, with coordinate ring $K[X]$. As a set, the *analytification* of X, which will be written X^{an}, is the collection of all *admissible seminorms* on the ring $K[X]$, that is, the collection of functions $\|\cdot\|\colon K[X] \to \mathbf{R}_{\geq 0}$ satisfying:

1. $\|fg\| = \|f\|\|g\|$ for all $f, g \in K[X]$.
2. $\|f+g\| \leq \|f\| + \|g\|$ for all $f, g \in K[X]$.
3. $\|a\| = |a|$ for all $a \in K \subset K[X]$.

It is not hard to show that any admissible seminorm satisfies the strong triangle inequality $\|f+g\| \leq \max\{\|f\|, \|g\|\}$.

In general, the Berkovich analytification X^{an} is quite large: there are many such admissible seminorms. The simplest examples come from the (closed) points of X itself. If $x \in X$ is a closed point, one associates to x an admissible seminorm $\|\cdot\|_x \in X^{\mathrm{an}}$, defined by $\|f\|_x = |f(x)|$. In this way, we view the set of closed points of X as a subset of X^{an}. These points are called the *classical points* of X^{an}.

In the special case when K is a trivially valued field, it is in fact possible to embed the non-closed scheme-theoretic points of X into X^{an} as well, in the following way. If $x \in X$ is a scheme-theoretic point corresponding to a prime ideal $\mathfrak{p}$ in $K[X]$, we associate to x the admissible seminorm $\|\cdot\|_x$ defined by

$$\|f\|_x := \begin{cases} 1 & f \notin \mathfrak{p}. \\ 0 & f \in \mathfrak{p}. \end{cases}$$

Such points of X^{an} will also be called *classical points*. Note that this definition agrees with the previous definition in the case when x is a closed point.

Notation 4.4.1. It is awkward to denote points in X^{an} using notation like $\|\cdot\|_x$. Instead, we will always denote points in X^{an} using letters like x, y, and z. If $\mathsf{x} \in X^{\mathrm{an}}$ is an admissible seminorm, its value on a function $f \in K[X]$ will be denoted $|f(\mathsf{x})|$. Such notation is typical in this subject.

So far we have only described X^{an} as a set, but it comes with a natural topology, namely, the weakest topology for which the evaluation maps $\mathsf{x} \in X^{\mathrm{an}} \mapsto |f(\mathsf{x})|$ are continuous for every $f \in K[X]$. This topology is called the *weak topology* on X^{an}, and it has a number of desirable properties:

Theorem 4.4.2 (Topological Properties)**.** *X^{an} is locally compact, Hausdorff, and locally path connected in its weak topology. If X is connected, then so is X^{an}. If K is not trivially valued, then the classical points of X^{an} are dense.*

Proofs of these statements can be found in [13]. Despite having these nice properties, the weak topology on X^{an} will not be metrizable in general. In fact, the topological structure of Berkovich analytic spaces is typically complicated to the point of being unwieldy. This structure is very well understood for analytifications of one-dimensional varieties (see [4]), but in higher dimensions is far more opaque. See the works [15, 16, 108, 83] for some results in the higher dimensional setting.

While we will not go into details here (see [13]) , it is possible to equip X^{an} with a sheaf of rings $\mathscr{O}_X$, called the *structure sheaf* of X^{an}, which makes X^{an} a locally ringed space. While the construction of $\mathscr{O}_X$ is very related to the construction of the structure sheaf in Tate's rigid analytic geometry, Berkovich's approach has the advantage of the structure sheaf being an actual sheaf, rather than a sheaf with respect to a Grothendieck topology.

Notation 4.4.3. In later chapters it will be important to distinguish the structure sheaf of X^{an} and the classical structure sheaf of X as a scheme. For this reason, we will denote the structure sheaf of the analytic space X^{an} by $\mathscr{O}_X$ and the structure sheaf of the scheme X by $\mathcal{O}_X$.

To every admissible seminorm $\mathsf{x} \in X^{\mathrm{an}}$, one can associate a prime ideal in $K[X]$, namely the ideal $\mathfrak{p}_{\mathsf{x}} = \{f \in K[X] : |f(\mathsf{x})| = 0\}$. This is called the *kernel* of x. In this way we obtain a canonical map $\pi\colon X^{\mathrm{an}} \to X$, where here X is viewed as a scheme, given by taking $\mathsf{x} \in X^{\mathrm{an}}$ to the scheme theoretic point corresponding to the kernel of x. The map π is continuous and is in fact a morphism of locally ringed spaces.

Suppose now that X and Y are two affine varieties over K, and $f\colon X \to Y$ is a morphism, corresponding to the homomorphism $f^*\colon K[Y] \to K[X]$ of coordinate rings. We then define a map $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ by sending a seminorm $\mathsf{x} \in X^{\mathrm{an}}$ to the seminorm $f^{\mathrm{an}}(\mathsf{x})$ defined by $|\varphi(f^{\mathrm{an}}(\mathsf{x}))| = |(f^*\varphi)(\mathsf{x})|$ for all $\varphi \in K[Y]$. The map f^{an} is (clearly) continuous, and agrees with $f\colon X \to Y$ on classical points of X^{an}. Abusing notation slightly, we will denote f^{an} simply by f.

We now describe the Berkovich analytification of general (not necessarily affine) varieties X over K. Let X be a variety over K, and let $U_1, \ldots, U_r$ be a finite cover of X by affine open subsets. The analytification X^{an} of X is obtained by gluing together the analytifications U_i^{an}, as we now describe. Let $\pi_i\colon U_i^{\mathrm{an}} \to U_i$ be the kernel map described above. We glue together U_i^{an} and U_j^{an} along the subsets $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ by identifying two points $\mathsf{x} \in \pi_i^{-1}(U_i \cap U_j)$ and $\mathsf{y} \in \pi_j^{-1}(U_i \cap U_j)$ if

1. $\pi_i(\mathsf{x}) = \pi_j(\mathsf{y})$, and

2. there exists an affine open neighborhood $U \subseteq U_i \cap U_j$ of $\pi_i(\mathsf{x}) = \pi_j(\mathsf{y})$ such that x and y induce the same seminorm on the coordinate ring $K[U]$.

It is not difficult to see that if condition 2 holds, then in fact it holds for *every* affine open neighborhood $U \subseteq U_i \cap U_j$ of $\pi_i(\mathsf{x}) = \pi_j(\mathsf{y})$. Using this, one sees that the space X^{an} obtained by these gluings is independent of the choice of affine cover $U_1, \dots, U_r$. The maps $\pi_i\colon U_i^{\mathrm{an}} \to U_i$ glue together to give a canonical map $\pi\colon X^{\mathrm{an}} \to X$, which is a morphism of locally ringed spaces.

All of our previous discussion of the Berkovich analytification of affine varieties now carries over easily to general varieties. The closed points of X embed naturally into X^{an}, and, if K is trivially valued, so do the non-closed scheme-theoretic points of X. Again, these are called the classical points of X^{an}. When K is not trivially valued, they are dense in X^{an}. If X is an irreducible *projective* variety (this is the case we will be concerned with later), then X^{an} is a compact, Hausdorff, path connected space. If $f\colon X \to Y$ is a morphism of varieties over K, then f induces a continuous map $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$, which agrees with f on classical points.

Finally, we point out that the Berkovich analytification $\pi\colon X^{\mathrm{an}} \to X$ of varieties X over K enjoy GAGA results completely analogous to the classical complex setting [13, §3.4-3.5]. Thus, for example, a morphism $f\colon X \to Y$ between varieties over K will be finite and/or flat if and only if the analytification $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is finite and/or flat.

4.5. Equidistribution of preimages in dimension one

Using the language of Berkovich analytic spaces, we can now reformulate the question of whether or not one has equidistribution of preimages over nonarchimedean fields.

Main Question. *Let K be an algebraically closed field, complete with respect to a nonarchimedean absolute value. Let $f\colon \mathbf{P}^r_K \to \mathbf{P}^r_K$ be an endomorphism of algebraic degree $d \geq 2$, and consider the extension $f\colon \mathbf{P}^{r,\mathrm{an}}_K \to \mathbf{P}^{r,\mathrm{an}}_K$ of f to the analytification $\mathbf{P}^{r,\mathrm{an}}_K$ of $\mathbf{P}^r_K$. Is there a probability measure μ_f on $\mathbf{P}^{r,\mathrm{an}}_K$ such that the iterated preimages of most points $\mathsf{x} \in \mathbf{P}^{r,\mathrm{an}}_K$ equidistribute to μ_f? If so, for which points x will the equidistribution hold, and for which will it not?*

It is worth reiterating that this question is now well-posed: since $\mathbf{P}^{r,\mathrm{an}}_K$ compact Hausdorff, we have the notion of Radon measures on $\mathbf{P}^{r,\mathrm{an}}_K$, and hence can talk about weak convergence of measures. There are of course still some details that need to be considered—for instance, we must define the multiplicity of an f-preimage of a point $\mathsf{x} \in \mathbf{P}^{r,\mathrm{an}}_K$, see §5.1—but we now have a meaningful question to study.

Remark 4.5.1. As mentioned in the Preface, there is a natural candidate for the measure μ_f. In [34], Chambert-Loir constructs for any polarized endomorphism f of a projective variety X over K a probability measure μ_f on X^{an}, similar to the pluripotential theoretic construction of the canonical measure μ_f for an endomorphism of $\mathbf{P}^r_{\mathbf{C}}$ discussed in §1.3. We will see the Chambert-Loir measure μ_f again in §4.6.

In dimension $r = 1$, this question was answered positively by Favre and Rivera-Letelier quite recently [67], see also [95, 86].

Theorem 4.5.2 (Favre and Rivera-Letelier)**.** *Let $f\colon \mathbf{P}^1_K \to \mathbf{P}^1_K$ be an endomorphism of degree $d \geq 2$. Then there is a probability measure μ_f on $\mathbf{P}^{1,\mathrm{an}}_K$ and a set $\mathscr{E}_f \subset \mathbf{P}^{1,\mathrm{an}}_K$ consisting entirely of classical points such that the iterated preimages of any $\mathsf{x} \in \mathbf{P}^{1,\mathrm{an}}_K \smallsetminus \mathscr{E}_f$ equidistribute to μ_f.*

The exceptional set $\mathscr{E}_f$, just like over the complex numbers, is the largest finite subset of $\mathbf{P}^1_K$ that is totally invariant for f, except in one special situation: if $\mathrm{char}(K) = p > 0$, and f is conjugate to an iterate of the Frobenius map, then $\mathscr{E}_f$ will be infinite, see §3.2. Apart from this possibility, one always has $\#\mathscr{E}_f \leq 2$, and generically $\mathscr{E}_f = \varnothing$, as was proved in §3.3.

The proof of Theorem 4.5.2 uses potential theoretic techniques, not at all dissimilar to the proof outlined in §1.2. Of course, in order to use such techniques, one first has to develop a potential theory on $\mathbf{P}^{1,\mathrm{an}}_K$. This has been accomplished over the course of the last decade by Favre-Jonsson [63] and Baker-Rumely [4] for $\mathbf{P}^{1,\mathrm{an}}_K$, and by Thuillier [107] for general Berkovich analytic curves. This development heavily uses the fact that $\mathbf{P}^{1,\mathrm{an}}_K$ has well understood tree structure, see [4, Chapters 1 and 2], [86, §3], or [13, Example 1.4.4] for discussions of this tree structure.

At the moment, there is no sufficiently strong pluripotential theory on the higher dimensional projective spaces $\mathbf{P}^{r,\mathrm{an}}_K$ to generalize Favre and Rivera-Letelier's proof of Theorem 4.5.2 to higher dimensions. However, initial steps in this direction have been taken [36, 22, 23]. It should also be noted that, in any dimension, there is currently no nonarchimedean analogue of the geometric proofs of the complex equidistribution of preimages theorem of Lyubich, Freire-Lopes-Mañé, and Briend-Duval [92, 70, 28].

Finally, it is worth pointing out that the support measure μ_f in Theorem 4.5.2 may be disjoint from the set of classical points of $\mathbf{P}^{1,\mathrm{an}}_K$. It can be shown [67] that the support of μ_f is the (appropriately defined) Julia set of f in $\mathbf{P}^{1,\mathrm{an}}_K$, which must therefore always be nonempty. The phenomenon discussed in §4.2 of there being maps f for which the (classical) Julia set is empty is now clarified: it is not that the Julia set is actually empty, but rather that it cannot be seen within the confines of the classical $\mathbf{P}^1_K$. One must go to the Berkovich projective line $\mathbf{P}^{1,\mathrm{an}}_K$ to see the Julia set of f.

4.6. Arithmetic equidistribution II

Now that we have the language of Berkovich analytic spaces, we can complete the discussion of Yuan's arithmetic equidistribution theorem that we started in §1.5. We begin by finishing the definition of the logarithmic Weil height $h\colon \mathbf{P}^r(\overline{\mathbf{Q}}) \to \mathbf{R}$, which at the moment we have only defined on $\mathbf{Q}$-points of $\mathbf{P}^r$. Recall that h is defined on $\mathbf{Q}$-points $x \in \mathbf{P}^r(\mathbf{Q})$ to be $h(x) = \max_i \log |x_i|_\infty$, where the x_i are integers with no prime factors in common such that $x = [x_0 : \cdots : x_r]$. Here $|\cdot|_\infty$ denotes the standard Euclidean absolute value. The key to extending this definition to $\overline{\mathbf{Q}}$-points is the following observation.

Observation 4.6.1. Let $x \in \mathbf{P}^r(\mathbf{Q})$, with $x = [x_0 : \cdots : x_r]$ any rational homogeneous coordinates of x. Then one has the identity

$$h(x) = \log \max_i |x_i|_\infty + \sum_p \log \max_i |x_i|_p,$$

where the sum is taken over all prime numbers p.

To prove the identity, first note that it holds when the x_i are chosen to be integers with no common prime factors, since $\max_i |x_i|_p = 1$ for every prime number p. Next, note that if we replace the x_i by λx_i for some $\lambda \in \mathbf{Q}$, then the right hand side of the identity becomes

$$\log \max_i |x_i|_\infty + \log |\lambda|_\infty + \sum_p \log \max_i |x_i|_p + \log |\lambda|_p.$$

But $\log |\lambda|_\infty + \sum_p \log |\lambda|_p = 0$ as consequence of the *product formula* $|\lambda|_\infty \prod_p |\lambda|_p = 1$ for absolute values on $\mathbf{Q}$. More generally, for any number field K, there is a product formula (stated below) for absolute values on K, see [32, §10.2].

Product Formula. *Let K be a number field, and let M_K be the collection of absolute values on K that extend one of the absolute values $|\cdot|_\infty$ or $|\cdot|_p$ on $\mathbf{Q}$. For any $v \in M_K$, let K_v and $\mathbf{Q}_v$ denote the completions of K and $\mathbf{Q}$ with respect to the absolute value $|\cdot|_v$, and let $n_v = [K_v : \mathbf{Q}_v]$. Then $\prod_{v \in M_K} |\lambda|_v^{n_v} = 1$ for all $\lambda \in K^\times$.*

If $x = [x_0 : \cdots : x_r] \in \mathbf{P}^r(K)$, we might try to mimic the definition of the Weil height on rational points by setting $h(x) = \sum_{v \in M_K} n_v \log \max_i |x_i|_v$. Though this sum is well-defined by the product formula, there is a problem with this definition. If L is a number field containing K, the two sums $h(x) = \sum_{v \in M_K} n_v \log \max_i |x_i|_v$ and $h(x) = \sum_{v \in M_L} n_v \log \max_i |x_i|_v$ will in general be different. To fix this problem, we modify the definition slightly:

$$h(x) := \frac{1}{[K : \mathbf{Q}]} \sum_{v \in M_K} n_v \log \max_i |x_i|_v \quad \text{for } x = [x_0 : \cdots : x_r] \in \mathbf{P}^r(K). \tag{4.1}$$

See [102, Ch. 3] for a proof that this gives a well-defined function $h\colon \mathbf{P}^r(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$ which satisfies the Northcott property and is Galois invariant.

Now let $f\colon \mathbf{P}^r(\overline{\mathbf{Q}}) \to \mathbf{P}^r(\overline{\mathbf{Q}})$ be an endomorphism of algebraic degree $d \geq 2$. Recall from §1.5 that the canonical height for f is the function $\hat{h}_f\colon \mathbf{P}^r(\overline{\mathbf{Q}}) \to \mathbf{R}$ defined by

$$\hat{h}_f(x) = \lim_{n\to\infty} d^{-n} h(f^n(x)).$$

Using Equation 4.1, we can derive an alternate expression for the canonical height. Suppose f is defined over a number field $K \subset \overline{\mathbf{Q}}$ containing x. Let $F\colon \mathbf{A}^{r+1}(K) \to \mathbf{A}^{r+1}(K)$ be any polynomial lift of f. Then Equation 4.1 yields

$$\hat{h}_f(x) = \frac{1}{[K:\mathbf{Q}]} \lim_{n\to\infty} d^{-n} \sum_{v\in M_K} n_v \log \|F^n(x_0,\ldots,x_r)\|_v,$$

where $\|(y_0,\ldots,y_r)\|_v := \max\{|y_0|_v,\ldots,|y_r|_v\}$. It can be shown [102, §5.9] that the limit can brought within the sum, giving

$$\hat{h}_f(x) = \frac{1}{[K:\mathbf{Q}]} \sum_{v\in M_K} n_v G_{F,v}(x_0,\ldots,x_r), \tag{4.2}$$

where $G_{F,v}\colon \mathbf{A}^{r+1}(K) \smallsetminus \{0\} \to \mathbf{R}$ is the (continuous) function

$$G_{F,v}(x_0,\ldots,x_r) := \lim_{n\to\infty} d^{-n} \log \|F^n(x_0,\ldots,x_r)\|_v. \tag{4.3}$$

One should think of Equation 4.2 as an analogue of Equation 4.1 for the canonical height function $\hat{h}_f$.

Expressions like Equation 4.3 have been around in complex dynamics in several variables since the first work on equidistribution of preimages in the early 1990s. Indeed, suppose that $v \in M_K$ is an archimedean absolute value coming from some embedding $\overline{K} \subset \mathbf{C}$. Let $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ be the induced extension of $f\colon \mathbf{P}^r(K) \to \mathbf{P}^r(K)$, and $F\colon \mathbf{A}^{r+1}_{\mathbf{C}} \to \mathbf{A}^{r+1}_{\mathbf{C}}$ the induced extension of $F\colon \mathbf{A}^{r+1}(K) \to \mathbf{A}^{r+1}(K)$. Then the limit in Equation 4.3 converges on all of $\mathbf{A}^{r+1}_{\mathbf{C}} \smallsetminus \{0\}$ to a continuous function $G_{F,v}$. This function $G_{F,v}$, sometimes called a *Green's function* of f, was first constructed by Fornæss-Sibony [69] and Hubbard-Papadopol [84], who used it define the *Green current* T_f of $f\colon \mathbf{P}^r_{\mathbf{C}} \to \mathbf{P}^r_{\mathbf{C}}$ and the canonical measure $\mu_f = T_f \wedge \cdots \wedge T_f$, as we discussed in §1.3. This connection explains the appearance of μ_f in the archimedean part of Yuan's arithmetic equidistribution theorem.

Suppose now that $v \in M_K$ is a nonarchimedean absolute value. Let $\mathbf{C}_v$ be the completion of the algebraic closure of K_v. Extend f and F to maps $\mathbf{P}^r_{\mathbf{C}_v} \to \mathbf{P}^r_{\mathbf{C}_v}$ and $\mathbf{A}^{r+1}_{\mathbf{C}_v} \to \mathbf{A}^{r+1}_{\mathbf{C}_v}$. Once again, the function $G_{F,v}$ now converges to a continuous function on all of $\mathbf{A}^{r+1}_{\mathbf{C}_v} \smallsetminus \{0\}$. One might hope there is a procedure analogous to the complex case for constructing a measure μ_f

on $\mathbf{P}^{r,\mathrm{an}}_{\mathbf{C}_v}$ from the data $G_{F,v}$. Such a procedure does exist (though it differs in some respects from the complex case), as was discovered by Chambert-Loir [34]. We will not take the time to discuss the construction here, but instead recommend the beautifully written survey [35]. Given that the measure μ_f exists, it should not be surprising that Yuan's theorem applies to this measure as well.

Yuan's arithmetic equidistribution theorem (Nonarchimedean part)**.** *Let $f\colon \mathbf{P}^r \to \mathbf{P}^r$ be an endomorphism of algebraic degree $d \geq 2$ defined over a number field K. Let $v \in M_K$ be a nonarchimedean absolute value on K, and let μ_f be the Chambert-Loir measure on $\mathbf{P}^{r,\mathrm{an}}_{\mathbf{C}_v}$ associated to the extension $f\colon \mathbf{P}^r_{\mathbf{C}_v} \to \mathbf{P}^r_{\mathbf{C}_v}$. Suppose $A_n \subseteq \mathbf{P}^r(\overline{K})$ is a sequence of finite sets such that:*

1. *Each A_n is $\mathrm{Gal}(\overline{K}/K)$-invariant.*
2. *The sequence A_n is Zariski generic in the sense that for all closed subvarieties $D \subsetneq \mathbf{P}^r_{\mathbf{C}_v}$ defined over K, one has $A_n \cap D = \varnothing$ for sufficiently large n.*
3. *The sequence $\max_{x \in A_n} \hat{h}_f(x)$ tends to 0 as $n \to \infty$.*

Then the A_n equidistribute to μ_f, that is, the probability measures $\mu_n = (\#A_n)^{-1}\sum_{x\in A_n}\delta_x$ converge weakly to μ_f as $n \to \infty$.

Our entire discussion of the relationship between the archimedean part of Yuan's theorem and the complex equidistribution of preimages theorem from §1.5 can now be carried over with no real change to the nonarchimedean setting. In particular, if f is defined over a number field K and $x \in \mathbf{P}^r(K)$ happens to be such that the preimage sets $A_n = f^{-n}(x)$ are Zariski generic, then the preimages of x equidistribute to the Chambert-Loir measure μ_f on $\mathbf{P}^{r,\mathrm{an}}_{\mathbf{C}_v}$ for any nonarchimdean $v \in M_K$.

Chapter 5

Equidistribution of preimages for maps of good reduction

In this final chapter, we will prove Theorem A, a nonarchimedean equidistribution theorem for maps of *good reduction.* The main ingredient in the proof is Theorem B, which we proved in §3.4. The idea is that if a dynamical system $f\colon \mathbf{P}^r_K \to \mathbf{P}^r_K$ has good reduction, then we can apply Theorem B to the reduction $\tilde{f}$ of f, and use this conclude Theorem A for f.

There are a few things that must be done before we can execute this strategy, however. First, we must briefly discuss multiplicities for finite maps between Berkovich analytic varieties, as well as pullbacks of measures by these maps. This is the focus of §5.1. In §5.2, we will say what it means for a polarized endomorphism f of a projective variety to have good reduction, and define a *reduction map.* After this, we will investigate how multiplicities and measures transform under the reduction map. This will put us into a position to (finally) prove Theorem A in §5.3. A complement to Theorem A is discussed in §5.4.

Let us fix once an for all an algebraically closed, complete nonarchimedean field K. Recall that K° denotes the ring of integers of K, and k denotes the residue field of K.

5.1. Multiplicities in Berkovich analytic spaces

In Chapter 2 we discussed in detail certain multiplicities associated to finite flat morphisms between classical algebraic varieties. We will now briefly define analogous multiplicities for finite flat morphisms between the Berkovich analytification of such varieties, and then relate the two notions. Recall that if X is a variety over K, then we will denote the structure sheaf of X^{an} as an analytic space by $\mathscr{O}_X$, and the structure sheaf of X as a classical algebraic variety by $\mathcal{O}_X$. Recall also that by the nonarchimedean GAGA principle [13, §3.4-3.5], a morphism $f\colon X \to Y$ between varieties over K will be finite (resp. flat) if and only if the corresponding map $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ between analytic spaces is finite (resp. flat).

Definition 5.1.1. Let X and Y be irreducible varieties over K, and suppose $f\colon X \to Y$ is a finite surjective morphism. Let $\mathsf{x} \in X^{\mathrm{an}}$ and $\mathsf{y} = f(\mathsf{x}) \in Y^{\mathrm{an}}$. Then the *multiplicity* of f at x is the integer

$$m_f(\mathsf{x}) := \dim_{\kappa(\mathsf{y})}(\mathscr{O}_{X,\mathsf{x}}/\mathfrak{m}_{\mathsf{y}}\mathscr{O}_{X,\mathsf{x}}),$$

where as usual $\mathscr{O}_{X,\mathsf{x}}$ is viewed as an $\mathscr{O}_{Y,\mathsf{y}}$-module via f.

In order to compare these multiplicities with the previously defined multiplicities for $f\colon X \to Y$, we will use the canonical maps $\pi_X\colon X^{\mathrm{an}} \to X$ and $\pi_Y\colon Y^{\mathrm{an}} \to Y$ defined in §4.4. It is straightforward to see these maps fit into the following commutative diagram:

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y^{\mathrm{an}} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

In particular, we note that if $\mathsf{y} \in Y^{\mathrm{an}}$ and $y = \pi_Y(\mathsf{y})$, then for any $\mathsf{x} \in f^{-1}(\mathsf{y})$ we must have that $\pi_X(\mathsf{x}) \in f^{-1}(y)$.

Proposition 5.1.2. *Let X and Y be irreducible varieties over K, and suppose $f\colon X \to Y$ is a finite surjective morphism. Let $x \in X$ and $f(x) = y$. Let $\mathsf{y} \in Y^{\mathrm{an}}$ be such that $\pi_Y(\mathsf{y}) = y$, and let $\mathsf{x}_1, \dots, \mathsf{x}_r$ be those f-preimages of y for which $\pi_X(\mathsf{x}_i) = x$. Then*

$$m_f(x) = \sum_{i=1}^{r} m_f(\mathsf{x}_i).$$

In particular, if x is the classical point of X^{an} corresponding to $x \in X$, then $m_f(x) = m_f(\mathsf{x})$.

Proof. As the statement is local, we may assume with no loss of generality that X and Y are affine. Let $\mathsf{x}_{r+1}, \dots, \mathsf{x}_s$ be those f-preimages of y with $\pi_X(\mathsf{x}_i) \neq x$. By [14, Prop. 2.6.10], there is natural isomorphism

$$\mathscr{O}_{Y,\mathsf{y}} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong \prod_{i=1}^{r} \mathscr{O}_{X,\mathsf{x}_i} \times \prod_{i=r+1}^{s} (\mathscr{O}_{X,\mathsf{x}_i})_{\mathfrak{p}_x},$$

where $\mathfrak{p}_x$ is the prime ideal in the coordinate ring $K[X]$ of X which corresponds to x. If we then tensor this expression over $\mathscr{O}_{Y,\mathsf{y}}$ with the residue field $\kappa(\mathsf{y})$, we obtain an isomorphism

$$\kappa(\mathsf{y}) \otimes_{\kappa(y)} (\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) \cong \prod_{i=1}^{r}(\mathscr{O}_{X,\mathsf{x}_i}/\mathfrak{m}_y\mathscr{O}_{X,\mathsf{x}_i}).$$

The $\kappa(\mathsf{y})$-dimension of the left hand side of this expression is $m_f(x)$, while the $\kappa(\mathsf{y})$-dimension of the right hand side is $\sum_{i=1}^{r} m_f(\mathsf{x}_i)$. This completes the proof. ☐

Corollary 5.1.3. *Suppose $f\colon X \to Y$ is a finite surjective flat morphism between irreducible varieties over K. Then every point $\mathsf{y} \in Y^{\mathrm{an}}$ has $[X :_f Y]$ preimages when counted according to their multiplicity. That is, $[X :_f Y] = \sum_{f(\mathsf{x})=\mathsf{y}} m_f(\mathsf{x})$.*

Proof. This is just Proposition 5.1.2 combined with Theorem 2.3.4 ☐

Proposition 5.1.4. *Let $f\colon X \to Y$ be a finite surjective flat morphism between irreducible varieties over K. Let V be an affinoid domain in Y^{an}, and $U = f^{-1}(V)$. Suppose that U has connected components $U_1, \dots, U_s$. Then there exist integers $n_1, \dots, n_s \geq 1$ such that every point $\mathsf{y} \in V$ has exactly n_i preimages in U_i, counted according to their multiplicity.*

Proof. This statement is a higher dimensional analogue of [67, Prop. 2.1]. First note that U is itself an affinoid domain [14, Prop. 3.1.7], as are the U_i [14, Cor. 2.2.7]. If $\mathcal{A}_V \to \mathcal{A}_U \cong \mathcal{A}_{U_1} \times \cdots \times \mathcal{A}_{U_s}$ is the corresponding map of affinoid algebras, then $\mathcal{A}_U$ is a finite Banach $\mathcal{A}_V$-module since f is finite. It follows immediately that each $\mathcal{A}_{U_i}$ is a finite Banach $\mathcal{A}_V$-module via the composite $\mathcal{A}_V \to \mathcal{A}_U \to \mathcal{A}_{U_i}$. Therefore $f|_{U_i}\colon U_i \to V$ is a finite map of K-analytic spaces. By assumption, f is flat, and hence $f|_{U_i}$ is flat for each i. It follows that $f_*(\mathscr{O}_X|_{U_i})$ is a coherent, locally free $\mathscr{O}_Y|_V$-module of some rank n_i. If $\mathsf{y} \in V$, then

$$f_*(\mathscr{O}_X|_{U_i})_{\mathsf{y}} \cong \bigoplus_{\mathsf{x} \in f^{-1}(\mathsf{y}) \cap U_i} \mathscr{O}_{X,\mathsf{x}},$$

and therefore

$$n_i = \dim_{\kappa(\mathsf{y})}(\kappa(\mathsf{y}) \otimes_{\mathscr{O}_{Y,\mathsf{y}}} f_*(\mathscr{O}_X|_{U_i})_{\mathsf{y}}) = \sum_{\mathsf{x} \in f^{-1}(\mathsf{y}) \cap U_i} \dim_{\kappa(\mathsf{y})}(\mathscr{O}_{X,\mathsf{x}}/\mathfrak{m}_y \mathscr{O}_{X,\mathsf{x}}) = \sum_{\mathsf{x} \in f^{-1}(\mathsf{y}) \cap U_i} m_f(\mathsf{x}).$$

This completes the proof. ☐

Using these results, we are now able to define a pullback operation on Radon measures, analogous to the pullback defined in Proposition 2.4.3. As we did there, the pullback is defined as the adjoint of a pushforward operation on functions.

Definition 5.1.5. Suppose $f\colon X \to Y$ is a finite surjective flat morphism between irreducible varieties over K. If $\varphi \in C^0(X^{\mathrm{an}})$, we define $f_*\varphi\colon Y^{\mathrm{an}} \to \mathbf{R}$ by

$$(f_*\varphi)(\mathsf{y}) := \sum_{f(\mathsf{x})=\mathsf{y}} m_f(\mathsf{x})\varphi(\mathsf{x}).$$

Proposition 5.1.6. *Suppose $f\colon X \to Y$ is a finite surjective flat morphism between irreducible varieties over K. Then the pushforward f_* defines a linear map $C^0(X^{\mathrm{an}}) \to C^0(Y^{\mathrm{an}})$. If we assume, in addition, that X and Y are projective, so that X^{an} and Y^{an} are compact, then f_* is a bounded linear operator between Banach spaces.*

Proof. Let $\varphi \in C^0(X^{\mathrm{an}})$, and let $\mathsf{y} \in Y^{\mathrm{an}}$. Let V be a small enough affinoid neighborhood of y such that $f^{-1}(V)$ is a disjoint union of components $U_1, \dots, U_s$, each containing exactly one preimage x_i of y. By shrinking V if necessary, we may assume that the variation of φ on U_i is at most ε for each i. By Proposition 5.1.4, if $\mathsf{y}' \in V$, then y' has exactly $m_f(\mathsf{x}_i)$ preimages in U_i when counted according to multiplicity. Thus

$$\begin{aligned} |(f_*\varphi)(\mathsf{y}') - (f_*\varphi)(\mathsf{y})| &\leq \sum_{i=1}^{s} \left| m_f(\mathsf{x}_i)\varphi(\mathsf{x}_i) - \sum_{\mathsf{x} \in f^{-1}(\mathsf{y}') \cap U_i} m_f(\mathsf{x})\varphi(\mathsf{x}) \right| \\ &\leq \sum_{i=1}^{s} \varepsilon m_f(\mathsf{x}_i) = [X :_f Y]\varepsilon. \end{aligned}$$

This proves that $f_*\varphi$ is continuous. In the case where X^{an} and Y^{an} are compact, and thus $C^0(X^{\mathrm{an}})$ and $C^0(Y^{\mathrm{an}})$ are Banach spaces, the fact that f_* is bounded is immediate from the easy estimate $|(f_*\varphi)(\mathsf{y})| \leq \|\varphi\| \sum_{f(\mathsf{x})=\mathsf{y}} m_f(\mathsf{x}) = [X :_f Y]\|\varphi\|$. $\square$

Definition 5.1.7. Let $f\colon X \to Y$ be a finite surjective flat morphism between irreducible projective varieties over K. Let $\mathcal{M}(X^{\mathrm{an}})$ and $\mathcal{M}(Y^{\mathrm{an}})$ denote the space of Radon measures on X^{an} and Y^{an}, respectively. We define the pullback operator $f^*\colon \mathcal{M}(Y^{\mathrm{an}}) \to \mathcal{M}(X^{\mathrm{an}})$ to be the adjoint of $f_*\colon C^0(X^{\mathrm{an}}) \to C^0(Y^{\mathrm{an}})$.

The following properties of the pullback $f^*\colon \mathcal{M}(Y^{\mathrm{an}}) \to \mathcal{M}(X^{\mathrm{an}})$ are now immediate from the definitions.

1. If μ is a positive Radon measure on Y^{an}, then $f^*\mu$ is positive as well. Moreover, if the total mass of μ is R, then $f^*\mu$ has total mass $[X :_f Y]R$.

2. If μ is any Radon measure on Y^{an}, then $f_*f^*\mu = [X :_f Y]\mu$, where here f_* denotes the usual pushforward operation on measures.

3. If $\mathsf{y} \in Y^{\mathrm{an}}$ and δ_{y} is the Dirac probability measure at y, then $f^*\delta_{\mathsf{y}} = \sum_{f(\mathsf{x})=\mathsf{y}} m_f(\mathsf{x})\delta_{\mathsf{x}}$.

5.2. Maps of good reduction

The last notions we need before being able to prove a nonarchimedean equidistribution of preimages theorem are the notions of *reduction* of Berkovich analytic varieties and of *good reduction* for morphisms between these varieties. These notions are expressed in terms of *models* of analytic spaces, an idea going back to foundational work of Raynaud in rigid analytic geometry (see [20, 21, 18] for detailed references).

Definition 5.2.1. Let X be an irreducible projective variety over K. A *model* of X is a flat projective scheme $\mathcal{X}$ over Spec K° with a specified isomorphism between X and the generic fiber $\mathcal{X}_K$ of $\mathcal{X}$.

In the case when K is equipped with the trivial absolute value, any model $\mathcal{X}$ of X is simply a variety over K that is isomorphic to X, and thus we lose no generality by taking $\mathcal{X} = X$. When K is equipped with a nontrivial absolute value, there is in general no canonical model of X, but some model $\mathcal{X}$ always exists.

Fix a model $\mathcal{X}$ of an irreducible projective variety X over K. The special fiber $\mathcal{X}_k$ of $\mathcal{X}$ is a projective variety over the residue field k of K, all of whose components have the same dimension as X. Let $\mathsf{x} \in X^{\mathrm{an}}$ be any point, and let $x = \pi(\mathsf{x}) \in X = \mathcal{X}_K$. The point x is an admissible seminorm on the coordinate ring of any affine open neighborhood of x, whose kernel consists of those functions which vanish at x. Therefore x defines an absolute value on the residue field $\kappa(x)$ of $\mathcal{X}_K$ at x.

Lemma 5.2.2. *If $\kappa(x)^\circ$ is the valuation ring of $\kappa(x)$ with respect to x, then the K-morphism* $\operatorname{Spec}\kappa(x) \to \mathcal{X}_K$ *corresponding to x extends uniquely to a K°-morphism* $\operatorname{Spec}\kappa(x)^\circ \to \mathcal{X}$.

Proof. By assumption, $\mathcal{X}$ is projective, hence proper over $\operatorname{Spec} K^\circ$. The lemma then follows immediately from the valuative criterion of properness [82, Theorem II.4.7]. $\square$

The special fiber of the K°-morphism $\operatorname{Spec}\kappa(x)^\circ \to \mathcal{X}$ given by Lemma 5.2.2 is then a morphism $\operatorname{Spec}\tilde{\kappa}(x) \to \mathcal{X}_k$, corresponding to some point $\xi \in \mathcal{X}_k$. Via this procedure, starting with a point $\mathsf{x} \in X^{\mathrm{an}}$ we have obtained a point $\xi \in \mathcal{X}_k$.

Definition 5.2.3. The map $\mathsf{red}\colon X^{\mathrm{an}} \to \mathcal{X}_k$ taking $\mathsf{x} \in X^{\mathrm{an}}$ to the point $\xi \in \mathcal{X}_k$ described above is called the *reduction map* with respect to the model $\mathcal{X}$. The point ξ is called the *reduction* of x in $\mathcal{X}$. In the literature ξ is sometimes also called the *center* of x, but we will not use this terminology here.

For any model $\mathcal{X}$ of X, the reduction map $\mathsf{red}\colon X^{\mathrm{an}} \to \mathcal{X}_k$ is *anticontinuous*, that is, the inverse image of a Zariski open subset of $\mathcal{X}_k$ is closed in X^{an}. It is always surjective, and has the property that the generic point of any irreducible component of $\mathcal{X}_k$ has exactly one preimage under red.

If K is equipped with the trivial absolute value, then as previously discussed all models of X are K-varieties isomorphic to X. Thus we obtain a canonical reduction $\mathsf{red}\colon X^{\mathrm{an}} \to X$. This map red should not be confused with the canonical map $\pi\colon X^{\mathrm{an}} \to X$ discussed in §4.4, as they are different. In general, $\mathsf{red}(\mathsf{x})$ specializes $\pi(\mathsf{x})$ for every $\mathsf{x} \in X^{\mathrm{an}}$, and one has equality $\mathsf{red}(\mathsf{x}) = \pi(\mathsf{x})$ if and only if x is a classical point of X^{an}. The unique classical point $\mathsf{x} \in X^{\mathrm{an}}$ such that $\mathsf{red}(\mathsf{x}) = \pi(\mathsf{x}) =$ the generic point of X is called the *Gauss point* of X^{an}.

Suppose now that X and Y are irreducible projective varieties over K (which we no longer assume is trivially valued), and let $\mathcal{X}$ and $\mathcal{Y}$ be models of X and Y, respectively. Suppose that $F\colon \mathcal{X} \to \mathcal{Y}$ is a finite flat K°-morphism. Then the generic and special fibers $F_K\colon X \to Y$ and $F_k\colon \mathcal{X}_k \to \mathcal{Y}_k$ are finite flat morphisms which are compatible with reduction in the sense that the following diagram commutes.

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, "F_K"] \arrow[d, "\mathsf{red}"'] & Y^{\mathrm{an}} \arrow[d, "\mathsf{red}"] \\
\mathcal{X}_k \arrow[r, "F_k"] & \mathcal{Y}_k
\end{tikzcd}$$

If $\mathcal{X}_k$ and $\mathcal{Y}_k$ are irreducible, then $[\mathcal{X}_k :_{F_k} \mathcal{Y}_k] = [X :_{F_K} Y]$. This compatibility with reduction is exactly the spirit of what we call *good reduction*.

Definition 5.2.4. Let X be an irreducible projective variety over K, and let $f\colon X \to X$ be a finite surjective flat morphism. Suppose that there exists a model $\mathcal{X}$ of X *with irreducible special fiber*, and a finite flat K°-morphism $F\colon \mathcal{X} \to \mathcal{X}$ such that $f = F_K$. Then we say that f has *good reduction* with respect to F. The map F_k is called the *reduction* of f.

Similarly, let $f\colon (X, L) \to (X, L)$ if a flat polarized morphism (see §3.1). If there is a model $\mathcal{X}$ of X, an ample line bundle $\mathcal{L}$ on $\mathcal{X}$ which models L, and a flat polarized morphism $F\colon (\mathcal{X}, \mathcal{L}) \to (\mathcal{X}, \mathcal{L})$ such that $f = F_K$, then we say f has *good reduction* with respect to F.

Observe that if K is trivially valued, then *every* finite surjective flat morphism $f\colon X \to X$ has good reduction, simply by taking $F = f$. On the other hand, if K is equipped with a nontrivial absolute value, then the notion of good reduction is rather restrictive.

Remark 5.2.5. In the case when $X = \mathbf{P}^r_K$, one sometimes says that a morphism $f\colon \mathbf{P}^r_K \to \mathbf{P}^r_K$ has good reduction without explicitly making mention of any model of $\mathbf{P}^r_K$, just as we did in the preface. In this case, it is implied that f has good reduction with respect to an endomorphism $F\colon \mathcal{X} \to \mathcal{X}$ of the model $\mathcal{X} = \mathbf{P}^r_{K^\circ}$ of $\mathbf{P}^r_K$. In dimension $r = 1$, the situation is rather simpler, in that every morphism $f\colon \mathbf{P}^1_K \to \mathbf{P}^1_K$ of good reduction with respect to some morphism $\mathcal{X} \to \mathcal{X}$ is, possibly after conjugating f by an automorphism of $\mathbf{P}^1_K$, induced from a morphism $\mathbf{P}^1_{K^\circ} \to \mathbf{P}^1_{K^\circ}$. One sometimes calls $f\colon \mathbf{P}^1_K \to \mathbf{P}^1_K$ a map of *potentially good reduction* if it is induced by some morphism $\mathcal{X} \to \mathcal{X}$, and a map of *good reduction* if it is induced by a morphism $\mathbf{P}^1_{K^\circ} \to \mathbf{P}^1_{K^\circ}$.

The following proposition is an analogue of Proposition 5.1.2 for the reduction map red.

Proposition 5.2.6. *Let X be an irreducible projective variety over K, and let $f\colon X \to X$ be a morphism which has good reduction with respect to a morphism $F\colon \mathcal{X} \to \mathcal{X}$. Let $\mathsf{y} \in X^{\mathrm{an}}$, and let $y = \mathsf{red}(\mathsf{y})$. Fix any $x \in \mathcal{X}_k$ with $F_k(x) = y$, and let $\mathsf{x}_1, \dots, \mathsf{x}_r$ be those f-preimages of y with $\mathsf{red}(\mathsf{x}_i) = x$. Then one has*

$$m_{F_k}(x) = \sum_{i=1}^{r} m_f(\mathsf{x}_i).$$

Proof. Let $\hat{\mathcal{O}}_{\mathcal{X},y}$ and $\hat{\mathcal{O}}_{\mathcal{X},x}$ be the completions of the local rings $\mathcal{O}_{\mathcal{X},y}$ and $\mathcal{O}_{\mathcal{X},x}$ with respect to their maximal ideals. Then $\hat{\mathcal{O}}_{\mathcal{X},x}$ is a finite free $\hat{\mathcal{O}}_{\mathcal{X},y}$-module via F, say of rank R. Because $y = \mathsf{red}(\mathsf{y})$, we have a natural K°-homomorphism $\hat{\mathcal{O}}_{\mathcal{X},y} \to \mathscr{H}(\mathsf{y})^{\circ}$, where here $\mathscr{H}(\mathsf{y})$ is the completed residue field of $\mathscr{O}_X$ at y. This homomorphism allows us to consider the tensor products

$$\hat{\mathcal{O}}_{\mathcal{X},x} \otimes_{\hat{\mathcal{O}}_{\mathcal{X},y}} \widetilde{\mathscr{H}}(\mathsf{y}) \quad \text{and} \quad \hat{\mathcal{O}}_{\mathcal{X},x} \otimes_{\hat{\mathcal{O}}_{\mathcal{X},y}} \mathscr{H}(\mathsf{y}),$$

which are then vector spaces of dimension R over $\widetilde{\mathscr{H}}(\mathsf{y})$ and $\mathscr{H}(\mathsf{y})$, respectively. Since

$$\hat{\mathcal{O}}_{\mathcal{X},x} \otimes_{\hat{\mathcal{O}}_{\mathcal{X},y}} \widetilde{\mathscr{H}}(\mathsf{y}) \cong (\mathcal{O}_{\mathcal{X}_k,x}/\mathfrak{m}_y \mathcal{O}_{\mathcal{X}_k,x}) \otimes_{\kappa(y)} \widetilde{\mathscr{H}}(\mathsf{y}),$$

one has $m_{F_k}(x) = R$. On the other hand,

$$\hat{\mathcal{O}}_{\mathcal{X},x} \otimes_{\hat{\mathcal{O}}_{\mathcal{X},y}} \mathscr{H}(\mathsf{y}) \cong \bigoplus_{i=1}^{r} (\mathscr{O}_{X,\mathsf{x}_i}/\mathfrak{m}_{\mathsf{y}} \mathscr{O}_{X,\mathsf{x}_i}) \otimes_{\kappa(\mathsf{y})} \mathscr{H}(\mathsf{y}),$$

so $R = \sum_{i=1}^{r} m_f(\mathsf{x}_i)$. This completes the proof. $\square$

5.3. Equidistribution of preimages

In this section we will prove our main nonarchimedean equidistribution of preimages theorem, valid for maps with good reduction. The setup for the entirety of this section is as follows. We fix an irreducible projective variety X over K and a polarized morphism $f\colon (X, L) \to (X, L)$ of algebraic degree $d \geq 2$, which has good reduction with respect to a polarized morphism $F\colon (\mathcal{X}, \mathcal{L}) \to (\mathcal{X}, \mathcal{L})$ modeling f. We denote by $\tilde{f}$ the reduction $\tilde{f}\colon (\mathcal{X}_k, \mathcal{L}_k) \to (\mathcal{X}_k, \mathcal{L}_k)$ of f. When K is equipped with the trivial absolute value, one has $(\mathcal{X}_k, \mathcal{L}_k) = (X, L)$ and $\tilde{f} = f$, but in the interest of keeping notation uniform we will still write $\tilde{f}$ and $(\mathcal{X}_k, \mathcal{L}_k)$.

The strategy behind our proof of the equidistribution theorem is to apply the equidistribution results proved in §3.4 to $\tilde{f}$, and then lift these results to f via the reduction map:

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "f"] \arrow[d, "\mathsf{red}"'] & X^{\mathrm{an}} \arrow[d, "\mathsf{red}"] \\ \mathcal{X}_k \arrow[r, "\tilde{f}"] & \mathcal{X}_k \end{tikzcd}$$

In order to make use of the results of §3.4 we will need to assume further that the reduction $\tilde{f}\colon (\mathcal{X}_k, \mathcal{L}_k) \to (\mathcal{X}_k, \mathcal{L}_k)$ satisfies Assumption 3.2.3.

We begin by using the canonical map $\pi\colon X^{\mathrm{an}} \to X$ and the reduction map $\mathsf{red}\colon X^{\mathrm{an}} \to \mathcal{X}_k$ to relate the measure theory of X^{an} to the measure theory of X of $\mathcal{X}_k$. Recall that π and red are continuous and anticontinuous, respectively. In particular, both maps are Borel measurable. If μ is a Radon measure on X^{an}, we are therefore able to consider the pushforward measures $\pi_*\mu$ and $\mathsf{red}_*\mu$. Our first goal is to prove that the pushforward operations π_* and red_* are compatible with pullbacks in the sense that $\pi_* f^* = f^*\pi_*$ and $\mathsf{red}_* f^* = \tilde{f}^*\mathsf{red}_*$.

Lemma 5.3.1.

1. *Let $V \subset \mathcal{X}_k$ be a nonempty proper irreducible closed subset of $\mathcal{X}_k$, and let $U = \mathsf{red}^{-1}(V)$. Then there is an increasing sequence of nonnegative continuous functions $\varphi_n\colon X^{\mathrm{an}} \to \mathbf{R}$ which converges pointwise to the characteristic function χ_U.*

2. *Let $V \subset X$ be a nonempty proper irreducible closed subset of X, and let $E = \pi^{-1}(V)$. Then there is a decreasing sequence of nonnegative continuous functions $\psi_n\colon X^{\mathrm{an}} \to \mathbf{R}$ which converges pointwise to the characteristic function χ_E.*

Proof. 1. Let $\{\mathcal{W}_\alpha\}$ be a finite affine open cover of $\mathcal{X}$, say $\mathcal{W}_\alpha = \mathrm{Spec}\ A_\alpha$. For each index α, the set $\mathscr{W}_\alpha := \mathsf{red}^{-1}(\mathcal{W}_{\alpha,k})$ is a closed subset of X^{an} (it is, in fact, an affinoid domain). Fix an α such that $\mathcal{W}_{\alpha,k}$ intersects V, and let $a_1, \ldots, a_r \in A_\alpha$ be elements whose images $\overline{a}_1, \ldots, \overline{a}_r$ in the reduction $\overline{A} = A \otimes_{K^\circ} k$ generate the prime ideal $\mathfrak{p}_V$ of V is $\mathcal{W}_{\alpha,k}$. Then $\mathsf{x} \in \mathscr{W}_\alpha$ lies in U if and only if $|a_i(\mathsf{x})| < 1$ for each i. Define $h_\alpha\colon \mathscr{W}_\alpha \to \mathbf{R}$ by $h_\alpha(\mathsf{x}) := \max_i |a_i(\mathsf{x})|$. This h_α is certainly continuous, nonnegative, and satisfies $h_\alpha(\mathsf{x}) \leq 1$, with equality if and only if $\mathsf{x} \notin U$. Moreover, h_α is independent of the choice of the a_i. In this way, we define h_α for each α. Since $h_\alpha = h_\beta$ on any intersection $\mathscr{W}_\alpha \cap \mathscr{W}_\beta$, the h_α glue together to give a continuous function $h\colon X^{\mathrm{an}} \to \mathbf{R}$ with the property that $0 \leq h(\mathsf{x}) \leq 1$ for all x, with $h(\mathsf{x}) < 1$ if and only if $\mathsf{x} \in U$. We can then define $\varphi_n := (1-h)^{1/n}$.

2. Let $\{U_\alpha\}$ be a finite affine open cover of X, say with $U_\alpha = \mathrm{Spec}\ A_\alpha$. For each index α, the set $U_\alpha^{\mathrm{an}} = \pi^{-1}(U_\alpha)$ is an open subset of X^{an}. Fix an α such that U_α intersects V, and let $a_1, \ldots, a_r \in A_\alpha$ be generators of the prime ideal $\mathfrak{p}_V$ of V in U_α. Then $\mathsf{x} \in U_\alpha^{\mathrm{an}}$ belongs to E if and only if $|a_i(\mathsf{x})| = 0$ for each i. Let $h_\alpha\colon U_\alpha^{\mathrm{an}} \to \mathbf{R}$ be the function $h_\alpha := \min\{1, \max_i |a_i(\mathsf{x})|\}$. This h_α is continuous and satisfies $h_\alpha(\mathsf{x}) \geq 0$, with equality if and only if $\mathsf{x} \in E$. Moreover, h_α is independent of the choice of the a_i. In this way, we define h_α for each index α. Since $h_\alpha = h_\beta$ on any overlap $U_\alpha^{\mathrm{an}} \cap U_\beta^{\mathrm{an}}$, the h_α can be glued together to give a continuous function $h\colon X^{\mathrm{an}} \to \mathbf{R}$ with the property that $0 \leq h(\mathsf{x}) \leq 1$ for all x, with $h(\mathsf{x}) = 0$ if and only if $\mathsf{x} \in E$. We can then define $\psi_n := 1 - h^{1/n}$. $\square$

Proposition 5.3.2. *Let μ be a positive Radon measure on X^{an}. Then the pushforward operations π_* and red_* are compatible with pullbacks in the sense that*

1. $\mathsf{red}_* f^*\mu = \tilde{f}^*\mathsf{red}_*\mu$, *and*

2. $\pi_* f^*\mu = f^*\pi_*\mu$.

Proof. 1. By Lemma A.2.1, it suffices to check that $(\mathsf{red}_* f^*\mu)(V) = (\tilde{f}^*\mathsf{red}_*\mu)(V)$ for every irreducible closed set $V \subseteq \mathcal{X}_k$. By Lemma 5.3.1, there is an increasing sequence $\varphi_n \colon X^{\mathrm{an}} \to \mathbf{R}$ of nonnegative continuous functions converging pointwise to $\chi_{\mathsf{red}^{-1}(V)}$. Then

$$(\mathsf{red}_* f^*\mu)(V) = (f^*\mu)(\mathsf{red}^{-1}(V)) = \lim_{n\to\infty} \int \varphi_n df^*\mu = \lim_{n\to\infty} \int f_*\varphi_n \, d\mu = \int f_*\mathsf{red}^*\chi_V \, d\mu.$$

On the other hand, we have

$$(\tilde{f}^*\mathsf{red}_*\mu)(V) = \int \tilde{f}_*\chi_V \, d\mathsf{red}_*\mu = \int \mathsf{red}^*\tilde{f}_*\chi_V \, d\mu.$$

It therefore suffices to show that $f_*\mathsf{red}^*\chi_V = \mathsf{red}^*\tilde{f}_*\chi_V$. If $\mathsf{y} \in X^{\mathrm{an}}$, then

$$(f_*\mathsf{red}^*\chi_V)(\mathsf{y}) = \sum_{f(\mathsf{x})=\mathsf{y}} m_f(\mathsf{x})\chi_V(\mathsf{red}(\mathsf{x})) = \sum_{\tilde{f}(x)=\mathsf{red}(\mathsf{y})} m_{\tilde{f}}(x)\chi_V(x) = (\mathsf{red}^*\tilde{f}_*\chi_V)(\mathsf{y}),$$

where the second equality is a consequence of Proposition 5.2.6. This completes the proof of statement 1. The argument for statement 2 is similar, except one uses Proposition 5.1.2 at the end instead of Proposition 5.2.6. ☐

Unfortunately, the push-forward operators π_* and red_* are *not* weakly continuous: if μ_n is a sequence of Radon measures on X^{an} which converge weakly to a measure μ, it is not necessarily the case that $\pi_*\mu_n$ converges weakly to $\pi_*\mu$, or that $\mathsf{red}_*\mu_n$ converges weakly to $\mathsf{red}_*\mu$. Indeed, it is not even necessarily the case that $\pi_*\mu_n$ and $\mathsf{red}_*\mu_n$ converge weakly to anything. The reason for this difficulty is that the weak topology for measures on X and $\mathcal{X}_k$ is defined in terms of semicontinuous functions, whereas the weak topology for Radon measures on X^{an} is defined in terms of continuous functions. The next proposition explores this phenomenon.

Proposition 5.3.3. *Let μ_n be a sequence of Radon probability measures on X^{an} which converges weakly to a measure μ. Then:*

1. *Suppose the measures $\nu_n := \mathsf{red}_*\mu_n$ converge weakly to a measure ν. Then one has the inequality $\nu(V) \geq (\mathsf{red}_*\mu)(V)$ for all irreducible closed subsets $V \subseteq \mathcal{X}_k$.*

2. *Suppose the measures $\nu_n := \pi_*\mu_n$ converge weakly to a measure ν. Then one has the inequality $\nu(V) \leq (\pi_*\mu)(V)$ for all irreducible closed subsets $V \subseteq X$.*

Proof. 1. Fix an irreducible closed subset $V \subseteq \mathcal{X}_k$. By Lemma 5.3.1, there exists an increasing sequence $\varphi_n \colon X^{\mathrm{an}} \to \mathbf{R}$ of nonnegative continuous functions converging pointwise to $\chi_{\mathsf{red}^{-1}(V)}$. Given $\varepsilon > 0$ and an index $N = N(\varepsilon)$ large enough, one then has

$$\begin{aligned}(\mathsf{red}_*\mu)(V) &= \lim_{n\to\infty} \int \varphi_n \, d\mu \leq \varepsilon + \int \varphi_N \, d\mu = \varepsilon + \lim_{m\to\infty} \int \varphi_N \, d\mu_m \\ &\leq \varepsilon + \liminf_{m\to\infty} \int \chi_{\mathsf{red}^{-1}(V)} \, d\mu_m = \varepsilon + \liminf_{m\to\infty} \nu_m(V) = \varepsilon + \nu(V).\end{aligned}$$

Letting $\varepsilon \to 0$ gives the desired inequality. The proof of statement 2 is similar. $\square$

We are now in a position to prove our main nonarchimedean equidistribution of preimages result.

Theorem 5.3.4. *Let X be an irreducible projective variety over K of dimension r, and let $f \colon (X, L) \to (X, L)$ be a polarized morphism of algebraic degree $d \geq 2$. Suppose that f has good reduction with respect to a morphism $F \colon (\mathcal{X}, \mathcal{L}) \to (\mathcal{X}, \mathcal{L})$. Finally, assume that the reduction $\tilde{f} \colon (\mathcal{X}_k, \mathcal{L}_k) \to (\mathcal{X}_k, \mathcal{L}_k)$ of f satisfies Assumption 3.2.3. Let $\mathscr{E}$ denote the exceptional set of $\tilde{f}$, as defined in §3.2. If μ is a Radon probability measure on X^{an} which gives no mass to $\mathsf{red}^{-1}(\mathscr{E})$, then the normalized pull-backs $d^{-rn} f^{n*}\mu$ converge weakly to the Dirac probability measure δ_{x} supported at the unique point $\mathsf{x} \in X^{\mathrm{an}}$ whose reduction is the generic point of $\mathcal{X}_k$.*

In the special case when $\mu = \delta_{\mathsf{y}}$ is the Dirac mass at a point $\mathsf{y} \in X^{\mathrm{an}}$ for which $\mathsf{red}(\mathsf{y}) \notin \mathscr{E}$, Theorem 5.3.4 gives that the iterated preimages of y equidistribute to δ_{x}. In particular, if $\mathscr{E} = \varnothing$ as is generically the case, then the preimages of *every* point $\mathsf{y} \in X^{\mathrm{an}}$ equidistribute to δ_{x}. Thus Theorem 5.3.4 is a complete analogue of the complex equidistribution of preimages theorem for generic maps of good reduction. However, when $\mathscr{E} \neq \varnothing$, Theorem 5.3.4 is strictly weaker than the desired result. Since $\mathsf{red}^{-1}(\mathscr{E})$ is an open neighborhood of the exceptional set of f, the theorem guarantees that the preimages of a point y that are "far" from the exceptional set of f equidistribute to δ_{x}; ideally we want to show that the preimages of *all* points y not lying the exceptional set of f equidistribute to δ_x.

Proof. Let $\mu_n = d^{-rn} f^{n*}\mu$ for each $n \geq 1$. It suffices to show that every weakly convergent subsequence of $\{\mu_n\}$ converges to δ_{x}. We therefore fix a weakly convergent subsequence μ_{n_i}, converging to some measure α. Let $\nu = \mathsf{red}_*\mu$ and $\nu_n = \mathsf{red}_*\mu_n$ for each n. The compatibility of red_* with pullbacks proved in Proposition 5.3.2 implies that $\nu_n = d^{-rn}\tilde{f}^{n*}\nu$ for each n. The assumption that μ does not give mass on $\mathsf{red}^{-1}(\mathscr{E})$ is equivalent to saying that ν does not give mass to $\mathscr{E}$. The equidistribution result Corollary 3.4.4 then implies that ν_n converges weakly to the Dirac mass at the generic point of $\mathcal{X}_k$ as $n \to \infty$. Proposition 5.3.3(1) then gives that $(\mathsf{red}_*\alpha)(V) = 0$ for all proper closed subsets $V \subsetneq \mathcal{X}_k$. We will use this property to conclude that $\alpha = \delta_{\mathsf{x}}$.

Let $\mathcal{A} \subseteq C^0(X^{\mathrm{an}})$ be the subalgebra consisting of functions which are constant away from a set of the form $\mathsf{red}^{-1}(V)$ for some proper closed set $V \subsetneq \mathcal{X}_k$. Clearly $\mathcal{A}$ contains all constant functions, and moreover $\mathcal{A}$ separates points by Lemma 5.3.1. We conclude by the Stone-Weierstrass theorem that $\mathcal{A}$ is dense in $C^0(X^{\mathrm{an}})$. Let $\varphi \in \mathcal{A}$, with $\varphi \equiv c$ away from a set $\mathsf{red}^{-1}(V)$ with $V \subsetneq \mathcal{X}_k$ closed. Then

$$\int \varphi \, d\alpha = c[1 - \alpha(\mathsf{red}^{-1}(V))] + \int_{\mathsf{red}^{-1}(V)} \varphi \, d\alpha = c$$

since by the previous paragraph we know $\alpha(\mathsf{red}^{-1}(V)) = 0$. This proves that α agrees with δ_{x} on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $C^0(X^{\mathrm{an}})$, we conclude that $\alpha = \delta_{\mathsf{x}}$ as Radon measures. □

In the special case when K is trivially valued, one can use the canonical map $\pi \colon X^{\mathrm{an}} \to X$ to obtain a more precise refinement of Theorem 5.3.4 that says what happens to the preimages of some points $\mathsf{y} \in X^{\mathrm{an}}$ whose reduction $\mathsf{red}(\mathsf{y})$ *does* lie in $\mathscr{E}$.

Theorem 5.3.5. *Let $f \colon (X, L) \to (X, L)$ be as in Theorem 5.3.4, only now assume K is trivially valued. Let $\mathsf{y} \in X^{\mathrm{an}}$ be any point, and assume that the smallest totally invariant closed subset of X containing $\pi(\mathsf{x})$ is the same as the smallest totally invariant closed subset of X containing $\mathsf{red}(\mathsf{x})$. Let V denote this set, and suppose V has irreducible decomposition $V = V_0 \cup \cdots \cup V_{s-1}$. Let y_i denote the classical point of X^{an} corresponding to V_i for each i. Then, up to relabeling the V_i if necessary, one has for each $i = 0, \ldots, s-1$ that $d^{-r(i+sn)} f^{(i+sn)*}\delta_{\mathsf{y}} \to \delta_{\mathsf{y}_i}$ weakly as $n \to \infty$.*

Proof. We will argue similarly to the proof of Theorem 5.3.4. Let $\mu_n = d^{-r(i+sn)} f^{(i+sn)*}\delta_{\mathsf{y}}$, and let α be a weak limit of a subsequence of the μ_n. By Corollary 3.4.3, up to relabeling the V_i if necessary, we know that the sequences $\pi_*\mu_n$ and $\mathsf{red}_*\mu_n$ both converge weakly to the Dirac mass δ_{y_i} at the generic point y_i of V_i. Applying Proposition 5.3.3, we conclude $\alpha(A) = 0$ for all sets A of the form $A = \pi^{-1}(U) \cup \mathsf{red}^{-1}(W)$ where $U \subseteq X$ is an open set disjoint from V_i and W is a proper closed subset of V_i. Let $\mathcal{A}$ denote the subalgebra of $C^0(X^{\mathrm{an}})$ consisting of all functions that are constant away from such a set $A = \pi^{-1}(U) \cup \mathsf{red}^{-1}(W)$. Again by Lemma 5.3.1, $\mathcal{A}$ contains all constant functions and separates points, and thus is dense in $C^0(X^{\mathrm{an}})$. If $\varphi \in \mathcal{A}$, say with $\varphi \equiv c$ outside of $A = \pi^{-1}(U) \cup \mathsf{red}^{-1}(W)$, then

$$\int \varphi \, d\alpha = c[1 - \alpha(A)] + \int_A \varphi \, d\alpha = c = \varphi(\mathsf{y}_i),$$

proving that α agrees with δ_{y_i} on the dense subalgebra $\mathcal{A}$. We conclude that $\alpha = \delta_{\mathsf{y}_i}$. $\square$

5.4. Equidistribution for tame valuations

The equidistribution theorems of the previous section are most applicable when K is trivially valued, since in this setting every dynamical system has good reduction. Even in this setting, however, the results are incomplete: there are plenty of points $\mathsf{y} \in X^{\mathrm{an}}$ whose reduction $\mathsf{red}(\mathsf{y})$ lies in the exceptional set, but whose preimages still equidistribute to the Dirac mass at the Gauss point of X^{an}. For instance, one would expect that the preimages of any point $\mathsf{y} \in X^{\mathrm{an}}$ whose kernel $\pi(\mathsf{y})$ is the generic point of X should equidistribute to the Dirac mass at the Gauss point of X^{an}. The goal of this section is to prove that this is often true.

The setup for this section is as follows. We assume K is an algebraically closed field equipped with the trivial absolute value. Let X be an irreducible *smooth* projective variety of dimension r over K, and let $f\colon (X, L) \to (X, L)$ be a polarized morphism of degree $d \geq 2$. Since X is smooth, f is automatically flat.

Let $\pi\colon X^{\mathrm{an}} \to X$ be the canonical map, and let $\mathsf{y} \in X^{\mathrm{an}}$ be any point such that $\pi(\mathsf{y})$ is the generic point of X. This condition says exactly that y is an admissible *norm* on the coordinate ring $K[U]$ some affine open subset $U \subseteq X$, that is, the kernel of y is 0. Therefore y extends to an absolute value on the fraction field of $K[U]$, which is of course just the function field $K(X)$ of X. This reasoning characterizes points y such that $\pi(\mathsf{y})$ is the generic point of X:

Proposition 5.4.1. *The points $\mathsf{y} \in X^{\mathrm{an}}$ with $\pi(\mathsf{y})$ equal to the generic point of X correspond to absolute values on the function field $K(X)$ of X which restrict to the trivial absolute value on K.*

In this trivially valued setting, it is standard not to work with absolute values on $K(X)$, but instead with their corresponding *valuations*. Recall that to any absolute value y on $K(X)$, one associates a valuation v on $K(X)$ by $v(\varphi) = -\log|\varphi(\mathsf{x})|$, and conversely, for any valuation v on $K(X)$ one gets an absolute value x defined by $|\varphi(\mathsf{x})| = e^{-v(\varphi)}$. Thus points $\mathsf{y} \in X^{\mathrm{an}}$ with $\pi(\mathsf{y})$ equal to the generic point of X can be viewed as valuations v on $K(X)$ with $v \equiv 0$ on $K^\times$. We will take this valuative perspective in this section.

Example 5.4.2. Because we have assumed X is smooth, for every point $\xi \in X$ the local ring $\mathcal{O}_{X,\xi}$ is regular. This implies that the function $\mathrm{ord}_\xi\colon \mathcal{O}_{X,\xi} \to \mathbf{N} \cup \{+\infty\}$ given by $\mathrm{ord}_\xi(\varphi) := \max\{n : \varphi \in \mathfrak{m}_\xi^n\}$ is a valuation on the ring $\mathcal{O}_{X,\xi}$. We may then extend ord_ξ to the fraction field $K(X) = \mathrm{Frac}\,\mathcal{O}_{X,\xi}$. This is a valuation on $K(X)$ that is $\equiv 0$ on $K^\times$.

Suppose that v is any valuation on $K(X)$ that is trivial on K. Unraveling the definition of the reduction map red, we see that $\mathsf{red}(v)$ is the (unique) point $\xi \in X$ such that $v \geq 0$ on the local ring $\mathcal{O}_{X,\xi} \subseteq K(X)$, with $v(\varphi) > 0$ if and only if $\varphi \in \mathfrak{m}_\xi$.

Definition 5.4.3. Let v be a valuation on $K(X)$ that is trivial on K, and let $\mathsf{red}(v) = \xi \in X$ be the reduction of v. We will say that v is a *tame valuation* if there is a constant $C > 0$ such that $v(\varphi) \leq C\mathrm{ord}_\xi(\varphi)$ for all $\varphi \in \mathcal{O}_{X,\xi}$.

Example 5.4.4. Tame valuations actually make up quite a bit of the space X^{an}, as the following examples are meant to illustrate.

1. The tame valuations in $\mathbf{P}^{1,\mathrm{an}}_K$ make up the so-called *hyperbolic space* $\mathbf{H} = \mathbf{P}^{1,\mathrm{an}}_K \setminus \mathbf{P}^1_K$ of non-classical points of $\mathbf{P}^{1,\mathrm{an}}_K$.

2. All *monomial valuations* are tame. A monomial valuation v on $K(X)$ is a valuation such that there exists a point $\xi \in X$, a system of parameters $t_1, \ldots, t_s$ of the completed local ring $\hat{\mathcal{O}}_{X,\xi} \cong \kappa(\xi)[\![t_1, \ldots, t_s]\!]$, and constants $\alpha_1, \ldots \alpha_s > 0$ such that
$$v\left(\sum\nolimits_{\beta \in \mathbf{N}^s} \lambda_\beta t^\beta\right) = \min\{\beta_1\alpha_1 + \cdots + \beta_s\alpha_s : \lambda_\beta \neq 0\}$$
on $\mathcal{O}_{X,\xi}$. It is easy to check that this valuation v has reduction ξ, and that v is tame.

3. All *divisorial valuations* are tame. A divisorial valuation v on $K(X)$ is a valuation such that there exists some blowup $p\colon X' \to X$ of X, an exceptional prime divisor E of p, and a real number $\lambda > 0$ such that $v(\varphi) = \lambda\mathrm{ord}_E(\varphi \circ p)$ for all $\varphi \in K(X)$. In this case, the reduction ξ of v is the generic point of $p(E)$ in X. Divisorial valuations are dense in X^{an}.

4. All *quasimonomial valuations* are tame. A valuation v on $K(X)$ is quasimonomial if there is some blowup $p\colon X' \to X$ of X in which v is monomial on $K(X') \cong K(X)$. This class of valuations actually includes both monomial and divisorial valuations. Quasimonomial valuations are studied in detail in [87], see also [56]. The fact that such valuations are tame is not obvious, and follows from a result of Tougeron [109, Lemma IX.1.3], though it is usually attributed to Izumi [85]. Sometimes they are called *Abhyankar valuations*, since they are precisely those valuations for which one has equality in the Abhyankar inequality. Alternatively, in Berkovich analytic language, they can be characterized as being Shilov boundaries of Weierstrass domains in X^{an}, see [98].

5. One should note that if $\dim X > 1$, then there exist tame valuations on $K(X)$ that are not quasimonomial, as well as valuations on $K(X)$ that are not tame.

The rest of the section is dedicated to proving that, so long as the dynamical system f is *separable*, i.e., the field extension $K(X)/f^*K(X)$ is separable, then the preimages of any tame valuation equidistribute to the Dirac mass at the Gauss point of X^{an}.

Proposition 5.4.5. *Let v be any valuation on $K(X)$ that is trivial on K, and let $w \in f^{-1}(v)$. Let L/F denote the field extension $K(X)/f^*K(X)$, so that w is a valuation on L extending v on F. If f is separable, then the multiplicity of f at w is given by $m_f(w) = [L_w : F_v]$, where L_w and F_v denote the completions of L and F with respect to w and v, respectively.*

Proof. Since $v, w \in X^{\mathrm{an}}$ are valuations, i.e., $\pi(v) = \pi(w) =$ the generic point of X, the local rings $\mathscr{O}_{X,w}$ and $\mathscr{O}_{X,v}$ are fields, and $m_f(w) = [\mathscr{O}_{X,w} : \mathscr{O}_{X,v}]$ by definition. The completed residue fields $\mathscr{H}(w)$ and $\mathscr{H}(v)$ of w and v are the completions of the fields $\mathscr{O}_{X,w}$ and $\mathscr{O}_{X,w}$ with respect to the valuations w and v, so necessarily $[\mathscr{H}(w) : \mathscr{H}(v)] \leq [\mathscr{O}_{X,w} : \mathscr{O}_{X,v}]$. Using the isomorphisms $\mathscr{H}(w) \cong L_w$ and $\mathscr{H}(v) \cong F_v$, we conclude $[L_w : F_v] \leq [\mathscr{O}_{X,w} : \mathscr{O}_{X,v}] = m_f(w)$. On the other hand, since L/F is separable,

$$F_v \otimes_F L \cong \bigoplus_{f(w)=v} L_w$$

by [24, Cor. VI.8.2/2]. The F_v-dimension of the left hand side is $[L : F]$, while the F_v-dimension of the right hand side is

$$\sum_{f(w)=v} [L_w : F_v] \leq \sum_{f(w)=v} m_f(w) = [L : F].$$

It follows that one has equality $m_f(w) = [L_w : F_v]$ for each $w \in f^{-1}(v)$. □

Corollary 5.4.6. *Let $N_f \colon K(X)^\times \to f^*K(X)^\times$ be the norm homomorphism of the field extension $K(X)/f^*K(X)$. Assuming f is separable, for any valuation v on $K(X)$ that is trivial on K and any $\varphi \in K(X)^\times$, one has*

$$\sum_{f(w)=v} m_f(w) w(\varphi) = v(N_f(\varphi)).$$

Proof. See [24, Cor. VI.8.5/3]. □

Proposition 5.4.7. *Let v be any valuation on $K(X)$ that is trivial on K. Assume that for each $\varphi \in K(X)^\times$ one has*

$$d^{-rn} \sum_{f^n(w)=v} m_{f^n}(w) |w(\varphi)| \to 0 \quad \text{as } n \to \infty.$$

Then the preimages of v equidistribute to the Dirac mass at the Gauss point of X^{an}.

Proof. Let $\mu_n = d^{-rn} f^{n*}\delta_v$ for each $n \geq 0$. It suffices to show that every weak limit μ of a subsequence μ_{n_i} is the Dirac mass at the Gauss point of X^{an}. Suppose for contradiction that such a μ was not the Dirac mass at the Gauss point. Then there is some irreducible proper closed set $E \subsetneq X$ such hat $\mu(\mathsf{red}^{-1}(E)) > 0$. Let $\psi\colon X^{\mathrm{an}} \to [0,1]$ be the continuous function

$$\psi(w) := \begin{cases} \min\{1, w(\mathfrak{m}_E)\} & \mathsf{red}(w) \in E \\ 0 & \mathsf{red}(w) \notin E \end{cases}$$

where $w(\mathfrak{m}_E) := \min\{w(\varphi) : \varphi \in \mathfrak{m}_E\}$. This function is strictly positive on $\mathsf{red}^{-1}(E)$ and 0 elsewhere, so $\int \psi \, d\mu > 0$. On the other hand, if $\varphi \in \mathfrak{m}_E$, then $\psi(w) \leq |w(\varphi)|$ for every $w \in X^{\mathrm{an}}$, which implies

$$0 < \int \psi \, d\mu = \lim_{i\to\infty} \int \psi \, d\mu_{n_i} \leq \lim_{i\to\infty} d^{-rn_i} \sum_{f^{n_i}(w)=v} m_{f^{n_i}}(w)|w(\varphi)| = 0,$$

a contradiction. This completes the proof. $\square$

Lemma 5.4.8. *Let $p \in X$ be a closed point, and let D be an effective divisor on X with local defining equation φ at p. Then $\mathrm{ord}_p(\varphi) \leq \deg_{L^s} D$, where $s \geq 1$ is an integer large enough that L^s is very ample.*

Proof. The lemma is trivial if $p \notin \mathrm{Supp}(D)$, so assume $p \in \mathrm{Supp}(D)$. Since L^s is very ample, there exist global sections $s_1, \ldots, s_r$ of L^s vanishing at p such that the germs $t_i := s_{i,p} \in \mathfrak{m}_p \subset \mathcal{O}_{X,p}$ generate the tangent space at p. Replacing the s_i by some K-linear combination of the s_i if necessary, the Weierstrass preparation theorem gives that φ can be decomposed in $\hat{\mathcal{O}}_{X,p}$ as $\varphi = uQ$, where u is a unit and

$$Q(t) = t_r^n + g_1(t_1, \ldots, t_{r-1})t_r^{n-1} + \cdots + g_n(t_1, \ldots, t_{r-1})$$

is a Weierstrass polynomial of degree $n = \mathrm{ord}_p(\varphi)$. It follows that

$$\dim_K \mathcal{O}_{X,p}/(\varphi, t_1, \ldots, t_{r-1}) = \dim_K K[t_m]/(t_m^n) = n = \mathrm{ord}_p(\varphi).$$

On the other hand, $\dim_K \mathcal{O}_{X,p}/(\varphi, t_1, \ldots, t_{r-1})$ is exactly the local intersection multiplicity of $D \cdot \mathrm{Div}(s_1) \cdot \cdots \cdot \mathrm{Div}(s_{r-1})$ at p. This is, of course, bounded above by the global intersection number $D \cdot \mathrm{Div}(s_1) \cdot \cdots \cdot \mathrm{Div}(s_{r-1}) = \deg_{L^s} D$. $\square$

Theorem 5.4.9. *Let X be a smooth irreducible projective variety over an algebraically closed, trivially valued field K. Suppose $f\colon (X, L) \to (X, L)$ is a separable polarized morphism of algebraic degree $d \geq 2$. Then the preimages of any tame valuation $v \in X^{\mathrm{an}}$ equidistribute to the Dirac mass at the Gauss point of X^{an}.*

Proof. Let $\varphi \in K^\times$ be a nonconstant function, and let D_1 and D_2 be effective divisors on X with $\mathrm{Div}(\varphi) = D_1 - D_2$. Let $\xi = \mathsf{red}(v)$. Fix an $n \geq 1$, and let $\psi_1, \psi_2 \in K(X)^\times$ be rational functions that are regular at every f^n-preimage of ξ such that $\varphi = \psi_1/\psi_2$. Then if $w \in f^{-n}(v)$, the fact that ψ_i is regular at the reduction of w gives $w(\psi_i) > 0$, and hence

$$|w(\varphi)| = |w(\psi_1) - w(\psi_2)| \leq w(\psi_1) + w(\psi_2).$$

Applying Corollary 5.4.6,

$$\sum_{f^n(w)=v} m_{f^n}(w)|w(\varphi)| \leq v(N_{f^n}(\psi_1)) + v(N_{f^n}(\psi_2)).$$

By construction, ψ_i is a local defining equation of D_i at each $\zeta \in f^{-n}(\xi)$ for $i = 1, 2$. Since $\mathrm{Div}(N_{f^n}(\psi_i)) = f^n_*\mathrm{Div}(\psi_i)$, it follows that $N_{f^n}(\psi_i)$ is regular at ξ and that it is a local defining equation for $f^n_* D_i$ at ξ.

By assumption v is tame, so there is a constant $C > 0$ such that $v \leq C\mathrm{ord}_\xi$ on $\mathcal{O}_{X,\xi}$. We therefore get the inequality

$$\sum_{f^n(w)=v} m_{f^n}(w)|w(\varphi)| \leq C\mathrm{ord}_\xi(N_{f^n}(\psi_1)) + C\mathrm{ord}_\xi(N_{f^n}(\psi_2)).$$

If $p \in X$ is a closed point specializing ξ at which both $N_{f^n}(\psi_1)$ and $N_{f^n}(\psi_2)$ are regular, then $\mathrm{ord}_\xi(N_{f^n}(\psi_i)) \leq \mathrm{ord}_p(N_{f^n}(\psi_i))$, giving

$$\sum_{f^n(w)=v} m_{f^n}(w)|w(\varphi)| \leq C\mathrm{ord}_p(N_{f^n}(\psi_1)) + C\mathrm{ord}_p(N_{f^n}(\psi_2)).$$

Since $N_{f^n}(\psi_i)$ is the local defining equation of $f^n_* D_i$, Lemma 5.4.8 implies that for any integer $s \geq 1$ such that L^s is very ample,

$$\begin{aligned}\sum_{f^n(w)=v} m_{f^n}(w)|w(\varphi)| &\leq C \deg_{L^s} f^n_* D_1 + C \deg_{L^s} f^n_* D_2 \\ &= C \deg_{f^{n*}L^s} D_1 + C \deg_{f^{n*}L^s} D_2 \\ &= Cd^{n(r-1)}(\deg_{L^s} D_1 + \deg_{L^s} D_2).\end{aligned}$$

This proves the estimate

$$d^{-rn} \sum_{f^n(w)=v} m_{f^n}(w)|w(\varphi)| = O(d^{-n}).$$

The theorem now follows from Proposition 5.4.7. $\square$

Appendix A

Measures and dynamics on Zariski spaces

This appendix is devoted to developing a basic theory of measures on Zariski spaces and using this theory to study dynamical systems on these spaces. Much can be said about this topic, but we restrict our focus here to results that are used in the main body of this thesis and instead refer to the author's paper [74] for a broader and more general development. It is worth pointing out that this theory provides a uniform framework for a variety of techniques that have found extensive use in nonarchimedean dynamics [97, 73, 40] and complex dynamics [60, 62, 29, 47, 53, 96, 105].

A.1. Zariski spaces

Recall that a topological space X is said to be *Noetherian* if every descending chain of closed subsets $E_1 \supseteq E_2 \supseteq \cdots$ is eventually constant. An equivalent and often useful definition is the following: X is Noetherian if and only if every nonempty collection of closed subsets has an element which is minimal under inclusion. A closed subset $E \subseteq X$ is said to be irreducible if it cannot be written as a union $E = E_1 \cup E_2$ of two proper closed sets $E_1, E_2 \subsetneq E$. In a Noetherian space, every closed set E can be written as a finite union $E = E_1 \cup \cdots \cup E_r$ of irreducible closed sets. Moreover, if one assumes that $E_i \not\subset E_j$ for $i \neq j$, this decomposition is unique; in this case, the E_i are called the irreducible components of E. The proofs of these facts can be found in [82, §I.1]. Our focus in this appendix is on a special class of Noetherian spaces, called *Zariski spaces*, defined below.

Definition A.1.1. A Noetherian space X is a *Zariski space* if every nonempty irreducible closed subset $E \subseteq X$ has a unique *generic point*, that is, a point which is dense in E.

Many of the Noetherian spaces encountered in practice are Zariski spaces, so this is a case of particular interest. For instance, if X is the underlying topological space of a Noetherian scheme, then X is a Zariski space. In the main body of the thesis, we will apply the results here only in the case with X is an algebraic variety, viewed as a scheme.

Fix a Zariski space X. Since in this appendix we will study Borel measures on X, it will be useful to understand in detail the Borel σ-algebra $\mathscr{B}(X)$ of X. In this section we give an explicit description of Borel subsets of X.

Definition A.1.2. Let $E \subseteq X$ be a nonempty irreducible closed set, and let $A \subseteq X$ be any set. We say that A has *type 1* intersection with E if there exist countably many closed sets $E_n \subsetneq E$ such that $E \smallsetminus \bigcup E_n \subseteq A \cap E$. We say that A has *type 2* intersection with E if there exist countably many closed sets $E_n \subsetneq E$ such that $A \cap E \subseteq \bigcup E_n$.

Observe that if $E \subseteq X$ is a nonempty irreducible closed set and A has type 1 intersection with E, then A contains the generic point of E. On the other hand, if A has type 2 intersection with E, then A does not contain the generic point of E. It is therefore impossible for a set to have both type 1 and type 2 intersection with E, though it is possible for A to have neither. Intuitively, one should think of type 1 intersections as being "thick," with $A \cap E$ containing "most" of E. Similarly type 2 intersections are "thin," with $A \cap E$ containing "hardly any" of E. A set A has type 1 intersection with E if and only if its complement $A^c = X \smallsetminus E$ has type 2 intersection with E.

Proposition A.1.3. *A set $A \subseteq X$ is a Borel set if and only if it has either type 1 or type 2 intersection with every nonempty irreducible closed set E.*

Proof. Let $\mathscr{A}$ be the collection of all sets $A \subseteq X$ that have either type 1 or type 2 intersection with every nonempty irreducible closed set $E \subseteq X$. Note that $\mathscr{A}$ contains all closed subsets of X. We start be showing that $\mathscr{A}$ is a σ-algebra, and hence that $\mathscr{B}(X) \subseteq \mathscr{A}$. Evidently $\mathscr{A}$ is closed under complements, so we only need to check that $\mathscr{A}$ is closed under countable unions. Let $A_1, A_2, \ldots \in \mathscr{A}$, and let $A = \bigcup A_n$. Fix a nonempty irreducible closed set E. If any of the A_n has type 1 intersection with E, then so does A. On the other hand, if all A_n have type 2 intersection with E, then so does A. Thus A has either type 1 or type 2 intersection with E, and we conclude that $A \in \mathscr{A}$. Therefore $\mathscr{B}(X) \subseteq \mathscr{A}$.

Suppose for contradiction that the reverse inclusion is false, say with $A \in \mathscr{A} \smallsetminus \mathscr{B}(X)$. Let T be the collection of all nonempty closed subsets $E \subseteq X$ for which $A \cap E \notin \mathscr{B}(E)$. Since $X \in T$, the collection T is nonempty. Because X is Noetherian, there is an element $E \in T$ which is minimal under inclusion. This E must be irreducible, since if it could be decomposed as a union $E = E_1 \cup E_2$ of proper closed subsets, then $A \cap E_i \in \mathscr{B}(E_i) \subseteq \mathscr{B}(E)$ by the minimality of E, and hence $A \cap E = (A \cap E_1) \cup (A \cap E_2) \in \mathscr{B}(E)$, a contradiction. Replacing A with A^c if necessary, we may assume that A has type 2 intersection with E, that is, $A \cap E \subseteq \bigcup E_n$ for closed sets $E_n \subsetneq E$. But again $A \cap E_n \in \mathscr{B}(E_n) \subseteq \mathscr{B}(E)$ by the minimality of E, so that $A \cap E = \bigcup (A \cap E_n) \in \mathscr{B}(E)$, a contradiction. $\square$

A.2. Classification of measures

In this section, we show that all finite Borel measures on Zariski spaces X are convergent sums of atomic measures. We use this structure theorem to construct a duality between the space of measures of X and certain family of functions on X, analogous to the duality between Radon measures and continuous functions on compact Hausdorff spaces. This duality allows us to topologize the space of measures on X; we then explore the compactness properties of measures in this topology. Throughout this section X will denote a fixed Zariski space, and $\mathcal{M}(X)$ will denote the real vector space of finite signed Borel measures on X. We will write $\mathcal{M}(X)_+ \subseteq \mathcal{M}(X)$ for the cone of positive measures on X.

Lemma A.2.1. *Let $\mu, \nu \in \mathcal{M}(X)_+$ be measures such that $\mu(E) = \nu(E)$ for all irreducible closed subsets $E \subseteq X$. Then $\mu = \nu$.*

Proof. Suppose for contradiction that there is a Borel set $A \in \mathscr{B}(X)$ with $\mu(A) \neq \nu(A)$. Let T be the collection all closed sets E such that $\mu(A \cap E) \neq \nu(A \cap E)$. Let $E \in T$ be a minimal element of T. We first note that E must be irreducible. Indeed, if E could be decomposed as a union $E = E_1 \cup E_2$ of proper closed subsets, then the minimality of E implies

$$\begin{aligned}\mu(A \cap E) &= \mu(A \cap E_1) + \mu(A \cap E_2) - \mu(A \cap E_1 \cap E_2) \\ &= \nu(A \cap E_1) + \nu(A \cap E_2) - \nu(A \cap E_1 \cap E_2) = \nu(A \cap E),\end{aligned}$$

a contradiction. By hypothesis, we then have $\mu(E) = \nu(E)$, which implies that $\mu(A^c \cap E) \neq \nu(A^c \cap E)$, and moreover that E is minimal among closed sets with this property. Therefore, replacing A with A^c if necessary, we may assume A has type 2 intersection with E, say $A \cap E \subseteq \bigcup E_n$. But then

$$\nu(A \cap E) = \lim_{N\to\infty} \mu\left(A \cap \bigcup_{n=1}^{N} E_n\right) = \lim_{N\to\infty} \left(A \cap \bigcup_{n=1}^{N} E_n\right) = \nu(A \cap E),$$

a contradiction. $\square$

Theorem A.2.2. *Every measure $\mu \in \mathcal{M}(X)$ can be written uniquely as an absolutely convergent sum of atoms $\mu = \sum_{x\in X} c_x \delta_x$, with coefficients $c_x \in \mathbf{R}$.*

Proof. Any measure μ can be written as a difference $\mu = \mu^+ - \mu^-$ of positive measures, so it suffices to prove the theorem for positive measures. We will prove that any $\mu \in \mathcal{M}(X)_+$ can be written uniquely as a convergent sum $\sum_{x\in X} c_x \delta_x$, where $c_x \geq 0$ for all $x \in X$.

First assume that μ can be written as a such a convergent sum of atoms $\mu = \sum c_x \delta_x$. Since $\mu(X) = \sum c_x < \infty$, at most countably many of the coefficients c_x can be nonzero. From this it follows immediately that if $E = \overline{\{x\}}$ for some $x \in X$, then

$$c_x = \inf\{\mu(E \smallsetminus F) : F \subsetneq E \text{ closed}\}. \tag{A.1}$$

Thus the coefficients c_x are determined by μ, proving the uniqueness statement. To prove the existence statement, we start by *defining* the constants $c_x \geq 0$ by Equation A.1. Suppose that $E_1, \dots, E_n$ are distinct nonempty irreducible closed subsets of X, with generic points $x_1, \dots, x_n$. Reindexing if necessary, we may assume $E_i \not\subset E_j$ for any $j < i$. Since the E_i are irreducible, this implies $E_i \not\subset E_1 \cup \dots \cup E_{i-1}$ for each i. Then

$$\begin{aligned}\infty > \mu(X) \geq \mu(E_1 \cup \dots \cup E_n) &= \mu(E_1) + \mu(E_2 \smallsetminus E_1) + \dots + \mu(E_n \smallsetminus [E_1 \cup \dots \cup E_{n-1}]) \\ &\geq c_{x_1} + \dots + c_{x_n}.\end{aligned}$$

This proves the sum $\sum c_x$ converges. We are thus justified in defining a measure $\nu \in \mathcal{M}(X)_+$ by $\nu = \sum c_x \delta_x$. To complete the proof, we must show $\nu = \mu$. Suppose for contradiction that $\nu \neq \mu$. By Lemma A.2.1, the collection T of closed sets E such that $\mu(E) \neq \nu(E)$ is nonempty. Let E be a minimal element of T. Then E is irreducible, since if E were reducible with $E = E_1 \cup E_2$, then the minimality of E would imply

$$\mu(E) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \nu(E_1) + \nu(E_2) - \nu(E_1 \cap E_2) = \nu(E).$$

Let x be the generic point of E, and choose a sequence $F_n \subsetneq E$ of closed sets such that $\mu(E \smallsetminus F_n) \to c_x$ and $\nu(E \smallsetminus F_n) \to c_x$. Again using the minimality of E,

$$\mu(E) - c_x = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} \nu(F_n) = \nu(E) - c_x,$$

a contradiction. □

Having classified measures on X, we now move on to characterizing them as dual to a space of functions on X. Recall that a function $f \colon X \to \mathbf{R}$ is said to be upper semicontinuous if $\{x \in X : f(x) \geq r\}$ is closed for all $r \in \mathbf{R}$. We will denote by $SC(X)$ the real vector space of function $f \colon X \to \mathbf{R}$ of the form $f = g - h$, where g and h are bounded upper semicontinuous on X. We equip $SC(X)$ with the sup norm $\|f\| = \sup_X |f|$, making it a normed linear space. We denote by $SC(X)^*$ the continuous dual space of $SC(X)$.

Definition A.2.3. A functional $\varphi \in SC(X)^*$ is *positive* if $\varphi(f) \geq 0$ whenever $f \geq 0$. The set of positive functionals forms a cone $SC(X)^*_+ \subseteq SC(X)^*$.

Proposition A.2.4. *Every $\varphi \in SC(X)^*$ can be written as $\varphi = \varphi^+ - \varphi^-$ where $\varphi^\pm \in SC(X)^*_+$ and $\|\varphi^\pm\| \le \|\varphi\|$.*

Proof. The proof will use the theory of Riesz spaces, see [25, Ch. II]. Specifically, the proposition follows from [25, Theorem II.2.1] if we can show $SC(X)$ is a Riesz space. Let $f \in SC(X)$, say $f = g - h$, where g and h are bounded upper semicontinuous functions. Then

$$\max(f, 0) = g - \min(g, h).$$

Since $\min(g, h)$ is upper semicontinuous, we see that $\max(f, 0) \in SC(X)$. If $f_1, f_2 \in SC(X)$, then $\max(f_1, f_2) = f_1 + \max(f_2 - f_1, 0)$ and $\min(f_1, f_2) = f_1 - \max(f_1 - f_2, 0)$, and hence $\max(f_1, f_2) \in SC(X)$ and $\min(f_1, f_2) \in SC(X)$. This proves $SC(X)$ is a Riesz space. $\square$

Lemma A.2.5. *Let $\mathcal{B}$ denote the collection of characteristic functions χ_E of nonempty irreducible closed sets E. Then $\mathcal{B}$ is a linearly independent family which spans a dense subspace of $SC(X)$. In particular, any $\varphi, \psi \in SC(X)^*$ which agree on $\mathcal{B}$ must be equal.*

Proof. We first prove linear independence. Suppose for contradiction that there is a linear dependence $c_1\chi_{E_1} + \cdots + c_r\chi_{E_r} = 0$ among elements of $\mathcal{B}$, with $c_i \neq 0$ for each i. Considering the supports of these functions, it follows that $E_i \subseteq \bigcup_{i \neq j} E_j$ for each i, contradicting the irreducibility of the E_i. Therefore $\mathcal{B}$ is a linearly independent family.

Next, we prove that for any closed (not necessarily irreducible) set F, the characteristic function χ_F lies in the span of $\mathcal{B}$. Suppose for contradiction that this is not the case, and let T be the collection of closed sets F for which χ_F does not lie in the span of $\mathcal{B}$. Choose a minimal element F of T. Clearly F cannot be irreducible, or else $\chi_F \in \mathcal{B}$ by definition. Let $F = F_1 \cup F_2$ be a nontrivial decomposition of F. But then $\chi_F = \chi_{F_1} + \chi_{F_2} - \chi_{F_1 \cap F_2}$ lies in the span of $\mathcal{B}$ by the minimality of F, a contradiction. Thus the span of $\mathcal{B}$ contains all functions χ_F for F closed.

To complete the proof, we need only show that the span of $\mathcal{B}' = \{\chi_F : F \text{ closed}\}$ is dense in $SC(X)$. Let $f \in SC(X)$, let $a = \inf_X f$, and let $b = \sup_X f$. We may assume $a \neq b$, as otherwise $f = a\chi_X \in \mathcal{B}'$. For any partition $\pi = \{a = r_0 < r_1 < \cdots < r_n = b\}$ of the interval $[a, b]$, let

$$f_\pi := r_0\chi_X + \sum_{i=1}^{n} (r_i - r_{i-1})\chi_{\{f \ge r_i\}}.$$

By construction $f_\pi \in \operatorname{span}(\mathcal{B}')$ for each partition π, and $\|f - f_\pi\| \le \operatorname{mesh}(\pi)$. Thus as we let $\operatorname{mesh}(\pi) \to 0$, the functions f_π converge uniformly to f. $\square$

Theorem A.2.6. *The linear map $\Lambda\colon \mathcal{M}(X) \to SC(X)^*$ given by integration,*

$$\Lambda(\mu)(f) := \int_X f \, d\mu,$$

is an isomorphism, thus inducing a duality $\mathcal{M}(X) \cong SC(X)^*$.

Proof. It is clear that Λ maps $\mathcal{M}(X)_+$ into $SC(X)^*_+$. By Lemma A.2.1 Λ is injective. By Proposition A.2.4, we only need to prove $\Lambda\colon \mathcal{M}(X)_+ \to SC(X)^*_+$ is surjective. Suppose $\varphi \in SC(X)^*_+$. For every nonempty irreducible closed set E, say with generic point x, define $c_x = \inf\{\varphi(\chi_E - \chi_F) : F \subsetneq E \text{ closed}\} \geq 0$. Suppose now that $E_1, \dots, E_n$ are any nonempty irreducible closed sets with generic points $x_1, \dots, x_n$. Reindexing if necessary, we may assume $E_i \not\subset E_j$ for $j < i$. Since the E_i are irreducible, this implies $E_i \not\subset E_1 \cup \cdots \cup E_{i-1}$ for all i. Then

$$\begin{aligned}\infty > \|\varphi\| \geq \varphi(\chi_{E_1\cup\cdots\cup E_n}) &= \varphi(\chi_{E_1}) + \varphi(\chi_{E_2} - \chi_{E_2\cap E_1}) + \cdots + \varphi(\chi_{E_n} - \chi_{E_n\cap(E_1\cup\cdots\cup E_{n-1})}) \\ &\geq c_{x_1} + \cdots + c_{x_n}.\end{aligned}$$

This proves the sum $\sum_{x\in X} c_x$ converges. We may then define $\mu = \sum c_x\delta_x \in \mathcal{M}(X)_+$. We will show that $\varphi = \Lambda(\mu)$. Suppose for contradiction that $\varphi \neq \Lambda(\mu)$. By Lemma A.2.5, the collection T of closed sets E for which $\varphi(\chi_E) \neq \Lambda(\mu)(\chi_E)$ is nonempty. Let E be a minimal element of T. This E is irreducible, since if E were reducible with decomposition $E = E_1 \cup E_2$, then the minimality of E would imply

$$\begin{aligned}\varphi(\chi_E) &= \varphi(\chi_{E_1}) + \varphi(\chi_{E_2}) - \varphi(\chi_{E_1\cap E_2}) \\ &= \Lambda(\mu)(\chi_{E_1}) + \Lambda(\mu)(\chi_{E_2}) - \Lambda(\mu)(\chi_{E_1\cap E_2}) = \Lambda(\mu)(\chi_E).\end{aligned}$$

Let x be the generic point of E, and let $F_n \subsetneq E$ be a sequence of closed sets such that $\varphi(\chi_E - \chi_{F_n}) \to c_x$ and $\Lambda(\mu)(\chi_E - \chi_{F_n}) = \mu(E \setminus F_n) \to c_x$. By the minimality of E,

$$\varphi(\chi_E) - c_x = \lim_{n\to\infty} \varphi(\chi_{F_n}) = \lim_{n\to\infty} \Lambda(\mu)(\chi_{F_n}) = \Lambda(\mu)(\chi_E) - c_x,$$

a contradiction. $\square$

The dual space $SC(X)^*$ has two natural topologies. First, one has the *strong topology*, which is induced by the operator norm on $SC(X)^*$. Second, one has the *weak topology*, that is, the topology of pointwise convergence. The latter is of most interest to us. Both topologies can be pulled back to $\mathcal{M}(X)$ via the isomorphism Λ. It is easy to verify using Lemma A.2.5 that a sequence μ_n of measures on X converges weakly to a measure μ if and only if $\mu_n(E) \to \mu(E)$ for all closed sets E.

Theorem A.2.7. *The set of probability measures on X is both compact and sequentially compact in the weak topology.*

The first part of Theorem A.2.7 (compactness) is an immediate corollary of the Banach-Alaoglu theorem, which is itself a nontrivial consequence of the axiom of choice. The second part of the theorem (sequential compactness) does not follow in any obvious way from compactness, and also requires a technical axiom of choice argument. Due to its length and technicality, we omit the proof here, and instead refer to [74, §4] for details.

A.3. Asymptotic behavior of orbits in Zariski dynamics

Now that we have developed a theory of measures on Zariski spaces, we can use it to study dynamical systems on these spaces. We will see in this section that such systems have very strong ergodic properties, and much can be said about the asymptotic behavior of orbits, both in forward and reverse time. In this section, the dynamical systems we consider are continuous self-maps $f\colon X \to X$ of Zariski spaces X.

We first consider the asymptotic behavior of the *forward* orbit of a point $x \in X$. The main result regarding this behavior is the following theorem of Favre [60].

Theorem A.3.1. *Let $f\colon X \to X$ be a continuous self-map of a Zariski space, and let $x \in X$ be any point. Then there exists a periodic cycle $y_1, \ldots, y_r \in X$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \delta_x \to \frac{1}{r}(\delta_{y_1} + \cdots + \delta_{y_r})$$

weakly as $n \to \infty$. Moreover, the y_i are the generic points of the irreducible components of the smallest closed set which contains $f^n(x)$ for all sufficiently large n.

By using the duality $\mathcal{M}(X) \cong SC(X)^*$, one can reformulate Theorem A.3.1 as a theorem about upper semicontinuous functions: for every bounded upper semicontinuous function $\tau\colon X \to \mathbf{R}$, the time averages along an orbit $n^{-1}\sum_{k=0}^{n-1}\tau(f^k(x))$ will *always* converge, namely to $r^{-1}(\tau(y_1) + \cdots + \tau(y_r))$. In this way, dynamical systems on Zariski spaces have strikingly strong ergodic properties. This formulation in terms of upper semicontinuous functions is how Favre presented his theorem originally. The new measure theoretic interpretation, as well as a simplified measure theoretic proof, are given in [74, §7]. We omit the proof here.

Example A.3.2. The dynamical Mordell-Lang conjecture states that if $f\colon X \to X$ is an endomorphism of a complex algebraic variety, $E \subseteq X$ is a subvariety, and $x \in X$, then the set $\{n \in \mathbf{N} : f^n(x) \in E\}$ is the union of a finite number of infinite arithmetic progressions and a finite set. As an essentially immediate corollary of Theorem A.3.1, one has the following weaker result: the set $\{n \in \mathbf{N} : f^n(x) \in E\}$ is the union of a finite number of infinite arithmetic progressions and a *zero density* set, cf. [41].

Example A.3.3. Fix an element $a \in \mathbf{G}_m^N \subseteq \mathbf{A}_k^N$, and let $T_a \colon \mathbf{G}_m^N \to \mathbf{G}_m^N$ be the translation map $T(x) = ax$. Assume that the orbit of 1 under T_a is Zariski dense. Then Theorem A.3.1 tells us that the averages $n^{-1} \sum_{i=0}^{n-1} \delta_{a^i}$ converge weakly the generic point of $\mathbf{G}_m^N$. This and Proposition 5.3.3 combine to prove a nonarchimedean analogue of Weyl's equidistribution theorem due to Petsche [97, Theorem 1].

In the setting of complex dynamics, Theorem A.3.1 has been a useful tool in several investigations of the equidistribution of preimages problem [62, 29, 96]. In other treatments of the problem [47, 53, 105], it has also proved useful to study the asymptotic behavior of *reverse* orbits. This is how we approach the problem in this thesis. The measure theoretic proof of Theorem A.3.1 can be adapted to give a theorem on the asymptotic behavior of backward orbits.

Theorem A.3.4. *Let $f \colon X \to X$ be a surjective continuous self-map of a Zariski space X. Fix a point $x \in X$ and a reverse orbit of x, that is, a sequence $\{x_{-n}\}_{n=0}^{\infty}$ of points such that $x_0 = x$ and $f(x_{-n}) = x_{-n+1}$ for all $n \geq 1$. Then there exists a periodic cycle $y_1, \ldots, y_r \in X$ of f such that*

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{x_{-k}} \to \frac{1}{r}(\delta_{y_1} + \cdots + \delta_{y_r})$$

weakly as $n \to \infty$. Moreover, the y_i are the generic points of the irreducible components of the closed set $\overline{\{x_{-k} : k \geq 0\}}$.

Again, see [74, §7] for the proof. In his work in complex dynamics, Dinh has given an alternative approach to studying reverse orbits [47], which we use extensively in Chapter 3. It is therefore worthwhile to adapt some of his results to our setting.

Fix a surjective continuous self-map $f \colon X \to X$ of a Zariski space X. Let $\tau \colon X \to \mathbf{R}$ be a bounded upper semicontinuous function on X. We can then define a sequence of bounded upper semicontinuous functions τ_n by $\tau_n := \sum_{k=0}^{n-1} \tau \circ f^k$. Theorem A.3.1 studies the asymptotic behavior of forward orbits by considering the limit $\tau_+ := \lim_{n\to\infty} \frac{1}{n}\tau_n$. To study reverse orbits, Dinh defines bounded upper semicontinuous functions τ_{-n} by

$$\tau_{-n}(x) := \sup_{f^n(y)=x} \tau_n(y)$$

for all $n \geq 1$. Dinh then considers the analogous limit

$$\tau_-(x) := \lim_{n\to\infty} \frac{1}{n} \tau_{-n}(x). \tag{A.2}$$

The most important properties of this function τ_- are summarized in the next theorem.

Theorem A.3.5. *The limit in Equation A.2 exists for any point $x \in X$, and the limit function τ_- satisfies the following properties:*

1. *For any $x \in X$, one has $\tau_-(x) = \max \tau_+(y)$, where the maximum is taken over periodic points y such that $x \in \overline{\{y\}}$.*

2. *For any $x \in X$, one has $\tau_-(x) = \max \lim_{n\to\infty} \tau_n(x_{-n})/n$, where the maximum is taken over all reverse orbits $\{x_{-n}\}_{n=1}^{\infty}$ of x.*

3. *The function $\tau_- \colon X \to \mathbf{R}$ is upper semicontinuous.*

4. *One has $\tau_-(x) \leq \tau_+(x)$ for all $x \in X$. If x is periodic, then equality holds.*

Each of the four statements in Theorem A.3.5 follow more or less immediately from the proof of the existence of the limit A.2. We will prove the limit exists in several steps.

Lemma A.3.6. *Let $c \in \mathbf{R}$. Then there is an $a < c$ such that $\tau(x) < a$ for every $x \in X$ such that $\tau(x) < c$.*

Proof. Observe that $\{x \in X : \tau(x) < c\}$ is the union of the increasing chain of open sets $U_n = \{x \in X : \tau(x) < c - 1/n\}$. Since X is Noetherian, this chain stabilizes, proving that $\{x \in X : \tau(x) < c\} = U_N$ for some N. $\square$

Proposition A.3.7. *Fix $c \in \mathbf{R}$, and let $Z = \{x \in X : \tau_n(x) \geq cn \text{ for all } n \geq 1\}$. Then there is a real number $b < c$ and an integer $N \geq 1$ with the following property: if $n \geq N$ and $x \in X$ are such that $f^k(x) \notin Z$ for all $k = 0, \dots, n$, then $\tau_n(x) \leq bn$.*

Proof. Let $V_n = \{x \in X : \tau_n(x) \geq cn\}$ for each $n \geq 1$, and let $U_n = X \smallsetminus V_n$. The V_n are closed, and by definition $Z = \bigcap V_n$. Since X is Noetherian, there is an integer $M \geq 1$ such that $Z = V_1 \cap \cdots \cap V_M$. By Lemma A.3.6 there is a real number $a < c$ such that $\tau_n(x) < an$ for any $x \in U_n$, where $n = 1, \dots, M$. We choose N large enough that $a + \|\tau\| M/N < c$, and let $b = a + \|\tau\| M/N$.

Suppose $n \geq N$ and $x \in X$ are such that $f^k(x) \notin Z$ for all $k = 0, \dots, n$. We recursively define a finite sequence k_i of integers as follows. First, set $k_1 = 0$. Assume now that k_i has been defined. If $n - k_i \leq M$, stop defining the k_i. If $n - k_i > M$, then by hypothesis $f^{k_i} \in U_j$ for some $j = 1, \dots, M$, and we set $k_{i+1} = k_i + j$. Suppose $k_1, \dots, k_\ell$ is the sequence constructed in this fashion. By construction, $0 \leq n - k_\ell \leq M$ and $\tau_{k_\ell} \leq a k_\ell$. It follows that

$$\begin{aligned}\tau_n(x) &= \tau_{k_\ell} + \tau_{n-k_\ell}(f^{k_\ell}(x)) \leq a k_\ell + \|\tau\|(n - k_\ell) \leq an + \|\tau\| M \\ &= (a + \|\tau\| M/n) n \leq bn,\end{aligned}$$

as desired. $\square$

Proposition A.3.8. *Suppose $y \in X$ is periodic and $x \in \overline{\{y\}}$. Then*

$$\liminf_{n\to\infty} \frac{\tau_{-n}(x)}{n} \geq \tau_+(y).$$

Proof. Let $y_0, y_1, \ldots, y_{r-1}$ be the orbit of y, where without loss of generality $y = y_0$ and $f(y_i) = y_{i-1}$, the indices taken modulo r. One may then choose a reverse orbit $\{x_{-n}\}_{n=0}^{\infty}$ of x such that $x_{-n} \in \overline{\{y_n\}}$ for all n, again with the indices for the y_n being taken modulo r. The proposition then follows from the inequalities $\tau_{-n}(x) \geq \tau_n(x_{-n}) \geq \tau_n(y_n)$ and the easy observation that $\tau_n(y_n)/n \to \tau_+(y)$ as $n \to \infty$. □

Proof of Theorem A.3.5. Let $c = \limsup_{n\to\infty} \tau_{-n}(x)/n$ and $Z = \{y \in X : \tau_n(y) \geq cn \text{ for all } n \geq 1\}$. Choose b and N as in Proposition A.3.7. Suppose Z has irreducible decomposition $Z = Z_1 \cup \cdots \cup Z_r$, and let z_i be the generic point of Z_i for each i. Let $L(z_i)$ denote the *ω-limit set* of z_i, that is, the smallest closed subset of X that contains $f^n(z_i)$ for all sufficiently large n. It is easy to see that $L(z_i)$ is an invariant closed set.

We begin by showing that $x \in L(z_i)$ for some i. Suppose for contradiction that x lies in no $L(z_i)$. Choose an integer s large enough that $f^s(Z_i) \in L(z_i)$ for each i. If $n \geq s$ and $y \in X$ are such that $f^n(y) = x$, it follows that $y \notin Z$. Furthermore, if $n \geq s + N$, then Proposition A.3.7 gives us that

$$\frac{\tau_{-n}(x)}{n} \leq \frac{1}{n}(b(n-s) + s\|\tau\|) \to b,$$

a contradiction of $\limsup_{n\to\infty} \tau_{-n}(x)/n = c > b$.

This proves that there must be some i with $x \in L(z_i)$. Let F be a component of $L(z_i)$ with $x \in F$, and let y be its generic point. Then y is periodic, and Theorem A.3.1 says precisely that $\tau_+(z_i) = \tau_+(y)$. By the definition of Z, we know $\tau_+(z_i) \geq c$. We conclude by Proposition A.3.8 that $\liminf_{n\to\infty} \tau_{-n}(x)/n \geq c$. This shows that the limit A.2 exists. We move on to proving statements 1. through 4., which are now easily deducible.

(1) We have already found a periodic point y such that $x \in \overline{\{y\}}$ and $\tau_+(y) \geq c$. On the other hand, if y' is a periodic point with $x \in \overline{\{y'\}}$, then $\tau_+(y') \leq c$ by Proposition A.3.8.

(2) By definition, $c = \tau_-(x) \geq \lim_{n\to\infty} \tau_n(x_{-n})/n$ for any given reverse orbit $\{x_{-n}\}_{n=0}^{\infty}$ of x. If we choose our reverse orbit so that it always lies in the closure of the (periodic) orbit of y, then in fact $c = \tau_+(y) \leq \lim_{n\to\infty} \tau_n(x_{-n})/n$ as well.

(3) We have proved that $\{x \in X : \tau_-(x) \geq c\}$ is the closed set $L(z_1) \cup \cdots \cup L(z_r)$.

(4) Since $x \in \overline{\{y\}}$, we have $L(y) \supseteq L(x)$, and hence $\tau_+(y) \leq \tau_+(x)$ by Theorem A.3.1. Since $\tau_-(x) = \tau_+(y)$, this gives the desired inequality. If x is periodic, we can take a periodic reverse orbit $\{x_{-n}\}_{n=0}^{\infty}$ of x, and conclude from 2. that $\tau_+(x) = \lim \tau_n(x_{-n})/n \leq \tau_-(x)$. □

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