The Rank Rigidity Theorem for Manifolds with No Focal Points

by

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Abstract

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We say that a Riemannian manifold M has $\operatorname{rank} M \geq k$ if every geodesic in M admits at least k parallel Jacobi fields. The Rank Rigidity Theorem of Ballmann and Burns-Spatzier, later generalized by Eberlein-Heber, states that a complete, irreducible, simply connected Riemannian manifold M of rank $k \geq 2$ (the "higher rank" assumption) whose isometry group Γ satisfies the condition that the Γ-recurrent vectors are dense in SM is a symmetric space of noncompact type. This includes, for example, higher rank M which admit a finite volume quotient. We adapt the method of Ballmann and Eberlein-Heber to prove a generalization of this theorem where the manifold M is assumed only to have no focal points. We then use this theorem to generalize to no focal points a result of Ballmann-Eberlein stating that for compact manifolds of nonpositive curvature, rank is an invariant of the fundamental group.

Chapter I

Introduction

This thesis aims to generalize certain "geometric rigidity" results in the theory of manifolds of nonpositive curvature by replacing the condition of nonpositive curvature with the weaker condition that the manifold has "no focal points". Before we proceed, however, we would like to give a brief introduction to rigidity results in geometry, and "rank rigidity" results in particular. This is accomplished in the present chapter. Chapter II gives a mathematical introduction to the material of this thesis, Chapter III develops some necessary background and tools for manifolds with no focal points, and our main theorems are proven in Chapters IV and V. The main results of this thesis have been published in [47].

1.1 Rigidity results in geometry

The term "rigidity result" has no technical meaning; examples serve to best illustrate the idea. The canonical example of a rigidity result is Mostow's celebrated Rigidity Theorem, proved in 1968:

Mostow's Rigidity Theorem ([37])**.** *Let M and N be compact Riemannian manifolds, each of dimension at least three, and with constant sectional curvature -1. Suppose the fundamental groups $\pi_1 M$ and $\pi_1 N$ are isomorphic; then M and N are isometric.*

In 1973 Prasad [41] generalized Mostow's theorem to the case where M and N have finite volume.

Mostow's proof works by lifting a homotopy $M \to N$ to a map between the "boundaries at infinity" of their universal covers, using arguments from geometry, dynamics, and analysis to show that this map on boundaries is conformal, and then arguing that such a map descends to an isometry $M \to N$. This idea of looking at the boundary of $\tilde{M}$ is a crucial component of many proofs of geometric rigidity results, including the main proof of this paper. We discuss this boundary for manifolds with no focal points in section 3.3.

If one thinks of a finite volume M equipped with a Riemmanian metric of constant negative curvature, one may interpret Mostow's result as saying that the metric on M is "rigid" in the sense that there do not exist any other metrics on M also having constant negative curvature. Note also that this result fails in dimension 2: The space of constant-curvature -1 metrics on a surface of genus $g \geq 2$ is $(6g-6)$-dimensional.

There is a large pool of rigidity results, including Mostow's, that concern themselves with semisimple Lie groups and symmetric spaces. The reader unfamiliar with the idea of semisimplicity may think of the Lie groups $SL(n, \mathbb{R})$; these exhibit much of the behavior of semisimple Lie groups in general. Some definitions: A semisimple Lie group is said to be *of noncompact type* if it has no compact factors (i.e., no nontrivial normal subgroup is compact). A *symmetric space of noncompact type* is a quotient G/K where G is a semisimple Lie group of noncompact type and K a maximal compact subgroup of G; there is a natural way to put a G-invariant metric on G/K making it into a Riemannian manifold. The algebraic nature of these manifolds makes them susceptible to a wide array of mathematical techniques, from dynamics to algebraic geometry to number theory; they are extensively studied. We sketch the necessary background in Section 2.2; the reader is encouraged to think of $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ and $SO(1,n)/SO(n)$ (which is isometric to $\mathbb{H}^n$) as primary examples.

Let us take a moment to rephrase Mostow's result in terms of symmetric spaces. It is well-known that the universal cover of a manifold of dimension n and constant curvature -1 is the hyperbolic space $\mathbb{H}^n$, and it is not difficult to show that $\mathbb{H}^n$ is isometric to the symmetric space $SO(1,n)/SO(n)$. The manifolds M and N of Mostow's theorem, having dimensions m and n, respectively, have fundamental groups $\pi_1(M) \subseteq SO(1,m)$ and $\pi_1(N) \subseteq SO(1,n)$. Hence we may write

$$M = \pi_1(M)\backslash SO(1,m)/SO(m) \text{ and } N = \pi_1(N)\backslash SO(1,n)/SO(n),$$

which realizes M and N as quotients of symmetric spaces. (Such quotients are called *locally symmetric spaces.*) It is natural in this context to wonder if Mostow's rigidity result applies to symmetric spaces coming from semisimple Lie groups other than $SO(1,n)$. This is indeed the case; the precise result, proved by Mostow in 1973, is as follows:

Mostow's Rigidity Theorem ([38])**.** *Let G, G' be connected semisimple Lie groups of noncompact type and trivial center, and let $\Gamma \subset G$ and $\Gamma' \subset G$ be discrete, cocompact subgroups. Assume Γ is irreducible and G is not isomorphic to $SL(2,\mathbb{R})$. Then any isomorphism $\pi : \Gamma \to \Gamma'$ extends to an isomorphism $\pi : G \to G'$.*

The condition that a semisimple group G be of noncompact type is equivalent to the geometric condition that G/K have nonpositive curvature; if G is simple, then this is equivalent to G being noncompact. The condition that Γ is irreducible is mildly technical and outside the scope of this thesis; it is meant to eliminate lattices like $\Gamma_1 \times \Gamma_2 \subset G_1 \times G_2$, and is satisfied trivially when G is simple.

If K is a maximal compact subgroup of G, then $\pi(K)$ is maximal compact in G', so π descends to a map of locally symmetric spaces $\pi : \Gamma\backslash G/K \to \Gamma'\backslash G'/\pi(K)$, and it isn't hard to check that this map is an isometry. Thus the result above specializes to Mostow's original theorem in the case $G = SO(1,n), G' = SO(1,m)$ for $n, m \geq 3$. Note that the condition that G not be isomorphic to $SL(2,\mathbb{R})$ eliminates the 2-dimensional counterexamples to Mostow's theorem.

Mostow's proof of the above result again makes critical use of the boundary of the symmetric space G/K. In addition, it is divided into two cases: the case where G/K has "rank one", which is similar in spirit to the 1968 result, and the case where it has "higher rank", which requires new insight. In particular, in the higher rank case Mostow makes use of a Tits building structure on the boundary of G/K; in the proof of our main result, some aspects of this structure are captured by the "Tits metric" on the boundary of our manifold, developed in section 3.4.

We give a geometric description of the *rank* of a symmetric space G/K of noncompact type, since this is the description that will be most useful to us later. Consider totally geodesic, isometric embeddings $f : \mathbb{R}^k \hookrightarrow G/K$. Such embeddings (or, equivalently, their images) are called k-flats in G/K. The maximal k such that G/K has a k-flat is called the *rank* of G/K. (For more general Riemannian manifolds, this definition of rank has to be modified; see section 2.1.) We remark that the rank of G/K is equal to the $\mathbb{R}$-rank of G in the sense of algebraic groups.

G/K is called *rank one* if its rank is equal to 1, and *higher rank* otherwise. A cornerstone result in the theory of higher rank symmetric spaces is Margulis' superrigidity theorem, proved by Margulis in the early 70's, which deals with extending maps out of lattice subgroups Γ of G to maps defined on all of G. One statement of a specific case of Margulis' superrigidity theorem is as follows:

Margulis' Superrigidity Theorem. *Let G be a connected, higher rank semisimple Lie group of noncompact type, and let Γ be an irreducible lattice in G. Let H be a simple, connected, noncompact real algebraic subgroup of $GL(n, \mathbb{R})$, and suppose we have a homomorphism $\pi : \Gamma \to H$ such that $\pi(\Gamma)$ is Zariski dense in H. Then π extends to a homomorphism $G \to H$.*

The precise statement of Margulis' theorem is somewhat more general than this; see the book by Margulis [35] or by R.J. Zimmer [51] for a statement as well as a proof of the general result. The Borel density theorem implies that if $\Gamma' \subset H$ is a lattice, then Γ' is Zariski dense in H; from this and superrigidity, one may deduce Mostow's rigidity theorem in the special case that G has higher rank. A second astounding consequence of superrigidity (which takes a bit more work to show) is Margulis' Arithmeticity Theorem, which states that lattices Γ in higher rank groups G must be "arithmetic", which very roughly means that they come from a nice algebraic-type construction; again, see [35] or [51] for a precise statement and proof.

From Mostow's theorem, one knows that a given subgroup Γ can only be the fundamental group of a unique locally symmetric space—said another way, Mostow rigidity implies that there is at most one locally symmetric metric on the smooth manifold $\Gamma \backslash G/K$ (where G is semisimple of noncompact type). In the early 80's Gromov proved a stronger result:

Theorem. *Let M be a nonpositively curved compact locally symmetric space whose universal cover $\widetilde{M}$ does not split as a product. Then the symmetric metric is the only nonpositively curved metric on M.*

Eberlein [21] obtained the same result under the added assumption that $\widetilde{M}$ splits as a product (but still without flat factors). We remark that Gromov's proof again relies heavily on various structural properties of the boundary of M at ∞ and its isometries.

1.2 Higher rank rigidity

To proceed, we require a more general definition of the *rank* of a Riemannian manifold M. Let $v \in SM$ be a unit tangent vector; the *rank* of v is the dimension of the space of parallel Jacobi fields along the geodesic γ_v through v. Note that if $\sigma : \mathbb{R}^k \to M$ is a totally geodesic isometric embedding, i.e., a k-flat in M, with v tangent to the image $\sigma(\mathbb{R}^k)$, then variations of geodesics parallel along the flat give rise to parallel Jacobi fields along γ_v. Thus our notion of "rank of v" is an infinitesimal version of the number "the largest k such that M has a k-flat through v". It's possible to show that this agrees with our prior definition of rank in the case G/K is a nonpositively curved symmetric space. The *rank* of M is then defined to be the minimum of rank(v) over all $v \in SM$.

In the mid-80's, building on an analysis of higher rank manifolds of nonpositive curvature carried out by Ballmann, Brin, Eberlein, and Spatzier in [5] and [6], Ballmann in [3] and Burns-Spatzier in [12] and [11] independently (and with different methods) proved the following higher rank rigidity theorem:

Rank Rigidity Theorem. *Let M be a complete, simply connected, irreducible Riemannian manifold of nonpositive curvature, rank $k \geq 2$, and curvature bounded below; suppose also M admits a finite volume quotient. Then M is a locally symmetric space of noncompact type.*

The theorem was later generalized by Eberlein-Heber in [22]. They removed the lower curvature bound, and also generalized the condition that M admit a finite volume quotient to the condition that a dense set of geodesics in M be Γ-recurrent;

they called this condition the "duality condition", for reasons not discussed here.

Our main result is a generalization of the Higher Rank Rigidity Theorem aabove. Before discussing it, however, we take a moment to survey two other "rank rigidity" theorems inspired by the above result.

In 1991, Hamenstädt gave the following definition: A Riemannian manifold M has *higher hyperbolic rank* if the sectional curvature of M is bounded above by -1, and along every geodesic γ there is a Jacobi field J making sectional curvature -1 with the geodesic (i.e. $\kappa(J(t), \dot{\gamma}(t)) = -1$ for all t, where $\kappa(v, w)$ denotes the sectional curvature of the plane spanned by v, w). She then proved [29] the following result:

Theorem. *A closed Riemannian manifold of higher hyperbolic rank is locally symmetric.*

The analogue of hyperbolic rank for positive curvature is called *spherical rank*; a Riemannian manifold M is said to have *positive spherical rank* if its sectional curvature is bounded above by 1, and every geodesic $\gamma : [0, \pi] \to M$ has a conjugate point at π. We remark that it follows from Rauch's comparison theorem and the curvature bound that no geodesic of M can have a conjugate point before π.

Using this notion, Shankar, Spatzier, and Wilking in 2005 proved the following [43]:

Theorem. *Let M be a complete simply connected Riemannian manifold of positive spherical rank. Then M is isometric to a compact rank one symmetric space.*

Note that both the hyperbolic and spherical rigidity results, like the higher rank rigidity result, assume both an upper curvature bound and also that geodesics satisfy some extremal condition with respect to that bound. One might similarly ask for rank-rigidity type theorems for manifolds that satisfy a *lower* curvature bound. Some questions in this area are still open. However, Heintze and Spatzier-Strake [46] have constructed (one-parameter families of) compact manifolds M with higher rank and nonnegative sectional curvature which are nonsymmetric, showing that the analog of higher rank rigidity in nonpositive curvature fails.

In the positive direction, Constantine [14] has shown the following:

Theorem. *Let M be a compact rank one manifold with nonpositive sectional curvature, and suppose that along every geodesic in M there is a parallel vector field making sectional curvature -1 with the geodesic. If M is odd dimensional, or if M is even dimensional and has sectional curvature κ pinched as $-\Lambda^2 < \kappa < -\lambda^2$ with $\lambda/\Lambda > .93$, then M has constant sectional curvature -1.*

Note that although an upper curvature bound is assumed, the condition on geodesics is not the extremal one implied by this bound.

1.3 Results of this thesis

Our aim in this thesis is to generalize the nonpositive curvature assumption of Ballmann, Burns-Spatzier, and Eberlein-Heber's result to a condition known as "no focal points". Precisely, M has no focal points if every Jacobi field J along a geodesic γ in M satisfying $J(0) = 0$ has $||J(t)||$ strictly increasing for $t > 0$. We investigate this definition more in Section 3.1; for the moment, we note that nonpositively curved manifolds have no focal points, so our result implies the higher rank rigidity theorem above. Specifically, Chapter IV proves the following:

Rank Rigidity Theorem. *Let M be a complete, simply connected, irreducible Riemannian manifold with no focal points and rank $k \geq 2$ with group of isometries Γ, and suppose that the Γ-recurrent vectors are dense in the unit tangent bundle SM. Then M is a symmetric space of noncompact type.*

When M admits a finite volume quotient, the Γ-recurrent vectors are dense in M. As a consequence we obtain the following corollary:

Corollary. *Let N be a complete, finite volume, irreducible Riemannian manifold with no focal points and rank $k \geq 2$; then N is locally symmetric.*

The conditions of no focal points and density of Γ-recurrent vectors pass nicely to de Rham factors; because of this, we will also get a decomposition theorem:

Corollary. *Let M be a complete, simply connected Riemannian manifold with no focal points and with group of isometries Γ, and suppose that the Γ-recurrent vectors are dense in SM. Then M decomposes as a Riemannian product*

$$M = E_r \times M_S \times M_1 \times \cdot \times M_l,$$

where E_r is a Euclidean space (of dimension r), M_S is a symmetric space of non-compact type and higher rank, and each factor M_i for $1 \leq i \leq l$ is an irreducible rank-one Riemannian manifold with no focal points.

In 1987, following the work of Prasad-Ragunathan [42], Ballmann and Eberlein in [8] defined the *rank* of an abstract group, and used the Higher Rank Rigidity Theorem in nonpositive curvature to show that, for nonpositively curved manifolds of finite volume, the rank of the manifold is equal to the rank of the fundamental group. Notice that Gromov's result then follows as a simple corollary of this and higher rank rigidity.

In Chapter V, we generalize their proof to the case of no focal points (but now adding in the assumption that the manifold is compact), obtaining the following:

Theorem. *Let M be a complete, simply connected Riemannian manifold without focal points, and let Γ be a discrete, cocompact subgroup of isometries of M acting freely and properly on M. Then* $\text{rank}(\Gamma) = \text{rank}(M)$.

As a corollary of this and the higher rank rigidity theorem, we find the following generalization of Gromov's theorem:

Corollary. *The locally symmetric metric is the unique Riemannian metric of no focal points on an irreducible, compact locally symmetric space of nonpositive curvature.*

(In the above corollary, "irreducible" means that the universal cover of M does not split as a product. However, we remark that if M has no flat factors, the results of chapter V imply that if $\widetilde{M}$ splits as a product, then so does some finite cover of M.)

The results of this thesis have been used by A. Zimmer [49] to show that compact asymptotically harmonic manifolds with no focal points are either flat or a rank one symmetric space of noncompact type. (Zimmer generalizes this result to other cases in [50].) In addition, Ledrappier and Shu [34] have used these results to obtain an entropy rigidity theorem for compact manifolds without focal points, showing that the equality of various notions of entropy on such a manifold M implies that M is locally symmetric.

1.4 A few questions

The Heintze and Spatzier-Strake counterexamples show that some additional hypotheses are needed in order to obtain a higher rank rigidity theorem. The most obvious generalization suggested by our results might be:

Question I.1. *Let M be a closed irreducible Riemannian manifold with higher rank and no conjugate points. Must M be locally symmetric?*

No counterexamples are known, but a proof would probably stray heavily from the methods of the current work. In particular, our proof relies heavily on the so-called Flat Strip Theorem (see Section 3.2), which fails in general for manifolds with no conjugate points, as shown by Burns in [10].

One might also ask whether it is possible to remove the assumption that Γ-recurrent vectors are dense in SM. Again, no counterexamples are known. However, the methods of Ballmann, Burns-Spatzier, Eberlein-Heber, and this thesis all rely heavily on analysing the dynamics of the geodesic flow on SM, and in particular, make considerable use of recurrence. One expects that a totally new approach would be required to tackle such a problem.

Another way to drop the curvature-type assumption might be to look at perturbations of locally symmetric metrics, motivating the following question:

Question I.2. *Let M be a closed manifold, and let g_t be a smooth one-parameter family of metrics on M such that (M, g_0) is an irreducible nonpositively curved higher rank locally symmetric space, and (M, g_t) is higher rank for all t. Must it be the case that each (M, g_t) is locally symmetric?*

Yet another avenue for generalization, and an active area of current research, is to replace the Riemannian manifold M by a "nonpositively curved" length space X. (One might assume X to be $CAT(0)$, or perhaps simply that its distance function is convex.) One then hopes that for an appropriate definition of "higher rank", one might classify higher rank nonpositively curved spaces X; for instance, Ballmann-Buyalo [7] have conjectured that such an X is either locally a product, isometric to a locally symmetric space, or isometric to a Euclidean building.

Chapter II

Preliminaries

2.1 Geodesic flows

Let M be a Riemannian manifold. All Riemannian manifolds in this work are assumed to be complete. We denote by TM and SM the tangent and unit tangent bundles of M, respectively, and by π the corresponding projection map. If v is a unit tangent vector to a manifold M, we let γ_v denote the (unique) geodesic with $\dot{\gamma}_v(0) = v$.

Central to our work is the geodesic flow on M, which is the flow $g^t : SM \to SM$ defined by

$$g^t v = \dot{\gamma}_v(t).$$

In this section, we establish some basic properties of g^t. First, a definition that is central to our entire paper:

Definition II.1. If $v \in SM$, the *rank* of v is the dimension of the space of parallel Jacobi fields along γ_v. The *rank* of M is the minimum of $\operatorname{rank} v$ over all $v \in SM$.

2.1.1 The Sasaki metric

We wish to construct a natural Riemannian metric (the Sasaki metric) on TM. We begin by noting the correspondence between paths in TM and vector fields along curves in M. Let $\sigma : \mathbb{R} \to TM$ be a (smooth) path. Then $\pi \circ \sigma$ is a path in M, and for each $t \in \mathbb{R}$, $\sigma(t)$ is a tangent vector to M at $\pi(\sigma(t))$. Thus σ gives a vector field along $\pi \circ \sigma$. Conversely, a vector field along a curve in M may be lifted to a path in TM.

With this in mind, let $v \in TM$, and let $X \in T_vTM$. Choose a path $\sigma : (-\epsilon, \epsilon) \to TM$ such that $\dot{\sigma}(0) = X$; such a σ gives rise to a vector field $V_\sigma(t)$ along $\pi \circ \sigma$. We define the *horizontal part* $KX \in T_{\pi(v)}M$ of X to be

$$KX = (D_t V_\sigma)(0),$$

where D_t indicates the covariant derivative. One can check that KX does not depend on the choice of path σ.

The map $K : T_vTM \to T_{\pi(v)}M$ is sometimes called the *connector*, and its kernel is called the *horizontal subspace* at v. There is, likewise, a *vertical subspace*, given by the kernel of the map $d\pi : T_vTM \to T_{\pi(v)}M$, and it isn't difficult to check that the map $(d\pi, K) : T_vTM \to T_{\pi(v)}M \oplus T_{\pi(v)}M$ is an isomorphism.

For $v \in T_pM$, there is a unique vector v^H in the horizontal subspace at v such that $d\pi(v^H) = v$, and a unique vector v^V in the vertical subspace at v such that $K(v^V) = v$. The vectors v^H and v^V are called the *horizontal lift* and *vertical lift* of v, respectively.

Definition II.2. The *Sasaki metric* on TM, as an inner product on T_vTM, is given by the pullback via $(d\pi, K)$ of the inner product on $T_{\pi(v)}M \oplus T_{\pi(v)}M$ determined by the Riemannian metric.

In terms of the connector K and the map $d\pi$, we may write the Sasaki metric on T_vTM as

$$\langle X, Y \rangle_{TM} = \langle d\pi X, d\pi Y \rangle_M + \langle KX, KY \rangle_M.$$

The restriction of the Sasaki metric on TM to a Riemannian metric on the submanifold SM is also called the Sasaki metric. In terms of the maps $d\pi$ and K, vectors in T_vTM tangent to SM are characterized by the following property:

Proposition II.3. *Let $v \in SM$. Then $X \in T_vTM$ is tangent to SM iff $\langle v, KX\rangle = 0$.*

Proof. Let $V(t)$ be a vector field along a path τ in M such that the associated path $\widetilde{V}(t)$ in TM is tangent to X. By construction $V(0) = v$ has unit norm. Thus $\widetilde{V}(t)$ is tangent to SM at v iff

$$0 = \frac{d}{dt}|_{t=0}||V(t)||^2 = 2\langle V(0), V'(0)\rangle = 2\langle v, KX\rangle.$$

□

It is of paramount importance to the theory of geodesic flows that the volume form given by the Sasaki metric is preserved by the geodesic flow on SM. Before establishing this, we must first discuss the relationship of the geodesic flow and Jacobi fields.

2.1.2 Jacobi fields and dg^t

Let $V(s)$ be a vector field along a path τ in M. As noted above, this vector field corresponds to a path $\widetilde{V}(s)$ in TM; then $\frac{d}{ds}\widetilde{V}(s)$ is an element of $T_{V(s)}TM$ and

$$d\pi\left(\tfrac{d}{ds}\widetilde{V}(s)\right) = \dot{\tau}(s)$$
$$K\left(\tfrac{d}{ds}\widetilde{V}(s)\right) = V'(s).$$

This allows us to write $\frac{d}{ds}\widetilde{V}(s)$ as an element of $T_{\tau(s)}M \oplus T_{\tau(s)}M$ using the isomorphism $(d\pi, K)$ discussed above.

We're interested in the behavior of the derivative dg^t of the geodesic flow, and hence we're interested in the behavior of such paths $\widetilde{V}(s)$ under the geodesic flow.

We may define a variation of geodesics by

$$\Gamma(t,s) = \exp(tV(s)).$$

In terms of Γ, the element of $T_{\tau(s)}M \oplus T_{\tau(s)}M$ corresponding to $\frac{d}{ds}\widetilde{V}(s)$ is

$$\big(\partial_s\Gamma(0,s), D_s\partial_t\Gamma(0,s)\big).$$

More generally, the element of $T_{\Gamma(t,s)}M \oplus T_{\Gamma(t,s)}M$ corresponding to the path $s \mapsto \partial_t\Gamma(t,s)$ in TM is

$$\big(\partial_s\Gamma(t,s), D_s\partial_t\Gamma(t,s)\big).$$

Note that $\Gamma(0,s) = V(s)$, and that $(s \mapsto \Gamma(t,s))$, as a path in TM, is the image of the path $V(s)$ under the geodesic flow g^t. Thus we have established that, under the identifications given by the maps $(d\pi, K)$, the derivative of the geodesic flow is determined by

$$dg^t\big(\partial_s\Gamma(0,0), D_s\partial_t\Gamma(0,0)\big) = \big(\partial_s\Gamma(t,0), D_s\partial_t\Gamma(t,0)\big).$$

The field $J(t) = \partial_s\Gamma(t,0)$ along $\Gamma(t,0)$ is a Jacobi field; furthermore, we have $D_s\partial_t\Gamma = D_t\partial_s\Gamma = J'(t)$. Hence the formula above reduces to the following:

Proposition II.4. *Let J be a Jacobi field along the geodesic γ. Then, under the identifications given by the maps $(d\phi, K)$,*

$$dg^t\big(J(0), J'(0)\big) = \big(J(t), J'(t)\big)$$

Since there is a (unique) Jacobi field J along γ satisfying $J(0) = v_1$ and $J'(0) = v_2$ for any choice of $v_1, v_2 \in T_pM$, the above proposition completely describes the derivative dg^t. In the future, we will feel free to make the identification $T_vTM \cong T_{\pi(v)}M \oplus T_{\pi(v)}M$ given by $(d\pi, K)$ without comment.

The connection between Jacobi fields and the geodesic flow is a major reason for studying the rank of a Riemannian manifold. In particular, if $v \in SM$ is rank one, there are no parallel Jacobi fields along v, and one hopes that one can glean from this that the geodesic flow has some sort of "hyperbolic" behavior along $g^t v$. An example of this kind of reasoning can be seen in Chapter V.

Conversely, if M has higher rank, one might hope that the parallel Jacobi fields along an arbitrary geodesic v of rank k come from a totally geodesic isometric embedding $\mathbb{R}^k \to M$, and then use the geometric structure of these embedded flats through every $v \in SM$ to say something about the structure of M. Indeed, this is the beginning of a proof of higher rank rigidity; the embedded flats are constructed for M complete and without focal points in section 4.1.

2.1.3 The contact form and the invariant metric

The manifold SM is equipped with a natural one-form α defined by

$$\alpha_v(X) = \langle v, d\pi X \rangle.$$

(Those familiar with symplectic geometry will recognize this as the restriction to SM of the pullback of the canonical one-form on T^*M by the isomorphism $TM \to T^*M$ given by the Riemannian metric.)

Of course, there is also a one-form on TM given by the same formula. However, it is necessary to restrict to SM to obtain the following proposition:

Proposition II.5. *α is invariant under the geodesic flow on SM.*

Proof. We write vectors in T_vSM as Jacobi fields $(J(t), J'(t))$. Thus we may calculate

$$\begin{aligned} ((g^t)^*\alpha)_v\big(J(0), J'(0)\big) &= \alpha_{g^tv}\big((g^t)_*(J(0), J'(0))\big) \\ &= \alpha_{g^tv}\big(J(t), J'(t)\big) \\ &= \langle g^tv, J(t)\rangle, \end{aligned}$$

so we reduce to the assertion that $\langle g^tv, J(t)\rangle$ is constant. We have

$$\frac{d}{dt}\langle g^tv, J(t)\rangle = \langle D_tg^tv, J(t)\rangle + \langle g^tv, J'(t)\rangle = 0,$$

where the first term vanishes since g^tv is the tangent vector field to a geodesic, and the second term vanishes by Proposition II.3. □

In fact, the one-form α is a *contact form*, which by definition means that the $(2n-1)$-form

$$\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha = \alpha \wedge (d\alpha)^{\wedge(n-1)}$$

is nonvanishing. To see this it helps to have the following nice formula for $d\alpha$:

Proposition II.6. $d\alpha(v, w) = \langle Kv, d\pi(w)\rangle - \langle d\pi(v), Kw\rangle$.

Proof. We establish this formula on TM, from which the formula on SM follows. Fix $v \in TM$. Let $E_1, \ldots, E_n$ be vector fields on M, defined locally around $\pi(v)$, such that $\{E_1(q), \ldots, E_n(q)\}$ is an orthonormal frame for T_qM for each q near $\pi(v)$. Choose local coordinates on TM such that the point $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ corresponds to the vector $\xi^iE_i(x^1, \ldots, x^n)$. (Here and throughout, we use the Einstein summation convention.)

We denote by ∂_i^x and ∂_i^ξ the (local) vector fields on TM given by differentiating in the direction of the x^i and ξ^i coordinates, respectively. Note that $d\pi\partial_i^\xi = 0$, and hence $\alpha(\partial_i^\xi) = 0$.

We use the formula

$$(\star) \qquad d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

to evaluate $d\alpha$ on the fields ∂_i^x and ∂_i^ξ. Since these are vector fields associated to coordinates, all terms involving brackets are equal to zero; by our observation above, all terms involving $\alpha(\partial_i^\xi)$ are also zero. Thus we need concern ourselves with computing the terms $\partial_i^x\alpha(\partial_j^x)$ and $\partial_i^\xi\alpha(\partial_j^x)$. First of all, we have

$$\begin{aligned}\alpha_v(\partial_j^x) &= \langle v, d\pi(\partial_j^x)\rangle \\ &= \langle v, \partial_j\rangle \\ &= \xi^k\langle E_k(x^1,\dots,x^n), \partial_j\rangle,\end{aligned}$$

where ∂_j denotes the vector field on M given by the x^j coordinate. We then have

$$\begin{aligned}\partial_i^x\alpha_v(\partial_j^x) &= \xi^k\partial_i^x\langle E_k(x^1,\dots,x^n), \partial_j\rangle \\ &= \xi^k\partial_i\langle E_k(x^1,\dots,x^n), \partial_j\rangle \\ &= \xi^k\langle E_k(x^1,\dots,x^n), \nabla_{\partial_i}\partial_j\rangle.\end{aligned}$$

From this, our formula $(\star)$, and the fact that $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$, we find $d\alpha(\partial_i^x, \partial_j^x) = 0$. With regard to ∂_i^ξ, we have

$$\partial_i^\xi\alpha_v(\partial_j^x) = \langle E_i(x^1,\dots,x^n), \partial_j\rangle.$$

Thus, we have computed $d\alpha$: it is the unique two-form satisfying

$$\begin{aligned}d\alpha(\partial_i^\xi, \partial_j^x) &= \langle E_i(x^1,\dots,x^n), \partial_j\rangle; \\ d\alpha(\partial_i^\xi, \partial_j^\xi) &= d\alpha(\partial_i^x, \partial_j^x) = 0.\end{aligned}$$

All that remains is to check that the two form given by $(v,w) \mapsto \langle Kv, d\pi(w)\rangle -$

$\langle d\pi(v), Kw\rangle$ satisfies the same equations, which follows immediately from the equations

$$d\pi\partial_i^\xi = 0 \qquad\qquad K\partial_i^\xi = E_i$$
$$d\pi\partial_i^x = \partial_i \qquad\qquad K\partial_i^x = 0.$$

□

Corollary II.7. *α is a contact form on SM.*

Proof. We must show that $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ is nonvanishing at each $v \in SM$. Fix such a v. Extend v to an orthonormal basis $v, w_1, \ldots, w_{n-1}$ of $T_{\pi(v)}M$. For each i, let w_i^H and w_i^V be the horizontal and vertical lifts of w_i, respectively; let v^H be the vertical lift of v. It's clear from Proposition II.3 that v^H, w_i^H, and w_i^V, for $1 \leq i \leq n-1$, are all in T_vSM. Furthermore, the previous proposition gives the following expressions for $d\alpha$:

$$d\alpha(w_i^H, w_j^V) = \delta_{ij}$$
$$d\alpha(w_i^H, w_j^H) = d\alpha(w_i^V, w_j^V) = 0$$
$$d\alpha(v^H, w_j^H) = d\alpha(v^H, w_j^V) = 0.$$

It follows that

$$\left(\alpha \wedge (d\alpha)^{\wedge(n-1)}\right)(v^H, w_1^H, w_1^V, \ldots, w_{n-1}^H, w_{n-1}^V) = 1,$$

and in particular that this form is nonvanishing. □

In fact, the previous proof shows more: Notice that the vectors v^H, w_i^H, w_i^V for $1 \leq i \leq (n-1)$ form a Sasaki-orthonormal basis for T_vSM. Since the form $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ takes this basis to 1, it is the volume form associated to the Sasaki metric. But since the geodesic flow leaves α invariant, it also leaves this form invariant. Thus we conclude that the volume associated to the Sasaki metric is invariant under the geodesic flow.

2.2 Symmetric spaces

In this section we give a very brief overview of symmetric spaces. As we do not require much from the theory of symmetric spaces other than the Berger-Simons holonomy theorem (stated in Section 2.3), we do not attempt to develop this theory here. The interested reader should see Helgason [30] or Kobayashi-Nomizu [33], the standard references.

2.2.1 Definitions

We begin with the simplest definition of a symmetric space. Let M be a complete Riemannian manifold and $p \in M$. Fix a neighborhood U of 0 in T_pM such that the restriction of $\exp_p$ to U is a diffeomorphism onto its image. Then we get a diffeomorphism $\sigma : \exp_p U \to \exp_p U$, called the *local geodesic symmetry* at p, by

$$\sigma = \exp_p \circ (-\mathrm{id}) \circ \exp_p^{-1} .$$

Definition II.8. A Riemannian manifold M is called *locally symmetric*, or is said to be a *locally symmetric space*, if for each $p \in M$, the local geodesic symmetry is an isometry. M is called (globally) *symmetric*, or is said to be a *symmetric space*, if it is locally symmetric and in addition each geodesic symmetry extends to an isometry of M.

There is an apparent ambiguity in the first half of this definition: namely, one might worry that it is possible that there be two neighborhoods U and V of 0 in T_pM such that $\exp_p$ restricts to a diffeomorphism on both U and V, but the geodesic symmetry is an isometry on $\exp_p U$ and not on $\exp_p V$. In fact this cannot happen under the assumption that *every* point have a neighborhood for which the local geodesic symmetry is an isometry.

Recall that for a Riemannian manifold there is a covariant derivative ∇ taking (p, q)-tensors to $(p+1, q)$-tensors determined by the Levi-Civita connection. We have a second characterization of locally symmetric spaces as follows:

Proposition. *M is locally symmetric iff $\nabla R = 0$, where R is the Riemannian curvature tensor of M.*

Along these same lines, one can also show that if M is locally symmetric and simply connected, then M is a symmetric space.

The above geometric definitions are convenient for their simplicity, but they do not really give the complete picture of the idea of a symmetric space. We are interested in nonpositively curved symmetric spaces; it turns out in fact that every nonpositively curved symmetric space is given by an algebraic quotient G/K, where G is a semisimple real Lie group and K is a maximal compact subgroup, equipped with a natural Riemannian metric for which the action of G on G/K is by isometries.

We defer the construction of this metric in the general case to Helgason [30]. However, we can carry out the construction for the special case $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ without too much abstraction; we do this presently. For ease of notation we let $G = SL(n, \mathbb{R})$ and $K = SO(n, \mathbb{R})$.

2.2.2 The symmetric space $SL(n, \mathbb{R})/SO(n, \mathbb{R})$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ of $SL(n, \mathbb{R})$ is the set of trace zero $n \times n$ matricies. $\mathfrak{sl}(n, \mathbb{R})$ has a natural bilinear form

$$B(X, Y) = \mathrm{tr}(XY).$$

and this form is nondegenerate; in fact, it is simple to see that it is positive definite on the space $\mathfrak{p}$ of trace zero symmetric matricies and negative definite on the space $\mathfrak{k} = \mathfrak{so}(n, \mathbb{R})$ of skew-symmetric matricies. We note also, for later reference, that

$$B(X, [Y, Z]) = B([X, Y], Z).$$

(This is easy to check.)

We let $\pi : G \to G/K$ be the projection map. Then $\mathfrak{k}$ is the kernel of $d\pi$, and since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we see that $d\pi$ identifies $\mathfrak{p}$ with the tangent space to the coset $1K \in G/K$. We denote the restriction of the form B to $\mathfrak{p}$ by $\langle \cdot, \cdot \rangle_{1K}$; it is an inner product on $\mathfrak{p}$.

G acts transitively on the left of G/K; we denote the action of $g \in G$ by $L_g : G/K \to G/K$. In particular, if $k \in K$ the map L_k fixes the coset $1K$, and so its derivative induces an automorphism dL_k of the tangent space $\mathfrak{p}$.

To derive an explicit formula for this automorphism, let $X \in \mathfrak{p}$; then $e^{tX}K$ is a curve through $1K$ with tangent vector X, and L_k takes this curve to the curve

$$ke^{tX}K = (ke^{tX}k^{-1})K;$$

differentiating at $t = 0$, we see that

$$dL_k(X) = kXk^{-1}.$$

Note that since $k \in K = SO(n, \mathbb{R})$, the matrix kXk^{-1} is again in $\mathfrak{p}$; this is the reason for introducing the factor of k^{-1}.

With this in mind, we may attempt to extend $\langle \cdot, \cdot \rangle_{1K}$ to a Riemannian metric on G/K by defining, for $g \in G$,

$$\langle dL_g X, dL_g Y \rangle_{gK} = \langle X, Y \rangle_{1K}$$

whenever $X, Y \in \mathfrak{p}$. Since dL_g gives an isomorphism of the tangent spaces at $1K$ and gK, this formula does indeed determine an inner product on the tangent space at gK. However, there is a possible ambiguity in this definition: We must check that if $k \in K$ our formula returns the original inner product on the coset $kK = 1K$. To do this, fix $X, Y \in \mathfrak{p}$ and $k \in K$ and calculate:

$$\begin{aligned}\langle dL_k X, dL_k Y \rangle_{1K} &= \operatorname{tr}((dL_k X)^* dL_k Y) \\ &= \operatorname{tr}(kX^* k^{-1} kY k^{-1}) \\ &= \operatorname{tr}(X^* Y) = \langle X, Y \rangle_{1K}.\end{aligned}$$

Thus our inner product $\langle \cdot, \cdot \rangle_{gK}$ is well-defined independent on the choice of coset representative g, and so we get a Riemannian metric on G/K, and (by construction) the left action of G is by isometries.

It is a fact of linear algebra that $G = \exp(\mathfrak{p})K$, and this allows us to write any point gK as $e^X K$, with $X \in \mathfrak{p}$. Then the geodesic symmetry at $1K$ is the map

$$\sigma_{1K} : e^X K \mapsto e^{-X} K,$$

and one can check that this is indeed an isometry. Since G/K is homogeneous, the geodesic symmetry at every point of G/K is then an isometry, i.e., G/K is in fact a symmetric space in the sense of our first definition.

2.2.3 The curvature tensor of G/K

Our next goal is to explicitly compute the curvature tensor of G/K. To do this, we note first that G is a pseudo-Riemannian manifold with the left-invariant metric determined by B, and this makes the projection $d\pi : G \to B$ into a pseudo-Riemannian submersion (that is, $d\pi$ is an isometry on $(\ker d\pi)^\perp$). G is equipped with a Levi-Civita connection satisfying the usual properties and determined by the Koszul formula.

In fact, the connections of G and G/K are related in a simple way. If X is a vector field on G/K we denote by $\widetilde{X}$ the unique vector field on G such that $\widetilde{X}$ and

X are π-related and $\widetilde{X}$ is everywhere orthogonal to the kernel of $d\pi$; we call $\widetilde{X}$ the *horizontal lift* of X. In addition, given a tangent vector W to G, we define $W^{\mathcal{H}}$ and $W^{\mathcal{V}}$ to be the unique orthogonal vectors such that $W = W^{\mathcal{H}} + W^{\mathcal{V}}$ and $d\pi(W^{\mathcal{V}}) = 0$.

Let $\widetilde{\nabla}$ be the connection on G and ∇ the connection on G/K. It is not hard to show, using the Koszul formula, that

$$\langle \nabla_X Y, Z\rangle = \langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}\rangle,$$

or, equivalently,

$$\widetilde{\nabla_X Y} = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})^{\mathcal{H}}.$$

One can use this to obtain a formula for the curvature tensor R on G/K in terms of the curvature tensor $\widetilde{R}$ on G:

O'Neill's Formula. *Let X, Y, V, W be horizontal fields on G. Then*

$$\begin{aligned}\langle R(d\pi X, d\pi Y)d\pi V, d\pi W\rangle =& \langle \widetilde{R}(X,Y), V, W\rangle - \frac{1}{2}\langle [X,Y]^{\mathcal{V}}, [V,W]^{\mathcal{V}}\rangle \\ & - \frac{1}{4}\Big(\langle [X,V]^{\mathcal{V}}, [Y,W]^{\mathcal{V}}\rangle - \langle [Y,V]^{\mathcal{V}}, [X,W]^{\mathcal{V}}\rangle\Big)\end{aligned}$$

For us the advantage here is that $\widetilde{\nabla}$, and hence $\widetilde{R}$, is easy to compute, again with the Koszul formula. Let X, Y, Z be left-invariant vector fields on G. The Koszul formula states

$$2\langle \widetilde{\nabla}_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle.$$

Since X, Y, Z are left invariant, $X\langle Y, Z\rangle = 0$. Furthermore

$$\langle X, [Y,Z]\rangle_g = B(X, [Y,Z]) = B([X,Y], Z) = \langle [X,Y], Z\rangle_g$$

and by a similar computation

$$\langle Y, [Z,X]\rangle = \langle [X,Y], Z\rangle.$$

It follows immediately that

$$\widetilde{\nabla}_X Y = \frac{1}{2}[X, Y].$$

It is now trivial to compute $\widetilde{R}$ straight from the definition. One finds that if $X, Y, Z \in \mathfrak{g}$, which we identify with the tangent space at the identity of G,

$$\widetilde{R}(X, Y)Z = -\frac{1}{4}\big[[X, Y], Z\big]$$

This, with O'Neill's formula, allows us to write down the curvature form of G/K at the coset $1K$. Let $X, Y, Z, W \in \mathfrak{p}$; denote also by X, Y, Z, W the associated left-invariant vector fields on G. Note then that since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, the brackets of these vector fields are left-invariant vector fields coming from $\mathfrak{k}$, which means they are vertical. Thus O'Neill's formula says

$$\langle R(X, Y)Z, W\rangle = \langle \widetilde{R}(X, Y)Z, W\rangle - \frac{1}{2}\langle [X, Y], [V, W]\rangle - \frac{1}{4}\langle [X, Z], [Y, W]\rangle + \frac{1}{4}\langle [Y, Z], [X, W]\rangle$$

and one repeatedly uses the fact that $B([X, Y], Z) = B(X, [Y, Z])$ as well as the Jacobi identity to compute

$$\langle R(X, Y)Z, W\rangle = -\langle [X, Y], [Z, W]\rangle.$$

Equivalently, we have

$$R(X, Y)Z = -\big[[X, Y], Z\big].$$

In particular, if X, Y are orthonormal, then

$$\langle R(X, Y)Y, X\rangle = -||[X, Y]||^2$$

is the sectional curvature of the plane spanned by X and Y, so we see that G/K has nonpositive curvature.

Moreover, we can investigate flats in G/K. Let $\mathfrak{h}$ be an abelian subalgebra of $\mathfrak{p}$; we know that for $A, B \in \mathfrak{h}$, the sectional curvature of the plane spanned by A, B is zero. Consider the submanifold $\exp(\mathfrak{h})K$ of G/K; we claim this submanifold is flat, and so we must compute its curvature at an arbitrary point $\exp(C)K$. Let $A \in \mathfrak{h}$; then

$$\begin{aligned} dL_{\exp(C)}A &= \frac{d}{dt}\Big|_{t=0} \exp(C)\exp(tA)K \\ &= \frac{d}{dt}\Big|_{t=0} \exp(C+tA)K, \end{aligned}$$

and this is tangent to $\exp(\mathfrak{h})$. Thus $dL_{\exp(C)}$ maps $\mathfrak{h}$ isometrically onto the tangent space to $\exp(\mathfrak{h})$ at $\exp(C)K$, and it follows that all sectional curvatures of $\exp(\mathfrak{h})$ at this point are zero, and therefore that $\exp(\mathfrak{h})$ is indeed flat.

Thus flats in G/K through the point $1K$ correspond to maximal abelian subalgebras of $\mathfrak{p}$. An example of such a subalgebra is the algebra $\mathfrak{h}$ of diagonal trace zero matricies; this has dimension $(n-1)$. Furthermore, for any $k \in K$ the set $k^{-1}\mathfrak{h}k$ is also an abelian subalgebra of $\mathfrak{p}$. Since any symmetric matrix can be orthogonally diagonalized, we see that for every $X \in \mathfrak{p}$ there exists a k such that $X \in k^{-1}\mathfrak{h}k$; geometrically, the hyperplane $k^{-1}\mathfrak{h}k$ integrates to an $(n-1)$-flat through the tangent vector X. Since G/K is homogeneous, every tangent vector to G/K is contained in an $(n-1)$-flat, and so we have shown:

Proposition. $\operatorname{rank}(SL(n,\mathbb{R})/SO(n,\mathbb{R})) = n-1$.

The argument above carries over to general semisimple Lie groups G and their maximal compact subgroups K. The largest change that needs to be made is in definition of the bilinear form B on $\mathfrak{g}$; in general this becomes the *Killing form*, defined by

$$B(X,Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y),$$

where $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is the endomorphism $Y \mapsto [X,Y]$. The Killing form is nondegenerate if G is semisimple, negative definite on the lie algebra $\mathfrak{k}$ of K. The subspace $\mathfrak{p}$

can then be defined to be the orthogonal complement to $\mathfrak{k}$ and is identified with the tangent space to G/K. Just as above one obtains the formula $R(X,Y) = -\operatorname{ad}[X,Y]$ for the curvature endomorphism, showing that G/K is nonpositively curved, and that flats in G/K correspond to maximal abelian subalgebras of $\mathfrak{p}$.

It is possible but slightly more complicated to give a description of the connection on G/K; again, one should see the standard references [30], [33]. Mautner [36] gives an explicit formula for the geodesic flow on G/K:

Proposition. *Let $g \in G, X \in \mathfrak{p}$. The geodesic $\gamma(t)$ through $dL_g X$ at time $t = 0$ is*

$$\gamma(t) = g \exp(tX)K.$$

The main result of Mautner's paper is that if G/K is higher rank, and Γ is a lattice in G, then the geodesic flow on $\Gamma\backslash G/K$ is not ergodic; he also gives an explicit description of the ergodic components of G. This involves algebraic machinery not developed here, so we refer the reader to his paper for that result.

2.3 Holonomy and reducibility

In this section we discuss the holonomy group of a Riemannian manifold, the deRham decomposition, and the Berger-Simons theorem. The main reference for the material on holonomy and deRham decomposition is the book by Kobayashi-Nomizu [32].

Let M be a complete orientable Riemannian manifold, and suppose $\gamma : [0,1] \to M$ is a nullhomotopic piecewise-$\mathscr{C}^1$ curve with $p := \gamma(0) = \gamma(1)$. Then parallel transport around γ defines an isometry $P_\gamma : T_pM \to T_pM$. The set of all such P_γ forms a subgroup of $SO(n)$. One can show that this subgroup is closed, and hence is a Lie subgroup of $SO(n)$.

Definition II.9. The *holonomy group* of M at p is the subgroup of $SO(n)$ consisting of all P_γ as above.

The group defined above is sometimes called the *restricted holonomy group* to distinguish it from the same construction with the assumption that the paths are nullhomotopic removed. We shall not consider the latter group, and we continue to use the slightly imprecise language "holonomy group" for the former.

If $\sigma : [0,1] \to M$ is a piecewise-$\mathscr{C}^1$ path with $\sigma(0) = p$ and $\sigma(1) = q$, then σ defines an isometry $P_\sigma : T_pM \to T_qM$ by parallel transport. P_σ evidently defines an isomorphism of the holonomy groups H_p and H_q of M at p and q respectively; in particular, we have $P_{\sigma\gamma\sigma^{-1}} = P_\sigma P_\gamma P_{\sigma^{-1}}$, where $\sigma\gamma\sigma^{-1}$ denotes concatenation of paths. For this reason, we often speak of *the* holonomy group of a connected manifold M without reference to a particular point; we mean any of the groups H_p, with the understanding that they are all isomorphic (though not naturally).

By definition, the holonomy group H_p acts on T_pM by isometries. It is easy to check that if $M = M_1 \times M_2$ is a (nontrivial) Riemannian product, then the holonomy group leaves the factors of M_1 and M_2 invariant; in other words, for each $p \in M$, the subspaces T_pM_1 and T_pM_2 are proper, holonomy-invariant subspaces of T_pM. What may be surprising is that the converse is also true, once we pass to the universal cover of M.

To be precise, suppose the representation of H_p on T_pM is reducible and let W_p be a proper invariant subspace. For every $q \in M$ the representation of H_q on T_qM is similarly reducible, and we obtain an invariant subspace $W_q = P_\sigma W_p$, where σ is any path from p to q; since W_p is invariant, it is easy to check that this is independent of the choice of path. Thus reducibility of the holonomy group of M implies the existence of a whole family of subspaces W_q related by the parallel transport isometries

P_σ. With this in mind, we can now state the following decomposition theorem of de Rham:

Theorem II.10 (de Rham)**.** *Let M be a simply connected complete Riemannian manifold, and suppose the holonomy group of M is reducible. Fix a family of subspaces W_p as above. Then M decomposes as a Riemannian product $M = M_1 \times M_2$, and for each $p \in M$ we have $T_pM_1 = W_p$ and $T_pM_2 = W_p^{\perp}$.*

If M is an arbitrary Riemannian manifold, $\widetilde{M}$ its universal cover, then it is clear that the holonomy group of $p \in M$ is isomorphic to the holonomy group of any lift $\widetilde{p} \in \widetilde{M}$ of p, and similarly that the representations of these groups on T_pM and $T_{\widetilde{p}}\widetilde{M}$ are isomorphic. Thus if M has reducible holonomy group, its universal cover splits as a product whose factors are tangent to the invariant subspaces as in the above theorem.

Definition II.11. A Riemmanian manifold is called *irreducible* if its holonomy group is irreducible. Equivalently, a Riemannian manifold is irreducible if its universal cover does not split as a (Riemannian) product.

As a consequence, any simply connected Riemannian manifold M splits as a Riemannian product $M = M_1 \times \cdots \times M_k$ where each M_i is irreducible. If none of the factors M_i is equal to $\mathbb{R}$, then M is said to have *no flat factors* or, equivalently, *no Euclidean factors.*

It is important to us that the decomposition above is essentially unique:

Theorem II.12 (de Rham)**.** *Any simply connected Riemannian manifold M splits as a Riemannian product*

$$M = E_r \times M_1 \times \cdots \times M_k,$$

where E_r is a Euclidean space of dimension r and each M_i is irreducible and non-flat. The number r and the factors M_i are unique up to order, in the following sense: any isometry ϕ of M decomposes as $\phi = \sigma\phi'$, where σ permutes the M_i and ϕ' preserves the factors of the decomposition, i.e.,

$$\phi' \in \text{Isom}(E_r) \times \text{Isom}(M_1) \times \cdots \times \text{Isom}(M_k).$$

Note that it is important to lump the Euclidean factors together for the uniqueness statement above. In particular, the decomposition $E_r = \mathbb{R} \times \cdots \times \mathbb{R}$ is not unique in de Rham's sense.

Definition II.13. Let $M = M_1 \times \cdots \times M_k$ (as Riemannian manifolds). We say that a subgroup Γ of isometries of M *preserves the factors of the decomposition* if

$$\Gamma \subseteq \text{Isom}(M_1) \times \cdots \times \text{Isom}(M_k).$$

In this case there exist obvious maps $\pi_i : \Gamma \to \text{Isom}(M_i)$, which we call the *associated projection maps.*

The following corollaries are essentially restatements of the uniqueness property above:

Corollary II.14. *Let Γ be a group of isometries of M, and let*

$$M = E_r \times M_1 \times \cdots \times M_k$$

be the de Rham decomposition of M. Then Γ has a finite index subgroup Γ^* preserving the factors of the decomposition.

Corollary II.15. *Let Γ be a group of isometries of $N \times L$, and suppose N has no flat factors. Then Γ has a finite index subgroup Γ^* preserving the factors of the decomposition.*

The first of these corollaries allows us to show that the property of Γ-recurrent vectors being dense in SM passes to de Rham factors when M has no focal points. The proof involves some technical arguments from dynamics and is not particularly enlightening to the current discussion, so we postpone it for the end of this section.

We remark that the notion of reducibility is crucial to any higher-rank rigidity result, since reducible Riemannian manifolds are automatically of higher rank, which follows from the fact that M and its universal cover $\tilde{M}$ have the same rank. However, we will use the holonomy group in an even more essential way, by making use of the following theorem of Berger-Simons:

Berger-Simons Holonomy Theorem ([9], [44]). *Let M be a complete irreducible Riemannian manifold. If the holonomy group of M is not transitive, then M is locally symmetric.*

Berger originally proved this result by classifying all possible holonomy groups of irreducible Riemannian manifolds; on non-symmetric manifolds, there are only five families and two exceptional groups, and all act transitively. Simons later gave a more direct proof of the theorem; his proof is quite algebraic, working with abstract properties of curvature tensors. Recently, Olmos [39] has given a geometric proof of the theorem that depends on so-called normal holonomy groups.

The Berger-Simons theorem was used crucially by Ballmann in his original proof of the higher rank rigidity theorem, and we use it in the same way: Our goal will be to show that any higher rank irreducible Riemannian manifold has a nontransitive holonomy group.

2.4 Γ-recurrence on the de Rham factors

In this section we give a proof of the following:

Proposition II.16. *Let M be a simply connected complete Riemannian manifold with de Rham decomposition*

$$M = E_r \times M_1 \times \cdots \times M_k.$$

Let Γ be a subgroup of isometries of M such that Γ-recurrent vectors are dense in SM, and let Γ^* be a finite index subgroup of Γ preserving the factors of the decomposition; denote by $\pi_i : \Gamma^* \to \mathrm{Isom}(M_i)$ the corresponding projection for $1 \leq i \leq k$. Then the $(\pi_i\Gamma^*)$-recurrent vectors are dense in SM_i.

The proof is in two steps: we show that Γ^*-recurrent vectors are dense in SM, and then that this condition passes to the factors. The latter of these steps is considerably easier, so we present it first:

Proposition II.17. *Let $M = M_1 \times M_2$, let Γ be a subgroup of isometries of M preserving the factors of the decomposition, and suppose Γ-recurrent vectors are dense in SM. Let π_i be the associated projection maps; then $\pi_i\Gamma$-recurrent vectors are dense in M_i.*

Proof. Let $v \in SM_i$. Lift v to a vector $\widetilde{v} \in SM$ such that $d\pi_i\widetilde{v} = v$, where (by abuse of notation) $\pi_i : M \to M_i$ is the projection. There exists a sequence $\widetilde{v}_n \in SM$ of Γ-recurrent vectors converging to $\widetilde{v}$. Let $v_n = d\pi_i\widetilde{v}_n$. Then it is clear that $v_n \to v$.

So we need only show v_n is $\pi_i\Gamma$-recurrent. Given n, we fix $\phi_m \in \Gamma$ and $t_m \to \infty$ such that $d\phi_m g^{t_m}\widetilde{v}_n \to \widetilde{v}_n$ (as $m \to \infty$). Then it is clear that $\pi_i(d\phi_m)g^t_m v_n \to v_n$. □

We know from Corollary II.15 that a subgroup Γ of isometries of M has a finite index subgroup preserving the factors of the de Rham decomposition. Thus Proposition II.16 is proved if we can show that whenever $\Gamma^* \subseteq \Gamma$ is finite index and Γ-recurrent vectors are dense, Γ^*-recurrent vectors are also dense.

Our proof uses a notion from dynamics that we avoid in the rest of the thesis:

Definition II.18. A vector $v \in SM$ is called Γ-*nonwandering* (for the geodesic flow) if for every neighborhood U of v and every $T > 0$, there exists $\phi \in \Gamma$ and $t > T$ such that

$$d\phi g^t(U) \cap U \neq \emptyset.$$

The set of Γ-nonwandering vectors is denoted by $\Omega(\Gamma)$.

It is clear that recurrent points are nonwandering and that the set of nonwandering vectors is closed. In fact we have the following proposition:

Proposition II.19. *Let M be a complete Riemannian manifold, Γ a group of isometries of M. Then Γ-recurrent vectors are dense in the set of Γ-nonwandering vectors.*

Proof. We give an argument from [17]. For each positive integer n, we let A_n be the set of vectors v such that there exists $t > n$ and $\phi \in \Gamma$ with

$$d(d\phi g^t v, v) < \frac{1}{n}.$$

Here d is the Sasaki metric (although this proof works for any $\mathbb{Z}$- or $\mathbb{R}$-action on a complete metric space). Clearly A_n is open; we claim it is also dense in the set of nonwandering vectors.

Let's show this. Fix n, let w be nonwandering, and let $B_\epsilon(w)$ the ϵ-ball about w. We may assume $\epsilon < 1/2n$. By definition, there exists $v \in B_\epsilon(w), t > n$, and $\phi \in \Gamma$ with $d\phi g^t v \in B_\epsilon(w)$. In particular

$$d(d\phi g^t v, v) \leq 2\epsilon < 1/n,$$

so that $v \in A_n$.

Since the set of nonwandering vectors is closed, it is in particular a complete metric space, and the Baire category theorem implies that $\bigcap A_n$ is dense in the set of nonwandering vectors. But $\bigcap A_n$ is just the set of recurrent vectors. □

We now turn to the main focus: proving recurrence for a finite-index subgroup. Our proof is again taken from Eberlein [19].

Lemma II.20. *Let $\Lambda \subseteq \Gamma$ be a normal subgroup of Γ. Let $\Sigma \subseteq SM$ denote the set of Λ-recurrent vectors. Then Σ is invariant under Γ.*

Proof. Let $v \in \Sigma$ and $\phi \in \Gamma$; we must show that $d\phi(v)$ is Λ-recurrent. Fix $\psi_n \in \Lambda$ and $t_n \to \infty$ realizing the recurrence for v—that is, such that $d\psi_n g^{t_n} v \to v$.

Set $\alpha_n = \phi\psi_n\phi^{-1} \in \Lambda$. Then

$$\begin{aligned} d\alpha_n g^{t_n} d\phi v &= (d\phi d\psi_n d\phi^{-1}) g^{t_n} d\phi v \\ &= d\phi d\psi_n g^{t_n} v \to d\phi v. \end{aligned}$$

□

Since recurrent vectors are dense in the set of nonwandering vectors, we have:

Corollary II.21. *Let $\Lambda \subseteq \Gamma$ be a normal subgroup of Γ. Then $\Omega(\Lambda)$ is invariant under Γ.*

Proposition II.22. *Let M be a complete Riemannian manifold, let Γ be a subgroup of isometries of M such that the Γ-recurrent vectors are dense in SM, and let Γ^* be a finite index subgroup of Γ. Then the Γ^*-recurrent vectors are dense in SM.*

Proof. Since Γ^* is finite index in Γ, there is a normal subgroup $\Lambda \subseteq \Gamma^*$ also of finite index in Γ, and it suffices to show that every vector of SM is Λ-nonwandering.

Fix a Γ-recurrent vector v and fix $\phi_n \in \Gamma$ and $t_n \to \infty$ realizing the recurrence. Since Λ is finite index, we may pass to a subsequence to assume that $\phi_n = \alpha\lambda_n$ for some fixed $\alpha \in \Gamma$, where $\lambda_n \in \Lambda$. Recurrence for v becomes

$$d\lambda_n g^{t_n} v \to \alpha^{-1} v.$$

We show that $\alpha^{-1}v$ is Λ-nonwandering. It follows from Corollary II.21 that v is Λ-nonwandering; since $\Omega(\Lambda)$ is closed and the set of Γ-recurrent vectors is dense, this will finish the proof.

We set $v_n = d\lambda_n g^{t_n} v$, so that $v_n \to v$. Fix an open neighborhood U of v, and choose N so that $v_n \in U$ for $n \geq N$. Fix some $T > 0$ and choose $n \geq N$ so that $t_n - t_N \geq T$. Then note that

$$d\lambda_n d\lambda_N^{-1} g^{t_n - t_N} v_N = v_n \in U,$$

which shows that v is nonwandering. $\square$

Chapter III

Manifolds without focal points

3.1 Definitions and basic results

Let M be a complete Riemannian manifold, p a point of M, and γ a geodesic through p. Recall that a point q on γ is said to be *conjugate* to p along γ if there exists a Jacobi field J along γ equal to zero at both p and q.

Two classical results show the importance of this condition. First, conjugate points are related to singularities of the exponential map $\exp : T_pM \to M$:

Proposition III.1. $\exp$ *is a local diffeomorphism at* $v \in T_pM$ *if and only if* $\exp(v)$ *is not conjugate to* p *along the geodesic* $\exp(tv)$.

In fact, the dimension of the kernel of $d\exp$ is exactly the dimension of the space of Jacobi fields J along $\exp(tv)$ equal to zero at both p and $\exp(v)$.

Second, fix $p, q \in M$ and a geodesic segment γ from p to q. For $s \in (-\epsilon, \epsilon)$, let σ_s be path from p to q such that $\sigma_0 = \gamma$, and such that the map $(s,t) \mapsto \sigma_s(t)$ is piecewise $\mathscr{C}^1$. Say that γ locally minimizes the length functional in the space of paths from p to q if for all such variations σ_s, the length functional taking a path to its length has a local minimum at $s = 0$ (that is, at γ). Then:

Proposition III.2. γ *locally minimizes the length functional in the space of paths from* p *to* q *if and only if no point of* γ *(between* p *and* q*) is conjugate to* p *along* γ*.*

In its most general form this second result is known as the Morse Index Theorem, which states that the dimension of the space of variations of paths for which γ fails to locally minimize the length functional is equal to the number of points between p and q conjugate to p along γ, counting multiplicity.

With the power of the above results in mind, we generalize the idea of a conjugate point in the following way. Let M be a Riemannian manifold, let N be a totally geodesic submanifold of M, and let γ be a geodesic of M with $\gamma(0) \in N$ and such that $\dot{\gamma}(0) \in T^{\perp}_{\gamma(0)}N$. We consider variations of geodesics $\gamma_s(t)$ with $\gamma_0 = \gamma$ and such that for all s we have both both (1) $\gamma_s(0) \in N$ and (2) $\dot{\gamma}_s(0) \in T^{\perp}_{\gamma(0)}N$.

Translating the conditions (1) and (2) into conditions on the Jacobi field of the variation $\gamma_s(t)$, we find that J satisfies the two conditions

$$J(0) \in T_{\gamma(0)}N \text{ and } J'(0) \in T^{\perp}_{\gamma(0)}N.$$

(The second of these conditions makes use of the fact that N is totally geodesic; in general, one has $J'(0) + S_{ga\dot{m}ma(0)}(J(0)) \in T^{\perp}_{\gamma(0)}N$, where $S_{\dot{\gamma}(0)}$ is the shape operator of N.)

Definition III.3. Let N be a totally geodesic submanifold of a Riemannian manifold M, and let γ be a geodesic with $\gamma(0) \in N$ and $\dot{\gamma}(0) \in T^{\perp}_{\gamma(0)}N$. Let $q = \gamma(a)$ be a point on γ. Then q is said to be a *focal point* of N (along γ) if there exists a Jacobi field J along γ satisfying both (1) $J(0) \in T_{\gamma(0)}N$ and (2) $J'(0) \in T^{\perp}_{\gamma(0)}N$, and such that J vanishes at q.

We will say that the totally geodesic submanifold N of M is *focal point free* in M if for every geodesic γ through N and orthogonal to N, N has no focal points along γ. Note that if $N = \{p\}$ is a single point, then N has no focal points if and only if p has no conjugate points along any geodesic through p.

Now, suppose N has a focal point at q along some geodesic γ and let J be a Jacobi field as in the definition. Either $J(0) = 0$ or $J(0) \neq 0$. In the former case, q is conjugate to $\gamma(0)$ along γ. In the latter case, we may consider the geodesic σ passing through $\gamma(0)$ with $\dot{\sigma}(0) = J(0)$; then it is evident that the submanifold $L = \{\sigma(t) : t \in \mathbb{R}\}$ also has a focal point at q along γ. This shows the equivalence of the two conditions in the following definition:

Definition III.4. M is said to have *no focal points* if either of the two equivalent conditions below hold:

1. Every totally geodesic submanifold N of M is focal point free in M; or
2. M has no conjugate points, and every geodesic γ of M, considered as a totally geodesic submanifold of M, is focal point free in M.

In particular, manifolds with no focal points have no conjugate points. We will show below that manifolds of nonpositive curvature have no focal points. Gulliver [28] shows that these inclusions are strict. On the other hand, many of the techniques used to study manifolds of nonpositive curvature can be adapted (often with relative ease) to the case of manifolds with no focal points; this thesis, as well as many of the results it cites, are a case in point.

It will be helpful to have a few different restatements of the no focal points condition. First, suppose γ and σ are geodesics intersecting orthogonally at $p = \gamma(a)$. Then $q = \gamma(0)$ is a focal point for σ along γ iff there is a Jacobi field J along γ such that $J(0) = 0$ and $\langle J(a), J'(a) \rangle = 0$. But this latter condition is equivalent to the statement that $\frac{d}{dt}|_{t=a} ||J(t)|| = 0$. Thus we have the following:

Proposition III.5. *M has no focal points if and only if for every geodesic γ and every Jacobi field J along γ with $J(0) = 0$ has $||J(t)||' \neq 0$ for all $t \neq 0$.*

The condition of having no focal points is often stated in terms of Proposition III.5. While this condition somewhat obscures the origin of the term "no focal points", it is often more technically useful than our definition. For instance, we can use this condition to easily prove the following two propositions:

Proposition III.6. *Let $M = N \times L$ have no focal points. Then N and L have no focal points.*

Proof. It suffices to show N has no focal points. Let γ be a geodesic in N, let $J(t)$ be a Jacobi field along γ satisfying $J(0) = 0$, and suppose $\langle J(a), J'(a)\rangle = 0$. Fix a variation of geodesics $\gamma_s(t)$ with variation field J. By fixing a point $q \in L$, we may lift γ_s to a variation $\widetilde{\gamma}_s$ of geodesics in M, where

$$\widetilde{\gamma}_s(t) = (\gamma_s(t), q) \in N \times L.$$

The Jacobi field $\widetilde{J}$ of this variation then clearly satisfies $\widetilde{J}(0) = 0$ and $\langle \widetilde{J}(a), \widetilde{J}'(a)\rangle = 0$, so that M has focal points. □

Proposition. *Manifolds of nonpositive curvature have no focal points.*

Proof. We use Proposition III.5. Let M have nonpositive curvature. Then let J be a Jacobi field and calculate

$$\begin{aligned} ||J(t)||'' = 2\langle J(t), J'(t)\rangle' &= 2\langle J(t), J''(t)\rangle + 2||J'(t)||^2 \\ &= -2\langle J(t), R(J(t), \dot{\gamma}(t))\dot{\gamma}(t)\rangle + 2||J'(t)||^2. \end{aligned}$$

Then note that the term $\langle J(t), R(J(t), \dot{\gamma}(t))\dot{\gamma}(t)\rangle$ is equal to the sectional curvature of the plane spanned by $\dot{\gamma}(t)$ and $J(t)$ multiplied by some positive constant, and is in particular nonpositive since M has nonpositive curvature. If now $J(0) = 0$ and J is nonzero, the above equation implies $||J(t)||'' > 0$, and hence $||J(t)||' > 0$, for all t. □

The second extremely important restatement of the no focal points condition is a generalization of Proposition III.1. Fix a totally geodesic submanifold N of M, and let $\nu^\perp N$ be the normal bundle to N in M. Denote by $\exp_N^\perp : \nu^\perp N \to M$ the restriction of the exponential map $\exp : TM \to M$. Then we have the following:

Proposition III.7. *Let N be a totally geodesic submanifold of M. Then $\exp_N^\perp$ is a local diffeomorphism at $v \in T_p^\perp N$ if and only if N does not have a focal point at $\exp(v)$ along the geodesic $\exp(tv)$. In particular, N is focal point free in M if and only if $\exp_N^\perp$ is a local diffeomorphism, and in this case $\exp_N^\perp : \nu^\perp N \to M$ is a covering map.*

The proof proceeds just as in the case for no conjugate points. We show one way this property may be useful. If M is a (complete) Riemannian manifold, a function $f : M \to \mathbb{R}$ is called *convex* if its restriction to every geodesic is a convex function $\mathbb{R} \to \mathbb{R}$.

Proposition III.8. *Let M be simply connected and without focal points, and let $p \in M$. Then the distance function $d_p : q \mapsto d(p, q)$ is convex.*

Proof. Let γ be a geodesic in M. If p lies on γ, the result is obvious, so assume otherwise. We consider the function $f(t) = d(p, \gamma(t))$ on $[0, 1]$; it suffices to show that f has no local maximum in $(0, 1)$.

Suppose to the contrary that f has a maximum at $t = \tau \in (0, 1)$. Then $f'(\tau) = 0$, and in particular the geodesic through p and $\gamma(\tau)$ is orthogonal to γ. On the other hand, $t \mapsto d(p, \gamma(t))$ must achieve a global minimum for some $t = \rho$, say, and we see that the geodesic through p and $\gamma(\rho)$ is also orthogonal to γ. Since $\exp_\gamma^\perp$ is a diffeomorphism, this cannot happen. □

Corollary III.9. *If M is simply connected and without focal points, then for every*

$p \in M$ and every $r > 0$ the geodesic ball $B_r(p)$ of radius r about p is convex.

Finally, we mention the following. If M is simply connected and nonpositively curved, it is well-known that any finite-order isometry of M must fix a point of M. The analogue of this fact for no focal points is easy to demonstrate and will be important to us in the proof of Theorem V.24:

Proposition III.10. *Let M be simply connected and without focal points, and let ϕ be a finite-order isometry of M. Then ϕ fixes a point of M.*

Proof. Note that M is diffeomorphic to $\mathbb{R}^n$ since it has no conjugate points. Given a subset $A \subset M$, we define the *convex closure* of A, denoted $\mathrm{cc}(A)$, to be the smallest closed, convex subset of M containing A. We note that if ϕ is an isometry of M, then $\phi(\mathrm{cc}(A)) = \mathrm{cc}(\phi(A))$, since ϕ preserves both convexity and closedness.

Since M has no focal points, the closed ball $B_r(p)$ of radius r about any point p is convex. Thus, if A is bounded, so is $\mathrm{cc}(A)$.

Now fix $p \in M$ and consider the orbit $S = \{\phi^n(p)\}$. This is a bounded subset of M since ϕ is finite order, and thus $\mathrm{cc}(S)$ is a compact convex subset of M that is taken to itself by the action of ϕ. The result now follows from Brouwer's fixed point theorem. □

3.2 Divergence of geodesics

Throughout this section, M is a simply connected Riemannian manifold with no focal points. We begin with a crucial definition.

Definition III.11. Let $v, w \in SM$. Then v, w are called *asymptotic* if $d(\gamma_v(t), \gamma_w(t))$ is bounded as $t \to \infty$, and v, w are called *parallel* if v, w are asymptotic and also $-v, -w$ are asymptotic. We say γ_v, γ_w are asymptotic or parallel when the same holds for v, w.

In section 3.3, we will use this notion to define a "boundary at infinity" of the manifold M. In this section we state some needed results on manifolds with no focal points. The main reference here is O'Sullivan's paper[1] [40].

First, we have the following two propositions, which often form a suitable replacement for convexity of the function $t \mapsto d(\gamma(t), \sigma(t))$ for geodesics γ, σ:

Proposition III.12 ([40] §1 Prop 2). *Let γ and σ be distinct geodesics with $\gamma(0) = \sigma(0)$. Then for $t > 0$, both $d(\gamma(t), \sigma)$ and $d(\gamma(t), \sigma(t))$ are strictly increasing and tend to infinity as $t \to \infty$.*

Proposition III.13 ([40] §1 Prop 4). *Let γ and σ be asymptotic geodesics; then both $d(\gamma(t), \sigma)$ and $d(\gamma(t), \sigma(t))$ are nonincreasing for $t \in \mathbb{R}$.*

O'Sullivan also proves an existence and uniqueness result for asymptotic geodesics:

Proposition III.14 ([40] §1 Prop 3). *Let γ be a geodesic; then for each $p \in M$ there is a unique geodesic through p and asymptotic to γ.*

Finally, O'Sullivan also proves a flat strip theorem (this result was also obtained, via a different method, by Eschenburg in [23]):

Flat Strip Theorem ([40] §2 Thm 1). *If γ and σ are parallel geodesics, then γ and σ bound a flat strip; that is, there is an isometric immersion $\phi : [0, a] \times \mathbb{R} \to M$ with $\phi(0, t) = \gamma(t)$ and $\phi(a, t) = \sigma(t)$.*

We will also need the following result, which is due to Eberlein [16]; a proof can also be found in [23].

Proposition III.15. *Bounded Jacobi fields are parallel.*

[1] Note that, as remarked by O'Sullivan himself, the relevant results in [40] are valid for *all* manifolds with no focal points (rather than only those with a lower curvature bound), since the condition $||J(0)|| \to \infty$ for all nontrivial initially vanishing Jacobi fields J is always satisfied for manifolds with no focal points, as shown by Goto [27].

(We remark that the simple argument in [23] shows the apparently weaker result that central Jacobi fields—that is, Jacobi fields which are both stable and unstable—are parallel. The above proposition follows then from the fact that any Jacobi field along γ bounded as $t \to \infty$ is a stable field, which itself follows from Goto's result [27] that the length of any initially vanishing Jacobi field must go to ∞ as $t \to \infty$.)

Finally, we have the following generalization of Proposition III.12:

Proposition III.16. *Let $p \in M$, let N be a totally geodesic submanifold of M through p, and let γ be a geodesic of M with $\gamma(0) = p$. Assume γ is not contained in N; then $d(\gamma(t), N)$ is strictly increasing and tends to ∞ as $t \to \infty$.*

Proof. Let σ_t be the unique geodesic segment joining $\gamma(t)$ to N and perpendicular to N; then (by a first variation argument) $d(\gamma(t), N) = L(\sigma_t)$, where $L(\sigma_t)$ gives the length of σ_t. Thus if $d(\gamma(t), N)$ is not strictly increasing, then we have $L'(\sigma_t) = 0$ for some t, and again a first variation argument establishes that then σ_t is perpendicular to γ, which is a contradiction since $\exp : \nu^{\perp}\sigma_t \to M$ is a diffeomorphism.

This establishes that $d(\gamma(t), N)$ is strictly increasing. To show it is unbounded we argue by contradiction. Suppose

$$\lim_{t \to \infty} d(\gamma(t), N) = C < \infty,$$

and choose sequences $t_n \to \infty$ and $a_n \in N$ such that $d(\gamma(t_n), N) = d(\gamma(t_n), a_n)$ and the sequence $d(\gamma(t_n), a_n)$ increases monotonically to C. We let w_n be the unit tangent vector at $\gamma(0)$ pointing at a_n; by passing to a subsequence, we may assume $w_n \to w \in T_{\gamma(0)}N$.

We claim $d(\gamma(t), \gamma_w) \leq C$ for all $t \geq 0$, contradicting Proposition III.12. Fix a time $t \geq 0$. For each n, there is a time s_n such that

$$d(\gamma(t), \gamma_{w_n}) = d(\gamma(t), \gamma_{w_n}(s_n)).$$

The triangle inequality gives

$$s_n \leq t + C.$$

Thus some subsequence of the points $\gamma_{w_n}(s_n)$ converges to a point $\gamma_w(s)$, and then clearly $d(\gamma(t), \gamma_w(s)) \leq C$, which establishes the result. □

3.3 The visual boundary

Again we assume M is a simply connected Riemannian manifold without focal points. We define for M a visual boundary $M(\infty)$, the *boundary of M at infinity*, a topological space whose points are equivalence classes of unit speed asymptotic geodesics in M.

If $\eta \in M(\infty)$, $v \in SM$, and γ_v is a member of the equivalence class η, then we say v (or γ_v) *points at η*. Alternatively, we may denote the equivalence class of the geodesic γ by $\gamma(\infty)$, and if τ is the inverse geodesic $\tau(t) = \gamma(-t)$, we may denote the equivalence class of τ by $\gamma(-\infty)$.

Proposition III.14 shows that for each $p \in M$ there is a natural bijection $S_pM \cong M(\infty)$ given by taking a unit tangent vector v to the equivalence class of γ_v. Thus for each p we obtain a topology on $M(\infty)$ from the topology on S_pM; in fact, these topologies (for various p) are all the same, which we now show.

Fix $p, q \in M$ and let $\phi : S_pM \to S_qM$ be the map given by taking $v \in S_pM$ to the unique vector $\phi(v) \in S_qM$ asymptotic to v. We wish to show ϕ is a homeomorphism, and for this it suffices to show:

Lemma III.17. *The map $\phi : S_pM \to S_qM$ is continuous.*

Proof. Let $v_n \in S_pM$ with $v_n \to v$, and let $w_n, w \in S_qM$ be asymptotic to v_n, v, respectively. We must show $w_n \to w$. Suppose otherwise; then, passing to a subsequence, we may assume $w_n \to u \neq w$. Fix $t \geq 0$. Choose n such that

$$d(\gamma_{w_n}(t), \gamma_u(t)) + d(\gamma_{v_n}(t), \gamma_v(t)) < d(p, q).$$

Then

$$d(\gamma_u(t), \gamma_w(t)) \leq d(\gamma_u(t), \gamma_{w_n}(t)) + d(\gamma_{w_n}(t), \gamma_{v_n}(t)) \\ + d(\gamma_{v_n}(t), \gamma_v(t)) + d(\gamma_v(t), \gamma_w(t)) < 3d(p, q),$$

the second and fourth terms being bounded by $d(p, q)$ by Proposition III.13. Since t is arbitrary, this contradicts Proposition III.12. □

We call the topology on $M(\infty)$ induced by the topology on any S_pM as above the *visual topology*. In fact, the visual topology on $M(\infty)$ extends to a topology on the visual compactification $\overline{M} = M \cup M(\infty)$, called the *cone topology*.

Definition III.18. For each $v \in SM$ and each $\epsilon > 0$, we define $C(v, \epsilon) \subseteq \overline{M}$ to be the set of those $x \in \overline{M}$ such that the geodesic from $\pi(v)$ to x makes angle less than ϵ with v. We define the *cone topology* on $\overline{M}$ to be the topology generated by the sets $C(v, \epsilon)$ and the open sets of M.

We remark that the tangent space T_pM to any point of M can also be viewed as a Riemannian manifold, and it therefore has a visual compactification $\overline{T_pM}$, equipped with the cone topology. The exponential map $\exp_p : T_pM \to M$ then has an obvious extension to a map $\exp_p : \overline{T_pM} \to \overline{M}$. The following result is due to Goto:

Theorem III.19 ([26])**.** *The cone topology on* $\overline{M} = M \cup M(\infty)$ *is the unique topology such that the map* $\exp_p : \overline{T_pM} \to \overline{M}$ *is a homeomorphism for each* $p \in M$.

We will be defining a second topology on $M(\infty)$ presently, so we take a moment to fix notation: If $\zeta_n, \zeta \in M(\infty)$ and we write $\zeta_n \to \zeta$, we *always* mean with respect to the visual topology unless explicitly stated otherwise.

If $\eta, \zeta \in M(\infty)$ and $p \in M$, then $\angle_p(\eta, \zeta)$ is defined to be the angle at p between v_η and v_ζ, where $v_\eta, v_\zeta \in S_pM$ point at η, ζ, respectively.

We now define a metric $\angle$ on $M(\infty)$, the *angle metric*, by

$$\angle(\eta, \zeta) = \sup_{p \in M} \angle_p(\eta, \zeta).$$

We note that the metric topology determined by $\angle$ is not in general equivalent to the visual topology. However, we do have:

Proposition III.20. *The angle metric is lower semicontinuous. That is, if $\eta_n \to \eta$ and $\zeta_n \to \zeta$ (in the visual topology), then*

$$\angle(\eta, \zeta) \leq \liminf \angle(\eta_n, \zeta_n).$$

Proof. It suffices to show that for all $\epsilon > 0$ and all $q \in M$, we have for all but finitely many n

$$\angle_q(\eta, \zeta) - \epsilon < \angle(\eta_n, \zeta_n).$$

Fixing $q \in M$ and $\epsilon > 0$, since $\eta_n \to \eta$ and $\zeta_n \to \zeta$, for all but finitely many n we have

$$\angle_q(\eta, \zeta) < \angle_q(\eta_n, \zeta_n) + \epsilon,$$

and this implies the inequality above. □

We also take a moment to establish a few properties of the angle metric.

Proposition III.21. *The angle metric $\angle$ is complete.*

Proof. For $\xi \in M(\infty)$, we denote by $\xi(p) \in S_pM$ the vector pointing at ξ. Let ζ_n be a $\angle$-Cauchy sequence in $M(\infty)$. Then for each p the sequence $\zeta_n(p)$ is Cauchy in the metric $\angle_p$, and so has a limit $\zeta(p)$; by Lemma III.17, the asymptotic equivalence class of $\zeta(p)$ is independent of p. We denote this class by ζ; it is now easy to check that $\zeta_n \to \zeta$ in the $\angle$ metric. (This follows from the fact that the sequences $\zeta_n(p)$ are Cauchy uniformly in p.) $\square$

Lemma III.22. *Let $v \in SM$ point at $\eta \in M(\infty)$, and let $\zeta \in M(\infty)$. Then $\angle_{\gamma_v(t)}(\eta, \zeta)$ is a nondecreasing function of t.*

Proof. This follows from Proposition III.13 and a simple first variation argument. $\square$

3.4 Asymptotic vectors, recurrence, and the angle metric

In this section we collect a number of technical lemmas. As a consequence we derive Corollary III.27, which says that the angle between the endpoints of recurrect vectors is measured correctly from any flat. (In nonpositive curvature, this follows from a simple triangle-comparison argument.)

Our first lemma allows us to compare the behavior of the manifold at (possibly distant) asymptotic vectors:

Lemma III.23. *Let $v, w \in SM$ be asymptotic. Then there exist sequences $t_n \to \infty$, $v_n \to v$, and $\phi_n \in \Gamma$ such that*

$$(d\phi_n \circ g^{t_n})v_n \to w$$

as $n \to \infty$.

Proof. First assume w is recurrent. Then we may choose $s_n \to \infty$ and $\phi_n \in \Gamma$ so that $(d\phi_n \circ g^{s_n})w \to w$. For each n let q_n be the footpoint of $g^{s_n}w$, and let v_n be the vector with the same footpoint as v such that the geodesic through v_n intersects q_n at some time t_n. Clearly $t_n \to \infty$. We now make two claims: First, that $v_n \to v$ and second, that $(d\phi_n \circ g^{t_n})v_n \to w$. Note that since v and w are asymptotic, Lemma III.22 gives

$$\angle_{\pi(v)}(v, v_n) \leq \angle_{q_n}(g^{t_n}v_n, g^{s_n}w).$$

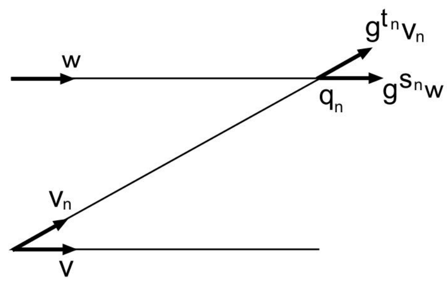

Figure 3.1: Lemma III.23, fig. 1

So if we show that the right-hand side goes to zero, both our claims are verified.

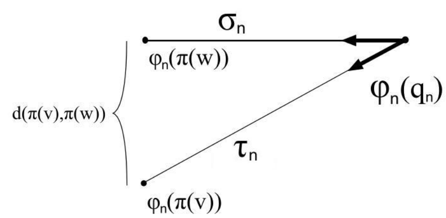

Figure 3.2: Lemma III.23, fig. 2

Consider the geodesic rays τ_n, σ_n through the point $\phi_n(q_n)$ satisfying

$$\dot{\tau}_n(0) = -d\phi_n(g^{t_n}v_n), \qquad \dot{\sigma}_n(0) = -d\phi_n(g^{s_n}w).$$

It suffices to show the angle between these rays goes to zero. Note $s_n, t_n \to \infty$. We claim that the distance between $\tau_n(t)$ and $\sigma_n(t)$ is bounded, independent of n, for $t \leq \max\{s_n, t_n\}$. To see this, first note that $|s_n - t_n| \leq d(\pi(v), \pi(w))$ by the triangle inequality. Suppose for example that $s_n \geq t_n$; then we find

$$d(\sigma_n(s_n), \tau_n(s_n)) \leq 2d(\pi v, \pi w),$$

and Proposition III.12 shows that for $0 \leq t \leq s_n$,

$$d(\sigma_n(t), \tau_n(t)) \leq 2d(\pi v, \pi w).$$

The same holds if $t_n \geq s_n$. Hence for fixed t, for all but finitely many n the above inequality holds. It follows that τ_n and σ_n converge to asymptotic rays starting at p. This establishes the theorem for recurrent vectors w.

We now do not assume w is recurrent; since recurrent vectors are dense in SM, we may take a sequence w_m of recurrent vectors with $w_m \to w$. For each m, there are sequences $v_{n,m} \to v$, $t_{n,m} \to \infty$, and $\phi_{n,m} \in \Gamma$ such that

$$(d\phi_{n,m} \circ g^{t_{n,m}})v_{n,m} \to w_m.$$

An appropriate "diagonal" argument now proves the theorem. □

As a corollary of the above proof we get the following:

Corollary III.24. *Let $v \in SM$ be recurrent and pointing at $\eta \in M(\infty)$; let $\zeta \in M(\infty)$. Then*

$$\angle(\eta, \zeta) = \lim_{t \to \infty} \angle_{\gamma_v(t)}(\eta, \zeta).$$

Proof. By Lemma III.22, the limit exists. Let $p = \pi(v)$, and fix arbitrary $q \in M$. Since v is recurrent, there exist $t_n \to \infty$ and $\phi_n \in \Gamma$ such that $(d\phi_n \circ g^{t_n})v \to v$. Let p_n be the footpoint of $g^{t_n}v$, and let γ_n be the geodesic from q to p_n. Define

$$v_n = g^{t_n}v \qquad \text{and} \qquad v_n' = \dot{\gamma}_n(p_n).$$

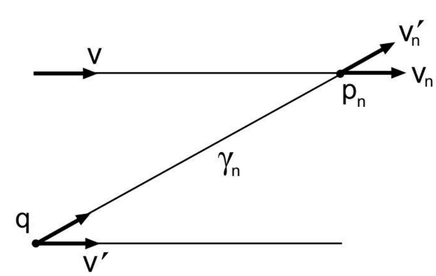

Figure 3.3: Corollary III.24

By the argument given in Lemma III.23, $\angle_{p_n}(v_n, v_n') \to 0$, and if we let $v' \in S_qM$ be the vector pointing at η, then $\dot\gamma_n(0) \to v'$. Thus

$$\angle_{p_n}(\zeta, v_n') \geq \angle_q(\zeta, \dot\gamma_n(0)) \to \angle_q(\zeta, \eta).$$

Since q was arbitrary, this proves the claim. □

In fact, the above corollary is true if v is merely asymptotic to a recurrent vector. To prove this we will need a slight modification to Lemma III.23, which is as follows:

Lemma III.25. *Let w be recurrent and v asymptotic to w. Then there exist sequences $w_n \to w$ and $s_n, t_n \to \infty$ such that $g^{t_n} w_n$ and $g^{s_n} v$ have the same footpoint q_n for each n, and*

$$\angle_{q_n}(g^{t_n} w_n, g^{s_n} v) \to 0.$$

Proof. First let $s_n \to \infty$, $\phi_n \in \Gamma$, be sequences such that

$$(d\phi_n \circ g^{s_n})w \to w.$$

Define $p = \pi(w), q = \pi(v)$, $p_n = \pi(g^{s_n} w)$, and $q_n = \pi(g^{s_n} v)$. Let w_n be the unit tangent vector with footpoint p such that there exists t_n such that $g^{t_n} w_n$ has footpoint q_n.

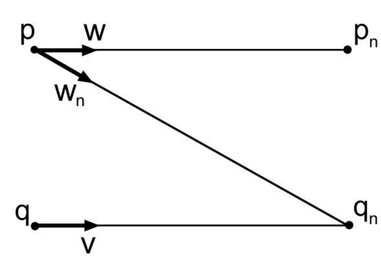

Figure 3.4: Lemma III.25

Note that for all n

$$\begin{aligned} d(\phi_n(q_n), p) &\leq d(\phi_n(q_n), \phi_n(p_n)) + d(\phi_n(p_n), p) \\ &\leq d(q_n, p_n) + K \\ &\leq d(q, p) + K, \end{aligned}$$

where K is some fixed constant. In particular, the points $\phi_n(q_n)$ all lie within bounded distance of p, and hence within some compact set. Therefore, by passing to a subsequence, we may assume we have convergence of the following three sequences:

$$r_n := \phi_n(q_n) \to r$$

$$w'_n := (d\phi_n \circ g^{t_n})w_n \to w'$$

$$v'_n := (d\phi_n \circ g^{s_n})v \to v'$$

for some r, w', v'. Then by the argument in the proof of Lemma III.23,

$$d(\gamma_{-w'_n}(t), \gamma_{-v'_n}(t)) \leq 2d(p, q)$$

for $0 \leq t \leq \max\{s_n, t_n\}$. It follows that $(-w')$ and $(-v')$ are asymptotic; since both have footpoint r, we see $w' = v'$. This gives the lemma. $\square$

We can now prove our previous claim:

Proposition III.26. *Let $w \in SM$ be recurrent, v asymptotic to w. Say v and w both point at $\eta \in M(\infty)$. Then for all $\zeta \in M(\infty)$*

$$\angle(\eta, \zeta) = \lim_{t\to\infty} \angle_{\gamma_v(t)}(\eta, \zeta).$$

Proof. Fix $\epsilon > 0$. By Corollary III.24, there exists a T such that

$$\angle_{\gamma_w(T)}(\eta, \zeta) \geq \angle(\eta, \zeta) - \epsilon.$$

We write $w' = g^T w$ and note that w' is also recurrent and asymptotic to v. Let p be the footpoint of w'. Choose by Lemma III.25 sequences $w_n \to w'$ and $s_n, t_n \to \infty$ such that

$$((*)) \qquad \angle_{\gamma_v(s_n)}(g^{t_n} w_n, g^{s_n} v) \to 0.$$

To fix notation, let w_n point at η_n. Then for large n

$$\begin{aligned}
\angle_{\gamma_v(s_n)}(\eta, \zeta) &\geq \angle_{\gamma_v(s_n)}(\eta_n, \zeta) - \epsilon && \text{by (*)} \\
&\geq \angle_p(\eta_n, \zeta) - \epsilon && \text{by Lemma III.22} \\
&\geq \angle_p(\eta, \zeta) - 2\epsilon && \text{by definition of the visual topology} \\
&\geq \angle(\eta, \zeta) - 3\epsilon && \text{by construction of } w'.
\end{aligned}$$

□

The key corollary of these results is:

Corollary III.27. *Let η be the endpoint of a recurrent vector w. Let F be a flat at $q \in M$, and $v, v' \in S_q F$ with v pointing at η. Say v' points at ζ; then*

$$\angle(\eta, \zeta) = \angle_q(\eta, \zeta).$$

In the next section we will establish the existence of plenty of flats; in section 4.2, this corollary will be one of our primary tools when we analyze the structure of the angle metric on $M(\infty)$.

3.5 Discrete groups of isometries

In this section M will denote a complete simply connected Riemannian manifold without focal points, and Γ will be a discrete subgroup of isometries of M. We discuss certain aspects of the action of Γ on M. This material generalizes to no focal points some of the work of Chen-Eberlein in [13] and Eberlein in [17], [18], [19], and [20]. The generalization to no focal points of [13] was carried out by Druetta in [15]; the other generalizations we give proceed exactly as in the nonpositive curvature case. We give complete proofs for the reader's convenience.

The material of this section will not be used until Chapter V, so the reader interested in only the proof of the higher rank rigidity theorem may safely skip it.

Definition III.28. A *Clifford translation* of M is an isometry ϕ of M such that $d(p, \phi(p))$ is constant over $p \in M$. If Γ is a group of isometries of M, we denote by $C(\Gamma)$ the subset of Γ consisting of Clifford translations.

We also denote by $Z(\Gamma)$ the center of Γ, and if $\Gamma^* \subseteq \Gamma$, we let $Z_\Gamma(\Gamma^*)$ be the centralizer of Γ^* in Γ.

We now present several important theorems from Druetta's work [15] on Clifford translations in manifolds without focal points, which generalizes [13]. First a definition:

Definition III.29. If ϕ is an isometry of M, the *associated vector field* X_ϕ is the unique vector field on M such that $\exp_p(X_\phi(p)) = \phi(p)$ for every $p \in M$.

Theorem III.30. *([15] Theorem 2.1) Write $M = E_r \times M_1$, where E_r is Euclidean and M_1 has no flat factors. Let ϕ be an isometry of M. The following are equivalent:*

1. *ϕ is a Clifford translation.*
2. *ϕ is bounded, i.e., $d(p, \phi(p))$ is bounded over $p \in M$.*
3. *The associated vector field X_ϕ is parallel.*
4. *ϕ decomposes as (ϕ_0, id), where ϕ_0 is a translation of E_r and id is the identity on M_1.*

Theorem III.31. *([15], Proposition 3.1 and Theorem 3.2) Let D be a subgroup of* $\mathrm{Isom}(M)$ *such that D-recurrent vectors are dense in SM. Then the centralizer $Z_{\mathrm{Isom}(M)}(D)$ consists of Clifford translations. If A is an abelian normal subgroup of D, then A consists of Clifford translations.*

In particular we note that abelian normal subgroups of Γ force M to have flat factors.

We also remark that, by Theorem III.30, the group $C(\Gamma)$ of Clifford translations is an abelian normal subgroup of Γ; thus if Γ-recurrent vectors are dense in SM, the above theorem shows that $C(\Gamma)$ is the unique maximal abelian normal subgroup of Γ. The following gives a partial converse as well as some further properties of $C(\Gamma)$ (and some of its subgroups):

Theorem III.32. *([15] Theorem 3.3) Suppose Γ acts freely on M and Γ-recurrent vectors are dense in SM. Let $A \subseteq \Gamma$ be an abelian normal subgroup. Then:*

1. *$A \cong \mathbb{Z}^s$ for some k with $1 \leq s \leq \dim(M)$;*
2. *M decomposes as $M = \mathbb{R}^s \times N$; and*
3. *M/Γ is foliated by compact totally geodesic flat submanifolds of dimension s, and Γ has a finite index normal subgroup with center of rank at least s.*

In fact [15] shows that the decomposition $M = \mathbb{R}^s \times N$ is obtained as follows: The vector fields X_a associated to the Clifford transformations $a \in C(\Gamma)$ determine an s-dimensional distribution D on M which is involutive, and its integral submanifolds form the $\mathbb{R}^k$ factor. We will be interested in the subgroup $\Gamma_0 = Z_\Gamma(C(\Gamma))$, which by Theorem III.30 is just the subgroup consisting of those elements of the form (γ_e, γ_1) where γ_e is a Euclidean translation of E_r. Since Γ_0 centralizes $C(\Gamma)$, one sees that Γ_0 preserves the distribution D, and it follows that Γ_0 preserves the factors of the decomposition $M = \mathbb{R}^k \times N$.

Proposition III.33. *Suppose* Γ *is a discrete group of isometries of* M *acting cocompactly on* M*. Then* $\Gamma_0 = Z_\Gamma(C(\Gamma))$ *is finite index in* Γ*.*

Proof. The proof follows the argument of Lemma 3 in Yau [48]. Each $\phi \in \Gamma$ acts on $C(\Gamma)$ by conjugation, $a \mapsto \phi^{-1} a \phi$. There is a natural distance function on $C(\Gamma)$ given by $||a|| = d(p, ap)$, which doesn't depend on the point p as a is a Clifford translation, and it is clear that the action of Γ by conjugation preserves this metric. $C(\Gamma)$ is a discrete group of translations of a Euclidean space, so there can be only finitely many such isometries. But if $\phi, \psi \in \Gamma$ give the same isometry, this says that

$$\phi^{-1} a \phi = \psi^{-1} a \psi$$

for all $a \in C(\Gamma)$; in other words, $\phi\psi^{-1}$ centralizes $C(\Gamma)$. □

This leads to the following generalization of a lemma in Eberlein [18]:

Lemma III.34. *Let* Γ *be a discrete group of isometries of* M *such that* Γ*-recurrent vectors are dense in* SM*. Then* Γ *admits a finite index subgroup* Γ_0 *such that for any finite index subgroup* Γ^* *of* Γ_0*, we have* $Z(\Gamma^*) = C(\Gamma^*)$*.*

Proof. Our proof is the same as Eberlein's. We let Γ_0 be the centralizer $Z_\Gamma(C(\Gamma))$, which is finite index in Γ by Proposition III.33. We have $C(\Gamma) \subseteq \Gamma_0$, and

$$C(\Gamma_0) = C(\Gamma) \subseteq Z(\Gamma_0).$$

Now let Γ^* be a finite index subgroup of Γ_0. Then Γ^*-recurrent vectors are also dense in SM, and so Theorem III.31 gives $Z(\Gamma^*) \subseteq C(\Gamma^*)$. On the other hand,

$$C(\Gamma^*) \subseteq C(\Gamma_0) \cap \Gamma^* \subseteq Z(\Gamma_0) \cap \Gamma^* \subseteq Z(\Gamma^*),$$

so $C(\Gamma^*) = Z(\Gamma^*)$. □

The last result we need is a generalization of a result of Eberlein [20], which states that if the discrete group Γ acts freely and cocompactly on M then the dimension of the Euclidean de Rham factor of M is equal to the rank of the maximal abelian normal subgroup of Γ. The proof is exactly as in [20], and will take us some time, since we first show that certain lemmas from nonpositive curvature carry over to no focal points. (However, all our proofs are exactly the same as in nonpositive curvature.) We begin by proving the following general lemma:

Lemma III.35. *Let $M = M_1 \times M_2$, and let Γ be a discrete, cocompact subgroup of isometries of M preserving the factors of the decomposition. Denote by $\pi_i : \Gamma :$ $\mathrm{Isom}(M_i)$ the projections, and assume $\pi_1\Gamma$ is discrete. Then $\ker \pi_1$ acts cocompactly on M_2.*

Proof. We construct a coarse compact fundamental domain for $\ker \pi_1$. Let F be a compact fundamental domain for the action of Γ; then $\pi_1 F$ is a compact coarse fundamental domain for the action of $\pi_1\Gamma$ on M_1. We let $H_1 \subseteq M_1$ be any compact coarse fundamental domain for $\pi_1\Gamma$ (for instance, set $H_1 = \pi_1 F$).

Since $\pi_1\Gamma$ is discrete, it acts properly on M_1; hence the set of all $a \in \pi_1\Gamma$ such that $aH_1 \cap \pi_1 F \neq \emptyset$ is finite. We denote its elements by $a_1, \ldots, a_k$, and we fix $b_1, \ldots, b_k \in \mathrm{Isom}(M_2)$ such that $(a_i, b_i) \in \Gamma$ for each i. Consider the compact set

$$K_2 = (a_1^{-1}, b_1^{-1})F \cup \cdots \cup (a_k^{-1}, b_k^{-1})F.$$

We claim $H_1 \times M_2 \subseteq (\ker \pi_1)K_2$. To see this let $(q_1, q_2) \in H_1 \times M_2$. There exists $(p_1, p_2) \in F$ and some $\gamma \in \Gamma$ such that $\gamma(p_1, p_2) = (q_1, q_2)$. In particular, we have $p_1 \in (\pi_1\gamma)^{-1}H_1 \cap \pi_1 F$, and this shows that γ has the form $\gamma = (a_i^{-1}, \gamma_2)$ for some $\gamma_2 \in \mathrm{Isom}(M_2)$. But then

$$(q_1, q_2) \in (1, \gamma_2 b_i)(a_i^{-1}, b_i^{-1})F \subseteq (\ker \pi_1)K_2.$$

□

The following is Proposition 2.2 of [17], whose proof is quite general:

Proposition III.36. *Let $M = M_1 \times M_2$, and let Γ be a cocompact subgroup of isometries of M preserving the factors. Suppose $\Gamma_1 = \pi_1\Gamma$ and $\Gamma_2 = \pi_2\Gamma$ are both discrete. Then Γ admits a finite index subgroup Γ^* splitting as $\Gamma^* = \Gamma_1^* \times \Gamma_2^*$.*

Proof. By Lemma III.35, $\ker \pi_1$ acts cocompactly on M_1 and $\ker \pi_2$ acts cocompactly on M_2. It follows that $\ker \pi_1 \times \ker \pi_2$ acts cocompactly on $M_1 \times M_2$, and hence is finite index in Γ. □

The following generalizes [17], Theorem 4.1:

Lemma III.37. *Let $M = M_1 \times M_2$. Let Γ be a discrete, cocompact subgroup of isometries of M preserving the decomposition. Suppose that $\pi_2\Gamma$ is discrete. Then either $\pi_1\Gamma$ is discrete or $N = \ker \pi_2$ contains nonidentity Clifford translations. Finally, if M_1 is Euclidean, then N contains nonidentity Clifford translation.*

Proof. The proof is as in [17]. Suppose $\pi_1\Gamma$ is not discrete, and let $G = \overline{\pi_1\Gamma}$ be its closure in $\mathrm{Isom}(M_1)$.

Clearly $\pi_1 N$ is normal in $\pi_1\Gamma$ and hence also in G. We let G_0 be the connected component of the identity of G. Then G_0 normalizes $\pi_1 N$, and hence must centralize $\pi_1 N$ since $\pi_1 N$ is discrete. In addition, we know from Lemma III.35 that $\pi_1 N$ acts cocompactly on M_1. It follows from Theorem III.31 that G_0 consists of Clifford translations.

We are now going to decompose M using the Clifford translations in G_0 into a product $M_\alpha \times M_\beta \times M_2$, where M_α is Euclidean, and G_0 acts by translations on M_α and by the identity on the other factors. By analyzing the action of G on this decomposition, we will show that the projection of Γ onto the second two factors is discrete, and this will allow us to find Clifford translations in the kernel of this projection.

We let D be the distribution given by

$$D_p = \mathrm{span}\{X_\phi(p) | \phi \in G_0\},$$

where $X_\phi(p)$ is determined by $\exp_p(X_\phi(p)) = \phi(p)$. The vector fields X_ϕ are parallel by Theorem III.30, and so (by the de Rham Theorem II.10) they determine a decomposition $M_1 = M_\alpha \times M_\beta$, where M_α, the integral manifold of D, is Euclidean. Since G_0 is normal in G, every $g \in G$ preserves the distribution D, and hence G preserves the decomposition. Moreover, G_0 acts as translations on M_α and the identity on M_β, as is clear from the definition of the vector fields X_ϕ and the fact that M_α is totally geodesic and Euclidean.

We let $\pi_\beta : G \to \mathrm{Isom}(M_\beta)$ be the projection; we claim $\pi_\beta(G)$ is discrete in $\mathrm{Isom}(M_\beta)$. Let us establish this claim. Suppose we have a sequence $\phi_n = (\alpha_n, \beta_n) \in G$ with $\beta_n \to \mathrm{id}$. We must show β_n is eventually constant. Fix $q_\alpha \in M_\alpha$; for each n there exists a translation $T_n \in G_0$ such that $\widetilde{\alpha}_n = T_n \alpha_n$ fixes q_α, and we define

$$\widetilde{\phi}_n = (\widetilde{\alpha}_n, \beta_n) \in G.$$

Fixing $q_\beta \in M_\beta$, it follows that $\widetilde{\phi}_n(q_\alpha, q_\beta) \to (q_\alpha, q_\beta)$ and thus, passing to a subsequence, we may assume $\widetilde{\phi}_n$ converges to an isometry $\widetilde{\phi} \in G$. In particular, $\widetilde{\phi}_n \widetilde{\phi}_m^{-1}$ converges as $(m,n) \to \infty$ to the identity of G, and hence is in G_0 for large m, n. But G_0 acts as the identity on M_β, which shows that β_n is constant for large n as desired. This proves our claim that $\pi_\beta(G)$ is discrete.

We now write $M = M_\alpha \times M_\beta \times M_2$, and note that Γ respects this decomposition. We have projections π_α, π_β, and π_2 from Γ into the isometry groups of these factors.

It follows from what we have just shown that $\pi_\beta(\Gamma)$ is discrete, and in particular that $\pi_\beta(\Gamma) \times \pi_2(\Gamma)$ is a discrete subgroup of isometries of $M_\beta \times M_2$. Thus Lemma III.35 shows that $N^* = \ker(\pi_\beta \times \pi_\Gamma)$ is a cocompact subgroup of $\mathrm{Isom}(M_\alpha)$. In particular, N^* must contain translations, which then lift to Clifford translations of M.

Thus we have shown that either $\pi_1\Gamma$ is discrete or N contains Clifford translations. To prove the last claim, we suppose now that M_1 is Euclidean; we would like to show that N contains Clifford translations. We have already shown this holds if $\pi_1\Gamma$ is not discrete, so we assume $\pi_1\Gamma$ is discrete. Then Γ admits a finite index subgroup Γ^* splitting as

$$\Gamma^* = \Gamma_1^* \times \Gamma_2^*.$$

Moreover, M_1/Γ_1^* is compact, so that Γ_1^*, and hence N, contains Clifford translations.

□

Corollary III.38. *Let $M = M_1 \times M_2$ have no flat factors, and suppose Γ is a discrete, cocompact subgroup of isometries of M. Suppose further that* $\mathrm{Isom}(M_2)$ *is discrete. Then Γ admits a finite index subgroup splitting as $\Gamma_1 \times \Gamma_2$.*

Proof. By the uniqueness of the de Rham decomposition, Γ admits a finite index subgroup Γ^* preserving the decomposition. Then $\pi_2\Gamma^*$ is discrete since $\mathrm{Isom}(M_2)$ is, and thus Lemma III.37 shows that $\pi_1\Gamma^*$ is discrete (since Clifford translations of M give rise to flat factors by Theorem III.30). Then Proposition III.36 shows that Γ^* admits a finite index subgroup that splits as a product. □

Recall that our goal is generalize Eberlein's result that the rank of the maximal abelian normal subgroup of M is the dimension of its Euclidean de Rham factor. Eberlein's proof relies on two crucial lemmas; the first is III.37 above. We prove the second presently:

Lemma III.39. *Let* $M = E_r \times M_2$*, where* E_r *is Euclidean and* M_2 *has no flat factors. Let* Γ *be a discrete subgroup of* $\mathrm{Isom}(M)$*. Suppose* Γ*-recurrent vectors are dense in* SM*; then* $\pi_2\Gamma$ *is discrete in* $\mathrm{Isom}(M_2)$*.*

Proof. Our proof is the same as Eberlein's [20]. Let A be the subgroup of $\mathrm{Isom}(M)$ consisting of translations of E_r, and let G be the closure of ΓA. As in [20], it follows from the Zassenhaus lemma (see p. 146 of [1]) that G_0, the connected component of the identity of G, is solvable, and so $\pi_2 G_0$ is also solvable.

We claim that $\pi_2 G_0$ is trivial, which we now show. Let A^* be the last nonidentity subgroup in the derived series for $\pi_2 G_0$. Then A^* is abelian. In addition, note that conjugation by an element of Γ gives an automorphism of G_0 since Γ normalizes A; such an automorphism must leave A^* invariant, and hence A^* is normalized by $\pi_2\Gamma$. If N denotes the normalizer of A^* in $\mathrm{Isom}(M_2)$, it follows that N-recurrent vectors are dense in SM_2, since N contains $\pi_2\Gamma$. Theorem III.31 shows that A^* consists of Clifford translations, and then Theorem III.30 shows that $A^* = \{\mathrm{id}\}$. Hence $\pi_2 G_0 = \{\mathrm{id}\}$.

We now complete the proof. Suppose $\phi_n = (\alpha_n, \beta_n)$ is a sequence of elements of Γ with $\beta_n \to \mathrm{id}$. Let T_n be the translation of E_r such that $\widetilde{\alpha}_n = \alpha_n T_n$ fixes $0 \in E_r$. Then $\widetilde{\phi}_n = (\widetilde{\alpha}_n, \beta_n) \in G$, and by passing to a subsequence we may assume that $\widetilde{\phi}_n$ converges in G. We set $\xi_n = \widetilde{\phi}_{n+1}\widetilde{\phi}_n^{-1}$; then $\xi_n \in G_0$ for large n. Since $\pi_2 G_0$ is trivial, β_n is constant for large n, and it follows that $\beta_n = \mathrm{id}$ for large n. □

The final lemma needed is a generalization of Lemma 5.1 in Eberlein's [17]:

Lemma III.40. *Let* $M = \mathbb{R}^k \times M_2$ *(where* M_2 *may have flat factors). Suppose* Γ *is a discrete subgroup of isometries of* M *preserving the factors of the decomposition and acting by translations on* $\mathbb{R}^k$*. Suppose also that the center* $Z(\Gamma)$ *of* Γ *is contained* *in* $\ker \pi_2$. *Then* $\Gamma_2 = \pi_2 \Gamma$ *is discrete.*

Proof. The proof is as in [17]. Let $G = \overline{\Gamma}_2$. We first claim that for each $\beta \in \Gamma_2$, there exists a neighborhood W_β of the identity in G such that if $\xi \in \Gamma_2 \cap W$, then ξ commutes with β.

Let us establish this claim. Fix $\beta \in \Gamma_2$, and lift to an element $\phi = (\alpha, \beta) \in \Gamma$. Let $\beta_n \in \Gamma_2$ be an arbitrary sequence converging to the identity. We lift these to elements $\phi_n = (\alpha_n, \beta_n) \in \Gamma$. Note that α_n, α are translations by assumption and hence commute. It follows that

$$[\phi_n, \phi] = \phi_n \phi \phi_n^{-1} \phi^{-1} = (\mathrm{id}, [\beta_n, \beta]) \in \Gamma_2,$$

and since $\beta_n \to \mathrm{id}$ it follows that $[\phi_n, \phi] \to \mathrm{id}$. Since Γ is discrete, ϕ_n and ϕ commute for large n, and hence so do β_n and β, which establishes the claim.

Second, we claim that if $X \in \mathfrak{g}$, the Lie algebra of G, then $\phi_t = \exp(tX)$ centralizes Γ_2 for all t.

We establish this claim: Let $\beta \in \Gamma_2$, and fix a neighborhood W as above. For small t, say $0 \leq t < \epsilon$, we have $\phi_t \in W_\beta$. Fixing such a t, we choose a sequence $\psi_n \in \Gamma_2$ converging to ϕ_t; then ψ_n commutes with β for all large n, and it follows by continuity that ϕ_t does as well. Thus we have proven the claim for $t < \epsilon$, and it is evident that the claim follows in general.

We now prove the lemma. Let U be a neighborhood of zero in $\mathfrak{g}$ such that $\exp : U \to V$ is a diffeomorphism, where V is a neighborhood of the identity in G. Suppose that $\beta \in \Gamma_2 \cap V$; then by the second claim above, β centralizes Γ_2. But then if we lift β to an element $\phi = (\alpha, \beta) \in \Gamma$, then the fact that α is a translation shows that ϕ is in the center of Γ, and hence, by assumption, that $\beta = \{\mathrm{id}\}$. Thus $\Gamma_2 \cap V = \{\mathrm{id}\}$ and it follows that G is zero-dimensional, i.e., that Γ_2 is discrete. $\square$

Finally we may prove Eberlein's main result from [20]:

Theorem III.41. *Let M be a closed Riemannian manifold without focal points, and let Γ be a discrete, cocompact subgroup of isometries of M acting freely. Then the dimension of the Euclidean de Rham factor of M equals the rank of the unique maximal abelian normal subgroup of Γ.*

Proof. We remark that by the discussion following Theorem III.31, the group $C(\Gamma)$ of Clifford translations is the unique maximal abelian normal subgroup of Γ. Our proof proceeds as in [20].

Let $M = E_r \times M_2$, where M_2 has no flat factors. If E_r is trivial, then Γ cannot have a nontrivial abelian normal subgroup by Theorem III.32. We therefore assume E_r is positive-dimensional. By Lemma III.39, $\pi_2\Gamma$ is discrete, and thus by Lemma III.37, Γ admits nonidentity Clifford translations.

We let Γ_0 be the centralizer of $C(\Gamma)$ in Γ. By Proposition III.33, Γ_0 has finite index in Γ. Thus Γ_0-recurrent vectors are dense in SM; since $Z(\Gamma_0)$ is an abelian normal subgroup of Γ_0, it follows from Theorem III.31 that $Z(\Gamma_0)$ consists of Clifford translations. It follows that $Z(\Gamma_0) = C(\Gamma_0) = C(\Gamma)$.

By III.32, we know that M decomposes as $M = \mathbb{R}^l \times N$, where $C(\Gamma) \cong \mathbb{Z}^l$ acts by translations on $\mathbb{R}^l$. Our goal is to show that N has no flat factors. The remarks following Theorem III.32 show that Γ_0 preserves the factors of this decomposition. In addition, we see from this description that Clifford translations of M act as the identity on N.

We let π_0, π_N denote the projection maps associated to the decomposition $M = \mathbb{R}^k \times N$. Then it is clear from the fact that $C(\Gamma)$ acts cocompactly by translations on $\mathbb{R}^k$ that $\pi_0\Gamma_0$ acts by translations on $\mathbb{R}^k$. Moreover, since the center of Γ_0 consists of Clifford translations, we have $\pi_N(Z(\Gamma_0)) = \{\mathrm{id}\}$. It follows from Lemma III.40

that $\pi_N\Gamma_0$ is discrete.

We now suppose N has a Euclidean de Rham factor, say $N = E_s \times N_\beta$. We know $\pi_N\Gamma_0$ is discrete, cocompact, and preserves the decomposition $E_s \times N_\beta$. By Lemma III.39, the projection of $\pi_N\Gamma_0$ to $\mathrm{Isom}(N_\beta)$ is discrete. If follows from Lemma III.37 that $\pi_N\Gamma_0$ contains nonidentity Clifford translations. However, if ϕ_2 is such a Clifford translation, then any lift ϕ of ϕ_2 to Γ_0 must be a Clifford translation, which is a contradiction since Clifford translations act as the identity on N. Thus N has no Euclidean de Rham factor, which completes the proof. $\square$

Chapter IV

Proof of the Rank Rigidity Theorem

4.1 Construction of flats

We repeat our standing assumption that M is a complete, simply connected, irreducible Riemannian manifold of higher rank and no focal points.

For a vector $v \in SM$, we let $\mathcal{P}(v) \subseteq SM$ be the set of vectors parallel to v, and we let P_v be the image of $\mathcal{P}(v)$ under the projection map $\pi : SM \to M$. Thus, $p \in P_v$ iff there is a unit tangent vector $w \in T_pM$ parallel to v. Our goal in this section will be to show that if v is a regular vector of rank m, that is, $v \in \mathcal{R}_m$, then the set P_v is an m-flat (a totally geodesic isometrically embedded copy of $\mathbb{R}^m$). To this end, we will first show that $\mathcal{P}(v)$ is a smooth submanifold of $\mathcal{R}_m$.

We begin by recalling that if $v \in SM$, there is a natural identification of T_vTM with the space of Jacobi fields along γ_v. In particular, the connection gives a decomposition of T_vTM into horizontal and vertical subspaces

$$T_vTM \cong T_{\pi(v)}M \oplus T_{\pi(v)}M,$$

and we may identify an element (x, y) in the latter space with the unique Jacobi field J along γ_v satisfying $J(0) = x, J'(0) = y$. Under this identification, T_vSM is identified with the space of Jacobi fields J such that $J'(t)$ is orthogonal to $\dot{\gamma}_v(t)$ for all t.

Define a distribution $\mathcal{F}$ on the bundle $TSM \to SM$ by letting $\mathcal{F}(v) \subseteq T_v SM$ be the space of parallel Jacobi fields along γ_v. The plan is to show that $\mathcal{F}$ is smooth and integrable on $\mathcal{R}_m$, and its integral manifold is exactly $\mathcal{P}(v)$. We note first that $\mathcal{F}$ is continuous on $\mathcal{R}_m$, since the limit of a sequence of parallel Jacobi fields is a parallel Jacobi field, and the dimension of $\mathcal{F}$ is constant on $\mathcal{R}_m$.

Lemma IV.1. *$\mathcal{F}$ is smooth as a distribution on $\mathcal{R}_m$.*

Proof. For $w \in SM$ let $\mathcal{J}_0(w)$ denote the space of Jacobi fields J along γ_w satisfying $J'(0) = 0$. For each $w \in \mathcal{R}_m$ and each $t > 0$, consider the quadratic form Q_t^w on $\mathcal{J}_0(w)$ defined by

$$Q_t^w(X,Y) = \int_{-t}^{t} \langle R(X,\dot{\gamma}_w)\dot{\gamma}_w, R(Y,\dot{\gamma}_w)\dot{\gamma}_w\rangle dt.$$

Since a Jacobi field J satisfying $J'(0) = 0$ is parallel iff $R(J,\dot{\gamma}_w)\dot{\gamma}_w = 0$ for all t, we see that $\mathcal{F}(w)$ is exactly the intersections of the nullspaces of Q_t^w over all $t > 0$. In fact, since the nullspace of Q_t^w is contained in the nullspace of Q_s^w for $s < t$, there is some T such that $\mathcal{F}(w)$ is exactly the nullspace of Q_T^w. We define $T(w)$ to be the infimum of such T; then $\mathcal{F}(w)$ is exactly the nullspace of $Q_{T(w)}^w$.

We claim that the map $w \mapsto T(w)$ is upper semicontinuous on $\mathcal{R}_m$. We prove this by contradiction. Suppose $w_n \to w$ with $w_n \in \mathcal{R}_m$, and suppose that $\limsup T(w_n) > T(w)$. Passing to a subsequence of the w_n, we may find for each n a Jacobi field Y_n along γ_{w_n} satisfying $Y_n'(0) = 0$ and such that Y_n is parallel along the segment of γ_{w_n} from $-T(w)$ to $T(w)$, but not along the segment from $-T(w_n)$ to $T(w_n)$.

We project Y_n onto the orthogonal complement to $\mathcal{F}(w_n)$, and then normalize so that $||Y_n(0)|| = 1$. Clearly Y_n retains the properties stated above. Then, passing to a further subsequence, we may assume $Y_n \to Y$ for some Jacobi field Y along γ_w. Then Y is parallel along the segment of γ_w from $-T(w)$ to $T(w)$. However, since $\mathcal{F}$

is continuous and Y_n is bounded away from $\mathcal{F}$, Y cannot be parallel along γ_w. This contradicts the choice of $T(w)$, and establishes our claim that $w \mapsto T(w)$ is upper semicontinuous.

To complete the proof, fix $w \in \mathcal{R}_m$ and choose an open neighborhood $U \subseteq \mathcal{R}_m$ of w such that $\overline{U}$ is compact and contained in $\mathcal{R}_m$. Since $T(w)$ is upper semicontinuous it is bounded above by some constant T_0 on U. But then the nullspace of the form $Q^u_{T_0}$ is exactly $\mathcal{F}(u)$ for all $u \in U$; since $Q^u_{T_0}$ depends smoothly on u and its nullspace is m-dimensional on U, its nullspace, and hence $\mathcal{F}$, is smooth on U. □

Our goal is to show that $\mathcal{F}$ is in fact integrable on $\mathcal{R}_m$; the integral manifold through $v \in \mathcal{R}_m$ will turn out to then be $\mathcal{P}(v)$, the set of vectors parallel to v. To apply the Frobenius theorem, we will use the following lemma, which states that curves tangent to $\mathcal{F}$ are exactly those curves consisting of parallel vectors:

Lemma IV.2. *Let $\sigma : (-\epsilon, \epsilon) \to \mathcal{R}_m$ be a curve in $\mathcal{R}_m$; then σ is tangent to $\mathcal{F}$ (for all t) iff for any $s, t \in (-\epsilon, \epsilon)$, the vectors $\sigma(s)$ and $\sigma(t)$ are parallel.*

Proof. First let $\sigma : (-\epsilon, \epsilon) \to \mathcal{R}_m$ be a curve tangent to $\mathcal{F}$. Consider the geodesic variation $\Phi : (-\epsilon, \epsilon) \times (-\infty, \infty) \to M$ determined by σ:

$$\Phi(s, t) = \gamma_{\sigma(s)}(t).$$

By construction and our identification of Jacobi fields with elements of TTM, we see that the variation field of Φ along the curve $\gamma_{\sigma(s)}$ is a Jacobi field corresponding exactly to the element $\dot{\sigma}(s) \in T_{\sigma(s)}TM$, and, by definition of $\mathcal{F}$, is therefore parallel. The curves $s \mapsto \Phi(s, t_0)$ are therefore all the same length L (as t_0 varies), and thus for any s, s' and all t

$$d(\gamma_{\sigma(s)}(t), \gamma_{\sigma(s')}(t)) \leq L.$$

Thus (by definition) $\sigma(s)$ and $\sigma(s')$ are parallel.

Conversely, let $\sigma : (-\epsilon, \epsilon) \to \mathcal{R}_m$ consist of parallel vectors and construct the variation Φ as before. We wish to show that the variation field $J(t)$ of Φ along $\gamma_{\sigma(0)}$ is parallel along $\gamma_{\sigma(0)}$, and for this it suffices, by Proposition III.15, to show that it is bounded.

Our assumption is that the geodesics $\gamma_s(t) = \Gamma(s, t)$ are all parallel (for varying s), and thus for any s the function $d(\gamma_0(t), \gamma_s(t))$ is constant (by the flat strip theorem). It follows that $||J(t)|| = ||J(0)||$ for all t, which gives the desired bound. □

Any curve $\sigma : (-\epsilon, \epsilon) \to \mathcal{R}_m$ defines a vector field along the curve (in M) $\pi \circ \sigma$ in the obvious way. It follows from the above lemma (and the symmetry $D_t \partial_s \Phi = D_s \partial_t \Phi$) for variations Φ) that if σ is a curve in $\mathcal{R}_m$ such that $\sigma(t)$ and $\sigma(s)$ are parallel for any t, s, then the associated vector field along $\pi \circ \sigma$ is a parallel vector field along $\pi \circ \sigma$.

We also require the following observation. Suppose that $p, q \in M$ are connected by a minimizing geodesic segment $\gamma : [0, a] \to M$, and let $v \in T_pM$. Then the curve $\sigma : [0, a] \to SM$ such that $\sigma(t)$ is the parallel transport of v along γ to $\gamma(t)$ is a minimizing geodesic in the Sasaki metric. It follows from this and the flat strip theorem that if v, w are parallel and connected by a unique minimizing geodesic in SM, then this geodesic is given by parallel transport along the unique geodesic from $\pi(v)$ to $\pi(w)$ in M and is everywhere tangent to $\mathcal{F}$.

Lemma IV.3. *$\mathcal{F}$ is integrable as a distribution on $\mathcal{R}_m$, and, if $v \in \mathcal{R}_m$, then the integral manifold through v is an open subset of $\mathcal{P}(v)$.*

Proof. To show integrability, we wish to show that $[X, Y]$ is tangent to $\mathcal{F}$ for vector fields X, Y tangent to $\mathcal{F}$. If ϕ_t, ψ_s are the flows of X, Y, respectively, then $[X, Y]_v =$

$\dot{\sigma}(0)$, where σ is the curve

$$\sigma(t) = \psi_{-\sqrt{t}}\phi_{-\sqrt{t}}\psi_{\sqrt{t}}\phi_{\sqrt{t}}(v).$$

From Lemma IV.2 we see that $\sigma(0)$ and $\sigma(t)$ are parallel for all small t, which, by the other implication in Lemma IV.2, shows that $[X, Y]_v \in \mathcal{F}(v)$ as desired. So $\mathcal{F}$ is integrable.

Now fix $v \in \mathcal{R}_m$ and let Q be the integral manifold of $\mathcal{F}$ through v. By Lemma IV.2, $Q \subseteq \mathcal{P}(v)$. Let $w \in Q$ and let U be a normal neighborhood of w contained in $\mathcal{R}_m$ (in the Sasaki metric); to complete the proof it suffices to show that $U \cap \mathcal{P}(v) \subseteq Q$. Take $u \in U \cap \mathcal{P}(v)$. Then (by the observation preceding the lemma) the SM-geodesic from w to u is contained in $\mathcal{R}_m$ and consists of vectors parallel to w, and hence to v. Thus $u \in Q$. □

For $v \in \mathcal{R}_m$ it now follows that $\mathcal{P}(v) \cap \mathcal{R}_m$ is a smooth m-dimensional submanifold of $\mathcal{R}_m$, and since the SM-geodesic between nearby points in $\mathcal{R}_m$ is contained in $\mathcal{P}(v)$, we see that $\mathcal{P}(v)$ is totally geodesic.

Consider the projection map $\pi : \mathcal{P}(v) \to P_v$; its differential $d\pi$ takes $(X, 0) \in \mathcal{F}(v) \subseteq T_v SM$ to $X \in T_{\pi(v)}M$. It follows that P_v is a smooth m-dimensional submanifold of M near those points $p \in M$ which are footpoints of vectors $w \in \mathcal{R}_m$ (and that π gives a local diffeomorphism of $\mathcal{P}(v)$ and P_v near such vectors w). We would like to extend this conclusion to the whole of P_v, and for this we will make use of Lemma III.23.

Proposition IV.4. *For every $v \in \mathcal{R}_m$, the set P_v is a convex m-dimensional smooth submanifold of M.*

Proof. Fix $v \in \mathcal{R}_m$. The flat strip theorem shows that P_v contains the M-geodesic between any two of its points, i.e., is convex. So we must show that P_v is an m-dimensional smooth submanifold of M.

For $u \in \mathcal{R}_m$, we let $C_\epsilon(u) \subseteq T_{\pi(u)}M$ be the intersection of the subspace $T_{\pi(u)}P_u$ with the ϵ-ball in $T_{\pi(u)}M$. Since $\mathcal{F}$ is smooth and integrable the foliation $\mathcal{P}$ is continuous with smooth leaves on $\mathcal{R}_m$; it follows that we may fix $\epsilon > 0$ and a neighborhood $U \subseteq \mathcal{R}_m$ of v such that for $u \in U$,

$$\exp_{\pi(u)} C_\epsilon(u) = P_u \cap B_\epsilon(\pi(u)),$$

where for $p \in M$ we denote by $B_p(\epsilon)$ the ball of radius ϵ about p in M.

By the flat strip theorem, the above equation is preserved under the geodesic flow; that is, for all t and all $u \in U$ we have

$$\exp_{\pi(g^t u)} C_\epsilon(g^t u) = P_{g^t u} \cap B_\epsilon(\pi(g^t u)).$$

This equation is also clearly also preserved under isometries.

Now fix $w \in \mathcal{P}(v)$; our goal is to show that P_v is smooth near $\pi(w)$. Choose by Lemma III.23 sequences $v_n \to v, t_n \to \infty$, and $\phi_n \in \Gamma$ such that $(d\phi_n \circ g^{t_n})v_n \to w$. We may assume $v_n \in U$ for all n. For ease of notation, let $w_n = (d\phi_n \circ g^{t_n})v_n$; then for all n we have $w_n \in \mathcal{R}_m$, and

$$\exp_{\pi(w_n)} C_\epsilon(w_n) = P_{w_n} \cap B_\epsilon(\pi(w_n)).$$

By passing to a subsequence if necessary, we may assume the sequence of m-dimensional subspaces $d\pi(\mathcal{F}(w_n))$ converges to a subspace $W \subseteq T_{\pi(w)}M$. Denote by W_ϵ the ϵ-ball in W. Then taking limits in the above equation we see that

$$\exp_{\pi(w)} W_\epsilon \subseteq P_w = P_v.$$

To complete the proof, we note that since P_v is convex (globally) and m-dimensional near v, P_v cannot contain an $(m+1)$-ball, for then convexity would show that it contains an $(m+1)$-ball near v. Thus if $U' \subseteq B_\epsilon(w)$ is a normal neighborhood of w, we must have

$$P_w \cap U' = \exp_{\pi(w)}(W_\epsilon) \cap U',$$

which shows that P_v is a smooth m-dimensional submanifold of M near w and completes the proof. □

Proposition IV.5. *For every $v \in \mathcal{R}_m$, the set P_v is an m-flat.*

Proof. Let $p = \pi(v)$. Choose a neighborhood U of v in $\mathcal{R}_m \cap T_p P_v$ such that for each $w \in U$, the geodesic γ_w admits no nonzero parallel Jacobi field orthogonal to P_v. We claim $P_w = P_v$ for all $w \in U$.

To see this, recall that $T_p P_w$ is the span of $Y(0)$ for parallel Jacobi fields $Y(t)$ along γ_w. If Y is such a field, then the component $Y^\perp$ of Y orthogonal to P_v is a bounded Jacobi field along γ_w, hence parallel, and therefore zero; it follows that $T_p P_w = T_p P_v$. Since P_v and P_w are totally geodesic, this gives $P_v = P_w$ as claimed.

But now take m linearly independent vectors in U; by the above we may extend these to m independent and everywhere parallel vector fields on P_v. Hence P_v is flat. □

Corollary IV.6. *For every $v \in SM$, there exists a k-flat F with $v \in S_{\pi(v)}F$.*

Proof. Let v_n be a sequence of regular vectors with $v_n \to v$. Passing to a subsequence if necessary, we may assume there is some $m \geq k$ such that $v_n \in \mathcal{R}_m$ for all n. For each n let W_n be the m-dimensional subspace of $T_{\pi(v_n)}M$ such that $\exp(W_n) = P_{v_n}$. Passing to a further subsequence, we may assume $W_n \to W$, where W is an m-dimensional subspace of $T_{\pi(v)}M$, and it is not difficult to see that $\exp W$ is an m-flat through v. □

4.2 The angle lemma, and an invariant set at ∞

The goal of the present section is to establish that $M(\infty)$ has a nonempty, proper, closed, Γ-invariant subset X. Our strategy is that of Ballmann [4] and Eberlein-Heber [22]. In section 4.3 we will use this set to define a nonconstant function f on SM, the "angle from X" function, which will be holonomy invariant, and this will show that the holonomy group acts nontransitively on M.

Roughly speaking X will be the set of endpoints of vectors of maximum singularity in SM; more precisely, in the language of symmetric spaces, it will turn out that X is the set of vectors which lie on the one-dimensional faces of Weyl chambers. To "pick out" these vectors from our manifold M, we will use the following characterization: For each $\zeta \in M(\infty)$, we may look at the longest curve $\zeta(t) : [0, \alpha(\zeta)] \to M(\infty)$ starting at ζ and such that

$$\angle_q(\zeta(t), \zeta(s)) = |t - s|$$

for every point $q \in M$; then ζ is "maximally singular" (i.e., $\zeta \in X$) if $\alpha(\zeta)$ (the length of the longest such curve) is as large as possible. One may check that in the case of a symmetric space this indeed picks out the one-dimensional faces of the Weyl chambers.

To show that the set so defined is proper, we will show that it contains no regular recurrent vectors; this is accomplished by demonstrating that every such path with endpoint at a regular recurrent vector extends to a longer such path in a neighborhood of that vector. For this we will need a technical lemma that appears here as Corollary IV.10.

We begin with the following lemma, which shows that regular geodesics have to "bend" uniformly away from flats:

Lemma IV.7. *Let $k = \operatorname{rank} M$, $v \in \mathcal{R}_k$, and let $\zeta = \gamma_v(-\infty), \eta = \gamma_v(\infty)$. Then there exists an $\epsilon > 0$ such that if F is a k-flat in M with $d(\pi(v), F) = 1$, then*

$$\angle(\zeta, F(\infty)) + \angle(\eta, F(\infty)) \geq \epsilon.$$

Proof. By contradiction. If the above inequality does not hold for any ϵ, we can find a sequence F_n of k-flats satisfying $d(\pi(v), F_n) = 1$ and

$$\angle(\zeta, F_n(\infty)) + \angle(\eta, F_n(\infty)) < 1/n.$$

By passing to a subsequence, we may assume $F_n \to F$ for some flat F satisfying $d(\pi(v), F) = 1$, and $\eta, \zeta \in F(\infty)$. In particular, F is foliated by geodesics parallel to v, so that $\mathcal{P}(v)$ is at least $(k+1)$-dimensional, contradicting $v \in \mathcal{R}_k$. □

This allows us to prove the following "Angle Lemma":

Lemma IV.8. *Let $k = \operatorname{rank} M$. Let $v \in \mathcal{R}_k$ be recurrent and suppose v points at $\eta_0 \in M(\infty)$. Then there exists $A > 0$ such that for all $\alpha \leq A$, if $\eta(t)$ is a path*

$$\eta(t) : [0, \alpha] \to M(\infty)$$

satisfying $\eta(0) = \eta_0$ and

$$\angle(\eta(t), \eta_0) = t$$

for all $t \in [0, \alpha]$, then $\eta(t) \in P_v(\infty)$ for all $t \in [0, \alpha]$.

Proof. Let $p = \pi(v)$ be the footpoint of v and let $\xi = \gamma_v(-\infty)$. By Lemma IV.7 we may fix $\epsilon > 0$ such that if F is a k-flat with $d(p, F) = 1$, then

$$\angle(\xi, F(\infty)) + \angle(\eta_0, F(\infty)) > \epsilon.$$

Choose $\delta > 0$ such that if $w \in S_pM$ with $\angle_p(v, w) < \delta$ then $w \in \mathcal{R}_k$, and set $A = \frac{1}{2}\min\{\delta, \epsilon\}$. Fix $\alpha \leq A$.

For the sake of contradiction, suppose there exists a path $\eta(t) : [0, \alpha] \to M(\infty)$ as above, but for some time $a \leq \alpha$

$$\eta(a) \notin P_v(\infty).$$

For $0 \leq s \leq a$, let $\eta_p(s) \in S_pM$ be the vector pointing at $\eta(s)$; since $\alpha < \delta$, we have $\eta_p(s) \in \mathcal{R}_k$. Fixing more notation, let $w = \eta_p(a)$.

We claim $\eta_0 \notin P_w(\infty)$. To see this, suppose $\eta_0 \in P_w(\infty)$; then by convexity $P_w(\infty)$ contains the geodesic γ_v, and since γ_v is contained in a unique k-flat, we conclude $P_w = P_v$, which contradicts our assumption that $\eta(a) \notin P_v(\infty)$.

It follows from Proposition III.16 that

$$d(\gamma_v(t), P_w) \to \infty \text{ as } t \to \infty.$$

Since v is recurrent, we may fix $t_n \to \infty$ and $\phi_n \in \Gamma$ such that the sequence $v_n = (d\phi_n \circ g^{t_n})v$ converges to v. By the above we may also assume $d(\gamma_v(t_n), P_w) \geq 1$ for all n. Then, since P_u depends continuously on $u \in \mathcal{R}_k$, there exists $s_n \in [0, a]$ such that

$$d(\gamma_v(t_n), P_{\eta_p(s_n)}) = 1.$$

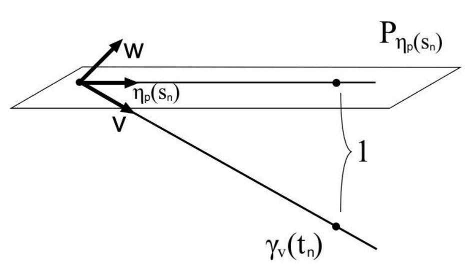

Figure 4.1: Lemma IV.8

We define a sequence of flats F_n by

$$F_n = \phi_n(P_{\eta_p(s_n)}).$$

Notice that F_n is indeed a flat, that $d(F_n, p) \to 1$, and that the geodesic $\gamma_{(-v_n)}$ intersects F_n at time t_n. By Proposition III.16, we have

$$d(\gamma_{(-v_n)}(t), F_n) \leq 1 \text{ for } 0 \leq t \leq t_n.$$

By passing to a subsequence, we may assume $F_n \to F$ for some k-flat F with $d(F, p) = 1$, and taking the limit of the above inequality, we see that $\gamma_{(-v)}(\infty) \in F(\infty)$. Thus Lemma IV.7 guarantees

$$\angle(\eta_0, F(\infty)) \geq \epsilon.$$

On the other hand, consider the sequence $\eta(s_n)$. By passing to a further subsequence, we may assume $\phi_n(\eta(s_n)) \to \mu$; since (by definition) $\phi_n(\eta(s_n)) \in F_n(\infty)$, we have $\mu \in F(\infty)$. Then

$$\begin{aligned}
\epsilon \leq \angle(\eta_0, F(\infty)) &\leq \angle(\eta_0, \mu) \\
&\leq \liminf_{n\to\infty} \angle(\phi_n(\eta_0), \phi_n(\eta(s_n)))) \\
&= \liminf_{n\to\infty} \angle(\eta_0, \eta(s_n)) \leq a \leq \alpha \leq \frac{\epsilon}{2},
\end{aligned}$$

where the inequality on the second line follows from Proposition III.20. This is the desired contradiction. □

As we did in section 3.4, we wish to extend this result not just to the k-flat F containing the regular recurrent vector v, but to every k-flat containing η_0 as an endpoint at ∞.

Proposition IV.9. *Let $v \in \mathcal{R}_k$ be recurrent and point at η_0, let A be as in Lemma IV.8 above, and let $\alpha \leq A$. Let F be a k-flat with $\eta_0 \in F(\infty)$, and suppose there exists a path*

$$\eta(t) : [0, \alpha] \to M(\infty)$$

with $\eta(0) = \eta_0$ *and*

$$\angle(\eta(t), \eta_0) = t \textit{ for all } t \in [0, \alpha].$$

Then $\eta(t) \in F(\infty)$ *for all* $t \in [0, \alpha]$.

Proof. Fix $q \in F$, and let $\eta_q \in S_q F$ point at η_0. Let $p = \pi(v)$, and let $\phi : S_q F \to S_p M$ be the map such that w and $\phi(w)$ are asymptotic. Denote by $B^F_\alpha(\eta_q)$ the restriction to F of the closed α-ball in the $\angle_q$-metric about η_q, and, similarly, denote by $B^{P_v}_\alpha(v)$ the restriction to P_v of the closed α-ball in the $\angle_p$-metric about v. We will show that ϕ gives a homeomorphism $B^F_\alpha(\eta_q) \to B^{P_v}_\alpha(v)$.

We first take a moment to note why this proves the proposition. We let $\eta_p(t) \in S_p M$ be the vector pointing at $\eta(t)$. Lemma IV.8 tells us that $\eta_p(t) \in B^{P_v}_\alpha(v)$ for $t \in [0, \alpha]$. Then since ϕ^{-1} takes $B^{P_v}_\alpha(v)$ into $B^F_\alpha(\eta_q)$, we see that $\eta(t) \in F(\infty)$ for such t.

So we've left to show ϕ gives such a homeomorphism. First, let's see that ϕ takes $B^F_\alpha(\eta_q)$ into $B^{P_v}_\alpha(v)$. Let $w \in B^F_\alpha(\eta_q)$ and let

$$\sigma : [0, \alpha] \to B^F_\alpha(\eta_q)$$

be the $\angle_q$-geodesic with $\sigma(0) = \eta_q$ and $\sigma(a) = w$ for some time a. Let

$$\tilde{\sigma} : [0, \alpha] \to M(\infty)$$

be the path obtained by projecting σ to $M(\infty)$. Then Corollary III.27 guarantees that $\tilde{\sigma}$ satisfies the hypotheses of Lemma IV.8, and so we conclude that $\tilde{\sigma}(t) \in P_v(\infty)$ for all t, from which it follows that ϕ maps $B^F_\alpha(\eta_q)$ into $B^{P_v}_\alpha(v)$ as claimed.

Now, note that for all $w \in B^F_\alpha(\eta_q)$ we have

$$\angle_q(w, \eta_q) = \angle_p(\phi(w), v),$$

again by Corollary III.27. Therefore for each $r \in [0, \alpha]$, ϕ gives an injective continuous map of the sphere of radius r in $B^F_\alpha(\eta_q)$ to the sphere of radius r in $B^{P_v}_\alpha(v)$; but any injective continuous map of spheres is a homeomorphism, and it follows that ϕ gives a homeomorphism of $B^F_\alpha(\eta_q)$ and $B^{P_v}_\alpha(v)$ as claimed. □

Corollary IV.10. *Let $v \in \mathcal{R}_k$ be recurrent and point at η_0, let A be as in Lemma IV.8, and let $\alpha \leq A$. Suppose we have a path*

$$\eta(t) : [-\alpha, \alpha] \to M(\infty)$$

with $\eta(0) = \eta_0$ and

$$\angle(\eta(t), \eta(0)) = t \text{ for all } t.$$

Then for all $q \in M$ and all $r, s \in [-\alpha, \alpha]$

$$\angle_q(\eta(r), \eta(s)) = \angle(\eta(r), \eta(s)).$$

Proof. Choose two points $q_1, q_2 \in M$. Then by Corollary IV.6 there are k-flats F_1, F_2 through q_1, q_2, respectively, with $\eta_0 \in F_1(\infty) \cap F_2(\infty)$. By Corollary IV.9, the path $\eta(t)$ lifts to paths $\eta_1(t) \subseteq S_{q_1}F_1$, $\eta_2(t) \subseteq S_{q_2}F_2$.

Fix $r, s \in [-\alpha, \alpha]$. Then for $i \in \{1, 2\}$ we have

$$d(\gamma_{\eta_i(r)}(t), \gamma_{\eta_i(s)}(t)) = 2t \sin\left(\tfrac{1}{2}\left(\angle_{q_i}(\eta(r), \eta(s))\right)\right).$$

Since $d(\gamma_{\eta_1(r)}(t), \gamma_{\eta_2(r)}(t))$ and $d(\gamma_{\eta_1(s)}(t), \gamma_{\eta_2(s)}(t))$ are both bounded as $t \to \infty$, we must have $\angle_{q_1}(\eta_1(r), \eta_1(s)) = \angle_{q_2}(\eta_2(r), \eta_2(s))$. Thus $\angle_q(\eta(r), \eta(s))$ is independent of $q \in M$, which gives the result. □

Proposition IV.11. *$M(\infty)$ contains a nonempty proper closed Γ-invariant subset.*

Proof. For each $\delta > 0$ define $X_\delta \subseteq M(\infty)$ to be the set of all $\xi \in M(\infty)$ such that there exists a path

$$\xi(t) : [0, \delta] \to M(\infty)$$

with $\xi(0) = \xi$ and

$$\angle_q(\xi(t), \xi(s)) = |t - s|$$

for all $t, s \in [0, \delta]$, and all $q \in M$.

Obviously X_δ is Γ-invariant. We claim it is closed. To this end, let $\xi_n \in X_\delta$ with $\xi_n \to \xi$, and choose associated paths

$$\xi_n(t) : [0, \delta] \to M(\infty).$$

By Arzela-Ascoli, some subsequence of these paths converges (pointwise, say) to a path $\xi(t)$, and this path satisfies

$$\angle_q(\xi(t), \xi(s)) = \lim_{n\to\infty} \angle_q(\xi_n(t), \xi_n(s)) = |t - s|,$$

so $\xi \in X_\delta$. Thus X_δ is closed; it follows that X_δ is compact.

We claim now that X_δ is nonempty for some $\delta > 0$. To see this choose a recurrent vector $v \in \mathcal{R}_k$, and say v points at η. Let A be as in Lemma IV.8, and let

$$\eta(t) : [0, A] \to M(\infty)$$

be the projection to $M(\infty)$ of any geodesic segment of length A starting at v in S_pP_v. Then by Corollary III.27, for all $t \in [0, A]$

$$\angle(\eta(t), \eta) = \angle_p(\eta(t), \eta) = t.$$

Thus by Corollary IV.10, $\angle_q(\eta(s), \eta(t))$ is independent of $q \in M$, and so in particular for any such q

$$\angle_q(\eta(s), \eta(t)) = \angle_p(\eta(s), \eta(t)) = |t - s|.$$

So $v \in X_A$.

A few remarks about the relationships between the various X_δ are necessary before we proceed. First of all, notice that if $\delta_1 < \delta_2$ then $X_{\delta_2} \subseteq X_{\delta_1}$. Furthermore, for any

δ, we claim that $\xi \in X_\delta$ iff $\xi \in X_\epsilon$ for all $\epsilon < \delta$. One direction is clear. To see the other, suppose $\xi \in X_{\epsilon_n}$ for a sequence $\epsilon_n \to \delta$. Then there exist paths

$$\xi_n(t) : [0, \epsilon_n] \to M(\infty)$$

satisfying the requisite equality, and again Arzela-Ascoli guarantees for some subsequence the existence of a pointwise limit

$$\xi(t) : [0, \delta] \to M(\infty)$$

which will again satisfy the requisite equality. Therefore, if we let

$$\beta = \sup\{\delta | X_\delta \text{ is nonempty}\}$$

then

$$X_\beta = \bigcap_{\delta<\beta} X_\delta.$$

In particular, being a nested intersection of nonempty compact sets, X_β is nonempty.

We now show that $\beta < \pi$. To see this, note that $\beta = \pi$ implies in particular that there exist two points ζ, ξ in $M(\infty)$ such that the angle between ζ and ζ when seen from any point is π. This implies that there exists a vector field Y on M such that for any point q, $Y(q)$ points at ζ and $-Y(q)$ points at ξ. The vector field Y is $\mathscr{C}^1$ by Theorem 1 (ii) in [23], and the flat strip theorem now shows that the vector field Y is holonomy invariant, so that M is reducible. Thus $\beta < \pi$.

We claim X_β is the desired set. We have already shown it is closed, nonempty, and Γ-invariant, so we have left to show that $X_\beta \neq M(\infty)$.

Fix a recurrent vector $v \in \mathcal{R}_k$; assume for the sake of contradiction that $v \in X_\beta$. Then there exists a path

$$\eta(t) : [0, \beta] \to M(\infty)$$

with $\eta(0) = \eta$ and $\angle_q(\eta(t), \eta(s)) = |t - s|$ for all $t, s \in [0, \beta]$. Let $p = \pi(v)$ be the footpoint of v, and let

$$\eta_p(t) : [0, \beta] \to S_p P_v$$

be the lift of $\eta(t)$. Then $\eta_p(t)$ is a geodesic segment in $S_p P_v$. We may choose $0 < \epsilon < A$, where A is as in Lemma IV.8, so that $\beta + \epsilon < \pi$. Thus we may extend $\eta_p(t)$ to a geodesic

$$\eta_p(t) : [-\epsilon, \beta] \to S_p P_v,$$

and we may use this to extend $\eta(t)$. By Corollaries III.27 and IV.10, we have for all $q \in M$

$$\angle_q(\eta(t), \eta(s)) = |t - s|,$$

and so $\eta(-\epsilon) \in X_{\beta+\epsilon}$, contradicting our choice of β. □

4.3 Completion of proof

We now fix a nonempty proper closed Γ-invariant subset $Z \subsetneq M(\infty)$ and define a function $f : SM \to \mathbb{R}$ by

$$f(v) = \min_{\zeta \in Z} \angle_{\pi(v)}(\gamma_v(\infty), \zeta).$$

It is clear that f is Γ-invariant, and Lemma III.22 gives that f is nondecreasing under the geodesic flow (that is, $f(g^t v) \geq f(v)$). We use the next four lemmas to prove that f is continuous, invariant under the geodesic flow, constant on equivalence classes of asymptotic vectors, and differentiable almost everywhere.

Lemma IV.12. *f is continuous.*

Proof. For each $\zeta \in M(\infty)$ define a function $f_\zeta : SM \to \mathbb{R}$ by

$$f_\zeta(v) = \angle_{\pi(v)}(\gamma_v(\infty), \zeta).$$

We will show that the family f_ζ is equicontinuous at each $v \in SM$, from which continuity of f follows.

Fix $v \in SM$ and $\epsilon > 0$. There is a neighborhood $U \subseteq SM$ of v and an $a > 0$ such that

$$d_a(u, w) = d(\gamma_u(0), \gamma_w(0)) + d(\gamma_u(a), \gamma_w(a))$$

is a metric on U giving the correct topology. Suppose $w \in U$ with $d_a(v, w) < \epsilon$. For $\zeta \in Z$, let $\zeta_{\pi(v)}, \zeta_{\pi(w)}$ be the vectors at $\pi(v), \pi(w)$, respectively, pointing at ζ. Then

$$|d_a(v, \zeta_{\pi(v)}) - d_a(w, \zeta_{\pi(w)})| \leq d_a(v, w) + d_a(\zeta_{\pi(v)}, \zeta_{\pi(w)}) \leq 3\epsilon,$$

by the triangle inequality for d_a for the first inequality, and Proposition III.13 for the second. This gives the desired equicontinuity at v. □

Lemma IV.13. *For $v \in SM$, we have $f(g^t v) = f(v)$ for all $t \in \mathbb{R}$.*

Proof. First assume v is recurrent. Fix $t_n \to \infty$ and $\phi_n \in \Gamma$ so that $d\phi_n g^{t_n} v \to v$. Then

$$f(d\phi_n g^{t_n} v) = f(g^{t_n} v)$$

and the sequence $f(g^{t_n} v)$ is therefore an increasing sequence whose limit is $f(v)$ and all of whose terms are bounded below by $f(v)$, so evidently $f(g^{t_n} v) = f(v)$ for all n, and it follows that $f(g^t v) = f(v)$ for all $t \in \mathbb{R}$.

Now we generalize to arbitrary v. Fix $t > 0$ and $\epsilon > 0$. By continuity of f and the geodesic flow, we may choose $\delta > 0$ so that if $u \in SM$ is within δ of v, then

$$|f(u) - f(v)| < \epsilon \text{ and } |f(g^t u) - f(g^t v)| < \epsilon.$$

Then choose u recurrent within δ of v to see that

$$|f(g^t v) - f(v)| \leq |f(g^t v) - f(g^t u)| + |f(g^t u) - f(u)| + |f(u) - f(v)| < 2\epsilon.$$

Since ϵ was chosen arbitrarily, $f(g^t v) = f(v)$. □

Lemma IV.14. *Let $v, w \in SM$ be arbitrary. If either v and w are asymptotic or $-v$ and $-w$ are asymptotic, then $f(v) = f(w)$.*

Proof. If v and w are asymptotic, fix by Lemma III.23 $t_n \to \infty$, $w_n \to w$, and $\phi_n \in \Gamma$, such that $(d\phi_n \circ g^{t_n})w_n \to v$. Then since f is continous,

$$f(w) = \lim f(w_n) = \lim f((d\phi_n \circ g^{t_n})w_n) = f(v).$$

On the other hand, if $-v$ and $-w$ are asymptotic, we may fix $t_n \to -\infty$, $w_n \to w$, and $\phi_n \in \Gamma$, such that $(d\phi_n \circ g^{t_n})w_n \to v$, and the exact same argument applies. □

Lemma IV.15. *f is differentiable almost everywhere.*

Proof. Fix $v \in SM$; there is a neighborhood U of v and an $a > 0$ such that

$$d_a(u, w) = d(\gamma_u(0), \gamma_w(0)) + d(\gamma_u(a), \gamma_w(a))$$

is a metric on U (giving the correct topology). Choose $u, w \in U$, and let $w' \in S_{\pi(u)}M$ be asymptotic to w. Then

$$|f(u) - f(w)| = |f(u) - f(w')| \leq \angle_{\pi(u)}(u, w') \leq C d_a(u, w'),$$

for some constant C. But note that

$$\begin{aligned} d_a(u, w') = d(\gamma_u(a), \gamma_{w'}(a)) &\leq d(\gamma_u(a), \gamma_w(a)) + d(\gamma_w(a), \gamma_{w'}(a)) \\ &\leq d(\gamma_u(a), \gamma_w(a)) + d(\gamma_w(0), \gamma_{w'}(0)) = d_a(u, w), \end{aligned}$$

by Proposition III.13. Therefore f is Lipschitz with respect to the metric d_a on U, and hence differentiable almost everywhere on U. □

From here on, the proof follows Ballmann [4], §IV.6, essentially exactly. We repeat his steps below for convenience.

We denote by $W^s(v), W^u(v) \subseteq SM$ the weak stable and unstable manifolds through v, respectively. Explicitly, $W^s(v)$ is the collection of those vectors asymptotic to v, and $W^u(v)$ the collection of those vectors w such that $-w$ is asymptotic to $-v$.

Lemma IV.16. *$T_vW^s(v) + T_vW^u(v)$ contains the horizontal subspace of T_vSM.*

Proof. Following Ballmann, given $w \in T_{\pi(v)}M$ we let $B^+(w)$ denote the covariant derivative of the stable Jacobi field J along γ_v with $J(0) = w$. That is, $B^+(w) = J'(0)$ where J is the unique Jacobi field with $J(0) = w$ and $J(t)$ bounded as $t \to \infty$. Similarly, $B^-(w)$ is the covariant derivative of the unstable Jacobi field along γ_v with $J(0) = w$. In this notation,

$$T_vW^s(v) = \{(w, B^+(w)) | w \in S_{\pi(v)}M\} \quad \text{and} \quad T_vW^u(v) = \{(w, B^-(w)) | w \in S_{\pi(v)}M\}.$$

Both B^+ and B^- are symmetric (as is shown in Eschenburg-O'Sullivan [24]). We let

$$E_0 = \{w \in T_{\pi(v)}M | B^+(w) = B^-(w) = 0\}.$$

Since B^+ and B^- are symmetric, they map $T_{\pi(v)}$ into the orthogonal complement $E_0^\perp$ of E_0.

The claim of the lemma is that any horizontal vector $(u, 0) \in T_vSM$ can be written in the form

$$(u, 0) = (w_1, B^+(w_1)) + (w_2, B^-(w_2)).$$

This immediately implies $w_2 = u - w_1$, so we are reduced to solving the equation

$$-B^-(u) = B^+(w_1) - B^-(w_1),$$

and for this it suffices to show the operator $B^+ - B^-$ surjects onto $E_0^\perp$, and for this it suffices to show that the restriction

$$B^+ - B^- : E_0^\perp \to E_0^\perp$$

is injective. Assuming $w \in E_0^\perp$, $B^+(w) = B^-(w)$ implies that the Jacobi field J with $J(0) = w$ and $J'(0) = B^+(w) = B^-(w)$ is both stable and unstable, hence bounded, hence, by Proposition III.15, parallel; thus $w \in E_0$ and it follows that $w = 0$. □

Corollary IV.17. *If c is a piecewise smooth horizontal curve in SM then $f \circ c$ is constant.*

Proof. Obviously it suffices to show the corollary for smooth curves c, so we assume c is smooth. By Lemma IV.15, f is differentiable on a set of full measure D. By the previous lemma and Lemma IV.14, if $\tilde{c}$ is a piecewise smooth horizontal curve such that $\tilde{c}(t) \in D$ for almost all t, then $f \circ \tilde{c}$ is constant (since $df(\dot{\tilde{c}}(t)) = 0$ whenever this formula makes sense).

Our next goal is to approximate c by suitable such curves $\tilde{c}$. Let l be the length of c, and parametrize c by arc length. Extend the vector field $\dot{c}(t)$ along c to a smooth horizontal unit vector field H in a neighborhood of c. Then there is some smaller neighborhood U of c which is foliated by the integral curves of H, and by Fubini (since $D \cap U$ has full measure in U), there exists a sequence of smooth horizontal curves $\tilde{c}_r$ such that $\dot{\tilde{c}}_r(t) \in D$ for almost all $t \in [0, l]$, and such that $\tilde{c}_r$ converges in the $\mathscr{C}_0$-topology to c. Since f is constant on each curve $\tilde{c}_t$ by the argument in the previous paragraph and f is continuous, we also have that f is constant on c. □

Finally, an appeal to the Berger-Simons holonomy theorem proves the result:

Rank Rigidity Theorem. *Let M be a complete irreducible Riemannian manifold with no focal points and rank $k \geq 2$. Assume that the Γ-recurrent geodesics are dense in M, where Γ is the isometry group of M. Then M is a symmetric space of noncompact type.*

Proof. By the previous corollary, the function f is invariant under the holonomy group of M. However, it is nonconstant. Thus the holonomy group of M is nontransitive and the Berger-Simons holonomy theorem implies that M is symmetric. □

Corollary IV.18. *Let M be a complete, simply connected Riemannian manifold with no focal points and with group of isometries Γ, and suppose that the Γ-recurrent vectors are dense in SM. Then M decomposes as a Riemannian product*

$$M = E_r \times M_S \times M_1 \times \cdot \times M_l,$$

where E_r is a Euclidean space of dimension r, M_S is a symmetric space of noncompact type, and each factor M_i for $1 \leq i \leq l$ is a nonsymmetric irreducible rank-one Riemannian manifold with no focal points.

Proof. Let

$$M = E_r \times N_1 \times \cdots \times N_s$$

be the de Rham decomposition of M. Proposition II.16 shows that the Isom(N_i)-recurrent vectors are dense in N_i, and III.6 shows that each N_i has no focal points. Thus each higher rank N_i is a symmetric space of noncompact type, and this gives the corollary. □

Chapter V

Fundamental Groups

In this section M is assumed to be a complete simply connected Riemannian manifold without focal points, and Γ a discrete, *cocompact* subgroup of isometries of M. We will also assume that Γ acts freely on M, so that M/Γ is a closed Riemannian manifold.

Following Prasad-Raghunathan [42] and Ballmann-Eberlein [8], define for each nonnegative integer i the subset $A_i(\Gamma)$ of Γ to be the set of those $\phi \in \Gamma$ such that the centralizer $Z_\Gamma(\phi)$ contains a finite index free abelian subgroup of rank no greater than i. We sometimes denote $A_i(\Gamma)$ simply by A_i when the group is understood.

Definition V.1 ([42])**.** $r(\Gamma)$ is the minimum i such that Γ can be written as a finite union of translates of A_i,

$$\Gamma = \phi_1 A_i \cup \cdots \cup \phi_k A_i,$$

for some $\phi_1, \ldots, \phi_k \in \Gamma$.

Definition V.2 ([8])**.** The *rank* of Γ is

$$\text{rank}(\Gamma) = \max\{r(\Gamma^*) : \Gamma^* \text{ is a finite index subgroup of } \Gamma\}.$$

Prasad-Ragunathan [42] show that $r(\Gamma) = \text{rank}(M)$ when M is a higher rank symmetric space; using this result, Ballmann-Eberlein [8] show that $\text{rank}(\Gamma) = \text{rank}(M)$

when M has nonpositive curvature. In this section, we generalize their result to no focal points:

Theorem. *Let M be a complete, simply connected Riemannian manifold with no focal points, and let Γ be a discrete, cocompact subgroup of isometries of M acting freely. Then* $\mathrm{rank}(\Gamma) = \mathrm{rank}(M)$.

This theorem is proved at the end of this chapter as Theorem V.26. We make some remarks on the plan of the proof. First, the Higher Rank Rigidity Theorem proved earlier in this thesis guarantees that M has a de Rham decomposition

$$M = M_S \times E_r \times M_1 \times \cdots \times M_l,$$

where M_S is a higher rank symmetric space, E_r is r-dimensional Euclidean space, and M_i is a nonsymmetric rank one manifold of no focal points, for $1 \leq i \leq l$.

We'd like to use this theorem to reduce to the rank one case. First of all, we have the following result from Ballmann-Eberlein:

Theorem V.3 ([8], Prop 2.1)**.** *Let Γ be an abstract group. Then:*

1. *If Γ^* is a finite index subgroup of Γ, then* $\mathrm{rank}\,\Gamma^* = \mathrm{rank}\,\Gamma$.
2. *If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$, then*

$$r(\Gamma) = \sum_{i=1}^{k} r(\Gamma_i) \text{ and } \mathrm{rank}(\Gamma) = \sum_{i=1}^{k} \mathrm{rank}(\Gamma_i).$$

What does this mean in the context of our de Rham decomposition? We will show that, in the case that the Euclidean factor is trivial, our group Γ has a finite-index subgroup Γ^* which splits as a product

$$\Gamma^* = \Gamma_S \times \Gamma_1 \times \cdots \times \Gamma_l,$$

where Γ_S and each Γ_i act cocompactly on the appropriate de Rham factor of M. Then Prasad-Ragunathan have shown that $\text{rank}(\Gamma_S) = \text{rank}(M_S)$; thus Theorem V.3 will allow us to reduce to computing the rank of Γ_i for $i \geq 1$, i.e., the case where M is a rank one manifold without focal points. In section 5.3 we will deal with the Euclidean factor of M, making use of the results of section 3.5.

Our goal in the remainder of this section is to show, under the assumption M has no flat factors, that Γ admits a finite index subgroup Γ^* splitting as above. Then, in sections 5.1 and 5.2, we will show $\text{rank}(\Gamma_i) = 1$ for the rank one factors. Finally, in section 5.3, we return to the general case to deal with the Euclidean part of M.

We begin with the following lemma:

Lemma V.4. *Let M have no flat factors, and let Γ be as above. Then M splits as a Riemannian product $M = M_S \times M_1$, where M_S is symmetric and M_1 has discrete isometry group.*

Proof. Let I_0 denote the connected component of the isometry group of M. By Theorem 3.3 of Druetta [15], Γ has no normal abelian subgroups. Then Proposition 3.3 of Farb-Weinberger [25] shows that I_0 is semisimple with finite center, and Proposition 3.1 of the same paper shows that M/Γ has a finite cover which decomposes as a Riemannian warped product

$$N \times_f B,$$

where N is locally symmetric of nonpositive curvature, and $\text{Isom}(B)$ is discrete. We claim that such a warped product must be trivial; this would show that M/Γ has a finite cover which decomposes as a Riemannian product $N \times B$, and thus M does as well.

Thus it suffices to show that a nontrivial compact Riemannian warped product must have focal points: Let $N \times_f B$ be a Riemannian warped product, where f :

$B \to \mathbb{R}_{>0}$ is the warping function. If f is not constant on B, there exists a geodesic γ in B such that f is not constant on γ. Let σ be a unit speed geodesic in N. Since $\{p\} \times B$ is totally geodesic in $N \times_f B$ for any $p \in N$ (as follows, for instance, from the Koszul formula), the variation $\Gamma(s,t) = (\sigma(s), \gamma(t))$ is a geodesic variation in $N \times_f B$. It is then easy to see that the variation field $J(t) = \partial_s \Gamma(0,t)$ of this variation satisfies

$$||J(t)|| = f(\gamma(t)),$$

which is bounded but nonconstant, so that $N \times_f B$ must have focal points. □

Corollary IV.18 guarantees that if M has no flat factors, then it admits a decomposition

$$M = M_S \times M_1 \times \cdots \times M_l,$$

where each of the $M_i, 1 \leq i \leq l$, has rank one and discrete isometry group. Then, by Lemmas V.4 and III.38, Γ has a finite index subgroup Γ^* splitting as

$$\Gamma^* = \Gamma_S \times \Gamma_1 \times \cdots \times \Gamma_l.$$

By the arguments above, to finish the proof in the case where M has no flat factors, we need to show that $\text{rank}(\Gamma) = 1$ in the case where M is irreducible, rank one, and has discrete isometry group. In section 5.2 we mimic the geometric construction of Ballmann-Eberlein to carry this out. Before doing this, however, we must first generalize a number of lemmas due to Ballmann [2] on rank one geodesics in manifolds of nonpositive curvature to the no focal points case; this is the work of the next section.

5.1 Rank one Γ-periodic vectors.

The following series of lemmas generalizes the work of Ballmann in [2]. As in that paper, we will be interested in geodesics γ that are Γ-periodic, i.e., such that there exists a $\phi \in \Gamma$ and some $a \in \mathbb{R}$ with $\phi \circ \gamma(t) = \gamma(t+a)$ for all t. Such a geodesic γ will be called *axial*, and ϕ will be called an *axis* of γ with *period* a.

Recall the following notation from section 3.3: $\overline{M} = M \cup M(\infty)$ is the visual compactification of M, and $C(v, \epsilon) \subseteq \overline{M}$ is the cone about v of angle ϵ. We are often interested in the following condition:

Definition V.5. We say that a geodesic γ *bounds a flat half-strip of with* c if there exists an isometric immersion $\Phi : \mathbb{R} \times [0, c) \to M$ such that $\Phi(t, 0) = \gamma(t)$, and that γ *bounds a flat half-plane* if there exists such Φ with $c = \infty$.

Note that if γ bounds a flat half-strip, then γ is higher rank. Note also that, in marked contrast to the higher-rank case, the implications

$$\gamma \text{ bounds a flat half plane } \to \gamma \text{ bounds a flat half strip } \to \gamma \text{ is higher rank}$$

are all strict. As a simple example, consider a negatively curved surface with a cusp; we cut off the cusp at some finite distance and reduce the curvature smoothly to zero, turning the cusp into a cylinder. Gluing such a surface to itself, we obtain a closed, rank one manifold; any geodesic wrapping around the central cylinder is higher rank, and in fact bounds a flat half strip, but does not bound a flat half plane. Letting the width of the central cylinder go to zero shows that a higher rank geodesic γ need not bound a flat half-strip.

If $p \in M$, $q \in \overline{M}$, we denote by γ_{pq} the unit speed geodesic through p and q with $\gamma(0) = p$. Note that if γ is a geodesic and $t_n \to \infty$, then $\gamma(t_n) \to \gamma(\infty)$ in the cone topology on $\overline{M}$. Moreover, if $p_n \in \overline{M}$ and $p_n \to \zeta \in M(\infty)$, then for $p \in M$ the geodesics γ_{pp_n} converge to $\gamma_{p\zeta}$. This follows from considering $\overline{T_pM}$ and Theorem III.19. More generally, we have the following lemma:

Lemma V.6. *Let $p, p_n \in M$ with $p_n \to p$, and let $x_n, \zeta \in \overline{M}$ with $x_n \to \zeta$. Then $\dot{\gamma}_{p_n x_n}(0) \to \dot{\gamma}_{p\zeta}(0)$.*

Proof. First pass to any convergent subsequence of $\dot{\gamma}_{p_n x_n}(0)$; say this subsequence converges to $\dot{\gamma}_{p\xi}(0)$, where $\xi \in M(\infty)$. Suppose for the sake of contradiction that $\xi \neq \zeta$. Let $c = d(\gamma_{p\zeta}(1), \gamma_{p\xi}(1)) > 0$. By the remarks preceding the lemma, we may choose n large enough so that each of

$$d(p_n, p),\ d(\gamma_{p_n x_n}(1), \gamma_{p\xi}(1)),\ \text{and } d(\gamma_{p x_n}(1), \gamma_{p\zeta}(1))$$

is strictly smaller than $c/3$. Proposition III.12 shows that $d(\gamma_{p_n x_n}(1), \gamma_{p x_n}(1)) < c/3$, and the triangle inequality gives the desired contradiction:

$$\begin{aligned} c &= d(\gamma_{p\zeta}(1), \gamma_{p\xi}(1)) \\ &\leq d(\gamma_{p\zeta}(1), \gamma_{p x_n}(1)) + d(\gamma_{p x_n}(1), \gamma_{p_n x_n}(1)) + d(\gamma_{p_n x_n}(1), \gamma_{p\xi}(1)) \\ &< c. \end{aligned}$$

□

Our next goal is to show that if γ does not bound a flat half plane and there exist geodesics γ_n with $\gamma_n(-\infty) \to \gamma(\infty)$ and $\gamma_n(\infty) \to \gamma(\infty)$, then in fact $\gamma_n \to \gamma$. The following lemma, which generalizes [2] Lemma 2.1 (i), does most of our work:

Lemma V.7. *Let γ be a geodesic, and suppose there exist*

$$p_k \in C(-\dot{\gamma}(0), 1/k) \cap M \qquad\qquad q_k \in C(\dot{\gamma}(0), 1/k) \cap M$$

such that $d(\gamma(0), \gamma_{p_k q_k}) \geq c > 0$ for all k. Then γ is the boundary of a flat half-strip of width c.

Proof. The idea is to show that the geodesic from p_k to q_k converges to a geodesic parallel to γ, and use the flat strip theorem. However, in fact the geodesics $\gamma_{p_k q_k}$

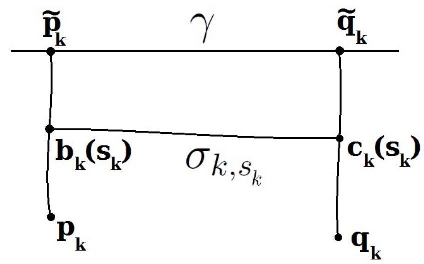

Figure 5.1: Lemma V.7

need not converge, so a slightly more technical argument is needed. For each k let $\tilde{p}_k, \tilde{q}_k$ be the points on γ closest to p_k, q_k, respectively. Let $b_k(s)$ be a smooth path with $b_k(0) = \tilde{p}_k,\ b_k(1) = p_k$, and similarly let $c_k(s)$ be a smooth path with $c_k(0) = \tilde{q}_k,\ c_k(1) = q_k$. We may further choose b_k so that the angle

$$\angle_{\gamma(0)}(\tilde{p}_k, b_k(s))$$

is an increasing function of s, and similarly for c_k. Finally, let $\sigma_{k,s}(t)$ be the unit speed geodesic through $b_k(s)$ and $c_k(s)$, parameterized so that $\sigma_{k,0}(0) = \gamma(0)$, and such that $s \mapsto \sigma_{k,s}(0)$ is a continuous path in M.

By hypothesis, $d(\sigma_{k,1}(0), \gamma(0)) \geq c$. Thus there exists s_k with $0 < s_k \leq 1$ and $d(\sigma_{k,s_k}(0), \gamma(0)) = c$. Passing to a subsequence, we may assume the geodesics σ_{k,s_k} converge as $k \to \infty$ to a geodesic σ with $d(\sigma(0), \gamma(0)) = c$.

Finally, any convergent subsequence of $b_k(s_k)$, or of $c_k(s_k)$, must converge to a point on γ, or one of the endpoints of γ. However, Lemma V.6, and the fact that $\sigma \neq \gamma$, shows that the only possibility is $b_k(s_k) \to \gamma(-\infty)$ and $c_k(s_k) \to \gamma(\infty)$. Another application of Lemma V.6 shows that σ is parallel to γ. The flat strip theorem now gives the result. □

In particular, if γ is rank one, it bounds no flat half strip of any width, so we obtain the following generalization of [2] Lemma 2.1 (ii):

Lemma V.8. *Let γ be rank one, and $c > 0$. Then there exists $\epsilon > 0$ such that if $x \in C(-\dot\gamma(0), \epsilon)$, $y \in C(\dot\gamma(0), \epsilon)$, then there is a geodesic connecting x and y.*

Furthermore, if σ is a geodesic with $\sigma(-\infty) \in C(-\dot\gamma(0), \epsilon)$ and $\sigma(\infty) \in C(\dot\gamma(0), \epsilon)$, then σ does not bound a flat half plane, and $d(\gamma(0), \sigma) \leq c$.

Proof. By Lemma V.7, there exists $\epsilon > 0$ such that $d(\gamma_{pq}, \gamma(0)) \leq c$ if $p \in C(-\dot\gamma(0), \epsilon) \cap M$ and $q \in C(\dot\gamma(0), \epsilon) \cap M$. We choose sequences $p_n \to x$ and $q_n \to y$; then some subsequence of $\gamma_{p_n q_n}$ converges to a geodesic connecting x and y.

To prove the second part, note that all geodesics τ with endpoints in $C(-\dot\gamma(0), \epsilon)$ and $C(\dot\gamma(0), \epsilon)$ satisfy $d(\gamma(0), \tau) \leq c$ by choice of ϵ. However, if σ bounds a flat half-plane then there are geodesics τ_n with the same endpoints as σ but with $\tau_n \to \infty$, a contradiction. □

As a corollary of the above, we see that indeed if γ is rank one and γ_n is a sequence of geodesics with $\gamma_n(-\infty) \to \gamma(-\infty)$ and $\gamma_n(\infty) \to \gamma(\infty)$, then $\gamma_n \to \gamma$.

Our next lemma is crucial, but technical. In preparation we state a lemma from nonpositive curvature.

Lemma V.9. *Let N be nonpositively curved, and let τ, σ be distinct asymptotic geodesics of N, and suppose that*

$$\angle(\dot\tau(0), \sigma(0)) + \angle(\dot\tau(0), \gamma(0)) = \pi.$$

Then τ, σ, and the geodesic segment from $\tau(0)$ to $\sigma(0)$ bound a flat half strip.

The standard proof of this lemma uses triangle comparison arguments, and we do not know whether it generalizes to no focal points. In keeping with the spirit of section 3.4, we attempt to replace it with a lemma that is somewhat less general and depends on the use of recurrence.

To see better the spirit of our generalization, suppose first that the geodesic γ connecting τ and σ above is axial for the isometry ϕ, and suppose further that $\sigma = \phi \circ \tau$. Then the fact that σ and τ are asymptotic says that ϕ has a fixed point ζ on $M(\infty)$ not equal to either of the endpoints of γ, and one might hope that the above lemma tells us in this case that the flat strips between the various translates $\phi^n\tau$ for $n \in \mathbb{Z}$ "glue up" to show that γ is the boundary of a flat half-plane F with $\zeta \in F(\infty)$.

In fact this is true; however, we will need a similar result in the case that γ is merely recurrent. Unfortunately there is then no single isometry ϕ realizing the recurrence, but rather a sequence of isometries (ϕ_n); correspondingly, we assume that there is some $x \in M(\infty)$ with $\phi_n(x)$ converging to somewhere other than the endpoints of γ. The precise statement follows:

Lemma V.10. *Let γ be a recurrent geodesic, and suppose ϕ_n is a sequence of isometries such that $d\phi_n(\dot\gamma(t_n)) \to \dot\gamma(0)$, where t_n increases to ∞. Further suppose that there exists $x, \zeta \in M(\infty)$ with $\phi_n(x) \to \zeta$, where $\zeta \neq \gamma(\infty)$ and $\zeta \neq \gamma(-\infty)$. Then γ is the boundary of a flat half plane F, and $\zeta \in F(\infty)$.*

Proof. For each $s \in \mathbb{R}$ let τ_s be the geodesic with $\tau_s(0) = \gamma(s)$ and $\tau_s(\infty) = x$, and let σ_s be the geodesic with $\sigma_s(0) = \gamma(s)$ and $\sigma_s(\infty) = \zeta$. Fix $t > 0$.

We first claim that for each $\epsilon > 0$, there exists an infinite subset $L(\epsilon) \subseteq \mathbb{N}$ such that for each $N \in L(\epsilon)$ there exists an infinite subset $L_N(\epsilon) \subseteq \mathbb{N}$ such that for $n \in L_N(\epsilon)$,

$$d(\tau_{t_N}(t), \tau_{t_n}(t)) \geq t_n - t_N - \epsilon.$$

Let us first show this claim.

By passing to a subsequence, we may assume

$$d(\phi_n\gamma(t_n), \gamma(0)) < \epsilon/3 \text{ and } d(\phi_n\tau_{t_n}(t), \sigma_0(t)) < \epsilon/3$$

for all $n \geq 1$; the second inequality follows from recurrence of γ, the fact that $\phi_n(x) \to \zeta$, and Proposition III.13.

Assume for the sake of contradiction that our claim is false; then again by passing to a subsequence, we may assume that for $m > n \geq 1$

$$d(\tau_{t_n}(t), \tau_{t_m}(t)) < t_m - t_n - \epsilon.$$

From this and the previous inequality, we conclude that for $m > n \geq 1$

$$d(\phi_n^{-1}\sigma_0(t), \phi_m^{-1}\sigma_0(t)) < t_m - t_n - \epsilon/3.$$

Choose l such that $l\epsilon/3 > 2t + \epsilon$. Then

$$\begin{aligned} d(\gamma(t_1), \gamma(t_l)) &\leq d(\gamma(t_1), \phi_1^{-1}\gamma(0)) + d(\phi_1^{-1}\gamma(0), \phi_1^{-1}\sigma_0(t)) + \sum_{i=1}^{l-1} d(\phi_i^{-1}\sigma_0(t), \phi_{i+1}^{-1}\sigma_0(t)) \\ &\quad + d(\phi_l^{-1}\sigma_0(t), \phi_l^{-1}\gamma(0)) + d(\phi_l^{-1}\gamma(0), \gamma(t_l)) \\ &< \epsilon/3 + t + \sum_{i=1}^{l-1}(t_{i+1} - t_i - \epsilon/3) + t + \epsilon/3 \\ &\leq 2t + \epsilon - l\epsilon/3 + t_l - t_1 \\ &< t_l - t_1, \end{aligned}$$

contradicting the fact that γ is length minimizing. This proves our claim.

The next step of the proof is to show that for $s > 0$

$$d(\sigma_0(t), \sigma_s(t)) = s.$$

Fix such s. Note that $d(\sigma_0(t), \sigma_s(t)) \leq s$ by Proposition III.13. Suppose for the sake of contradiction that

$$d(\sigma_0(t), \sigma_s(t)) = s - 3\epsilon$$

for some $\epsilon > 0$. Choose $N \in L(\epsilon)$ large enough such that

$$d(\phi_N \tau_{t_N}(t), \sigma_0(t)) < \epsilon \text{ and } d(\phi_N \tau_{t_N+s}(t), \sigma_s(t)) < \epsilon.$$

As before, that this can be done follows from recurrence of γ, the fact that $\phi_n(x) \to \zeta$, and Proposition III.13. Then if $n \in L_N(\epsilon)$ with $t_n > t_N + s$, we find

$$\begin{aligned} d(\tau_{t_N}(t), \tau_{t_n}(t)) &= d(\phi_N \tau_{t_N}(t), \phi_N \tau_{t_n}(t)) \\ &\leq d(\phi_N \tau_{t_N}(t), \sigma_0(t)) + d(\sigma_0(t), \sigma_s(t)) \\ &+ d(\sigma_s(t), \phi_N \tau_{t_N+s}(t)) + d(\phi_N \tau_{t_N+s}(t), \phi_N \tau_{t_n}(t)) \\ &< \epsilon + (s - 3\epsilon) + \epsilon + t_n - (t_N + s) \\ &= t_n - t_N - \epsilon, \end{aligned}$$

contradicting the definitions of $L(\epsilon), L_N(\epsilon)$. Hence

$$d(\sigma_s(t), \sigma_0(t)) = s$$

as claimed. In fact, the above argument shows that for all $r, s \in \mathbb{R}$

$$d(\sigma_r(t), \sigma_s(t)) = |r - s|.$$

We now complete the proof. Lemma 2 in O'Sullivan [40] shows that the curves θ_t defined by $\theta_t(s) = \sigma_s(t)$ are geodesics, and they are evidently parallel to γ. Thus the flat strip theorem guarantees for each t the existence of a flat F_t containing γ and θ_t; since F_t is totally geodesic, it contains each of the geodesics σ_s. (We remark, of course, that all the F_t coincide.) □

As indicated above, Lemma V.10 often works as a good enough replacement in no focal points for Lemma V.9. For our present purposes, we use it to generalize Lemma 2.4 in [2] in the following two corollaries:

Corollary V.11. *Let γ be a recurrent geodesic, and suppose there exists $x \in M(\infty)$ such that $\angle_{\gamma(t)}(x, \gamma(\infty)) = \epsilon$ for all t, where $0 < \epsilon < \pi$. Then γ is the boundary of a flat half-plane.*

Proof. If ϕ_n is a sequence of isometries such that $\phi_n \dot{\gamma}(t_n) \to \dot{\gamma}(0)$, for $t_n \to \infty$, one sees that any accumulation point ζ of $\phi_n(x)$ in $M(\infty)$ must satisfy $\angle_{\gamma(0)}(\gamma(\infty), \zeta) = \epsilon$, and so the previous lemma applies. □

Corollary V.12. *Let ϕ be an isometry with axis γ and period a. Suppose $B \subseteq M(\infty)$ is nonempty, compact, $\phi(B) \subseteq B$, and neither $\gamma(\infty)$ nor $\gamma(-\infty)$ is in B. Then γ bounds a flat half plane.*

Proof. Take $\phi_n = \phi^n$ and $t_n = na$, along with the recurrent geodesic $-\gamma$, in Lemma V.10. □

This allows us to prove the following generalization of [2], Lemma 2.5, in exactly the same manner as Ballmann:

Lemma V.13. *Let ϕ be an isometry with rank one axis γ and period a. Then for all ϵ, δ with $0 < \epsilon < \pi$ and $0 < \delta < \pi$, and all $t \in \mathbb{R}$, there exists s with*

$$\overline{C(\dot{\gamma}(s), \delta)} \subseteq C(\dot{\gamma}(t), \epsilon).$$

Proof. Suppose otherwise; then there exists such ϵ, δ, t such that for all s the above inclusion does not hold. In particular we may choose for each n a point z_n with

$$z_n \in \overline{C(\dot{\gamma}(na), \delta)} \qquad\qquad z_n \notin C(\dot{\gamma}(t), \epsilon).$$

Then if we set $x_n = \phi^{-n}(z_n)$, we have $x_n \in \overline{C(\dot{\gamma}(0), \delta)}$, and none of $x_n, \phi(x_n), \ldots, \phi^n(x_n)$ is in $C(\dot{\gamma}(t), \epsilon)$.

Thus if we let B be the set

$$B = \{x \in M(\infty) \cap \overline{C(\dot{\gamma}(0), \delta)} : \phi^n(x) \notin C(\dot{\gamma}(t), \epsilon) \text{ for all } n\},$$

we see that B is nonempty (it contains any accumulation point of x_n) and satisfies the other requirements of Corollary V.12, so γ is the boundary of a flat half plane. $\square$

Finally, we obtain a generalization of (parts (i)-(iii) of) [2] Proposition 2.2, one of the main results of that paper. Again, the proof is exactly as in [2].

Theorem V.14. *Let ϕ be an isometry with axis γ and period a. The following are equivalent:*

1. *γ is not the boundary of a flat half plane;*
2. *Given $\overline{M}$-neighborhoods U of $\gamma(-\infty)$ and V of $\gamma(\infty)$, there exists $N \in \mathbb{N}$ with $\phi^n(\overline{M} - U) \subseteq V$ and $\phi^{-n}(\overline{M} - V) \subseteq U$ whenever $n \geq N$; and*
3. *For any $x \in M(\infty)$ with $x \neq \gamma(\infty)$, there exists a geodesic joining x and $\gamma(\infty)$, and none of these geodesics are the boundary of a flat half plane.*

Proof. ($1 \Rightarrow 2$) By Lemma V.13 we can find $s \in \mathbb{R}$ with

$$\overline{C(-\dot{\gamma}(-s), \pi/2)} \subseteq U, \qquad \overline{C(\dot{\gamma}(s), \pi/2)} \subseteq V.$$

If $Na > 2s$ then for $n \geq N$

$$\begin{aligned} \phi^n(\overline{M} - U) &\subseteq \phi^n(\overline{M} - C(-\dot{\gamma}(-s), \pi/2)) \\ &\subseteq \overline{C(\dot{\gamma}(s), \pi/2)} \subseteq V, \end{aligned}$$

and analogously for U and V swapped.

($1 \Rightarrow 3$) By Lemma V.8 we can find $\epsilon > 0$ such that for $y \in C(-\dot{\gamma}(0), \epsilon)$ there exists a geodesic from y to $\gamma(\infty)$ which does not bound a flat half plane. But by (2) we can find n such that $\phi^{-n}(x) \in C(-\dot{\gamma}(0), \epsilon)$.

($2 \Rightarrow 1$) and ($3 \Rightarrow 1$) are obvious (by checking the contrapositive). $\square$

We now consider the specific case where Γ is a subgroup of isometries of M satisfying the duality condition. Our proof is again a straightforward generalization of Ballmann's proof of Proposition 2.13 in [2].

Proposition V.15. *Assume Γ-recurrent vectors are dense in SM. If γ is rank one and U, V are neighborhoods of $\gamma(-\infty)$ and $\gamma(\infty)$, then there exists an isometry $\phi \in \Gamma$ with rank one axis σ, where $\sigma(-\infty) \in U$ and $\sigma(\infty) \in V$.*

Proof. Since Γ-recurrent vectors are dense in SM, we may assume γ is recurrent, and take $\phi_n \in \Gamma$, $t_n \to \infty$, such that $d\phi_n \dot{\gamma}(t_n) \to \dot{\gamma}(0)$. We define $v_n = d\phi_n g^{t_n} v$.

Fix $\epsilon > 0$ and $c > 0$ by Lemma V.8. We replace U and V by $U_\epsilon = C(-v, \epsilon) \cap U$ and $V_\epsilon = C(v, \epsilon) \cap V$. Then for any $x \in U_\epsilon, y \in V_\epsilon$, there exists a unique rank one geodesic joining x and y.

We claim that for sufficiently large n, ϕ_n has fixed points in U_ϵ, V_ϵ. We first claim

$$\phi_n^{-1}(\overline{V}_\epsilon) \subseteq V_\epsilon \text{ for large } n.$$

We prove this claim by contradiction. Suppose $x_n \in \overline{V}$ with $\phi_n^{-1}(x_n) \notin V$. By passing to a subsequence, we may also assume that x_n converges (to an unnamed point) and that $\phi_n^{-1}(x_n) \to x$. Since $x_n \in V$ and $v_n \to v$ we have

$$\lim \angle_{\pi(v_n)}(v_n, x_n) \leq \epsilon,$$

from which we conclude

$$\lim \angle_{\gamma(t_n)}(\gamma(\infty), \phi_n^{-1}(x_n)) \leq \epsilon.$$

In addition, by construction we have

$$\lim \angle_{\gamma(0)}(\gamma(\infty), \phi_n^{-1}(x_n) \geq \epsilon.$$

By Lemma III.22, for $0 \leq t \leq t_n$ we have

$$\angle_{\gamma(0)}(\gamma(\infty), \phi_n^{-1}(x_n)) \leq \angle_{\gamma(t)}(\gamma(\infty), \phi_n^{-1}(x_n)) \leq \angle_{\gamma(t_n)}(\gamma(\infty), \phi_n^{-1}(x_n)).$$

It follows from these equations that for all t

$$\angle_{\gamma(t)}(\gamma(\infty), x) = \epsilon.$$

Corollary V.11 shows that γ bounds a flat half plane, which is a contradiction. Thus $\phi_n^{-1}(\overline{V}_\epsilon) \subseteq V_\epsilon$ for large n.

A similar argument shows that $\phi_n(\overline{U}_\epsilon) \subseteq U_\epsilon$ for large n. We provide this argument for completeness. The claim is shown by contradiction; we assume $y_n \in \overline{U}_\epsilon$ with $\phi_n(y_n) \notin U_\epsilon$. We may assume $y_n \to y$ by passing to a subsequence, and that $\phi_n(y_n)$ converges (to an unnamed point). Then we find

$$\lim \angle_{\gamma(t_n)}(\gamma(-\infty), y_n) \geq \epsilon$$

while at the same time

$$\lim \angle_{\gamma(0)}(\gamma(-\infty), y_n) \leq \epsilon$$

and again Lemma III.22 shows that

$$\angle_{\gamma(t)}(\gamma(-\infty), y) = \epsilon,$$

again contradicting Corollary V.11.

We have thus shown that ϕ_n has fixed points $\eta_n \in U_\epsilon$ and $\zeta_n \in V_\epsilon$, and hence must fix the geodesic σ_n through these points. The only thing left to argue is that ϕ_n translates σ_n, i.e., that ϕ_n does not fix σ_n pointwise.

But note that $d(\phi_n\gamma(0), \gamma(0)) \to \infty$, while by our choice of ϵ and c, $d(\sigma_n, \gamma(0))$ is uniformly bounded. Thus ϕ_n cannot fix σ_n pointwise for large n. □

Corollary V.16. *Rank one Γ-periodic vectors are dense in the set of rank one vectors.*

5.2 The geometric construction.

Our goal in this subsection is to prove the following:

Theorem V.17. *Let M have rank one, and let Γ be a discrete subgroup of isometries of M such that Γ-recurrent vectors are dense in M. Then $r(\Gamma) = 1$.*

Our method is simply to show that the Ballmann-Eberlein construction works equally well in the setting of no focal points. With the work of section 5.1 in hand, our proof is nearly identical to theirs, with some simple modifications. For completeness, we present the details.

Define

$$B_1(\Gamma) = \{\phi \in \Gamma : \phi \text{ translates a rank one geodesic }\}.$$

Lemma V.18. $B_1(\Gamma) \subseteq A_1(\Gamma)$.

Proof. For $\phi \in B_1(\Gamma)$ translating γ, the flat strip theorem guarantees that γ is the unique rank one geodesic translated by ϕ. Thus every element of $Z_\Gamma(\phi)$ leaves γ invariant. Since Γ is discrete, $Z_\Gamma(\phi)$ must therefore contain an infinite cyclic group of finite index. □

We will show there exist elements $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \Gamma$ such that

$$(\star) \qquad \Gamma = \phi_{11}^{-1} B_1 \cup \phi_{12}^{-1} B_1 \cup \psi_{21}^{-1} B_1 \cup \psi_{22}^{-1} B_2,$$

which implies $r(\Gamma) \leq 1$, after which we will make a separate argument for equality.

As in Ballmann-Eberlein a point $x \in M(\infty)$ is called *hyperbolic* if for any $y \neq x$ in $M(\infty)$, there exists a rank one geodesic joining y to x. By Theorem V.14, any rank one axial geodesic has hyperbolic endpoints; thus Corollary V.16 implies that the set of hyperbolic points is dense in the open set of $M(\infty)$ consisting of endpoints of rank one vectors.

The following generalizes Lemmas 3.5 and 3.6 in [8]:

Lemma V.19. *Let $p \in M$, let $x \in M(\infty)$ be hyperbolic, and let U^* be a neighborhood of x in $\overline{M}$. Then there exists a neighborhood U of x in $\overline{M}$ and $R > 0$ with the following property: For all $u \in U$ and $v \in \overline{M} - U^*$, there is a unique rank one geodesic σ connecting u and v, and moreover $d(p, \sigma) \leq R$.*

Proof. The proof is identical to that of [8]. Fix $q \in \overline{M} - U^*$. Since x is hyperbolic there exists a rank one geodesic γ from x to q; then Lemma V.8 shows that there exists a neighborhood V_q of q and U_q of p such that any $u \in U_q$ and $v \in V_q$ can be connected by a unique rank one geodesic σ, and moreover that σ lies within some bounded distance of $\gamma(0)$, and in particular, within some bounded distance R_q of p.

Now, $\overline{M} - U^*$ is compact, so we may cover it with finitely many of the V_q,

$$\overline{M} - U^* \subseteq V_{q_1} \cup \cdots \cup V_{q_k}.$$

Then $U = U_{q_1} \cap \cdots \cap U_{q_k}$ satisfies the conclusion of the lemma with $R = \max\{R_{q_i}\}$. □

We now generalize [8] Lemma 3.8:

Lemma V.20. *Let x, y be distinct points in $M(\infty)$ with x hyperbolic, and suppose U_x and U_y are neighborhoods of x and y, respectively. Then there exists an isometry $\phi \in \Gamma$ with*

$$\phi(\overline{M} - U_x) \subseteq U_y \text{ and } \phi^{-1}(\overline{M} - U_y) \subseteq U_x.$$

Proof. Since x is hyperbolic, there is a rank one geodesic from x to y. By Proposition V.15 there is therefore a Γ-periodic geodesic with endpoints in U_x and U_y, and the result now follows from Theorem V.14. □

The following is an imperfect generalization of [8] Lemma 3.9 which suffices for our purposes:

Lemma V.21. *Let $x \in M(\infty)$ be hyperbolic, $U^* \subseteq \overline{M}$ a neighborhood of x, and $p \in M$. Then there exists a neighborhood $U \subseteq \overline{M}$ of x such that if ϕ_n is a sequence of isometries with $\phi_n(p) \to z \in M(\infty) - U^*$, then*

$$\sup_{u \in U} \angle_{\phi_n(p)}(p, u) \to 0 \text{ as } n \to \infty.$$

Proof. By Lemma V.19 there exists $R > 0$ and a neighborhood $U \subseteq \overline{M}$ of x such that if σ is a geodesic with endpoints in U and $\overline{M} - U^*$ then $d(p, \sigma) \leq R$.

Let $x_n \in U$ be an arbitrary sequence, and for each n let σ_n be the geodesic through x_n with $\sigma_n(0) = \phi_n(p)$. Denote by b_n be the point on σ_n closest to p, and let γ_n be the geodesic through p with $\gamma_n(0) = \phi_n(p)$.

By construction $d(p, b_n) \leq R$, and so we also have $d(\phi_n^{-1}(p), \phi_n^{-1}(b_n)) \leq R$. It follows that any subsequential limit of $\phi_n^{-1}\sigma_n$ is asymptotic to any subsequential limit of $\phi_n^{-1}\gamma_n$. In particular

$$\angle_{\phi_n(p)}(p, x_n) = \angle_p(\phi_n^{-1}(p), \phi_n^{-1}(x_n)) \to 0,$$

from which the lemma follows. □

Finally we generalize [8] Lemma 3.10:

Lemma V.22. *Fix $p \in M$. Let x_1, x_2 be hyperbolic points in $M(\infty)$, and let A_1, A_2 be open subsets of $M(\infty)$ that are δ-separated when viewed from p, i.e., $\angle_p(a_1, a_2) \geq \delta$ for all $a_1 \in A_1, a_2 \in A_2$. Then there exist neighborhoods V_1 of x_1 and V_2 of x_2 such that for all $\phi \in \Gamma$, one of the four intersections $\phi(V_i) \cap A_j$ (for $i, j \in \{1, 2\}$) is empty.*

Proof. The proof is identical to that of [8]. We begin by fixing disjoint neighborhoods W_1 of x_1 and W_2 of x_2 in $\overline{M}$, and let U_1^*, U_2^* be neighborhoods of x_1, x_2 in $\overline{M}$ such that $\overline{U_i^*} \subseteq W_i$. By Lemma V.21, there exist neighborhoods $U_i \in U_i^*$ of x_i such that for any sequence of isometries $\phi_n \in \Gamma$ with $\phi_n(p) \to z \in M(\infty)$, we have

$$\sup_{u \in U} \angle_{\phi_n(p)}(p, u) \to 0.$$

We proceed by contradiction. Thus, we assume there exists a neighborhood basis V_i^n of x_i and a sequence $\phi_n \in \Gamma$ such that for every n, the intersections $\phi_n(V_i^n) \cap A_j$ are all nonempty. By passing to a subsequence, we may assume that $V_i^n \subseteq U_i$ for every n, and we may assume that both sequences $\{\phi_n(p)\}$ and $\{\phi_n^{-1}(p)\}$ converge, say to y and z in $\overline{M}$, respectively.

We claim first that $y \in M(\infty)$. For suppose $y \in M$; then, passing to a further subsequence, we may assume the isometries ϕ_n converge to an isometry ψ of M (which need not be in Γ). But then for any fixed neighborhood O of $\psi(x_1)$, the sets $\phi_n(V_1^n)$ must all eventually lie in O, and in particular, they cannot intersect both the δ-separated sets A_1 and A_2. This establishes the claim, and we remark that therefore $z \in M(\infty)$ as well.

Now, either $z \notin W_1$ or $z \notin W_2$. Suppose for instance $z \notin W_1$. Then $\phi_n^{-1}(p) \in M - U_1^*$ for large n, and thus by construction

$$\sup_{u \in U_1} \angle_p(\phi_n(p), \phi_n(u)) = \sup_{u \in U_1} \angle_{\phi_n^{-1}(p)}(p, u) \to 0.$$

It follows that $\phi_n(U_1)$ is eventually contained in any open neighborhood of y, and in particular, can not meet both A_1 and A_2, which is the desired contradiction. □

Proposition V.23. *If M is a rank one manifold without focal points and Γ is a discrete subgroup of isometries of M, then $r(\Gamma) \leq 1$.*

Proof. The proof is identical to that of [8]. Fix $p \in M$. Fix also distinct points $x_1, x_2, y_1, y_2 \in M(\infty)$ such that x_1, x_2 are hyperbolic; finally, fix $\delta > 0$ and neighborhoods C_1, C_2 of x_1, x_2 and A_1, A_2 of y_1, y_2 such that any two of these four neighborhoods are δ-separated when viewed from p. By making C_1, C_2 smaller if necessary, we may assume these satisfy the conclusion of Lemma V.22 relative to A_1, A_2. Finally, we fix by Lemma V.19 neighborhoods V_1, V_2 of x_1, x_2 so that both (a) $\overline{V}_i \subseteq C_i$ and (b) any point in V_i is connected to any point in $\overline{M} - C_i$ by a unique rank one geodesic.

Using Lemma V.20, we may choose for each $i, j \in \{1, 2\}$ an element $\phi_{ij} \in \Gamma$ with

$$\phi_{ij}(\overline{M} - A_j) \subseteq V_i \text{ and } \phi_{ij}^{-1}(\overline{M} - V_i) \subseteq A_j.$$

See figure, where we have drawn lines for the $\phi_i j$ to indicate that ϕ_{ij} may be thought of as translation of a rank one geodesic with endpoints in V_i and A_j.

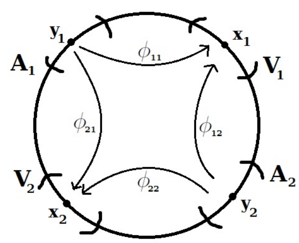

Figure 5.2: Proposition V.23

We claim that

$$\Gamma = \phi_{11}^{-1} B_1 \cup \phi_{12}^{-1} B_1 \cup \psi_{21}^{-1} B_1 \cup \psi_{22}^{-1} B_2.$$

To see this, let $\xi \in \Gamma$. By construction, there is some i, j such that $\xi(C_i) \cap A_j = \emptyset$; we fix this i, j for the remainder of the proof.

Consider the map $\phi_{ij}\xi$; we have

$$\phi_{ij}\xi(C_i) \subseteq \phi_{ij}(\overline{M} - A_j) \subseteq V_i \subseteq \overline{V}_i \subseteq C_i$$

from which it follows that $\phi_{ij}\xi$ fixes a point v in V_i. Similarly,

$$(\phi_{ij}\xi)^{-1}(\overline{M} - C_i) \subseteq \xi^{-1} A_j \subseteq \overline{M} - C_i,$$

so that $(\phi_{ij}\xi)^{-1}$, and therefore also $\phi_{ij}\xi$, has a fixed point u in $\overline{M} - C_i$. It follows that $\phi_{ij}\xi$ fixes the unique rank one geodesic γ from v to u.

We would like to show that in fact $\phi_{ij}\xi$ translates the geodesic γ and thus that $\phi_{ij}\xi \in B_1(\Gamma)$. In fact, $\phi_{ij}\xi$ fixes no point of M. To see this, note that $\phi_{ij}\xi(C_i)$ is a proper subset of C_i; but this is impossible if $\phi_{ij}\xi$ fixes any point of M, which is clear by considering the action of $\phi_{ij}\xi$ on $M(\infty)$ as seen from the fixed point.

Thus we have shown that $\phi_{ij}\xi$ translates a rank one geodesic, and it follows, as claimed, that

$$\Gamma = \phi_{11}^{-1}B_1 \cup \phi_{12}^{-1}B_1 \cup \psi_{21}^{-1}B_1 \cup \psi_{22}^{-1}B_2,$$

and hence that $r(\Gamma) \leq 1$. $\square$

Theorem V.24. *If M is a rank one manifold without focal points and Γ is a discrete cocompact subgroup of isometries of M, then $r(\Gamma) = 1$.*

Proof. Again, the proof is as in [8]. In light of the previous result, we must show $r(\Gamma) \neq 0$. Suppose otherwise; then there exist $\xi_1, \ldots, \xi_k \in \Gamma$ with

$$\Gamma = \xi_1 A_0 \cup \cdots \xi_k A_0.$$

We remark that A_0 is the set of elements whose centralizer is finite. In particular, any element of A_0 has finite order and hence fixes a point of M by Proposition III.10.

We wish to construct x_1, x_2, y_1, y_2 and neighborhoods C_i of x_i and A_i of y_i as in the previous proof, but now satisfying the following property: for each j we should have $\xi_j(C_1) \cap A_1 = \emptyset$. To do this, first choose y_1, y_2 and neighborhoods A_1, A_2 so that the complement of the union of the sets $\xi_j^{-1}(A_1)$ has nonempty interior; we may then choose x_1, x_2 in this interior and proceed as in the previous proof.

Then with ϕ_{11} as above, we have shown that $\phi_{11}\xi_j$ fixes no point of M. However, we must have $\phi_{11}^{-1} \in \xi_j A_0$ for some j, and then $(\phi_{11}\xi_j)^{-1} \in A_0$ and hence has a fixed point, which is a contradiction. □

5.3 Completion of the proof.

We complete the generalization of Ballmann-Eberlein's Theorem on the rank of the fundamental group in this section by dealing with the flat factors of M. We work with Clifford translations; the necessary theory for these isometries, for manifolds without focal points, was developed in section 3.5. Our proofs in this section are identical to the proof of Theorem 3.11 in [8].

Lemma V.25. *Let M be a complete, simply connected Riemannian manifold with no focal points and without flat factors, and let Γ be a discrete, cocompact subgroup of isometries of M. Then* $\text{rank}(\Gamma) = \text{rank}(M)$.

Proof. By Corollary IV.18, M decomposes as

$$M = M_S \times M_1 \times \cdots \times M_l,$$

where M_S is a symmetric space of noncompact type, and each M_i is rank one and nonsymmetric. By Lemma V.4, each M_i has discrete isometry group. Then Lemma III.38 shows that Γ has a finite index subgroup Γ^* splitting as a product

$$\Gamma^* = \Gamma_S \times \Gamma_1 \times \cdots \times \Gamma_l.$$

Then Theorem V.24 shows that $r(\Gamma_i) = 1$ for each i. In fact it follows from that theorem that $r(\Gamma_i^*) = 1$ for any finite index subgroup Γ_i^* of Γ_i, so that $\text{rank}(\Gamma_i) = 1$. Meanwhile, Prasad-Raghunathan [42] have shown $\text{rank}(\Gamma_s) = \text{rank}(M_S)$, so that by Theorem V.3

$$\begin{aligned}\text{rank}(\Gamma) = \text{rank}(\Gamma^*) &= \text{rank}(\Gamma_S) + \text{rank}(\Gamma_1) + \cdots + \text{rank}(\Gamma_l)\\ &= \text{rank}(M_S) + l = \text{rank}(M).\end{aligned}$$

$\square$

Theorem V.26. *Let M be a complete, simply connected Riemannian manifold with no focal points, and let Γ be a discrete, cocompact subgroup of isometries of M acting freely. Then* $\mathrm{rank}(\Gamma) = \mathrm{rank}(M)$.

Proof. Write $M = E_s \times M_2$ where M_2 has no flat factors E_s is an s-dimensional Euclidean space. By Lemma III.34 there is a finite index subgroup Γ_0 of Γ such that for any finite index subgroup $\Gamma^* \subseteq \Gamma_0$, we have $Z(\Gamma^*) = C(\Gamma^*)$.

We now fix such a finite index subgroup $\Gamma^* \subseteq \Gamma_0$. By Theorem III.41, $Z(\Gamma^*)$ is a free abelian group of rank s, i.e., $E_s/Z(\Gamma^*)$ is a flat s-torus.

We let $\pi_E : \Gamma^* \to \mathrm{Isom}(E_s)$ and $\pi_2 : \Gamma^* \to \mathrm{Isom}(M_2)$ be the projections, and denote the images by Γ^*_E and Γ^*_2. Note that Γ^*_E consists of translations of E_s since $Z(\Gamma^*)$ is a lattice of translations of E_s. In addition, Γ^*_2 is discrete by Lemma III.39. It follows from Lemma V.25 that

$$\mathrm{rank}(\Gamma^*_2) = \mathrm{rank}(M_2) = \mathrm{rank}(M) - s.$$

We claim that $\pi_2(A_i(\Gamma^*)) = A_{i-s}(\Gamma^*_2)$, and moreover that $A_i(\Gamma^*) = \pi_2^{-1}(A_{i-s}(\Gamma^*_2))$. To check these equalities, fix $\phi = (\phi_1, \phi_2) \in \Gamma^*$. Since Γ^*_E is a group of translations, it's easy to see that

$$\pi_2^{-1}(Z_{\Gamma^*_2}(\phi_2)) = Z_\Gamma(\phi),$$

and therefore that $\pi_2 : Z_\Gamma(\phi) \to Z_{\Gamma^*_2}(\phi_2)$ is surjective with kernel $Z(\Gamma^*)$. Moreover, $\pi_2^{-1}(A) \subseteq \Gamma^*$ is abelian iff $A \subseteq \Gamma^*_2$ is abelian. The claim now follows from the definition of A_i, noticing that $Z(\Gamma^*) = C(\Gamma^*)$ is a free abelian group of rank s.

One now sees easily that if $\{\phi_1, \dots, \phi_l\}$ is a finite subset of Γ^*, then

$$\Gamma^* = \bigcup_{\alpha=1}^{l} \phi_\alpha A_i(\Gamma^*)$$

if and only if

$$\Gamma_2^* = \bigcup_{\alpha=1}^{l} \pi_2(\phi_\alpha) A_{i-s}(\Gamma_2^*).$$

In particular, it follows that $r(\Gamma^*) = r(\Gamma_2^*) + s$. We conclude that

$$\mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma^*) = \mathrm{rank}(\Gamma_2^*) + s = \mathrm{rank}(M).$$

□

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