A Mirror Theorem for the Mirror Quintic

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2013

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to Mom, Dad, and Lauren

ACKNOWLEDGEMENTS

First and foremost I would like to express my thanks to Yongbin Ruan, for his patience, vision, and guidance. Over the years he has shared much more than just his mathematical expertise with me. Through him I have learned how math research is done, and how to enjoy doing it. For all of this I am grateful.

I would also like to thank Y.P. Lee, my collaborator and mentor. Not only is much of this thesis based on joint work with him, but I have also greatly enjoyed and benefitted from the numerous conversations we've had, both mathematical and otherwise.

Renzo Cavalieri has been a surrogate advisor to me, and my visits to Fort Collins were always both more productive and more fun because of him.

The Mathematics Department at Michigan has provided a wonderfully supportive environment, and I have loved the time I've spent here. I am grateful to the many professors and postdoctoral researchers who have generously shared their time and knowledge with me, in particular Khalid Bou-Rabee, Igor Dolgachev, Bill Fulton, Mircea Mustaţă, Matt Satriano, Peter Scott, Ian Shipman, and Karen Smith. I thank the administrative faculty, who work so hard and have made my life easier in many ways. I am also grateful to those who run the graduate student instructor program so effectively, and in particular Karen Rhea, from whom I learned to love teaching. I would also like to thank all the other graduate students for making Ann Arbor a fun place to live for five years.

Finally, thanks to my academic siblings (and extended cousins) Emily Clader, Steffen Marcus, Nathan Priddis, Dusty Ross, and Yefeng Shen. I am so lucky to have had such a great group of people to learn from and learn with.

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CHAPTER I

Introduction

Mirror symmetry, introduced by physicists over 20 years ago, predicts a complex relationship between certain pairs of Calabi-Yau manifolds V and V°. Originally arising out of string theory, it caught the attention of the mathematical community in 1990, when Candelas-de la Ossa-Green-Parkes [6] used mirror symmetry to predict the number of rational curves on a quintic hypersurface M in $\mathbb{P}^4$. These predictions were proven in 1997 by Givental [16] and Lian-Liu-Yau [21], in what is generally referred to as the mirror theorem. In this thesis we prove a corresponding statement for the mirror $\mathcal{W}$ of the quintic hypersurface. This proves that mirror symmetry is a true duality.

1.1 Mirror symmetry

Given a three dimensional complex Calabi–Yau manifold (or orbifold) V, one can sometimes associate to V a so-called *mirror manifold* V°. In the case when V is a toric hypersurface, for example, Batyrev gives a method for constructing V° via a combinatorial construction involving polytopes [3]. Mirror symmetry predicts a deep relationship between V and V°; in the language of physics, there should be a correspondence between the *A model* of V and the *B model* of V°. Mathematically this translates, roughly speaking, as saying that information about the

Kähler deformations of V should correspond to the complex deformations of V°. The Kähler deformations of a manifold X are parametrized by $H^{1,1}(X)$ (all cohomology groups are with complex coefficients unless otherwise specified). If X is a Calabi–Yau three-fold, the dimension of the space of complex deformations is $h^{2,1}(X)$. Consequently, an immediate prediction of mirror symmetry is that if V and V° are a mirror pair,

$$h^{1,1}(V) = h^{2,1}(V^\circ).$$

But the mirror symmetry prediction goes far beyond a correspondence at the level of cohomology. In fact the A model of V involve also encodes enumerative information about V, and is defined mathematically in terms of the Gromov–Witten theory (GWT) of V. The B model of a space, on the other hand, is formulated in terms of the variation of Hodge structures (VHS) associated to the complex deformations of that space. This VHS is computed via period integrals. We thus arrive at the following (somewhat vague) conjecture:

Conjecture I.1 (mirror conjecture)**.** *The Gromov–Witten theory of V is related to the period integrals over a family of deformations of V°.*

This correspondence was surprising to mathematicians; not only does it allow one to predict the Gromov–Witten invariants of V by relating them to the (more easily computed) VHS of V°, but it also indicates that there exist complex recursions among these invariants which were previously unknown. We will reformulate the above conjecture more precisely in what follows.

1.2 A mirror pair

In the current work we focus our attention on one particular mirror pair. Let Q be the Fermat quintic polynomial

$$Q(x) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5.$$

Let M be the projective hypersurface defined by

$$M := \{Q(x) = 0\} \subset \mathbb{P}^4.$$

This smooth complex variety is Calabi–Yau by the adjunction formula. Its mirror is the Deligne–Mumford stack $\mathcal{W}$ defined as a quotient

$$\mathcal{W} := [M/\bar{G}] = \{Q(x) = 0\} \subset [\mathbb{P}^4/\bar{G}],$$

where $\bar{G} \cong (\mathbb{Z}/5\mathbb{Z})^3$ is a (finite abelian) subgroup of the big torus of $\mathbb{P}^4$ acting via generators e_1, e_2, e_3:

$$e_1[x_0, x_1, x_2, x_3, x_4] = [\zeta x_0, x_1, x_2, x_3, \zeta^{-1} x_4]$$
$$e_2[x_0, x_1, x_2, x_3, x_4] = [x_0, \zeta x_1, x_2, x_3, \zeta^{-1} x_4]$$
$$e_3[x_0, x_1, x_2, x_3, x_4] = [x_0, x_1, \zeta x_2, x_3, \zeta^{-1} x_4].$$

The pair $(M, \mathcal{W})$ were predicted to be a mirror pair. In what follows, we will refer to M simply as the quintic, and to $\mathcal{W}$ as the mirror quintic. The original mirror theorem (Theorem I.2) describes a correspondence

$$A \text{ model of } M \equiv B \text{ model of } \mathcal{W}.$$

In order for the mirror symmetry to be a true duality, one must also show that

$$B \text{ model of } M \equiv A \text{ model of } \mathcal{W}.$$

This is the main result of this thesis.

1.3 The A model

The A model of a space X is described mathematically in terms of Gromov–Witten theory (see Definition II.3). Gromov–Witten theory aims to study X by considering spaces of maps from complex curves into X. Let $\overline{\mathscr{M}}_{g,k}(X,d)$ denote the moduli space of maps $f : C \to X$ where C is a complex curve of genus g with k marked points $\{p_1, \ldots, p_k\}$, and f is a map of degree d. Gromov–Witten invariants of X are defined as integrals of certain specified cohomology classes over these spaces. They can be viewed as giving a count of the number of maps satisfying specified incidence and tangency conditions. In some cases these numbers have been shown to correspond to enumerative information on X, but this is not true in general.

Specifically, for $1 \leq i \leq k$, consider the evaluation map $ev_i : \overline{\mathscr{M}}_{g,k}(X,d) \to X$ defined by sending the point $(f : C \to X) \in \overline{\mathscr{M}}_{g,k}(X,d)$ to $f(p_i) \in X$, the image of the i^{th} marked point under f. We obtain cohomology classes on $\overline{\mathscr{M}}_{g,k}(X,d)$ by pulling back classes from X under these evaluation maps. Another source of cohomology classes comes from line bundles on $\overline{\mathscr{M}}_{g,k}(X,d)$. For $1 \leq i \leq k$, let L_i denote the line bundle with fiber $T^*_{p_i}C$ over the point $(f : C \to X)$. We define the *ψ-classes* of X as $\psi_i := c_1(L_i)$. Gromov–Witten invariants for X are defined as integrals

$$\langle \alpha_1 \psi_1^{k_1}, \ldots, \alpha_n \psi_n^{k_n} \rangle^X_{g,n,d} := \int_{[\overline{\mathscr{M}}_{g,n}(X,d)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i) \psi_i^{k_i},$$

where $\alpha_i \in H^*_{CR}(X)$. Here $[\overline{\mathscr{M}}_{g,n}(X,d)]^{vir}$ denotes the so-called *virtual fundamental class*. In general the moduli space $\overline{\mathscr{M}}_{g,n}(X,d)$ may consist of several irreducible components of various dimensions. The virtual fundamental class is a homology class on $\overline{\mathscr{M}}_{g,n}(\mathcal{X},d)$ of pure dimension used to make the intersection theory better behaved.

It is often useful to organize the Gromov–Witten invariants of X in the form of a generating function. In this way recursive relations between the Gromov–Witten invariants can be expressed as differential equations satisfied by the generating function. Although in theory it is possible to compute a large class of Gromov–Witten invariants, expressing these generating functions in a nice form is often a difficult problem.

For the purposes of mirror symmetry, the most interesting Gromov–Witten generating function is Givental's J-function, $J^X(\mathbf{t}, z)$. Let $\{T_i\}$ be a basis for $H^*(X)$ and let $\{T^i\}$ denote the dual basis. Let $\mathbf{t}$ denote a point in $H^*(X)$. Then define

$$J^X(\mathbf{t}, z) := 1 + \frac{\mathbf{t}}{z} + \sum_d \sum_{n \geq 0} \sum_i \frac{q^d}{n!} \left\langle \frac{T_i}{z - \psi_1}, 1, \mathbf{t}, \ldots, \mathbf{t} \right\rangle^X_{0,2+n,d} T^i,$$

where $\frac{1}{z-\psi_1}$ represents the sum $1/z \sum_{k \geq 0} (\psi/z)^k$. The J-function is then a function from $H^*(X)$ to $H^*(X)[[1/z]]$. We define the *small J-function* by restricting our input $\mathbf{t}$ to lie in $H^2(X)$:

$$J^X_{small}(\mathbf{t}, z) := J^X(\mathbf{t}, z)|_{\mathbf{t} \in H^2(X)} : H^2(X) \to H^*(X)[[1/z]].$$

Motivation for this particular generating function will be given in section III. For now we remark only that the J-function is small enough that it can often be computed explicitly, and large enough that it allows one to recover a large amount of information about the Gromov–Witten theory of X. For instance, in many cases one can recover all genus-zero Gromov–Witten invariants of X from J^X

1.4 Orbifold Gromov–Witten theory

Gromov–Witten invariants are defined not only for smooth varieties, but also for smooth Deligne–Mumford stacks, henceforth referred to as orbifolds. This is relevant for the study of mirror symmetry, as often one or both of a mirror pair (V, V°) is an orbifold.

One important difference in this case is the use of Chen–Ruan cohomology in defining Gromov–Witten invariants. If $\mathcal{X}$ is an orbifold, the Chen–Ruan cohomology of $\mathcal{X}$ is isomorphic as a vector space to the cohomology of the *inertia orbifold*, $I\mathcal{X}$, of $\mathcal{X}$:

$$H^*_{CR}(\mathcal{X}) := H^*(I\mathcal{X}).$$

If $\mathcal{X} = [V/G]$ is a global quotient of a nonsingular variety V by a finite group G, $I\mathcal{X}$ takes a particularly simple form. Let S_G denote the set of conjugacy classes (g) in G, then

$$I[V/G] = \coprod_{(g)\in S_G} [V^g/C(g)],$$

where V^g is set of points in V fixed by g, and $C(g)$ is the centralizer of g. Note that $[V/G]$ can be identified with the connected component $[V^e/G]$ of $I[V/G]$ indexed by the identity e in G. This holds for a general orbifold $\mathcal{X}$. Under this identification, $\mathcal{X} \subset I\mathcal{X}$ is referred to as the *untwisted sector* of $I\mathcal{X}$, and we obtain an inclusion $H^*(\mathcal{X}) \subseteq H^*_{CR}(\mathcal{X})$.

We may define Gromov–Witten invariants $\langle \alpha_1\psi^{k_1}, \dots, \alpha_n\psi^{k_n}\rangle^{\mathcal{X}}_{g,n,d}$ as in the non-orbifold case, but due to the orbifold structure of $\mathcal{X}$, the natural target of the evaluation map ev_i is in fact $I\mathcal{X}$[1], rather than $\mathcal{X}$ (see Section 2.2.1 for details). Consequently, the classes $\alpha_1, \dots, \alpha_n$ are cohomology classes in $H^*_{CR}(\mathcal{X})$ (this is in fact one of the major motivations for defining Chen–Ruan cohomology). In analogy to the above, we may define a J-function

$$J^{\mathcal{X}} : H^*_{CR}(\mathcal{X}) \to H^*_{CR}(\mathcal{X})[[1/z]].$$

[1]Technically ev_i maps to the *rigidified inertia stack*, see [1] and [11] for details.

1.5 The B model

Mathematically the B model of a space X is defined in terms of the variation of Hodge structures on a family of complex deformations of X (see Definition V.2). Let X_t be a smooth family of deformations of X depending on a parameter $t \in S$. We can study the variation of Hodge structure of X_t via period integrals. Let ω_t be a local section of $R^3\pi_*\mathbb{C} \otimes \mathscr{O}_S$, i.e., for each t on which it is defined, $\omega_t \in H^3(X_t)$. Integrating this family over a basis of locally constant cycles $\gamma_i(t)$ in $H_3(X_t)$ defines the *period integrals* of ω_t:

$$\pi_i(t) = \int_{\gamma_i(t)} \omega_t.$$

Given a choice of ω_t, the corresponding period integrals satisfy a set of differential equations called the *Picard–Fuchs equations* of ω_t. These differential equations and their solutions can often be calculated explicitly via the Griffiths–Dwork method (section 5.2).

In the case of the mirror quintic, the space of deformations is one-dimensional and can be described explicitly. Consider the deformation of the Fermat polynomial $Q(x)$ given by

$$Q_\psi(x) = \sum_{i=0}^{4} x_i^5 - \psi x_0 x_1 x_2 x_3 x_4.$$

$Q_\psi(x)$ is invariant under the action of $\bar{G}$, so we may define a family of Deligne–Mumford stacks

$$\mathcal{W}_\psi = \{Q_\psi(x) = 0\} \subset [\mathbb{P}^4/\bar{G}].$$

At $\psi = 0$ we recover the mirror quintic $\mathcal{W}$. Let ω denote the family of holomorphic $(3,0)$-forms on $\mathcal{W}_\psi$,

$$\omega = \operatorname{Res}\left(\frac{\psi}{Q_\psi(x)}\Omega_0\right) \tag{1.1}$$

where $\Omega_0 = \sum_{i=0}^{4}(-1)^i x_i dx_0 \cdots \hat{dx_i} \cdots dx_4$. We can express the periods of ω in the form of a hypergeometric series (Section 6.2). Let

$$I^{\mathcal{W}_\psi}(t,z) = e^{tH/z}\left(\sum_{d\geq 0} e^{dt}\frac{\prod_{m=1}^{5d}(5H+mz)}{\prod_{b=1}^{d}(H+bz)^5}\right) \mod H^4,$$

then letting $t = -5\log(\psi)$ and expanding $I^{\mathcal{W}_\psi}$ in terms of H/z gives a basis of solutions for the Picard–Fuchs equation of ω.

1.6 A model of $M \equiv B$ model of $\mathcal{W}$

The classical mirror theorem relates the Gromov–Witten theory of M to period integrals over $\mathcal{W}_\psi$. In the formulation given by Givental [16], this takes the form of a correspondence between J^M_{small} and $I^{\mathcal{W}_\psi}$. Let $H \in H^2(M)$ denote the pullback of the hyperplane class from $\mathbb{P}^4$, and let t denote the dual coordinate to H. Under this identification, we can view $I^{\mathcal{W}_\psi}(t,z)$ as a function from $H^2(M)$ to $H^*(M)[[1/z]]$, exactly as in the case of $J^M_{small}(t,z)$:

$$J^M_{small}(t,z),\ I^{\mathcal{W}_\psi}(t,z) : H^2(M) \to H^*(M)[[1/z]].$$

The mirror theorem then says

Theorem I.2 (= Theorem VII.4)**.** *$J^M_{small}(t,z)$ is equal to $I^{\mathcal{W}_\psi}(t,z)$ after an explicit change of variables.*

The above change of variables is usually referred to as a *mirror transformation* or mirror map.

This theorem not only allows us to calculate $J^M_{small}(t,z)$ explicitly, but shows that the Gromov–Witten invariants of M have a complicated recursive structure, reflected in the fact that $I^{\mathcal{W}_\psi}$ satisfies the Picard–Fuchs differential equation for ω.

Remark I.3. Note that $\dim(H^3(\mathcal{W}_\psi)) = 4$, so it is natural to ask whether the periods of other families of three-forms over $\mathcal{W}_\psi$ can be related to the Gromov–Witten theory of M. In fact one can show that period integrals for any family of three-forms over $\mathcal{W}_\psi$ can be expressed as linear combinations of derivatives of periods of ω and can thus be expressed in terms of derivatives of J^M up to a mirror transformation. In this sense the above theorem implies a *full correspondence*, that is, a correspondence relating *all* periods of $\mathcal{W}_\psi$ to the Gromov–Witten theory of M. This is explained in detail in section 7.2, where the A model and B model are reinterpreted in terms of flat connections on certain vector bundles, and a "full correspondence" is understood to be an isomorphism of vector bundles which identifies the two connections. The following corollary of Theorem I.2 is the mirror theorem in its complete form.

Corollary I.4 (= Theorem VII.6)**.** *The fundamental solutions of the Gauss–Manin connection for $\mathcal{W}_\psi$ are equivalent, up to a mirror map, to the fundamental solutions of the Dubrovin connection for M, when restricted to $H^2(M)$.*

1.7 A model of $\mathcal{W} \equiv B$ model of M

In the present work, we relate the Gromov–Witten theory of the mirror quintic $\mathcal{W}$ to period integrals over a family of deformations of the quintic M. Immediately however we encounter a technical difficulty. The dimension of the space of complex deformations of M is $h^{2,1}(M) = 101$, thus our Picard–Fuchs equations would be PDEs in 101 variables, the calculation of which is unfeasible. There is a similar difficulty in the calculation of the small J-function of $\mathcal{W}$, a generating function in 101 variables. For this reason we will restrict our attention to a one-dimensional deformation family of M, and restrict the inputs of $J^{\mathcal{W}}_{small}$ to a one-dimensional subspace of $H^2_{CR}(\mathcal{W})$.

In the A model of $\mathcal{W}$, we restrict the small J-function of $\mathcal{W}$ to the one-dimensional subspace $H^2(\mathcal{W}) \subset H^2_{CR}(\mathcal{W})$ supported on the untwisted section of $I\mathcal{W}$. For the B model of M, we consider the family $M_\psi = \{Q_\psi(x) = 0\} \subset \mathbb{P}^4$, where $Q_\psi(x)$ is the same polynomial as was defined before, but now viewed as a homogeneous function on $\mathbb{P}^4$. Again we may consider periods of the family of three-forms $\omega = \mathrm{Res}\left(\frac{\psi}{Q_\psi(x)}\Omega_0\right)$ now viewed as a family on M_ψ. As before we may construct a function I^{M_ψ}, whose components give a basis of solutions to the periods of ω.

A first observation is an exact analog to Theorem I.2, namely, the functions $J^{\mathcal{W}}$ and I^{M_ψ} coincide up to a change of variables.

Theorem I.5. *$J^{\mathcal{W}}|_{H^2(\mathcal{W})}$ is equal to I^{M_ψ} after a mirror transformation.*

But as in Remark I.3, one would like to obtain a correspondence relating the periods of *any* family of three-forms over M_ψ to generating functions of Gromov–Witten invariants of $\mathcal{W}$, and here the situation is more complicated. In this case $\dim(H^3(M_\psi)) = 204$, and it is no longer true that period integrals for any family of three-forms over M_ψ can be expressed as linear combinations of derivatives of periods of ω.

To obtain a full correspondence, we define new generating functions of Gromov–Witten invariants of $\mathcal{W}$. These functions are analogous to Givental's J-function, but reflect the orbifold structure of $\mathcal{W}$. Let us first write $I\mathcal{W}$ in terms of its connected components,

$$I\mathcal{W} = \coprod_g \mathcal{W}_g.$$

For each component $\mathcal{W}_g$, let $\mathbb{1}_g \in H^*_{CR}(\mathcal{W})$ denote the fundamental class on $\mathcal{W}_g$. Define the generating function

$$J_g^{\mathcal{W}}(\mathbf{t}, z) := \mathbb{1}_g + \mathbb{1}_g \cdot \frac{\mathbf{t}}{z} + \sum_d \sum_{n \geq 0} \sum_i \frac{q^d}{n!} \left\langle \frac{T_i}{z - \psi_1}, \mathbb{1}_g, \mathbf{t}, \ldots, \mathbf{t} \right\rangle^X_{0,2+n,d} T^i.$$

For certain g, these can be related to the periods of other families of three-forms over M_ψ. Namely, there exists a set of families of three-forms $\{\omega_g\}$ indexed by certain components of $I\mathcal{W}$, and for each ω_g, there exists a function I_g^B whose components give the periods of ω_g. Our main theorem may be phrased as follows.

Theorem I.6 (= Corollary VII.3)**.** *For each ω_g in the above set, $J_g^{\mathcal{W}}|_{H^2(\mathcal{W})}$ is equal to I_g^B after applying the mirror transformation.*

Although the set $\{\omega_g\}$ does not generate all of $R^3\pi_*\mathbb{C} \otimes \mathscr{O}_S$, in analogy to Remark I.3, the period integrals for any section may be expressed as linear combinations of derivatives of periods of the ω_g. This implies our mirror theorem in its complete form.

Corollary I.7 (= Theorem VII.8)**.** *The fundamental solutions of the Gauss–Manin connection for $\{M_\psi\}$ are equivalent, up to a mirror transformation, to the fundamental solutions of the Dubrovin connection for $\mathcal{W}$, when restricting to $H^2(\mathcal{W})$.*

Remark I.8. The material in this thesis is the result of collaborative work with Y.-P. Lee, and appears also in the preprint [20].

CHAPTER II

Quantum Cohomology

In this section we give a brief review of Chen–Ruan cohomology and quantum orbifold cohomology, with the parallel goal of setting notation. A more detailed general review can be found in [11].

Conventions II.1. We work in the algebraic category. The term *orbifold* means "smooth separated Deligne–Mumford stack of finite type over $\mathbb{C}$."

The various dimensions are complex dimensions. On the other hand, the degrees of cohomology are all in real/topological degrees.

Unless otherwise stated all cohomology groups have coefficients in $\mathbb{C}$.

2.1 Chen–Ruan cohomology groups

Let $\mathcal{X}$ be a stack. Its inertia stack $I\mathcal{X}$ is the fiber product

$$\begin{tikzcd} I\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

where Δ is the diagonal map. The fiber product is taken in the 2-category of stacks. One can think of a point of $I\mathcal{X}$ as a pair (x, g) where x is a point of $\mathcal{X}$ and $g \in \mathrm{Aut}_{\mathcal{X}}(x)$. There is an involution $I : I\mathcal{X} \to I\mathcal{X}$ which sends the point (x, g) to

(x, g^{-1}). It is often convenient to call the components of $I\mathcal{X}$ for which $g \neq e$ the *twisted sectors*.

If $\mathcal{X} = [V/G]$ is a global quotient of a nonsingular variety V by a finite group G, $I\mathcal{X}$ takes a particularly simple form. Let S_G denote the set of conjugacy classes (g) in G, then

$$I[V/G] = \coprod_{(g) \in S_G} [V^g / C(g)].$$

The *Chen–Ruan orbifold cohomology groups* $H^*_{CR}(X)$ ([8]) of a Deligne–Mumford stack $\mathcal{X}$ are the cohomology groups of its inertia stack

$$H^*_{CR}(\mathcal{X}) := H^*(I\mathcal{X}).$$

Let (x, g) be a geometric point in a component $\mathcal{X}_i$ of $I\mathcal{X}$. By definition $g \in \mathrm{Aut}_{\mathcal{X}}(x)$. Let r be the order of g. Then the g-action on $T_x\mathcal{X}$ decomposes as eigenspaces

$$T_x\mathcal{X} = \bigoplus_{0 \leq j < r} E_j$$

where E_j is the subspace of $T_x\mathcal{X}$ on which g acts by multiplication by $\exp(2\pi\sqrt{-1}j/r)$. Define the age of $\mathcal{X}_i$ to be

$$\mathrm{age}(\mathcal{X}_i) := \sum_{j=0}^{r-1} \frac{j}{r} \dim(E_j).$$

This is independent of the choice of geometric point $(x, g) \in \mathcal{X}_i$.

Let α be an element in $H^p(\mathcal{X}_i) \subset H^*(I\mathcal{X})$. Define the age-shifted degree of α to be

$$\deg_{CR}(\alpha) := p + 2\,\mathrm{age}(\mathcal{X}_i).$$

This defines a grading on $H^*_{CR}(\mathcal{X})$.

When $\mathcal{X}$ is compact the *orbifold Poincaré pairing* is defined by

$$(\alpha_1, \alpha_2)^{\mathcal{X}}_{CR} := \int_{I\mathcal{X}} \alpha_1 \cup I^*(\alpha_2),$$

where α_1 and α_2 are elements of $H^*_{CR}(\mathcal{X})$. It is easy to see that when α_1 and α_2 are homogeneous elements, $(\alpha_1, \alpha_2)_{CR} \neq 0$ only if $\deg_{CR}(\alpha_1) + \deg_{CR}(\alpha_2) = 2\dim(\mathcal{X})$.

2.2 Orbifold Gromov–Witten theory

2.2.1 Gromov–Witten invariants

We follow the standard references [9] and [1] of orbifold Gromov–Witten theory.

Given an orbifold $\mathcal{X}$, there exists a moduli space $\overline{\mathscr{M}}_{g,n}(\mathcal{X}, d)$ of stable maps from n-marked genus g pre-stable orbifold curves to $\mathcal{X}$ of degree $d \in H_2(\mathcal{X}; \mathbb{Q})$, which we describe below. Each source curve $(\mathcal{C}, p_1, \dots, p_n)$ has non-trivial orbifold structure only at the nodes and marked points: At each (orbifold) marked point it is a cyclic quotient stack and at each node a *balanced* cyclic quotient. That is, étale locally isomorphic to

$$\left[\mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(xy)}\right) / \mu_r\right],$$

where $\zeta \in \mu_r$ acts as $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$. The maps are required to be representable at each node.

Each marked point p_i is étale locally isomorphic to $[\mathbb{C}/\mu_{r_i}]$. There is an induced homomorphism

$$\mu_{r_i} \to \mathrm{Aut}_{\mathcal{X}}(f(p_i)).$$

Maps in $\overline{\mathscr{M}}_{g,n}(\mathcal{X}, d)$ are required be representable, which amounts to saying that these homomorphisms be injective (see [2], Definition 2.44). For each marked point p_i, one can associate a point (x_i, g_i) in $I\mathcal{X}$ where $x_i = f(p_i)$, and $g_i \in \mathrm{Aut}_{\mathcal{X}}(x_i)$ is the image of $\exp(2\pi\sqrt{-1}/r_i)$ under the induced homomorphism.

Given a family $\mathcal{C} \to S$ of marked orbifold curves, there may be nontrivial gerbe structure above the locus defined by the i-th marked point. For this reason there is generally not a well defined map

$$ev_i : \overline{\mathscr{M}}_{g,n}(\mathcal{X}, d) \to I\mathcal{X}.$$

However, as explained in [1] and [11] Section 2.2.2, it is still possible to define maps

$$ev_i^* : H^*_{CR}(\mathcal{X}) \to H^*(\overline{\mathscr{M}}_{g,n}(\mathcal{X}, d))$$

which behave *as if the evaluation maps* ev_i *are well defined*.

Let X denote the coarse underlying space of the stack $\mathcal{X}$. There is a *reification map*

$$\overline{\mathscr{M}}_{g,n}(\mathcal{X}, d) \to \overline{\mathscr{M}}_{g,n}(X, d),$$

which forgets the orbifold structure of each map. For each marked point there is an associated line bundle, the i^{th} universal cotangent line bundle,

$$\begin{array}{c} L_i \\ \downarrow \\ \overline{\mathscr{M}}_{g,n}(X, d) \end{array}$$

with fiber $T^*_{p_i}C$ over $\{f : (C, p_1, \ldots, p_n) \to X\}$. Define the i-th ψ-class by $\psi_i := r^*(c_1(L_i))$.

As in the non-orbifold setting, there exists a virtual fundamental class $[\overline{\mathscr{M}}_{g,n}(\mathcal{X}, d)]^{vir}$. *Orbifold Gromov-Witten invariants* for $\mathcal{X}$ are defined as integrals

$$\langle \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \rangle^{\mathcal{X}}_{g,n,d} := \int_{[\overline{\mathscr{M}}_{g,n}(\mathcal{X},d)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i) \psi_i^{k_i},$$

where $\alpha_i \in H^*_{CR}(\mathcal{X})$.

Let $\overline{\mathscr{M}}_{g,(g_1,\dots,g_n)}(\mathcal{X},d)$ denote the open and closed substack of $\overline{\mathscr{M}}_{g,n}(\mathcal{X},d)$ such that ev_i maps to a component $\mathcal{X}_{g_i}$ of $I\mathcal{X}$. The space $\overline{\mathscr{M}}_{g,(g_1,\dots,g_n)}(\mathcal{X},d)$ has (complex) virtual dimension

$$n + (g-1)(\dim \mathcal{X} - 3) + \langle c_1(T\mathcal{X}), d\rangle - \sum_{i=0}^{n} \mathrm{age}(\mathcal{X}_{g_i}). \tag{2.1}$$

In other words, for homogeneous classes $\alpha_i \in H^*(\mathcal{X}_{g_i})$ the Gromov-Witten invariant $\langle \alpha_1, \dots, \alpha_n \rangle_{g,n,d}^{\mathcal{X}}$ will vanish unless

$$\sum_{i=1}^{n} \deg_{CR}(\alpha_i) = 2\left(n + (g-1)(\dim \mathcal{X} - 3) + \langle c_1(T\mathcal{X}), d\rangle\right).$$

2.2.2 Quantum cohomology and the Dubrovin connection

Let $\{T_i\}_{i\in I}$ be a basis for $H^*_{CR}(\mathcal{X})$ and $\{T^i\}_{i\in I}$ its dual basis. We can represent a general point in coordinates by

$$\mathbf{t} = \sum_i t^i T_i \in H^*_{CR}(\mathcal{X}).$$

Gromov-Witten invariants allow us to define a family of product structures parameterized by $\mathbf{t}$ in a formal neighborhood of 0 in $H^*_{CR}(\mathcal{X})$. The *(big) quantum product* $*_{\mathbf{t}}$ is defined as

$$\alpha_1 *_{\mathbf{t}} \alpha_2 := \sum_d \sum_{n\geq 0} \sum_i \frac{q^d}{n!} \langle \alpha_1, \alpha_2, T_i, \mathbf{t}, \dots, \mathbf{t} \rangle_{0,3+n,d}^{\mathcal{X}} T^i, \tag{2.2}$$

where the first sum is over the Mori cone M of effective curve classes and the variables q^d are in an appropriate Novikov ring Λ used to guarantee formal convergence of the sum (generally Λ is defined as a completion of the semigroup ring of effective curve classes in M). The *WDVV equations* ([12], Section 8.2.3) imply the associativity of the product. The *small quantum product* is defined by restricting the parameter of the quantum product to divisors $\mathbf{t} \in H^2(\mathcal{X})$ supported on the *non-twisted sector*.

One can interpret $*_{\mathbf{t}}$ as defining a product structure on the tangent bundle $TH^*_{CR}(\mathcal{X};\Lambda)$, such that for a fixed $\mathbf{t}$ the quantum product defines a (Frobenius) algebra structure on $T_{\mathbf{t}}H^*_{CR}(\mathcal{X};\Lambda)$. This can be rephrased in terms of the *Dubrovin connection*, defined by:

$$\nabla^z_{\frac{\partial}{\partial t^i}}\left(\sum_j a_j T_j\right) = \sum_j \frac{\partial a_j}{\partial t^i} T_j - \frac{1}{z}\sum_j a_j T_i *_{\mathbf{t}} T_j.$$

This defines a z-family of connections on $TH^*_{CR}(\mathcal{X};\Lambda)$.

Remark II.2. Note that when $\mathbf{t}$, T_i and T_j are in $H^{even}_{CR}(\mathcal{X})$, a simple dimension count using (2.1) shows that $T_i *_{\mathbf{t}} T_j$ will be also be supported in even degree. Thus ∇^z restricts to a connection on $TH^{even}_{CR}(\mathcal{X};\Lambda)$. When restricted to $TH^{even}_{CR}(\mathcal{X};\Lambda)$, the quantum product is commutative.

Definition II.3. For the purpose of this paper, we clarify here what we mean by "A model of $\mathcal{X}$". Let $H := H^{even}_{CR}(\mathcal{X};\Lambda)$. The (genus zero part of) the *A model of $\mathcal{X}$* is defined to be the tangent bundle TH together with its natural (flat) fiberwise pairing and the Dubrovin connection restricted to $H^{1,1}_{CR}(\mathcal{X})$.

The commutativity and associativity of the quantum product implies that the Dubrovin connection is flat. The *topological recursion relations* allow us to explicitly describe solutions to ∇^z. Define

$$s_i(\mathbf{t},z) = T_i + \sum_d \sum_{n\geq 0}\sum_j \frac{q^d}{n!}\left\langle \frac{T_i}{z-\psi_1}, T^j, \mathbf{t},\ldots,\mathbf{t}\right\rangle^{\mathcal{X}}_{0,2+n,d} T_j \tag{2.3}$$

where $1/(z-\psi_1)$ should be viewed as a power series in $1/z$. The sections s_i form a basis for the ∇^z-flat sections; see e.g. [12], Proposition 10.2.1. Thus we obtain a fundamental solution matrix $S = S(\mathbf{t},z) = (s_{ij})$ given by

$$s_{ij}(\mathbf{t},z) = (T^i, s_j)^{\mathcal{X}}_{CR}. \tag{2.4}$$

If one restricts the base to divisors $\mathbf{t} \in H^2(\mathcal{X})$, the *divisor equation* ([12] Section 10.1.2) allows a substantial simplification of the formula for s_i

$$s_i(\mathbf{t},z)|_{\mathbf{t}\in H^2(\mathcal{X})} = e^{\mathbf{t}/z}\left(T_i + \sum_{d>0}\sum_j q^d e^{d\cdot\mathbf{t}}\left\langle \frac{T_i}{z-\psi_1}, T^j\right\rangle_{0,2,d}^{\mathcal{X}} T_j\right).$$

2.3 Generating functions

Given an orbifold $\mathcal{X}$, Givental's (big) J-function is the first row vector of the fundamental solution matrix, obtained by pairing the solution vectors of the Dubrovin connection with 1.

$$\begin{aligned} J_{big}^{\mathcal{X}}(\mathbf{t},z) &:= \sum_i (s_i(\mathbf{t}),1)_{CR}^{\mathcal{X}}\, T^i \\ &= 1 + \sum_d \sum_{n\geq 0}\sum_i \frac{q^d}{n!}\left\langle \frac{T_i}{z-\psi_1}, 1, \mathbf{t}, \dots, \mathbf{t}\right\rangle_{0,2+n,d}^{\mathcal{X}} T^i \\ &= 1 + \frac{\mathbf{t}}{z} + \sum_d \sum_{n\geq 0}\sum_i \frac{q^d}{n!}\left\langle \frac{T_i}{z(z-\psi_1)}, \mathbf{t}, \dots, \mathbf{t}\right\rangle_{0,1+n,d}^{\mathcal{X}} T^i, \end{aligned}$$

The last equality follows from the *string equation* (see [12] where it is referred to as the Fundamental Class Axiom). It is also easy to see that the fundamental solution matrix $S(\mathbf{t},z)$ of (2.4) is equal to $z\nabla J_{big}$. As such, J_{big} encodes all information about quantum cohomology.

However, the big J-function is often impossible to calculate directly. In the non-orbifold Gromov–Witten theory, when the cohomology is generated by divisors, the *small J-function* proves much more computable, while powerful enough to solve many problems; see e.g. [16, 17]. The small J-function for a nonsingular *variety* X is a function on $\mathbf{t} \in H^2(X)$:

$$\begin{aligned} J_{small}^{X}(\mathbf{t},z) &:= J_{big}^{X}(\mathbf{t},z)|_{\mathbf{t}\in H^2(X)} \\ &= e^{\mathbf{t}/z}\left(1 + \sum_{d>0}\sum_i q^d e^{d\cdot\mathbf{t}}\left\langle \frac{T_i}{z-\psi_1}, 1\right\rangle_{0,2,d}^{X} T^i\right). \end{aligned}$$

In orbifold theory, however, the Chen–Ruan cohomology is never generated by divisors except for trivial cases, due to the presence of the twisted sectors. Therefore, the knowledge of the small J-function alone is often not enough to reconstruct significant information about the orbifold quantum cohomology. (Note however that in Section 5 of [11], one way was found to circumvent this obstacle for weighted projective spaces.)

We propose the following definition of *small J-matrix for orbifolds*.

Definition II.4. For $\mathbf{t} \in H^2(\mathcal{X})$, define $J_g^{\mathcal{X}}$ as the cohomology-valued function

$$
\begin{aligned}
J_g^{\mathcal{X}}(\mathbf{t},z)|_{\mathbf{t}\in H^2(\mathcal{X})} &:= \sum_i \left(s_i(\mathbf{t})|_{\mathbf{t}\in H^2(X)}, \mathbb{1}_g\right)_{CR}^{\mathcal{X}} T^i \\
&= e^{\mathbf{t}/z}\left(\mathbb{1}_g + \sum_{d>0}\sum_i q^d e^{d\cdot \mathbf{t}} \left\langle \frac{T_i}{z-\psi_1}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{X}} T^i \right),
\end{aligned} \tag{2.5}
$$

where $\mathbb{1}_g$ is the fundamental class on the component $\mathcal{X}_g$ of $I\mathcal{X}$.

The *small J-matrix* is the matrix-valued function

$$
J_{small}^{\mathcal{X}}(\mathbf{t},z) = \left[J_{g,i}^{\mathcal{X}}(\mathbf{t},z)\right]_{g\in G, i\in I} = \left[(J_g^{\mathcal{X}}(\mathbf{t},z), T_i)_{CR}^{\mathcal{X}}\right]_{g\in G, i\in I},
$$

where G is the index set of the components of $I\mathcal{X}$, I the index for the basis $\{T_i\}_{i\in I}$ of $H_{CR}^*(\mathcal{X})$ and $J_{g,i}^{\mathcal{X}}(\mathbf{t},z)$ the coefficient of T^i in $J_g^{\mathcal{X}}(\mathbf{t},z)$.

Remark II.5. We believe that the small J-matrix is the right replacement of the small J-function in the orbifold theory, for its computability and structural relevance.

Structurally equation (2.4) shows that one needs to specify "two-points" (i.e. a matrix) in the generating function in order to form the fundamental solutions of the Dubrovin connection. Ideally, one would like to get the full $|I| \times |I|$ fundamental solution matrix $S = z\nabla J_{big}$ restricted to $\mathbf{t} \in H^2(\mathcal{X})$. This would give all information about the *small* quantum cohomology. Unfortunately, a direct computation of $S(\mathbf{t})|_{\mathbf{t}\in H^2(\mathcal{X})}$ is mostly out of reach in the orbifold theory.

In the (non-orbifold) case when $H^*(X)$ is generated by divisors, as shown by Givental, the small J-function is often enough to determine the essential information for small quantum cohomology. One can think of the small J-function as a a submatrix of size $1 \times |I|$, indeed the first row vector, of S.

However, in the orbifold theory, the above matrix is not enough to determine useful information about small quantum cohomology except in the trivial cases. We believe that the smallest useful submatrix of S is the small J-matrix (of size $|G| \times |I|$) defined above. We will show that it is both computable and relevant to the structure of orbifold quantum cohomology. In this paper we are able to calculate the small J-matrix of the toric orbifold $\mathcal{Y} = [\mathbb{P}^4/\bar{G}]$, and we use a submatrix of the small J-matrix $J^{\mathcal{W}}_{small}$ to fully describe the solution matrix $S(\mathbf{t})|_{\mathbf{t} \in H^2(\mathcal{X})}$ of the mirror quintic $\mathcal{W}$.

CHAPTER III

J-function of $[\mathbb{P}^4/\bar{G}]$

3.1 Inertia orbifold of $[\mathbb{P}^4/\bar{G}]$

Let $[x_0, x_1, x_2, x_3, x_4]$ be the homogeneous coordinates of $\mathbb{P}^4$. Denote

$$\zeta := \zeta_5 = e^{2\pi\sqrt{-1}/5}.$$

Let the group $\bar{G} \cong (\mathbb{Z}/5\mathbb{Z})^3$ be a (finite abelian) subgroup of the big torus of $\mathbb{P}^4$ acting via generators e_1, e_2, e_3:

$$\begin{aligned} e_1[x_0, x_1, x_2, x_3, x_4] &= [\zeta x_0, x_1, x_2, x_3, \zeta^{-1} x_4] \\ e_2[x_0, x_1, x_2, x_3, x_4] &= [x_0, \zeta x_1, x_2, x_3, \zeta^{-1} x_4] \\ e_3[x_0, x_1, x_2, x_3, x_4] &= [x_0, x_1, \zeta x_2, x_3, \zeta^{-1} x_4]. \end{aligned} \tag{3.1}$$

Let $\mathcal{Y} = [\mathbb{P}^4/\bar{G}]$. As explained in the introduction, the mirror quintic is defined as a hypersurface inside $\mathcal{Y}$. It is therefore not surprising that this orbifold plays an instrumental role in the calculations that follow. We give here a detailed presentation of its corresponding inertia orbifold.

The group $\bar{G}$ can be described alternatively as follows. Let

$$G := \{(\zeta^{r_0}, \ldots, \zeta^{r_4}) \mid \sum_{i=0}^{4} r_i \equiv 0 \pmod 5\}$$

and

$$\bar{G} \cong G/\langle(\zeta, \ldots, \zeta)\rangle.$$

The $\bar{G}$-action on $\mathbb{P}^4$ comes from coordinate-wise multiplication. By a slight abuse of notation, we will represent a group element $g \in G$ by the power of ζ in each coordinate:

$$G = \{(r_0, \ldots, r_4) \mid \sum_{i=0}^{4} r_i \equiv 0 \pmod 5, 0 \leq r_i \leq 4\, \forall i\}.$$

For an element $g \in G$, denote $[g]$ the corresponding element in $\bar{G}$.

Fix an element $\bar{g} \in \bar{G}$. Let $g = (r_0, \ldots, r_4) \in G$ be such that $[g] = \bar{g}$. Define

$$I(g) := \left\{ j \in \{0,1,2,3,4\} \mid r_j = 0 \right\},$$

then

$$\mathbb{P}^4_g := \left\{ x_j = 0 \right\}_{j \notin I(g)} \subset \mathbb{P}^4$$

is a component of $(\mathbb{P}^4)^{\bar{g}}$. From this we see that each element $g \in G$ such that $[g] = \bar{g}$ corresponds to a connected component $\mathcal{Y}_g := [\mathbb{P}^4_g/\bar{G}]$ of $I\mathcal{Y}$. Note that if g has no coordinates equal to zero then $\mathbb{P}^4_g$ is empty, and so is $\mathcal{Y}_g$. This gives us a convenient way of indexing components of $I\mathcal{Y}$ and of describing its cohomology. We will let H denote the class in $H^*([\mathbb{P}^4/\bar{G}])$ which pulls back to the hyperplane class in $H^*(\mathbb{P}^4)$.

We summarize the above discussions in the following lemma.

Lemma III.1.

$$I\mathcal{Y} = \coprod_{g \in S} \mathcal{Y}_g,$$

where

$$\mathcal{Y}_g = \{(x, [g]) \in I\mathcal{Y} \mid x \in [\mathbb{P}^4_g/\bar{G}]\}$$

is a connected component and S denotes the set of all $g = (r_0, \ldots, r_4)$ such that at least one coordinate r_i is equal to 0.

Consequently, a convenient basis $\{T_i\}$ *for* $H^*_{CR}(\mathcal{Y})$ *is*

$$\bigcup_{g\in S}\{\mathbb{1}_g, \mathbb{1}_g\tilde{H}, \ldots, \mathbb{1}_g H^{\dim(\mathcal{Y}_g)}\}.$$

3.2 J-functions

Recalling a basic fact about global quotient orbifolds, a map of orbifolds $f : \mathcal{C} \to [\mathbb{P}^4/\bar{G}]$ can be identified with a principal $\bar{G}$-bundle C, and a $\bar{G}$-equivariant map $\tilde{f} : C \to \mathbb{P}^4$ such that the following diagram commutes: [1]

(3.2)
$$\begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d, "\pi_C"'] & \mathbb{P}^4 \arrow[d, "\pi_{\mathbb{P}^4}"'] \\ \mathcal{C} \arrow[r, "f"] & {[\mathbb{P}^4/\bar{G}].} \end{tikzcd}$$

Lemma III.2. *(i) The map* f *is representable if and only if* C *is a nodal curve with each irreducible component a smooth variety.*

(ii) There do not exist representable orbifold morphisms $f : \mathcal{C} \to \mathcal{Y}$ *from a genus* 0 *orbifold curve* $\mathcal{C}$ *with only one orbifold marked point.*

Proof. (i) follows from the definition of representability (Theorem 2.45 of [2]).

(ii) follows from (i): In the case $\mathcal{C}$ is irreducible, this is because there do not exist smooth covers of genus 0 orbifold curves with only one point with nontrivial isotropy. An induction argument then shows that the same is true of reducible curves with only one orbifold marked point (we assume always that our nodes be balanced). □

A line bundle on $[\mathbb{P}^4/\bar{G}]$ can be identified with a $\bar{G}$-equivariant line bundle on $\mathbb{P}^4$. Therefore, the Picard group on $[\mathbb{P}^4/\bar{G}]$ is a $\bar{G}$-extension of $\mathbb{Z}$. Let H be the hyperplane class on $\mathbb{P}^4$. Let L be any fixed choice of line bundle on $\mathcal{Y}$ such

[1]Technically f is identified with an equivalence class of such objects ([2], Corollary .246).

that $\pi_{\mathbb{P}^4}^*(L) = H$. Even though there are as many as $|\bar{G}|$ choices of L, they are topologically equivalent and will serve the same purpose in our discussion. By (3.2) and the projection formula, we have the following equality

$$\int_{\mathcal{C}} f^*(L) = \frac{1}{125}\int_C \tilde{f}^*(H).$$

We define the degree of a map $f : \mathcal{C} \to \mathcal{Y}$ by

$$d := \frac{1}{125}\int_C \tilde{f}^*(H).$$

Conventions III.3. By an abuse of notation, we will denote by H any fixed choice of L on $\mathcal{Y}$ such that $\pi_{\mathbb{P}^4}^*(L) = H$.

Given $h = (r_0(h), \ldots, r_4(h))$ and $g = (r_0(g), \ldots, r_4(g))$ in G, this also allows us to determine necessary conditions on the triple (d, h, g) such that

$$\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y}, d) := \overline{\mathscr{M}}_{0,2}(\mathcal{Y}, d) \cap ev_1^{-1}(\mathbb{1}_h) \cap ev_2^{-1}(\mathbb{1}_g)$$

to be nonempty.

Proposition III.4. *The space $\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y}, d)$ is nonempty only if*

(i) $[h] = [g]^{-1}$ *in $\bar{G}$;*

(ii) $r_i(h) + r_i(g) \equiv 5d \pmod 5$ *or equivalently* $\langle d \rangle = \langle (r_i(h) + r_i(g))/5 \rangle$ *for* $0 \le i \le 4$.

Proof. We will first consider the case where the source curve is irreducible. Assume that there exists a map $\{f : \mathcal{C} \to \mathcal{Y}\}$ in $\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y}, d)$ such that $\mathcal{C}$ is non-nodal. Consider the principal $\bar{G}$-bundle $\pi_C : C \to \mathcal{C}$ from (3.2). After choosing a generic base point $x \in \mathcal{C}$ and a point $\tilde{x}$ in $\pi_C^{-1}(x)$, we obtain a homomorphism $\phi : \pi_1(\mathcal{C}, x) \to \bar{G}$. We can specify generators ρ_1, and ρ_2 of $\pi_1(\mathcal{C}, x)$ such that ρ_i

is the class of loops wrapping once around p_i in the counterclockwise direction. Then $\phi(\rho_1) = [h]$ and $\phi(\rho_2) = [g]$. Because $\rho_1 \cdot \rho_2 = 1$ in $\pi_1(\mathcal{C}, x)$, it must be the case that $[h] \cdot [g] = 1$ in $\bar{G}$. This proves (i) for $\mathcal{C}$ non-nodal.

Next we will show (ii) in the case where $\mathcal{C}$ is non-nodal. To see this, note that the only smooth connected cover of $\mathcal{C}$ is isomorphic to $\mathbb{P}^1$. This cover is degree $r := |[h]|$, so C must consist of $|\bar{G}|/r$ components, each isomorphic to $\mathbb{P}^1$. In the case $h = (0,0,0,0,0)$, this implies that C has 125 components, and so d is an integer. Thus Condition (ii) holds trivially.

If $h \neq (0,0,0,0,0)$, then $r = 5$. First note that (i) implies that $r_i(h) + r_i(g) \pmod 5$ is the same for any i. Thus, we only need to prove the statement for one i. Let $C' \cong \mathbb{P}^1$ be one component of C and let

$$f' := \tilde{f}\big|_{C'} : C' \to \mathbb{P}^4$$

be the $\langle [h] \rangle$-equivariant morphism induced from the $\bar{G}$-equivariant morphism $\tilde{f} : C \to \mathbb{P}^4$. $(f')^*(\mathscr{O}(1))$ is a degree $5d$ line bundle on $C' = \mathbb{P}^1$. Therefore, any lifting of the torus action on $\mathbb{P}^1$ will have *weights* $(w, w+5d)$ at the fibers of the 2 fixed points. Call these two fixed points p'_1 and p'_2. Since $\langle [h] \rangle$ is a subgroup of the torus, the *characters* of the $[h]$-action at the fibers of the 2 fixed points must be (ζ^w, ζ^{w+5d}), for some w in $\{0, \ldots, 4\}$.

Let $q_1 := f'(p'_1)$ and $q_2 := f'(p'_2)$. By assumption, $q_1 \in \mathbb{P}^4_h$, $q_2 \in \mathbb{P}^4_g$. Choose an $i \in I(h)$ and $j \in I(g)$ such that $i \neq j$, $x_i(q_1) \neq 0$ and $x_j(q_2) \neq 0$. The action of $[h]$ on the fiber over q_1 and q_2 can be chosen to be $(\zeta^{r_i(h)}, \zeta^{-r_j(h)})$. By the above weight/character arguments,

$$r_i(h) - (-r_j(h)) \equiv 5d \pmod 5.$$

Since $j \in I(g)$ and $i \in I(h)$,

$$r_j(h) = r_j(h) - r_i(h) = r_i(g) - r_j(g) = r_i(g),$$

so we can rewrite the above as $r_i(h) + r_i(g) \equiv 5d \pmod 5$.

The nodal case follows similarly. Consider a nodal curve $f : \mathcal{C} \to \mathcal{Y}$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be the irreducible components connecting p_1 to p_2. It follows from Lemma III.2, each of these components will have 2 orbifold points (at either nodes or marked points) and these will be the only points in $\mathcal{C}$ with nontrivial orbifold structure. The above calculation for irreducible components plus the condition that all nodes be balanced in this situation then implies the claim. □

Once condition (i) is satisfied, the degree of maps allowed is thus determined by the quantity

$$d(h,g) := \langle (r_i(h) + r_i(g))/5 \rangle.$$

Note that this number remains constant as i varies.

We will define generating functions related to the J-functions $J_g^{\mathcal{Y}}$ which isolate the 2-point invariants of $\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y}, d)$. Let

$$S(d,h) := \{(b,k) \mid 0 < b \leq d, \ \ 0 \leq k \leq 4, \ \ \langle b \rangle = r_k(h)/5\},$$

and let

$$c(d,h) := \big|S(d,h)\big|.$$

Given $h, g \in G$ such that $[h] = [g]^{-1}$, define

$$Z_{h,g} := \sum_d Q^{c(d,h)} \sum_i \left\langle \frac{T_i^h}{z - \psi_1}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{Y}} T_h^i,$$

where $\{T_i^h\}$ is a basis for $H^*(\mathcal{Y}_h)$, and $\{T_h^i\}$ is the dual basis under the Chen-Ruan orbifold pairing. (The motivation behind this choice of exponent for Q will become clear in what follows: it is chosen to simplify the recursion satisfied by our generating function). Notice that by the above lemma, the only degrees which contribute to $Z_{h,g}$ are d such that $\langle d\rangle = d(h,g)$. Finally, let

$$Z_g := \mathbb{1}_g + \sum_{\{h|\,[h]=[g]^{-1}\}} Z_{h,g}.$$

Let $T = \mathbb{C}^*$ act on $\mathbb{C}^5$ with (generic) weights $-\lambda_0, \dots, -\lambda_4$. This induces an action on $\mathbb{P}^4$ and $\mathcal{Y}$. Furthermore there is an induced T-action on the inertia orbifold $I\mathcal{Y}$ and on $\overline{\mathscr{M}}_{0,2}(\mathcal{Y}, d)$. We will consider an equivariant analogue Z_g^T of Z_g defined by replacing the coefficients of Z_g with their equivariant counterparts:

$$Z_{h,g}^T := \sum_{d,i} Q^{c(d,h)} \left\langle \frac{T_i^h}{z-\psi_1}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{Y},T} T_h^i, \quad Z_g^T := \mathbb{1}_g + \sum_{\{h|\,[h]=[g]^{-1}\}} Z_{h,g}^T.$$

where $\{T_i^h\}$ is now a basis of the equivariant cohomology $H_T^*(\mathcal{Y}_h)$ and $\langle -, - \rangle_{0,2,d}^{\mathcal{Y},T}$ denotes the corresponding integral on $\overline{\mathscr{M}}_{0,2}(\mathcal{Y}, d)^T$.

Consider the cohomology valued functions

$$Y_{h,g}^T := \sum_{\{d|\langle d\rangle = d(h,g)\}} Q^{c(d,h)} \frac{\mathbb{1}_{h^{-1}}}{\prod\limits_{(b,k)\in S(d,h)} (bz + H - \lambda_k)}, \tag{3.3}$$

where

$$h^{-1} := (-r_0(h), \dots, -r_4(h)) \,(\mathrm{mod}\, 5).$$

As with Z, let

$$Y_g^T := \mathbb{1}_g + \sum_{\{h|\,[h]=[g]^{-1}\}} Y_{h,g}^T. \tag{3.4}$$

Theorem III.5. *We have the equality in equivariant cohomology:*

$$Z_g^T = Y_g^T.$$

In particular, taking the nonequivariant limit, we conclude that $Z_g = Y_g$, (where Y_g is the non-equivariant limit $\lambda_i \mapsto 0$ of Y_g^T.)

Remark III.6. For those who are familiar with the computation of the small J-function for toric manifolds [17], the generating functions Z, as indicated above, play the role of the J-function. The hypergeometric-type functions Y then take the place of the I-function. Recall that one way of formulating the computation of genus zero GW invariants is to say that the J-function is equal to the I-function after a change of variables, called the *mirror map*. In the present case, the mirror map is trivial.

3.3 Proof of Theorem III.5

The proof follows from a localization argument similar in spirit to that in [17]. The strategy is to apply the Localization Theorem (after inverting the equivariant characters $\lambda_0, \ldots, \lambda_4$ in the ring $H^*_{CR,T}(\mathcal{Y})$) on the equivariant generating functions to determine a recursion satisfied by Z_g^T. This recursion relation in fact determines Z_g^T up to the constant term in the Novikov variables. We then show that Y_g^T satisfies the same recursion. Since Z_g^T and Y_g^T have the same initial term and the same recursion relation, $Z_g^T = Y_g^T$.

3.3.1 a lemma on $c(d,h)$

We will first explain the seemingly strange appearance of the exponents $c(d,h)$ in the definition of $Z_{h,g}$.

Lemma III.7. *Let*

$$m_d = \dim(\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y}, d)),$$

then if $[h] = [g]^{-1}$ *and* $\langle d \rangle = d(h,g)$, *we have*

$$c(d,h) = m_d - \dim(\mathcal{Y}_h) + 1.$$

Proof. The standard formula for virtual dimension gives

$$m_d = 5d + 3 - \mathrm{age}(h) - \mathrm{age}(g).$$

Note that for any presentation $g = (r_0(g), \ldots, r_4(g))$, $\mathrm{age}(g) = \sum_{i=0}^{4} r_i(g)/5$. Because $[h] = [g]^{-1}$, we have that

$$r_i(g) - r_j(g) \equiv r_j(h) - r_i(h) \pmod 5.$$

This allows us to write

$$\frac{r_k(g)}{5} = \begin{cases} -r_k(h)/5 + d(h,g) & d(h,g) \geq r_k(h)/5 \\ 1 - r_k(h)/5 + d(h,g) & d(h,g) < r_k(h)/5 \end{cases},$$

which gives

$$\begin{aligned} m_d &= 5d + 3 - 5d(h,g) - |\{k \,|d(h,g) < r_k(h)/5\}| \\ &= 5\lfloor d \rfloor + |\{k \,|d(h,g) \geq r_k(h)/5\}| - 2. \end{aligned}$$

Now, for a fixed k,

$$|\{b \,|0 \leq b \leq d, \;\; \langle b \rangle = r_k(h)/5\}| = \left\{ \begin{array}{cc} \lfloor d \rfloor & d(h,g) < r_k(h)/5 \\ 1 + \lfloor d \rfloor & d(h,g) \geq r_k(h)/5 \end{array} \right\}.$$

Summing over all k, we get that

$$m_d = |\{(b,k) \,|0 \leq b \leq d, \quad 0 \leq k \leq 4, \quad \langle b \rangle = r_k(h)/5\}| - 2.$$

Finally,

$$\dim(\mathcal{Y}_g) = |\{k \,|\, 0 = r_k(h)/5\}| - 1,$$

which gives the desired equality. $\square$

3.3.2 Setting up the localization

The action of T on $\overline{\mathscr{M}}_{0,h,g}(\mathcal{Y},d))$ allows us to reduce integrals on the moduli space to sums of integrals on the fixed point loci with respect to the torus action. As usual, this reduces us to considering integrals of certain graph sums (here the graph is the dual graph to a generic source curve in the fixed locus, together with decorations describing where marked points and contracted components are mapped, see [22] for more details). The generating function Z_g^T consists of integrals where the first insertion is the pull back of a class on

$$\coprod_{\{h|[h]=[g]^{-1}\}} \mathcal{Y}_h.$$

We will now express Z_g in terms of a new basis for this space which interacts nicely with the localization procedure. For each coordinate $0 \leq i \leq 4$, i is in $I(h)$ for exactly one h in $\{h|[h] = [g]^{-1}\}$. (For $h, h' \in \{h|[h] = [g]^{-1}\}$, $r_i(h) = r_i(h')$ if and only if $h = h'$). Then for $i \in I(h)$, let q_i be the T-fixed point of $\mathcal{Y}_h$ obtained by setting all coordinates $\{j|j \neq i\}$ equal to zero. Then, for $i \in I(h)$, let

$$\phi_i = \mathbb{1}_h \cdot \prod_{j \in I(h)-i} \left(H - \lambda_j\right).$$

If we pair Z_g^T with ϕ_i, we obtain the function

$$Z_{i,g}^T = \frac{\delta^{i,I(g)}}{125} + \sum_d Q^{c(d,h)} \left\langle \frac{\phi_i}{z-\psi_1}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{Y},T},$$

where $\delta^{i,I(g)}$ equals 1 if $i \in I(g)$ and 0 otherwise. The fixed point set of $\mathcal{Y}_h$ consists of $\{q_j|j \in I(h)\}$. Note that under the inclusion $i_j : \{q_j\} \to \mathcal{Y}_h$, H pulls back to λ_j. Therefore $i_j^*(\phi_i) = 0$ unless $i = j$. From this we see that the coefficients of $Z_{i,g}^T$ consist of integrals over graphs such that the first marked point is mapped to q_i.

We divide the remaining graphs into *two types*: the first type of graph contains maps $(f : \mathcal{C} \to \mathcal{Y})$ such that the first marked point is on an irreducible component which is contracted under f, the second type contains maps in which the first marked point is on a noncontracted component.

Claim III.8. *There is no contribution from graphs of the first type.*

Proof. The proof is a dimension count. We will show that the contributions from graphs of the first type must contain as a multiplicative factor integrals of the form $\int_M \Psi$ such that $\deg_{\mathbb{C}}(\Psi) > \dim(M)$, and hence the vanishing claim.

The complex degree of ϕ_i is $\dim(\mathcal{Y}_h)$, so the invariant $\langle \phi_i \psi_1^k, \mathbb{1}_g \rangle_{0,2,d}^{\mathcal{Y},T}$ vanishes unless $k \geq m_d - dim(\mathcal{Y}_h)$. Thus we can simplify our expression for $Z_{i,g}^T$:

$$\begin{aligned}
Z_{i,g}^T &= \frac{\delta^{i,I(g)}}{125} + \sum_d Q^{c(d,h)} \left\langle \frac{\phi_i}{z - \psi_1}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{Y},T} \\
&= \frac{\delta^{i,I(g)}}{125} + \sum_d Q^{c(d,h)} \frac{1}{z} \sum_{k=0}^{\infty} \langle \phi_i (\psi_1/z)^k, \mathbb{1}_g \rangle_{0,2,d}^{\mathcal{Y},T} \\
&= \frac{\delta^{i,I(g)}}{125} + \sum_d Q^{c(d,h)} \frac{1}{z} \sum_{k=c(d,h)-1}^{\infty} \langle \phi_i (\psi_1/z)^k, \mathbb{1}_g \rangle_{0,2,d}^{\mathcal{Y},T} \\
&= \frac{\delta^{i,I(g)}}{125} + \sum_d \left(\frac{Q}{z}\right)^{c(d,h)} \left\langle \frac{\phi_i \psi_1^{c(d,h)-1}}{1 - (\psi_1/z)}, \mathbb{1}_g \right\rangle_{0,2,d}^{\mathcal{Y},T}.
\end{aligned}$$

Here the third equality follows from Lemma III.7.

Now consider a fixed point graph M_Γ such that p_1 is on a contracted component. At the level of virtual classes, we can write

$$[M_\Gamma] = F(\Gamma) \cdot \prod_k [M_{v_k}], \tag{3.5}$$

where each M_{v_k} represents a contracted component of the graph isomorphic to a component of $\overline{M}_{0,n}(B\mathbb{Z}_r, 0)$, and $F(\Gamma)$ is a factor determined by Γ. Let M_{v_0} be the component containing p_1. M_{v_0} contains at most 2 orbifold marked points, and the number of non-orbifold marked points is restricted by d. In particular, each non-orbifold marked point corresponds to a (non-orbifold) edge of the dual graph. Each of these edges must have degree at least 1, so if the total degree of the map is d, then there can be at most $\lfloor d \rfloor$ nontwisted marked points. Thus the dimension of M_{v_0} is at most $\lfloor d \rfloor - 1$. Now, the proof of Lemma III.7 shows that

$$c(d,h) - 1 = 5\lfloor d \rfloor + |\{k \mid r_k(h)/5 \leq d(h,g)\}| - 2 - \dim(\mathcal{Y}_h).$$

But $\dim(\mathcal{Y}_h)$ is exactly $|\{k \mid r_k(h) = 0\}| - 1$, which implies that

$$c(d,h) - 1 \geq 5\lfloor d \rfloor - 1.$$

If $d \geq 1$, the above quantity is strictly greater than $\lfloor d \rfloor - 1$. Because there do not exist graphs such that p_1 is on a non-contracted component for $d < 1$, we have that for M_Γ, $c(d,h) - 1 \gneq \dim(M_{v_0})$. But $\psi_1^{c(d,I)-1}$ must therefore vanish on these graphs, proving the claim. $\square$

3.3.3 Contributions from a graph of the second type

Now let us consider the contribution to $\langle \frac{\phi_i}{z-\psi_1}, \mathbb{1}_g \rangle_{0,2,d}^{\mathcal{Y},T}$ from a particular graph Γ of the second type. In particular, we know that p_1 is on a noncontracted component. Call this component $\mathcal{C}_0$, and denote the rest of the graph Γ'. Γ' and $\mathcal{C}_0$ connect at a node p', which maps to some $q_k \in \mathcal{Y}$. Let d' be the degree of one connected component of the principal $\bar{G}$-bundle above $\mathcal{C}_0$. We know from Proposition III.4 that $\langle d' \rangle = r_k(h)/5$. By identifying $p' \in \Gamma'$ as a marked point (replacing p_1 on $\mathcal{C}_0$), we can view $M_{\Gamma'}$ as a fixed point locus in $\overline{\mathscr{M}}_{0,h',g}(\mathcal{Y}, d - d')$, where $[h] = [h']$, but $r_k(h') = 0$. Our plan will be to express integrals on M_Γ in terms of integrals on $M_{\Gamma'}$, thus reducing the calculation to one involving maps of strictly smaller degree. This will give us a recursion.

The factor $F(\Gamma)$ in Equation 3.5 is composed of three contributions: the automorphisms of the graph Γ itself, a contribution from each edge of Γ (the noncontracted components of curves in M_Γ), and a contribution from certain flags of

Γ (the nodes of curves in M_Γ). The edge corresponding to $\mathcal{C}_0$ maps to the line $q_{ik} \cong \mathbb{P}^1/\bar{G}$ connecting q_i and q_k. (Note that the $\bar{G}$-action is a subgroup of the big torus $(\mathbb{C}^*)^4$ of $\mathbb{P}^4$, $\bar{G}$ naturally acts on $(\mathbb{C}^*)^4$ orbits.) The degree of the map upstairs is $5d'$. Thus there is a contribution of $1/(5d')$ to $F(\Gamma)$ from the automorphism of M_Γ coming from rotating the underlying curve. The edge also contributes a factor of $1/25$ due to the fact that q_{ik} is a $(\mathbb{Z}/5\mathbb{Z})^2$-gerbe. So the total contribution to $F(\Gamma)$ from the edge containing p_1 is $1/(125d')$. The contribution from the node p' is $125/r$. (Recall $r = |[h]|$, which is equal to the order of the isotropy at p'). There will be an additional factor of r appearing when we examine deformations of M_Γ, thus canceling the r in the denominator. We finally arrive at the relation

$$[M_\Gamma] = F(\Gamma) \cdot \prod_{\text{vertices } v \in \Gamma} [M_v] \;=\; \frac{F(\Gamma')}{d'} \cdot \prod_{\text{vertices } v \in \Gamma'} [M_v] \;=\; \frac{1}{d'}\,[M_{\Gamma'}]\,.$$

By examining the localization exact sequence (see [22]), we have the following identity:

$$e(N_\Gamma) = \frac{e(H^0(\mathcal{C}_0, f^*T\mathcal{Y})^m)(\text{node smoothing at } p')}{e(H^0(p', f^*T\mathcal{Y})^m)e(H^1(\mathcal{C}_0, f^*T\mathcal{Y})^m)e((H^0(\mathcal{C}_0, T\mathcal{C}_0)^m)} e(N_{\Gamma'}) \tag{3.6}$$

where e denotes the equivariant Euler class, and as is standard we identify certain vector bundles with their fibers. Here the superscript m denotes the moving part of the vector bundle with respect to the torus action. Let us calculate the factors in (3.6).

- (node smoothing at p'): The node smoothing contributes a factor of

$$\left(\frac{\lambda_k - \lambda_i}{rd'} - \frac{\psi_1'}{r}\right) = \frac{1}{r}\left(\frac{\lambda_k - \lambda_i}{d'} - \psi_1'\right),$$

where ψ_1' is the ψ-class corresponding to p_1' on M_Γ'. This factor of r is what cancels with the previous factor mentioned above.

• $e(H^0(\mathcal{C}_0, T\mathcal{C}_0)^m)$: Let C be the principal $\bar{G}$-bundle over $\mathcal{C}_0$ induced from $f|_{\mathcal{C}_0}$: $\mathcal{C}_0 \to [\mathbb{P}^4/\bar{G}]$. As was argued in Proposition III.4, C consists of $(|\bar{G}|/r)$ copies of $\mathbb{P}^1$. Let C_0 be one of these copies. Then C_0 is a principal $\langle [h] \rangle$-bundle over $\mathcal{C}_0$ and

$$H^0(\mathcal{C}_0, T\mathcal{C}_0) = H^0(C_0, TC_0)^{\langle [h] \rangle}.$$

The $\langle [h] \rangle$-invariant part of $H^0(C_0, TC_0)$ is one dimensional. It is fixed by the torus action, thus the moving part of $H^0(\mathcal{C}_0, T\mathcal{C}_0)$ is trivial and $e(H^0(\mathcal{C}_0, T\mathcal{C}_0)^m) = 1$.

• $e(H^1(\mathcal{C}_0, f^*T\mathcal{Y})^m)$: Let C_0 be as in the previous bullet, then

$$H^1(\mathcal{C}_0, f^*T\mathcal{Y}) = H^1(C_0, \tilde{f}^*T\mathbb{P}^4)^{\langle [h] \rangle} = 0.$$

Therefore $e(H^1(\mathcal{C}_0, f^*T\mathcal{Y})^m) = 1$.

• $e(H^0(\mathcal{C}_0, f^*T\mathcal{Y})^m)$: To calculate this term, note that

$$H^0(\mathcal{C}_0, f^*T\mathcal{Y})^m \cong \left(H^0(C_0, \tilde{f}^*T\mathbb{P}^4)^{\langle [h] \rangle} \right)^m.$$

We will look at the $\langle [h] \rangle$ invariant part of the short exact sequence

$$0 \to \mathbb{C} \to H^0(\mathscr{O}_{C_0}(rd')) \otimes V \to H^0(\tilde{f}^*T\mathbb{P}^4) \to 0,$$

where $\mathbb{P}^4 = \mathbb{P}(V)$ and $V \cong \mathbb{C}^5$. The exact sequence comes from the pullback of the Euler sequence for $\mathbb{P}^4$ to C_0. (Note that the degree of $\tilde{f} : C_0 \to \mathbb{P}^4$ is rd'). The action of $[h]$ on the first term in the sequence is trivial.

Recall that $\mathbb{P}(V)$ has coordinates $[x_0, \ldots, x_4]$. Let $[s,t]$ be homogeneous coordinates on $C_0 \cong \mathbb{P}^1$, such that the preimage of p_1 in C_0 is $[0,1]$ and the preimage of p' in C_0 is $[1,0]$. Then the middle term of the sequence is spanned by elements of the form $s^a t^b \frac{\partial}{\partial x_l}$ where $0 \leq l \leq 4$ and $a + b = rd'$. The action is given by

$$[h].(s^a t^b \frac{\partial}{\partial x_l}) = e^{2\pi\sqrt{-1}(-a + r_l(h))/r} s^a t^b \frac{\partial}{\partial x_l},$$

and so this summand is invariant under the $\langle [h] \rangle$-action if and only if $r_l(h)/r = \langle a/r \rangle$. The $\mathbb{C}^*$-action on this term has weight

$$\left(a/rd'\right)\lambda_k + \left(b/rd'\right)\lambda_i - \lambda_l,$$

so we finally arrive at

$$\begin{aligned} & e(H^0(\mathcal{C}_0, f^*T\mathcal{Y})^m) \\ = & \prod_{\substack{\{(a,l) | 0 \le a \le rd' \;\; 0 \le l \le 4 \;\; r_l(h)/r = \langle a/r \rangle\} \\ \setminus \{(0,i),\, (rd',k)\}}} \left(\frac{a}{rd'}\lambda_k + \frac{rd'-a}{rd'}\lambda_i - \lambda_l \right) \\ = & \prod_{\substack{\{(a,l) | 0 \le a \le rd' \;\; 0 \le l \le 4 \;\; r_l(h)/r = \langle a/r \rangle\} \\ \setminus \{(0,i),\, (rd',k)\}}} \left(a\left(\frac{\lambda_k - \lambda_i}{rd'}\right) + \lambda_i - \lambda_l \right). \end{aligned}$$

• $e(H^0(p', f^*T\mathcal{Y})^m)$: Similarly, the node p' is isomorphic to $B\mathbb{Z}_r$, and each of the $|\bar{G}|/r$ points lying in the principal $\bar{G}$-bundle over p' is a principal $\langle [h] \rangle$-bundle over p'. Thus $H^0(p', f^*T\mathcal{Y})^m \cong \left((T_{q_k}\mathbb{P}^n)^{\langle [h] \rangle} \right)^m$ and

$$e(H^0(p', f^*T\mathcal{Y})^m) = \prod_{l \in I(h') \setminus \{k\}} (\lambda_k - \lambda_l).$$

Finally note that $ev_1^*(\phi_i) = \prod_{l \in I(h) - i}(\lambda_i - \lambda_l)$. We can do one further simplification. On the graphs which we consider, namely those where p_1 is on a noncontracted component, ψ_1 restricts to $\frac{\lambda_k - \lambda_i}{d'}$. (In fact $e(T^*_{p_1}\mathcal{C}) \cong \frac{\lambda_k - \lambda_i}{rd'}$, but because we are following the convention that ψ-classes are pulled back from the reification, we must multiply this by a factor of r).

These calculations plus (3.6) then give us the contribution to $\left\langle \frac{\phi_i \psi_1^{c(d,h)-1}}{1 - \psi_1/z}, \mathbb{1}_g \right\rangle^{\mathcal{Y},T}_{0,2,d}$

from the graph M_Γ:

$$\begin{aligned}
&\int_{[M_\Gamma]} \frac{ev_1^*(\phi_i)\psi_1^{c(d,I)-1}}{e(N_\Gamma)\,(1-\psi_1/z)} \\
=&\frac{\frac{\lambda_k-\lambda_i}{d'}^{c(d,I)-1}\prod_{l\in I(h)\setminus\{i\}}(\lambda_i-\lambda_l)e(H^1(\mathcal{C}_0,f^*T\mathcal{Y})^m)}{e(H^0(\mathcal{C}_0,f^*T\mathcal{Y})^m)(1-\frac{\lambda_k-\lambda_i}{d'z})} \\
&\cdot\frac{1}{d'}\int_{[M'_\Gamma]}\frac{e(H^0(p',f^*T\mathcal{Y})^m)}{(\text{node smoothing at }p')e(N_{\Gamma'})} \\
=&\frac{\frac{\lambda_k-\lambda_i}{d'}^{c(d,h)-1}\prod_{l\in I(h)\setminus\{i\}}(\lambda_i-\lambda_l)}{(d'-\frac{\lambda_k-\lambda_i}{z})\prod_{\substack{\{(a,l)|0\le a\le rd'\ 0\le l\le 4\ r_l(h)/r=\langle a/r\rangle\}\\ \setminus\{(0,i),\ (rd',k)\}}}\left(a\left(\frac{\lambda_k-\lambda_i}{rd'}\right)+\lambda_i-\lambda_l\right)} \\
&\cdot\int_{[M_{\Gamma'}]}\frac{\prod_{l\in I(h')\setminus\{k\}}(\lambda_k-\lambda_l)}{(\frac{\lambda_k-\lambda_i}{d'}-\psi_1)e(N_{\Gamma'})}.
\end{aligned}$$

3.3.4 Recursion relations

We will formulate the above computations into a recursion relation. To do that, the following regularity lemma is needed.

Lemma III.9 (Regularity Lemma)**.** *$Z_{i,g}^T$ is an element of $\mathbb{Q}(\lambda_i,z)[[Q]]$. The coefficient of each Q^D is a rational function of λ_i and z which is regular at $z=(\lambda_i-\lambda_j)/k$ for all $j\neq i$ and $k\geq 1$.*

Proof. This follows from a standard localization argument, see e.g. Lemma 11.2.8 in [12]. □

Using the Regularity Lemma, the above computation simplifies to

$$\begin{aligned}
&\left(\left\langle\frac{\phi_i\psi_1^{c(d,h)-1}}{1-\psi_1/z},\mathbb{1}_g\right\rangle_{0,2,d}^{\mathcal{Y},T}\right)_{M_\Gamma} \\
&=C_{d'}^{i,k}\cdot\left(\frac{\lambda_k-\lambda_i}{d'}\right)^{c(d,h)-1-(c(d',h)-1)}\cdot\left(\left\langle\frac{\phi_k}{z-\psi_1},\mathbb{1}_g\right\rangle_{0,2,d-d'}^{\mathcal{Y},T}\right)_{M_{\Gamma'}}\Bigg|_{z\mapsto\frac{\lambda_k-\lambda_i}{d'}},
\end{aligned}$$

where

$$C^{i,k}_{d'} = \frac{1}{(d' - \frac{\lambda_k - \lambda_i}{z}) \prod_{\{(a,l) \in S(d',h) \setminus \{(d',k)\}\}} \left(a + d'\left(\frac{\lambda_i - \lambda_l}{\lambda_k - \lambda_i}\right)\right)}$$

and $(-)_{M_\Gamma}$ means the contribution of the fixed component M_Γ to the expression in parentheses.

Due to the fact that $r_k(h)/5 = \langle d' \rangle$, one can check that

$$c(d,h) - c(d',h) = c(d-d',h')$$

(see (3.8)). We arrive at the expression

$$C^{i,k}_{d'} \cdot \left(Q^{c(d-d',k)} \left\langle \frac{\phi_k}{z - \psi_1}, \mathbb{1}_g \right\rangle^{\mathcal{Y},T}_{0,2,d-d'} \right)_{M_{\Gamma'}} \Bigg|_{z \mapsto \frac{\lambda_k - \lambda_i}{d'}, Q \mapsto \frac{\lambda_k - \lambda_i}{d'}}.$$

After summing over all possible graphs, we obtain the recursion:

(3.7)
$$Z^T_{i,g} = \frac{\delta^{i,I(g)}}{125} + \sum_{\{(d',k) | \frac{r_k(h)}{5} = \langle d' \rangle, k \neq i, d' \neq 0\}} \left(\frac{Q}{z}\right)^{c(d',h)} C^{i,k}_{d'} \cdot Z^T_{k,g} \Bigg|_{z \mapsto \frac{\lambda_k - \lambda_i}{d'}, Q \mapsto \frac{Q}{z} \frac{\lambda_k - \lambda_i}{d'}}.$$

Although we have suppressed this in the notation, recall that in the above summand, h is the presentation such that ϕ_i is supported on $\mathcal{Y}_h$ ($i \in I(h)$).

We will now turn our attention to Y^T_g. Let us define the function $Y^T_{i,g}$ analogously to that of $Z^T_{i,g}$,

$$Y^T_{i,g} := (\phi_i, Y^T_g)^{\mathcal{Y}}_{CR}.$$

For $i \in I(h)$,

$$Y^T_{i,g} = \frac{1}{125} \left(\delta^{i,I(g)} + \sum_{\langle d \rangle = d(h,g)} Q^{c(d,h)} \frac{1}{\prod_{(b,k) \in S(d,h)} (bz + \lambda_i - \lambda_k)} \right).$$

Claim III.10. *$Y^T_{i,g}$ satisfy the same recursion as $Z^T_{i,g}$ in* (3.7).

Proof. Consider the summand of $Y_{i,g}^T$ of degree $c(d,h)$ in Q, which we will denote $(Y_{i,g}^T)^{c(d,h)}$.

$$
\begin{aligned}
(Y_{i,g}^T)^{c(d,h)} =& \frac{1}{125}\left(\frac{Q}{z}\right)^{c(d,h)} \frac{1}{\prod_{(b,k)\in S(d,h)}(b+(\lambda_i-\lambda_k)/z)} \\
=& \frac{1}{125}\left(\frac{Q}{z}\right)^{c(d,h)} \sum_{\{(b,k)|r_k(h)/5=\langle b\rangle, k\neq i, b\neq 0\}} \frac{1}{(b+(\lambda_i-\lambda_k)/z)} \\
& \cdot \frac{1}{\prod\limits_{(m,l)\in S(d,h)\setminus\{(b,k)\}} (b(\lambda_i-\lambda_l)/(\lambda_k-\lambda_i)+m)} \\
=& \frac{1}{125}\left(\frac{Q}{z}\right)^{c(d,h)} \sum_{\{(b,k)|r_k(h)/5=\langle b\rangle, k\neq i, b\neq 0\}} \\
& \left(\frac{1/\left(b+(\lambda_i-\lambda_k)/z\right)}{\prod\limits_{\{(m,l)\in S(d,h)\setminus\{(b,k)\}|m\leq b\}} (b(\lambda_i-\lambda_l)/(\lambda_k-\lambda_i)+m)}\right. \\
& \left.\cdot \frac{1}{\prod\limits_{\{(m,l)\in S(d,h)\setminus\{(b,k)\}|m> b\}} (b(\lambda_i-\lambda_l)/(\lambda_k-\lambda_i)+m)}\right).
\end{aligned}
$$

The last product from above can be rewritten as

$$
\prod_{(n,l)\in S(d-b,h')} \left(n + b\frac{\lambda_k-\lambda_l}{\lambda_k-\lambda_i}\right),
$$

where h' is chosen such that $[h]=[h']$ and $k\in I(h')$. To see this note that if (b,k) and (m,l) are both in $S(d,h)$, then by definition $r_k(h)/5=\langle b\rangle$ and $r_l(h)/5=\langle m\rangle$. If $k\in I(h')$, then

$$
\begin{aligned}
\frac{r_l(h')}{5} =& \frac{r_l(h')}{5} - \frac{r_k(h')}{5} \\
\equiv& \frac{r_l(h)}{5} - \frac{r_k(h)}{5} \equiv \langle m\rangle - \langle b\rangle \equiv \langle m-b\rangle \pmod 1.
\end{aligned}
$$

In other words $r_l(h')/5=\langle m-b\rangle$. This proves that if $(b,k)\in S(d,h)$, and h' is chosen as above, then for pairs (m,l) with $b<m\leq d$,

$$
(m,l)\in S(d,h) \text{ if and only if } (m-b,l)\in S(d-b,h'). \tag{3.8}
$$

We arrive at the relation

$$\left(Y_{i,g}^{T}\right)^{c(d,h)} = \sum_{\{(b,k)|r_k(h)/5=\langle b\rangle,k\neq i,b\neq 0\}} \left(\frac{Q}{z}\right)^{c(b,h)} C_b^{i,k} \left(Y_{k,g}^{T}\right)^{c(d-b,h')}\Bigg|_{z\mapsto \frac{\lambda_k-\lambda_i}{b},Q\mapsto \frac{Q}{z}\frac{\lambda_k-\lambda_i}{b}}.$$

We conclude that $Y_{i,g}^{T}$ satisfy the same recursion as $Z_{i,g}^{T}$. $\square$

The recursion relation and initial conditions imply $Y_{i,g}^{T} = Z_{i,g}^{T}$. The proof of Theorem III.5 is now complete.

Remark III.11. As a corollary one may easily obtain an explicit formula for the small J-matrix $J_{small}^{\mathcal{Y}}(t,z)$ by isolating coefficients of the various $Z_g^{\mathcal{Y}}$. We give an explicit expression for certain specified rows of $J_{small}^{\mathcal{Y}}(t,z)$ in Corollary IV.8.

CHAPTER IV

A model of the mirror quintic $\mathcal{W}$

4.1 Fermat quintic and its mirror

Let $M \subset \mathbb{P}^4$ be the Fermat quintic defined by the equation $Q_0(x) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$

$$M := \{Q_0(x) = 0\} \subset \mathbb{P}^4.$$

The Greene–Plesser *mirror construction* [23] gives the *mirror orbifold* as the quotient stack

$$\mathcal{W} := [M/\bar{G}].$$

Note that the $\bar{G}$-action on $\mathbb{P}^4$ (3.1) preserves the quintic equation $Q_0(x)$ and therefore induces an action on M. Equivalently,

$$\mathcal{W} = \{Q_0 = 0\} \subset \mathcal{Y} = [\mathbb{P}^4/\bar{G}]. \tag{4.1}$$

Remark IV.1. Gromov–Witten theory is invariant under deformation (this property is called the *deformation axiom* in [12], or alternatively, describes a part of what is referred to as the *composition law* in [24]). Since in this section we will only be interested in the Gromov–Witten theory of $\mathcal{W}$, we will only speak of the mirror orbifold instead of the mirror family.

Recall in Lemma III.1 the inertia orbifold of $\mathcal{Y} = [\mathbb{P}^4/\bar{G}]$ is indexed by $g \in G$. For a particular g, the dimension of $\mathcal{Y}_g$ is equal to $\left|\{j|r_j = 0\}\right| - 1$, and can be identified with a linear subspace of $\mathcal{Y}$. The age shift of $\mathcal{Y}_g$ is $\mathrm{age}(g) = \sum_{i=0}^4 r_i/5$.

The inertia orbifold of the mirror quintic $\mathcal{W}$ can be described by that of $\mathcal{Y}$. $\mathcal{W}$ intersects nontrivially with $\mathcal{Y}_g$ exactly when $\left|\{j|r_j = 0\}\right| \geq 2$. (that is, $\dim \mathcal{Y}_g \geq 1$.) Let

$$\bar{S} := \left\{ g = (r_0, \ldots, r_4) \in G \middle| 2 \leq \left|\{j|r_j = 0\}\right| \right\}.$$

(Note that $\bar{S}$ contains $e = (0, \ldots, 0)$.) Then

$$I\mathcal{W} = \coprod_{g \in \bar{S}} \mathcal{W}_g , \qquad \mathcal{W}_g := \mathcal{W} \cap \mathcal{Y}_g.$$

All nontrivial intersections are transverse, so

$$\dim(\mathcal{W}_g) = \dim(\mathcal{Y}_g) - 1 = \left|\{j|r_j = 0\}\right| - 2.$$

It follows that the age shift of $\mathcal{W}_g$ is equal to the age shift of $\mathcal{Y}_g$. The cohomology of $\mathcal{W}$ is given by

$$H^*_{CR}(\mathcal{W}) = \bigoplus_{g \in \bar{S}} H^{*-2\,\mathrm{age}(g)}(\mathcal{W}_g).$$

In the sequel, we will only be interested in the subring of $H^*_{CR}(\mathcal{W})$ consisting of classes of even (real) degree. We will denote this ring as $H^{even}_{CR}(\mathcal{W})$. It can be checked via a direct calculation that if $i : \mathcal{W} \hookrightarrow \mathcal{Y}$ is the inclusion,

$$H^{even}_{CR}(\mathcal{W}) = i^* H^*_{CR}(\mathcal{Y}).$$

Conventions IV.2. By a further abuse of notation, we will also denote by H the induced class on $\mathcal{W}$ pulled back from $\mathcal{Y}$.

A convenient basis $\{T_i\}$ for $H^{even}_{CR}(\mathcal{W})$ is

$$\bigcup_{g \in \bar{S}} \{\mathbb{1}_g, \mathbb{1}_g H, \ldots, \mathbb{1}_g H^{\dim(\mathcal{W}_g)}\}. \tag{4.2}$$

We also note that $H^{even}_{CR}(\mathcal{W}) \subset H^*_{CR}(\mathcal{W})$ is a self-dual subring with respect to the Poincaré pairing of $H^*_{CR}(\mathcal{W})$. Furthermore, this basis is self-dual (up to a constant factor). Given $g = (r_0, \ldots, r_4) \in S$, let

$$g^{-1} := (-r_1, \ldots, -r_4) \pmod 5.$$

Then the Poincaré dual elements can be easily calculated:

$$\left(\mathbb{1}_g H^k\right)^\vee = 25 \left(\mathbb{1}_{g^{-1}} H^{\dim(\mathcal{W}_g)-k}\right).$$

4.2 J-functions of $\mathcal{W}$

Conventions IV.3. By the matrix J-function of $\mathcal{W}$, we will mean the matrix consisting of the collection of $H^{even}_{CR}(\mathcal{W})$-valued functions with variable $\mathbf{t} = tH$.

$$J^{\mathcal{W}}_g(t,z) := e^{tH/z} \left(\mathbb{1}_g + \sum_{d,i} q^d e^{dt} \left\langle \frac{T_i}{z-\psi_1}, \mathbb{1}_g \right\rangle^{\mathcal{W}}_{0,2,d} T^i \right), \tag{4.3}$$

where the basis $\{T_i\}$ is for $H^{even}_{CR}(\mathcal{W})$, as in (4.2). Here as in Section III, by the degree d of a map $f : \mathcal{C} \to \mathcal{W}$ we mean

$$d := \int_{\mathcal{C}} f^*(H).$$

Note that if we extend the basis $\{T_i\}$ to a full basis of $H^*_{CR}(\mathcal{W})$, the classes of odd (real) degree will not contribute to $J^{\mathcal{W}}_g(t,z)$, and thus (4.3) is equal to the J_g-function of (2.5).

As has been shown in Proposition III.4, for an orbi-curve $\mathcal{C}$ with two marked points, the degree must be a multiple of $1/5$. Recall also from Proposition III.4 that the only nonzero contribution to the terms in $J^{\mathcal{W}}_g$ comes from elements T_i supported on some $\mathcal{W}_h$ such that $[h] = [g^{-1}]$. From the definition of $\bar{S}$, it is required that

$$\left|\{j | r_j = 0\}\right| \geq 2, \quad \sum_j r_j \equiv 0 \pmod 5. \tag{4.4}$$

We will enumerate all possible cases.

It follows from the conditions (4.4) that $\left|\{j|r_j = 0\}\right|$ must be equal to 2, 3 or 5. That is, $\dim(\mathcal{W}_g)$ is equal to 0, 1 or 3.

If $\dim(\mathcal{W}_g) = 3$, $g = e = (0,0,0,0,0)$ and $\mathbb{1}_e = 1$. The only basis elements which contribute to $J_e^{\mathcal{W}}$ come from the nontwisted sector. We have

$$J_e^{\mathcal{W}}(t,z) = e^{tH/z}\left(1 + \sum_{d>0} q^d e^{dt}\left\langle \frac{H^i}{z-\psi_1}, 1\right\rangle_{0,2,d}^{\mathcal{W}} (25H^{3-i})\right). \tag{4.5}$$

If $\dim(\mathcal{W}_g) = 1$, then up to a permutation of the entries, $g = (0,0,0,r_1,r_2)$ with $r_1 \neq r_2$. By definition of $\bar{S}$, other than g there is no $h \in \bar{S}$ such that $[h] = [g]$. Therefore, the two basis elements which contribute nontrivially to $J_g^{\mathcal{W}}$ are $\mathbb{1}_{g^{-1}}$ and $\mathbb{1}_{g^{-1}}H$. We arrive at

$$\begin{aligned} J_g^{\mathcal{W}}(t,z) = e^{tH/z}\Bigg(&\mathbb{1}_g + \\ &\sum_{d>0} q^d e^{dt}\left(\left\langle \frac{\mathbb{1}_{g^{-1}}}{z-\psi_1}, \mathbb{1}_g\right\rangle_{0,2,d}^{\mathcal{W}} (25\mathbb{1}_g H) + \left\langle \frac{\mathbb{1}_{g^{-1}}H}{z-\psi_1}, \mathbb{1}_g\right\rangle_{0,2,d}^{\mathcal{W}} (25\mathbb{1}_g)\right)\Bigg). \end{aligned} \tag{4.6}$$

If $\dim(\mathcal{W}_g) = 0$, then up to a permutation of the entries, $g = (0,0,r_1,r_1,r_2)$, with $r_1 \neq r_2$. There is only one other $g_1 \in \bar{S}$ such that $[g_1] = [g]$, namely, $g_1 = (-r_1,-r_1,0,0,r_2-r_1) \pmod 5$. The two basis elements which contribute nontrivially to the invariants of $J_g^{\mathcal{W}}$ are $\mathbb{1}_{g^{-1}}$ and $\mathbb{1}_{(g_1)^{-1}}$. Thus we can express $J_g^{\mathcal{W}}(t,z)$ as

$$\begin{aligned} J_g^{\mathcal{W}}(t,z) = e^{tH/z}\Bigg(&\mathbb{1}_g + \\ &\sum_{d>0} q^d e^{dt}\left(\left\langle \frac{\mathbb{1}_{g^{-1}}}{z-\psi_1}, \mathbb{1}_g\right\rangle_{0,2,d}^{\mathcal{W}} (25\mathbb{1}_g) + \left\langle \frac{\mathbb{1}_{(g_1)^{-1}}}{z-\psi_1}, \mathbb{1}_g\right\rangle_{0,2,d}^{\mathcal{W}} (25\mathbb{1}_{g_1})\right)\Bigg). \end{aligned} \tag{4.7}$$

Thus for each twisted component $\mathcal{W}_g$, the J-function $J_g^{\mathcal{W}}$ has two components.

We will relate the functions $J_g^{\mathcal{W}}$ to certain hypergeometric functions, called I-functions. To start with, let us introduce "bundled-twisted" Gromov–Witten invariants. Let $E \to \mathcal{X}$ be a line bundle over the orbifold $\mathcal{X}$. We have the following diagram

$$\begin{tikzcd} & E \arrow[d] \\ \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{X} \\ \overline{\mathscr{M}}_{0,n}(\mathcal{X},d). & \end{tikzcd}$$

The E-twisted Gromov–Witten invariants are defined to be

$$\langle \alpha_1\psi^{k_1},\ldots,\alpha_n\psi^{k_n}\rangle_{0,n,d}^{\mathcal{X},\mathrm{tw}} = \int_{[\overline{\mathscr{M}}_{0,n}(\mathcal{X},d)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{k_i} \cup e(E_{0,n,d}),$$

where

$$E_{o,n,d} := \pi_* f^*(E)$$

and $e(E_{0,n,d})$ is the Euler class of the K-class. We can define a twisted pairing on $H^*_{CR}(\mathcal{X};\Lambda)$ by

$$(\alpha_1,\alpha_2)_{CR}^{\mathcal{X},\mathrm{tw}} = \int_{\mathcal{X}} \alpha_1 \cup I^*(\alpha_2) \cup e(E).$$

With this, we can define a twisted J-function

$$J^{\mathcal{X},\mathrm{tw}}(\mathbf{t},z) = 1 + \mathbf{t}/z + \sum_d \sum_{n\geq 0} \sum_i \frac{q^d}{n!}\left\langle \frac{T_i}{z-\psi_1}, 1, \mathbf{t},\ldots,\mathbf{t}\right\rangle_{0,2+k,d}^{\mathcal{X},\mathrm{tw}} T^i.$$

Here T_i is a basis for $H^*_{CR}(\mathcal{X};\Lambda)$ and T^i is the dual basis with respect to the twisted pairing.

The twisted invariants are related to invariants on the hypersurface. In our case, $\mathcal{X} = \mathcal{Y} = [\mathbb{P}^4/\bar{G}]$, and $E = \mathscr{O}(5) \to \mathcal{Y}$. One can check that $\dim\left(H^0(f^*(\mathscr{O}(5))\right)$ is constant on connected components of $\overline{\mathscr{M}}_{0,n}(\mathcal{Y},d)$. It follows that $E_{0,n,d} = R^0\pi_*f^*(\mathscr{O}(5))$ is a vector bundle. The embedding $i : \mathcal{W} \hookrightarrow \mathcal{Y}$ induces a morphism $\iota : \overline{\mathscr{M}}_{0,n}(\mathcal{W},d) \hookrightarrow \overline{\mathscr{M}}_{0,n}(\mathcal{Y},d)$. As is shown in e.g. [13] [1],

$$\iota_*[\overline{\mathscr{M}}_{0,n}(\mathcal{W},d)]^{\mathrm{vir}} = e(E_{0,n,d}) \cap [\overline{\mathscr{M}}_{0,n}(\mathcal{Y},d)]^{\mathrm{vir}}. \tag{4.8}$$

[1] That proof, given in the non-orbifold setting, can be readily modified to the orbifold setting.

This relates the twisted invariants on $\mathcal{Y}$ to the invariants on $\mathcal{W}$. Assume that $\mathbf{t}$ is restricted to $H^{even}_{CR}(\mathcal{Y})$, then by the projection formula,

$$J^{\mathcal{W}}(\mathbf{t}, z) = i^* J^{\mathcal{Y},tw}(\mathbf{t}, z).$$

Let us now further restrict $\mathbf{t}$ to $H^2_{CR}(\mathcal{Y})$. In our setting we may write an element of $H^2_{CR}(\mathcal{Y})$ as

$$\mathbf{t} = tH + \sum_{\{g \mid \operatorname{age}(g)=1\}} t^g \mathbb{1}_g. \tag{4.9}$$

Write the J-function of $\mathcal{Y}$ as

$$J^{\mathcal{Y}}(\mathbf{t}) = \sum_d q^d J^{\mathcal{Y}}_d(\mathbf{t}).$$

For each d, define the modification factor

$$M^{E/\mathcal{Y}}_d := \prod_{m=1}^{5d} (5H + mz).$$

(Note that we have taken the $\lambda = 0$ limit in [10].)

Definition IV.4. Define the *twisted I-function* by

$$I^E(\mathbf{t}) := \sum_d q^d M^{E/\mathcal{Y}}_d J^{\mathcal{Y}}_d(\mathbf{t}).$$

Write

$$\begin{aligned} I^E(\mathbf{t}, z) =& I^E_e(t, z) + \frac{1}{z}\left(\sum_{\{g \mid \operatorname{age}(g)=1\}} t^g I^E_g(t, z)\right) \\ &+ \frac{1}{z}\left(\sum_{\{g_1, g_2 \mid \operatorname{age}(g_i)=1\}} t^{g_1} t^{g_2} I^E_{g_1, g_2}(t, z) + \dots\right). \end{aligned} \tag{4.10}$$

For g such that $\operatorname{age}(g) \leq 1$ (including $g = e$), define the A model hypergeometric functions

$$I^A_g(t, z) = i^*\left(I^E_g(t, z)\right). \tag{4.11}$$

The theorem below is our main result from the A model which will be needed to prove the mirror theorem.

Theorem IV.5. *Given $g = (r_0, \dots, r_4)$ such that the age shift of $\mathcal{W}_g$ is at most 1, there exist functions $F_0(t)$, $G_0(t)$, and $H_g(t)$, determined explicitly by $I_g^E(t,z)$ such that F_0 and H_g ($g \neq 0$) are invertible, and*

$$J_g^{\mathcal{W}}(\tau(t), z) = \frac{I_g^A(t,z)}{H_g(t)} \qquad \textit{where } \tau(t) = \frac{G_0(t)}{F_0(t)}. \tag{4.12}$$

Remark IV.6. In the statement of the theorem, $F_0(t)$ and $G_0(t)$ do not depend on g, so the *mirror map* $t \mapsto \tau(t) = G_0(t)/F_0(t)$ is well defined.

4.3 Proof of Theorem IV.5

There are two key ingredients in the proof. The first one is the version of *quantum Lefschetz hyperplane theorem* (QLHT) for orbifolds proved in [10]. By Equation (4.1), $\mathcal{W}$ is a hyperplane section of $\mathcal{Y}$ and hence $J^{\mathcal{W}}(\mathbf{t}, z)$ can be calculated by QLHT. Corollary 5.1 in [10] in particular implies the following:

Theorem IV.7 ([10])**.** *Let the setting be as above, with $E = \mathscr{O}(5) \to \mathcal{Y}$. Then*

$$I^E(\mathbf{t}, z) = F(\mathbf{t}) + \frac{G(\mathbf{t})}{z} + O(z^{-2}) \tag{4.13}$$

for some F and G with F scalar valued and invertible, and

$$J^{\mathcal{Y},\mathrm{tw}}(\tau(\mathbf{t}), z) = \frac{I^E(\mathbf{t}, z)}{F(\mathbf{t})} \qquad \textit{where } \tau(\mathbf{t}) = \frac{G(\mathbf{t})}{F(\mathbf{t})}. \tag{4.14}$$

The second ingredient is the explicit formula of $J_g^{\mathcal{Y}}$ from Section III. Note that we are only concerned with those g such that $i^*\mathbb{1}_g \neq 0$ and $\mathrm{age}(\mathbb{1}_g) \leq 1$. Therefore only those $J_g^{\mathcal{Y}}$ are listed. The following is a straightforward corollary of Theorem III.5, (3.3) and (3.4) by equating the terms $Q^{c(d,h)}\mathbb{1}_{h^{-1}}H^k$ of Z_g with the terms $q^d e^{dt}\mathbb{1}_{h^{-1}}H^k$ of $J_g^{\mathcal{Y}}$.

Corollary IV.8. *The functions $J_g^{\mathcal{Y}}(t,z)$ are given by the following formulas.*

(i) If $g = e = (0,0,0,0,0)$,

$$J_e^{\mathcal{Y}} = e^{tH/z}\left(1 + \sum_{\langle d\rangle=0} q^d e^{dt}\frac{1}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^5}\right). \tag{4.15}$$

(ii) If $g = (0,0,0,r_1,r_2)$, let $g_1 = (-r_1,-r_1,-r_1,0,r_2-r_1) \pmod 5$ and let $g_2 = (-r_2,-r_2,-r_2,r_1-r_2,0) \pmod 5$. Then

$$\begin{aligned}
J_g^{Y} =& e^{tH/z}\mathbb{1}_g\left(1 + \sum_{\langle d\rangle=0}\frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^3\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{r_2}{5}\right\rangle}}(H+bz)\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{r_1}{5}\right\rangle}}(H+bz)}\right)\\
&+e^{tH/z}\mathbb{1}_{g_1}\left(\sum_{\langle d\rangle=\left\langle\frac{r_1}{5}\right\rangle}\frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{r_1}{5}\right\rangle}}(H+bz)^3\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{2r_1}{5}\right\rangle}}(H+bz)}\right)\\
&+e^{tH/z}\mathbb{1}_{g_2}\left(\sum_{\langle d\rangle=\left\langle\frac{r_2}{5}\right\rangle}\frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{r_2}{5}\right\rangle}}(H+bz)^3\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=\left\langle\frac{2r_2}{5}\right\rangle}}(H+bz)\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)}\right).
\end{aligned} \tag{4.16}$$

(iii) If $g = (0,0,r_1,r_1,r_2)$, let $g_1 = (-r_1,-r_1,0,0,r_2-r_1) \pmod 5$ and let $g_2 =$

$(-r_2, -r_2, r_1 - r_2, r_1 - r_2, 0) \pmod 5$. *Then*

(4.17)

$$
\begin{aligned}
J_g^Y =& e^{tH/z}\mathbb{1}_g \left(1 + \sum_{\langle d\rangle = 0} \frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = 0}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{3r_2}{5}\right\rangle}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{2r_1}{5}\right\rangle}} (H+bz)} \right) \\
&+ e^{tH/z}\mathbb{1}_{g_1} \left(\sum_{\langle d\rangle = \left\langle \frac{r_1}{5}\right\rangle} \frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{r_1}{5}\right\rangle}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = 0}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{r_2}{5}\right\rangle}} (H+bz)} \right) \\
&+ e^{tH/z}\mathbb{1}_{g_2} \left(\sum_{\langle d\rangle = \left\langle \frac{r_2}{5}\right\rangle} \frac{q^d e^{dt}}{\prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{r_2}{5}\right\rangle}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = \left\langle \frac{2r_1}{5}\right\rangle}} (H+bz)^2 \prod\limits_{\substack{0<b\leq d\\ \langle b\rangle = 0}} (H+bz)} \right).
\end{aligned}
$$

In case (ii), up to permutation $(r_1, r_2) = (2,3)$ *or* $(1,4)$. *Due to the age requirement, in case (iii) only* $(r_1, r_2) = (1,3)$ *or* $(2,1)$ *are possible.*

Lemma IV.9. *There are scalar valued functions* $F_0(t), G_0(t)$ *and* $G_g(t)$ *for each* g *with* $\operatorname{age}(g) = 1$, *such that*

$$
i^*\left(I^E(\mathbf{t}, z)\right) = F_0(t) + \frac{G_0(t)H}{z} + \sum_{\operatorname{age}(g)=1} \frac{t^g G_g(t)\mathbb{1}_g}{z} + R,
$$

where R *denotes the* remainder, *consisting of terms with either the degrees in* t^g*'s greater than or equal to* 2 *or the degree in* z^{-1} *greater than or equal to* 2. *In other words, if we write* $G(\mathbf{t})$ *from* (4.13) *as*

$$
G(\mathbf{t}) = \overline{G_0}(\mathbf{t})H + \sum_g \overline{G_g}(\mathbf{t})\mathbb{1}_g
$$

and denote $O(2)$ *the terms with the degrees in* t^g*'s greater or equal to* 2, *then*

$$
F(\mathbf{t}) = F_0(t) + O(2), \quad \overline{G_0}(\mathbf{t}) = G_0(t) + O(2), \quad \overline{G_g}(\mathbf{t}) = t^g G_g(t) + O(2).
$$

Proof. The proof of this lemma follows from Corollary IV.8 together with the following observations. First, in case (ii) $i^*(\mathbb{1}_{g_1}) = i^*(\mathbb{1}_{g_2}) = 0$ due to dimensional reasons. Similarly with $i^*(\mathbb{1}_{g_2}) = 0$ in case (iii). Secondly, in case (iii) the $\mathbb{1}_{g_1}$ term has higher z^{-1} power: The modification factor contributes terms of z^{5d} plus lower order (in z) terms. $i^* J_g^{\mathcal{Y}}$ contributes $z^{-(5d+1)}$ plus higher order (in z^{-1}) terms. The combined contribution goes to the remainder R. $\square$

With all this preparation, it is easy to prove Theorem IV.5.

Proof of Theorem IV.5. Start by pulling back the equation (4.14) to $\mathcal{W}$. Setting all $t^g = 0$ we get (4.12) for the case $g = e$ if we let $H_e = F_0$:

$$I_e^A(t) = i^* I_e^E(t) = i^* I^E(\mathbf{t})|_{\mathbf{t}=tH}.$$

Here by $\mathbf{t} = tH$ we mean that setting all $t^g = 0$ in (4.9). In the case $g \neq e$, take the partial derivative of (4.14) with respect to t^g and then set all $t^g = 0$. Note that from (4.10), we have

$$I_g^A(t) = i^* I_g^E(t) = z\frac{\partial}{\partial t^g} i^* I^E(\mathbf{t})|_{\mathbf{t}=tH}.$$

By Lemma IV.9 all the "extra terms" vanish and (4.12) follows for $g \neq e$ after letting $H_g(t) = G_g(t)$. The proof is now complete. $\square$

CHAPTER V

Periods and Picard–Fuchs equations

The theory of variation of Hodge structures (VHS) is closely related to the *B model* of a Calabi–Yau variety X, which encodes information about the deformations of complex structures on X. By the local Torelli theorem for Calabi–Yau's, the Kodaira–Spencer spaces inject to the tangent spaces of period domains and one can investigate the deformations of X via VHS, which can be described by a system of flat connections on cohomology vector bundles.

For the benefit of the readers who come from the GWT side of mirror symmetry, we give a brief and self-contained summary of the parts of VHS theory which are related to our work: the Gauss–Manin connection and the associated notions of the period matrix and Picard–Fuchs equations. For a more detailed introduction the reader may consult [19], [18].

5.1 Gauss–Manin connections, periods, and Picard–Fuchs equations

Over a smooth family of projective varieties $\pi : \mathscr{X} \to S$ of relative dimension n, we can consider the higher direct image sheaf (tensored with $\mathscr{O}_S$) on S:

$$R^n \pi_* \mathbb{C} \otimes \mathscr{O}_S.$$

The fiber over a point $t \in S$ of this sheaf is $H^n(X_t)$. This sheaf is locally free, and is naturally endowed with a *flat* connection ∇^{GM}, the *Gauss–Manin* connection. It can be defined in terms of the flat sections given by the lattice $R^n\pi_*\mathbb{Z}$ in $R^n\pi_*\mathbb{C} \to S$, a *local system*. The Hodge filtration can be described fiberwise by

$$(\mathscr{F}^p)_t \cong \oplus_{a \geq p} H^{a,n-a}(X_t).$$

We will be particularly interested in the case when the base S is one dimensional. Suppose now S is an open curve and the family π extends to a flat family over a proper curve $\bar{S}$. The vector bundle $R^n\pi_*\mathbb{C} \otimes \mathscr{O}_S$ extends to a vector bundle $\mathscr{H} \to \bar{S}$ whose fiber over t in S consists of the middle cohomology group $H^n(X_t)$. While it is not true that ∇^{GM} extends to a connection on all of $\mathscr{H}$, the singularities which arise are at worst a regular singularities [14]. This means that after choosing local coordinates, the connection matrix acquires at worst logarithmic poles at points of $\bar{S} \setminus S$. Nevertheless we may still speak of flat (multi-valued) sections of ∇^{GM}, controlled by the monodromy.

Let $\{\gamma_i\}$ be a basis of $H_n(X_{t_0})$. Since $\pi : \mathscr{X} \to S$ is smooth, it is a locally trivial fibration and n-cycles γ_i can be extended to *locally constant* cycles $\gamma_i(t)$. Let ω_t be a (local) section of $\mathscr{H}$. The functions $\int_{\gamma_i(t)} \omega_t$ are called the *periods* and by the local constancy of $\gamma_i(t)$

$$\frac{d}{dt}\left(\int_{\gamma_i(t)} \omega_t\right) = \int_{\gamma_I(t)} \nabla_t^{GM} s(t).$$

The periods satisfy the *Picard–Fuchs equations*, defined as follows. Taking successive derivatives of ω_t with respect to the connection gives a sequence of sections

$$\omega_t, \nabla_t^{GM}\omega_t, \ldots, \left(\nabla_t^{GM}\right)^k \omega_t, \ldots.$$

Because the rank of $\mathscr{H}$ is finite, for some k there will exist a relation between these sections of the form

$$\left(\nabla_t^{GM}\right)^k \omega_t + \sum_{i=0}^{k-1} f_i(t) \left(\nabla_t^{GW}\right)^i \omega_t = 0.$$

The corresponding differential equation

$$\left(\left(\frac{d}{dt}\right)^k + \sum_{i=0}^{k-1} f_i(t) \left(\frac{d}{dt}\right)^i\right) \left(\int_{\gamma(t)} \omega_t\right) = 0 \tag{5.1}$$

is the Picard–Fuchs equation for ω_t. The situation when the dimension of S is greater than one is essentially the same, but (5.1) is replaced by a PDE.

Let $\{\phi_i\}_{i\in I}$ be a basis of sections of $\mathscr{H}$. Then if $\{\gamma_i\}_{i\in I}$ is a basis of locally constant n-cycles, we can write the fundamental solution matrix of the Gauss-Manin connection in coordinates as

$$S = \left(s_{ij}\right) \text{ with } s_{ij} = \int_{\gamma_j} \phi_i.$$

With this choice of basis, we see that the i^{th} row of S gives the periods for the section ϕ_i.

Remark V.1. In the literature, often (but not always) the term *periods* are reserved for the case when $\phi(t)$ is a holomorphic n-form, i.e. a section of $\mathscr{F}^n$, and Picard–Fuchs equations are defined only for periods in this restricted sense. Here, we choose to use these terms in the more general sense described above. Note, however, by the results in [5], for Calabi–Yau threefolds the general Picard–Fuchs equations can be determined from the restricted ones.

Definition V.2. Let U denote the Kuranishi space of the Calabi-Yau n-fold X. For the purpose of this paper, we define the (genus zero part of) *B model of X* as the vector bundle $\mathscr{H} \to U$ with the natural (flat) fiberwise pairing and the Gauss–Manin connection.

5.2 Griffiths–Dwork method

Let us assume now that the family X_t is a family of hypersurfaces defined by homogeneous polynomials Q_t of degree d in $\mathbb{P}^{n+1}$. In this case the *Griffiths–Dwork method* can be employed to explicitly calculate the Picard–Fuchs equations. We summarize the relevant results of [18] here.

The method relies on Griffiths' work in [18] showing that one can calculate the period integrals on X_t in terms of *rational* forms on $\mathbb{P}^{n+1}$. *For the time being, let us fix t and suppress it in the notation.* Griffiths first shows that in fact any class Ω in $H^{n+1}(\mathbb{P}^{n+1} \setminus X)$ can be represented in cohomology by a *rational* $n+1$ form. In particular, let Ω_0 be the canonical $n+1$-form on $\mathbb{P}^{n+1}$: $\Omega_0 = \sum_{i=0}^{n+1}(-1)^i x_i dx_0 \cdots \hat{dx_i} \cdots dx_{n+1}$. We can represent any class Ω by a rational form with poles along X,

$$\Omega = \frac{P(x)}{Q(x)^k}\Omega_0$$

where $P(x)$ is a homogeneous polynomial with degree $kd-(n+2)$.

The rational $n+1$ forms are then related to regular n forms on X via the residue map. More precisely, let $A^n_k(X)$ denote the space of rational $(n+1)$-forms on $\mathbb{P}^{n+1}$ with poles of order at most k on X, and let

$$\mathcal{H}_k(X) := A^{n+1}_k(X)/dA^n_{k-1}(X).$$

This gives an obvious filtration

$$\mathcal{H}_1(X) \subset \mathcal{H}_2(X) \subset \cdots \subset \mathcal{H}_{n+1}(X) =: \mathcal{H}(X).$$

This description of rational forms interacts nicely with the Hodge filtration F^p of the *primitive classes*. Griffiths proves that the following diagram is commutative:

$$
\begin{array}{ccccccc}
\mathcal{H}_1(X) & \subset & \mathcal{H}_2(X) & \subset & \cdots & \subset & \mathcal{H}_{n+1}(X) \\
\downarrow \text{Res} & & \downarrow \text{Res} & & & & \downarrow \text{Res} \\
F^n & \subset & F^{n-1} & \subset & \cdots & \subset & F^0
\end{array}
\tag{5.2}
$$

and that each vertical arrow is surjective. In particular,

$\mathcal{H}_{k+1}(X)/\mathcal{H}_k(X) \cong F^{n-k}/F^{n-k+1}$.

Now, for each n-cycle γ in $H_n(X)$, let

$$T : H_n(X) \to H_{n+1}(\mathbb{P}^{n+1} \setminus X)$$

be the *tube map* where $T(\gamma)$ is a sufficiently small S^1-bundle around γ in $\mathbb{P}^{n+1} \setminus X$. Griffiths then shows that the tube map is *surjective* in general and also *injective when n is odd*.

Theorem V.3. *All primitive classes on X can be represented as residues of rational forms on* $\mathbb{P}^{n+1}$ *with poles on X. This representation is unique when n is odd.*

This follows from the surjectivity/injectivity of Res and T, as well as the residue formula

$$\frac{1}{2\pi i}\int_{T(\gamma)} \Omega = \int_\gamma \text{Res}(\Omega).$$

Next Griffiths relates the rational forms to the Jacobian ring. Let

$J(Q) = \langle \partial Q/\partial x_0, \ldots, \partial Q/\partial x_{n+1} \rangle$ be the Jacobian ideal of Q. The key relationship between rational forms is given by the following formula ((4.5) in [18])

$$\frac{\Omega_0}{Q(x)^k}\sum_{j=0}^{n+1} B_j(x)\frac{\partial Q(x)}{\partial x_j} = \frac{1}{k-1}\frac{\Omega_0}{Q(x)^{k-1}}\sum_{j=0}^{n+1}\frac{\partial B_i(x)}{\partial x_j} + d\phi, \tag{5.3}$$

where $\phi \in A^n_{k-1}$. Thus, the order of the pole of a form $\frac{P(x)}{Q(x)^k}\Omega_0$ can be lowered if and only if $P(x)$ is contained in $J(Q)$. By identifying the form $\text{Res}\left(\frac{P(x)}{Q(x)^k}\Omega_0\right)$ with the homogeneous polynomial P, one obtains the following theorem.

Theorem V.4.

$$\mathbb{C}[x_0, \ldots, x_{n+1}]_{dk-n-1}/J(Q) \cong F^{n-k}/F^{n+1-k} \subseteq PH^{n-k,k}(X). \tag{5.4}$$

The above results allow one to explicitly calculate the Picard–Fuchs equations for certain families of forms ω_t on X_t. As before, let X_t be a family of hypersurfaces defined by a degree d homogeneous family of polynomials Q_t. Then we can represent a family of forms as $\omega_t = \mathrm{Res}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right)$. Let γ_t be a locally constant n cycle as before, then

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\gamma_t}\omega_t =& \frac{\partial}{\partial t}\int_{\gamma_t}\mathrm{Res}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right) = \frac{\partial}{\partial t}\int_{T(\gamma_t)}\frac{P_t(x)}{Q_t(x)^k}\Omega_0 \\ =& \int_{T(\gamma_t)}\frac{\partial}{\partial t}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right) = \int_{\gamma_t}\mathrm{Res}\left(\frac{\partial}{\partial t}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right)\right).\end{aligned}$$

The third equality follows because a small change in $T(\gamma(t))$ will not change its homology class. In other words, letting ∇^{GM} denote the Gauss–Manin connection,

$$\nabla_t^{GM}\mathrm{Res}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right) = \mathrm{Res}\left(\frac{\partial}{\partial t}\left(\frac{P_t(x)}{Q_t(x)^k}\Omega_0\right)\right),$$

allowing one to obtain the Picard–Fuchs equations of ω_t via explicit calculations of the polynomials (in the Jacobian rings). An explicit example is given in the next section.

CHAPTER VI

B model of the Fermat quintic M

We now turn to the specific case of the Fermat quintic threefold M in $\mathbb{P}^4$. It has been shown (see e.g. [3]) that the Hodge diamonds of M and $\mathcal{W}$ are mirror symmetric

$$h^{p,q}(M) = h^{3-p,q}(\mathcal{W}).$$

In particular, the deformation family of $\mathcal{W}$ is one-dimensional while for M the deformation is 101 dimensional.

Recall in our study of the A model of $\mathcal{W}$, we restrict the Dubrovin connection (i.e. Frobenius structure) to to the "small" parameter t corresponding to the hyperplane class H. In the following discussions of the complex moduli of M, we will also study the full period matrix for the Gauss–Manin connection, but restricted to a particular deformation parameter.

Let

$$Q_\psi(x) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - \psi x_0 x_1 x_2 x_3 x_4, \tag{6.1}$$

and define the family $M_\psi = \{Q_\psi(x) = 0\} \subset \mathbb{P}^4$.

6.1 Picard–Fuchs equations for M_ψ

In the specific case of the family M_ψ, there is a "diagrammatic technique", pioneered in [7] and refined in [15], which utilizes the symmetry of Q_ψ and P to simplify the bookkeeping.

The starting point is the equation (5.3). Consider the rational form

$$\omega_\psi = \frac{P(x)}{Q_\psi(x)^k}\Omega_0, \quad P(x) = x_0^{r_0}\cdots x_4^{r_4}, \quad \text{with} \sum_{i=0}^{4} r_i = 5(k-1).$$

Fix i between 0 and 4, and set $B_j = \delta_{ij}x_iP(x)$ for $0 \le j \le 4$. Noting that

$$\frac{\partial}{\partial x_i}Q_\psi(x) = 5x_i^4 - \psi x_0\cdots\hat{x}_j\cdots x_4,$$

and applying (5.3) with these choices of B_j (and k replaced by $k+1$), we arrive at

$$5\int_{T(\gamma)}\frac{(x_i^5)\,P}{Q_\psi^{k+1}}\Omega_0 - \psi\int_{T(\gamma)}\frac{(x_0\ldots x_4)\,P}{Q_\psi^{k+1}}\Omega_0 = \frac{1+r_i}{k}\int_{T(\gamma)}\frac{P}{Q_\psi^k}\Omega_0 \tag{6.2}$$

for any choice of cycle $\gamma \in H_n(X)$. Note, however, that there is a degenerate case in the above setting: in the case when $P(x)$ is independent of x_i, let $B_j = \delta_{ij}P(x)$. Then in (5.3) we get

$$5\int_{T(\gamma)}\frac{(x_i^4)\,P}{Q_\psi^{k+1}}\Omega_0 - \psi\int_{T(\gamma)}\frac{(x_0\ldots\hat{x}_i\ldots x_4)\,P}{Q_\psi^{k+1}}\Omega_0 = 0. \tag{6.3}$$

We can interpret this equation as allowing $r_i = -1$ in (6.2).

Furthermore, $\frac{\partial}{\partial\psi}Q_\psi = -x_0\cdots x_4$, and so we have the relationship

$$\frac{\partial}{\partial\psi}\int_{T(\gamma)}\frac{P}{Q_\psi^k}\Omega_0 = k\int_{T(\gamma)}\frac{(x_0\cdots x_4)\,P}{Q_\psi^{k+1}}\Omega_0. \tag{6.4}$$

The authors in [7, 15] apply (6.2) (6.3) and (6.4) recursively to get relations of the periods, hence the Picard–Fuchs equations. For convenience of bookkeeping, one can keep track of the polynomial $P(x)$ by its exponents $(r_0,\ldots,r_4)$. (6.2) can be understood symbolically as a relation between $(r_0, \ldots, r_4)$, $(r_0, \ldots, r_i + 5, \ldots, r_4)$ and $(r_0 + 1, \ldots, r_4 + 1)$.

Consider for example the case $P = 1$ corresponding to $(0, \ldots, 0)$. Applying (6.4) four times, one may write the fourth derivative of $(0, \ldots, 0)$ as a multiple of $(4, \ldots, 4)$. This may then be related to $(5, 5, 5, 5, 0)$ by (6.3). Applying (6.2) to relate $(r_0, \ldots, r_4)$ to a linear combination of $(r_0, \ldots, r_i - 5, \ldots, r_4)$ and $(r_0 + 1, \ldots, r_i - 4, \ldots, r_4 + 1)$ repeatedly, one can reduce to terms with $r_i \leq 4$ for all i. In fact, eventually all terms will be of the form $\{(r, r, \ldots, r)\}$ for $r = 0, \ldots, 4$. This can be seen by noting that *none of* (6.2), (6.2) *or* (6.4) *changes* $r_i - r_j \pmod 5$. Hence, we have found a relation between the fourth derivative of $(0, \ldots, 0)$ and $\{(r, \ldots, r)\}$ for $r = 0, \ldots, 4$. By (6.4), the various $(r, \ldots, r)$ are r-th derivatives of $(0, \ldots, 0)$, and we obtain a fourth order ODE in ψ for the period corresponding to $P = 1$. (See Table 1 below for the equation.) Other cases can be computed similarly. These arguments can be illuminated by diagrams in [7, 15], hence the name *diagrammatic technique*.

Now we apply this method to calculate the Picard–Fuchs equations for the period integrals we are interested in. For every $g = (r_0, \ldots, r_4) \in G$ (defined in Section 3.1), define

$$P_g(x) = x_0^{r_0} \cdots x_4^{r_4}$$

and

$$k = \left(\sum_{i=0}^{4} \frac{r_i}{5} \right) + 1 = \operatorname{age}(g) + 1.$$

We will consider specific families of the form

$$\omega_g(\psi) := \operatorname{Res} \left(\frac{\psi P_g(x)}{Q_\psi(x)^k} \Omega_0 \right) \tag{6.5}$$

For our purposes, *it will be sufficient to consider families ω_g such that P_g satisfies* age$(g) \leq 1$ *(i.e.* $\sum_{i=0}^{4} r_i \leq 5$*) and at least two of the* r_i*'s equal* 0. We remark that these conditions on g match the conditions on A model computation in Section IV perfectly. In Claim VII.7 it is shown that the derivatives of these families generate all of $\mathscr{H}$.

Table 1 below gives the Picard–Fuchs equation satisfied by each of the above-mentioned forms. We label the forms by the corresponding 5-tuple $g = (r_0, \ldots, r_4)$. Note that permuting the r_i's does not effect the differential equation, so we do not distinguish between permutations. Here

$$t = -5\log(\psi).$$

The same computation was done in [7, 15]. We note however that *there are several*

type	Picard–Fuchs equation
$(0,0,0,0,0)$	$(\frac{d}{dt})^4 - 5^5e^t(\frac{d}{dt}+\frac{1}{5})(\frac{d}{dt}+\frac{2}{5})(\frac{d}{dt}+\frac{3}{5})(\frac{d}{dt}+\frac{4}{5})$
$(0,0,0,1,4)$	$(\frac{d}{dt})^2 - 5^5e^t(\frac{d}{dt}+2/5)(\frac{d}{dt}+3/5)$
$(0,0,0,2,3)$	$(\frac{d}{dt})^2 - 5^5e^t(\frac{d}{dt}+1/5)(\frac{d}{dt}+4/5)$
$(0,0,1,1,3)$	$(\frac{d}{dt})(\frac{d}{dt}-1/5) - 5^5e^t(\frac{d}{dt}+1/5)(\frac{d}{dt}+3/5)$
$(0,0,2,2,1)$	$(\frac{d}{dt})(\frac{d}{dt}-2/5) - 5^5e^t(\frac{d}{dt}+1/5)(\frac{d}{dt}+2/5)$

Table 6.1: The Picard–Fuchs equations for forms ω_g.

differences between the period integrals we consider, and those of [15]. First, our family M_ψ differs from that in [15] by a factor of 5 in the first term. Second, the forms we consider (6.5) differ slightly from those considered in [15] by an extra factor of ψ in the numerator (see remark VI.1). Finally, our final equations use different coordinates than in [15]. However the same methods used in their paper can easily be modified to obtain the formulas we present here.

Remark VI.1. The factor of ψ in the numerator of (6.5) might appear unnatural at the first glance, but it can be considered as a way to change the form of the Picard-Fuchs equation, as

$$\frac{d}{dt}e^{-t/5}f(t) = e^{-t/5}\left(-\frac{1}{5}+\frac{d}{dt}\right)f(t).$$

In the comparison of the A model and B model this modification will simplify the I-functions from both sides. It is also used in the Mirror Theorem for the Fermat quintic.

6.2 B model I-functions

We can solve the above Picard-Fuchs equations with hypergeometric series. As in Section III, we will organize these solutions in the form of an I-function. For each of the above forms ω_g, I_g^B will be a function taking values in $H^*_{CR}(\mathcal{W}) \cong H^*(I\mathcal{W})$, whose components give solutions to the corresponding Picard–Fuchs equation.

Proposition VI.2. *For the g listed in table 6.1, the components of $I_g^B(t,1)$ give a basis of solutions to the Picard–Fuchs equations for ω_g, where $I_g^B(t,z)$ is given below.*

(i) If $g = e = (0,0,0,0,0)$,

$$I_e^B(t,z) = e^{tH/z}\left(1+\sum_{\langle d\rangle=0} e^{dt}\frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^5}\right) \tag{6.6}$$

(ii) If $g = (0,0,0,r_1,r_2)$*,*

$$I_g^B(t,z) = e^{tH/z}\mathbb{1}_g$$

$$\left(1 + \sum_{\langle d\rangle = 0} e^{dt} \frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle = 0}}(H+bz)^3 \prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{r_2}{5}\right\rangle}}(H+bz) \prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{r_1}{5}\right\rangle}}(H+bz)}\right) \tag{6.7}$$

(iii) If $g = (0,0,r_1,r_1,r_2)$*, let* $g_1 = (-r_1,-r_1,0,0,r_2-r_1) (\operatorname{mod} 5)$*. Then*

$$I_g^B(t,z) = \tag{6.8}$$

$$e^{tH/z}\mathbb{1}_g \left(1 + \sum_{\langle d\rangle = 0} e^{dt} \frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle = 0}}(H+bz)^2 \prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{3r_2}{5}\right\rangle}}(H+bz)^2 \prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{2r_1}{5}\right\rangle}}(H+bz)}\right)$$

$$+\, e^{tH/z}\mathbb{1}_{g_1} \left(\sum_{\langle d\rangle = \left\langle \frac{r_1}{5}\right\rangle} e^{dt} \frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{r_1}{5}\right\rangle}}(H+bz)^2 \prod_{\substack{0<b\le d\\ \langle b\rangle = 0}}(H+bz)^2 \prod_{\substack{0<b\le d\\ \langle b\rangle = \left\langle \frac{r_2}{5}\right\rangle}}(H+bz)}\right)$$

Remark VI.3. Note that the functions $I_g^B(t,z)$ in equations (6.6), (6.7), and (6.8), are supported on spaces of dimension 3, 1, and 0 respectively (in particular, $H \equiv 0$ in (6.8)). So for each g, the number of components of $I_g^B(t,z)$ equals the order of the corresponding Picard–Fuchs equation as desired.

CHAPTER VII

Mirror Theorem for the mirror quintic: $A(\mathcal{W}) \equiv B(M)$

In this section, we will show the "mirror dual" version of (the mathematical version of) the *mirror conjecture* by Candelas–de la Ossa–Greene–Parkes [6]. More specifically, we will show that the A model of $\mathcal{W}$ is equivalent to the B model of M, up to a mirror map.

We start in 7.1 by stating a "classical" mirror theorem relating the GWT of $\mathcal{W}$ with the periods of M_ψ on the level of generating functions. This is exactly analogous to Givental's original formulation in [16]. In 7.2 we give a brief explanation of how Givental's original statement of the mirror theorem implies a full correspondence between the A model of M and the B model of $\mathcal{W}$. Finally in 7.3 we use similar methods as in 7.2 to prove a mirror theorem equating the A model of $\mathcal{W}$ to the B model of M.

7.1 A correspondence of generating functions

We will first show that the I-functions I_g^A of the A model of $\mathcal{W}$ (Definition IV.4) are identical to the I-functions I_g^B of the B model of M_ψ defined in Section 6.2.

Remark VII.1. Note that in the formula I_g^A, the Novikov variable q always appears next to e^t. There is therefore no harm in setting $q = 1$. We apply this specialization in what follows.

Proposition VII.2. *Let $g = (r_0, \dots, r_4) \in G$ satisfies the conditions* $\mathrm{age}(g) \le 1$ *and that at least two of r_i's are equal to zero. We have an A-interpretation of g as parameterizing a component of $\mathcal{W}_g$ in $I\mathcal{W}$. We have also a B-interpretation of g in ω_g* (6.5) *where P_g denote the polynomial $x_0^{r_0} \cdots x_4^{r_4}$. Then*

$$I_g^A(t,z) = I_g^B(t,z).$$

Proof. This follows from a direct comparison of formulas (4.15), (4.16), and (4.17) from Corollary IV.8 with formulas (6.6), (6.7), and (6.8) respectively. □

Combining Proposition VII.2 with Theorem IV.5, we conclude that some periods from VHS of M correspond to the Gromov–Witten invariants of $\mathcal{W}$.

Corollary VII.3 ([20])**.** *For each $g = (r_0, \dots, r_4) \in G$ such that* $\mathrm{age}(g) \le 1$ *and $\mathcal{W}_g$ is nonempty (i.e. at least two r_i's vanish), we have*

$$J_g^{\mathcal{W}}(\tau(t), z) = \frac{I_g^B(t,z)}{H_g(t)} \qquad \textit{where } \tau(t) = \frac{G_0(t)}{F_0(t)}.$$

In other words, under the mirror map

$$t \mapsto \tau = \frac{G_0(t)}{F_0(t)},$$

the periods of $\frac{\omega_g}{H_g(t)}$ are equal to the coefficients of $J_g^{\mathcal{W}}(\tau, 1)$.

This theorem should be viewed as an analogue of Givental's original mirror theorem VII.4 stated below.

7.2 Mirror theorem for the Fermat quintic revisited

To obtain some insight into the full correspondence, we return to the "classical" mirror theorem for the Fermat quintic threefold. While this is not strictly necessary for the logical flow of the proof, we feel that it illuminates our approach in a simpler setting. We also strive to clarify certain points which are not entirely clear in the literature.

Let $J^M(t,z)$ denote the small J-function for M where t is the coordinate of $H^2(M)$ dual to the hyperplane class H. Let $\mathcal{W}_\psi$ denote the one dimensional deformation family defined by the vanishing of Q_ψ (see (6.1)) in $\mathcal{Y}$.

$$\mathcal{W}_\psi := \{Q_\psi(x) = 0\} \subset \mathcal{Y}. \tag{7.1}$$

Let

$$\omega = \operatorname{Res}\left(\frac{\psi\Omega_0}{Q_\psi(x)}\right).$$

As in section VI there exists an $H^*(M)$-valued I-function, $I^B_{\mathcal{W}_\psi}(t,z)$, such that the components of $I^B_{\mathcal{W}_\psi}(t,1)$ give a basis of solutions for the Picard–Fuchs equations for ω_ψ, where $t = -5\log\psi$.

Theorem VII.4 (mirror theorem [16, 21, 4])**.** *There exist explicitly determined functions $F(t)$ and $G(t)$, such that F is invertible, and*

$$J^M(\tau(t),z) = \frac{I^B_{\mathcal{W}_\psi}(t,z)}{F(t)} \qquad \textit{where } \tau(t) = \frac{G(t)}{F(t)}.$$

We will show how Theorem VII.4 implies a correspondence between the fundamental solution matrix of the Dubrovin connection for M and that of the Gauss–Manin connection for $\mathcal{W}_\psi$. In order to emphasize the symmetry between the A model and B model, we will denote the respective pairings as $(-,-)^A$ and $(-,-)^B$.

Let

$$s = \psi^{-5},$$

and consider the flat family $\mathcal{W}_s$ over $S = \operatorname{Spec}(\mathbb{C}[s])$. Then if we let $t = \log(s)$, $I^B_{cW_s} = I^B_{cW_\psi}$. In the Calabi–Yau case, the H expansion of I^B always occurs in the form of a function of H/z, in particular $I^B_{\mathcal{W}_s}$ is homogeneous of degree zero if one sets $\deg(z) = 2$. The same is true of J^M. Thus, one may set $z = 1$ without loss of information. $I^B_{\mathcal{W}_s}(t,1)$ gives a basis of solutions for the Picard–Fuchs equations of ω. In other words after an appropriate choice of basis $\{s^B_0(t), \ldots, s^B_3(t)\}$ of solutions of ∇^{GM},

$$(s^B_i(t), \omega)^B = \left(I^B_{\mathcal{W}_s}\right)_i(t,1),$$

where $\left(I^B_{\mathcal{W}_s}\right)_i(t,z)$ is the H^i coefficient of $I^B_{\mathcal{W}_s}(t,z)$.

By the same argument, if we choose an appropriate basis $\{s^A_0(\tau), \ldots, s^A_3(\tau)\}$ of solutions for ∇^z, Section II shows that the coefficients $J^M_i(\tau,1)$ of the function $J^M(\tau,1)$ give us the functions

$$(s^A_i(\tau), 1)^A = J^M_i(\tau,1).$$

Thus we can interpret Theorem VII.4 as saying that after choosing correct bases of flat sections and applying the mirror map

$$t \mapsto \tau = \frac{G(t)}{F(t)},$$

we have the equality

$$(s^B_i(t), \omega/F(t))^B = \frac{\left(I^B_{\mathcal{W}_s}\right)_i(t,1)}{F(t)} = J^M_i(\tau,1) = (s^A_i(\tau),1)^A.$$

To show the full correspondence between the solution matrix for the Dubrovin connection for M and the solution matrix of the Gauss–Manin connection on S, we must find a basis $\phi_0, \ldots, \phi_3$ of sections of $\mathscr{H}$ and a basis $T_0, \ldots, T_3$ of sections of $H^{even}(M)$ such that for all i and j,

$$(s^B_i, \phi_j)^B = (s^A_i, T_j)^A \tag{7.2}$$

As one might expect, we set $\phi_0 = \omega/F(t)$ and $T_0 = 1$.

Claim VII.5.

$$\phi_j = \left(\nabla_t^{GM}\right)^j \phi_0 \text{ for } 0 \le j \le 3$$

gives a basis of sections for $\mathscr{H}$.

Proof. This follows from standard Hodge theory for Calabi–Yau threefolds, but in this case can be explicitly calculated.

$$\begin{aligned}
\nabla_t^{GM}\phi_0 =& \frac{d}{dt}\left(\frac{1}{F(t)}\right)\omega + \frac{1}{F(t)}\nabla_t^{GM}\omega \\
=& -\frac{F'(t)}{F(t)}\phi_0 + \frac{1}{F(t)} \operatorname{Res}\left(\frac{d}{dt}\frac{\psi\Omega_0}{Q_\psi}\right) \\
=& -\frac{F'(t)}{F(t)}\phi_0 + \frac{1}{F(t)} \operatorname{Res}\left(s\frac{d}{ds}\frac{\psi\Omega_0}{Q_\psi}\right) \\
=& -\frac{F'(t)}{F(t)}\phi_0 + \frac{1}{F(t)} \operatorname{Res}\left(\frac{-\psi}{5}\frac{d}{d\psi}\frac{\psi\Omega_0}{Q_\psi}\right) \\
=& -\frac{F'(t)}{F(t)}\phi_0 + \frac{-\psi}{5F(t)} \operatorname{Res}\left(\frac{\Omega_0}{Q_\psi} + \frac{x_0\cdots x_4}{Q_\psi^2}\Omega_0\right).
\end{aligned} \tag{7.3}$$

Because of the last term in the above sum, the image of $\left(\nabla_t^{GM}\right)\phi_0$ in $\mathscr{F}^2/\mathscr{F}^3$ is nonzero by (5.4). Similarly, the image of $\left(\nabla_t^{GM}\right)^j\phi_0$ in $\mathscr{F}^{3-j}/\mathscr{F}^{3+1-j}$ for $1 \le j \le 3$ is nonzero, thus the sections $\phi_0, \ldots, \phi_3$ must be linearly independent. □

Note that

$$\begin{aligned}
&(s_i^B, \phi_1)^B = (s_i^B, \nabla_t^{GM}\phi_0)^B = \frac{\partial}{\partial t}(s_i^B, \phi_0)^B = \\
&\frac{\partial}{\partial t}(s_i^A, T_0)^A = \left(\frac{\partial \tau}{\partial t}\right)\frac{\partial}{\partial \tau}(s_i^A, T_0)^A = \left(s_i^A, \left(\frac{\partial\tau}{\partial t}\right)\nabla_\tau^z T_0\right)^A.
\end{aligned} \tag{7.4}$$

Therefore, if we set

$$T_1 = \frac{\partial(G/F)}{\partial t}\nabla_\tau^z T_0,$$

we have the relationship

$$(s_i^B, \phi_1)^B = (s_i^A, T_1)^A.$$

If we similarly set

$$T_k = \frac{\partial(G/F)}{\partial t} \nabla^z_\tau T_{k-1},$$

(7.2) follows.

This shows that after identifying the section ϕ_i with T_i the mirror map lifts to an isomorphism of vector bundles, which preserves the connection. Indeed, the fundamental solution of the Gauss–Manin connection is a 4 by 4 matrix, where 4 is the rank of $H^3(\mathcal{W})$. On the other hand, the fundamental solution of the Dubrovin connection is also a 4 by 4 matrix, where 4 is the rank of $H^{even}(M)$. We recall that the J-function can be thought of as the first row vectors of the fundamental solution matrix, as discussed in Section II. The above discussion shows that we can extend the correspondence between the first row of the fundamental solution to the full fundamental solution.

We summarize the above in the following theorem.

Theorem VII.6. *The fundamental solutions of the Gauss–Manin connection for $\mathcal{W}_s$ are equivalent, up to a mirror map, to the fundamental solutions of the Dubrovin connection for M, when restricted to $H^2(M)$.*

7.3 Mirror theorem for the mirror quintic

In this subsection, we will extend the partial correspondence in Section 7.1 between the periods of M_ψ and the A model of $\mathcal{W}$ to the full correspondence, generalizing the ideas in Section 7.2.

Similar to the above, consider the flat family M_s over $S = \mathrm{Spec}(\mathbb{C}[s])$ defined by (6.1), where $s = e^t = \psi^{-5}$. Corollary VII.3 states that some periods of M_s correspond to Gromov–Witten invariants on $\mathcal{W}$. We would like to extend this result to all periods.

First, we must choose a basis of sections of $\mathscr{H} \to S$. Let ω_e denote the holomorphic family of (3,0)-forms corresponding to $g = e = (0, \ldots, 0)$ in (6.5). It is no longer true that derivatives of $\omega_e / F_0(t)$ with respect to the Gauss–Manin connection generate a basis of sections of $\mathscr{H}$, thus it becomes necessary to consider the other forms ω_g satisfying the conditions formulated in Corollary VII.3. Namely, let $\phi_e = \omega / F_0(t)$ and let $\phi_g = \omega_g / H_g(t)$ where g satisfies $\mathrm{age}(g) = 1$. Consider the set of sections

$$\{\phi_0, \nabla_t^{GM}\phi_0, (\nabla_t^{GM})^2\phi_0, (\nabla_t^{GM})^3\phi_0\} \cup \{\phi_g, \nabla_t^{GM}\phi_g\}.$$

Claim VII.7. *These forms comprise a basis of the Hodge bundle $\mathscr{H}$.*

Proof. The proof is similar to Claim VII.5. We note that in the last four rows in Table 1, corresponding to age one type, the dimensions are 20, 20, 30, and 30. Thus $|\{\phi_g\}| = 100$, and there are exactly 204 forms in the above set. One can check via (5.4) and another argument like in (7.3) that these sections are in fact linearly independent. $\square$

Then, as in (7.4) the periods of $(\nabla_t^{GM})^k\phi_0$ correspond to the derivatives $\left(\frac{d}{dt}\right)^k J_e^{\mathcal{W}}(\tau, 1)$, and the periods of $\nabla_t^{GM}\phi_g$ correspond to $\left(\frac{d}{dt}\right) J_g^{\mathcal{W}}(\tau, 1)$.

Let $T_0 = 1$, and $T_k = \frac{\partial(G_0/F_0)}{\partial t}\nabla_\tau^z T_{k-1}$ for $0 \le k \le 3$. Let $T_g = \mathbb{1}_g$ and $T_g' = \frac{\partial(G_0/F_0)}{\partial t}\nabla_\tau^z \mathbb{1}_g$. Then if we choose the correct basis of flat sections $\{s_i^B\}$ and $\{s_i^A\}$, we have that

$$(s_i^B, (\nabla_t^{GM})^k\phi_0)^B = (s_i^A, T_k)^A,$$

$$(s_i^B, \phi_g)^B = (s_i^A, T_g)^A \text{ and}$$

$$(s_i^B, \nabla_t^{GM}\phi_g)^B = (s_i^A, T_g')^A.$$

This implies that the set

$$\{T_0, T_1, T_2, T_3\} \cup \{T_g, T_g'\},$$

is a basis of $TH^{even}_{CR}(\mathcal{W})$, and that with these choices of bases the solution matrices for the two respective connections are identical after the mirror transformation. Thus we obtain the full correspondence.

In terms of the language of Theorem VII.6, we can formulate our final result in the following form. On the side of the A model of $\mathcal{W}$, let t be the dual coordinate of H; on the side of B model of M_s, let $t = \log(s)$. Then we have

Theorem VII.8 ([20])**.** *The fundamental solution matrix of the Gauss–Manin connection ∇^{GM}_t for M_s is equal, up to a mirror map, to the fundamental solution matrix of the Dubrovin connection ∇^z_t for $\mathcal{W}$ restricted to $tH \in H^2(\mathcal{W})$.*

Remark VII.9*.* Even though the base direction is constrained to one dimension instead of the full 101-dimension deformation space, our fundamental solutions are full 204 by 204 matrices, as both ranks of $H^3(M)$ and $H^{even}(\mathcal{W})$ are 204.

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