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SOCIAL INDIFFERENCE CURVES AND AGGREGATE DEMAND

by

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Abstract. Suppose that the government continually maximizes a fixed social welfare function through the use of various policy instruments. (These instruments may be of a variety of sorts _other_ than classical lump sum transfers such as indirect commodity taxes, nonlinear income taxes and so on.) Then under certain conditions the aggregate demand data for this economy will appear as though it were generated by a single representative consumer.

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It is a common practice in applied welfare economics to use aggregate demand data to construct measures of economic welfare. This is usually done not because it is thought to be a correct or particularly desirable procedure; if we had access to data on individual consuming units most economists would agree that analysis of this individual data would be preferable to analysis of aggregate data. But individual data is often not available, so we typically have no real alternative but to use the aggregate data.

In this paper I want to consider some justifications for this practice of treating aggregate data as though it came from a representative individual, and in particular, the practice of using aggregate data to make welfare judgements. In my mind, there is no question that analysis of individual data is to be preferred to analysis of aggregate data. But if that possibility is not open to us, it is of interest to consider the best possible case for the treatment of data that is available.

1. The Representative Consumer Model

Suppose that we have T observations on k-vectors of prices and quantities (p^t, x^t) $t = 1, \ldots, T$. We will say a nonsatiated utility function $u(x)$ <u>rationalizes</u> this data if $u(x^t) \geqq u(x)$ for all x such that $p^t x^t \geqq p^i x$, for all $t = 1, \ldots T$. (Any finite set of data can be rationalized by a constant utility function; hence the nonsatiation requirement is needed to avoid this trivial case.)

Suppose that we consider a situation with n consumers who generate choices $(x_1^t, \ldots, x_n^t)$ when faced with prices p^t, and these individual are consistent with individual utility maximization; that is, there exist n nonsatiated individual utility functions $u_1(x), \ldots, u_n(x)$ that rationalize the data $(p^t, x_1^t), \ldots, (p^t, x_n^t)$ respectively.

We now consider the aggregate demands X^t, $t = 1, \ldots, n$ defined in the obvious way by:

$$X^t = \sum_{i=1}^{n} x_i^t \qquad t = 1, \ldots, T \tag{1}$$

and the aggregate data (P^t, X^t) $t = 1, \ldots, T$ where $P^t \equiv p^t$ for typographic convenience. We might ask when does there exist a "representative" utility function $U(x)$ that rationalizes the observed aggregate data?

There are basically three answers to this question that have been suggested in the literature.[1] They are:

(1) All consumers have identical preferences and incomes.

(2) All consumers have homothetic preferences and proportional endowments.

(3) In each period there is a lump sum transfer of income between consumers that maximizes some fixed concave social welfare function.

The first condition needs little comment. Although it is quite unlikely that this hypothesis holds exactly, it is perhaps not so unreasonable that it holds approximately. If the aggregate demand data is dominated by a large "middle class" of consumers with roughly similar tastes and incomes, the aggregate data would presumably be rationalizable by the common utility function of these "representative consumers". This is especially true if the categories of goods are very broadly defined categories of goods such as "food", "clothing", "housing" and so on, since it might be expected that at this level of commodity aggregation tastes may be rather similar.

The second condition originates in the work of Gorman (1953) and has subsequently been generalized in a variety of ways by Chipman (1974), Muellbauer (1976), Shapiro (1977) and several others. The particular statement given here is due to Chipman (1974). There seem to be two problems with using this condition as a general rationalization for the representative consumer model. The first is that it is that the conditions are quite nonrobust; if one relaxes the proportional endowments requirement only a small amount, one loses the result. In fact, Mantel (1976) has shown that any excess demand function for k goods (that satisfies certain regularity conditions) can be rationalized by k homothetic consumers with "almost" proportional endowments.

The second problem is that it is easy to show that if condition (2) holds then the aggregate demand data has to be consistent with homotheticity. This is a testable restriction, and it is typically rejected. If one rejects a homothetic specification for the aggregate utility function then one can hardly use (2) as a justification for aggregate demand analysis.

The third condition originates in the work of Samuelson (1956), (1964).[2] In certain ways it seems rather plausible: certainly governments do attempt to redistribute wealth when relative prices change, and there seems to be some consistency to the direction of redistribution. The approximate maximization of some fixed social welfare function may not be such a totally unrealistic hypothesis. It is rather more difficult to accept the hypothesis that this redistribution takes place through lump sum transfers. Instead a variety of instruments - taxes, subsidies, rationing, quotas, etc. - seem to be used to achieve the policy goals of social decision makers. One of the goals of this paper is to extend the Samuelson result to situations where a variety of instruments are used to maximize welfare. We proceed in the following manner.

In Section 2 we present a brief proof of the Samuelson result. In Section 3 we pose the welfare maximization problem in a somewhat different way to expose the essence of the necessary assumptions. The crucial condition turns out to involve the concavity of the "social utility function"; this curvature property in turn rests on the flexibility of the instruments at the disposal of the government. We examine these issues in Sections 4, 5 and 6. Section 7 describes another set of conditions under which the Samuelson result holds. Finally Section 8 summarizes our main findings.

2. Samuelson's Theorem on Social Indifference Curves

Here we present a formal treatment of Samuelson's original result; the proof technique seems to be new, but it is not particularly difficult. The main trick, due originally to Samuelson, is to use revealed preference theory to show that the aggregate data behaves like individual data. We begin with a brief statement of the revealed preference terminology and results that we will use.

Let (P^t, X^t) $t = 1, \ldots, T$ be some demand data. We say observation t is revealed preferred to observation s, written $X^t \, R \, X^s$ if there is some sequence of observations $(X^i, X^j, \ldots, X^k)$ such that:

$$P^t X^t \geq P^t X^i, \; P^i X^i \geq P^i X^j, \ldots, \; P^k X^k \geq P^k X^s$$

We say a set of data satisfies the Generalized Axiom of Revealed Preference (GARP) if:

$$X^s \, R \, X^t \;\; \underline{\text{implies}} \;\; P^t X^t \leq P^t X^s \tag{2}$$

The generalized Axiom of Revealed Preference is, of course, a generalization of Houthaker's (1950) Strong Axiom of Revealed Preference which in turn rests on Samuelson's (1947) work. GARP is also closely related - in fact equivalent to - Afriat's (1967) condition of cyclical consistency. Afriat's (1967) arguments can be modified slightly to yield the following theorem.

AFRIAT'S THEOREM *The following conditions are equivalent:*

(1) *There exists a nonsatiated utility function that rationalizes the demand data.*

(2) *The demand data satisfies* GARP.

(3) *There exists a nonsatiated, continuous, concave monotonic utility function that rationalizes the data.*

For a proof of this theorem, see Varian (1980), Diewert (1973) or Afriat (1967), (1973), (1976). We have stated the theorem in a way that emphasizes the fact that (1) and (3) are in fact equivalent: if the data can be rationalized at all - i.e. if the data satisfies GARP - then it can be rationalized by a very nice well behaved utility function.

Let us now turn to the social welfare maximization problem. We let $W(x_1,\dots, x_n)$ be the fixed welfare function regarded as a function of the allocation $x = (x_1,\dots, x_n)$. This function does not necessarily have to be of the Bergson -Samuelson form $W(u_1(x_1),\dots, u_n(x_n))$ although this is obviously a special case of some interest.

Let $F(X, t)$ be the transformation function available to society at observation t, where $X = (X^1,\dots, X^k)$ are the aggregate amounts of goods $1,\dots, k$. We assume that the allocation at observation t, $x(t)$, maximizes some fixed social welfare function subject to the technological constraints represented by $F(X, t)$:

$$W(x(t)) \equiv \max_x W(x_1,\dots, x_n) \tag{3}$$

$$\text{s.t.} \quad F(X, t) \leqq 0 \tag{4}$$

$$X = \sum_{i=1}^{n} x_i \tag{5}$$

We will assume that $W(x)$ and $F(X, t)$ are differentiable and that all optima are interior so that we satisfy the familiar first order conditions:

$$\frac{\partial W(x(t))}{\partial x_i^j} - \lambda(t)\frac{\partial F(X(t), t)}{\partial X^j} = 0 \qquad \begin{array}{l} j = 1,\dots, k \\ i = 1,\dots, n \\ t = 1,\dots, T \end{array} \tag{6}$$

where $\lambda(t)$ is a strictly positive Lagrange multiplier. From our point of view the main content of these conditions is that $\partial W(x^t)/\partial x_i^j$ is independent of i - it only depends on the technological conditions involving good j.

To highlight this fact, we define the shadow price of the j^{th} good at time t by:

$$P^j(t) = \partial F(X(t), t)/\partial X^j \tag{7}$$

and let:

$$P(t) = (P^1(t), \ldots, P^k(t)) \tag{8}$$

We suppose that it is these shadow prices (or any vector proportioned to these prices) that are observed in the data (P(t), X(t)), so that (6) can be writted as:

$$\frac{\partial W(x(t))}{\partial x_i^j} = \lambda(t)P^j(t) \tag{9}$$

Note that under conditions of competitive profit maximization, P(t) will simply be the equilibrium producer prices. If the social welfare function is of the Bergson-Samuelson sort and the social planner can use lump sum transfers to effect maximization of this social welfare function, then these producer prices will also be the consumer prices. We will discuss this point further in the next section. At this point we can provide a simple statement and proof of the Samuelson Theorem:

THEOREM 2. Suppose that:

(1) W(x) is a differentiable concave function of x;

and

(2) the data (P^t, X^t) were generated by the welfare maximization problem and therefore satisfies (9).

Then:

(3) The data satisfy the Generalized Axiom of Revealed Preference;

and, therefore

(4) there exists a representative utility function that rationalizes the observed data.

Proof. The concavity requirement on $W(x)$ implies that for all observations t and s,

$$W(x(s)) \leq W(x(t)) + \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{\partial W(x(t))}{\partial x_i^j}(x_i^j(s) - x_i^j(t)) \tag{10}$$

Using (9) we can rewrite this as:

$$W(x(s)) \leq W(x(t)) + \lambda(t) \sum_{j=1}^{k} P^j(t) \sum_{i=1}^{n} (x_i^j(s) - x_i^j(t)) \tag{11}$$

or, in vector notation:

$$W(x(s)) \leq W(x(t)) + \lambda(t)P(t)(X(s) - X(t)) \tag{12}$$

Hence if $P(t)X(t) \geq P(t)X(s)$ then $W(x(t)) \geq W(x(s))$, and if $P(t)X(t) > P(t)X(s)$ then $W(x(t)) > W(x(s))$. Referring to the definition of the revealed preference relation R in (2) we see that: we can repeat this argument for $(i, j, \ldots, k)$ to establish:

$$\text{if} \quad X^t \, R \, X^s, \text{ then } W(x(t)) \geq W(x(s)) \tag{13}$$

It is now easy to see that the data $(P(t), X(t))$ must satisfy GARP. Suppose not; then there are some observations t and s such that:

$$X(t) \, R \, X(s) \text{ and,} \tag{14}$$

$$P(s)\, X(s) > P(s)\, X(t) \tag{15}$$

But (14) implies $W(x(t)) \geq W(x(s))$ and (15) implies $W(x(s)) > W(x(t))$, a contradiction. Hence the data satisfies GARP and the conclusion follow. □

3. The Social Utility Function

Let us seek a generalization of Samuelson's theorem by approaching the welfare maximization problem in a somewhat different way. Let us suppose that the social decision maker can take two sorts of actions in his welfare maximization activities. The first set of actions, A, are purely redistributive actions. An action a in A affects the allocation x - so we write x(a) - but it does not affect society's production possibilities. Examples of purely redistributive actions would be the lump sum transfers of wealth of classical welfare economics, or taxation of consumer goods considered in the recent public finance literature.

The second set of actions, B, are the purely technological actions. An action b in B affects society's production possibilities, so we represent it by a shift in the transformation function F(X, b, t). Examples of purely technological actions might be subsidized research and development, or government provision of other sorts of services or products.

Using this notation, the social welfare maximization problem can be written as:

$$\max_{\substack{a \text{ in } A \\ b \text{ in } B}} W(x(a)) \tag{16}$$

$$F(X, b, t) \leqq 0 \tag{17}$$

$$\sum_{i=1}^{n} x_i(a) = X \tag{18}$$

In writing the problem this way we have implicitly assumed that the redistributive actions a do not influence the technological conditions given by F(X, b, t). This will allow us to decompose the social welfare maximization problem in a useful way.

Let us define the "social utility function" U(X) by:

$$U(X) = \max_{a \text{ in } A} W(x(a)) \tag{19}$$

$$\text{s.t.} \sum_{i=1}^{n} x_i(a) = X \tag{20}$$

The social utility function as defined here might be called an "indirect welfare function". It measures the maximum level of welfare as a function of the (fixed) amounts of goods 1, ..., k available.

We can now represent the overall maximization problem facing the social planner by:

$$\max_{X,\, b} U(X) \tag{21}$$

$$\text{s.t. } F(X, b, t) \leq 0 \tag{22}$$

It is easy to see that solving (21)-(22) and (19)-(20) is equivalent to solving (16)-(18), under our assumption that the actions a are purely redistributive and the actions b are purely technological.

It is clear that the social utility function is the right measure of aggregate welfare for benefit-cost analysis. If we are considering some proposed policies b or b' that result in aggregate bundles X or X', then the maximal social welfare associated with (X, b) and (X', b') is given by U(X) and U(X'). Hence X is preferable to X' if and only if U(X) > U(X').

Of course this assumes that: (1) the social decision maker now accepts the social welfare function used during the time during which the data was generated, and (2) that the redistributive actions implied by (19)-(20) will indeed take place.

Given these two somewhat heroic assumptions the major question remaining is how to estimate the social utility function given in (19)-(20) from the observed data.

Let us suppose that the social utility function $U(X)$ is differentiable and the maximization problem given in (21)-(22) is well behaved so that we satisfy the first order conditions:

$$\frac{\partial U(X(t))}{\partial X^j} = \lambda(t) \frac{F(X(t)), b, t)}{\partial X^j} \qquad (23)$$

$$0 = \frac{\partial F(X(t), b, t)}{\partial b} \qquad (24)$$

As before we suppose that the producer shadow prices (or something proportional to them) are observable so that (23) can be written as:

$$\frac{\partial U(X(t))}{\partial X^j} = \lambda(t)\, P^j(t) \qquad (25)$$

and that our observed aggregate data is $(X(t), P(t))$ $t = 1, \ldots, T$. Note that the prices associated with the consumer goods $X(t)$ are in fact the producer prices $P(t)$. It is these prices that reflect the marginal rates of transformation facing the social planner in his solution of problem (21)-(22).

We can now state an appropriate generalization of Samuelson's theorem:

THEOREM 3 Suppose that:

(1) $U(X)$ is a concave differentiable function of X; and

(2) The data (P^t, X^t) were generated by the welfare maximization problem (16)-(18), or equivalently (19)-(22), and therefore satisfies (23).

Then:

(1) The data satisfy the Generalized Axiom of Revealed Preference; and, therefore,

(2) There exists a utility function $V(X)$ that rationalizes the observed data.

Proof. The proof is virtually the same as before. Concavity of $U(X)$ implies

$$U(X(s)) \leqq U(X(t)) + \sum_{j=1}^{k} \frac{\partial U(X(t))}{\partial X^j} (X^j(t) - X^j(s)) \tag{26}$$

Using (23) we can write this as:

$$U(X(s)) \leqq U(X(t)) + \lambda(t) P(t) (X(s) - X(t)) \tag{27}$$

Hence, following the lines of the previous argument $X(t) \; R \; X(s)$ implies $U(X(t)) \geqq U(X(s))$, and the data therefore satisfies GARP. □

Theorem 3 shows that the crucial condition for Samuelson's theorem has nothing to do with the lump sum transfers hypothesis; rather, it has to do with the curvature of the social utility function.[3] It turns out that if lump sum transfers are a feasible instrument and that the social welfare function is concave, then the social utility function will be concave. But this is only a special case of a more general result we will establish in section 6.

The curvature of the social utility function really is the critical issue. Figure 1 gives an example of how things can go wrong if the social utility function is not concave. Here we see that point $X(t)$ maximizes social utility over

$F(X, t)$ and point $X(s)$ maximizes social utility over $F(X, s)$ ... but $X(t)$ and $X(s)$ are each revealed preferred to the other at their supporting market prices.

Theorem 3 gives us general conditions when aggregate data can be rationalized by some "representative" utility function $V(X)$. But what is the relationship between the true social utility function $U(X)$ and the rationalizing utility function $V(X)$? Does a utility function estimated from aggregate demand data have any welfare significance?

We can investigate this question in two different ways. The first way is via the traditional approach to revealed preference theory, where we think of our observed market data consisting of an entire demand function - that is the infinite set of ordered pairs of prices and associated demanded bundles. In this case the welfare significance of $V(X)$ boils down to the uniqueness of the preference ordering that rationalizes some observed market behaviour. This is examined in section 4.

The second approach is via the finite approach to revealed preference theory originated by Afriat (1967), (1973), (1976) and extended by Varian (1980a), (1980b). In this approach our observed market data consists of a finite set of ordered pairs of prices and demanded bundles. Here we cannot hope to recover a unique preference ordering to rationalize the observed data; but we can hope to describe explicitly the set of orderings that can rationalize some observed data. Hence some "partial" welfare judgments can be made even when only a finite amount of data is available. We examine this approach in section 5.

4. The Welfare Significance of Aggregate Demand Functions

Let us now suppose that we have an entire aggregate demand function $X(P)$ that we believe was generated by the social utility/social welfare maximization process described in the last section.[4] We test the demand function for consistency with GARP and find out that it satisfies this condition, so we know that there exists some utility function $V(X)$ that rationalizes the observed data $(P, X(P))$. What is the relationship between $V(X)$ and the "true" underlying social utility function $U(X)$?

First we note that the social utility function $U(X)$ must also rationalize the demand data. For we know by concavity and social utility maximization that $X(t) R X$ implies $U(X(t)) \geq U(X)$ from the proof of Theorem 3. Hence $P(t) X(t) \geq P(t) X$ implies *a fortiori* that $U(X(t)) \geq U(X)$.

So now the question becomes: suppose we have two utility functions $U(X)$ and $V(X)$ that rationalize some demand function $X(P)$. What is the relationship between $U(X)$ and $V(X)$?

The definitive answer to this question was given by Mas-Collel (1976). He showed that under weak conditions $U(X)$ must equal $V(X)$ (up to a monotoni transformation of course). Thus the true underlying preference ordering is uniquely recoverable from the market data. Mas-Collel's proof, however, is rather indirect. Other existing direct proofs of this fact such as in Uzawa (1971) or Stigum (1973), which require stronger conditions than Mas-Colle are direct but rather complicated. We therefore provide a simple and elegant proof of this result below. The conditions we impose are not nearly as general as those used in the above cited works; but the simplicity of the proof should compensate for that to some degree.

THEOREM 4. Let $U(X)$ be a strictly quasi-concave differentiable utility function that rationalizes some differentiable demand function $X(p)$, and let $X^o = X(p^o)$ and $X^1 = X(p^1)$. Then

(1) $X^1 \, R \, X^o$, $X^1 \neq X^o$ implies $U(X') > U(X')$

(2) if $U(X^1) > U(X^o)$, then there exists some finite sequence of observations such that $X^1 \, R \, X^o$, $X^1 \neq X^o$.

Proof. The first part is trivial. The second part follows from the following construction.

Let the compensated (Hicksian) demand function associated with $U(X)$ be given by $h(p, u)$. Let $u^1 = u(x^1)$ and $u^o = u(x^o)$ and define the following path from x^o to x^1.

$$x: [0, 1] \to R^k_+$$

$$x(t) = h(tp^1 + (1-t)p^o, \; tu^1 + (1-t)\, u^o)$$

From this construction it is clear that

$$x(0) = x^o$$

$$x(1) = x^1$$

and:

$$\frac{du(x(t))}{dt} > 0$$

Expanding this derivative we have:

$$\frac{du(x(t))}{dt} = \sum_{i=1}^{k} \frac{\partial u(x(t))}{\partial x_i} \frac{dx_i(t)}{dt} > 0$$

or

$$= \lambda(t) \sum_{i=1}^{k} p_i(t) \frac{dx_i}{dt}(t) > 0$$

Using the definition of the derivative we can write:

$$\sum_{i=1}^{k} p_i(t) \frac{[x_i(t) - x_i(t-\delta_t)]}{\delta_t} > 0$$

for some (small) δ_t's. Using the compactness of the unit interval we can pick a finite number of overlapping intervals $t_1, \ldots, t_n$ where this inequality holds, so we have

$$\sum_{i=1}^{k} p_i(t_j) \, [x_i(t_j) - x_i(t_{j-1})] > 0$$

for $j = 2, \ldots, n$. Hence $x(t_j) \; R^o \; x(t_{j-1})$ for $j = 2, \ldots, n$, and therefore $x^1 \; R \; x^o$, as was to be shown. □

This result shows that the revealed preference relation contains all of the ordinal information in the utility function - at least under appropriate regularity conditions. Thus if $U(X)$ and $V(X)$ both rationalize the demand data $X(p)$, they are ordinally equivalent.

In the context of aggregate demand analysis, a utility function that rationalizes aggregate demand behaviour generated by social welfare maximization must in fact be a social utility function.

5. The Welfare Significance of Aggregate Demand Observations

Let us suppose that we have some finite set of aggregate demand data (P^i, X^i) $i = 1, \ldots, n$ that is consistent with GARP. We are given two new points X^o and X' that we wish to compare. Since we only have a partial ordering on the observed data, we can not hope for a complete comparison of all bundles X^o and X'. However we might be able to make certain sorts of incomplete comparisons.

Varian (1980a) has suggested the following sort of procedure. Let us define the set of prices $p^o \geqq 0$ that support a bundle x^o by:

$$S(x^o) = \{p^o : (p^i, x^i)\ i = 0, \ldots, n \text{ satisfy GARP}\}$$

Thus $S(x^o)$ is simply the set of prices at which x^o could be demanded and still be consistent with the rest of the data.

Then define:

$$RW(x^o) = \{x : \text{for all } p^o \text{ in } S(x^o),\ p^o x^o \geqq p^o x^j \text{ for some } x^j \text{ R } x\}$$

and:

$$RP(x') = \{x : \text{for all } p \text{ in } S(x),\ px \geqq px^j \text{ for some } x^j \text{ R } x'\}$$

Now we can present the main result of this section:

THEOREM 5. Let $U(X)$ be any monotonic quasi-concave utility function that rationalizes the data. Then if X' is in $RW(x^o)$, $U(X^o) \geqq U(X')$ and similarly for $RP(x^o)$.

Proof. Since $U(X)$ is monotonic quasi-concave, there is a $p^o \geqq 0$ such that p^o is a supporting hyperplane for $\{X : U(X) \geqq U(X^o)\}$ at X^o. Clearly p^o is in $S(x^o)$; since X^o R X', then any utility function that rationalizes the data must ensure $U(X^o) \geqq U(X')$. □

Let us note that $RW(x^o)$ and $RP(x')$ are quite operational concepts. Varian (1980) shows that whether x' is in $RW(x^o)$ or not can be checked by solving a simple linear program.

The relevance of the above result is this: If we believe the welfare maximization model described earlier, then the estimated revealed preference ordering can be used to make welfare judgments.

Note that the quasi-concavity of the utility function plays a central role. If we considered some X^o on a non-convex portion of the indifference curve, then there would typically not exist the $P^o \geqq 0$ required in the theorem.

6. The Concavity of the Social Utility Function

As the last three sections have made clear, the concavity of the social utility function plays an important role in the arguments for consistency with the representative consumer model, and for the welfare significance of the estimated utility function. In this section we attempt to relate this curvature condition to some more fundamental considerations.

Suppose the welfare function $W(x)$ is concave in x. One interesting situation where this happens is when the welfare function is of the Bergson-Samuelson form $W(u_1, \ldots, u_n)$ and:

(1) W is concave in $(u_1 \ldots u_n)$

(2) $u_i(x_i)$ is quasi-concave in x_i for $i = 1, \ldots, n$.

For a proof of this, see Gorman (1959).

Let us further suppose that the set of actions open to the social decision maker have the following "convex range" property:

CRP: <u>Let</u> $W(x(a))$ <u>and</u> $W(x(a'))$ <u>be two allocations resulting from actions</u> a <u>and</u> a'. <u>Then there exists an action</u> a'' <u>in</u> A <u>such that:</u>

$$W(x(a'')) \geqq W[tx(a) + (1-t)x(a')]$$

<u>with</u> $$\sum_{i=1}^{n} x_i(a'') = t \sum_{i=1}^{n} x_i(a) + (1-t) \sum_{i=1}^{n} x_i(a').$$

This property is basically concerned with the flexibility of the actions open to the social decision maker. If he has actions at his disposal that can achieve any pareto efficient allocation, then he clearly satisfies CRP. But it seems likely that CRP can be satisfied in much more general circumstances.

The chief interest in CRP lies in the following theorem.

THEOREM 6. Suppose $W(x)$ is concave and A satisfies CRP. Then the social utility function is concave.

Proof. Let X and X' be given and let a and a' be the optimal social actions associated with X and X', so

$$U(X) = W(x(a))$$
$$U(X') = W(x(a'))$$

Now:

$$tU(X) + (1-t)U(X') = tW(x(a)) + (1-t)W(x(a'))$$
$$\leq W[tx(a) + (1-t)x(a')]$$

By CRP we have that for some a'' in A:

$$\leq W[x(a'')]$$

for some $x(a'')$ with $\sum_{i=1}^{n} x_i(a'') = tX + (1-t)X'$

Now let a^* be the optimal action given $tX + (1-t)X'$. Then certainly:

$$\leq W[x(a^*)]$$
$$= U[tX + (1-t)X'].$$

Hence $U(X)$ is concave. □

7. Technological Constraints

The preceding sections have shown that if the social utility function obeys certain curvature restrictions then aggregate data will behave in a way consistent with the representative consumer model. Here we show that under certain technological restrictions the data will behave as if generated by a representative consumer, regardless of the curvature of $U(X)$.

Let us recall the production possibilities set given by $F(X, t)$. We suppose this set is generated by an economy with:

(1) only one nonproduced input to production;

(2) no joint production;

(3) constant returns to scale.

Then Samuelson's nonsubstitution theorem shows that $F(X, t) \leq 0$ will be a half space - the production possibilities frontier is a hyperplane. If all goods are produced in equilibrium, this means that the production possibilities set <u>coincides</u> with the social budget constraint.

Hence we can write the social utility maximization problem as:

$$\max U(X)$$

$$\text{s.t.} \quad P^t X = Y^t$$

where $U(X)$ is the social utility function in (19)-(20).

We now note that aggregate demand data generated by the above model will *a fortiori* satisfy GARP and hence be consistent with the representative consumer model. No curvature assumptions about $U(X)$ are needed for this result.[5]

However, if we want to recover $U(X)$ from the observed data we need some sort of curvature restrictions on $U(X)$. If we want to recover $U(X)$ completely as in Theorem 4, quasiconcavity of $U(X)$ is a necessary assumption; and even if we want only an incomplete recovery as in Theorem 5 a similar assumption is necessary.

8. Summary

If we are willing to postulate that some aggregate economic data was generated by social welfare maximization with sufficiently flexible instruments then this aggregate economic data will appear as though it were generated by a single individual. Furthermore, utility function for this representative consumer will be a "social utility function" and can serve as a guide to welfare policy.

An interesting outcome of these investigations concerns the relevant data for aggregate demand studies. If one adopts the above viewpoint then the appropriate prices for analysis of aggregate consumer demand data are in fact the prices facing the producers of the goods. For it is these prices that reflect the technological scarcities facing the social welfare maximizer.

FOOTNOTES

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1. See the excellent review by Shafer and Sonnenscheim (1979).
2. Robert Pollack (1976), (1980) has recently investigated the Samuelson result in an index number context. In his description of what he calls the "maximizing society" he provides a new proof of the Samuelson theorem, but does not consider the extensions described in sections 3-7 of this paper.
3. Note that the proof of Theorem 3 only uses the fact that problem (21)-(22) is maximized. That is, it does not require that U(X) necessarily derives from a maximization problem itself. The allocative procedure can be arbitrary; but the technological choices must arise from constrained maximization.
4. In particular, we will assume that X(P) is a function so that there is a unique demanded bundle associated with each price vector.
5. We can generalize the assumptions somewhat in the following manner. Suppose there are several nonproduced goods but all are available on world markets at constant prices. Then the technological constraints facing the social planner still define a halfspace and the above argument will work.

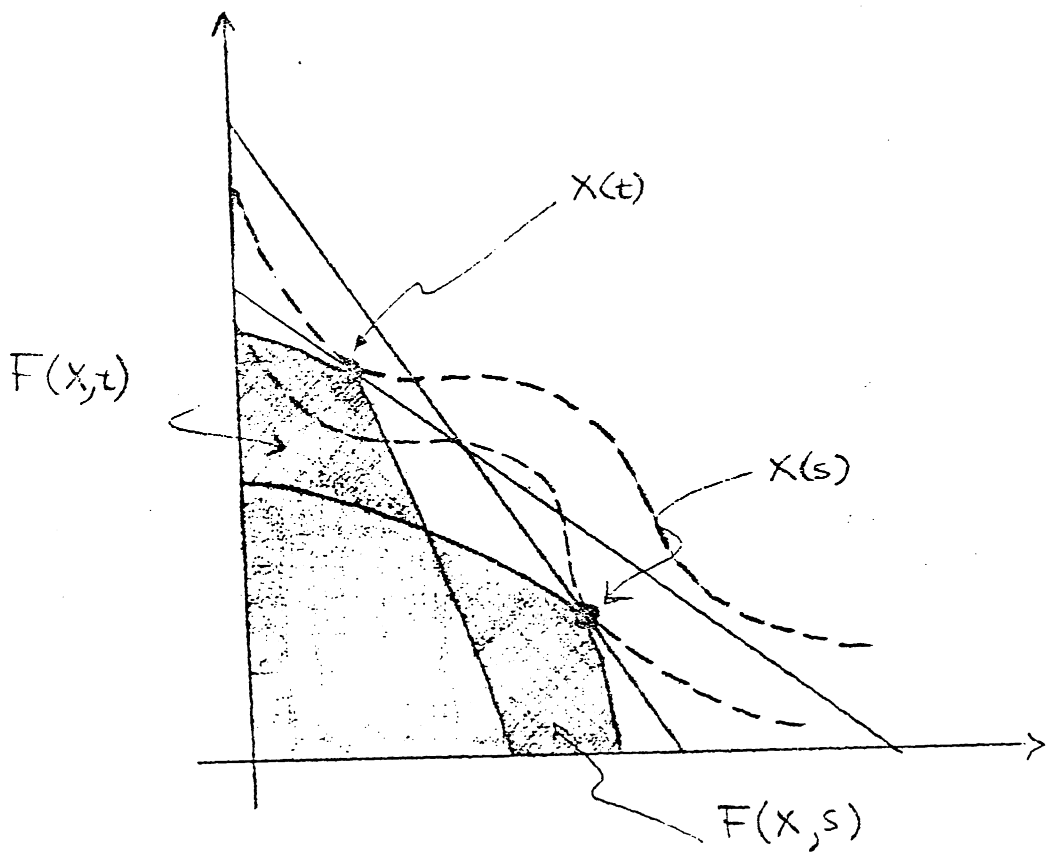

Fig 1. Violation of GARP with Nonconcave Preferences

References

Afriat, S. (1967) 'The Construction of a Utility Function from Expenditure Data', International Economic Review, 8, 67-77.

Afriat, S. (1973) 'On a System of Inequalities in Demand Analysis: an Extension of the Classical Method', International Economic Review, 14, 460-472.

Afriat, S. (1976) The Combinatorial Theory of Demand, Input-Output Publishing Company, London.

Chipman, J. (1974) 'Homothetic Preferences and Aggregation', Journal of Economic Theory, 8, 26-38.

Chipman, J. and J. Moore (1979) 'On Social Welfare Functions and the Aggregation of Preferences', Journal of Economic Theory, 21, 1, 111-139.

Diewert, E. (1973) 'Afriat and Revealed Preference Theory', Review of Economic Studies, 40, 419-426.

Gorman, W. (1953) 'Community Preference Fields', Econometrica, 21, 63-80.

Gorman, W. (1959) 'Are Social Indifference Curves Convex?' Quarterly Journal of Economics, LXXIII, 485-498.

Mantel, R. (1976) 'Homothetic Preferences and Community Excess Demand Functions', Journal of Economic Theory, 12, 197-201.

Mas-Collel, A. (1978) 'On Revealed Preference Analysis', Review of Economic Studies, XLV, 121-132.

Mas-Collel, A. (1976) 'On the Recoverability of Consumer's Preferences from Market Demand Behaviour', Econometrica, 44.

Muellbauer, J. (1976) 'Community Preferences and the Representative Consumer', Econometrica, 44, 979-999.

Pollack, R. (1976) "The Social Cost of Living Index," disc. paper 378, University of Pennsylvania.

Pollack, R. (1980) "Group Cost of Living Indexes," American Economic Review, 70, 2, 273-278.

Samuelson, P. (1956) 'Social Indifference Curves', Quarterly Journal of Economics, LXX, 1-22. Reprinted in The Collected Scientific Papers of Paul A. Samuelson, Vol. II (J. Stiglitz, ed.) MIT Press, Cambridge, Mass.

Samuelson, P. (1964) 'Principles of Efficiency - Discussion', American Economic Review, 54, 93-96.

Shafer, W. and H. Sonnenschein (1979) 'Market Demand and Excess Demand Functions', in K. Arrow and M. Intriligator, Handbook of Mathematical Economics, North Holland Press, Amsterdam.

Shapiro, P. (1977) 'Aggregation and the Existence of a Social Utility Function', Journal of Economic Theory, 16, 475-480.

Stigurn, B. (1973) 'Revealed Preference: A Proof of Houthakker's Theorem', Econometrica, 41, 411-423.

Uzaura, H. (1971) 'Preference and Rational Choice in the Theory of the Consumer', in Chipman et.al., Preferences, Utility and Demand, Harcourt, Brace, Jovanovich, Inc., New York.

Varian, H. (1978) Microeconomic Analysis, W.W. Norton and Company, New York.

Varian, H. (1980) 'The Nonparametric Approach to Demand Analysis', University of Michigan working paper.

Varian, H. (1980a) 'The Nonparametric Approach to Demand Analysis', working paper.

Varian, H. (1980b) 'Nonparametric Tests of Consumer Behavior', working paper.