The structure of W-graphs arising in Kazhdan-Lusztig theory

by

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ABSTRACT

The structure of W-graphs arising in Kazhdan-Lusztig theory

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Chair: John R. Stembridge

This thesis is primarily about the combinatorial aspects of Kazhdan-Lusztig theory. Central to this area is the notion of a W-graph, a certain weighted directed graph which encodes a representation of the Iwahori-Hecke algebra of a Coxeter group. The most important examples were given in the original work of Kazhdan and Lusztig in 1979; from these graphs the Kazhdan-Lusztig polynomials are obtained via a weighted path count. In the first part, we consider "parallel transport" relations among edge weights. Some of these relations, namely those coming from simply-laced Weyl groups, appeared in the same paper of Kazhdan and Lusztig. We introduce additional ones corresponding to doubly-laced Weyl groups, and, as an application, prove Green's $0 - 1$ conjecture in type B. In the second part we clarify the structure of W-graphs corresponding to minuscule and quasi-minuscule quotients of finite Weyl groups. The W-graphs for minuscule quotients can be deduced, on a case-by-case basis, from previous work on the associated Kazhdan-Lusztig polynomials; we give a type-independent proof of a weaker result that these graphs can be characterized by simple combinatorial rules. For quasi-minuscule quotients, we compute the graphs for all finite Weyl groups except for Lie type D (where we give a conjectural answer). We also compute the parabolic Kazhdan-Lusztig polynomials for the type A quasi-minuscule quotient. The last part concerns the conjecture that in Lie type A, the only strongly connected W-graphs which satisfy a weak set of conditions known as "admissibility" are the Kazhdan-Lusztig examples. We prove a partial result that the symmetrically weighted edges of such a graph are the same as the symmetrically weighted edges of some Kazhdan-Lusztig examples.

CHAPTER 1

Introduction

This thesis deals with combinatorial constructions called W-graphs. They are weighted directed graphs, with additional structure that encodes certain representations of the Iwahori-Hecke algebra $\mathcal{H}_W$ of a Coxeter group W.

The definition originated in Kazhdan-Lusztig theory. In [KL79], Kazhdan and Lusztig introduced the canonical basis $\{C_w \mid w \in W\}$ of $\mathcal{H}_W$. The transition matrix between this basis and the standard basis is conveniently expressible in terms of the so-called Kazhdan-Lusztig polynomials. One of the primary reasons for introducing these polynomials is the Kazhdan-Lusztig Conjecture which states that their evaluation at 1 gives the multiplicities of Verma modules in the simple modules of the principal block of category $\mathcal{O}$. The conjecture has been proven, shortly after being stated, by Beilinson and Bernstein [BB81], and Brylinski and Kashiwara [BK81] independently. The Kazhdan-Lusztig polynomials are also intricately connected with the geometry of Schubert varieties; they are Poincaré polynomials of the local intersection cohomology of a Schubert variety at a point in a given Schubert cell. A survey of this area can be found in Chapter 8 of [Hum08].

Kazhdan and Lusztig wrote down formulas for the action of the generators of $\mathcal{H}_W$ on the canonical basis (in terms of some coefficients of the Kazhdan-Lusztig polynomials; see μ-coefficients in Section 1.2.1). The information needed to write down the action can be conveniently captured by a graph which we refer to as the Kazhdan-Lusztig W-graph. The vertices of this graph are in bijection with the elements of W. Since only a few coefficients of the Kazhdan-Lusztig polynomials are used in the action, it seems that the W-graph contains much less information than the polynomials. However it turns out that in case W is finite, it is fairly computationally easy to obtain all the polynomials from the W-graph (see the end of Section 1.2.3). The same computation from the definition is prohibitively hard, so it is interesting to find alternative ways of constructing the Kazhdan-Lusztig W-graph. This leads one to consider the class of graphs which encode representations of $\mathcal{H}_W$, i.e. W-graphs.

It turns out that the strongly connected components of a W-graph also encode representations, and hence are themselves W-graphs. Strongly connected W-graphs are called

cells. The partition of W into cells of the Kazhdan-Lusztig W-graph has been a subject of numerous papers. It has been known since [KL79] that in Lie type A, the cells are given by the Robinson-Schensted correspondence. In a series of papers [Gar90, Gar92, Gar93, Gar] Garfinkle introduced a domino insertion algorithm and described its relationship with the cells in Lie types B and D. The conjectures in [BGIL10] suggest that the similarity between types A and B is better seen if one considers the Hecke algebra with unequal parameters. The cell breakdown in affine Weyl groups has been extensively studied by Lusztig ([Lus85, Lus87a, Lus87b]) and Shi ([Shi86, Shi90, Shi91]).

Another development we would like to mention is that of parabolic induction. In [Cou99] Couillens showed that the representation induced from a one-dimensional one always has a W-graph structure. This idea was further developed by Howlett and Yin ([HY03, HY04]); they showed that a representation induced from one given by a W-graph also has a W-graph structure.

We would like to study W-graphs from a combinatorial viewpoint, however the class of all W-graphs is too large to be approached in this way. Stembridge [Ste08a] isolated several properties of the Kazhdan-Lusztig W-graph (which he called "admissibility") and combinatorially characterized admissible W-graphs. In the same paper he asked three fundamental questions. The first was whether for finite W there are finitely many admissible W-cells. This question was answered affirmatively in his later paper [Ste12]. The second was whether two cells of the Kazhdan-Lusztig graph which define isomorphic representations must in fact be isomorphic as W-graphs. We know of no progress with regard to this question. The last question was whether all admissible W-cells in type A are cells of the Kazhdan-Lusztig W-graph. We present some results concerning this question in Chapter 4. Stembridge's notes [Ste08b] describe the methods he developed to construct all admissible W-cells "from scratch" and used to do empirical computations. In particular, the notion of molecular graphs introduced there will be heavily used in the thesis.

Below we give a chapter-by-chapter summary.

In the introduction we describe both the elements of Kazhdan-Lusztig theory, as well as the combinatorial approach of Stembridge. The only part of this which may be new is the relationship between the Kazhdan-Lusztig theory of parabolic quotients and the classical Kazhdan-Lusztig theory via W-graphs. It has been known since Deodhar's introduction ([Deo87]) of the parabolic variant that the Kazhdan-Lusztig polynomials in the parabolic case are related in a nontrivial fashion to the classical ones. We observe, relying on the computations of Couillens ([Cou99]) that the associated W-graphs are, in fact, related in a very easy way (see discussion around Propositions 1.2.14 and 1.2.17).

In Chapter 2, we discuss two versions of "parallel transport" of edge weights in W-graphs.

It is a set of relations among edge weights (normally of the form "one edge weight is equal to another"). These relations turn out to be quite powerful and often allow us to find most edge weights relatively easily. The version which we refer to as Kazhdan-Lusztig transport has been introduced via the $*$-operator in [KL79]. Its counterpart in the combinatorial picture of Stembridge is most likely the so-called Local Polygon Rule. In simply laced types we find generating relations for the Local Polygon Rule (to which we refer as arc transport) which show a similarity between the two (Theorems 2.1.2 and 2.1.3). It is easy to see that in simply laced types Kazhdan-Lusztig transport is at least as strong as arc transport. We then extend this theory to Coxeter groups that have a double bond. In Definition 2.2.7, we give additional relations which may be considered a version of Kazhdan-Lusztig transport for double bonds and proceed to show that these relations indeed hold in the Kazhdan-Lusztig W-graph. We also show that, with this new definition, Kazhdan-Lusztig transport implies the Local Polygon Rule. This proof is somewhat cumbersome since we are unable to isolate a simple generating set of the relations implied by the Local Polygon Rule. As an application of the double bond version of Kazhdan-Lusztig transport, we prove, in Lie type B, the conjecture of Green [Gre09] that the weights of the edges in the Kazhdan-Lusztig graph which start at fully commutative elements must be 0 or 1.

In Chapter 3 we clarify the structure of W-graphs for certain important parabolic quotients, namely the minuscule and quasi-minuscule ones. For the minuscule case we show that the W-graph is "not too complicated," in the sense that it can be reconstructed fully via either Local Polygon Rule or Kazhdan-Lusztig transport. The Kazhdan-Lusztig polynomials have been computed before (see [Boe88, LS81, Bre09, Bre02]), so it is not surprising that we can get our hands on this graph. However what is interesting about our proof is that it is entirely independent of the Lie type. In the quasi-minuscule case we are able to compute the W-graphs for all the finite Weyl groups, except for the type D_n. In this case we conjecture the answer, but are unable to prove it as of yet. In type A we can go further and give an explicit formula for (one of the two variants of) the parabolic Kazhdan-Lusztig polynomials.

In Chapter 4 we specialize entirely to type A. We attempt to resolve the conjecture that all the admissible cells come from the Kazhdan-Lusztig graph. While we are unable to resolve it completely, we prove that certain subgraphs (so called "molecules") out of which all admissible cells are built do, in fact, arise in the Kazhdan-Lusztig graph. To extend the result to cells we formulated a conjecture that molecules inside a molecular graph (a slightly larger class of graphs than W-graphs) cannot form a cycle. This turns out to be false; we exploit a limitation of the Kazhdan-Lusztig transport to find such examples in type A_{35}. A computerized search by Stembridge found a smaller example in A_{13}.

Most of the results in this thesis concern finite crystallographic root systems. To set the

Figure 1.1: Dynkin diagrams of finite crystallographic root systems.
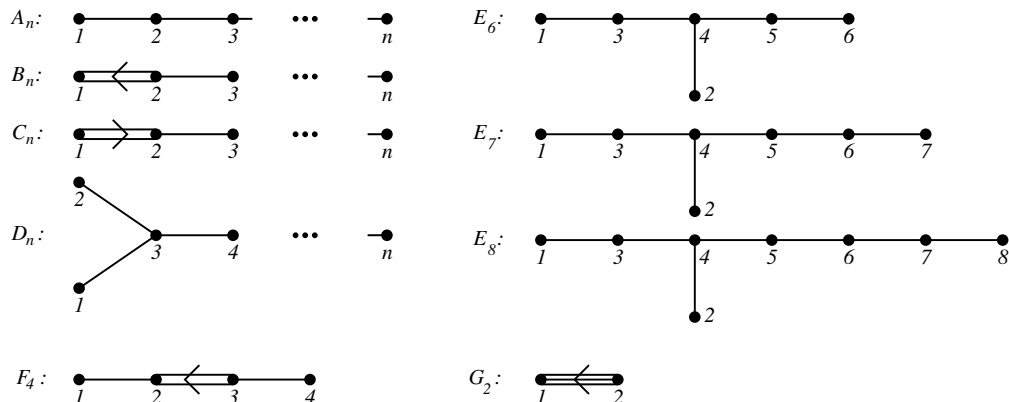
notation, we give a list of Dynkin diagrams for these in Figure 1.1. We will refer to this root numbering as "standard." With one exception (section 2.3), we will use this numbering of roots. The edges of the Coxeter graph will be referred to as bonds to distinguish them from edges of other graphs. We will say that s_i is bonded to s_j (denoted $s_i \sim s_j$) if there is a bond between the corresponding vertices of the Coxeter graph.

1.1 W-graphs and W-molecules

We start by recalling Stembridge's combinatorial approach to Kazhdan-Lusztig theory.

1.1.1 Admissible S-labeled graphs and sBCS graphs

In this section we describe the kind of graphs enhanced with additional structure which we will deal with throughout the thesis.

Let (W, S) be a Coxeter system.

An *admissible S-labeled graph* is a tuple $G = (V, m, \tau)$, where V is a finite set (vertices), $m : V \times V \to \mathbb{Z}^{\geqslant 0}$, and $\tau : V \to 2^S$ are maps such that

1. as a directed graph (with edges given by pairs of vertices with non-zero m value), G is bipartite,

2. if $\tau(u) \subseteq \tau(v)$ then $m(u, v) = 0$,

3. if $\tau(u)$ and $\tau(v)$ are incomparable, then $m(u, v) = m(v, u)$.

The function τ will be referred to as the *τ-invariant*. We will most of the time omit the word "admissible" since we consider no other S-labeled graphs.

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From the definition we can see that either one of $m(u, v)$ and $m(v, u)$ is zero, or they are equal. By a *simple edge* we mean a pair of vertices (v_1, v_2) such that neither $m(v_1, v_2)$ nor $m(v_2, v_1)$ are 0. In diagrams we draw these as undirected edges (see the left side of Figure 1.6). By an *arc* $v_1 \to v_2$ we mean a pair of vertices (v_1, v_2) such that $m(v_1, v_2) \neq 0$, but $m(v_2, v_1) = 0$. Thus if $u \to v$ is an arc, then $\tau(u) \supset \tau(v)$. If (u, v) is a simple edge then $\tau(u)$ and $\tau(v)$ are incomparable, and $m(u, v) = m(v, u)$.

Arcs will not come into play much until section 1.1.3.

A simple edge (u, v) *activates* a bond (i, j) if precisely one of $\tau(u)$ and $\tau(v)$ contains i, and precisely the other one contains j.

Let (i, j) be a bond, and let G be an S-labeled graph. An (i, j)-*string* is a path $v_1, \ldots, v_l$ in G without self-intersections such that

- for each $1 \leqslant k \leqslant l - 1$, the edge (v_k, v_{k+1}) activates the bond (i, j),

- no other edges adjacent to $v_1, \ldots, v_l$ activate the bond (i, j).

An S-labeled graph is called *uniform* if all the τ-invariants have the same size.

Definition 1.1.1. An S-labeled graph is called an *sBCS graph* if it satisfies

(SR) If (u, v) is a simple edge then $m(u, v) = m(v, u) = 1$. Thus we will omit the weights of simple edges in our diagrams.

(CR) If $u \to v$ is an edge, i.e. $m(u, v) \neq 0$, then every $i \in \tau(u) \setminus \tau(v)$ is bonded to every $j \in \tau(v) \setminus \tau(u)$.

(sBR) Suppose (i, j) is a bond of order m_{ij} in the Coxeter graph of (W, S), i.e. the order of $s_i s_j$ in W is m_{ij}. For each vertex u, precisely one of the following holds:

- $i, j \in \tau(u)$,

- $i, j \notin \tau(u)$,

- u is part of an (i, j)-string of length $m_{ij} - 1$.

The rules are called, respectively, simplicity rule, compatibility rule, and strong bonding rule.

Remark 1.1.2. The original version of the bonding rule (BR) in [Ste08a] allowed more general graphs than just strings; more precisely the restriction of the graph to vertices which contain precisely one of i and j is a union of ADE Dynkin diagrams whose Coxeter numbers divide m. In particular, a path of length $m - 1$ is such a diagram. The notion "strong bonding rule" is used here since the graphs that come from Kazhdan-Lusztig theory only involve paths.

Remark 1.1.3. For the case of a simple bond (i, j), sBR just says that any vertex u with $i \in \tau(u)$ and $j \notin \tau(u)$ is connected to a unique vertex v with $i \notin \tau(v)$ and $j \in \tau(v)$, and vice versa. In this case (BR) is equivalent to (sBR).

1.1.2 Classification of sBCS graphs of some Coxeter groups of small rank

In this section we classify the sBCS-graphs for the Coxeter groups of types A_3 and B_3. We also introduce a "flatness" condition and classify all sBCS-graphs which satisfy it for the Coxeter systems of types A_4, B_4, and F_4.

Notice that there are no conditions on the weights of the arcs (besides ones imposed by admissibility). So, for the purposes of this section, we may assume that all the arcs have weight 0.

There are two symmetries which we will exploit. First, a symmetry of the Coxeter graph induces a symmetry of an sBCS graph (just act on the τ-invariants). We will refer to this as a "diagram symmetry." Both the Coxeter graphs for A_n and for F_4 have such a symmetry. The second is that if you complement the τ-invariants of an sBCS graph (and, in general, reverse all the arcs), you get again an sBCS graph. We will refer to this as the "complementation symmetry." It holds for any Coxeter system.

1.1.2.1 Rank 3

Example 1.1.4. Let us classify the connected A_3 sBCS graphs. Because of admissibility, a vertex whose τ-invariant is $\varnothing$ cannot be connected to any other vertex by a simple edge. Similarly for a vertex whose τ-invariant is $\{1, 2, 3\}$.

Suppose we have a vertex v_1, whose τ-invariant is $\{1\}$. By sBR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 2, but not 1. By CR, $3 \notin \tau(v_2)$, and hence $\tau(v_2) = \{2\}$. By sBR, v_2 is connected by a simple edge to a vertex v_3 whose τ-invariant contains 3, but not 2. We already know $v_3 \neq v_1$. By sBR, $\tau(v_3) = \{3\}$. There are no other simple edges possible, and this is a complete sBCS graph. By symmetry, the same analysis works for v_1 having τ-invariants of $\{3\}, \{1, 2\}, \{2, 3\}$.

Suppose we have a vertex v_1, whose τ-invariant is $\{2\}$. By sBR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 1, but not 2. The case of $\tau(v_2) = \{1\}$ was described above, so the only choice is $\tau(v_2) = \{1, 3\}$. This yields a complete sBCS graph. By symmetry, the same argument works for v_1 having τ-invariant of $\{1, 3\}$.

This completes the classification. The results are shown in Figure 1.2.
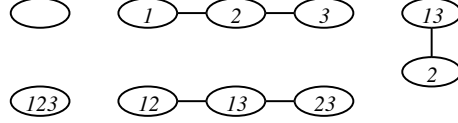
Figure 1.2: Connected sBCS graphs of type A_3.

Example 1.1.5. Let us classify the connected B_3 sBCS graphs. Because of admissibility, a vertex whose τ-invariant is $\varnothing$ cannot be connected to any other vertex by a simple edge. Similarly for a vertex whose τ-invariant is $\{1, 2, 3\}$.

Suppose we have a vertex v_1, whose τ-invariant is $\{1\}$. By sBR, it is part of a $(1, 2)$-string. There are two cases, depending on whether or not v_1 is in the middle of its string. Suppose first that it is in the middle, i.e. the string is a path v_0, v_1, v_2 and $1 \notin \tau(v_0) \cup \tau(v_2)$, $2 \in \tau(v_0) \cap \tau(v_2)$. By CR, $3 \notin \tau(v_0) \cup \tau(v_2)$, so $\tau(v_0) = \tau(v_1) = \{2\}$. By sBR, there exists v_3 connected to v_2 with $3 \in \tau(v_3)$ and $2 \notin \tau(v_3)$, and there exists v_{-1} connected to v_0 with the same restriction. Now $v_3 \neq v_{-1}$ since otherwise we would violate sBR with respect to the bond $(2, 3)$. Also, $v_3 \neq v_1$ (since $\tau(v_1) = \{1\}$), and hence by sBR, $1 \notin \tau(v_3)$. Similarly, $1 \notin \tau(v_{-1})$. Hence $\tau(v_3) = \tau(v_{-1}) = \{3\}$. This completes the graph.

Now suppose v_1 is not in the middle, i.e. the string is a path v_1, v_2, v_3, $\tau(v_2)$ contains 2 but not 1, and $\tau(v_3)$ contains 1 but not 2. By CR, $3 \notin \tau(v_2)$, hence $\tau(v_2) = \{2\}$. By sBR, we know that v_2 must be adjacent to a vertex with 3 in its τ-invariant. If that vertex is v_3, then we have a complete sBCS graph. Otherwise, there must exist v_4 with $3 \in \tau(v_4)$ and $2 \notin \tau(v_4)$. By sBR, $1 \notin \tau(v_4)$, and hence $\tau(v_4) = \{3\}$. This is again a complete sBCS graph.

Thus we have classified all possible B_3 sBCS graphs which have a vertex with τ-invariant $\{1\}$; there are three of them. By symmetry, we have also classified B_3 sBCS graphs which have a vertex with τ-invariant $\{2, 3\}$.

Suppose we have a vertex v_1, whose τ-invariant is $\{2\}$, but the graph has no vertex with τ-invariant $\{1\}$ or $\{2, 3\}$. Clearly, v_1 cannot be in the middle of its $(1, 2)$-string, since otherwise both ends of the string would have τ-invariants $\{1, 3\}$ which would violate sBR with respect to the bond $(2, 3)$. Hence the string is a path v_1, v_2, v_3. Now $1 \in \tau(v_2)$ and $2 \notin \tau(v_2)$, so, by assumption, $\tau(v_2) = \{1, 3\}$. Then the edge (v_1, v_2) activates the bond $(2, 3)$. By sBR, the edge (v_2, v_3) must not activate this bond, so $\tau(v_3) = \{2, 3\}$. This is a contradiction; there is no graph with τ-invariant of $\{2\}$ but no τ-invariant of $\{1\}$ or $\{2, 3\}$. By symmetry, we know that there is no sBCS graph with a τ-invariant of $\{1, 3\}$ but no τ-invariant of $\{1\}$ or $\{2, 3\}$.

Suppose we have a vertex v_1, whose τ-invariant is $\{3\}$, but the graph has no vertex with τ-invariant $\{1\}$ or $\{2\}$. By sBR, there exists v_2 with $2 \in \tau(v_2)$ and $3 \notin \tau(v_2)$. Now $1 \notin \tau(v_2)$
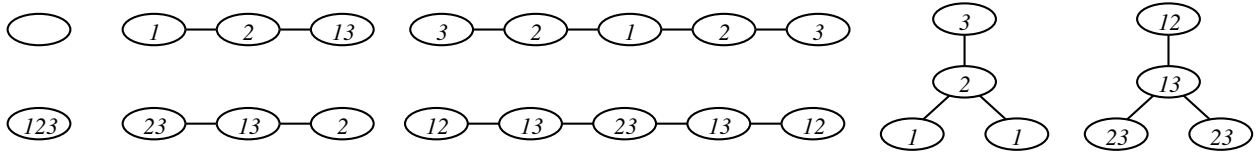
Figure 1.3: Connected sBCS graphs of type B_3.
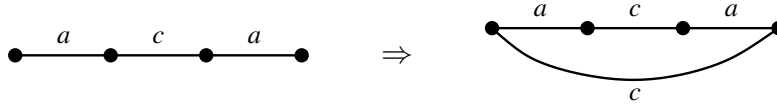
Figure 1.4: Definition of "flat 4-cycles".

by CR. So $\tau(v_2) = \{2\}$, which is a contradiction.

This completes the classification. The results are shown in Figure 1.3.

1.1.2.2   The compatibility graph and flatness

Before dealing with the rank 4 root systems, we need to introduce two notions. The first is that of an sBCS graph with flat 4-cycles. This is a technical condition which simplifies the classification and will later be implied by other rules of W-graphs and W-molecules (specifically the Polygon Rule and the Local Polygon Rule). The second is that of the compatibility graph; it is a graph that will allow us to systematically impose CR, thus reducing the complexity of the arguments.

Definition 1.1.6. Let a and c be non-adjacent bonds. An sBCS graph *has flat 4-cycles* if whenever there are three edges in the configuration on the left side of Figure 1.4, where edges are labeled by the bonds they activate, and none of the above edges activate a bond that is adjacent to both a and c, then there is a fourth edge activating the bond c.

Now let us describe the construction of the compatibility graph, following [Ste08a]. The graph $\text{Comp}(W, S)$ is a directed graph whose vertices are all the subsets of S and for $J, K \subseteq S$, $J \to K$ is an edge precisely when $J \not\subseteq K$ and any $s \in J \setminus K$ is bonded to any $t \in J \setminus K$.

Some of the edges of the graph are trivial, namely those of the form $J \to K$ for $J \supsetneq K$. If we do not draw these edges, the compatibility graph for the case when the Coxeter graph is a path with four vertices (such as A_4, B_4, and F_4) is shown in Figure 1.5

Now CR may be rephrased as saying that there is a graph homomorphism from the S-labeled graph into $\text{Comp}(W, S)$ which maps a vertex to its τ-invariant.

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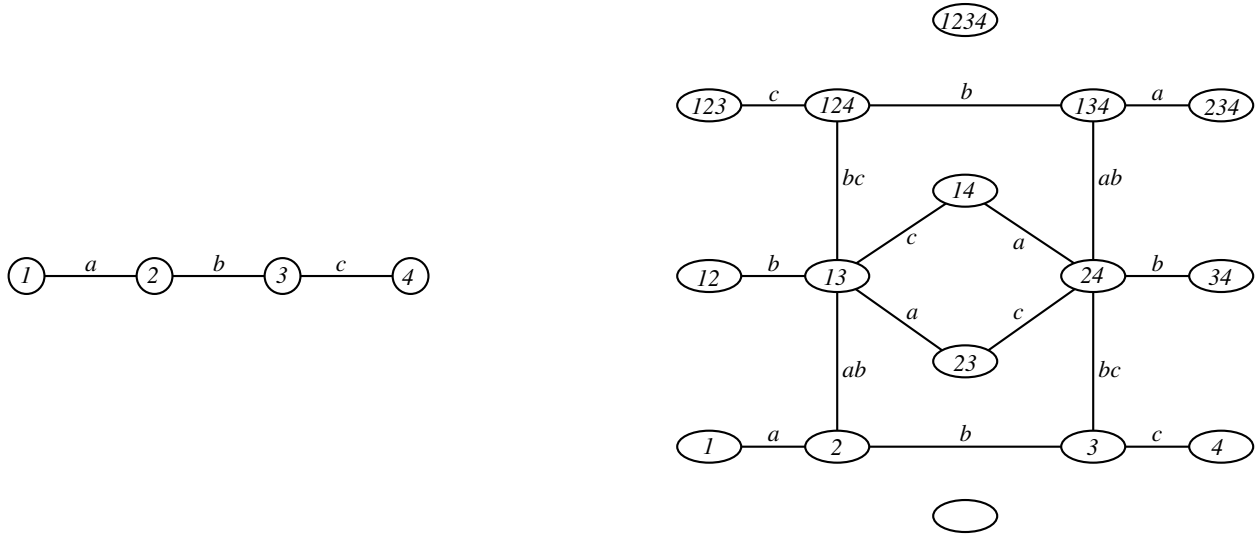
Figure 1.5: Coxeter graph of a root system and the corresponding compatibility graph.

Remark 1.1.7. For any of the Coxeter systems whose graph is a path on four vertices, the condition of having flat 4-cycles implies that whenever a graph contains a path with τ-invariants $14 - 24 - 23$ or $14 - 13 - 23$, then it must contain the entire (flat) 4-cycle.

1.1.2.3 Rank 4

Throughout this section we will be frequently using the compatibility graph in Figure 1.5.

Proposition 1.1.8. *The connected sBCS graphs of type A_4 with flat 4-cycles are shown in Figure 1.6.*

Proof. Suppose G is a connected sBCS graph of type A_4 with flat 4-cycles. As always, there are two possibilities for G to have only one vertex. Thus from now on we assume that is not the case.

Suppose G has a vertex v_1 with τ-invariant $\{1\}$. It is only possible to activate a, so by sBR, v_1 is connected to v_2 with $\tau(v_2) = \{2\}$. Now the bond b must be activated along some edge from v_2, and that edge cannot activate the bond a. Hence v_2 is connected to v_3 with $\tau(v_3) = \{3\}$. By the same argument, v_3 is connected to v_4 with $\tau(v_4) = \{4\}$. The result is a complete sBCS graph. Using the complementation and diagram symmetries, we have classified the sBCS graphs which contain a vertex with τ-invariant $\{4\}$, $\{2, 3, 4\}$,or $\{1, 2, 3\}$.

Suppose G has a vertex v_1 with τ-invariant $\{2\}$, but no vertex with τ-invariant $\{1\}$. Looking at the compatibility graph we see that v_1 is connected to a vertex v_2 with $\tau(v_2) = \{1, 3\}$. There must be an edge out of v_2 which activates c, however it cannot activate a or b

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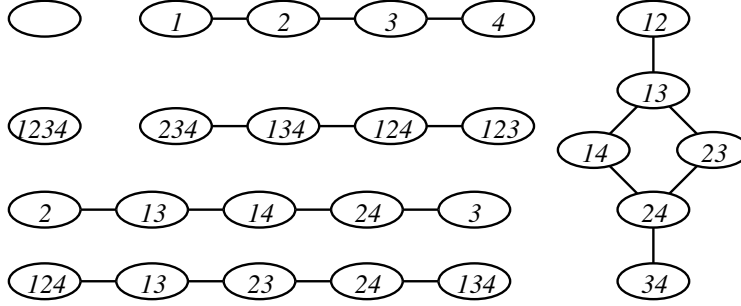
Figure 1.6: Connected sBCS graphs of type A_4 with flat 4-cycles.

since those are already activated along (v_1, v_2). Thus v_2 is connected to v_3 with $\tau(v_3) = \{1, 4\}$, which in turn is connected to a vertex v_4 with $\tau(v_4) = \{2, 4\}$. Now there must be an edge from v_4 to v_5 which activates c. If $\tau(v_5) = \{2, 3\}$ then the flatness assumption would imply that v_5 and v_2 are connected, which is impossible. Hence $\tau(v_5) = \{3\}$. The result is a complete sBCS graph. Thus, by exploiting the symmetries, we have classified all sBCS graphs with flat 4-cycles that contain vertices with τ-invariants of size other than 2.

By similar arguments one can see that if G contains only vertices whose τ-invariants are of size 2, then G must be isomorphic to the induced subgraph of $\mathrm{Comp}(W, S)$ on two-element subsets. $\qquad\square$

Proposition 1.1.9. *The connected sBCS graphs of type B_4 with flat 4-cycles are shown in Figure 1.8.*

Proof. Suppose G is a connected sBCS graphs of type B_4 with flat 4-cycles. There are, as always, two one-vertex sBCS graphs.

In this case edges activating a must come in adjacent pairs.

Suppose G has a vertex v_1 with τ-invariant $\{1\}$. First suppose v_1 is in the middle of its $(1, 2)$-string v_0, v_1, v_2. All instances of activation of a around these vertices have been used up, and there is a unique way of completing this graph to an sBCS graph (it yields the path with singleton τ-invariants in Figure 1.8).

Now suppose v_1 is at the end of its $(1, 2)$-string v_1, v_2, v_3. We know that $\tau(v_2) = \{2\}$, however there are two choices for $\tau(v_3)$. If $\tau(v_3) = \{1\}$, then the graph completes uniquely to an sBCS graph with singleton τ-invariants which looks like the Coxeter graph of the D_5 root system. Assume $\tau(v_3) = \{1, 3\}$. Since b and c are simple bonds, we get a unique possible continuation of the path to v_4 with $\tau(v_4) = \{1, 4\}$.

Now v_4 must be part of a $(1, 2)$-string. Suppose first that it is in the middle of the string v_5, v_4, v_6. Then $\tau(v_5) = \tau(v_6) = \{2, 4\}$. There must be an edge out of v_5, to say v_7, which

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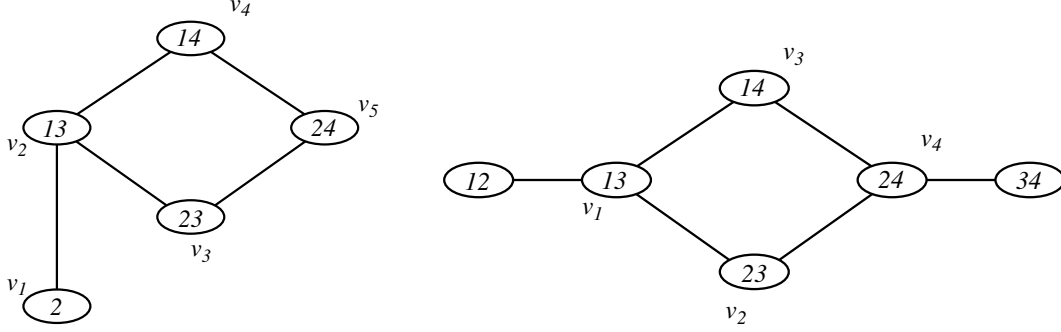
Figure 1.7: Partial sBCS graphs of type B_4.

activates c. Then $\tau(v_7) = \{3\}$, since if $\tau(v_7)$ were $\{2,3\}$ then we would have issues with flatness of 4-cycles. Similarly, v_6 must be connected to a vertex with τ-invariant $\{3\}$. This yields a complete sBCS graph.

Now assume v_4 is at the end of its $(1,2)$-string v_4, v_5, v_6. Then $\tau(v_5) = \{2,4\}$. By the same argument as before, v_5 is connected to v_7 with $\tau(v_7) = \{3\}$. Hence $\tau(v_6) = \{1,4\}$ (as b is already activated on the edge (v_5, v_7)). This completes uniquely to the sBCS graph in the lower left-hand corner of Figure 1.8.

This finished the classification of graphs which contain a vertex with τ-invariant $\{1\}$. By symmetry (only complementation is applicable to type B), we have also classified ones which have a vertex with τ-invariant $\{2,3,4\}$.

Suppose G has a vertex v_1 with τ-invariant $\{2\}$, but none with τ-invariants $\{1\}$ or $\{2,3,4\}$. Now v_1 cannot be at the center of its $(1,2)$-string since both other vertices would have to have τ-invariant $\{1,3\}$ (contradicting sBR). Using flatness we can uniquely reconstruct the partial graph on the left of Figure 1.7. The vertices v_4 and v_5 may require more edges coming out of them.

If v_5 is the center of its $(1,2)$-string, then it must be connected to v_6 with $1 \in \tau(v_6)$. Then $\tau(v_6) = \{1,3,4\}$ since otherwise we will contradict the flatness assumption. The result is a complete sBCS graph. If, on the other hand, v_4 is the center of the $(1,2)$-string, then it must be connected to a vertex v_6 with $\tau(v_6) = \{2,4\}$. Now v_5 must be incident to an edge activating b, and v_6 must be incident to an edge activating b and an edge activating c. It is clear that v_5 must be connected to a vertex with τ-invariant $\{3,4\}$. To avoid contradicting flatness, v_6 must be connected to a vertex with τ-invariant $\{3\}$. This finishes the classification of sBCS graphs with a vertex with τ-invariant $\{2\}$ (or, by symmetry, $\{1,3,4\}$), but no vertex with τ-invariants $\{1\}$ or $\{2,3,4\}$.

By a similar argument one can see that it is impossible to have an sBCS graph with τ-invariant $\{3\}$ but no τ-invariant $\{2\}$. One clearly cannot have an sBCS graph with τ-

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invariant $\{4\}$ but no τ-invariant $\{3\}$.

The only possibilities we have not yet considered are the graphs all of whose vertices have two elements in their τ-invariants. It is fairly clear that all of them must contain the graph on the right of Figure 1.7. The only freedom we have is whether v_1 or v_2 is the center of the $(1, 2)$ string (the analogous follows for v_3 and v_4 via flatness; see Remark 1.1.7). This gives us the two sBCS graphs in the bottom right part of Figure 1.8. $\qquad\square$

Proposition 1.1.10. *The connected non-uniform sBCS graphs of type F_4 with flat 4-cycles are shown in Figure 1.12.*

Proof. Suppose G is a connected nonuniform sBCS graph of type F_4 with flat 4-cycles. Up to symmetry we may conclude that G has a vertex v_1 with τ-invariant $\{2\}$ and a vertex v_2 with τ-invariant $\{1, 3\}$. There are three choices to extend this graph via sBR on the bond b; these are shown in Figure 1.9.

The last of these is a complete sBCS graph. For the first two there is a unique extension by applying sBR with respect to a and c; it is shown in Figure 1.10.

The only vertex which will have more edges out of it is v_3. To activate the bond c, v_3 must be connected to v_4 with $\tau(v_4) = \{3\}$ (otherwise we will contradict flatness). We then face a choice of whether v_3 or v_4 is the middle of the $(2, 3)$-string. The remainder completes uniquely so we get the four possibilities shown in Figure 1.11.

This finishes the proof. $\qquad\square$

Remark 1.1.11. The two graphs at the bottom of Figure 1.12 are of little interest for us since the stronger rules which will replace flatness (Polygon Rule and Local Polygon Rule) will not be satisfied by these graphs regardless of the arrangement of arcs.

Proposition 1.1.12. *The connected uniform sBCS graphs of type F_4 with flat 4-cycles are shown in Figure 1.13.*

Proof. As always, there are two one-vertex sBCS graphs; these are all the ones with empty or full τ-invariants. It is fairly easy to see (using the same methods as in the previous proofs) that the only graphs whose τ-invariants are singletons are the ones shown in Figure 1.13. By symmetry, the same is true for the graphs whose τ-invariants contain three elements.

Suppose G is a connected sBCS graph of type F_4 with flat 4-cycles, and all τ-invariants have two elements. A vertex with a τ-invariant $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, or $\{2, 4\}$ must automatically be part of a 4-cycle. A vertex with τ-invariant $\{1, 3\}$ must either be connected to two vertices with τ-invariants $\{1, 2\}$ (both which are not connected to anything else), or it must

Figure 1.8: Connected sBCS graphs of type B_4 with flat 4-cycles.

Figure 1.9: The three choices for the beginning of a nonuniform sBCS graph to type F_4.
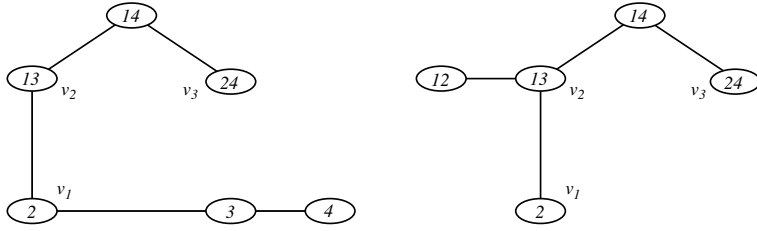
Figure 1.10: The possibilities for the continuation of a nonuniform sBCS graph to type F_4.
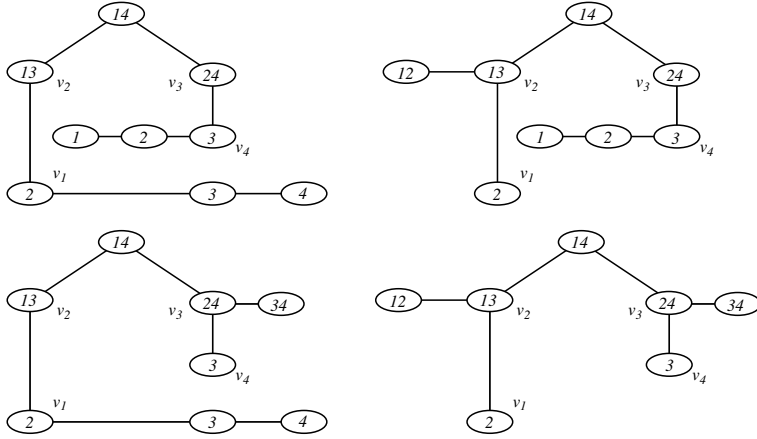
Figure 1.11: The possibilities for the ending of a nonuniform sBCS graph to type F_4.
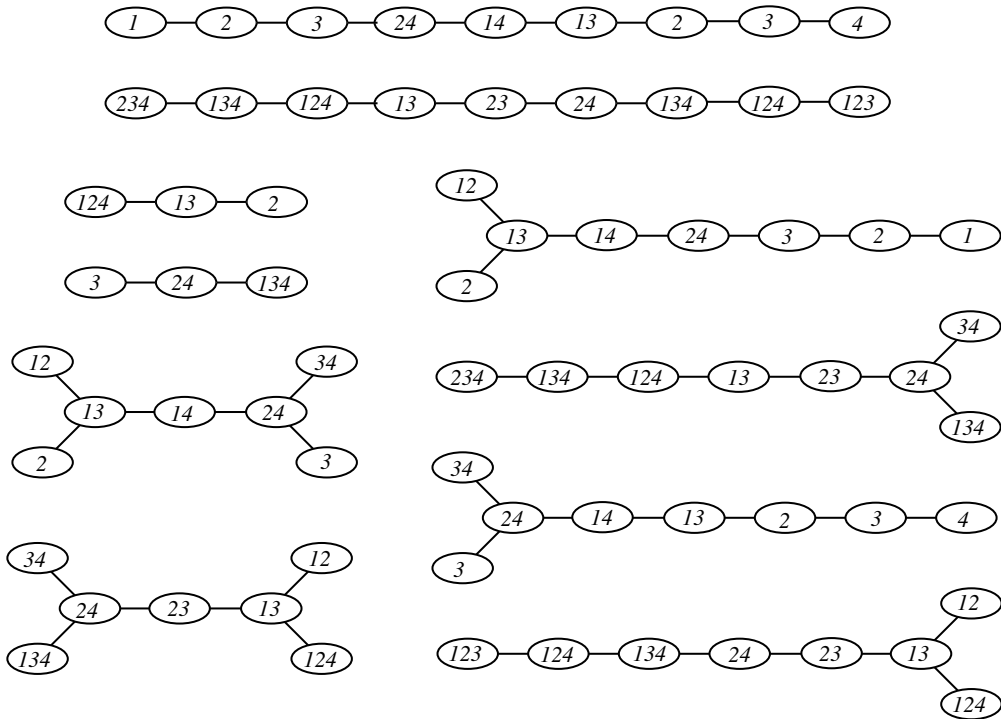
Figure 1.12: Non-uniform sBCS graphs of type F_4 with flat 4-cycles.

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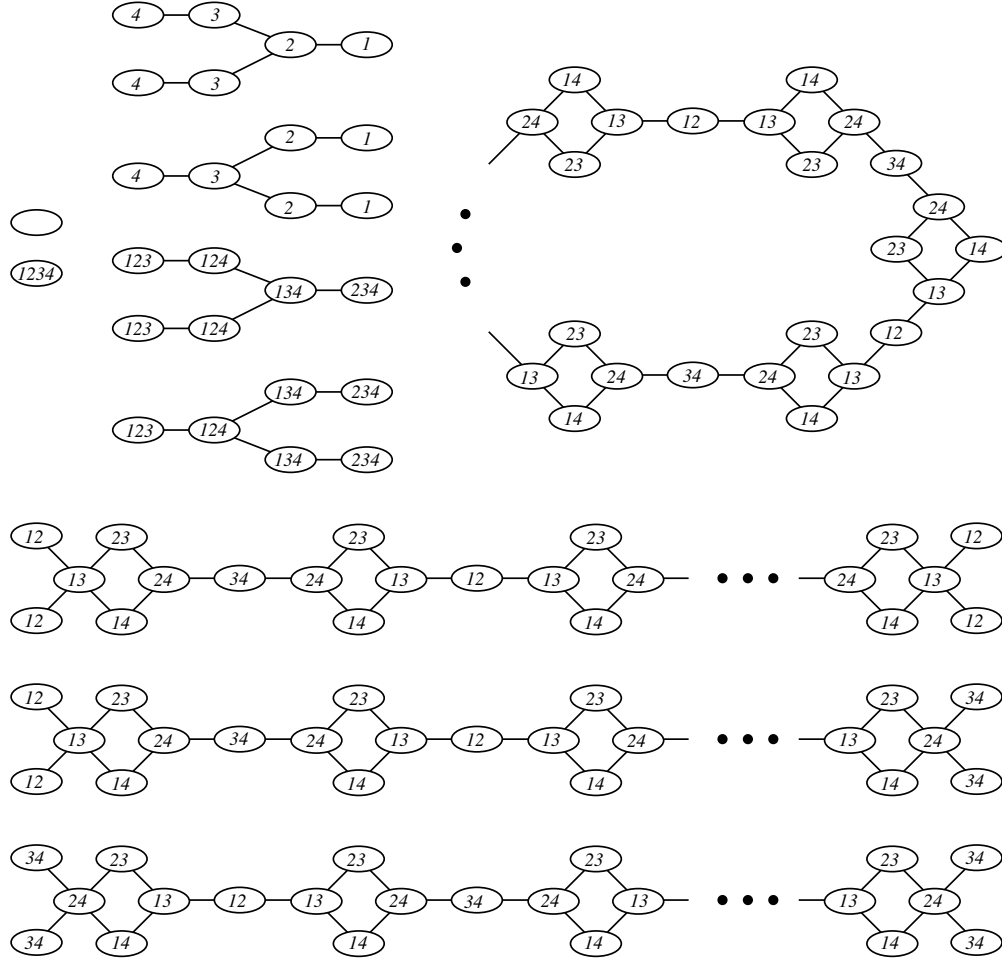
Figure 1.13: Uniform sBCS graphs of type F_4 with flat 4-cycles.

be connected to one vertex with τ-invariant $\{1, 2\}$ which in turn is connected to another vertex with τ-invariant $\{1, 3\}$. Similarly for a vertex with τ-invariant $\{3, 4\}$.

This is a complete classification; the possibilities are

- a path of 4-cycles linked by (13)—(12)—(13) and (24)—(34)—(24), with a fork on each end,

- a cycle of 4-cycles linked by the same connectors.

□

1.1.3 W-graphs

Definition 1.1.13. An admissible S-labeled graph is called an *admissible W-graph* if the formal span (over the ground ring $\mathbb{Z}[q^{\pm 1/2}]$) of its vertices carries a representation of the

Iwahori-Hecke algebra, with the action of the generators given by

$$
T_s(u) = \begin{cases} qu, & s \notin \tau(u), \\ -u + q^{1/2} \sum_{v:s\notin\tau(v)} m(u,v)v, & s \in \tau(u), \end{cases}
$$

(see section 1.2.1 for the defining relations of the Iwahori-Hecke algebra).

Remark 1.1.14. Notice that transposing the matrix for the action of each T_s will preserve the relations of the Hecke algebra. Hence in the above definition we could have used the transposed formula:

$$
T_s(u) = \begin{cases} qu + q^{1/2} \sum_{v:s\in\tau(v)} m(v,u)v, & s \notin \tau(u), \\ -u, & s \in \tau(u). \end{cases}
$$

This is the version which appeared in the papers of Kazhdan and Lusztig. The version in the definition was used by Stembridge since it is more natural from a combinatorial point of view.

We now describe the combinatorial characterization of admissible W-graphs following [Ste08a].

Suppose G is an S-labeled graph. For distinct $i,j \in S$, a directed path (possibly involving simple edges) $u \to v_1 \to v_2 \to \cdots \to v_{r-1} \to v$ in G is *alternating of type* (i,j) if

- $i,j \in \tau(u)$ and $i,j \notin \tau(v)$,

- $i \in \tau(v_k), j \notin \tau(v_k)$ for odd k,

- $i \notin \tau(v_k), j \in \tau(v_k)$ for even k.

Let

$$
N_{ij}^r(G; u, v) := \sum_{v_1,\dots,v_{r-1}} m(u,v_1)m(v_1,v_2)\dots m(v_{r-1},v),
$$

where the sum is over the set of alternating paths of type (i,j) from u to v and length r.

Theorem 1.1.15. *[Ste08a, Theorem 4.9] Suppose (W,S) is braid-finite (i.e. for any $s,t \in S$, the order of st in W is finite). An admissible S-labeled graph is a W-graph if and only if it satisfies SR, CR, BR (see Remark 1.1.2) and*

(PR) Suppose $i,j \in S$ with s_is_j of order p. For any $u,v \in V$ and any $2 \leqslant r \leqslant p$, we have

$$
N_{ij}^r(G; u, v) = N_{ji}^r(G; u, v).
$$

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This rule is called the Polygon Rule.

Definition 1.1.16. A *strongly admissible W-graph* is an sBCS graph which satisfies PR.

Remark 1.1.17. The Polygon Rule does put restriction on the possible arcs. In fact, in the two sBCS graphs at the bottom of Figure 1.12, no valid arrangement of arcs can satisfy the Polygon Rule. It is also easy to see that the Polygon Rule implies the flatness of 4-cycles assumption as defined in 1.1.6.

Now we introduce the "smallest" objects in the category of W-graphs.

Definition 1.1.18. A *W-cell* is a strongly connected W-graph. A *cell* of a W-graph is a subgraph induced by a strongly connected component of the W-graph.

All the admissible A_4-cells are shown in Figure 1.14.

Remark 1.1.19. There is no contradiction in the above terminology in that a cell of a W-graph is a W-cell. Indeed, it encodes a subquotient of the representation given by the W-graph as follows. All the vertices that can be reached out of the cell span a subrepresentation. All the vertices outside the cell span a subrepresentation of that. It is easy to see from the formulas for the action that the cell encodes the quotient representation.

1.1.4 Matrices for generators and paths

We now interpret combinatorially, in terms of paths in the W-graph, the matrix entries of the action of products of generators of the Iwahori-Hecke algebra in the W-graph basis. Some of these matrix entries will later be seen to be the Kazhdan-Lusztig polynomials.

Definition 1.1.20. Let $\mathbf{s} = (s_1, \ldots, s_l)$ be a word in S. A *path $u \to v$ of type $\mathbf{s}$* is a path $u = u_0 \to u_1 \to u_2 \cdots \to u_{l-1} \to u_l = v$, possibly with repeated entries, such that if $u_{i-1} \neq u_i$ then $s_i \in \tau(u_{i-1})$ and $s_i \notin \tau(u_i)$. The *weight of a step $u_{i-1} \to u_i$* in such a path is

$$
wt_i := \begin{cases} -1, & u_{i-1} = u_i, s_i \in \tau(u_i) \text{ (i.e. the path could have left } u_{i-1}), \\ q, & u_{i-1} = u_i, s_i \notin \tau(u_i) \text{ (i.e. the path could not have left } u_{i-1}), \\ q^{1/2} m(u_{i-1}, u_i), & \text{otherwise.} \end{cases}
$$

The *weight of a path* is the product of the weights of all the steps.

Proposition 1.1.21. *Let $\mathbf{s} = (s_1, \ldots, s_l)$ be a word in S. Let $M_{\mathbf{s}}$ be the matrix for the action of $T_{s_1} \ldots T_{s_l}$ on the W-graph basis in the KL convention (see Remark 1.1.14). Then the matrix entry $[M_{\mathbf{s}}]_{u,v}$ is the weighted sum of paths of type $(s_1, \ldots, s_l)$ from u to v.*

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Proof. For matrices $A_1 = [a_{ij}^1], \ldots, A_k = [a_{ij}^k]$ we have

$$[A_1 \ldots A_k]_{ij} = \sum_{i_1, \ldots, i_{k-1}} a_{i,i_1}^1 a_{i_1,i_2}^2 \ldots a_{i_{k-1},j}^k.$$

Apply this to our scenario. $\qquad\square$

1.1.5 W-molecules

We now consider smaller pieces of W-graphs which are sometimes easier to work with. Given a W-graph, an induced subgraph connected by simple edges must certainly be part of one cell (since it is strongly connected). This subgraph does not, in general, have to encode a representation. However it satisfies a restricted version of the Polygon Rule. We will now study the graphs which satisfy this restricted rule.

Definition 1.1.22. An admissible S-labeled graph is called an *admissible molecular graph* if it satisfies SR, CR, BR, and

(LPR2) For any $i, j \in S$ for any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$ and $\tau(v) \setminus \tau(u) \neq \varnothing$, we have
$$N_{ij}^2(G; u, v) = N_{ji}^2(G; u, v).$$

(LPR3) Let $k, i, j, l \in S$ be a copy of A_4 in the Coxeter graph: $k - i - j - l$. For any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$, $k, l \notin \tau(u)$, $k, l \in \tau(v)$, we have
$$N_{ij}^3(G; u, v) = N_{ji}^3(G; u, v).$$

The rules are collectively called the Local Polygon Rule. Similarly a *strongly admissible molecular graph* is an sBCS graph which satisfies LPR2 and LPR3.

Definition 1.1.23. An S-labeled graph is called a *molecule* if it is a molecular graph, and there is a path of simple edges between any pair of vertices.

Example 1.1.24. The admissible A_4-molecules (remember that for simply-laced types "admissible" and "strongly admissible" are the same) are shown in Figure 1.14. To classify these one starts with a classification of sBCS graphs in Figure 1.6 (flatness of 4-cycles is implied by LPR2) and draws in all possible arcs remembering that if $u \to v$ is an arc then $\tau(u) \supsetneq \tau(v)$ and the resulting graph is bipartite. The only possibilities for arcs arise in the two paths of length 5. Now imposing LPR2 from the vertex with τ-invariant $\{1, 4\}$ to the vertex with τ-invariant $\{3\}$ tells us that the weight of the arc from $\{1, 3\}$ to $\{3\}$ must be 1. Similarly for the other arcs. Thus to classify admissible A_4 molecules we only needed LPR2.

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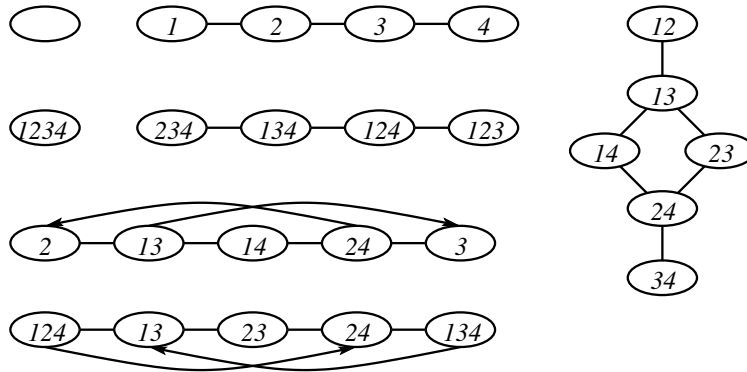
Figure 1.14: Admissible A_4-molecules (the arcs have weight 1).

When talking about an instance of LPR2 we will refer to an element of $\tau(v) \setminus \tau(u)$ as a *witness*. Thus the instance of LRP2 used in the example is of type $(1,4)$ with witness 3. Similarly when talking about an instance of LPR3 we will refer to elements k and l as *witnesses*.

The *simple part* of a molecule is the graph formed by erasing all the arcs. We usually view it as an undirected graph. A *morphism of molecules* $\varphi : M \to N$ is a map between the vertex sets which

1. is a graph morphism of the simple parts,

2. preserves τ-invariants.

Notice that a morphism does not need to respect arcs.

Remark 1.1.25. We would like to comment on the structure of alternating paths involved in the Local Polygon Rule. Apriori only the first and the last edges of an alternating path could be arcs. In fact, the additional assumptions on the τ-invariants of the starting and ending vertices force at least one of these edges to be simple. So any alternating path involved in the Local Polygon Rule contains at most one arc. This proves that an induced subgraph of a W-graph which is connected by simple edges is indeed a W-molecule.

1.1.6 Binding spaces

We now explore the question of how two molecular graphs can be glued by arcs to form another molecular graph. We follow [Ste08b]. Because the union should be bipartite, it is not clear which of the two sets of edges are allowed. To deal with this issue we extend the category to *molecular graphs with parity*; an object is a molecular graph with a bipartition into even vertices and odd vertices. Suppose G_1 and G_2 are two molecular graphs with parity.
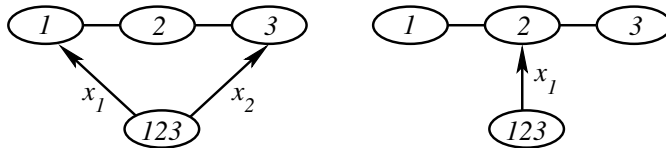
Figure 1.15: An example of binding spaces.

Suppose there are N possible arcs from G_1 to G_2 which respect parity (namely the arcs must go from even vertices of G_1 to odd vertices of G_2 and vice versa). For each $1 \leqslant i \leqslant N$ choose a variable x_i for the weight of the i-th arc. Since any path used in the Local Polygon Rule uses at most one arc, the weights of the arcs between G_1 and G_2 in no way affect the fact that the graphs are molecular graphs. Moreover, imposing Local Polygon Rule between vertices of G_1 and G_2 gives linear conditions on $x_1, \ldots, x_N$.

Definition 1.1.26. Suppose G_1 and G_2 are two molecular graphs with parity, and $x_1, \ldots, x_N$ are the weights of all the possible arcs which respect parity. Then the *binding space* is a linear subspace of $\mathbb{R}^N$ (with coordinate functions $x_1, \ldots, x_N$) cut out by the equations from the Local Polygon Rule.

Example 1.1.27. The binding spaces from ⬭123 to ①—②—③ are, respectively 2-dimensional and 1-dimensional; this is illustrated in Figure 1.15. There are no instances of Local Polygon Rule here since there are no witnesses.

The binding spaces from ①—②—③ to ⬭123 are both 0 since there are no possible arcs.

Note that $(0, \ldots, 0)$ is always in the binding space. The binding space is 0-dimensional precisely when there is no way to join the two graphs into one besides disjoint union.

Sometimes of particular interest are binding spaces from a molecule to itself. There are two of these: the *even binding space* when the parities of the molecules are the same, and the *odd binding space* when the parities of the molecules are different. We may also consider the *self binding space* of a molecular graph; namely the most general solution to the Local Polygon Rule in a given molecular graph. The self binding space of a molecular graph is an affine translate of the even binding space.

When trying to build cells out of molecules, it may be useful to construct a *binding graph*. The vertices of the graph are molecules with parity. The edges are present whenever the pairwise binding space is nonzero. This may be used to show that certain molecules cannot be connected to others inside a cell.

1.1.7 Restriction

Let $J \subseteq S$ and let W_J be the corresponding parabolic subgroup (i.e. the subgroup generated by J).

Let $M = (V, m, \tau)$ be an S-labeled graph. The W_J-*restriction* of M is $N = (V, m', \tau')$, with

1. for all $v \in V$, $\tau'(v) = \tau(v) \cap J$,

2. for all $u, v \in V$,

$$m'(u, v) = \begin{cases} 0, & \text{if } \tau'(u) \subseteq \tau'(v), \\ m(u, v), & \text{otherwise.} \end{cases}$$

If M is a W-molecular graph, then its W_J-restriction is a W_J-molecular graph. If M is a W-graph, then its W_J-restriction is a W_J-graph.

Suppose M is a W-molecular graph. A W_J-*submolecule* of M is a W_J-molecule (i.e. component connected by simple edges) of the W_J-restriction of M. There is a natural inclusion map of a W_J-submolecule into the original molecular graph. Sometimes, abusing notation, we refer to the image of this map as a W_J-submolecule. The sense in which we use the word should be clear from the context.

Remark 1.1.28. It is sometimes convenient to think of the Local Polygon Rule in terms of restriction. Suppose we are looking at an instance of LPR2 of type (i, j) with witness k. Then this instance of LPR2 holds if and only if it holds for the $W_{\{i,j,k\}}$ restriction. Similarly, an instance of LPR3 of type (i, j) with witnesses k, l holds if and only if it holds for the $W_{\{i,j,k,l\}}$ restriction.

1.2 Kazhdan-Lusztig Theory

The primary reason for introducing the definition of a W-graph was to study the examples of the construction which arise in Kazhdan-Lusztig theory.

This section summarizes the basics of this theory. The standard references are [KL79, Hum92].

1.2.1 Basic definitions

Let (W, S) be a Coxeter system of rank n. Then $S = \{s_1, \ldots, s_n\}$. The length function is denoted by $l(\cdot)$. The Bruhat order is denoted by $\leqslant$. For $w \in W$ denote by $\tau(w)$ (resp. $\tau_R(w)$) the left (resp. right) descent set of w.

For $s_i, s_j \in S$, let $m_{i,j}$ be the order of $s_i s_j$ in W. The Iwahori-Hecke algebra is a q-analogue of the group algebra of W; it is an algebra over the ground ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ generated by $T_1, \ldots, T_n$. The relations are

$$T_i^2 = (q-1)T_i + q,$$

as well as

$$(T_i T_j)^{m_{i,j}/2} = (T_j T_i)^{m_{i,j}/2} \qquad \text{if } m_{i,j} \text{ is even,}$$

$$(T_i T_j)^{(m_{i,j}-1)/2} T_i = (T_j T_i)^{(m_{i,j}-1)/2} T_j \qquad \text{if } m_{i,j} \text{ is odd.}$$

We let $\mathcal{H}$ denote this algebra.

Of interest to us are two well-known bases for $\mathcal{H}$: $\{T_w\}_{w \in W}$, and $\{C_w\}_{w \in W}$. The first is given by

$$T_w = T_{i_1} \ldots T_{i_r},$$

where $w = s_{i_1} \ldots s_{i_r}$ is a reduced expression of w. It is called the *standard basis* and is the analogue of the standard basis in the group algebra. The second basis is called the *Kazhdan-Lusztig basis*; it takes slightly more effort to define.

Notice that each T_i is invertible ($T_i^{-1} = q^{-1}(T_i - (q-1))$). Hence each T_w is also invertible. An involution $\bar{\cdot}$ on $\mathcal{H}$ may be given by

$$\overline{q^{1/2}} = q^{-1/2},$$

$$\overline{T_w} = (T_{w^{-1}})^{-1},$$

and extending to linear combinations.

Proposition 1.2.1. *For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ such that*

$$\overline{C_w} = C_w,$$

$$C_w = (-1)^{l(w)} q^{l(w)/2} \sum_{x \leqslant w} (-1)^{l(x)} q^{-l(x)} \overline{P_{x,w}} T_x,$$

for some collection of polynomials $P_{x,w} \in \mathbb{Z}[q]$ satisfying

$$P_{x,x} = 1, \text{ for all } x \in W, \qquad deg(P_{x,w}) \leqslant \frac{l(w)-l(x)-1}{2}, \text{ for all } x, w \in W.$$

Proof. See [KL79, Theorem 1.1]. $\qquad \square$

Let $d(x,w) = \frac{l(w)-l(x)-1}{2}$.

Notice that since the transition matrix between C_w and T_w is triangular (with powers of q on the diagonal), $\{C_w\}_{w \in W}$ must be a basis of $\mathcal{H}$. The polynomials $P_{x,w}$ are called the *Kazhdan-Lusztig polynomials*.

The cases when the Kazhdan-Lusztig polynomials reach the above degree bound are of particular importance. For $x \leqslant w$ define $\mu(x, w)$ as the coefficient of $q^{d(x,w)}$ in $P_{x,w}$. For $x > w$ define $\mu(x, w) = \mu(w, x)$. When x and w are not related in the Bruhat order, by convention let $P_{x,w} = 0$.

We now recall some formulas about Kazhdan-Lusztig polynomials that will be of use later.

Proposition 1.2.2. *If the length difference between x and w is 1 then $P_{x,w} = 1$ whenever $x < w$ (in the Bruhat order).*

Proof. See [KL79, Lemma 2.6 (iii)]. □

The Kazhdan-Lusztig polynomials may be computed recursively using the following formula:

Proposition 1.2.3. *Suppose $x, w \in W$ and $s \in \tau(w)$. Let*

$$c = \begin{cases} 1 & s \in \tau(x), \\ 0 & s \notin \tau(x). \end{cases}$$

Then

$$P_{x,w} = q^{1-c} P_{sx,sw} + q^c P_{x,sw} - \sum_{z < sw, s \in \tau(z)} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{x,z}.$$

Proof. See [KL79, (2.2.c)]. □

Proposition 1.2.4. *Suppose $x, w \in W$, $x < w$, and $s \in \tau(w) \setminus \tau(x)$. Then $P_{x,w} = P_{sx,w}$.*

Proof. See [KL79, (2.3.g)]. □

Proposition 1.2.5. *Suppose $x, w \in W$, $x < w$, and $s \in \tau(w) \setminus \tau(x)$. Then $\mu_{x,w} \neq 0$ if an only if $w = sx$.*

Proof. See [KL79, (2.3.e)]. □

1.2.2 Connection with W-graphs

One of the reasons that the μ values are important is that they encode all the information necessary to write down the left (or right) action of the generators of $\mathcal{H}$ on the Kazhdan-Lusztig basis:

Proposition 1.2.6. *Let* $w \in W$, $s \in S$. *If* $s \in \tau(w)$ *then*

$$T_s C_w = -C_w,$$

while if $s \notin \tau(w)$ *then*

$$T_s C_w = q C_w + q^{1/2} C_{sw} + q^{1/2} \sum_{z < w, s \in \tau(z)} \mu(z, w) C_z.$$

Notice that the above formulas look like the the formulas defining a W-graph (see Remark 1.1.14). More precisely,

Proposition 1.2.7. *Construct an S-labeled graph* $G = (V, \tau, m)$ *by:*

   1. $V = W$,

   2. for $v \in V$, $\tau(v)$ is the left descent set of v,

   3. for $v, w \in V$,

$$m(v, w) = \begin{cases} \mu(v, w) & \text{, if } \tau(v) \nsubseteq \tau(w) \\ 0 & \text{, otherwise.} \end{cases}$$

Then G is an admissible W-graph.

Proof. Most axioms of admissibility are clearly satisfied. The reason the resulting graph is bipartite is that $\mu(v, w)$ can only be nonzero when v and w have different signs (since the Kazhdan-Lusztig polynomials are polynomials in q and not $q^{1/2}$. It is a very nontrivial fact that the μ-values are nonnegative integers. It has been shown in the case of finite (and affine) Weyl groups by geometrical methods ([KL80]). It has recently been shown for all Coxeter groups by Elias and Williamson ([WE12]).

The formulas do give the regular representation in the Kazhdan-Lusztig convention. $\qquad\square$

Remark 1.2.8. It can be shown using basic properties of Coxeter groups that G is in fact strongly admissible. Moreover all strings lie vertically with respect to the Bruhat order.

From the results of Kazhdan and Lusztig we can describe certain properties of this W-graph.

Proposition 1.2.9. *Suppose (v, w) is an arc. Then either*

   1. $v < w$,

   2. $v = sw$ and $\tau(v) \setminus \tau(w) = \{s\}$.

Suppose (v, w) is a simple edge and $v < w$, then $\tau(w) \setminus \tau(v) = \{s\}$ for some s, and $v = sw$.

Proof. Both of these follow from Proposition 1.2.5. $\qquad\square$

Thus any arc that is oriented downward in the Bruhat order follows a weak order covering. We have seen in Proposition 1.2.2 that the weight of such an arc is always 1. Moreover, any weak order covering has a downward edge following it (sometimes the upward edge is also present, in which case we have a simple edge). Hence the Hasse diagram for the weak order consists of all the simple edges and all the downward arcs.

By the same reasoning, a Bruhat covering $v < w$ which is not a weak covering must either be an arc $v \to w$ or we must have $\tau(v) = \tau(w)$. All the remaining arcs are directed upward and have a length difference between top and bottom of at least three. Sometimes we refer to these as *surprising arcs*.

1.2.3   The inverse change-of-basis matrix

This section deals with a result of Kazhdan and Lusztig which allows one to express T_w in the C_w basis, provided that the group W is finite. Assume W is such, and let w_0 be the longest element.

Proposition 1.2.10. *If $x, w \in W$ and $x \leqslant w$, then*

$$\sum_{x \leqslant z \leqslant w} (-1)^{l(x)+l(z)} P_{x,z} P_{w_0 w, w_0 z} = \delta_{x,w}.$$

Proof. See [KL79, Theorem 3.1]. $\qquad\square$

Corollary 1.2.11. *For $w \in W$,*

$$T_w = \sum_{x \leqslant w} q^{l(w)-l(x)/2} \overline{P_{w_0 w, w_0 x}} C_x.$$

Proof.

$$
\begin{aligned}
\sum_{x \leqslant w} q^{l(w)-l(x)/2} \overline{P_{w_0 w, w_0 x}} C_x &= \sum_{x \leqslant w} q^{l(w)-l(x)/2} \overline{P_{w_0 w, w_0 x}} (-1)^{l(x)} q^{l(x)/2} \sum_{y \leqslant x} (-1)^{l(y)} q^{-l(y)} \overline{P_{y,x}} T_y \\
&= \sum_{y \leqslant w} q^{l(w)-l(y)} \left(\sum_{y \leqslant x \leqslant w} (-1)^{l(x)+l(y)} \overline{P_{y,x} P_{w_0 w, w_0 x}} \right) T_y \\
&= \sum_{y \leqslant w} q^{l(w)-l(y)} \delta_{y,w} T_y \\
&= T_w.
\end{aligned}
$$

$\qquad\square$

If e is the identity of W then C_e is the identity of $\mathcal{H}$. Thus

$$T_w C_e = \sum_{x \leqslant w} q^{l(w)} q^{-l(x)/2} \overline{P_{w_0 w, w_0 x}} C_x.$$

For $w \in W$, let $\left[M_w^{KL} \right]_{u,v \in W}$ be the matrix for the action of T_w with respect to the C_w basis. Then

$$P_{x,w} = q^{l(w_0 x) - l(w_0 w)/2} \overline{\left[M_{w_0 x}^{KL} \right]_{w_0 w, e}}.$$

We have seen in Proposition 1.1.21 that the matrix (the matrix for the action T_w on the W-graph basis is, in this case, exactly the same as M_w^{KL}) entry can be calculated from the W-graph. Hence in the case of a finite Coxeter group, the W-graph carries enough information to reconstruct the Kazhdan-Lusztig polynomials.

1.2.4 Parabolic variant

Deodhar ([Deo87]) generalized the above theory to parabolic quotients. Recall that above we constructed a basis for the regular representation of the Iwahori-Hecke algebra so that the action is described by a W-graph. For each parabolic subgroup W_J, Deodhar did this for two different representations, indexed by a parameter u which can take on the values -1 and q. Both of these have vertices in bijection with the set of unique shortest coset representatives W^J. The representations are induced from the sign representation of W_J (one-dimensional representation where each T_i acts by -1) and the "trivial representation" of W_J (one-dimensional representation where each T_i acts by q).

Detailed calculations, including the generalizations of the results in the previous section, were carried out by Couillens ([Cou99]). She worked in a slightly more general setting, namely with representations induced from any one-dimensional representations of W_J. This work was further generalized by Howlett and Yin [HY03, HY04] to inducing any representation given by a W-graph. While, as we will see further in the section, the W-graphs coming from Deodhar's construction are admissible, the more general induced graphs are not. We can see this already in Couillens' work: in Table IV of the appendix we see that the length difference between ts and $stuts$ is 3 and the corresponding polynomial is $-q$. Hence the μ values in this generalization are no longer nonnegative.

1.2.4.1 Notation

We begin by reviewing the notation. For $J \subseteq S$, W_J is the subgroup generated by J. Then W^J is the set of shortest coset representatives of W/W_J (see [Hum92, Chapter 5]). Let $w_{0,J}$

denote the longest element of W_J. Let $W^{J,max}$ denote the set of longest coset representatives of W/W_J:

$$W^{J,max} = \{ww_{0,J}|w \in W^J\}.$$

For $w \in W^J \cup W^{J,max}$, by $\tau(w)$ we continue to denote the left descent set *inside* W, i.e. $\tau(w) = \{s \in S|sw < w\}$. When dealing with parabolic quotients we will also need notation for the ascent set of an element of $w \in W^J$:

$$\text{Asc}(w) = \{s \in S : sw \in W^J \text{ and } sw > w\}.$$

Recall that W^J is an order ideal with respect to the left weak order, so if $w \in W^J$ and $s \in \tau(w)$ then automatically $sw \in W^J$. This is why the definition of the descent set remains unchanged. Similarly, $W^{J,max}$ is an order filter with respect to the weak order, so that $w \in W^{J,max}, s \notin \tau(w) \Rightarrow sw \in W^{J,max}$.

For the two representations of $\mathcal{H}$ mentioned in the introduction to this section ($u = -1$, and $u = q$), Deodhar constructed canonical bases $\left\{C_w^{[-1]}\right\}_{w \in W^J}$, and $\left\{C_w^{[q]}\right\}_{w \in W^J}$. Following Couillens, we will use the letter χ to denote either $[-1], [q]$; in this case we will use the notation ${}^j\chi$ to denote the opposite one.

The two representations also have standard bases: $\{T_w^\chi\}_{w \in W^J}$. These bases satisfy $T_w^\chi = T_w C_e^\chi$, where e is the identity in W. The change of bases matrix is given in terms of analogues of Kazhdan-Lusztig polynomials $\left\{P_{x,w}^\chi\right\}_{x,w \in W^J}$. These are polynomials in q which satisfy the same degree bound as the usual Kazhdan-Lusztig polynomials: $\deg(P_{x,w}^\chi) \leqslant d(x,w)$. The μ coefficients $\{\mu^\chi(x,w)\}_{x,w \in W^J}$ are defined in the same way as in the regular Kazhdan-Lusztig case.

The Kazhdan-Lusztig polynomials satisfy a similar recurrence to the regular ones:

Proposition 1.2.12. *Suppose $x, w \in W^J$ and $s \in \tau(w)$. Then*

$$P_{x,w}^{[-1]} = \widetilde{P} - \sum_{x \leqslant z \leqslant sw, s \notin \text{Asc}(z)} \mu(z,sw) q^{\frac{l(w)-l(z)}{2}} P_{x,z}^{[-1]},$$

where

$$\widetilde{P} = \begin{cases} P_{sx,sw}^{[-1]} + qP_{x,sw}^{[-1]}, & \text{if } s \in \tau(x), \\ qP_{sx,sw}^{[-1]} + P_{x,sw}^{[-1]}, & \text{if } s \in \text{Asc}(x), \\ (1+q)P_{x,sw}^{[-1]}, & \text{otherwise.} \end{cases}$$

Proof. See [Deo87, Proposition 3.9]. □

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Proposition 1.2.13. *Suppose* $x, w \in W^J$ *and* $s \in \tau(w)$. *Then*

$$P_{x,w}^{[q]} = \widetilde{P} - \sum_{x \leqslant z \leqslant sw, s \notin \tau z} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{x,z}^{[-1]},$$

where

$$\widetilde{P} = \begin{cases} P_{sx,sw}^{[-1]} + q P_{x,sw}^{[-1]}, & \textit{if } s \in \tau(x), \\ q P_{sx,sw}^{[-1]} + P_{x,sw}^{[-1]}, & \textit{if } s \in \mathrm{Asc}(x), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. See [Deo87, Proposition 3.10].   □

The polynomials $P_{x,w}^\chi$ are zero unless $x \leqslant w$. If $x \leqslant w$ then the constant coefficient of $P_{x,w}^{[-1]}$ is 1. This fails in the case $\chi = [q]$.

1.2.4.2   Action formulas

First we give the formulas for the action of $\mathcal{H}$ on the two bases:

Proposition 1.2.14. *[Cou99, Théorème 4.3] Let* $s \in S$, $w \in W^J$. *Then*

$$T_s C_w^{[q]} = \begin{cases} q C_w^{[q]} + q^{1/2} C_{sw}^{[q]} + q^{1/2} \sum\limits_{sz < z < w} \mu^{[q]}(z, w) C_z^{[q]}, & s \in \mathrm{Asc}(w), \\ q C_w^{[q]} + q^{1/2} \sum\limits_{sz < z < w} \mu(z, w) C_z^{[q]}, & s \notin \mathrm{Asc}(w) \cup \tau(w), \\ -C_w^{[q]}, & s \in \tau(w), \end{cases}$$

$$T_s C_w^{[-1]} = \begin{cases} q C_w^{[-1]} + q^{1/2} C_{sw}^{[-1]} + q^{1/2} \sum\limits_{\substack{z < w \\ s \notin \mathrm{Asc}(z)}} \mu^{[-1]}(z, w) C_z^{[-1]}, & s \in \mathrm{Asc}(w), \\ -C_w^{[-1]}, & s \notin \mathrm{Asc}(w). \end{cases}$$

This involves the parabolic μ coefficients, whose meaning we can clarify:

Proposition 1.2.15. *[Cou99, Corollaire 4.2] If* $x, w \in W^J$ *then*

$$\mu^{[q]}(x, w) = \mu(x, w), \quad \mu^{[-1]}(x, w) = \mu(x w_{0,J}, w w_{0,J}).$$

The first action formula tells us that the W-graph for the $u = q$ representation is precisely the induced subgraph of the full Kazhdan-Lusztig W-graph on W^J. The second formula has a similar interpretation, except we have to index the basis elements by $W^{J,max}$ instead of W^J. Namely for $w \in W^{J,max}$, let

$$\widetilde{C}_w^{[-1]} = C_{w w_{0,J}}^{[-1]}.$$

It can be seen that for $w \in W^{J,max}$, we have $s \in \tau(w)$ if and only if $s \notin \mathrm{Asc}(ww_{0,J})$. This follows from the following basic fact about Coxeter groups:

Fact 1.2.16. *If $w \in W^J$ and $s \in S$ then either $sw \in W^J$ or $sw = wt$ for some $t \in J$.*

Proof. Suppose $sw \notin W^J$. Then $swt < sw$ for some $t \in J$. Suppose $s_1 \ldots s_r = w$ is a reduced expression for w. By the Exchange Property, either $swt = w$ or $swt = ss_1 \ldots \widehat{s_i} \ldots s_r t$. The second case would contradict the assumption that the expression for w was reduced. Hence only the first case is possible, as desired. $\qquad\square$

Rewriting the action in these terms gives

Proposition 1.2.17. *Let $s \in S$, $w \in W^{J,max}$. Then*

$$
T_s \widetilde{C}_w^{[-1]} = \begin{cases} q\widetilde{C}_w^{[-1]} + q^{1/2}\widetilde{C}_{sw}^{[-1]} + q^{1/2} \displaystyle\sum_{\substack{z \in W^{J,max} \\ sz < z < w}} \mu(z,w)\widetilde{C}_z^{[-1]}, & s \notin \tau(w), \\[2em] -\widetilde{C}_w^{[-1]}, & s \in \tau(w). \end{cases}
$$

This tells us that the W-graph for the $u = -1$ representation is precisely the induced subgraph of the full Kazhdan-Lusztig W-graph on $W^{J,max}$.

1.2.4.3 Inversion

Deodhar showed (see [Deo87, Proposition 3.2]) that for $w \in W^J$,

$$
C_w^\chi = \sum_{\substack{x \in W^J \\ x \leqslant w}} (-1)^{l(x)+l(w)} q^{l(w)/2-l(x)} \overline{P_{x,w}^\chi} T_x^\chi.
$$

The analogue of the inversion formula was given by Couillens:

Proposition 1.2.18. *[Cou99, Proposition 6.1] For $x, w \in W^J$ with $x \leqslant w$,*

$$
\sum_{\substack{z \in W^J \\ x \leqslant z \leqslant w}} \varepsilon_z \varepsilon_w P_{x,z}^\chi P_{w_0 w w_{0,J}, w_0 z w_{0,J}}^{j\chi} = \delta_{x,w}.
$$

This may be turned into an explicit formula for expressing T_w^χ in terms of the canonical basis

Corollary 1.2.19. *Let $w \in W^J$. Then*

$$
T_w^\chi = \sum_{x \leqslant w} q^{l(w)-l(x)/2} \overline{P_{w_0 w w_{0,J}, w_0 x w_{0,J}}^{j\chi}} C_x^\chi.
$$

Proof.

$$\sum_{x \leqslant w} q^{l(w)-l(x)/2} \overline{P^{j\chi}_{w_0 w w_{0,J}, w_0 x w_{0,J}}} C^\chi_x =$$

$$= \sum_{\substack{x \in W^J \\ x \leqslant w}} q^{l(w)-l(x)/2} \overline{P^{j\chi}_{w_0 w w_{0,J}, w_0 x w_{0,J}}} \sum_{\substack{y \in W^J \\ y \leqslant x}} (-1)^{l(x)+l(y)} q^{l(x)/2-l(y)} \overline{P^\chi_{y,x}} T^\chi_y =$$

$$= \sum_{\substack{x \in W^J \\ x \leqslant w}} \sum_{\substack{y \in W^J \\ y \leqslant x}} (-1)^{l(x)+l(y)} q^{l(w)-l(y)} \overline{P^\chi_{y,x} P^{j\chi}_{w_0 w w_{0,J}, w_0 x w_{0,J}}} T^\chi_y =$$

$$= \sum_{\substack{y \in W^J \\ y \leqslant w}} (-1)^{l(w)+l(y)} q^{l(w)-l(y)} \sum_{\substack{x \in W^J \\ y \leqslant x \leqslant w}} (-1)^{l(x)+l(w)} \overline{P^\chi_{y,x}} \overline{P^{j\chi}_{w_0 w w_{0,J}, w_0 x w_{0,J}}} T^\chi_y =$$

$$= \sum_{\substack{y \in W^J \\ y \leqslant w}} (-1)^{l(w)+l(y)} q^{l(w)-l(y)} \delta_{y,w} T^\chi_y =$$

$$= T^\chi_w$$

$\square$

Thus, if e is the identity in W and $w \in W^J$ then

$$T_w C^\chi_e = \sum_{x \leqslant w} q^l(w) q^{-l(x)/2} \overline{P^{j\chi}_{w_0 w w_{0,J}, w_0 x w_{0,J}}} C^\chi_x.$$

For $w \in W^J$, let $\left[M^{\chi,KL}_w \right]_{u,v \in W}$ be the matrix for the action of T_w with respect to the C^χ_w basis. Then

$$P^\chi_{x,w} = q^{l(w_0 x w_{0,J})-l(w_0 w w_{0,J})/2} \overline{\left[M^{j\chi,KL}_{w_0 x w_{0,J}} \right]_{w_0 w w_{0,J}, e}}.$$

We have seen in Proposition 1.1.21 that the matrix entry can be calculated from the W-graph. It is somewhat surprising, however, that to get a parabolic Kazhdan-Lusztig polynomial of one parabolic representation one should need to do a weighted count of paths in a W-graph associated to a different parabolic representation. We will address this issue in the next section by showing that the two graphs are related.

1.2.4.4 Relationship between W-graphs on W^J and $W^{J,max}$

Consider the map on W^J given by $w \mapsto w_0 w$.

Proposition 1.2.20. *The above map gives a graph isomorphism between the W-graph on W^J and the one on $W^{J,max}$ which behaves nicely with respect to τ-invariants, namely:*

- *the map is a bijection between W^J and $W^{J,max}$,*

- *for $x, w \in W^J$, $\mu(x, w) = \mu(w_0 w, w_0 x)$,*

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- *for $w \in W^J, s \in S$, we have $s \in \tau(w) \Leftrightarrow w_0 s w_0 \notin \tau(w_0 w)$.*

Proof. First,

$$w \in W^J \Leftrightarrow \forall s \in J, ws > w \Leftrightarrow \forall s \in J, w_0 ws < w_0 w \Leftrightarrow w_0 w \in W^{J,max}.$$

Now $w \mapsto w_0 w$ is an involution, so this implies it is a bijection. The second part is the statement of Corollary 3.2 of [KL79]. Suppose $w \in W^J, s \in S$. Then

$$s \in \tau(w) \Leftrightarrow sw < w \Leftrightarrow w_0 sw > w_0 w \Leftrightarrow (w_0 sw_0)w_0 w > w_0 w \Leftrightarrow w_0 sw_0 \notin \tau(w_0 w).$$

$\square$

The matrix entries for the $u = q$ and the $u = -1$ cases are the same, namely:

Proposition 1.2.21. *Let $w \in W$, $w = s_1 \ldots s_r$ a reduced expression. Let M_w^χ be the matrix for the action of T_w on the parabolic W-graph basis of type χ, in the KL convention. If $\chi = [-1]$ this differs from $M_w^{\chi,KL}$ by reindexing of rows and columns with $W^{J,max}$ as opposed to W^J. Then*

$$[M_w^\chi]_{u,v} = [M_{w_0 w^{-1} w_0}^{j\chi}]_{w_0 v, w_0 u}.$$

Proof. By Proposition 1.1.21, $[M_w^\chi]_{u,v}$ is the weighted count of paths of type $\mathbf{s} = (s_1, \ldots, s_r)$ from u to v, where $s_1 s_2 \ldots s_r = w$ is a reduced expression. Then

$$w_0 w^{-1} w_0 = (w_0 s_r w_0) \ldots (w_0 s_1 w_0)$$

is a reduced expression of $w_0 w^{-1} w_0$.

Suppose we have a path

$$u_0 = u \to u_1, \to \cdots \to u_r = v$$

of the type $\mathbf{s}$. We claim that the path

$$w_0 v \to w_0 u_{r-1} \to \cdots \to w_0 u_0$$

has type $(w_0 s_r w_0, \ldots, w_0 s_1 w_0)$. We just need to check that if $w_0 u_{r-i} \neq w_0 u_{r-i-1}$ then $w_0 s_{r-i} w_0 \in \tau(w_0 u_{r-i})$ and $w_0 s_{r-i} w_0 \notin \tau(w_0 u_{r-i-1})$. But if $w_0 u_{r-i} \neq w_0 u_{r-i-1}$ then $u_{r-i} \neq u_{r-i-1}$. So $s_{r-i} \in \tau(u_{r-i-1})$ and $s_{r-i} \notin \tau(u_{r-i})$. An application of Proposition 1.2.20 finishes the verification.

The previous discussion gives us a bijection between the sets of paths in question. It is easy to see that the bijection is weight preserving. $\square$

Corollary 1.2.22. *For* $x, w \in W^J$,

$$P_{x,w}^{[q]} = q^{l(w_0 x w_{0,J}) - l(w_0 w w_{0,J})/2} \overline{\left[M_{w_0^J x^{-1}}^{[q]} \right]}_{w_0^J, w},$$

$$P_{x,w}^{[-1]} = q^{l(w_0 x w_{0,J}) - l(w_0 w w_{0,J})/2} \overline{\left[M_{w_0^J x^{-1}}^{[-1]} \right]}_{w_0, w w_{0,J}},$$

where w_0^J *is the longest element of* W^J.

CHAPTER 2

Parallel Transport

The term parallel transport will encompass two concepts. The first of these is arc transport: a collection of relations between edge weights (see below) that must hold in an admissible molecular graph. They are referred to as arc transport because each of them claims that the weight of an edge is equal to the weight of some other edge. The second is Kazhdan-Lusztig transport (originally proven in [KL79, Theorem 4.2]): a similar collection of relations which is known to hold in the Kazhdan-Lusztig graph.

We start by defining arc transport and showing that, in simply laced types, it is equivalent to the Local Polygon Rule. It is then easy to see that in simply-laced types an sBCS graph satisfying Kazhdan-Lusztig transport must satisfy the Local Polygon Rule. The converse is, in general, false; the quasi-minuscule sBCS graph for D_4 can be made into a molecular graph in infinitely many ways, but only one of these satisfies Kazhdan-Lusztig transport. We conjecture that the two are equivalent in type A.

We next move to study finite Coxeter systems with double bonds. The original definition of Kazhdan-Lusztig transport was based only on simple bonds. We find new relations which correspond to the double bond (see Definition 2.2.7) and show that these hold in the Kazhdan-Lusztig graph. Although these relations do not look quite like transporting an edge weight, we still call them the double-bond version of Kazhdan-Lusztig transport. We proceed to show that the new set of relations implies the Local Polygon Rule. Unlike the simply-laced case we do not have a good version of arc transport, so the proof is somewhat cumbersome.

Finally we give an application of the new version of Kazhdan-Lusztig transport. In [Gre09], Green conjectured that, in the Kazhdan-Lusztig graph of a Coxeter system, the weights of all arcs which begin at fully commutative elements are either 0 or 1, and proved it for type (affine) A. Later Gern in [Ger13] proved this conjecture for type D. We use the new relations, Stembridge's characterization of fully commutative elements in the type B Coxeter groups, and some structural results of Shi to prove the conjecture for type B.
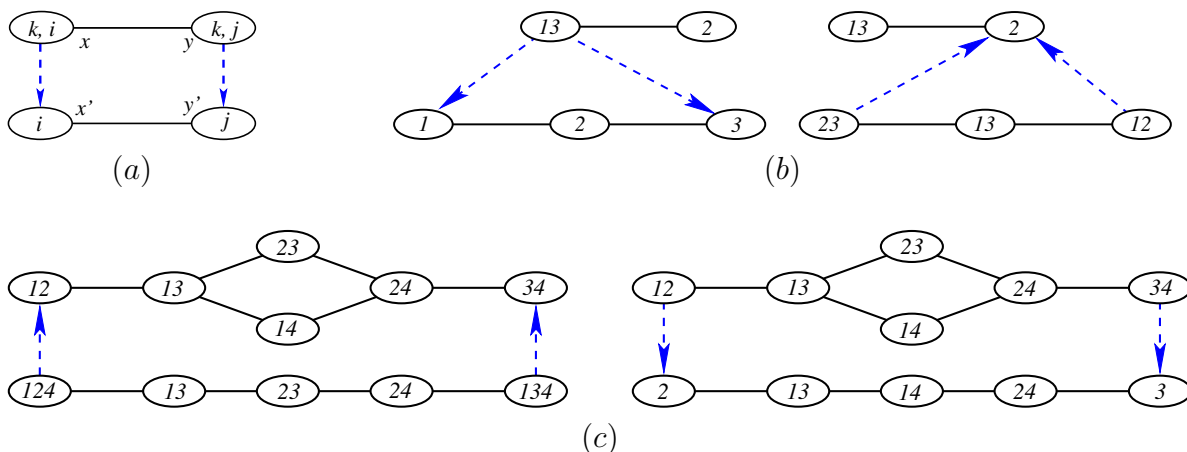
Figure 2.1: Arc transport.

2.1 Arc transport

Let (W, S) be a simply laced, finite Coxeter system.

Definition 2.1.1. Let G be an sBCS graph.

1. Suppose (x, y) and (x', y') are simple edges of G that activate the same bond, say (i, j). Without loss of generality, $i \in \tau(x) \cap \tau(x')$ and $j \in \tau(y) \cap \tau(y')$. Suppose moreover that there exists $k \in S$ such that $k \in \tau(x) \cap \tau(y)$ and $k \notin \tau(x') \cup \tau(y')$. If for all pairs of such edges we have $m(x, x') = m(y, y')$, then G is said to *satisfy AT1*.

   In picture notation (after restriction to the parabolic subgroup generated by $J = \{i, j, k\}$), the blue edges in Figure 2.1(a) must have the same weight:

2. Restrict G to any copy of A_3. If whenever one of the two simple edge configurations in Figure 2.1(b) occurs, the weights of the blue edges are equal, then G is said to *satisfy AT2*.

3. Restrict G to any copy of A_4. If whenever one of the two simple edge configurations in Figure 2.1(c) occurs, the weights of the blue edges are equal, then G is said to *satisfy AT3*.

As the following propositions demonstrate, the conditions AT1, AT2, and AT3 are equivalent to the Local Polygon Rule.

Theorem 2.1.2. *Suppose G is a molecular graph. Then G satisfies AT1, AT2, and AT3.*
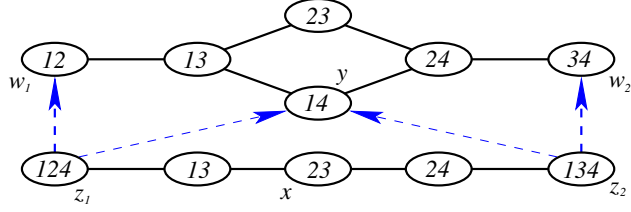
Figure 2.2: A labeling of some vertices of an instance of AT3.

Proof. First we prove that AT1 is satisfied. There are two evident instances of LPR2, namely

$$N^2_{ki}(G, x, y') = N^2_{ik}(G, x, y'),$$

and

$$N^2_{kj}(G, y, x') = N^2_{jk}(G, y, x').$$

Consider the first of these. Let us analyze when could there be other possible alternating paths besides the ones pictured. First look at alternating paths of type (k, i). They must pass through a vertex $z \neq y$ with $k \in \tau(z)$ and $i \notin \tau(z)$. Since $z \neq y$ and (x, z) is an edge, BR tells us that $j \notin \tau(z)$. Now (z, y') must be an edge, so by CR we have that (j, k) is a bond. Hence $N^2_{ki}(G, x, y') = m(y, y')$ unless (j, k) is a bond.

Now look at alternating paths of type (i, k). They must pass through a vertex $z \neq x'$ with $i \in \tau(z)$ and $k \notin \tau(z)$. Since $z \neq x'$ and (z, y') is and edge, BR tells us that $j \in \tau(z)$. Now (x, z) must be an edge, so by CR we have that (j, k) is a bond. Hence $N^2_{ik}(G, x, y') = m(x, x')$ unless (j, k) is a bond.

Thus the first instance of LPR2 gives the desired result unless (j, k) is a bond. By the same argument with i and j switched, the second instance of LPR2 gives the desired result unless (i, k) is a bond. But the Coxeter graph cannot contain triangles, so at least one of (i, k) and (j, k) cannot be a bond. Hence AT1 holds.

In case of AT2, apply the only evident instance of LPR2. By Remark 1.1.25 we are seeing all the possible paths involved, so the desired equality follows.

For AT3, we will just treat the left picture; the right one is done in the same way.

Label the vertices as shown in Figure 2.2

Applying LPR3 with regard to paths from x to y gives $m(z_1, y) = m(z_2, y)$; as above we can see all the possible paths. Since we know that G satisfies AT2, we have $m(z_1, y) = m(z_1, w_1)$ (we now restrict G to the parabolic subgroup generated by $J = \{2, 3, 4\}$), and $m(z_2, y) = m(z_2, w_2)$. This finishes the proof.

$\square$
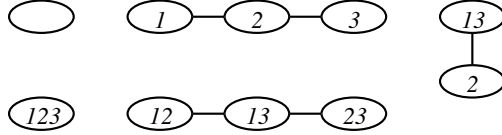
Figure 2.3: Admissible sBCS graphs of type A_3.

Theorem 2.1.3. *Suppose G is an sBCS graph which satisfies AT1, AT2, and AT3. Then G is a molecular graph.*

Proof. We wish to show that G satisfies the Local Polygon Rule. First we take care of the length 2 paths. Suppose $i, j, k \in S$, $u, w \in G$, $i, j \in \tau(u) \setminus \tau(w)$, $k \in \tau(w) \setminus \tau(u)$.

If k is not bonded to either i or j, then no alternating paths are possible so the local polygon relation is trivially satisfied. Without loss of generality, assume $k \sim j$. We will construct a weight-preserving bijection between ij-alternating paths and ji-alternating paths.

First assume that k and i are not bonded. Suppose $u \to v \to w$ is an ij-alternating path. Then $k \in \tau(v)$ and the edge (u, v) is simple and activates the bond $k - j$. Let $v' \in G$ be the unique vertex connected to w such that $j \in \tau(v')$ and $k \notin \tau(v')$. Then $i \notin \tau(v')$ since k and i are not bonded. Since G satisfies AT1 , we know that $\mu(v, w) = \mu(u, v')$ and $u \to v' \to w$ is a ji-alternating path. Applying a symmetric argument to $u \to v' \to w$ yields $u \to v \to w$ (this depends on v being the unique vertex connected to u by a $j - k$ activating bond, which in turn depends on simplicity). This gives the desired bijection, finishing this case.

Now suppose k is bonded to both i and j. Then i, k, j form a copy of A_3 in the Dynkin diagram; clearly the instance of the Local Polygon Rule holds if and only if it holds in this restriction to A_3 (we relabel i to 1, k to 2, and j to 3). The only admissible sBCS graphs of type A_3 are the molecules; they are reproduced in Figure 2.3.

The vertex u could only be in the two molecules with a vertex with descent set $\{1, 3\}$. Similarly w could only be in the two molecules with a vertex with descent set $\{2\}$. This leaves four cases to analyze. One of these is trivial since there are no possible arcs between two copies of the two-vertex molecule. Since AT1 and AT2 are satisfied, the matching color edges in the diagrams in Figure 2.4 have the same weight. Since all vertices of any alternating path lie in the two molecules, one clearly sees that the Local Polygon Rule is indeed satisfied.

Thus we have shown that the length 2 local polygon relations are satisfied.

Consider an instance of the Local Polygon Rule of length 3; namely, after restricting to the corresponding copy of A_4, vertices u, w have $\tau(u) = \{2, 3\}$ and $\tau(v) = \{1, 4\}$. The only connected sBCS graphs of type A_4 which satisfy length 2 Local Polygon Rule are the molecules (see Example 1.1.24); they are reproduced in Figure 2.5.

The vertex u could only be in the bottom-left (non-uniform) molecule or the 2-uniform
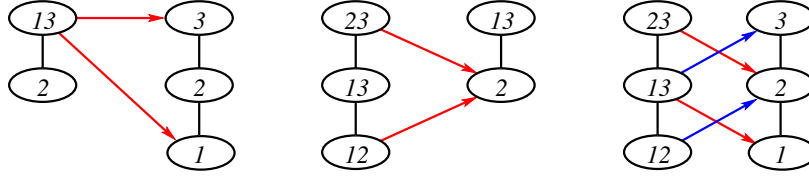
Figure 2.4: Relations imposed by AT1 and AT2. Edges of the same color have equal weights.
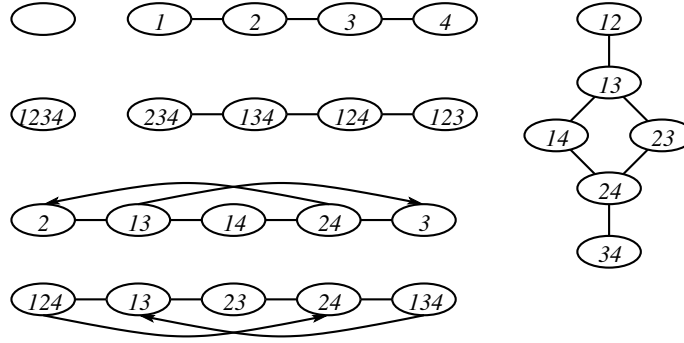
Figure 2.5: Admissible sBCS graphs of type A_4 which satisfy LPR2.

molecule, as these are the only molecules with a vertex with descent set $\{2, 3\}$. Similarly w could only be in the other non-uniform molecule or in another copy of the 2-uniform molecule. This leaves four cases to analyze. However since there are no descent containments from the 2-uniform molecule to itself, that case is trivial. Since AT1, AT2, and AT3 hold, the matching color edges in the diagrams in Figure 2.6 have the same weight. Since all vertices of any alternating path lie in the two molecules, one clearly sees that the Local Polygon Rule is indeed satisfied.

$\square$
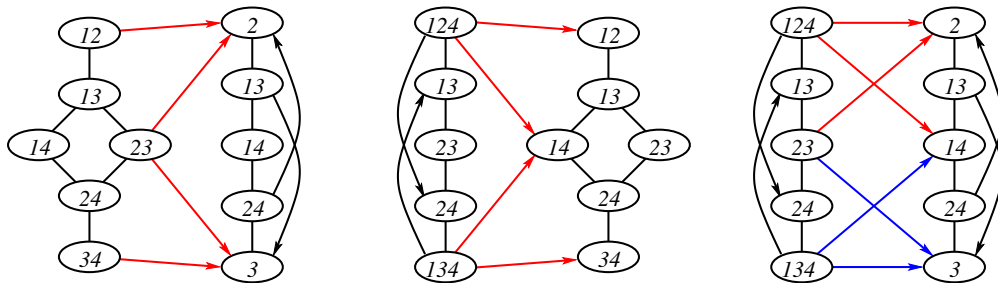
Figure 2.6: Relations imposed by AT1, AT2, and AT3. Edges of the same color have equal weights.

2.2 Kazhdan-Lusztig transport

2.2.1 Simply laced case

Definition 2.2.1. An sBCS graph $G = (V, \tau, m)$ *satisfies Kazhdan-Lusztig transport* if there exists a function $\mu : G \times G \to \mathbb{Z}^{\geq 0}$ such that

1. for $v, w \in G$, $\mu(v, w) = \mu(w, v)$,

2. if $v, w \in G$ and $\tau(v) \nsubseteq \tau(w)$ then $m(v, w) = \mu(v, w)$,

3. if $v_1 - w_1$ and $v_2 - w_2$ are two simple edges in G which activate the same bond, then $\mu(v_1, v_2) = \mu(w_1, w_2)$.

Remark 2.2.2. The original paper of Kazhdan and Lusztig proves that the Kazhdan-Lusztig W-graph satisfies Kazhdan-Lusztig transport with their μ ([KL79, Theorem 4.2]).

Remark 2.2.3. The reason the Kazhdan-Lusztig transport does not fit well with the admissible W-graph paradigm is that we can have $\mu(x, w) \neq 0$ even when $\tau(x) = \tau(w)$. Thus even edges which are irrelevant for the representation can in fact participate. As mentioned in the introduction, this can sometimes lead to Kazhdan-Lusztig transport implying more relations than Local Polygon Rule.

Corollary 2.2.4. *An sBCS graph G that satisfies Kazhdan-Lusztig transport is a molecular graph.*

Proof. By Theorem 2.1.3 it is sufficient to prove that the three versions of arc transport hold. This is obvious. $\qquad\square$

Remark 2.2.5. We know that the converse is false. In fact, even the full Polygon Rule does not imply Kazhdan-Lusztig transport; the quasi-minuscule sBCS graph for E_8 can be extended to three distinct admissible graph structures ([Ste]), but only one satisfies Kazhdan-Lusztig transport.

Conjecture 2.2.6. In type A, an sBCS graph G satisfies Kazhdan-Lusztig transport if and only if it is a molecular graph.

2.2.2 Double bond

In this section we formulate the notion of Kazhdan-Lusztig transport for finite Coxeter groups with a double bond, and prove the analogues of the above results.
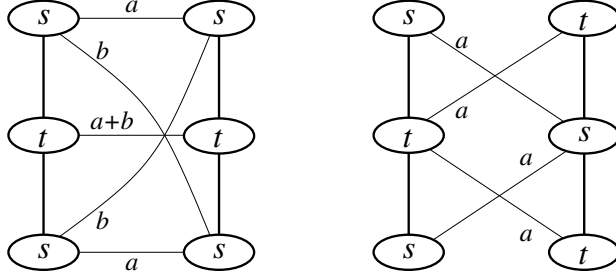
Figure 2.7: Kazhdan-Lusztig transport relations associated to a double bond.

Let (W, S) is a finite Coxeter system with a double bond (e.g. type B, type F). Suppose $s, t \in S$ form the double bond. Supposed G is an sBCS graph for this system. Then the parabolic restriction to the subgroup generated by s, t consists of the following connected components:

1. isolated vertices with both s and t in their τ-invariant,

2. isolated vertices with neither s nor t in their τ-invariant,

3. sts - strings,

4. tst - strings.

Definition 2.2.7. An *sBCS* graph *satisfies Kazhdan-Lusztig transport* if there exists a function $\mu : G \times G \to \mathbb{Z}^{\geq 0}$ such that

1. for $v, w \in G$, $\mu(v, w) = \mu(w, v)$,

2. if $v, w \in G$ and $\tau(v) \not\subseteq \tau(w)$ then $m(v, w) = \mu(v, w)$,

3. if (v_1, w_1) and (v_2, w_2) are two simple edges in G which activate the same simple bond, then $\mu(v_1, v_2) = \mu(w_1, w_2)$.

4. tst-strings and sts-strings are linked as shown in Figure 2.7, with edges labeled by the μ-values.

2.2.3   The case of the Kazhdan-Lusztig W-graph

We now prove the analogue of Remark 2.2.2, namely that the Kazhdan-Lusztig W-graph satisfies the Kazhdan-Lusztig transport with μ defined in terms of Kazhdan-Lusztig polynomials.
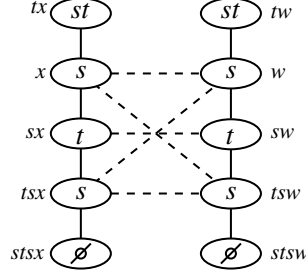
Figure 2.8: Two *sts*-strings.

We start by some calculations using the Kazhdan-Lusztig recurrence. Define $\nu(x, w)$ to be the coefficient of $q^{d(x,w)}$ in $P_{x,w}$. Thus $\nu(x, w) = \mu(x, w)$ if $x < w$ and $\nu(x, w) = 0$ otherwise.

Lemma 2.2.8. *Suppose $x, w \in W$ are both topmost (in Bruhat order) elements of sts-strings. Then*

1. *if $x < w$ then $\nu(x, w) = \nu(sx, sw) - \nu(x, tsw)$,*

2. *if $sx < sw$ then $\nu(sx, sw) = \nu(tsx, tsw) + \nu(x, tsw)$,*

3. *if $tsx < w$ then $\nu(tsx, w) = \nu(sx, sw) - \nu(tsx, tsw)$.*

Proof. While doing the calculations it may be helpful to keep in mind the picture of the two strings (see Figure 2.8).

(1) Suppose that $x < w$. Then we may apply the Kazhdan-Lusztig recurrence (Proposition 1.2.3) with respect to s:

$$P_{x,w} = P_{sx,sw} + qP_{x,sw} - \sum_{\substack{z:s\in\tau(z)\\x\leqslant z<sw}} \nu(z, sw)q^{-l(z)/2}q^{l(w)/2}P_{x,z}.$$

Now since $t \in \tau(sw) \setminus \tau(x)$ we have $P_{x,sw} = P_{tx,sw}$. Considering the coefficients of $q^{d(x,w)}$ yields:

$$\nu(x, w) = \nu(sx, sw) + \nu(tx, sw) - \sum_{\substack{z:s\in\tau(z)\\x\leqslant z<sw}} \nu(x, z)\nu(z, sw).$$

At least one of the W-graph edges (x, z) and (z, sw) must be simple since $\tau(x) \not\supseteq \tau(sw)$. If $t \in \tau(z)$ then it is the first one and $z = tx$. If not, then it is the second one and $z = tsw$. Both tx and tsw have s in their τ-invariants, so the equation becomes

$$\nu(x, w) = \nu(sx, sw) + \nu(tx, sw) - 1 \cdot \nu(tx, sw) - \nu(x, tsw) \cdot 1 = \nu(sx, sw) - \nu(x, tsw).$$

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(2) Suppose that $sx < sw$. Applying the Kazhdan-Lusztig recurrence with respect to t yields:

$$P_{sx,sw} = P_{tsx,tsw} + qP_{sx,tsw} - \sum_{\substack{z:t\in\tau(z)\\ sx\leqslant z<tsw}} \nu(z,tsw)q^{-l(z)/2}q^{l(sw)/2}P_{sx,z}.$$

Now since $s\in\tau(tsw)\setminus\tau(sx)$ we have $P_{sx,tsw} = P_{x,tsw}$. Consider the coefficients of $q^{d(sx,sw)}$:

$$\nu(sx,sw) = \nu(tsx,tsw) + \nu(x,tsw) - \sum_{\substack{z:t\in\tau(z)\\ sx\leqslant z<tsw}} \nu(sx,z)\nu(z,tsw).$$

At least one of the W-graph edges (sx,z) and (z,tsw) must be simple since $\tau(sx)\not\supseteq\tau(tsw)$. If $s\in\tau(z)$ then it is the first one and $z=x$. If not, then it is the second one and $z=stsw$. Neither tx nor tsw have t in their τ-invariants, so the equation becomes

$$\nu(sx,sw) = \nu(tsx,tsw) + \nu(x,tsw).$$

(3) Suppose that $tsx < w$. Applying the Kazhdan-Lusztig recurrence with respect to s yields:

$$P_{tsx,w} = P_{stsx,sw} + qP_{tsx,sw} - \sum_{\substack{z:s\in\tau(z)\\ tsx\leqslant z<sw}} \nu(z,sw)q^{-l(z)/2}q^{l(w)/2}P_{tsx,z}.$$

Now since $t\in\tau(sw)\setminus\tau(stsx)$ and $t\in\tau(sw)\setminus\tau(tsx)$ we have $P_{stsx,sw} = P_{tstsx,sw}$ and $P_{tsx,sw} = P_{sx,sw}$. The first of these implies that the coefficient of $q^{d(tsx,w)}$ in $P_{stsx,sw}$ is 0. Hence taking the coefficients of $q^{d(tsx,w)}$ in the above equation gives:

$$\nu(tsx,w) = \nu(sx,sw) - \sum_{\substack{z:s\in\tau(z)\\ tsx\leqslant z<sw}} \nu(tsx,z)\nu(z,sw).$$

At least one of the W-graph edges (tsx,z) and (z,sw) must be simple since $\tau(tsx)\not\supseteq\tau(sw)$. If $t\in\tau(z)$ then it is the first one and $z=sx$. If not, then it is the second one and $z=tsw$. Now $s\notin\tau(sx)$ while $s\in\tau(tsw)$, so the equation becomes

$$\nu(tsx,w) = \nu(sx,sw) - \nu(tsx,tsw).$$

$\square$

We will also need some calculations for when the two strings are different:

Lemma 2.2.9. *Suppose $x\in W$ is a top element of an sts-string, while $w\in W$ is a top element of a tst-string. Then*
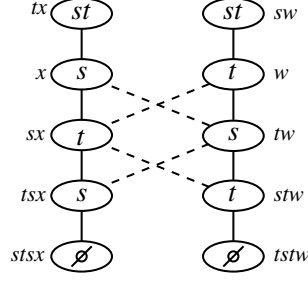
Figure 2.9: An *sts*-string and a *tst*-string.

1. *if $x < tw$ then $\nu(x, tw) = \nu(sx, stw)$,*

2. *if $sx < w$ then $\nu(sx, w) = \nu(tsx, tw) + \nu(x, tw) - \nu(sx, stw)$,*

3. *if $tsx < tw$ then $\nu(tsx, tw) = \nu(sx, stw)$.*

Proof. While doing the calculations it may be helpful to keep in mind the picture of the two strings (see Figure 2.9).

(1) Suppose $x < tw$. Applying the Kazhdan-Lusztig recurrence with respect to s yields:

$$P_{x,tw} = P_{sx,stw} + qP_{x,stw} - \sum_{\substack{z:s\in\tau(z) \\ x\leqslant z<stw}} \nu(z, stw)q^{-l(z)/2}q^{l(tw)/2}P_{x,z}.$$

Now since $t \in \tau(stw) \setminus \tau(x)$ we have $P_{x,stw} = P_{tx,stw}$. Consider the coefficients of $q^{d(x,tw)}$:

$$\nu(x, tw) = \nu(sx, stw) + \nu(tx, stw) - \sum_{\substack{z:s\in\tau(z) \\ x\leqslant z<stw}} \nu(x, z)\nu(z, stw).$$

At least one of the W-graph edges (x, z) and (z, stw) must be simple since $\tau(x) \not\supseteq \tau(stw)$. If $t \in \tau(z)$ then it is the first one and $z = tx$. If not, then it is the second one and $z = tstw$. Now $s \in \tau(tx)$ while $s \notin \tau(tstw)$, so the equation becomes

$$\nu(x, tw) = \nu(sx, stw) + \nu(tx, stw) - \nu(tx, stw) = \nu(sx, stw).$$

(2) Suppose that $sx < w$. Then we may apply the Kazhdan-Lusztig recurrence (Proposition 1.2.3) with respect to t:

$$P_{sx,w} = P_{tsx,tw} + qP_{sx,tw} - \sum_{\substack{z:t\in\tau(z) \\ sx\leqslant z<tw}} \nu(z, tw)q^{-l(z)/2}q^{l(w)/2}P_{sx,z}.$$

Now since $s \in \tau(tw) \setminus \tau(sx)$ we have $P_{sx,tw} = P_{x,tw}$. Considering the coefficients of $q^{d(sx,w)}$

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yields:

$$\nu(sx, w) = \nu(tsx, tw) + \nu(x, tw) - \sum_{\substack{z:t\in\tau(z)\\ sx\leqslant z<tw}} \nu(sx, z)\nu(z, tw).$$

At least one of the W-graph edges (sx, z) and (z, tw) must be simple since $\tau(sx) \not\supseteq \tau(tw)$. If $s \in \tau(z)$ then it is the first one and $z = x$. If not, then it is the second one and $z = stw$. Now $t \notin \tau(x)$ while $t \in \tau(stw)$, so the equation becomes

$$\nu(sx, w) = \nu(tsx, tw) + \nu(x, tw) - \nu(sx, stw).$$

(3) Suppose that $tsx < tw$. Applying the Kazhdan-Lusztig recurrence with respect to s yields:

$$P_{tsx,tw} = P_{stsx,stw} + qP_{tsx,stw} - \sum_{\substack{z:s\in\tau(z)\\ tsx\leqslant z<stw}} \nu(z, stw)q^{-l(z)/2}q^{l(tw)/2}P_{tsx,z}.$$

Now since $t \in \tau(stw) \setminus \tau(stsx)$ and $t \in \tau(stw) \setminus \tau(tsx)$ we have $P_{stsx,stw} = P_{tstsx,stw}$ and $P_{tsx,stw} = P_{sx,stw}$. Using the first of these equalities yields that $\deg P_{stsx,stw} < d(tsx, tw)$. Consider the coefficients of $q^{d(tsx,tw)}$ in the recurrence equation:

$$\nu(tsx, tw) = \nu(sx, stw) - \sum_{\substack{z:s\in\tau(z)\\ tsx\leqslant z<stw}} \nu(tsx, z)\nu(z, stw).$$

At least one of the W-graph edges (tsx, z) and (z, stw) must be simple since $\tau(tsx) \not\supseteq \tau(stw)$. If $t \in \tau(z)$ then it is the first one and $z = sx$. If not, then it is the second one and $z = tstw$. Neither sx nor $tstw$ have s in their τ-invariants, so the equation becomes

$$\nu(tsx, tw) = \nu(sx, stw).$$

$\square$

Now we can take care of the main part of the proof that Kazhdan-Lusztig transport is satisfied.

Lemma 2.2.10. *For any $x, w \in W$, if x, w are both top elements of sts-strings, then*

- $\mu(x, w) = \mu(tsx, tsw)$,

- $\mu(x, tsw) = \mu(tsx, w)$,

- $\mu(sx, sw) = \mu(x, w) + \mu(x, tsw)$.

If $x \in W$ is a top element of an sts-string, while $w \in W$ is a top element of a tst-string, then

$$\mu(x, tw) = \mu(sx, w) = \mu(sx, stw) = \mu(tsx, tw).$$

Proof. First suppose that x, w are both top elements of *sts*-strings. Suppose x and w are incomparable in the Bruhat order. Then so are the pairs (sx, sw) and (tsx, tsw). Thus $\mu(x, w) = \mu(sx, sw) = \mu(tsx, tsw) = 0$. We cannot have $tsx \geqslant w$ since otherwise x and w would be related. If $tsx < w$ then using Lemma 2.2.8 we have $\mu(tsx, w) = \nu(tsx, w) = 0$. If tsx and w are incomparable then also $\mu(tsx, w) = 0$. We cannot have $tsw \geqslant x$ since otherwise x and w would be related. If $tsw < x$ then using Lemma 2.2.8 (with x and w interchanged) we have $\mu(x, tsw) = \nu(tsw, x) = 0$. If tsw and x are incomparable then also $\mu(x, tsw) = 0$. Thus all the equalities hold since the relevant μ values are all 0.

We can now assume that x and w are related, and, without loss of generality that $x < w$. Then $sx < sw$, $tsx < tsw$, and $tsx < w$. So $\mu(x, w) = \nu(x, w), \mu(sx, sw) = \nu(sx, sw), \mu(tsx, tsw) = \nu(tsx, tsw), \mu(tsx, w) = \nu(tsx, w)$. If $x < tsw$, then $\mu(x, tsw) = \nu(x, tsw)$ and all the desired statements follow from Lemma 2.2.8.

Suppose $x \not< tsw$. Then Lemma 2.2.8 tells us that $\mu(x, w) = \mu(sx, sw) = \mu(tsx, tsw)$ and $\mu(tsx, w) = 0$. If $x \not> tsw$, then $\mu(x, tsw) = 0$ and all the necessary equalities hold. Hence suppose $x > tsw$. Then, since $tsx < tsw$, (tsw, x) must be a Bruhat covering. But s is a descent for both, so $(stsw, sx)$ is also a Bruhat covering. However $t \in \tau(sx) \setminus \tau(stsw)$, so we must have $stsw = tsx$. This is obviously false since the τ-invariants are different. Thus we have a contradiction, finishing the first half of the proof.

Now suppose x is a top element of an *sts*-string, while w is a top element of a *tst*-string.

Suppose first that x and tw are incomparable in the Bruhat order. Then so are sx and stw, and hence $\mu(x, tw) = \mu(sx, stw) = 0$. If tsx and tw are incomparable, then so are sx and w, and hence the statement of the lemma holds since the relevant μ values are all 0. So we can assume tsx and tw are comparable. Now $tsx \not> tw$ since otherwise x and tw would be related. So $tsx < tw$ (and hence also $sx < w$). Using the third part of Lemma 2.2.9 we then have $\mu(tsx, tw) = \nu(tsx, tw) = \nu(sx, stw) = 0$. Now using the second part of Lemma 2.2.9 we also have $\mu(sx, w) = \nu(sx, w) = 0$. Thus we have shown that if x and tw are incomparable then all the relevant μ values are 0 and the lemma holds. By symmetry, the same is true if sx and w are incomparable.

We thus assume that x and tw are comparable and that sx and w are comparable. Apriori there are four cases to be analyzed. However the case $x < tw$ and $sx > w$ is impossible due to partial order properties, and the cases $x < tw$, $sx < w$ and $x > tw$, $sx > w$ differ by exchanging s and t. We treat the cases $x < tw$, $sx < w$ and $x > tw$, $sx < w$ below.

Suppose $x < tw$, $sx < w$ (and hence $sx < stw$, and $tsx < tw$). Then $\mu(x, tw) = \nu(x, tw)$,

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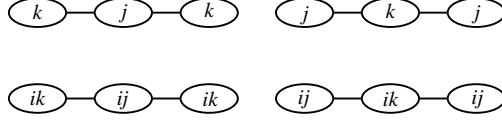
Figure 2.10: The connected sBCS graphs of type $B_2 \times A_1$.

$\mu(sx, w) = \nu(sx, w)$, $\mu(tsx, tw) = \nu(tsx, tw)$, $\mu(sx, stw) = \nu(sx, stw)$. Then the statement of the lemma follows from Lemma 2.2.9.

Suppose $x > tw, sx < w$ (and hence $sx > stw$ and $tsx < tw$). In this case all these relations have to be Bruhat coverings (since the length differences must be 1). Then $\mu(x, tw) = \mu(sx, w) = \mu(sx, stw) = \mu(tsx, tw) = 1$. $\qquad\qquad\square$

Finally, we have

Theorem 2.2.11. *The Kazhdan-Lusztig W-graph satisfies Kazhdan-Lusztig transport.*

Proof. The function μ is symmetric by definition. Part (2) of the definition follows from the definition of the Kazhdan-Lusztig W-graph. Part (3) was shown in [KL79, Theorem 4.2], while part (4) is proved in Lemma 2.2.10. $\qquad\qquad\square$

2.2.4 Graphs satisfying Kazhdan-Lusztig transport are molecular graphs

Now we prove an analogue of Corollary 2.2.4.

Theorem 2.2.12. *An sBCS graph G (for a finite Coxeter group with at most double bonds) which satisfies Kazhdan-Lusztig transport is a molecular graph.*

Proof. Suppose G satisfies Kazhdan-Lusztig transport.

First we check that LPR2 is satisfied. Suppose we have an instance of LPR2, namely $i, j, k \in S$, $u, v \in G$ such that $i, j \in \tau(u) \setminus \tau(v)$, and $k \in \tau(v) \setminus \tau(u)$.

If $k \not\sim i$ and $k \not\sim j$ then there are no alternating paths possible and LPR2 is vacuously satisfied. Thus we may suppose that $k \sim j$.

First consider the case when $k \not\sim i$. If (k, j) is a simple bond, then the case is handled as in Theorem 2.1.3. Suppose (k, j) is a double bond. To check that LPR2 holds we restrict to the parabolic subgroup generated by i, j, k, which is isomorphic to $B_2 \times A_1$. There are only four possible $B_2 \times A_1$ sBCS graphs; they are shown in Figure 2.10.

Hence the situation looks as one of the four cases in Figure 2.11. The green edges are the possible directed edges which will be involved in the instance of LPR2, with variables

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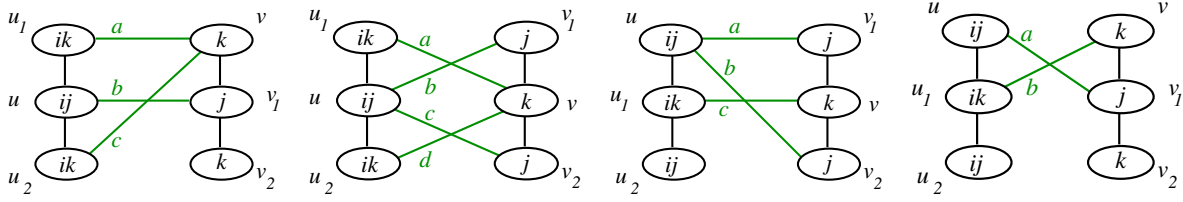
Figure 2.11: All possible instances of LPR2 for graphs of type $B_2 \times A_1$.
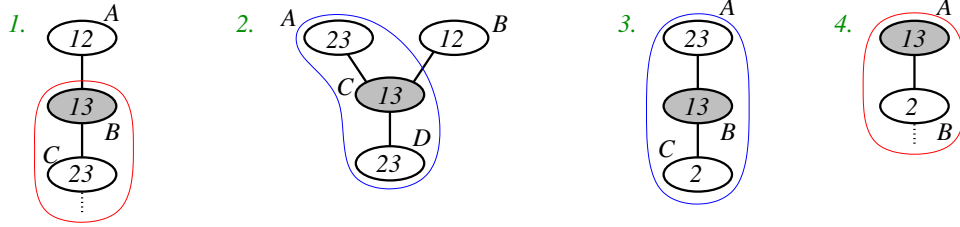
Figure 2.12: All the possible neighborhoods of u.

giving their weights. The equations which need to be satisfied are, respectively, $b = a + c$, $a + d = b + c$, $c = a + b$, $a = b$. It is clear that these follow from part (4) of Definition 2.2.7.

Now assume $k \sim j$ and $k \sim i$. If both bonds are simple, then we can use the same argument as in the simply-laced case. Thus we may assume that precisely one of the bonds is double and hence i, j, k generate a copy of B_3. Restrict G to this parabolic subgroup and rename the generators as in the standard B_3 (so k becomes 2 while i and j become 1 and 3 in whichever order appropriate). The sBCS graphs for B_3 were classified in Example 1.1.5. For the purposes of LPR2, we are only interested in the vertices adjacent to to u and v. So the only possible neighborhoods of u are shown in Figure 2.12. The shaded vertex is u, the red loops indicate 121-strings and the blue loops indicate 212-strings. Similarly, the possible neighborhoods of v are shown in Figure 2.13.

Apriori we need to check that LPR2 holds for every pair of neighborhoods. However, there is a symmetry which allows us to reduce the number of cases by a factor of 2. Taking
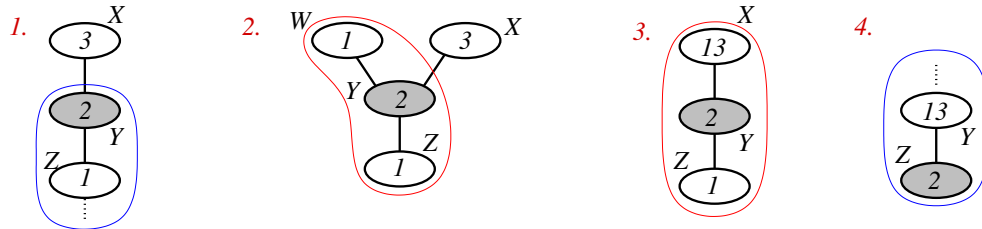
Figure 2.13: All the possible neighborhoods of v.

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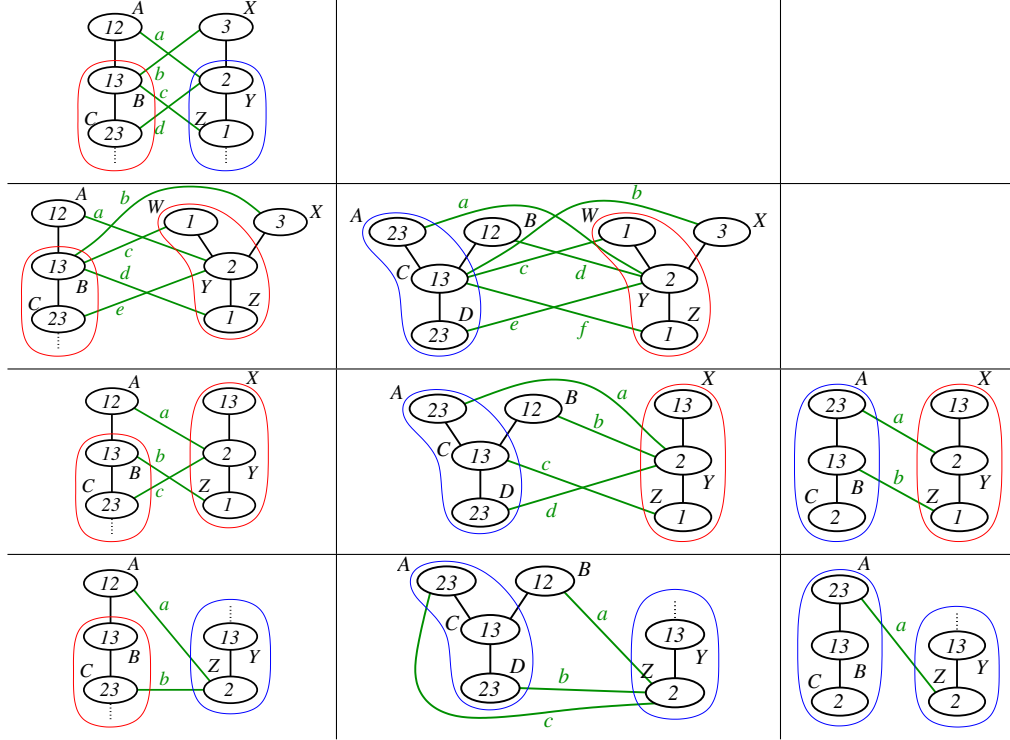
Figure 2.14: Possible instances of LPR2 among B_3 sBCS graphs.

complements of τ-invariants and reversing the arrows preserves the LPR equations which need to be satisfied, and preserves the Kazhdan-Lusztig transport moves which are necessary to show that the equations are satisfied. Thus, given the numbering of the neighborhoods above, LPR2 is satisfied from the i-th neighborhood of u to the j-th neighborhood of v if and only if it is satisfied from the j-th neighborhood of u to the i-th neighborhood of v. Finally, there are no arcs possible between the 4-th neighborhood of u and the 4-th neighborhood of v. The cases that arise are shown in Figure 2.14. The checking is done in Table 2.1. In each cell we give the equation dictated by LPR2 and the steps of Kazhdan-Lusztig transport necessary to see that it is satisfied. We label an equality by s if it follows from the single bond Kazhdan-Lusztig transport (part (3) of Definition 2.2.7), and label it by d if it follows from the double bond Kazhdan-Lusztig transport (part (4) of Definition 2.2.7).

Thus G satisfies LPR2.

Now we check that LPR3 is satisfied. Consider an instance of LPR3, namely $i, j, k, l \in S$ with $k \sim i, i \sim j, j \sim l$ and $u, v \in G$ with $i, j \in \tau(u) \setminus \tau(v)$ and $k, l \in \tau(v) \setminus \tau(u)$. There are four possibilities depending on whether $k - i - j - l$ generate a copy of A_4, B_4, or F_4.

The first case is handled in the same way as the simply-laced case. First assume k, i, j, l generate a copy of B_4 with (k, i) being the double bond. Restrict G to this copy of B_4 and rename k to 1, i to 2, etc. The only B_4 sBCS graphs which satisfy LPR2 were shown in

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LPR2 : $a + c = b + d$ $a \overset{s}{=} b, c \overset{d}{=} d$		
LPR2 : $a + c + d = b + e$ $a \overset{s}{=} b, c + d \overset{d}{=} e$	LPR2 : $a + b + e = c + d + f$ $a \overset{d}{=} c \overset{d}{=} e \overset{d}{=} f, b \overset{s}{=} d$	
LPR2 : $a + b = c$ $a \overset{s}{=} BX, c \overset{d}{=} b + CX$	LPR2 : $a + d = b + c$ $b \overset{s}{=} CX, a \overset{d}{=} CX \overset{d}{=} c \overset{d}{=} d$	LPR2 : $a = b$ $a \overset{d}{=} b$
LPR2 : $a = b$ $a \overset{s}{=} BY, BY \overset{d}{=} b$	LPR2 : $a = b + c$ $a \overset{s}{=} CY, CY \overset{d}{=} b + c$	LPR2 : $a = 0$ $BY \overset{d}{=} CZ + a, BY \overset{s}{=} CZ$

Table 2.1: Checking that instances of LPR2 are satisfied among B_3 sBCS graphs.

Proposition 1.1.9. The vertex u has τ-invariant $\{2, 3\}$, and as far as LPR3 is concerned, we are only interested in a neighborhood of radius 2. The possible neighborhoods of u are shown in Figure 2.15. The shaded vertex is u, the red loops indicate 121-strings and the blue loops indicate 212-strings. We omit any information about the internal arcs since it is irrelevant for the pairwise binding space.

Similarly, the possible neighborhoods of v are shown in Figure 2.16.

Again, we need to check that LPR3 holds for every pair of possible neighborhoods, but simplifications can be made as for LPR2. There are no edges from the i-th neighborhood of u to the j-th neighborhood of v if both i and j are greater than 4, and we only need to consider half of the cases for the same reason. The cases to be checked are shown in Figure 2.17. We check that LPR3 holds in each case in Table 2.2.

Finally assume i, j, k, l generate a copy of F_4 with (i, j) being the double bond. Restrict G to this copy of F_4 and rename k to 1, i to 2, etc. The only F_4 sBCS graphs which satisfy LPR2 were shown in Propositions 1.1.10 and 1.1.12. The possible neighborhoods of u are shown in Figure 2.18. The shaded vertex is u, the red loops indicate 121-strings and the blue loops indicate 212-strings. We omit any information about the internal arcs since it is irrelevant for the pairwise binding space.

Similarly, the possible neighborhoods of v are shown in Figure 2.19.

As above, we check all the cases in Figure 2.20. In this case there is an additional symmetry which comes from the symmetry of the Coxeter graph (everything is preserved under the automorphism mapping $1 \leftrightarrow 4, 2 \leftrightarrow 3$). This symmetry allows us to omit the third column of the table.

$\square$
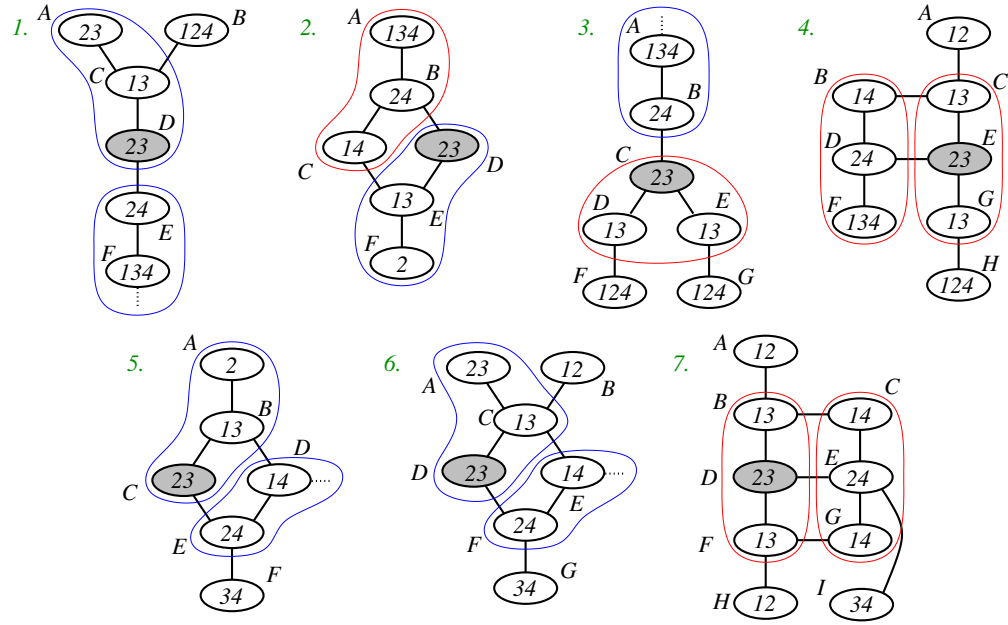
Figure 2.15: All the possible neighborhoods of u (type B_4).
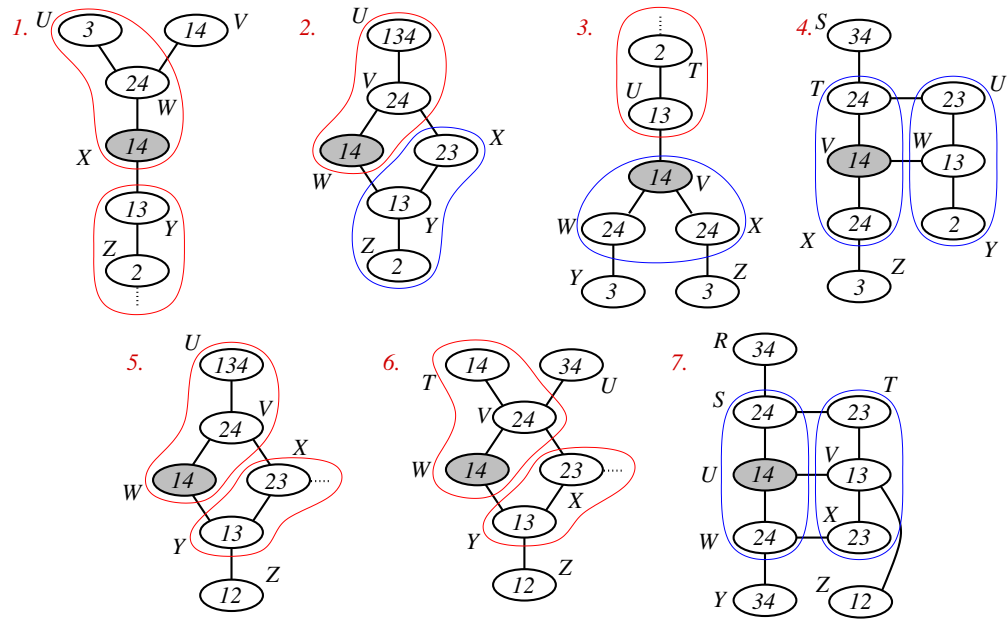
Figure 2.16: All the possible neighborhoods of v (type B_4).
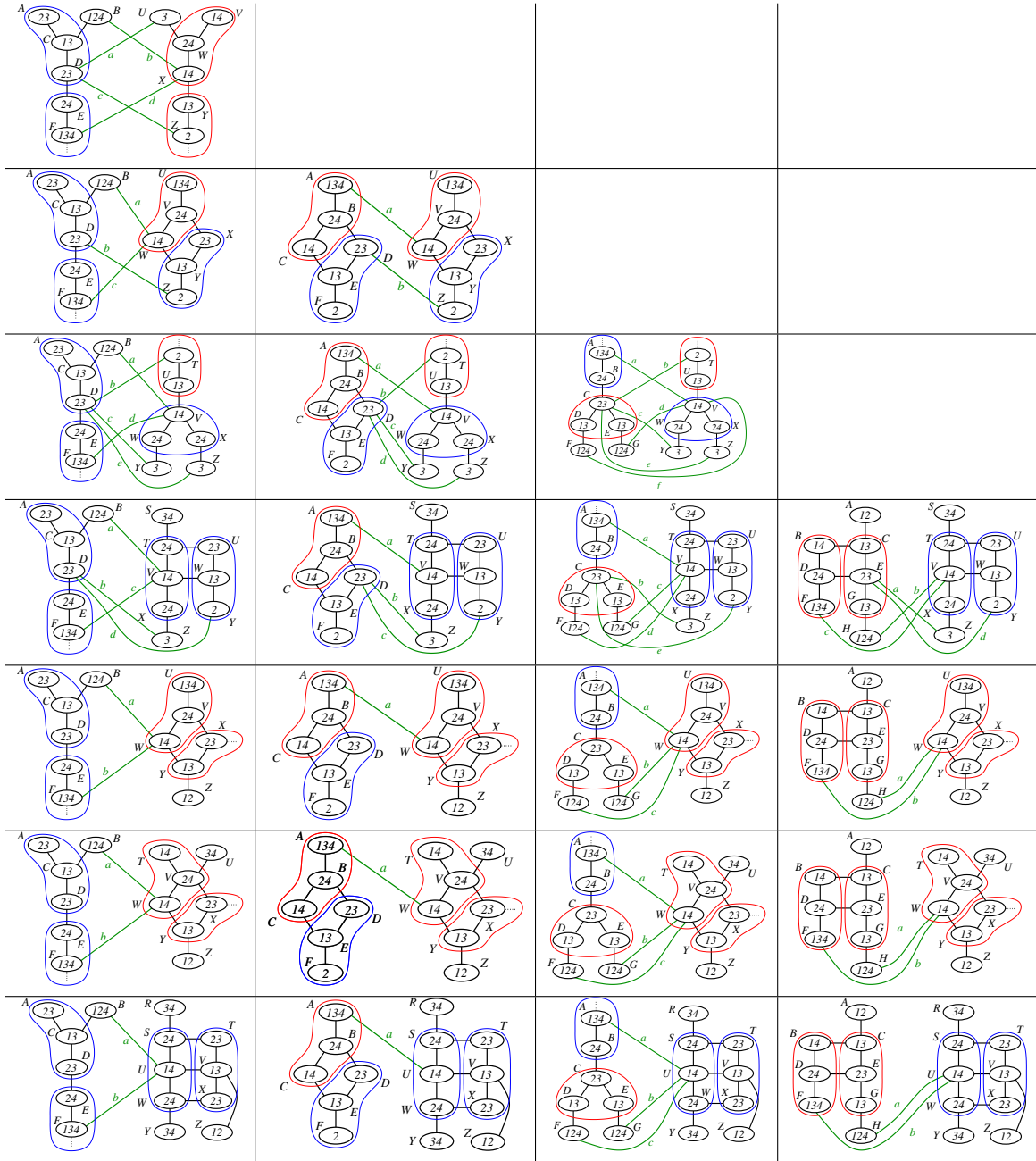
Figure 2.17: Possible instances of LPR3 among B_4 sBCS graphs.

$LPR3 : a + b = c + d$ $a \stackrel{s}{=} EW \stackrel{d}{=} d$ $b \stackrel{s}{=} CY \stackrel{d}{=} c$			
$LPR3 : a = b + c$ $a \stackrel{s}{=} CY \stackrel{d}{=} b + DX$ $c \stackrel{d}{=} EW \stackrel{s}{=} DX$	$LPR3 : a + b = 0$ $a + AU \stackrel{d}{=} BV \stackrel{s}{=} AU$ $b + FZ \stackrel{d}{=} EY \stackrel{s}{=} FZ$		
$LPR3 : a + c + e = b + d$ $b \stackrel{d}{=} CU \stackrel{s}{=} a$ $d \stackrel{d}{=} EW + EX \stackrel{s}{=} c + e$	$LPR3 : a + b = c + d$ $a \stackrel{d}{=} BW \stackrel{s}{=} c$ $a \stackrel{d}{=} BX \stackrel{s}{=} d$ $b \stackrel{d}{=} EU \stackrel{s}{=} CV \stackrel{d}{=} a$	$LPR3 : a + b = c + d + e + f$ $a \stackrel{d}{=} BW + BX \stackrel{s}{=} c + e$ $b \stackrel{d}{=} DU + EU \stackrel{s}{=} d + f$	
$LPR3 : a + b = c + d$ $a \stackrel{s}{=} CW \stackrel{d}{=} d + DU$ $c \stackrel{d}{=} ET + EX \stackrel{s}{=} ET + b \stackrel{s}{=} DU + b$	$LPR3 : b = a + c$ $b \stackrel{s}{=} BX \stackrel{d}{=} a$ $c + FY \stackrel{d}{=} EW \stackrel{s}{=} FY$	$LPR3 : b + c + d = a + e$ $c \stackrel{s}{=} EW \stackrel{d}{=} e$ $d \stackrel{s}{=} DW \stackrel{d}{=} e$ $a \stackrel{d}{=} BT + BX \stackrel{s}{=} CU + b \stackrel{d}{=} e + b$	$LPR3 : a + b = c + d$ $a \stackrel{s}{=} DX \stackrel{d}{=} c$ $b \stackrel{s}{=} GW \stackrel{d}{=} d$
$LPR3 : a = b$ $a \stackrel{s}{=} CY \stackrel{d}{=} DX \stackrel{s}{=} EV \stackrel{d}{=} b$	$LPR3 : a = 0$ $a + AU \stackrel{d}{=} BV \stackrel{s}{=} AU$	$LPR3 : a = b + c$ $a \stackrel{d}{=} BV \stackrel{s}{=} CX \stackrel{d}{=} DY + EY$ $b \stackrel{s}{=} EY, c \stackrel{s}{=} DY$	$LPR3 : a = b$ $a \stackrel{s}{=} GY \stackrel{d}{=} EX - CY \stackrel{s}{=} DV - CY$ $b \stackrel{d}{=} DV - BW \stackrel{s}{=} DV - CY$
$LPR3 : a = b$ $a \stackrel{s}{=} CY \stackrel{d}{=} DX \stackrel{s}{=} EV \stackrel{d}{=} b$	$LPR3 : a = 0$ $a \stackrel{d}{=} BV - CW \stackrel{s}{=} DX - EY \stackrel{d}{=} 0$	$LPR3 : a = b + c$ $a \stackrel{d}{=} BV \stackrel{s}{=} CX \stackrel{d}{=} DY + EY$ $b \stackrel{s}{=} EY, c \stackrel{s}{=} DY$	$LPR3 : a = b$ $a \stackrel{s}{=} GY \stackrel{d}{=} EX - CY \stackrel{s}{=} DV - CY$ $b \stackrel{d}{=} DV - BW \stackrel{s}{=} DV - CY$
$LPR3 : a = b$ $a \stackrel{s}{=} CV \stackrel{d}{=} DT + DX$ $b \stackrel{d}{=} ES + EW \stackrel{s}{=} DT + DX$	$LPR3 : a = 0$ $a \stackrel{d}{=} BS \stackrel{s}{=} DT$ $a \stackrel{d}{=} CU \stackrel{s}{=} EV$ $a \stackrel{d}{=} BW \stackrel{s}{=} DX$ $EV \stackrel{d}{=} DT + DX$	$LPR3 : a = b + c$ $a \stackrel{d}{=} BS + BW \stackrel{s}{=} CT + CX$ $CT \stackrel{d}{=} CX \stackrel{d}{=} DV \stackrel{d}{=} EV$ $b \stackrel{s}{=} EV, c \stackrel{s}{=} DV$	$LPR3 : a = b$ $a \stackrel{s}{=} GV \stackrel{d}{=} EX \stackrel{s}{=} DW \stackrel{d}{=} b$

Table 2.2: Checking that instances of LPR3 are satisfied among B_4 sBCS graphs.
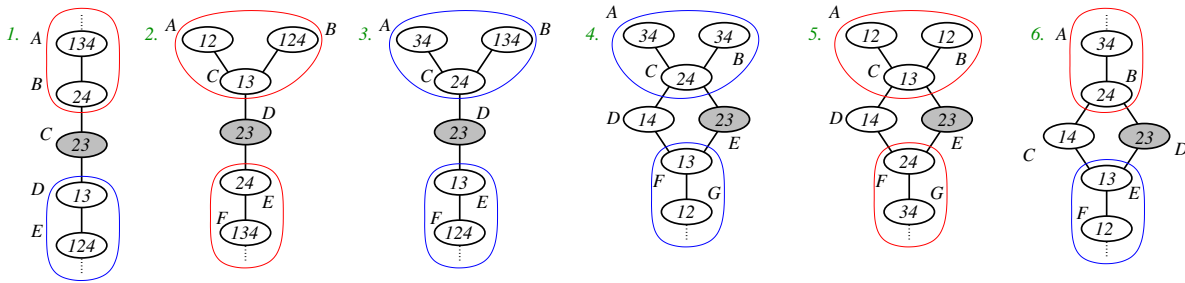
Figure 2.18: All the possible neighborhoods of u (type F_4).
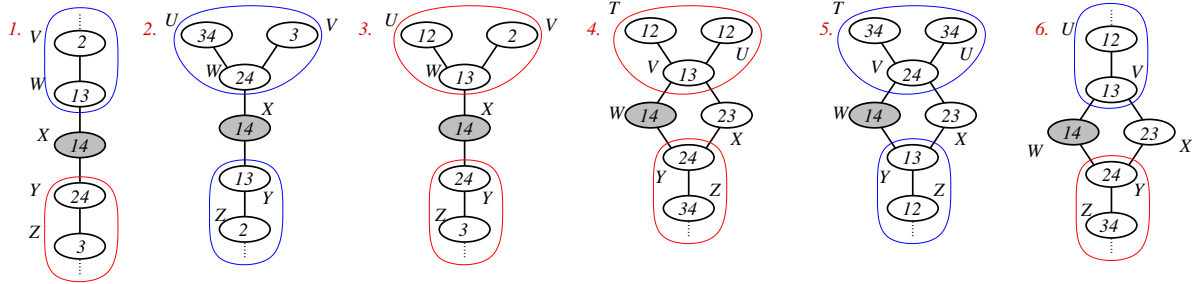
Figure 2.19: All the possible neighborhoods of v (type F_4).

2.3 Application: arcs beginning at fully commutative elements

In this section we are concerned with the Kazhdan-Lusztig W-graph of type B_n, so (W, S) is the Coxeter system of type B_n with the standard set of generators. We number the generators in a different fashion than usually in this case (see Figure 2.21). This agrees better with the explicit realization we will be using. So

$$S = \{s_0, \dots, s_{n-1}\}.$$

We complete the proof that the weight of an arc whose tail is a fully commutative element (defined originally in [Ste96]; see also Section 2.3.1.2) is either 0 or 1. This was shown for type $\widetilde{A}$ in [Gre09] (the same proof holds for the finite type A), and was conjectured to hold for all Coxeter groups. It was shown for type D in [Ger13].

2.3.1 Preliminaries

2.3.1.1 Explicit realization of the B_n Coxeter system

First, we explicitly realize the Coxeter group of type B_n as signed permutations of n elements. A detailed account of this construction may be found in section 8.1 of [BB05]. For example, the permutation

$$
\begin{array}{cccccc}
-3 & -2 & -1 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & -3 & 2 & -2 & 3 & -1
\end{array}
$$

will be written in one-line notation as $\overline{2}3\overline{1}$.

$$LPR3 : a+b=c+d$$
$$a \stackrel{s}{=} BY \stackrel{s}{=} c$$
$$b \stackrel{s}{=} DW \stackrel{s}{=} d$$

$$LPR3 : a+c=b+d \qquad LPR3 : a+b=c+d$$
$$a \stackrel{s}{=} BW \stackrel{s}{=} b \qquad a \stackrel{s}{=} CY \stackrel{s}{=} c$$
$$c \stackrel{s}{=} DY \stackrel{s}{=} d \qquad b \stackrel{s}{=} EW \stackrel{s}{=} d$$

$$LPR3 : a+b=c+d \qquad LPR3 : a+c=b+d$$
$$a \stackrel{s}{=} BY \stackrel{s}{=} c \qquad a \stackrel{s}{=} CW \stackrel{s}{=} b$$
$$b \stackrel{s}{=} DW \stackrel{s}{=} d \qquad c \stackrel{s}{=} EY \stackrel{s}{=} d$$

$$LPR3 : a=b \qquad LPR3 : a=b$$
$$a \stackrel{s}{=} BY \stackrel{s}{=} CX \stackrel{s}{=} \qquad a \stackrel{s}{=} CV \stackrel{s}{=} DX \stackrel{s}{=}$$
$$DV \stackrel{s}{=} b \qquad EY \stackrel{s}{=} b$$

$$LPR3 : a=b \qquad LPR3 : a=b$$
$$a \stackrel{s}{=} BV \stackrel{s}{=} CX \stackrel{s}{=} \qquad a \stackrel{s}{=} CY \stackrel{s}{=} DX \stackrel{s}{=}$$
$$DY \stackrel{s}{=} b \qquad EV \stackrel{s}{=} b$$

$$LPR3 : a=b \qquad LPR3 : a=b$$
$$a \stackrel{s}{=} BY \stackrel{s}{=} CX \stackrel{s}{=} \qquad a \stackrel{s}{=} CV \stackrel{s}{=} DX \stackrel{s}{=}$$
$$DV \stackrel{s}{=} b \qquad EY \stackrel{s}{=} b$$

Figure 2.20: Possible instances of LPR3 among F_4 sBCS graphs.

B_n:
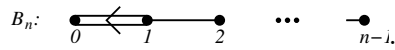
Figure 2.21: Alternative numbering of roots in the Dynkin diagram of B_n.

There is a good description of the right descent set $\tau_R(w)$ of a signed permutation w:

Proposition 2.3.1. *[BB05, Proposition 8.1.2] Suppose $w \in W$. Then*

$$\tau_R(w) = \{s_i \in S : w(i) > w(i+1)\},$$

where $w(0) := 0$.

For the above permutation we have $w(0) = 0$, $w(1) = -2$, $w(2) = 3$, and $w(3) = -1$. So $\tau_R(w) = \{0, 2\}$.

The description of the left descent set is not as nice in this realization, so throughout this section we deal with the "right" version of the Kazhdan-Lusztig graph as opposed to the "left" version (for example we will use τ_R, the right descent set, instead of τ). The results remain valid in the "left" version via the map $w \mapsto w^{-1}$.

2.3.1.2 Fully commutative elements

The fully commutative elements are a subset of a Coxeter group which enjoys numerous interesting combinatorial properties; they were introduced in [Ste96]. It is well known that one can move between reduced expressions of any $w \in W$ by using only the braid relations

$$sts \cdots = tst \ldots,$$

hence not changing the length at any point. An element $w \in W$ is *fully commutative* if the only braid relations necessary have of the form $st = ts$ (i.e. when s and t are commuting generators).

Fully commutative elements can be described in terms of their so-called heaps. Given a word $\mathbf{s} = (s_1, \ldots, s_l)$ in the simple reflections, the heap of $\mathbf{s}$ is a poset on $\{1, \ldots l\}$ which is the transitive closure of relations $i \prec j$, where $i \prec j$ if

1. $i < j$, and

2. $s_i = s_j$ or $s_i s_j \neq s_j s_i$.

One thinks of a heap as a "labeled poset," where i is labeled by s_i. If $w \in W$ is fully commutative, then all reduced expressions will yield isomorphic heaps. Moreover, in this case reduced expressions are precisely the linear orderings of the heap.

Proposition 2.3.2. *[Ste96, Proposition 2.3] The heap P of a word $\mathbf{s} = (s_1, \ldots, s_l)$ is a heap of a fully commutative element if and only if*

1. *there is no convex chain $i_1, \ldots, i_m$ in P such that $i_1 = s_3 = \cdots = s$, $s_2 = s_4 = \cdots = t$ and the order of st in W is $m \geqslant 3$,*

2. *there is no covering relation $i < j$ in P with $s_i = s_j$.*

One interesting property of the set of fully commutative elements which follows from this classification is that it is a union of Kazhdan-Lusztig molecules. Since fully commutative elements form an order ideal of the weak order ([Ste96, Proposition 1.4]), the only thing we need to check is that if there is a simple edge between x and w, for $x < w$ and x fully commutative, then w is fully commutative. This easily follows from the above proposition.

Hence Kazhdan-Lusztig molecules split into fully commutative ones and non-fully commutative ones. Note that this is not always an intrinsic feature of a molecule; for example in type D there exist pairs of isomorphic molecules, one of which is fully commutative and the other one is not. As we will see in a moment, this cannot happen in type B.

Another feature of fully commutative elements is a restriction on their descent set. We call $J \subset S$ *commutative* if for any $s, t \in J$ we have $st = ts$. The right descent set of a fully commutative element is commutative: any element in the right descent set of a fully commutative element must be a label of a maximal element in its heap. By the definition of a heap, no two maximal elements can have bonded labels.

The last feature makes it very convenient to apply Kazhdan-Lusztig transport with respect to a simple bond; more precisely:

Proposition 2.3.3. *Suppose $x \leqslant w$, $\tau_R(x) \supset \tau_R(w)$, and x is fully commutative. Suppose w' is connected to w by a simple edge which activates a simple bond. Then there exists x' connected by a simple edge to x with $\mu(x, w) = \mu(x', w')$.*

Proof. Suppose (s, t) is the bond in question and $s \in \tau_R(w)$. Then $s \in \tau_R(x)$. Since x is fully commutative, $t \notin \tau_R(x)$. So there exists an edge from x to some x' which activates the bond (s, t). The result follows by Kazhdan-Lusztig transport. $\qquad \square$

We will need a result of Shi ([Shi03]):

Proposition 2.3.4. *Suppose $w \in W$ is not fully commutative. Then any minimal element of the molecule of w in the Kazhdan-Lusztig graph has non-commutative $\tau_R(w)$.*

Proof. This is the content of Lemma 3.1 from [Shi03]. Two remarks are in order. First, the statement of the lemma has a typo, and the assumption is meant to be that J is non-commutative. Second, the proof is carried out in a more general context of affine $\widetilde{C_n}$, but all the proofs can be easily specialized to the corresponding finite type B_n. $\qquad \square$
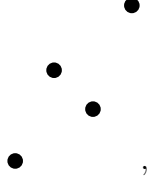
Figure 2.22: The descent structure relevant to Proposition 2.3.6.

The next result we need is a pattern-avoidance type characterization of fully commutative elements due to Stembridge ([Ste97]). Let us clarify what we mean by pattern avoidance with an example. We say that w avoids the pattern $(2, -1, -3)$ if there exist no integers $i < j < k$ such that $-w(k) < w(i) < -w(j) < 0$. Namely there exists three entries in the one-word notation for w out of which the second is smallest in absolute value and negative, the third is largest in absolute value and negative, while the first is medium in absolute value and positive. There is a notational discrepancy with [Ste97], namely we say that w avoids a pattern when [Ste97] would say w^{-1} avoids it. However since the set of fully commutative elements is preserved under taking inverse, this will have no affect on our situation.

Proposition 2.3.5. *An element $w \in W$ is fully commutative if and only if it avoids the following patterns:*

$$\overline{12}, 321, 32\overline{1}, \overline{3}21, \overline{3}2\overline{1}, 31\overline{2}, \overline{3}1\overline{2}, 21\overline{3}, 2\overline{3}1, 2\overline{3}\overline{1}, 1\overline{3}\overline{2}.$$

Proof. This is an explicitly expanded version of Theorem 5.1 of [Ste97]. $\square$

2.3.1.3 Miscellany

We will need the following purely combinatorial statement:

Proposition 2.3.6. *Suppose $(x_i)_{i=1}^k$ (for some k) is a sequence such that if $x_i > x_{i+1}$ then*

$$x_{i+2} > x_i > x_{i+1}$$

provided $i + 2 \leqslant k$, and

$$x_i > x_{i+1} > x_{i-1}$$

provided $i - 1 \geqslant 1$. In picture notation, every descent locally looks like Figure 2.22, except the boundary is trimmed for the first and last descents. Then (x_i) is a union of two increasing subsequences.
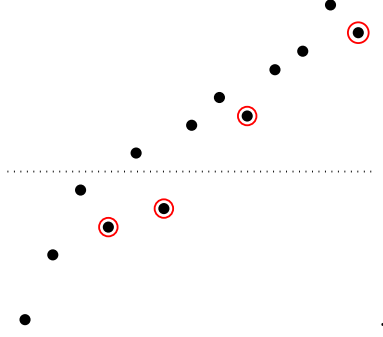
Figure 2.23: A sequence as in Proposition 2.3.6 splits into two increasing sequences.

Proof. Let $(i_1, \ldots, i_r)$ be the sequence of indices of descents (so $x_{i_j} > x_{i_j+1}$). For $1 \leqslant j \leqslant r$, let $y_j = x_{i_j+1}$. It is easy to see that both $(y_j)_{j=1}^r$ and its complement in $(x_i)_{i=1}^k$ are increasing (see Figure 2.23. $\square$

We wold like to slightly refine the notation for (s,t)-strings (introduced in section 1.1.1) for the double bond $(0,1)$. In particular we want to differentiate between a 010-string and a 101-string. We would also like to mention that the strings in the Kazhdan-Lusztig graph lie vertically in the weak order (i.e. form chains in that order). This follows from the fact that the graph is strongly admissible, and Propositions 1.2.5 and 1.2.2.

Now we will clarify why using Kazhdan-Lusztig transport to equate two edge weights with "incorrect" τ-invariant containment implies that the edge weights must be 0 or 1. This is the main idea for the proof of the $0 - 1$ conjecture.

Proposition 2.3.7. *Suppose $x < w$ and $\tau_R(x) \supsetneq \tau_R(w)$. Moreover suppose l instances of Kazhdan-Lusztig transport (possibly using the double bond) yield sequences $x = x_0, x_1, \ldots, x_l$ and $w = w_0, w_1, \ldots, w_l$ with $\mu(x_0, w_0) = \mu(x_1, w_1) = \cdots = \mu(x_l, w_l)$. Finally suppose $\tau_R(x_l) \not\supset \tau_R(w_l)$. Then $\mu(x,w) \in \{0,1\}$.*

Proof. Notice that x_i and x_{i+1} differ by at most two simple edges (they sometimes differ by two if we are using the double bond version of Kazhdan-Lusztig transport). If x_i and w_i differ in length by 1 for some i, then we are automatically done. Hence we will be looking for such a pair. Suppose x_l and w_l are not such a pair. Then, by Proposition 1.2.5 we know that either $\mu(x_l, w_l) = 0$ or $x_l > w_l$. Choose i such that $x_i < w_i$ but $x_{i+1} > w_{i+1}$. Hence x_i and w_i they differ in length by at least 3. But the length difference between x_i and x_{i+1} is at most two, and similarly for w_i and w_{i+1}. Thus the length difference between x_i and w_i must be 3, and the length difference between x_{i+1} and w_{i+1} is 1. This finishes the proof. $\square$

2.3.2 The $0-1$ conjecture

We would like to prove

Theorem 2.3.8. *Suppose $x, w \in W$ and $x < w$. If x is fully commutative then $\mu(x, w)$ is either 0 or 1.*

Some pairs (x, w) require technical arguments; we analyze these before getting to the main proof.

Definition 2.3.9. A pair $(x, w) \in W \times W$ is 010-*bad* if all of the following hold:

- $x < w$,

- x and w lie in the middle of 010-strings,

- $\tau(x) \supset \tau(w)$,

- x is fully commutative, while w is not,

- for all $s \in \tau_R(w)$, the edge between w and sw does not activate any simple bond (this edge may be directed or even phantom, i.e. have the same τ-invariants at both ends).

Similarly for 101-*bad*.

Lemma 2.3.10. *There are no* 010-*bad pairs.*

Proof. Suppose (x, w) is a 010-bad pair. We will describe what w looks like as a signed permutation and conclude, using Proposition 2.3.5, that w is fully commutative.

Consider what it means if w is in the middle of a 010-string. We know that $0 \notin \tau_R(w)$, so
$$w(1) > 0.$$

We also know that $0 \in \tau_R(ws_1)$, so
$$w(2) < 0.$$

Moreover $1 \notin \tau_R(ws_0)$, so
$$-w(1) < w(2).$$

Now suppose $i \in \tau_R(w)$ and $i \geqslant 1$. We know that $\tau_R(w)$ is commutative, so $i - 1, i + 1 \notin \tau_R(w)$ (assuming they make sense). Since (x, w) is bad, we must have $i - 1, i + 1 \notin \tau_R(ws_i)$. Hence $w(i - 1) < w(i + 1)$ and $w(i + 2) > w(i)$. Combining these with the assumption that $i \in \tau_R(w)$ gives
$$w(i + 2) > w(i) > w(i + 1) > w(i - 1).$$

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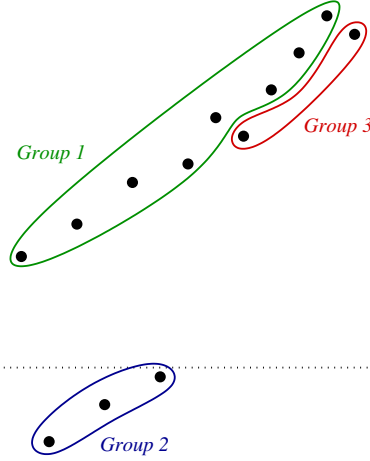
Figure 2.24: A pictorial representation of w for a 010-bad pair (x, w).

Thus the sequence $(w(1), \ldots, w(n))$ satisfies the conditions of Lemma 2.3.6 and hence can be split into two increasing sequences. A pictorial representation of w is shown in Figure 2.24.

Notice that the first sequence only has positive elements and that the negative elements form an initial segment of the second sequence. Also, any negative element is *in absolute value* smaller than any element of the first sequence. We split the sequence into three groups: 1) the elements of the first sequence, 2) the negative elements, and 3) the positive elements of the second sequence.

Now we can show that w is fully commutative. As we have mentioned above, w avoids the pattern $1\bar{2}$. The negative elements form an increasing sequence, so w avoids $\bar{1}\bar{2}$. The only remaining patterns are

$$321, 32\bar{1}, \bar{3}21, \bar{3}2\bar{1}.$$

Now 321 and $32\bar{1}$ are avoided because our sequence is a union of two increasing subsequences. Suppose we have an instance of the pattern $\bar{3}21$. Since Group 1 is in absolute value larger than Group 2, the 2 and 1 must correspond to Group 3. This is again a contradiction since Group 3 forms an increasing sequence. Finally, suppose we have an instance of the pattern $\bar{3}2\bar{1}$. Then the two negatives must correspond to Group 2 and hence the 2 must correspond to Group 1. This is again impossible since Group 1 is in absolute value larger than Group 2. □

Lemma 2.3.11. *If (x, w) is a 101-bad pair, then $\mu(x, w) \in \{0, 1\}$.*

Proof. Suppose (x, w) is a 101-bad pair.

Since $0 \in \tau_R(w)$, we have $w(1) < 0$. Since $0 \notin \tau_R(ws_1)$, we have $w(2) > 0$. As in the

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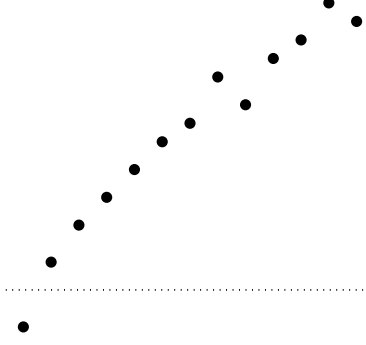
Figure 2.25: A pictorial representation of w for a 101-bad pair (x, w).

previous lemma, if $i \in \tau_R(w)$ and $i \geqslant 1$ then

$$w(i + 2) > w(i) > w(i + 1) > w(i - 1),$$

whenever the inequalities make sense. Hence the sequence $(w(1), \ldots, w(n))$ splits into two increasing sequences.

We now show that $2 \notin \tau_R(w)$ and $w(k) > 0$ for all $k > 1$. Suppose $2 \in \tau_R(w)$. Then $2 \in \tau_R(x)$. Now x is in the middle of its 101-string, so $1 \in \tau_R(xs_0)$. But a descent set can only lose one element when following a downward weak-order covering, so $2 \in \tau_R(xs_0)$. Hence $\{1, 2\} \subseteq \tau_R(xs_0)$. But xs_0 should be fully commutative since it is part of the molecule of x. This is a contradiction, so $2 \notin \tau_R(x)$. Now it is clear the first of the two sequences contains only one negative entry while the second sequence is completely positive. A pictorial representation of w is shown in Figure 2.25.

Now we will use parallel transport to find a pair $(\widetilde{x}, \widetilde{w})$ with the same μ value but with $\tau_R(\widetilde{x}) \not\supseteq \tau_R(\widetilde{w})$. This will finish the proof by Proposition 2.3.7. Notice that $\tau_R(w) \neq \{0\}$ (otherwise x would have to be the identity, which would contradict the assumption that (x, w) is bad). Let k be the smallest nonzero element of $\tau_R(w)$. Consider the sequence $(w_1, \ldots, w_k)$, where $w_1 = w$ and $w_i = ws_{k-1}s_{k-2} \ldots s_{k-i+1}$ for $i > 1$. From the picture it is clear that (w_i, w_{i+1}) is a simple edge activating the bond $(k - i, k - i + 1)$.

Construct the sequence $(x_1, \ldots, x_k)$ inductively. Let $x_1 = x$. Using proposition 2.3.3, let x_2 be such that $\mu(x_2, w_2) = \mu(x_1, w_1)$. If $\tau_R(x_2) \not\supseteq \tau_R(w_2)$ then we have found the necessary $\widetilde{x}$ and $\widetilde{w}$, and we do not need to continue. Also, if $x_2 \not\leqslant w_2$ then $\mu(x_2, w_2) \in \{0, 1\}$ (either x_2 and w_2 are incomparable, or one of the pairs (x_1, w_1), (x_2, w_2) has elements which differ in length by 1), and again we do not need to continue. Otherwise we may repeat this step to find x_3. Continue the process until we are either of the above special cases occurs, or until we find x_k such that $\mu(x_k, w_k) = \mu(x, w)$, $x_k < w_k$, and $\tau_R(x_k) \supset \tau_R(w_k)$.
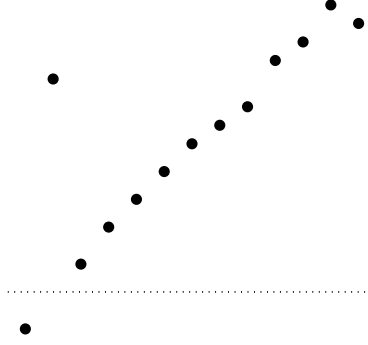
Figure 2.26: A pictorial representation of w_k for a 101-bad pair (x, w).

A pictorial representation of w_k is shown in Figure 2.26. We show next that $w_k(2) \leqslant -w_k(1)$ since otherwise w_k (and hence w) is fully commutative. We will use the pattern avoidance characterization from Proposition 2.3.5. We know that the only negative element of the sequence $(w_k(1), \ldots, w_k(n))$ is the first one. So none of the patterns with a negative entry beyond the first one can appear. The only remaining possibilities are 321 and $\overline{3}21$. It is fairly easy to see that the sequence is a union of two increasing subsequences (the sequence corresponding to w was such a union; now $w_k(2)$ will go into the first sequence while the entries it jumped over will go into the second). So the pattern 321 is avoided. Thus the pattern $\overline{3}21$ must arise. The entries smaller than $w_k(2)$ form an increasing subsequence, hence the 2 in the pattern must correspond to an entry $\geqslant w_k(2)$. Thus $-w_k(1) \geqslant w_k(2)$, as desired.

We claim that x_k must be part of a 010-string. Indeed, both 0 and 2 must be in $\tau(x_k)$. If it was in the middle of a 101-string then at least one of its neighbors would need to have both 1 and 2 in its τ-invariant, which would contradict that it is fully commutative. Now since the types of strings of x_k and w_k are different, an application of the Kazhdan-Lusztig transport gives $\mu(x_k, w_k) = \mu(x_{k+1}, w_k s_0)$ for some x_{k+1} in the molecule of x. By the discussion in the previous paragraph we know that 1 and 2 are both in $\tau(w s_0)$. Letting $\widetilde{x} = x_{k+1}$ and $\widetilde{w} = w s_0$ finishes the proof (by Proposition 2.3.7). $\square$

Remark 2.3.12. The 101-bad pairs do indeed arise; for example:

$$x = \overline{2}14365, w = \overline{6}13245.$$

Proof of Theorem 2.3.8. Another paper of Green, [Gre07], provides the proof in case both x and w are fully commutative. So we are interested in the case when w is not fully commutative.

We proceed by induction with respect to the Bruhat order of the molecule of w. We

can assume $\tau(x) \supsetneq \tau(w)$, since otherwise $\mu(x, w) \in \{0, 1\}$ by basic Kazhdan-Lusztig theory. The base case (namely the case when w is a minimal element of its molecule) follows from Proposition 2.3.4 as well as the fact that $\tau(x)$ is commutative.

Now suppose w is not a minimal element of its molecule. There exists $s \in \tau(w)$ such that the edge between w and sw is simple. Thus there exists $t \in \tau(sw) \setminus \tau(w)$; it must be bonded to s. We then know that $s \in \tau(x)$, and, since $\tau(x)$ is commutative, $t \notin \tau(x)$.

Suppose first that (s, t) is a simple bond. Let u be the vertex adjacent to x along a simple edge activating the bond (s, t). Since the graph satisfies Kazhdan-Lusztig transport, $\nu(x, w) = \mu(x, w) = \mu(u, sw)$. If $u \leqslant sw$ then we are done by induction. If $u \geqslant sw$ then the length difference between u and sw must be 1, and hence $\mu(u, sw) = 1$. Otherwise $\mu(u, sw) = 0$.

Now suppose that (s, t) is the double bond. If w is at the top of its (s, t)-string, then an analogous argument to the one above (except using part (4) of Definition 2.2.7) completes the proof. By construction, w is not at the bottom of its (s, t)-string. Either (x, w) is a bad (in the sense of Definition 2.3.9) pair, or the types of (s, t)-strings of x and w are different. The first case was treated in Lemmas 2.3.10 and 2.3.11, and the second one follows by induction and Kazhdan-Lusztig transport as before. $\qquad\square$

CHAPTER 3

W-graphs of minuscule and quasi-minuscule representations

In this chapter we study, in detail, two important examples of parabolic Kazhdan-Lusztig theories: the minuscule quotients and the quasi-minuscule quotients. We may wish to compute the W-graph, and, where possible, the Kazhdan-Lusztig polynomials.

The computations for the minuscule case have been done before. For the case $u = -1$ the Kazhdan-Lusztig polynomials were computed by Lascoux and Schützenberger ([LS81]) for the symmetric group and by Boe ([Boe88]) for the other Lie types. For the case $u = q$ they were computed by Brenti ([Bre02, Bre09]). Our main result here (Theorem 3.1.16) is that the W-graph is "not too complicated," in the sense that one only needs the Local Polygon Rule to determine the edge weights (in particular, the $0 - 1$ conjecture holds in this case). The main merit of our approach as compared to the previous work is that it is independent of the Lie type.

For the quasi-minuscule quotient, we carry out some of the above calculations. We compute the W-graph for all the (finite, irreducible) Weyl groups except for type D (where we conjecture what the answer is but are unable to prove it). We also compute the Kazhdan-Lusztig polynomials in the case $u = -1$ for Lie type A.

3.1 Minuscule quotients

3.1.1 Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra, Φ its root system, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base. Define s_i to be the simple reflection is α_i. Then $S = \{s_1, \ldots, s_n\}$ is the set of simple reflections. Let W be the Weyl group, i.e. the group generated by S. For $\gamma \in \Phi$ let $\gamma^\vee = \frac{\gamma}{\langle \gamma, \gamma \rangle}$ be the corresponding coroot.

A dominant integral weight $\lambda \neq 0$ is minuscule if the weights of the corresponding irreducible representation of $\mathfrak{g}$ form a single W-orbit. This is equivalent to

Definition 3.1.1. A dominant integral weight $\lambda \neq 0$ is *minuscule* if $\langle \lambda, \beta^\vee \rangle \in \{0, \pm 1\}$ for any root β.

Suppose λ is a minuscule weight. The stabilizer of λ is a parabolic subgroup W_J generated by $J \subseteq S$. The quotient W/W_J is referred to as a *minuscule quotient*. Let W^J denote the set of shortest coset representatives. Let w_0^J be the longest element of W^J. Then W^J is a subinterval $[1, w_0^J]$ in the left-weak order of W. Every element in W^J is "dominant minuscule" in the sense of [Ste01a]. By [Ste01a, Proposition 2.1], any element of W^J is fully commutative. On the subinterval, the Bruhat order and the left weak order coincide ([Ste96, Theorems 2.1 and 6.1]).

Proposition 3.1.2. *Suppose λ is a minuscule weight (corresponding to an irreducible root system). Then λ is a fundamental weight ω_s for some $s \in S$ (namely, $\langle \lambda, \alpha_t^\vee \rangle = \delta_{st}$).*

Proof. Since λ is dominant, we know that $\langle \lambda, \alpha_t^\vee \rangle \geqslant 0$ for all $t \in S$. Suppose, toward a contradiction, that $\langle \lambda, \alpha_s^\vee \rangle = \langle \lambda, \alpha_t^\vee \rangle = 1$. Let $s = s_0, s_1, \ldots, s_{k-1}, s_k = t$ be a simple path in the Dynkin diagram from s to t (it exists since the root system is irreducible). Then

$$\langle \lambda, s_k \ldots s_1 \alpha_s^\vee \rangle = \left\langle \lambda, s_k \ldots s_2 (\alpha_{s_0}^\vee - \left\langle \alpha_{s_0}^\vee, \alpha_{s_1} \right\rangle \alpha_{s_1}^\vee) \right\rangle = \ldots$$
$$= \left\langle \lambda, \alpha_{s_0}^\vee - a_{s_0,s_1} \alpha_{s_1}^\vee + a_{s_0,s_1} a_{s_1,s_2} \alpha_{s_2}^\vee - \cdots + (-1)^k a_{s_0,s_1} \ldots a_{s_{k-1},s_k} \alpha_{s_k}^\vee \right\rangle,$$

where $a_{s,t} = \langle \alpha_s^\vee, \alpha_t \rangle$. Now $a_{s_i,s_{i+1}} \leqslant -1$ since s_i and s_{i+1} are bonded, so

$$\left\langle \lambda, \alpha_{s_0}^\vee - a_{s_0,s_1} \alpha_{s_1}^\vee + a_{s_0,s_1} a_{s_1,s_2} \alpha_{s_2}^\vee - \cdots + (-1)^k a_{s_0,s_1} \ldots a_{s_{k-1},s_k} \alpha_{s_k}^\vee \right\rangle$$
$$\geqslant \left\langle \lambda, \alpha_{s_0}^\vee \right\rangle + \left\langle \lambda, \alpha_{s_1}^\vee \right\rangle + \cdots + \left\langle \lambda, \alpha_{s_k}^\vee \right\rangle \geqslant 2.$$

This is a contradiction. $\square$

We will use the following result about the structure of reduced expressions of elements of W^J.

Proposition 3.1.3. *Suppose $w = s_{i_1} \ldots s_{i_r} \in W^J$ is a reduced expression. Then between any two occurrences of an element s_i (with no other occurrences of s_i between them)*

- *there are exactly two terms that do not commute with s_i such that the corresponding simple roots are short relative to α_i, or,*

- *the is one term s_j that does not commute with s_i such that $\langle \alpha_j, \alpha_i^\vee \rangle = -2$.*

Proof. This follows from [Ste01a, Proposition 2.3]. $\qquad\square$

3.1.2 Heap of w_0^J

Let P be the heap of w_0^J (recall the discussion in section 2.3.1). The order filters of P correspond precisely to the elements of W^J as follows. Given an order filter we take a linear extension and multiply the corresponding labels. Given $w \in W^J$ we take its heap to get the order filter $fl(w)$. The linear extensions of $fl(w)$ are precisely the reduced expressions of w. The weak (equivalently Bruhat) ordering on W^J corresponds to the containment ordering on the order filters of P.

In this section we describe, in considerable detail, the structure of P.

The heap is a ranked poset by [Ste01a, Corollary 3.4]. Let r be the rank of P.

We will write $Q_{k_1, k_2, \ldots k_n}$ for the restriction of ranked poset Q to levels $k_1, k_2, \ldots k_n$.

Proposition 3.1.4. *P has a greatest element and a least element.*

Proof. Let $\lambda = \omega_s$ be the minuscule weight. Suppose w_0^J has reduced expressions $s_{i_1} \ldots s_{i_r}$. Now $s_{i_r} \notin J$, so $s_{i_r}\lambda \neq \lambda$. So $\left\langle \lambda, \alpha_{s_{i_r}}^\vee \right\rangle \neq 0$. But λ is dominant and minuscule, so $\left\langle \lambda, \alpha_{s_{i_r}}^\vee \right\rangle = 1$ and $s_{i_r} = s$. Hence any maximal element of P is labeled s. Since all elements labeled s are related in P, we conclude that the maximal element is unique, and thus is the greatest element.

Now prove that P has a least element. Suppose $t \in \tau(w_0^J)$. Then

$$-1 = \left\langle w_0^J \lambda, \alpha_t^\vee \right\rangle = \langle w_0 \lambda, \alpha_t^\vee \rangle = \langle \lambda, w_0 \alpha_t^\vee \rangle = \langle \lambda, -\alpha_{t'}^\vee \rangle = -\delta_{st'},$$

where $t' = w_0 t w_0$. Hence $\tau(w_0^J) = \{w_0 s w_0\}$ has just one element. Hence any minimal element of P has to be labeled by $w_0 s w_0$. Hence the minimal element is unique and thus is the least element. $\qquad\square$

Remark 3.1.5. This proposition, together with the fact that P is ranked, implies that P is actually a graded poset.

We will need the following statement about the classification of intervals between adjacent elements of the same label:

Proposition 3.1.6. *In the heap of a dominant minuscule element, every interval between two elements labeled i (with no elements labeled i between them) is isomorphic as a labeled poset to the heap of $s_k \ldots s_3 s_1 s_2 s_3 \ldots s_k$ in D_k, or to the heap of $s_k \ldots s_2 s_1 s_2 \ldots s_k$ in C_k. Examples of these are shown in Figure 3.1.*
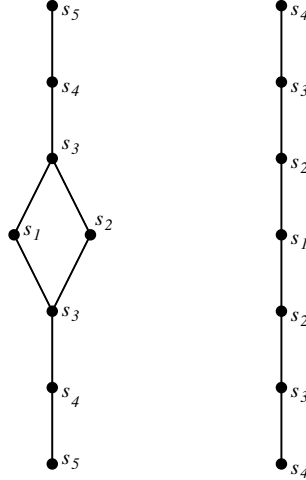
Figure 3.1: Intervals of type D_5 and C_4.

Proof. This follows from [Ste01a, Proposition 3.3]. □

Notice that the above proposition is similar to Proposition 3.1.3, however whereas the word version does make a statement about relative length of roots, the new version asserts isomorphism of labeled posets.

Proposition 3.1.7. *If $y, z < x$ are covers in P then there exists $w \in P$ covered by y and z, and the labels of x and w coincide. Similarly, if $y, z > x$ are covers in P then there exists $w \in P$ covering y and z, and the labels of x and w coincide.*

Proof. Suppose $y, z < x$ are covers in P. Let s be the label of x and t, t' be the labels of y and z, respectively. Let $y \to x_1 \to x_2 \to \cdots \to x_k \to z$ be a path in P formed by descending from y to the least element and coming up to z. The labels trace a path in the Dynkin diagram from t to t'. Since the Dynkin diagram has no cycles and s is adjacent to both t and t', some x_i is labeled s. Now $x_i < x$ so we may define w to be the maximal element labeled by s which is smaller than x. Now $y, z \in [w, x]$, so by Proposition 3.1.6, w is covered by y and z.

The second case follows by a similar argument. □

Proposition 3.1.8. *Any $x \in P$ covers at most two elements and is covered by at most two elements.*

Proof. Suppose x, labeled by s, covers three elements: y_1, y_2, y_3. By Proposition 3.1.7 there exist z_1, z_2 labeled by s such that z_1 is covered by y_1, y_2 and z_2 is covered by y_2, y_3. Then z_1 and z_2 are in the same grading level, but are labeled by the same letter and hence comparable. So $z_1 = z_2 =: z$. This contradicts Proposition 3.1.6. □

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Definition 3.1.9. A *fence* is a graded poset Q of rank 2 such that the Hasse diagram is connected and no vertex has degree bigger than 2. We distinguish three types, based on the relative sizes of the two levels:

- *V-fence*: ,

- *Λ-fence*: ,

- *N-fence*: .

An enumeration of a rank-level L *consistent with the fence* is an ordering $L = (x_1, \ldots, x_n)$ such that the distance in the Hasse diagram between adjacent entries is 2.

The Hasse diagram of a fence can be embedded into the plane. Given an embedding, we can distinguish between an N-fence

and an $\overline{N}$-fence

Proposition 3.1.10. *The restriction of P to two consecutive rank levels is a fence. Moreover the Hasse diagram of P can be embedded into the plane so that the restriction to two consecutive levels is a V-fence, a Λ-fence, an N-fence or an $\overline{N}$-fence.*

Proof. Suppose we are looking at the levels $r - i$ and $r - i - 1$. Proceed by induction on i.

Suppose $i = 0$. The only element of rank r is the maximal element. It covers at most two elements. So the restriction of P to the levels r and $r - 1$ is a fence. On its own, it can clearly be embedded into the plane.

Suppose $P_{r-i+1,r-i}$ is a fence. Enumerate the elements of P_{r-i} by $x_1, \ldots, x_n$ so that they are consistent with $P_{r-i+1,r-i}$. By Proposition 3.1.7, for $j = 1, \ldots, n - 1$, there exists an element $y_j \in P_{r-i-1}$ which is covered by x_j and x_{j+1}.

Since every element of P_{r-i-1} is covered by an element of P_{r-i}, there are at most two other elements of P_{r-i-1}: y_0 covered by x_1 and y_n covered by x_n. If they exist, they are easily seen to be distinct. So $P_{r-i,r-i-1}$ is again a fence. Moreover, it is clear that one can extend the embedding of $P_{r,r-1,\ldots,r-i}$ to an embedding of $P_{r,r-1,\ldots,r-i-1}$. $\square$

From now on we pick an embedding of the Hasse diagram of P into the plane.

Remark 3.1.11. All labels in the fence must be distinct. Indeed two labels in the same rank level cannot coincide by definition and two labels in adjacent ranks cannot coincide by parity considerations.

We call a vertex of the Dynkin diagram *distinguished* if it is either the long end of a double bond or if it is the middle of a fork. There must be at most one distinguished vertex in the Dynkin diagram of Φ.

Lemma 3.1.12. *Let $k \in Z^{\geqslant 0}$. Suppose either the first or last vertex of P_k is four-valent. If the label of that vertex is s then either s is the long end of a double bond, or s is the middle of a fork in the Coxeter graph.*

Proof. All four of the labels of the adjacent vertices must be bonded to s. If they are all distinct, then s must be a quadruple vertex of the Dynkin diagram. This is of no interest to us since Φ was assumed to be finite. If there are three distinct ones, then s is the middle of a fork. If there are two distinct ones, then for some t, tst will be a subword of the reduced word or w_0^J. By Proposition 3.1.3, this means that s is the long end of a double bond. $\qquad\square$

Lemma 3.1.13. *Suppose an interval $[i, f]$ in P, with both i and f labeled by $s \in S$ is a chain. Then it contains an element in its interior which is labeled by the distinguished vertex (more specifically the long end of a double bond).*

Proof. Without loss of generality we may assume there are no other elements labeled s in the interval (else just take the subinterval from the highest of them to f). Thus the interval must be of type C according to Proposition 3.1.6. Consider the middle three elements; suppose their labels are t_1, t_2, t_1. Since the elements from an interval, their labels must form a subword $t_1 t_2 t_1$ of of a reduced expression of W_0^J. By Proposition 3.1.3, t_2 must correspond to a long end of a double bond. $\qquad\square$

Proposition 3.1.14. *The heap P is rank unimodal.*

Proof. Suppose P is not unimodal. Then, restricting to some consecutive ranks of P we have, from top to bottom, a V-fence followed by some N- and $\overline{N}$-fences followed by a Λ-fence (see Figure 3.2). Denote the rank levels by $r_i, r_i + 1, \ldots, r_f$.

First we consider three special cases. The first two take care of the case when there are no N- and $\overline{N}$-fences (i.e. $r_f = r_i + 2$), while the last one is the case when the V- and Λ-fences are as small as possible.

Case 1. Suppose $r_f = r_i + 2$ and both rank levels r_i and r_f have only two vertices (see Figure 3.3 (a)).

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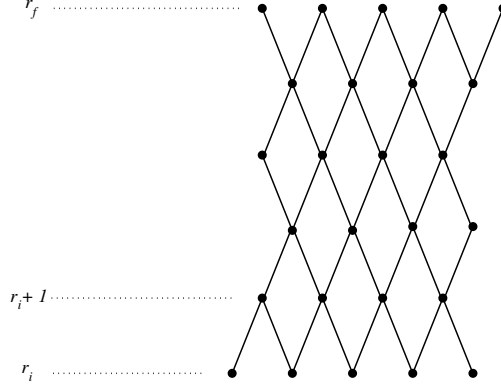
Figure 3.2: A section of a non-unimodal heap.

By Lemma 3.1.12, a must be distinguished. However, since all four expressions *bad*, *bae*, *cad*, *cae* possible subwords of a reduced expression of w_0^J, we have one of the following scenarios:

- if b, c, d, e are all distinct then a is a quadruple vertex of the Dynkin diagram,

- if there is one match among b, c, d, e (without loss of generality, $b = d$) then a is the middle of a fork *and* the long end of a double bond,

- if there are two matches among b, c, d, e (without loss of generality, $b = d$ and $c = e$) then a is the long end of two double bonds.

All of these configurations contradict the finiteness of the Coxeter system. Note that there cannot be more matches among b, c, d, e since elements with the same label must be comparable. This finishes the case.

Case 2. Suppose $r_f = r_i + 2$ and both rank levels r_i and r_f have more than two vertices (see Figure 3.3 (b)).

By Lemma 3.1.12, a and b must both be distinguished vertices of the Dynkin diagram. Since the corresponding elements of the heap are incomparable, a and b must be distinct. This contradicts the finiteness of the Coxeter system.

Case 3. Suppose $r_f > r_i + 2$ and both rank levels r_i and r_f have only two vertices (see Figure 3.3 (c)).

While there are no 4-valent vertices here, a and b must still be distinguished vertices of the Dynkin diagram. Indeed, if c, d, e are distinct then a is the middle of a fork, while otherwise (without loss of generality $c = e$) it is the long end of a double bond. Similarly b is also distinguished. Due to finiteness, $a = b$. But the interval (in the heap) between the corresponding vertices is a line. By Lemma 3.1.13, it contains a distinguished vertex in its
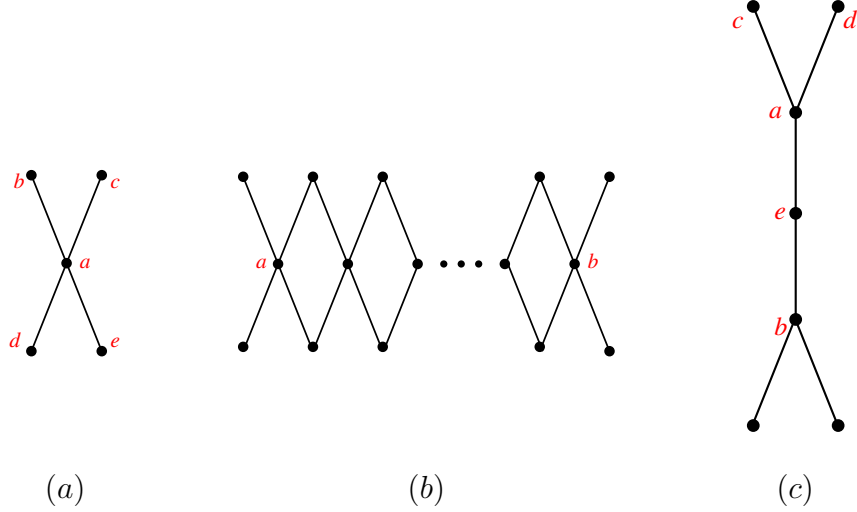
Figure 3.3: The special cases of nonunimodal heaps.

interior. Hence an interval between two vertices labeled by a distinguished simple reflection cannot be a line.

Now we consider the general larger case (i.e. rank levels r_i and r_f have more than two vertices and some N- or $\overline{N}$-fences are present). Without loss of generality, P_{r_f-1,r_f-2} is an N-fence. We claim that the next level down must also be an N-fence. Indeed, otherwise we are in the situation in Figure 3.4 (a). In this case the vertices x and y must both be labeled by the distinguished simple reflection a. However we can see a path of odd length between these vertices. Since a path in the heap induces a path in the Dynkin diagram, any path between two vertices with the same labels must have even length.

Thus we know that P_{r_f-1,r_f-2} and P_{r_f-2,r_f-3} are N-fences. Suppose P_{r_f-3,r_f-4} is not an $\overline{N}$ fence. Then the picture looks like Figure 3.4 (b). All the labels shown are distinct except for the possibility that $b = f$ (this follows from the fact that vertices with the same label must be comparable, Proposition 3.1.6, and Lemma 3.1.13). If $b = f$ then the diagram in Figure 3.5 (a) is a subdiagram of the Dynkin diagram. Otherwise the diagram in Figure 3.5 (b) is a subdiagram of the Dynkin diagram. Neither of these is allowed by finiteness. Hence P_{r_f-1,r_f-2} and P_{r_f-2,r_f-3} are N-fences while P_{r_f-3,r_f-4} is an $\overline{N}$ fence. It must also be the case that P_{r_f-4,r_f-5} is an $\overline{N}$ fence since otherwise we would have two vertices labeled a in the fence P_{r_f-3,r_f-4}.

The picture now looks like Figure 3.4 (c). The labels in the topmost fence are distinct, so the path from x to z corresponds to a path from a to b. The straight line path from y to z gives a path from a to b of length 2. There are no cycles in the Dynkin diagram, so these paths must be the same. Hence the rank level r_f must only have three vertices (see Figure 3.4 (d)).

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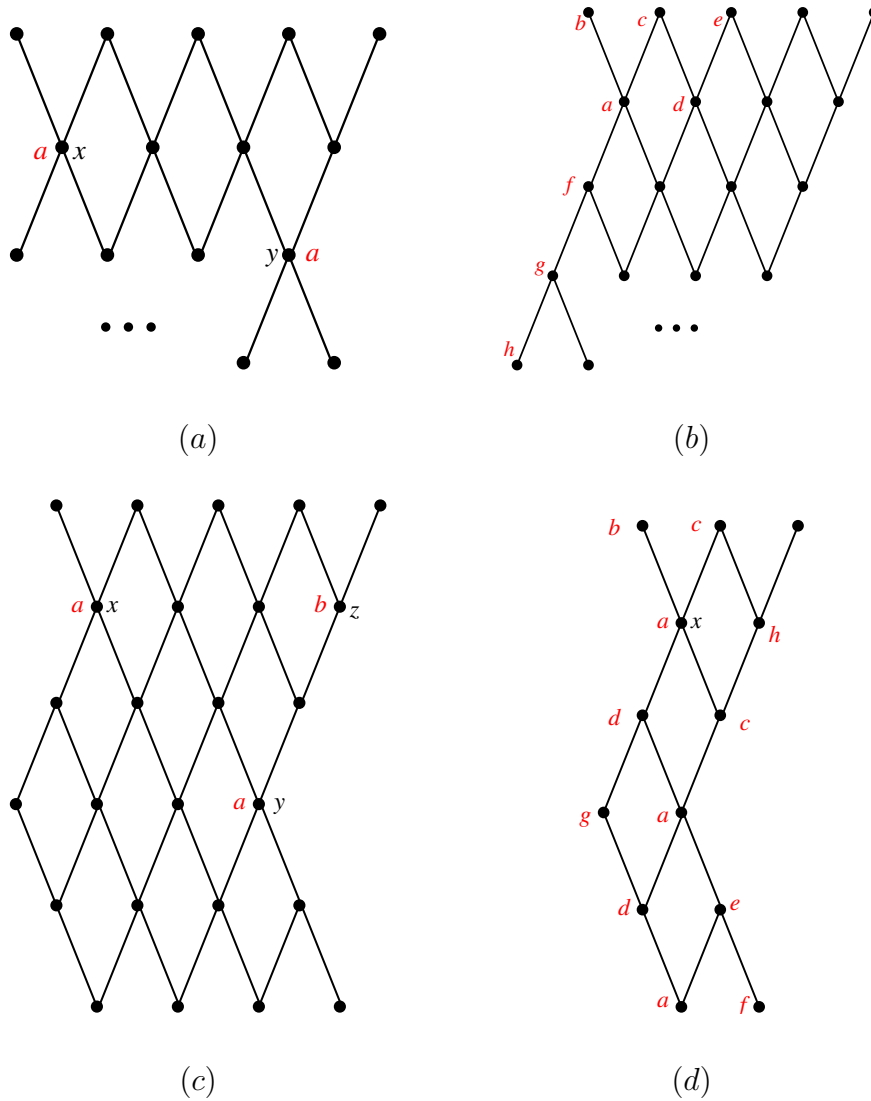
Figure 3.4: Heaps encountered in the proof of Proposition 3.1.14.
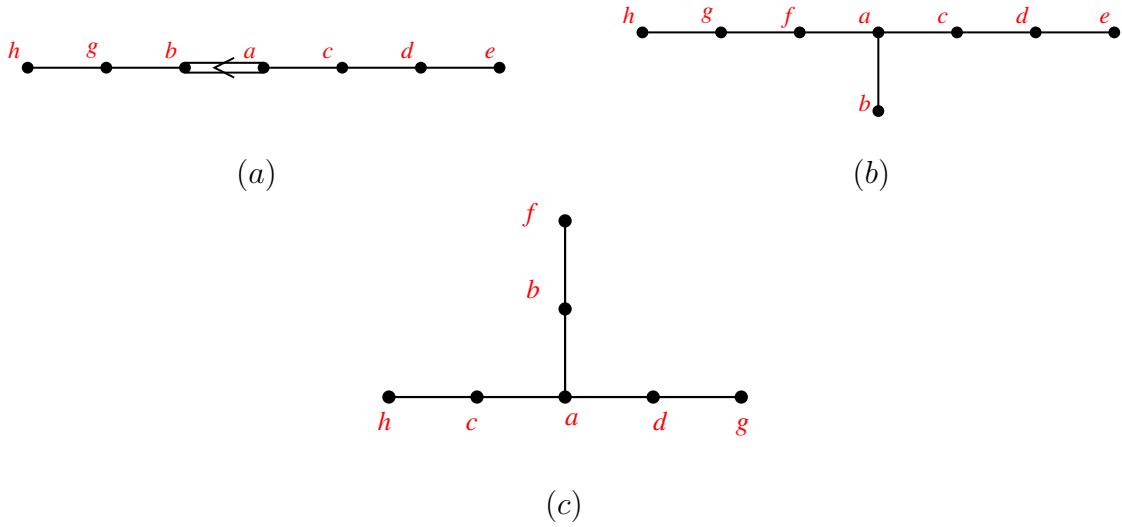
Figure 3.5: Dynkin diagrams encountered in the proof of Proposition 3.1.14.

Notice that $b \neq d$; this would imply that a is the long end of a double bond, but the interval between the vertices labeled d contains a and another vertex whose label is adjacent to d (this contradicts Proposition 3.1.6). Thus a is a fork of the Dynkin diagram. Thus $e = b$. However all the other labels shown are distinct. Hence the diagram in Figure 3.5 (c) is a subdiagram of the Dynkin diagram. This contradicts finiteness, finishing the proof. □

3.1.3 W-graph for a minuscule quotient

Recall that and $v \in W^J$ is fully commutative, and hence its τ-invariant must be commutative.

Lemma 3.1.15. *Suppose that for some k, $P_{k+1,k}$ is a Λ-fence. If $w \in W^J$ and $fl(w) \subseteq P_{r,r-1,\ldots,k}$ then $|\tau(w)| \leqslant |P_k|$ and equality holds if and only if $fl(w) = P_{r,r-1,\ldots,k}$.*

Proof. Let $x_1, \ldots, x_n$ be the left-to-right enumeration of P_k consistent with the embedding. Choose $i_0 \in \{1, \ldots, n\}$. Since P is unimodal, we know that for $j > k$, $P_{j-1,j}$ is either a Λ-fence or N-fence. Using these fences we can define injections $\phi_j : P_j \to P_{j-1}$. Since $P_{k+1,k}$ is a Λ-fence we can make sure that $x_{i_0} \notin \phi_{k+1}(P_{k+1})$. For $i = 1, \ldots, n$ define $S_i := \{x \in P_{r,r-1,\ldots k} : \phi_{k+1} \circ \phi_{k+2} \circ \cdots \circ \phi_{\mathrm{rank}(x)}(x) = x_i\}$ These n chains partition $P_{r,r-1,\ldots k}$. The minimal elements of $fl(w)$ (whose labels comprise $\tau(w)$) form an antichain, so there are at most n of them.

If there are n of them then $fl(w)$ contains at least one element from each S_i, and in particular contains x_{i_0} (since $S_{i_0} = \{x_{i_0}\}$). But i_0 was an arbitrary index, so $fl(w) \supset P_k$. So $fl(w) = P_{r,r-1,\ldots,k}$. □

Theorem 3.1.16. *Let Γ be an admissible molecular graph (for W) with vertices in bijection with W^J such that*

1. *for any vertex v of Γ, $\tau(v)$ is the left descent set of v as an element of W^J,*

2. *if $v \to w$ is an edge of Γ then v and w are related in left weak (equivalently, Bruhat) order,*

3. *if $v < w$ are vertices of Γ and $l(w) - l(v) = 1$ then $v \to w$ is an edge of Γ,*

4. *if $v \to w$ is an edge of Γ and $|l(v) - l(w)| \neq 1$ then $v < w$ and the lengths of v and w have different parities.*

Then Γ is the W-graph of the parabolic representation of W with respect to W_J.

Lemma 3.1.17. *Suppose that Γ is an admissible molecular graph satisfying the conditions of Theorem 3.1.16, and $v \to w$ is an arc of Γ for some $v < w$. Then there exists $u < w$ which is connected by a simple edge to w.*

Proof. Suppose not. Let k be the lowest level of $fl(w)$, and $x_1, \ldots x_n$ be an enumeration of P_k consistent with the embedding. Suppose that $x_i \in fl(w)$ is labeled by s (so s is in the descent set of w). Then $fl(sw)$ is obtained from $fl(w)$ by removing x_i. The edge from w to sw cannot be simple by assumption, so x_{i-1} and x_{i+1} must exist and belong to $fl(w)$. Repeating this argument yields that $P_k \subset fl(w)$. Moreover, the same argument shows that $P_{k,k+1}$ is a Λ-fence. Since $v \to w$ is an arc, we have $|\tau(v)| > |\tau(w)| = n$. However $v < w$, and so $fl(v) \subsetneq fl(w)$. This contradicts Lemma 3.1.15. $\square$

Proof of Theorem 3.1.16. We know that the parabolic W-graph satisfies the four conditions, so it suffices to prove that such an admissible molecular graph is unique.

Clearly all simple edges of Γ and non-surprising arcs are determined by conditions $(1) - (4)$ together with admissibility of Γ. Suppose $v \to w$ is a surprising arc of Γ. We wish to show that $m(v \to w)$ is uniquely determined. We will show that the weight of $v \to w$ is determined by the weights of arcs which have a lower head (in the Bruhat order). This will finish the proof by induction.

Since $v \to w$ is an arc, we may fix $s \in \tau(v) \setminus \tau(w)$. Now for any $s' \in \tau(w)$, we have $s \notin \tau(s'w)$. Indeed, if $(w, s'w)$ is an arc, then $s \notin \tau(w) \supset \tau(s'w)$. If $(w, s'w)$ is a simple edge then any element in $\tau(s'w) \setminus \tau(w)$ must be bonded to s'. However s and s' are not bonded since they both belong to $\tau(v)$.

Moreover, whenever $(w, s'w)$ is a simple edge, we can apply LPR2 with type (s, s') from v to $s'w$. We know that $s \notin \tau(s'w)$, so we need to only make sure that there is a witness.
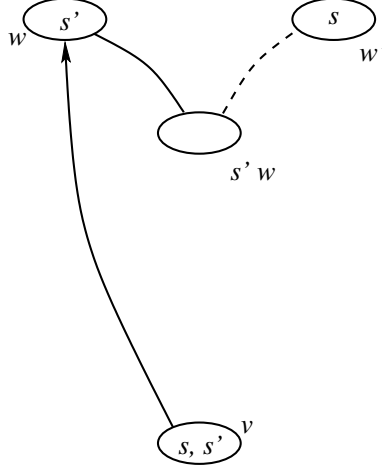
Figure 3.6: Structure of a possible surprising arc in the W-graph of a minuscule quotient.

Now there exists $k \in \tau(s'w) \setminus \tau(w)$, and it must be bonded to s'. Thus $k \notin \tau(v)$, and we can apply LPR2. We will next show that there exists $s' \in \tau(w)$ such that the edge $(w, s'w)$ is simple, and the instance of LPR2 from v to $s'w$ described above involves no alternating paths that go as high as w.

Suppose, toward a contradiction, that this is not the case. Notice that condition (4) prevents any path from going above w. So for every s' such that $(w, s'w)$ is simple, any instance of LRP2 from v to $s'w$ involves an (s, s')- or (s', s)-alternating path $v \to w' \to s'w$ with $l(w) = l(w')$. The edge $w' \to s'w$ is directed downward. Because of Proposition 1.2.4, the path must have type (s, s'), and we must have $w' = ss'w$ (see Figure 3.6).

Let k be the lowest level of $fl(w)$ in P, and let $x_1, \ldots, x_n$ be an enumeration of P_k consistent with the embedding.

Suppose first that $fl(w)_k \subsetneq P_k$ (see Figure 3.7). Then, without loss of generality, for some i we have $x_i \in fl(w)$ and $x_{i+1} \notin fl(w)$. Let s' be the label of x_i. Let $y \in P_{k+1}$ be the common cover of x_i and x_{i+1} and t be its label. Thus $s' \in \tau(w) \setminus \tau(s'w)$ and $t \in \tau(s'w) \setminus \tau(w)$ (since $fl(s'w)$ is obtained from $fl(w)$ be removing x_i). So $w \to s'w$ is a simple edge. By assumption, there exists an ss'-alternating path $v \to w' \to s'w$ with $l(w) = l(w')$. Now $t \sim s'$, so $t \notin \tau(v)$. Since $v \to w'$ is an arc, we know that $t \notin \tau(w')$. Thus the heap of w' is obtained from the heap of $s'w$ by attaching x_{i+1}. But since $s \in \tau(w')$ and $s \notin \tau(s'w)$, we know that the label of x_{i+1} was s.

Suppose in the situation of the last paragraph that $i + 1 < n$. Let $y' \in P_{k+1}$ be the common cover of x_{i+1} and x_{i+2} and let t' be its label. We know that $y' \in fl(s'w)$; otherwise we would have $l(w') > l(s'w) + 1 = l(w)$. So $y' \in fl(w)$. However $t' \sim s$, so $t' \notin \tau(v) \supset \tau(w)$, so y' is not minimal in $fl(w)$. Since $x_{i+1} \notin fl(w)$, we must have $x_{i+2} \in fl(w)$.
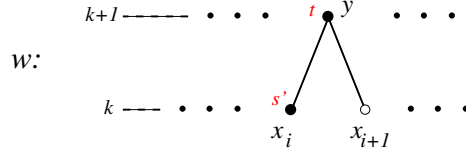
Figure 3.7: Rank levels k and $k + 1$ of the heap of w.

Recalling that no two elements in a rank level have the same label, the observations in the last two paragraphs (and their mirror images) imply that $fl(w)$ contains all but, perhaps, one element of P_k, and that element is labeled s. Moreover the argument implies that all covers of that element are in $fl(w)$ (hence $P_{k+1} \subset fl(w)$). Then $\tau(v)$ contains the set of labels of the entire P_k.

If $P_{k+1,k}$ is a Λ-fence, then Lemma 3.1.15 implies that $fl(v) \supseteq P_{r,\dots,k} \supseteq fl(w)$, which contradicts the fact that $v \leqslant w$. So there exists $\widetilde{y} \in P_{k+1}$ (labeled $\widetilde{t}$) which covers only one element $\widetilde{x}$ (labeled $\widetilde{s}$) of P_k.

Suppose $\widetilde{s} = s$. Now $\widetilde{x} \notin fl(w)$, since otherwise we would have $s \in \tau(w)$. Then $\widetilde{t} \in \tau(w)$. However this contradicts the fact that $\widetilde{t} \notin \tau(v) \supset \tau(w)$ (since $\widetilde{t} \sim s$). Thus $\widetilde{s} \neq s$ and $\widetilde{x} \in fl(w)$. Now $w \to \widetilde{s}w$ is a simple edge and an $s\widetilde{s}$-alternating path $v \to w' \to \widetilde{s}w$ with $l(w) = l(w')$ is impossible. Indeed, we know that w' would need to be $s\widetilde{s}w$, and since there is no way to cover $\widetilde{y}$ from the bottom other than by $\widetilde{x}$, $\widetilde{t} \in \tau(w')$. This is impossible because $\widetilde{t} \sim s$ and hence $\widetilde{t} \notin \tau(v)$. This finishes the proof. $\qquad\square$

3.2 Quasi-minuscule quotients

3.2.1 Preliminaries

Let (W, S) be an irreducible finite Weyl group, (Φ, Δ) the corresponding root system, $\overline{\alpha}$ the dominant short root.

Let $\Phi_s := W\overline{\alpha}$ be the orbit of short roots. Let $J \subset S$ be the set of simple reflections that fix $\overline{\alpha}$. The quotient W/W_J is referred to as the *quasi-minuscule quotient*.

There is a bijection between W^J and Φ_s given by $w \mapsto w\overline{\alpha}$. Since we will be primarily thinking in terms of roots, we use this bijection to define, for $\gamma = w\overline{\alpha}$, $\tau(\gamma) = \tau(w)$, $l(\gamma) = l(w)$, etc.

In [Ste01b], Stembridge introduces a partial order on short roots called the Cayley order as the transitive closure of relations of the form

$$\beta > \gamma \text{ when for some } s_i \in S, \ s_i\beta = \gamma \text{ and } \langle \beta, \alpha_i \rangle > 0.$$

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He proceeds to show that, in the case of a finite crystallographic root system, this ordering *restricted to positive short roots* is identical to the standard one ([Ste01b, Proposition 3.2]), i.e. $\beta > \gamma$ if $\beta - \gamma$ is a sum of positive simple roots.

He also shows ([Ste01b, Proof of Theorem 2.6]) that the map $w \mapsto -w\overline{\alpha}$ gives an order isomorphism between the order ideal of the left weak order $\{w \in W \mid 1 \leqslant_L w \leqslant_L \sigma_{\overline{\alpha}}\}$, where $\sigma_{\overline{\alpha}}$ is the reflection in $\overline{\alpha}$, and Φ_s.

Now by [Ste01b, Theorem 2.6(c)], we know that $\sigma_{\overline{\alpha}} \in W^J$. Since W^J is an ideal with respect to the weak order, we know that $w \mapsto -w\overline{\alpha}$ is an order isomorphism $W^J \to \Phi_s$. Hence $w \mapsto w\overline{\alpha}$ is an order anti-isomorphism $W^J \to \Phi_s$.

The Bruhat order on Φ_s only requires the additional covering relations $\alpha_i < -\alpha_j$ when i and j are bonded, as compared to the left weak order. We reserve the sign $\leqslant$ for the Bruhat order on Φ_s, even though most of the pictures will be based on the Cayley order and hence look "upside-down."

Below we collect a few more general facts about quasi-minuscule quotients before moving on to a type-by-type analysis.

Proposition 3.2.1. *If $\alpha \in \Phi_s$ and $\beta \in \Phi$ then $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1, \pm 2\}$. Moreover $\langle \alpha, \beta^\vee \rangle = \pm 2$ if and only if $\alpha = \pm \beta$.*

Proof. See [Bou02, VI.1.3]. □

Proposition 3.2.2. *For $\gamma \in \Phi_s$, $\tau(\gamma) = \{s \in S : \langle \gamma, \alpha_s^\vee \rangle < 0\}$.*

Proof. This is a consequence of the fact that the Cayley order is reverse-graded by length. □

Proposition 3.2.3. *Suppose $\gamma \in \Phi_s$, $s_i, s_j \in \tau(\gamma)$, and s_i and s_j do not commute. Then $\gamma = -\alpha_i - \alpha_j$.*

Proof. We know that $\langle \gamma, \alpha_i^\vee \rangle \leqslant -1$, $\left\langle \gamma, \alpha_j^\vee \right\rangle \leqslant -1$, and $\langle \alpha_j, \alpha_i^\vee \rangle \leqslant -1$. So

$$\langle \gamma, s_j \alpha_i^\vee \rangle = \left\langle \gamma, \alpha_i^\vee - \langle \alpha_i^\vee, \alpha_j \rangle \alpha_j^\vee \right\rangle = \langle \gamma, \alpha_i^\vee \rangle - \langle \alpha_i^\vee, \alpha_j \rangle \left\langle \gamma, \alpha_j^\vee \right\rangle \leqslant -2.$$

By Proposition 3.2.1, we know that $\langle \gamma, s_j \alpha_i^\vee \rangle \geqslant -2$. So

$$\langle \gamma, \alpha_i^\vee \rangle = -1, \quad \left\langle \gamma, \alpha_j^\vee \right\rangle = -1, \quad \langle \alpha_j, \alpha_i^\vee \rangle = -1,$$

and $\gamma = -(s_j \alpha_i^\vee)^\vee = -s_j \alpha_i = -\alpha_i - \alpha_j$. □

We now review the finite crystallographic root systems for each Lie type. For the series we review the explicit root system constructions, give the descent sets for all the short roots, and

show a representative Cayley order. For the exceptional types, we just present the Cayley order since finding the W-graph is a computer calculation. The vertices of the Cayley graphs are labeled by descent sets and by coordinates with respect to the simple roots.

3.2.1.1 Type A_n

This root system is constructed in the subspace of $\mathbb{R}^{n+1} = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}$ given by $\varepsilon_1 + \cdots + \varepsilon_{n+1} = 0$. The simple roots are

$$\Delta = \{\varepsilon_{i+1} - \varepsilon_i \mid i \in \{1, \ldots, n\}\}.$$

Denote $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$. The Weyl group $W = S_{n+1}$ permutes the basis vectors. So the set of roots is

$$\Phi = \Phi_s = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, n\}\}.$$

Denote $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$. The positive roots are $\Phi^+ = \{\alpha_{i,j} \mid i > j\}$.

Next we can find the descent sets. If i and j are not adjacent (and $1 \leqslant i, j \leqslant n+1$) then

$$\tau(\alpha_{i,j}) = \begin{cases} \{s_i, s_{j-1}\}, & \text{if } i < n+1 \text{ and } j > 1, \\ \{s_i\}, & \text{if } i < n+1 \text{ and } j = 1, \\ \{s_{j-1}\}, & \text{if } i = n+1 \text{ and } j > 1, \\ \varnothing, & \text{if } i = n+1 \text{ and } j = 1. \end{cases}$$

If $1 \leqslant i \leqslant n$ then

$$\tau(\alpha_{i+1,i}) = \begin{cases} \{s_{i-1}, s_{i+1}\}, & \text{if } 1 < i < n, \\ \{s_{n-1}\}, & \text{if } i = n, \\ \{s_2\}, & \text{if } i = 1. \end{cases}$$

If $1 \leqslant i \leqslant n$ then

$$\tau(\alpha_{i,i+1}) = \{s_i\}.$$

The ascent set of a root consists of elements which are not in the descent set, but which nevertheless alter the root (i.e. the simple generators for the stabilizer of the root are precisely those simple reflections which are neither ascents nor descents).

On can also describe the length fairly easily:

$$l(\alpha_{i,j}) = n - (i-j), \text{ if } i > j, \qquad l(\alpha_{i,j}) = n - (i-j) - 1, \text{ if } i < j.$$
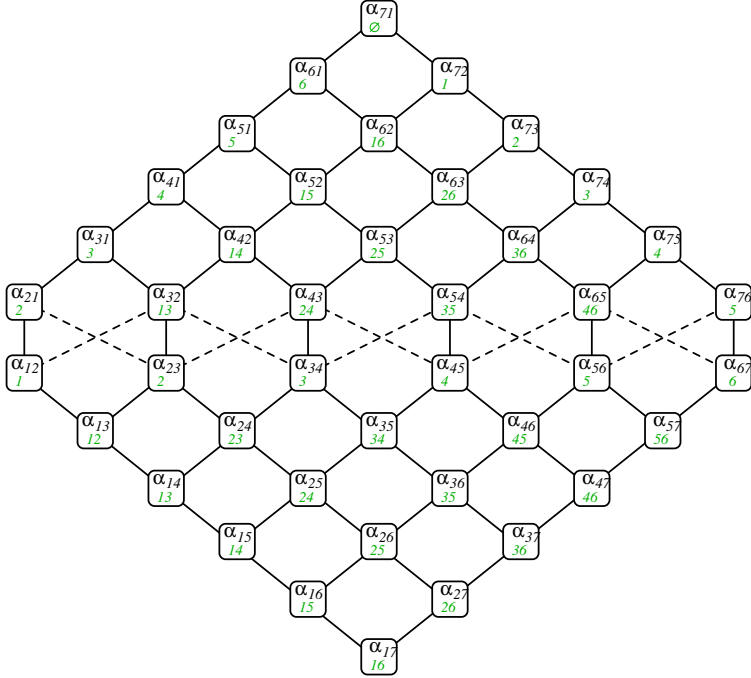
Figure 3.8: The Cayley order on the short roots of A_6. The τ-invariants are labeled in green. The dashed lines are additional Bruhat order coverings.

Finally one can describe the Bruhat order. If $i > j$ then

$$[\overline{\alpha}, \alpha_{i,j}] = \{\alpha_{k,l} \mid l \leqslant j < i \leqslant k\},$$

and if $i < j$ then

$$[\overline{\alpha}, \alpha_{i,j}] = \{\alpha_{k,l} \mid i \leqslant k < l \leqslant j\} \cup \{\alpha_{k,l} \mid j - 1 \leqslant k > l \leqslant i + 1\}.$$

A representative example of the Hasse diagram for the Cayley order is shown in Figure 3.8

3.2.1.2 Type B_n

This root system is constructed in $\mathbb{R}^n = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$. The simple roots are

$$\Delta = \{\varepsilon_1\} \cup \{\varepsilon_{i+1} - \varepsilon_i \mid i \in \{1, \ldots, n-1\}\}.$$

Denote $\alpha_1 = \varepsilon_1, \alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 < i \leqslant n$. The Weyl group W is the group of permutations and sign changes of the basis vectors. So the set of roots is

$$\Phi = \{\pm\varepsilon_i \mid i = 1, \ldots, n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j \in \{1, \ldots, n\}\},$$

and the set of short roots is

$$\Phi_s = \{\pm\varepsilon_i \mid i = 1, \ldots, n\}.$$

The descent sets are

$$\tau(\varepsilon_i) = \begin{cases} s_{i+1}, & \text{if } i < n, \\ \varnothing, & \text{if } i = n, \end{cases}$$

and

$$\tau(-\varepsilon_i) = \{s_i\}.$$

A representative example of the Hasse diagram for the Cayley order is shown in Figure 3.9

3.2.1.3 Type C_n

This root system is constructed in $\mathbb{R}^n = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$. The simple roots are

$$\Delta = \{2\varepsilon_1\} \cup \{\varepsilon_{i+1} - \varepsilon_i \mid i \in \{1, \ldots, n-1\}\}.$$

Denote $\alpha_1 = 2\varepsilon_1, \alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 < i \leqslant n$. The Weyl group W is the group of permutations and sign changes of the basis vectors. So the set of roots is

$$\Phi = \{\pm 2\varepsilon_i \mid i = 1, \ldots, n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j \in \{1, \ldots, n\}\},$$

and the set of short roots is

$$\Phi_s = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j \in \{1, \ldots, n\}\}.$$

For $i < j$, let $\alpha_{i,j}^{+;-} = \varepsilon_i - \varepsilon_j$, and similarly for other signs.
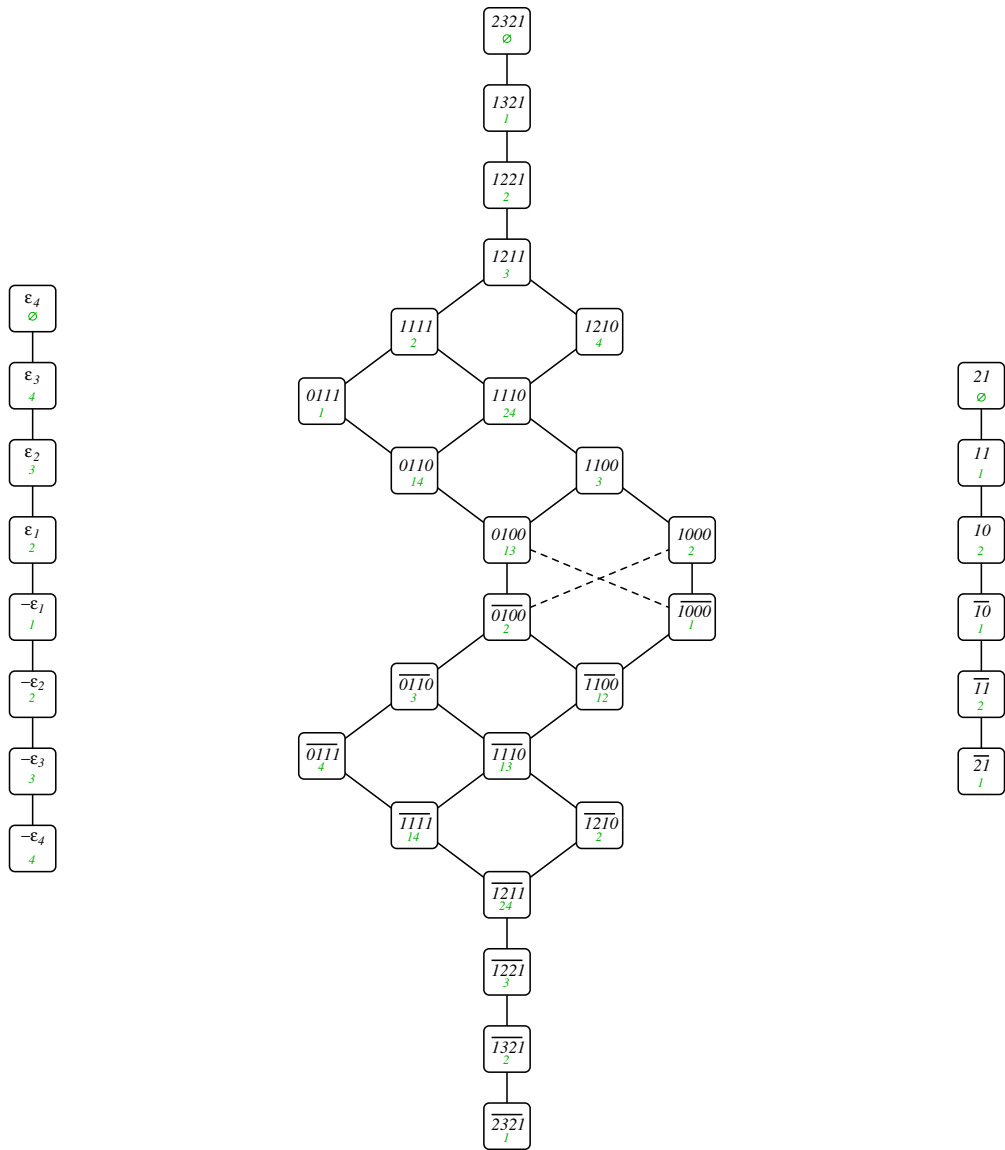
Figure 3.9: The Cayley orders on the short roots of B_4, F_4, and G_2. The dashed lines are additional Bruhat order coverings. For the last two, the roots are labeled with their coordinates in the simple roots. A bar indicates a negative root.

The descent sets are:

$$
\tau(\alpha_{i,j}^{+;+}) = \begin{cases}
\{s_{i+1}, s_{j+1}\}, & \text{if } j < n, j \neq i+1, \\
\{s_{i+1}\}, & \text{if } j = n, j \neq i+1, \\
\{s_{j+1}\}, & \text{if } j < n, j = i+1, \\
\varnothing, & \text{if } j = n, i = n-1,
\end{cases}
$$

$$
\tau(\alpha_{i,j}^{+;-}) = \begin{cases}
\{s_{i+1}, s_j\}, & \text{if } j \neq i+1, \\
\{s_{i+1}\}, & \text{if } j = i+1,
\end{cases}
$$

$$
\tau(\alpha_{i,j}^{-;+}) = \begin{cases}
\{s_i, s_{j+1}\}, & \text{if } j < n, \\
\{s_i\}, & \text{if } j = n,
\end{cases}
$$

$$
\tau(\alpha_{i,j}^{-;-}) = \begin{cases}
\{s_i, s_j\}, & \text{if } j \neq i+1, \\
\{s_i\}, & \text{if } j = i+1.
\end{cases}
$$

A representative example of the Hasse diagram for the Cayley order is shown in Figure 3.10

3.2.1.4 Type D_n

This root system is constructed in $\mathbb{R}^n = \text{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$. The simple roots are

$$
\Delta = \{\varepsilon_1 + \varepsilon_2\} \cup \{\varepsilon_{i+1} - \varepsilon_i \mid i \in \{1, \ldots, n-1\}\}.
$$

Denote $\alpha_1 = \varepsilon_1 + \varepsilon_2, \alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 < i \leqslant n$. The Weyl group W is the group of permutations and sign changes *of an even number of signs* of the basis vectors. So the set of roots is

$$
\Phi = \Phi_s = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j \in \{1, \ldots, n\}\}.
$$

For $i < j$, let $\alpha_{i,j}^{+;-} = \varepsilon_i - \varepsilon_j$, and similarly for other signs.

The descent sets are:

$$
\tau(\alpha_{i,j}^{+;+}) = \begin{cases}
\{s_{i+1}, s_{j+1}\}, & \text{if } j < n, j \neq i+1, \\
\{s_{i+1}\}, & \text{if } j = n, j \neq i+1, \\
\{s_{j+1}\}, & \text{if } j < n, j = i+1, \\
\varnothing, & \text{if } j = n, i = n-1,
\end{cases}
$$

81
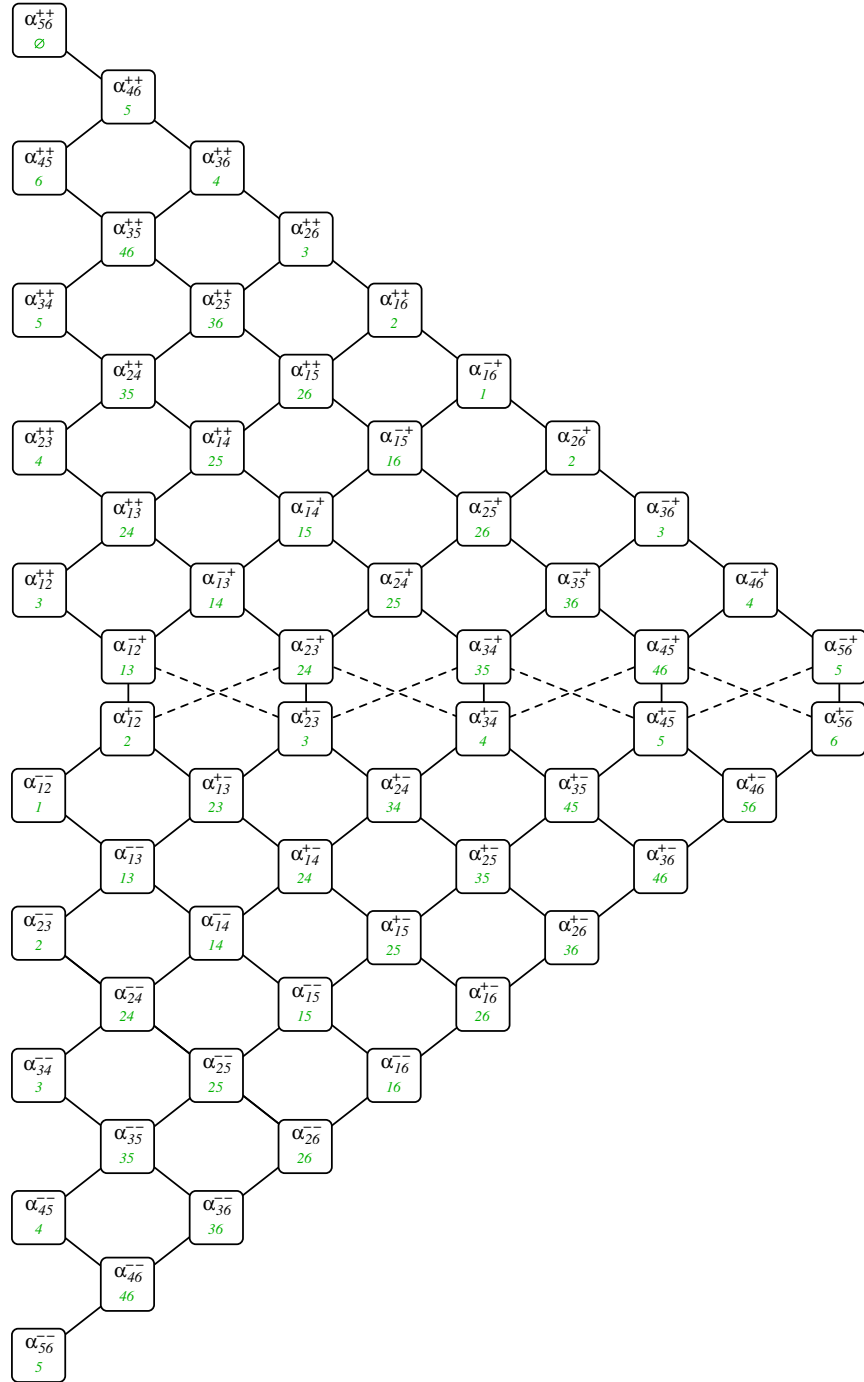
Figure 3.10: The Cayley order on the short roots of C_6. The dashed lines are additional Bruhat order coverings.

$$\tau(\alpha_{i,j}^{+;-}) = \begin{cases} \{s_{i+1}, s_j\}, & \text{if } j \neq i+1, \\ \{s_{i+1}\}, & \text{if } j = i+1, \end{cases}$$

$$\tau(\alpha_{i,j}^{-;+}) = \begin{cases} \{s_i, s_{j+1}\}, & \text{if } j < n, i > 2, \\ \{s_i\}, & \text{if } j = n, i > 2, \\ \{s_1, s_2, s_{j+1}\}, & \text{if } j < n, i = 2, \\ \{s_1, s_2\}, & \text{if } j = n, i = 2, \\ \{s_1, s_{j+1}\}, & \text{if } 2 < j < n, i = 1, \\ \{s_3\}, & \text{if } j = 2, i = 1, \\ \{s_1\}, & \text{if } j = n, i = 1, \end{cases}$$

$$\tau(\alpha_{i,j}^{-;-}) = \begin{cases} \{s_i, s_j\}, & \text{if } j \neq i+1, i \neq 2, \\ \{s_i\}, & \text{if } j = i+1, i \neq 2, \\ \{s_1, s_2, s_j\}, & \text{if } j \neq 3, i = 2, \\ \{s_1, s_2\}, & \text{if } j = 3, i = 2. \end{cases}$$

A representative example of the Hasse diagram for the Cayley order is shown in Figure 3.11

3.2.1.5   Type E_6

The Hasse diagram for the Cayley order is shown in Figure 3.12. The vertices are labeled by the coordinates of the corresponding roots with respect to simple roots, and by descent sets.

3.2.1.6   Type E_7

The Hasse diagram for the Cayley order is shown in Figure 3.13. The vertices are labeled by the coordinates of the corresponding roots with respect to simple roots, and by descent sets.

3.2.1.7   Type E_8

The Hasse diagram for the Cayley order is shown in figure 3.14. The vertices are labeled by the coordinates of the corresponding roots with respect to simple roots, and by descent sets.
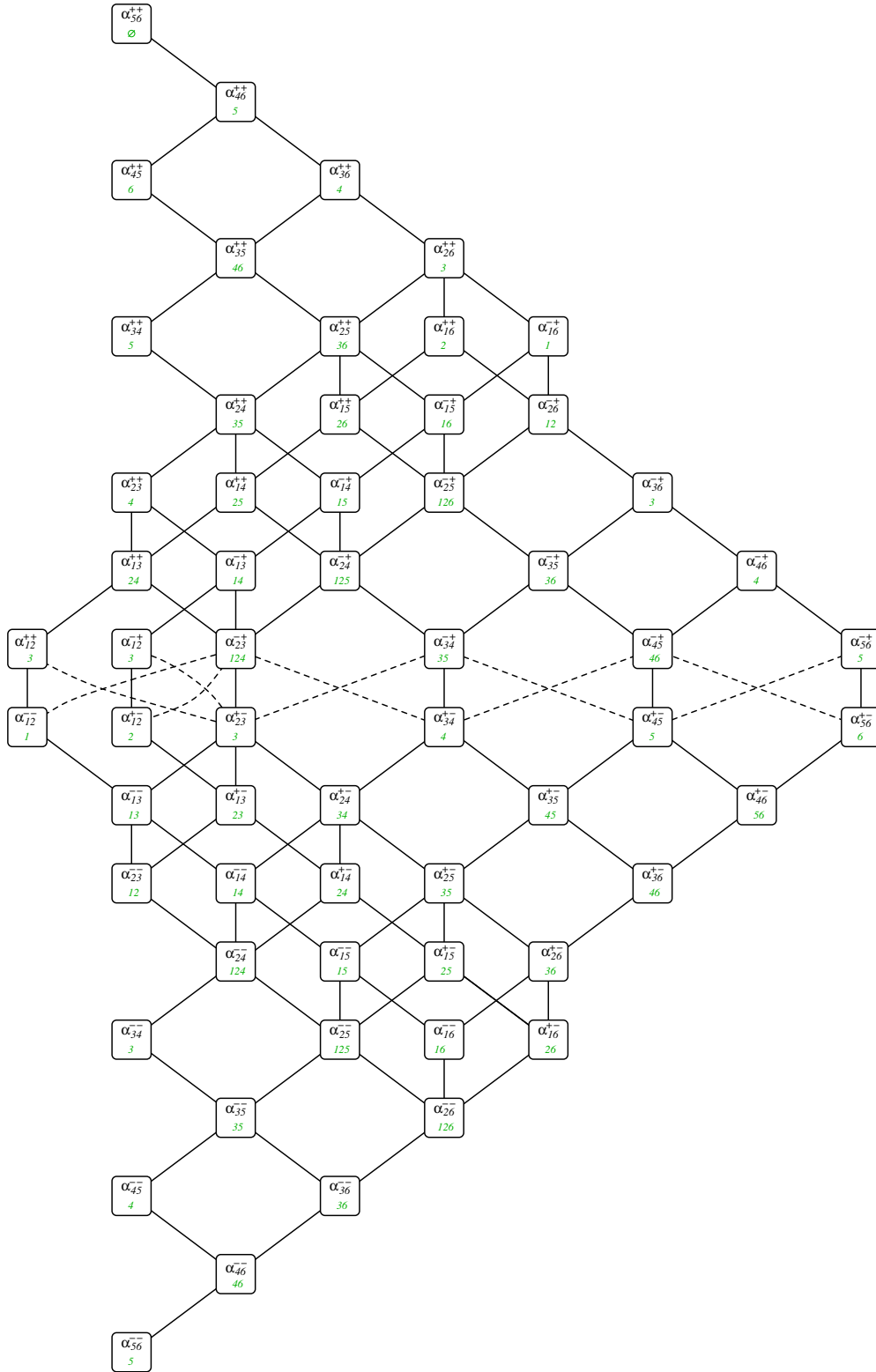
Figure 3.11: The Cayley order on the short roots of D_6. The dashed lines are additional Bruhat order coverings.
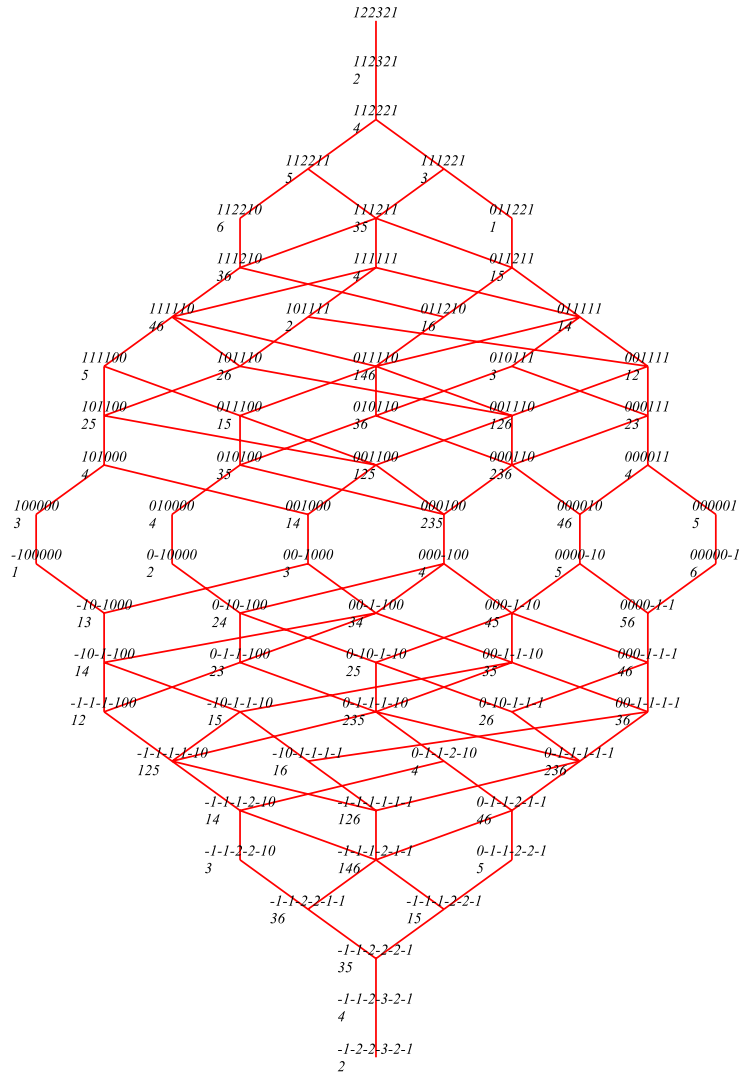
Figure 3.12: The Cayley order on the short roots of E_6.

2234321

1234321
1
1224321
3
1223321
4
1223221   1123321
5   2
1223211   1123221
6   25
1223210   1123211   1122221
7   26   4
1123210   1122211   1112221
27   46   3
1122210   1122111   1112211   0112221
47   5   36   1
1122110   1112210   1112111   0112211
57   37   35   16
1122100   1112110   1111111   0112210   0112111
6   357   4   17   15
1112100   1111110   1011111   0112110   0111111
36   47   2   157   14
1111100   1011110   0112100   0111110   0101111   0011111
46   27   16   147   3   12
1111000   1011100   0111100   0101110   0011110   0001111
5   26   146   37   127   23
1011000   0111000   0101100   0011100   0001110   0000111
25   15   36   126   237   4
1010000   0101000   0011000   0001100   0000110   0000011
4   35   125   236   47   5
1000000   0100000   0010000   0001000   0000100   0000010   0000001
3   4   14   235   46   57   6
-1000000   0-100000   00-10000   000-1000   0000-100   00000-10   000000-1
1   2   3   4   5   6   7
-10-10000 0-10-1000 00-1-1000 000-1-100 0000-1-10 00000-1-1
13   24   34   45   56   67
-10-1-1000 0-1-1-1000 0-10-1-100 00-1-1-100 000-1-1-10 0000-1-1-1
14   23   25   35   46   57
-1-1-1-1000 -10-1-1-100 0-1-1-1-100 0-10-1-1-10 00-1-1-1-10 000-1-1-1-1
12   15   235   26   36   47
-1-1-1-1-100 -10-1-1-1-10 0-1-1-2-100 0-1-1-1-1-10 0-10-1-1-1-1 00-1-1-1-1-1
125   16   4   236   27   37
-1-1-1-2-100 -1-1-1-1-1-10 -10-1-1-1-1-1 0-1-1-2-1-10 0-1-1-1-1-1-1
14   126   17   46   237
-1-1-1-2-2-100 -1-1-1-2-1-10 -1-1-1-1-1-1-1 0-1-1-2-2-10 0-1-1-2-1-1-1
3   146   127   5   47
-1-1-2-2-1-10 -1-1-1-2-2-10 -1-1-1-2-1-1-1 0-1-1-2-2-1-1
36   15   147   57
-1-1-2-2-2-10 -1-1-2-2-1-1-1 -1-1-1-2-2-1-1 0-1-1-2-2-2-1
35   37   157   6
-1-1-2-3-2-10 -1-1-2-2-2-1-1 -1-1-1-2-2-2-1
4   357   16
-1-2-2-3-2-10 -1-1-2-3-2-1-1 -1-1-2-2-2-2-1
2   47   36
-1-2-2-3-2-1-1 -1-1-2-3-2-2-1
27   46
-1-2-2-3-2-2-1 -1-1-2-3-3-2-1
26   5
-1-2-2-3-3-2-1
25
-1-2-2-4-3-2-1
4
-1-2-3-4-3-2-1
3
-2-2-3-4-3-2-1
1

Figure 3.13: The Cayley order on the short roots of E_7.

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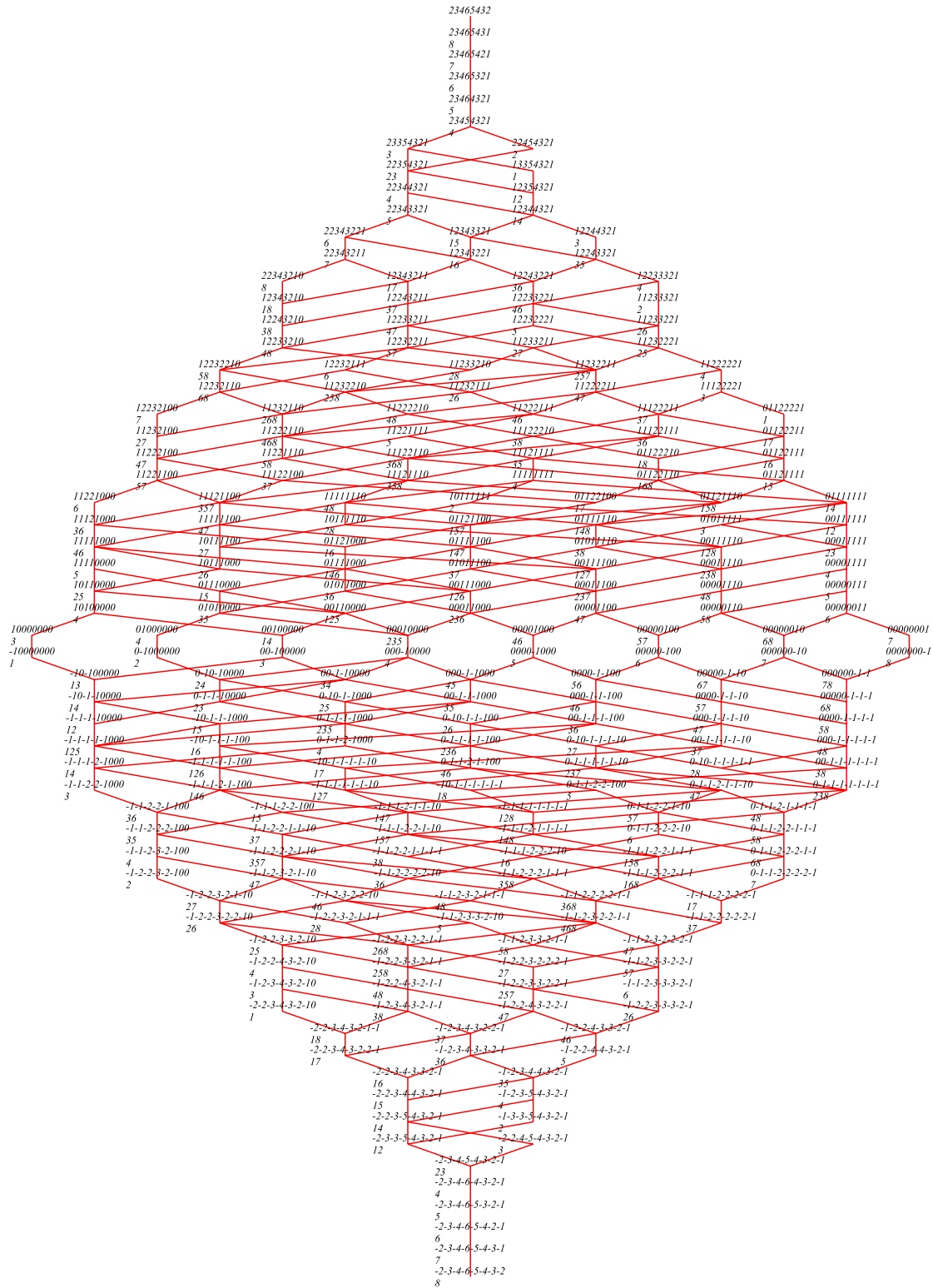
Figure 3.14: The Cayley order on the short roots of E_8.

3.2.1.8 Type F_4

The Hasse diagram for the Cayley order is shown in Figure 3.9 The vertices are labeled by the coordinates of the corresponding roots with respect to simple roots, and by descent sets.

3.2.1.9 Type G_2

The Hasse diagram for the Cayley order is shown in Figure 3.9 The vertices are labeled by the coordinates of the corresponding roots with respect to simple roots, and by descent sets.

3.2.2 Results

3.2.2.1 Type A_n

In this case we can describe both the W-graph and the parabolic Kazhdan-Lusztig polynomials for the case $u = -1$.

Theorem 3.2.4. *The W-graph of the quasi-minuscule representation of type A_n has no surprising edges (in the sense of section 1.2.2). See Figure 3.15 for an example of the resulting W-graph.*

Proof. The size of the τ-invariant is limited to $0, 1$, or 2, and the only element with empty τ-invariant is the lowest one in Bruhat order. Thus if $\beta \to \gamma$ is an arc, then $|\tau(\beta)| = 2, |\tau(\gamma)| = 1$. We know all the τ-invariants from the explicit construction; the only roots with a singleton descent sets are

$$\{\alpha_{i,i+1}\}_{i=1}^n, \quad \{\alpha_{i,1}\}_{i=2}^n, \quad \{\alpha_{n+1,j}\}_{j=2}^n.$$

The first group is the negative simple roots, and the last two are the sides of the top pyramid in the picture of the Cayley order. These last two groups, together with $\overline{\alpha}$, form an order filter of the Cayley order on positive roots, and hence an order ideal of the Bruhat order. Since there are no elements with larger τ-invariants below them, no arc points at these elements.

Suppose $\beta \to \gamma$ is a surprising arc. Then $\gamma = \alpha_{i,i+1} = -\alpha_i$ for some $1 \leqslant i \leqslant n$, and $\tau(\gamma) = \{s_i\}$. All the roots below γ in Bruhat order are positive. The positive roots whose τ-invariants properly contain $\{s_i\}$ are $\{\alpha_{i,j}, j = 2, \ldots, i-1\}$ and $\{\alpha_{j,i+1}, j = i+2, \ldots, n\}$. Out of these, at most two $(\alpha_{i,i-1}, \alpha_{i+2,i+1})$ are one level away from γ while the rest are further away. So

$$\beta \in \{\alpha_{i,j}, j = 2, \ldots, i-2\} \cup \{\alpha_{j,i+1}, j = i+3, \ldots, n\}.$$
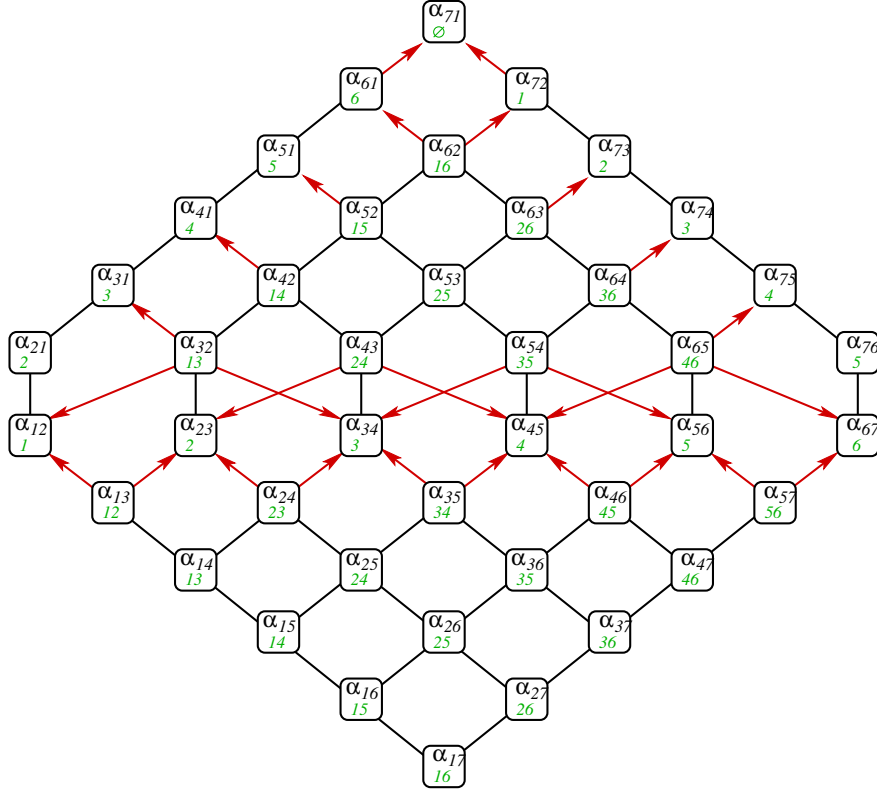
Figure 3.15: The W-graph for the quasi-minuscule quotient of A_6. The τ-invariants are labeled in green.

First, consider $2 \leqslant i \leqslant n-1$. Then there is a simple edge between γ and α_i, and $\tau(\alpha_i) = \{s_{i-1}, s_{i+1}\}$. Suppose $\beta = \alpha_{i,j}$ for some $2 \leqslant j \leqslant i-2$. Then $\tau(\beta) = \{s_i, s_{j-1}\}$. The simple edge starting at β which activates the bond $(i, i+1)$ goes to $\alpha_{i+1,j}$ whose τ-invariant is $\{s_{i+1}, s_{j-1}\}$. By AT1 (with s_{j-1} as the witness), $m(\beta, \gamma) = m(\alpha_{i+1,j}, \alpha_i)$. But $m(\alpha_{i+1,j}, \alpha_i) = 0$ since both τ-invariants have the same size. The case $\beta = \alpha_{j,i+1}$ for some $i+3 \leqslant j \leqslant n$ is handled in the same way.

Now consider the two edge cases for i. Suppose $i = 1$. There is a simple edge between γ and α_1, and $\tau(\alpha_1) = \{s_2\}$. We must have $\beta = \alpha_{j,2}$ for some $4 \leqslant j \leqslant n$, and $\tau(\beta) = \{s_1, s_j\}$. The simple edge from β activating the bond $(1, 2)$ goes to $\alpha_{j,3}$, whose τ-invariant is $\{s_2, s_j\}$. By AT1, $m(\beta, \gamma) = m(\alpha_{j,3}, \alpha_1)$. But $\alpha_{j,3} \not\geqslant \alpha_1$ since α_1 is a maximal positive root. Also $\alpha_{j,3} \not\leqslant \alpha_1$ since the only roots smaller than α_1 have singleton (or empty) τ-invariants. Hence $m(\alpha_{j,3}, \alpha_1) = 0$. This finishes the case $i = 1$. The case $i = n$ is handled in the same way.

Thus any potential surprising arc must have weight 0. $\qquad\square$

Now we calculate the Kazhdan-Lusztig polynomials for the case $u = -1$. We use the shorthand notation $P_{k,l;i,j}$ for $P^{[-1]}_{\alpha_{k,l}, \alpha_{i,j}}$.

Theorem 3.2.5. *If $i > j$, then*

$$P_{k,l;i,j} = \begin{cases} 1 & , \ \textit{if } l \leqslant j < i \leqslant k, \\ 0 & , \ \textit{otherwise.} \end{cases}$$

Hence the Kazhdan-Lusztig polynomials restricted to the positive roots are 1 precisely when the two roots are related in Bruhat order.

Proof. We know that $\alpha_{k,l} \leqslant \alpha_{i,j}$ precisely when $l \leqslant j < i \leqslant k$. Hence we know that if the inequality does not hold then the Kazhdan-Lusztig polynomial is zero. Hence from now on we assume that the inequality holds. We proceed by induction on the position of $\alpha_{i,j}$ in the Bruhat order. There are five cases.

Case 1. Suppose $i < n+1$ and $k = i$. Then $\tau(\alpha_{i,j}) = \{s_i, s_{j-1}\}$ and $s_i \in \tau(\alpha_{k,l})$. Hence

$$P_{i,l;i,j} = P_{i+1,l;i+1,j} + qP_{i,l;i+1,j} - \sum_{\substack{\alpha_{i,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{i,l;z}\mu^{[-1]}(z, \alpha_{i+1,j})q^{(l(\alpha_{i,j})-l(z))/2}.$$

Now $P_{i+1,l;i+1,j} = 1$ by induction, $P_{i,l;i+1,j} = 0$ since $\alpha_{i,l} \not\leqslant \alpha_{i+1,j}$, and the sum is over an empty interval. So the original Kazhdan-Lusztig polynomial is 1.

Case 2. Suppose $i = n+1$, $k = i$ and $l = j-1$. In particular, $j > 1$. Then $\tau(\alpha_{i,j}) = \{s_{j-1}\}$ and $s_{j-1} \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{n+1,j-1;n+1,j} = qP_{n+1,j;n+1,j-1} + P_{n+1,j-1;n+1,j-1}$$
$$- \sum_{\substack{\alpha_{n+1,j-1} \leqslant z \leqslant \alpha_{n+1,j-1} \\ s_{j-1} \notin \mathrm{Asc}(z)}} P_{n+1,j-1;z}\mu^{[-1]}(z, \alpha_{n+1,j-1})q^{(l(\alpha_{n+1,j})-l(z))/2}.$$

Now $P_{n+1,j-1;n+1,j-1} = 1$ by induction, $P_{n+1,j;n+1,j-1} = 0$ since the roots are not correctly related in Bruhat order, and the sum is over an empty set (the interval contains only one element and s_{j-1} is in its ascent set).

Case 3. Suppose $i = n+1$, $k = i$ and $l < j - 1$. Then $\tau(\alpha_{i,j}) = \{s_{j-1}\}$ and $s_{j-1} \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{n+1,l;n+1,j} = (1+q)P_{n+1,l;n+1,j-1} - \sum_{\substack{\alpha_{n+1,l} \leqslant z \leqslant \alpha_{n+1,j-1} \\ s_{j-1} \notin \mathrm{Asc}(z)}} P_{n+1,l;z}\mu^{[-1]}(z, \alpha_{n+1,j-1})q^{(l(\alpha_{n+1,j})-l(z))/2}.$$

Now $P_{n+1,l;n+1,j-1} = 1$ by induction. Also, the only element below $\alpha_{n+1,j-1}$ for which the μ-coefficient is nonzero (in fact, it is 1) is $\alpha_{n+1,j}$, and s_{j-1} is not in its ascent set (it is a descent). So $P_{n+1,l;n+1,j} = 1 + q - q^{2/2} = 1$.

Case 4. Suppose $k = i+1$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{i+1,l;i,j} = qP_{i,l;i+1,j} + P_{i+1,l;i+1,j} - \sum_{\substack{\alpha_{i+1,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{i+1,l;z} \mu^{[-1]}(z, \alpha_{i+1,j}) q^{(l(\alpha_{i,j}) - l(z))/2}.$$

Now $P_{i,l;i+1,j} = 0$ since the two roots are not correctly related in the Bruhat order, and $P_{i+1,l;i+1,j} = 1$ by inductive assumption. The only element of the interval with nonzero μ-coefficient is $\alpha_{i+1,j-1}$ and $s_i \in \mathrm{Asc}(\alpha_{i+1,j-1})$. So $P_{i+1,l;i,j} = 1$.

Case 5. Suppose $k > i+1$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{k,l;i,j} = (1+q)P_{k,l;i+1,j} - \sum_{\substack{\alpha_{k,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{k,l;z} \mu^{[-1]}(z, \alpha_{i+1,j}) q^{(l(\alpha_{i,j}) - l(z))/2}.$$

Now $P_{k,l;i+1,j} = 1$ by inductive assumption. The only elements below $\alpha_{i+1,j}$ with nonzero μ-coefficients are $\alpha_{i+2,j}$ and $\alpha_{i+1,j-1}$, but $s_i \in \mathrm{Asc}(\alpha_{i+1,j-1})$. So $P_{k,l;i,j} = 1 + q - q = 1$. $\quad\square$

Theorem 3.2.6. *If $i < j$, then*

$$P_{k,l;i,j} = \begin{cases} 1 + q^{j-i} & , \text{ if } l < i < j < k, \\ 1 & , \text{ if } i+1 \geqslant l < k \in \{j, j-1\}, \\ 1 & , \text{ if } j-1 \leqslant k > l \in \{i, i+1\}, \\ 1 & , \text{ if } i \leqslant k < l \leqslant j, \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof. We already know the last case since that is precisely when $\alpha_{k,l} \not\leqslant \alpha_{i,j}$. Let us first handle the case when $\alpha_{k,l}$ is also a negative root, i.e. $i \leqslant k < l \leqslant j$. If $i = k$ and $j = l$ then the result is trivial. Without loss of generality (by symmetry of the Coxeter graph) we may assume $i < k$. Two cases are required here:

Case 1. Suppose $k = i+1$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{i+1,l;i,j} = qP_{i,l;i+1,j} + P_{i+1,l;i+1,j} - \sum_{\substack{\alpha_{i+1,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{i+1,l;z} \mu^{[-1]}(z, \alpha_{i+1,j}) q^{(l(\alpha_{i,j}) - l(z))/2}.$$

Now $P_{i,l;i+1,j} = 0$ since the elements are not correctly related in the Bruhat order, and $P_{i+1,l;i+1,j} = 1$ by induction. Also, the only element in the interval for which the μ-coefficient is nonzero is $\alpha_{i+1,j-1}$ and s_i is in its ascent set. So $P_{i+1,l;i,j} = 1$.

Case 2. Suppose $k > i + 1$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{k,l;i,j} = (1+q)P_{k,l;i+1,j} - \sum_{\substack{\alpha_{k,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{k,l;z} \mu^{[-1]}(z, \alpha_{i+1,j}) q^{(l(\alpha_{i,j})-l(z))/2}.$$

Now $P_{k,l;i+1,j} = 1$ by induction. Also, the only element in the interval for which the μ-coefficient is nonzero is $\alpha_{i+2,j}$ and s_i is not in its ascent set (nor is it a descent). So $P_{n+1,l;n+1,j} = 1 + q - q = 1$.

Now we assume $l < i < j < k$. So $\alpha_{k,l}$ is a positive root. This will require three cases. Before proceeding, we mention, for use in induction, that the nonzero μ-coefficient appears precisely when $k = j+1, l = i-1$ (this is not manifested in the W-graph since the τ-invariants of the two roots coincide).

Case 1. Suppose $j = i + 1$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{k,l;i,i+1} = (1+q)P_{k,l;i+1,i} - \sum_{\substack{\alpha_{k,l} \leqslant z \leqslant \alpha_{i+1,i} \\ s_i \notin \mathrm{Asc}(z)}} P_{k,l;z} \mu^{[-1]}(z, \alpha_{i+1,i}) q^{(l(\alpha_{i,i+1})-l(z))/2}.$$

Now $P_{k,l;i+1,i} = 1$ by the previous theorem. Also, the only elements in the interval for which the μ-coefficients are nonzero are $\alpha_{i+2,i}$ and $\alpha_{i+1,i-1}$. Both of these have s_i in their ascent sets. So $P_{k,l;i,i+1} = 1 + q$.

Case 2. Suppose $j = i + 2$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{k,l;i,i+2} = (1+q)P_{k,l;i+1,i+2} - \sum_{\substack{\alpha_{k,l} \leqslant z \leqslant \alpha_{i+1,i+2} \\ s_i \notin \mathrm{Asc}(z)}} P_{k,l;z} \mu^{[-1]}(z, \alpha_{i+1,i+2}) q^{(l(\alpha_{i,i+2})-l(z))/2}.$$

Now $P_{k,l;i+1,i+2} = 1 + q$ by induction. Also, the elements below $\alpha_{i+1,i+2}$ for which the μ-coefficients are nonzero are $\alpha_{i+2,i+1}$, $\alpha_{i+1,i}, \alpha_{i+3,i+2}$ and $\alpha_{i+3,i}$. However $\alpha_{i+1,i}$ and $\alpha_{i+3,i}$ have s_i in their ascent sets. So $P_{k,l;i,i+1} = (1+q)(1+q) - q - q = 1 + q^2$.

Case 3. Suppose $j > i + 2$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \notin \mathrm{Asc}(\alpha_{k,l}) \cup \tau(\alpha_{k,l})$. Hence

$$P_{k,l;i,j} = (1+q)P_{k,l;i+1,j} - \sum_{\substack{\alpha_{k,l} \leqslant z \leqslant \alpha_{i+1,j} \\ s_i \notin \mathrm{Asc}(z)}} P_{k,l;z} \mu^{[-1]}(z, \alpha_{i+1,j}) q^{(l(\alpha_{i,j})-l(z))/2}.$$

Now $P_{k,l;i+1,j} = 1 + q^{j-i-1}$ by induction. Also, the elements below $\alpha_{i+1,j}$ for which the μ-coefficients are nonzero are $\alpha_{i+2,j}$, $\alpha_{i+1,j-1}$, and $\alpha_{j+1,i}$. However $\alpha_{i+1,j-1}$ and $\alpha_{j+1,i}$ have s_i in their ascent sets. So $P_{k,l;i,i+1} = (1+q)(1+q^{j-i-1}) - (1+q^{j-i-2})q = 1 + q^{j-i}$.

Now we assume $i + 1 \geqslant l < k \in \{j, j-1\}$. This will require six cases. The third case from the theorem statement follows from this one via the diagram symmetry, so we will not

treat it explicitly.

Case 1. Suppose $j = k = i + 1$. Then $l < i + 1$. However the case $l = i$ is trivial since then $\alpha_{i,j}$ and $\alpha_{k,l}$ are a distance 1 apart. So we may assume $l < i$. Then $s_i \in \tau(\alpha_{i,j})$ and $s_i \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{i+1,l;i,i+1} = qP_{i,l;i+1,i} + P_{i+1,l;i+1,i} - \sum_{\substack{\alpha_{i+1,l} \leqslant z \leqslant \alpha_{i+1,i} \\ s_i \notin \mathrm{Asc}(z)}} P_{i+1,l;z}\mu^{[-1]}(z, \alpha_{i+1,i})q^{(l(\alpha_{i,i+1}) - l(z))/2}.$$

Now $P_{i,l;i+1,i} = 0$ since the roots are not correctly related in Bruhat order, and $P_{i+1,l;i+1,i} = 1$ by induction. Also, the only element in the interval for which the μ-coefficient is nonzero is $\alpha_{i+1,i-1}$ and s_i is in its ascent set. So $P_{i+1,l;i,j} = 1$.

Case 2. Suppose $j = k + 1 = i + 1$. Then $s_i \in \tau(\alpha_{i,j})$, $s_i \in \tau(\alpha_{k,l})$ and $l < i$. Hence

$$P_{i,l;i,i+1} = P_{i+1,l;i+1,i} + qP_{i,l;i+1,i} - \sum_{\substack{\alpha_{i,l} \leqslant z \leqslant \alpha_{i+1,i} \\ s_i \notin \mathrm{Asc}(z)}} P_{i,l;z}\mu^{[-1]}(z, \alpha_{i+1,i})q^{(l(\alpha_{i,i+1}) - l(z))/2}.$$

Now $P_{i+1,l;i+1,i} = 1$ by induction, $P_{i,l;i+1,i} = 0$ since the roots are not correctly related in Bruhat order, and the sum is over an empty interval. So $P_{i,l;i,i+1} = 1$.

Case 3. Suppose $j = k = i + 2$. Then $l < i + 2$. Then $s_{i+1} \in \tau(\alpha_{i,j})$ and $s_{i+1} \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{i+2,l;i,i+2} = qP_{i+1,s_{i+1}(l);i,i+1} + P_{i+2,l;i,i+1} - \sum_{\substack{\alpha_{i+2,l} \leqslant z \leqslant \alpha_{i,i+1} \\ s_{i+1} \notin \mathrm{Asc}(z)}} P_{i+2,l;z}\mu^{[-1]}(z, \alpha_{i,i+1})q^{(l(\alpha_{i,i+2}) - l(z))/2}.$$

Now by induction

$$P_{i+1,s_{i+1}(l);i,i+1} = \begin{cases} 0, & \text{if } l = i + 1, \\ 1, & \text{if } l < i + 1, \end{cases} \qquad P_{i+2,l;i,i+1} = \begin{cases} 1, & \text{if } l = i + 1 \text{ or } l = i, \\ 1 + q, & \text{if } l < i. \end{cases}$$

The only elements below $\alpha_{i,i+1}$ for which the μ-coefficient is nonzero are $\alpha_{i+1,i}$, $\alpha_{i+2,i+1}$, $\alpha_{i,i-1}$ and $\alpha_{i+2,i-1}$. Now s_{i+1} is in the ascent set of $\alpha_{i+2,i+1}$ and $\alpha_{i+2,i-1}$. Also $\alpha_{i+1,i}$ is above $\alpha_{i+2,l}$ precisely when $l < i + 1$, and $\alpha_{i,i-1}$ is above $\alpha_{i+2,l}$ precisely when $l < i$. So $P_{i+2,l;i,i+2} = 1$.

Case 4. Suppose $j = k + 1 = i + 2$. Then $l < i + 1$. Then $s_{i+1} \in \tau(\alpha_{i,j})$ and $s_{i+1} \in \tau(\alpha_{k,l})$. Hence

$$P_{i+1,l;i,i+2} = P_{i+2,l;i,i+1} + qP_{i+1,l;i,i+1} - \sum_{\substack{\alpha_{i+1,l} \leqslant z \leqslant \alpha_{i,i+1} \\ s_{i+1} \notin \mathrm{Asc}(z)}} P_{i+1,l;z}\mu^{[-1]}(z, \alpha_{i,i+1})q^{(l(\alpha_{i,i+2}) - l(z))/2}.$$

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Now by induction

$$P_{i+2,l;i,i+1} = \begin{cases} 1, & \text{if } l = i, \\ 1+q, & \text{if } l < i, \end{cases} \qquad P_{i+1,l;i,i+1} = 1.$$

The only elements below $\alpha_{i,i+1}$ for which the μ-coefficient is nonzero are $\alpha_{i+1,i}$, $\alpha_{i+2,i+1}$, $\alpha_{i,i-1}$ and $\alpha_{i+2,i-1}$. However $\alpha_{i+2,i+1}$ and $\alpha_{i+2,i-1}$ and not above $\alpha_{i+1,l}$. Also $\alpha_{i+1,i}$ is above $\alpha_{i+1,l}$ precisely when $l < i+1$, and $\alpha_{i,i-1}$ is above $\alpha_{i+1,l}$ precisely when $l < i$. So $P_{i+1,l;i,i+2} = 1$.

Case 5. Suppose $j = k > i+2$. Note that $l \leqslant i+1 < j-1$. Then $s_{j-1} \in \tau(\alpha_{i,j})$ and $s_{j-1} \in \mathrm{Asc}(\alpha_{k,l})$. Hence

$$P_{j,l;i,j} = qP_{j-1,l;i,j-1} + P_{j,l;i,j-1} - \sum_{\substack{\alpha_{j,l} \leqslant z \leqslant \alpha_{i,j-1} \\ s_{j-1} \notin \mathrm{Asc}(z)}} P_{j,l;z}\mu^{[-1]}(z,\alpha_{i,j-1})q^{(l(\alpha_{i,j})-l(z))/2}.$$

Now by induction

$$P_{j-1,l;i,j-1} = 1; \qquad P_{j,l;i,j-1} = \begin{cases} 1, & \text{if } l = i \text{ or } l = i+1, \\ 1+q^{j-i-1}, & \text{if } l < i. \end{cases}$$

The only elements below $\alpha_{i,j-1}$ for which the μ-coefficient is nonzero are $\alpha_{i+1,j-1}$, $\alpha_{i,j-2}$, and $\alpha_{j,i-1}$. Now s_{j-1} is in the ascent set of $\alpha_{i+1,j-1}$ and of $\alpha_{j,i-1}$. So $P_{j,l;i,j} = qP_{j-1,s_{j-1}(l);i,j-1} + P_{j,l;i,j-1} - qP_{j,l;i,j-2} = 1$, since, by induction,

$$P_{j,l;i,j-2} \begin{cases} 1, & \text{if } l = i \text{ or } l = i+1, \\ 1+q^{j-i-2}, & \text{if } l < i. \end{cases}$$

Case 6. Suppose $j = k+1 > i+2$. Then $s_{j-1} \in \tau(\alpha_{i,j})$ and $s_{j-1} \in \tau(\alpha_{k,l})$. Hence

$$P_{j-1,l;i,j} = P_{j,l;i,j-1} + qP_{j-1,l;i,j-1} - \sum_{\substack{\alpha_{j-1,l} \leqslant z \leqslant \alpha_{i,j-1} \\ s_{j-1} \notin \mathrm{Asc}(z)}} P_{j-1,l;z}\mu^{[-1]}(z,\alpha_{i,j-1})q^{(l(\alpha_{i,j})-l(z))/2}.$$

Now by induction

$$P_{j,l;i,j-1} = \begin{cases} 1, & \text{if } l = i \text{ or } l = i+1, \\ 1+q^{j-i-1}, & \text{if } l < i, \end{cases} \qquad P_{j-1,l;i,j-1} = 1.$$

The only elements below $\alpha_{i,j-1}$ for which the μ-coefficient is nonzero are $\alpha_{i+1,j-1}$, $\alpha_{i,j-2}$, and

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$\alpha_{j,i-1}$. However s_{j-1} is in the ascent set of $\alpha_{i+1,j-1}$ and of $\alpha_{j,i-1}$. Now

$$P_{j-1,l;i,j-2} = \begin{cases} 1, & \text{if } l = i \text{ or } \mathrm{l} = \mathrm{i{+}1}, \\ 1 + q^{j-i-2}, & \text{if } l < i, \end{cases}$$

so $P_{j-1,l;i,j} = 1$. □

3.2.2.2 Type B_n, $n \geqslant 2$

In this case the Cayley order is a chain and all the τ-invariants (besides the one of $\overline{\alpha}$) are singletons. Hence there is no possibility for surprising arcs and the W-graph is is fully described by the Cayley order. An example of the resulting W-graph is shown in Figure 3.16.

3.2.2.3 Type C_n, $n \geqslant 3$

In this case we can fully describe the W-graph:

Theorem 3.2.7. *If $n = 3$ then from the Hasse diagram of the Cayley order it is clear that no surprising arcs are possible in the W-graph of the quasi-minuscule representation of type C_n. If $n > 3$ then the surprising arcs are:*

1. $\alpha_{i,i+2}^{+;+} \to \alpha_{i-1,i}^{+;+}, \quad 2 \leqslant i \leqslant n-3,$

2. $\alpha_{i,i+2}^{-;-} \to \alpha_{i+2,i+3}^{-;-}, \quad 1 \leqslant i \leqslant n-3.$

3. $\alpha_{1,3}^{+;+} \to \alpha_{1,2}^{+;-},$

4. $\alpha_{1,3}^{-;+} \to \alpha_{1,2}^{-;-},$

See Figure 3.17 for an example.

Proof. First show that the weight of each arc mentioned above is 1, and then prove that no other possible surprising arc has nonzero weight.

Suppose $2 \leqslant i \leqslant n-3$. Then $\tau(\alpha_{i,i+2}^{+;+}) = \{s_{i+1}, s_{i+3}\}$ and $\tau(\alpha_{i-1,i}^{+;+}) = \{s_{i+1}\}$. The bond $(i+1, i+2)$ can be activated from both roots; from $\alpha_{i,i+2}^{+;+}$ it is activated on an edge to $\alpha_{i,i+1}^{+;+}$, and from $\alpha_{i-1,i}^{+;+}$ on the edge to $\alpha_{i-1,i+1}^{+;+}$. By Kazhdan-Lusztig transport,

$$\mu(\alpha_{i,i+2}^{+;+}, \alpha_{i-1,i}^{+;+}) = \mu(\alpha_{i,i+1}^{+;+}, \alpha_{i-1,i+1}^{+;+}) = 1,$$

where the last equality holds since the pair of roots are a left weak order covering. Hence the first group of arcs indeed has weight 1.
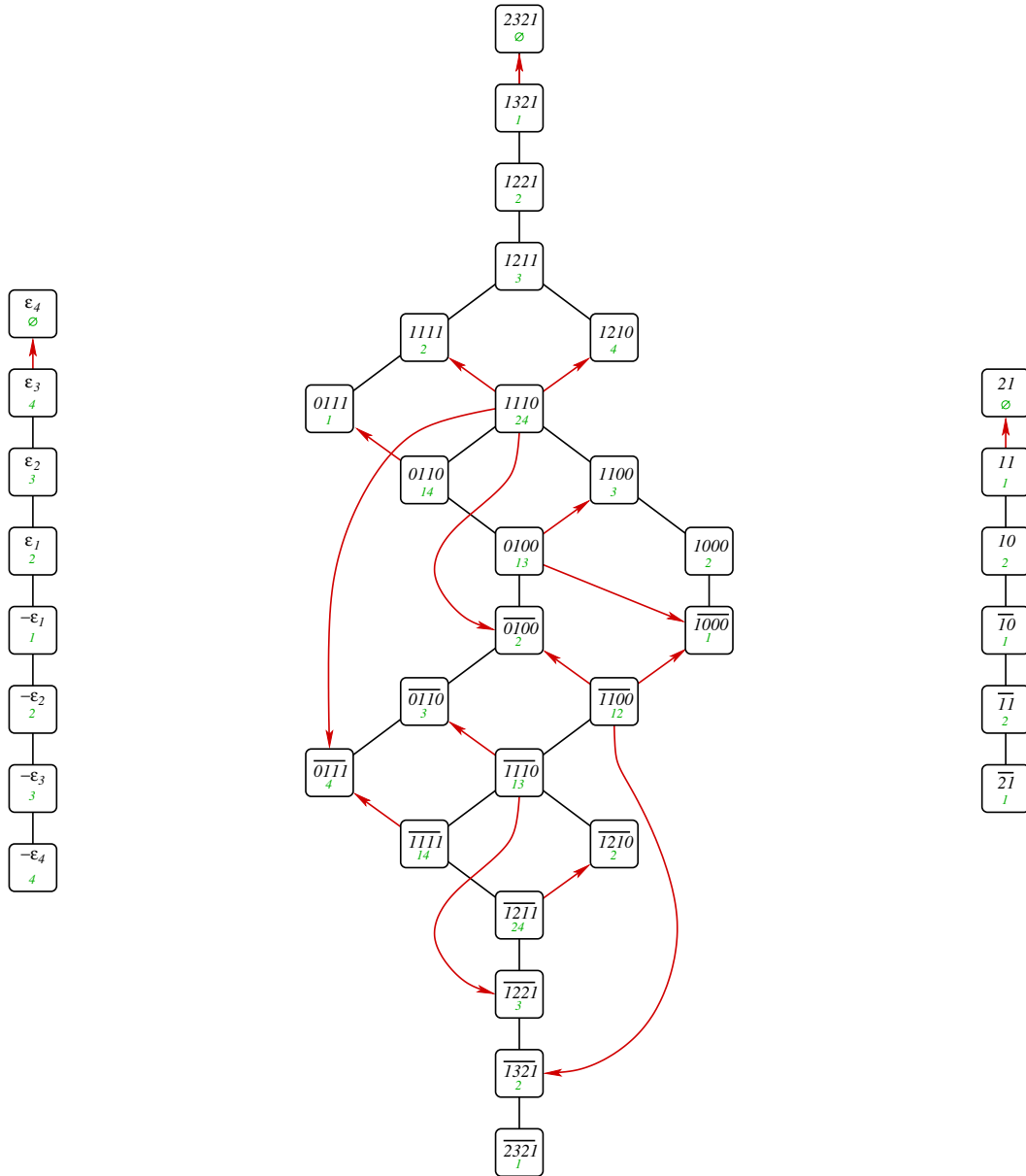
Figure 3.16: The W-graphs for the quasi-minuscule quotients of B_4, F_4, and G_2. In the case of B_4 the roots are labeled in accordance with the explicit description in section 3.2.1. In the other cases, the roots are labeled by coordinates in the simple roots. The bar denotes the negative of the root.

Suppose $1 \leqslant i \leqslant n-3$. We have $\tau(\alpha_{i,i+2}^{-;-} = \{s_i, s_{i+2}\}$ and $\tau(\alpha_{i+2,i+3}^{-;-}) = \{s_{i+2}\}$. By Kazhdan-Lusztig transport (with respect to the bond $(i+1, i+2)$),

$$\mu(\alpha_{i,i+2}^{-;-}, \alpha_{i+2,i+3}^{-;-}) = \mu(\alpha_{i+1,i+2}^{-;-}, \alpha_{i+1,i+3}^{-;-}) = 1,$$

where the last equality holds since the pair of roots are a left weak order covering. Hence the fourth group of arcs indeed has weight 1.

The remaining two arc weights can be determined from a single use of the Kazhdan-Lusztig transport corresponding to the double bond. The $(1, 2)$-string of $\alpha_{1,3}^{-;+}$ also contains $\alpha_{2,3}^{-;+}$ and $\alpha_{1,3}^{+;+}$. The $(1, 2)$-string of $\alpha_{1,2}^{-;-}$ also contains $\alpha_{1,2}^{+;-}$ and $\alpha_{1,2}^{-;+}$. By Kazhdan-Lusztig transport,

$$\mu(\alpha_{1,3}^{-;+}, \alpha_{1,2}^{-;-}) = \mu(\alpha_{1,3}^{+;+}, \alpha_{1,2}^{+;-}) = \mu(\alpha_{1,3}^{-;+}, \alpha_{1,2}^{-;+}) = 1,$$

where the last equality holds since the pair of roots are a left weak order covering. The reason these arcs appear only for $n > 3$ is that if $n = 3$ then the τ-invariants of the vertices joined by them coincide. This finishes the check that all the arcs claimed are indeed present.

Now we need to show that no other surprising arcs have nonzero weight. Since the maximal size of the τ-invariants is 2, any surprising arc must start at a root with two elements in the τ-invariant and end at a root with one element in the τ-invariant. Hence we list the elements with singleton τ-invariants:

$$\alpha_{i,n}^{+;+}, 1 \leqslant i \leqslant n-1, \qquad \alpha_{i,i+1}^{+;+}, 1 \leqslant i \leqslant n-2, \qquad \alpha_{i,i+1}^{+;-}, 1 \leqslant i \leqslant n-1,$$

$$\alpha_{i,n}^{-;+}, 1 \leqslant i \leqslant n-1, \qquad \alpha_{i,i+1}^{-;-}, 1 \leqslant i \leqslant n-1.$$

Notice that $\{\alpha_{i,n}^{\pm;+} \mid 1 \leqslant i \leqslant n-1\} \cup \{\overline{\alpha}\}$ is an order ideal in the weak (hence Bruhat, since all these roots are positive) order. Hence these roots cannot be ends of surprising arcs. The remaining possible endpoints need to be examined carefully.

Let $1 \leqslant i \leqslant n-2$, and examine the root $\alpha_{i,i+1}^{+;+}$. We have $\tau(\alpha_{i,i+1}^{+;+}) = \{i+2\}$. Since the root is positive, the only roots below it in Bruhat order must be positive. The possibilities are

$$\alpha_{i+1,j}^{+;+}, \quad i+2 < j < n; \quad \alpha_{j,i+1}^{+;+}, \quad 1 \leqslant j < i;$$
$$\alpha_{i+2,j}^{-;+}, \quad i+2 < j < n; \quad \alpha_{j,i+1}^{-;+}, \quad 1 \leqslant j < i+1.$$

Since weak order and standard order are dual on positive roots, we see that only $\alpha_{i+1,j}^{+;+}$ for $i+2 < j < n$ are, in fact, lower in the Bruhat order. Thus we consider the arc $\alpha_{i+1,j}^{+;+} \to \alpha_{i,i+1}^{+;+}$. The case $j = i+3$ has already been handled (there indeed is an arc there), so we assume

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$j > i + 3$. By Kazhdan-Lusztig transport,

$$m(\alpha_{i+1,j}^{+,+}, \alpha_{i,i+1}^{+,+}) = m(\alpha_{i,j}^{+,+}, \alpha_{i,i+2}^{+,+}).$$

The τ-invariants of $\alpha_{i,j}^{+,+}$, and $\alpha_{i,i+2}^{+,+}$ are, respectively, $\{s_{i+1}, s_{j+1}\}$ and $\{s_{i+1}, s_{i+3}\}$. These are incomparable and the two roots are not related by a simple reflection (since $j > i + 3$). Hence the weight of the arc must be 0.

Let $1 \leqslant i \leqslant n - 1$, and examine the root $\alpha_{i,i+1}^{+,-}$, whose τ-invariant is $\{s_{i+1}\}$. Since the root is a negative simple one, the only roots below it in Bruhat order must be positive. The possibilities are

$$\begin{array}{llll}
\alpha_{i,j}^{+,+}, & i+1 < j < n; & \qquad \alpha_{j,i}^{+,+}, & 1 \leqslant j < i-1; \\
\alpha_{i+1,j}^{-,+}, & i+1 < j < n; & \qquad \alpha_{j,i}^{-,+}, & 1 \leqslant j < i.
\end{array}$$

The cases that have already been treated (there are arcs there) are: $j = i + 2$ in the bottom left case, $j = i - 1$ in the bottom right case, and $i = 1, j = 3$ in the top left case. We now assume we are not in any of these.

If $i > 1$ then an application of Kazhdan-Lusztig transport in each of the cases yields

$$\begin{array}{ll}
\mu(\alpha_{i,j}^{+,+}, \alpha_{i,i+1}^{+,-}) = \mu(\alpha_{i-1,j}^{+,+}, \alpha_{i,i+1}^{-,+}) = 0; & \mu(\alpha_{j,i}^{+,+}, \alpha_{i,i+1}^{+,-}) = \mu(\alpha_{j,i+1}^{+,+}, \alpha_{i,i+1}^{-,+}) = 0; \\
\mu(\alpha_{i+1,j}^{-,+}, \alpha_{i,i+1}^{+,-}) = \mu(\alpha_{i,j}^{-,+}, \alpha_{i,i+1}^{-,+}) = 0; & \mu(\alpha_{j,i}^{-,+}, \alpha_{i,i+1}^{+,-}) = \mu(\alpha_{j,i+1}^{-,+}, \alpha_{i,i+1}^{-,+}) = 0.
\end{array}$$

The above arc weights must be zero since the pairs of roots have incomparable τ-invariants, but are not related by a simple reflection.

If $i = 1$ then the second column of cases does not exist and an application of the double bond Kazhdan-Lusztig transport handles both rows of the first column:

$$m(\alpha_{1,j}^{+,+}, \alpha_{1,2}^{+,-}) = m(\alpha_{2,j}^{-,+}, \alpha_{1,2}^{+,-}) = m(\alpha_{1,j}^{-,+}, \alpha_{1,2}^{-,+}) = 0.$$

The last equality holds since the pairs of roots have incomparable τ-invariants, but are not related by a simple reflection (remember that the case $j = 3$ has been handled).

Let $1 \leqslant i \leqslant n - 1$ and examine the root $\alpha_{i,i+1}^{-,-}$, whose τ-invariant is $\{s_i\}$.

The possibilities for the beginnings of the arcs are (ignoring Bruhat order issues for a

while.):

$$
\begin{array}{llll}
\alpha_{i-1,j}^{+;+}, & i<j<n; & \alpha_{j,i-1}^{+;+}, & 1\leqslant j<i-2; \\
\alpha_{i,j}^{-;+}, & i<j<n; & \alpha_{j,i-1}^{-;+}, & 1\leqslant j<i-1; \\
\alpha_{i-1,j}^{+;-}, & i<j\leqslant n; & \alpha_{j,i}^{+;-}, & 1\leqslant j<i-1; \\
\alpha_{i,j}^{-;-}, & i+1<j\leqslant n; & \alpha_{j,i}^{-;-}, & 1\leqslant j<i-1.
\end{array}
$$

Now the bottom four of these have negative roots, and on the negative roots the Cayley and standard orderings coincide and they are both dual to the Bruhat order. This allows us to eliminate the bottom left group and most of the 3-rd row left group. The only element of that group which is below $\alpha_{i,i+1}^{-;-}$ is $\alpha_{i-1,j}^{+;-}$. However these two elements are an even distance apart, so there is no possibility for an arc. Similarly by parity we can eliminate $\alpha_{i-2,i}^{+;-}$.

Of the remaining, the cases that have already been treated (there are arcs there) are: $j=i-2$ in the bottom right case, and $i=1, j=3$ in the case in row 2, column 1. We now assume we are not in any of these.

If $i>1$ then an application of Kazhdan-Lusztig transport in each of the cases yields

$$
\mu(\alpha_{i-1,j}^{+;+},\alpha_{i,i+1}^{-;-})=\begin{cases}\mu(\alpha_{i-2,j}^{+;+},\alpha_{i-1,i+1}^{-;-}), & i>2;\\ \mu(\alpha_{2,j}^{-;+},\alpha_{1,3}^{+;-}), & i=2;\end{cases}
\qquad
\mu(\alpha_{j,i-1}^{+;+},\alpha_{i,i+1}^{-;-})=\mu(\alpha_{j,i}^{+;+},\alpha_{i-1,i+1}^{-;-});
$$

$$
\mu(\alpha_{i,j}^{-;+},\alpha_{i,i+1}^{-;-})=\begin{cases}\mu(\alpha_{i-1,j}^{-;+},\alpha_{i-1,i+1}^{-;-})=0, & i>2;\\ \mu(\alpha_{1,j}^{+;+},\alpha_{1,3}^{+;-}), & i=2;\end{cases}
\qquad
\mu(\alpha_{j,i-1}^{-;+},\alpha_{i,i+1}^{-;-})=\mu(\alpha_{j,i}^{-;+},\alpha_{i-1,i+1}^{-;-});
$$

$$
\mu(\alpha_{j,i}^{+;-},\alpha_{i,i+1}^{-;-})=\mu(\alpha_{j,i+1}^{+;-},\alpha_{i-1,i+1}^{-;-});
$$

$$
\mu(\alpha_{j,i}^{-;-},\alpha_{i,i+1}^{-;-})=\mu(\alpha_{j,i+1}^{-;-},\alpha_{i-1,i+1}^{-;-}).
$$

All the μ-values on the right-hand sized of the equalities are 0.

If $i=1$ then the only possibilities for the start of the arc are $\alpha_{1,j}^{-;+}$ for $1<j<n$. The case $j=2$ is eliminated by parity and the case $j=3$ has been handled. For $j>4$, Kazhdan-Lusztig transport (the double bond version) gives

$$
\mu(\alpha_{1,j}^{-;+},\alpha_{1,2}^{-;-})=\mu(\alpha_{1,j}^{-;+},\alpha_{1,2}^{-;+})=0.
$$

This finishes the proof. $\qquad\square$
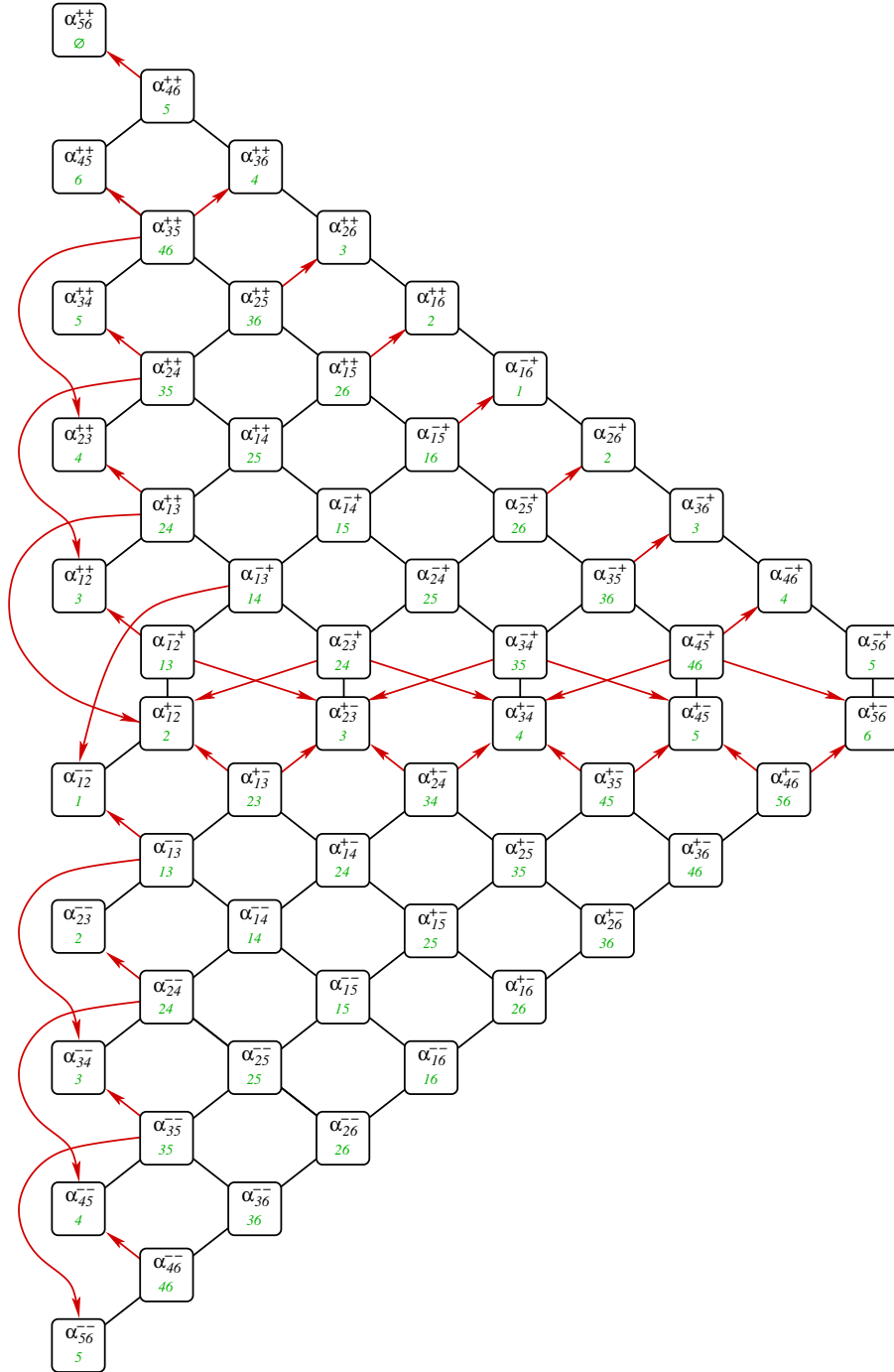
Figure 3.17: The W-graph for the quasi-minuscule quotient of C_6. The τ-invariants are labeled in green.

3.2.2.4   Type D_n, $n \geqslant 4$

In this case we give a conjecture for what the W-graph is, but we cannot prove it at this point.

Conjecture 3.2.8. For any $n \geqslant 4$, the following are surprising arcs in the W-graph of the quasi-minuscule representation of type D_n:

1. For $1 \leqslant k \leqslant n - 3$,

$$
\begin{array}{ccc}
\alpha_{k,k+2}^{--} & & \\
& \searrow & \\
\alpha_{k,k+2}^{+-} & \to & \alpha_{k,k+1}^{--} \\
& \nearrow & \\
\alpha_{k+1,k+3}^{+-} & &
\end{array}
$$

2. For $4 \leqslant k \leqslant n$,

$$
\begin{array}{ccc}
& & \alpha_{1,k}^{--} \\
& \nearrow & \\
\alpha_{2,k-1}^{-+} & & \\
& \searrow & \\
& & \alpha_{1,k}^{+-}
\end{array}
$$
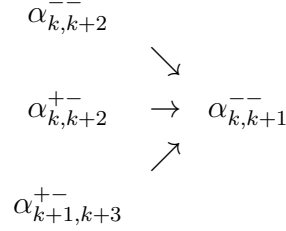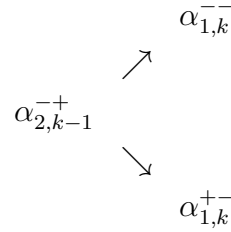
3. $\alpha_{2,3}^{-+} \to \alpha_{2,3}^{--}$.

   For $n \geqslant 5$ the following are also surprising arcs:

4. $\alpha_{k-1,k+1}^{++} \to \alpha_{k-2,k-1}^{++}$, $3 \leqslant k \leqslant n - 2$,

5. $\alpha_{1,4}^{-+} \to \alpha_{1,2}^{--}$,

6. $\alpha_{1,4}^{++} \to \alpha_{1,2}^{+-}$,

7. $\alpha_{2,4}^{++} \to \alpha_{1,2}^{-+}$.

Se Figure 3.18 for an example.

3.2.2.5   Type E_m, $m \in \{6, 7, 8\}$

This is, for the most part, a computer calculation. The computer-generated data is shown in Appendix A.
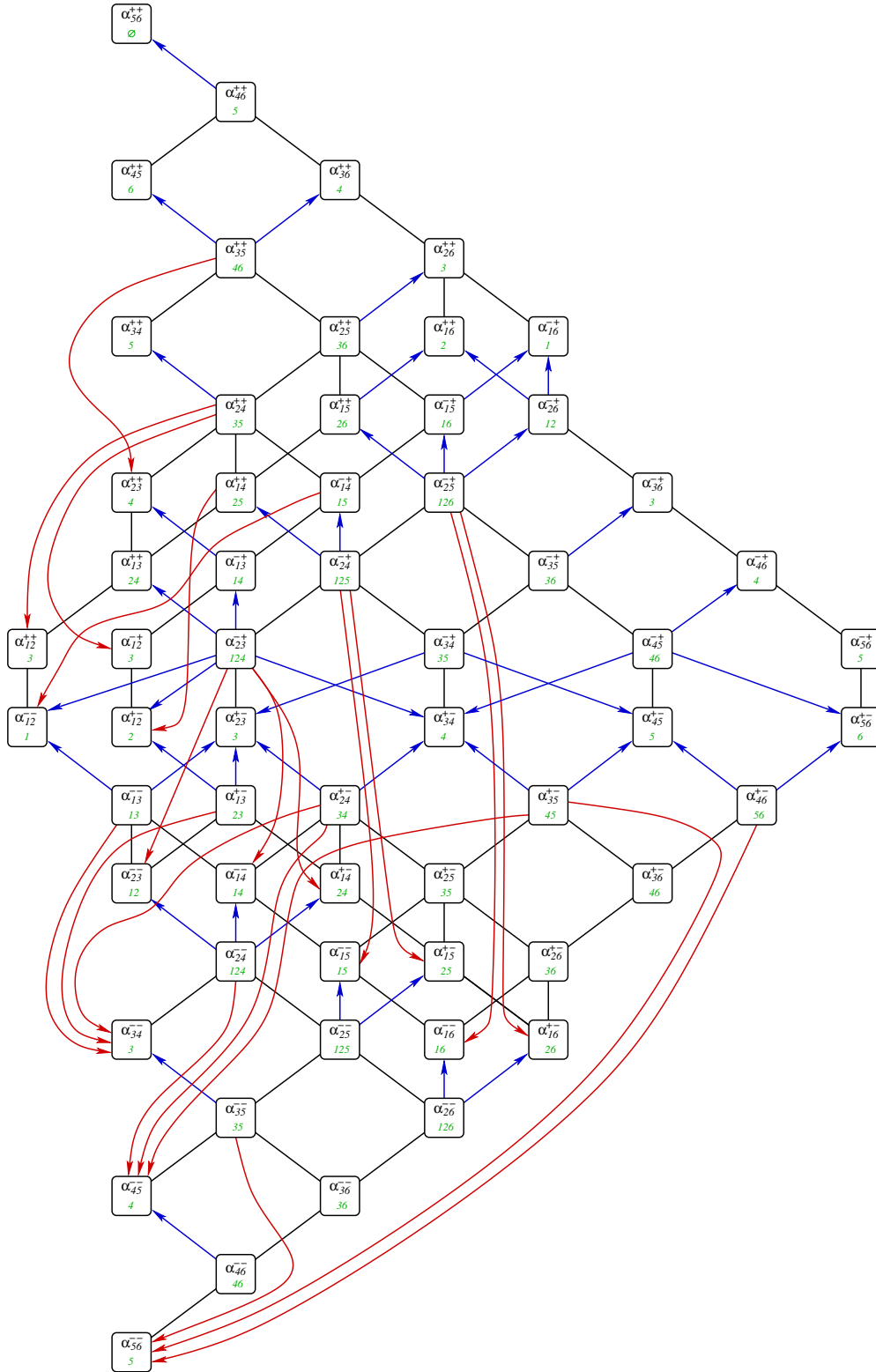
Figure 3.18: The W-graph for the quasi-minuscule quotient of D_6. The τ-invariants are labeled in green.

3.2.2.6 Type F_4

A computer calculation imposing Kazhdan-Lusztig transport yields the graph in Figure 3.16. We note that the double-bond version of Kazhdan-Lusztig transport is necessary to nail down the weights (all of them are either 0 or 1). Alternatively one can achieve the same goal by using the (full) Polygon Rule.

3.2.2.7 Type G_2

The situation is similar to the one for type B; the graph is shown in Figure 3.16.

CHAPTER 4

Type A molecules are Kazhdan-Lusztig

There are no known examples of admissible A_n-cells besides the Kazhdan-Lusztig cells (Stembridge has checked it up to $n = 9$). Hence one may formulate

Conjecture 4.0.9. The only possible admissible type A cells are the Kazhdan-Lusztig ones.

A possible strategy of proof is as follows:

1. Show that any A_n-molecule is isomorphic to a molecule in the Kazhdan-Lusztig graph.

2. It is known that each Kazhdan-Lusztig A_n-cell has only one molecule, and the simple edges are well understood (they are called *dual Knuth moves*). The second step is to prove that no cell may have multiple molecules. The fact that no admissible A_n-cell may contain two or more Kazhdan-Lusztig molecules has been checked for $n \leqslant 12$ ([Ste]).

3. The last part is to prove that there can be only one A_n-graph with a given underlying molecule. For Kazhdan-Lusztig molecules this has been checked for $n \leqslant 13$ ([Ste]).

In this section we complete the first part of the above program. Together with the above computations, this result implies that all admissible A_n-cells up to $n = 12$ are Kazhdan-Lusztig. The main ingredient of the proof is the Assaf's axiomatization of graphs on tableaux generated by dual Knuth moves ([Ass08]). Five of the axioms follow easily from the molecules axioms, but the last one presents a challenge. Recently Roberts suggested a revised version of the last axiom ([Rob13]). Using it one can give a short computerized proof of our result.

The results were announced in the proceedings of FPSAC 2013 ([Chm13a], [Chm13b]).

4.1 Dual equivalence graphs

This section summarizes the relevant definitions and results of [Ass08]. The results are restated to make the similarity with the W-molecule world more apparent.

Fix $n \in \mathbb{Z}^{>0}$. Let (W, S) be a Coxeter system of type A_n. Identify S in a natural way with $\{1, \ldots, n\}$. Define b_i to be the bond $(i, i+1)$. Then $B := \{b_1, \ldots, b_{n-1}\}$ is the set of all bonds. For examples with small n we will use the notation $a, b, c, \ldots$ instead of $b_1, b_2, b_3, \ldots$.

Definition 4.1.1. A *signed colored graph of type $n+1$* is a tuple (V, E, τ, β), where (V, E) is a finite undirected simple graph, $\tau : V \to 2^S$, and $\beta : E \to 2^B$.

Denote by E_i the set of edges with label i (i.e. such that the corresponding value of β contains i); we call these *i-colored* edges. This is a slight reindexing from Assaf's original definition; in the original E_i was the set of edges whose label contains $i - 1$.

We start by constructing a family, indexed by partitions, of signed colored graphs.

4.1.1 "Standard" dual equivalence graphs

Let λ be a partition of $n + 1$. Let $SYT(\lambda)$ be the set of standard Young tableaux of shape λ. Using the English convention for tableaux, the left-descent set of a tableau T is

$$\tau(T) := \{1 \leqslant i \leqslant n : i \text{ is located in a higher row than } i + 1 \text{ in } T\}.$$

The set of vertices of our graph is $V := SYT(\lambda)$ (see Example 4.1.2).

By a diagonal of a tableau we mean a $NW - SE$ diagonal. A *dual Knuth move* is the exchange of i and $i+1$ in a standard tableau, provided that either $i-1$ or $i+2$ lies (necessarily strictly) between the diagonals containing i and $i+1$. This corresponds to dual Knuth moves on the symmetric group via, for example, the "content reading word" (reading each diagonal from top to bottom, and concatenating in order of increasing height of the diagonals). The dual Knuth moves define the edges of our graph:

$$E := \{(T, U) : T \text{ and } U \text{ are related by a dual Knuth move}\}.$$

A dual Knuth move between tableaux T and U *activates the bond b_i* if i lies in precisely one of $\tau(T)$ and $\tau(U)$, and $i + 1$ lies precisely in the other. Denote this condition by $T \overset{b_i}{-} U$. For $(T, U) \in E$, let
$$\beta(T, U) := \{b_i \in B : T \overset{b_i}{-} U\}.$$

The graph $G_\lambda := (V, E, \tau, \beta)$ is a signed colored graph of type $n + 1$.

One can give a slightly more explicit description of activations on tableaux. Notice that $i, i + 1$, and $i + 2$ have to lie on three distinct diagonals in any tableau. We have $T \overset{b_i}{-} U$ precisely when T and U differ by switching the two of the above entries on the outside diagonals, provided that the middle diagonal does not contain $i + 1$.
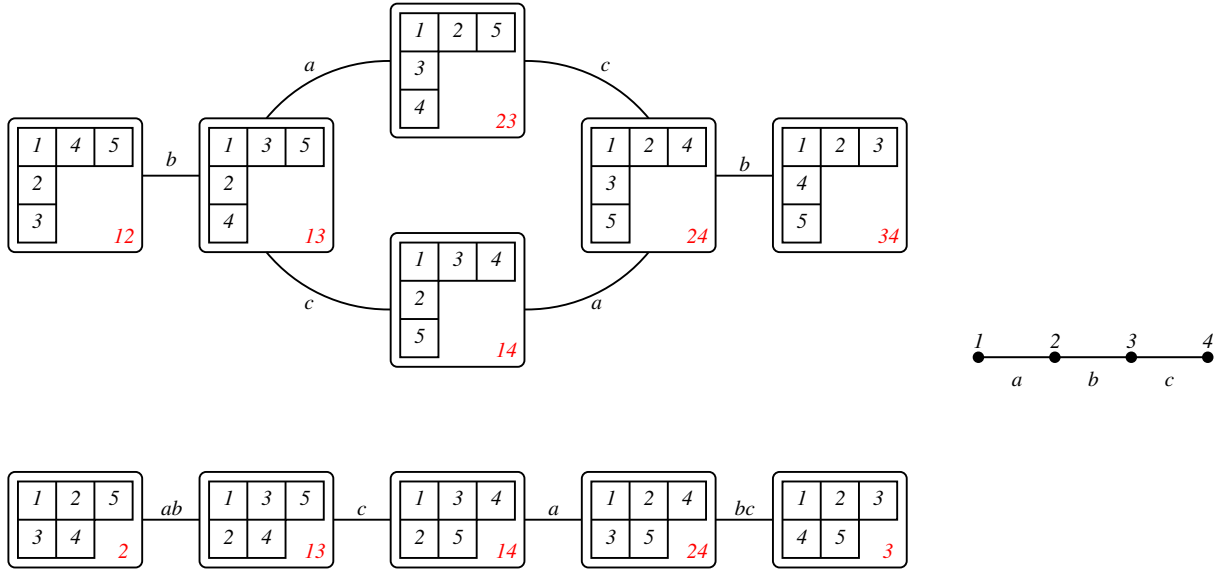
Figure 4.1: The standard dual equivalence graphs corresponding to the shapes 311 and 32.

Example 4.1.2. Two standard dual equivalence graphs, corresponding to the shapes 311 and 32, are shown in Figure 4.1. The values of τ are shown in red in the lower right-hand corner of each vertex.

4.1.2 Axiomatics

Now we review Assaf's axiomatization of the above construction.

A vertex w of a signed colored graph is said to *admit an i-neighbor* if precisely one of i and $i + 1$ lies in $\tau(w)$.

Definition 4.1.3. A *dual equivalence graph of type $n+1$* is a signed colored graph (V, E, τ, β) such that for any $1 \leqslant i < n$:

1. For $w \in V$, w admits an i-neighbor if and only if there exists $x \in V$ which is connected to w by an edge of color i. In this case x must be unique.

2. Suppose (w, x) is an i-colored edge. Then $i \in \tau(w)$ iff $i \notin \tau(x)$, $i + 1 \in \tau(w)$ iff $i + 1 \notin \tau(x)$, and if $h < i - 1$ or $h > i + 2$ then $h \in \tau(w)$ iff $h \in \tau(x)$.

   In other words, going along an i-colored edge switches i and $i + 1$ in the τ-invariant, and does not affect any labels except $i - 1, i, i + 1$, and $i + 2$.

3. Suppose (w, x) is an i-colored edge. If $i - 1 \in \tau(w)\Delta\tau(x)$ then $(i - 1 \in \tau(w)$ iff

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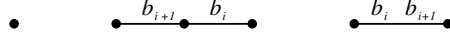
Figure 4.2: Possible connected components of restrictions of a dual equivalence graph to i- and $(i+1)$-colored edges.
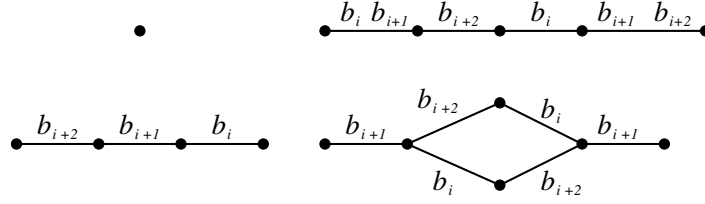
Figure 4.3: Possible connected components of restrictions of a dual equivalence graph to i-, $(i+1)$- and $(i+2)$-colored edges.

$i + 1 \in \tau(w)$), where Δ is the symmetric difference. If $i + 2 \in \tau(w)\Delta\tau(x)$ then ($i + 2 \in \tau(w)$ iff $i \in \tau(w)$).

4. If $i < n - 2$, consider the subgraph on all the vertices and i- and $(i+1)$-colored edges. Each of its connected components has the form shown in Figure 4.2. If $i < n - 3$, consider the subgraph on all the vertices and i-, $(i+1)$- and $(i+2)$-colored edges. Each of its connected components has the form shown in Figure 4.3.

5. Suppose $(w, x) \in E_i, (x, y) \in E_j$, and $|i - j| \geqslant 3$. Then there exists $v \in V$ such that $(w, v) \in E_j, (v, y) \in E_i$.

6. Consider a connected component of the subgraph on all the vertices and edges of colors $\leqslant i$. If we erase all the i-colored edges it breaks down into several components. Any two of these are connected by an i-colored edge.

Examples of A_4 dual equivalence graphs can be found on the right of Figure 4.5.

A *morphism* of signed colored graphs is a map on vertex sets which preserves τ and β.

Proposition 4.1.4. *The graph G_λ is a dual equivalence graph. Moreover, $\{G_\lambda\}_\lambda$ is a complete collection of isomorphism class representatives of connected dual equivalence graphs.*

Proof. The references are to [Ass08]. The first statement is Proposition 3.5. The second is a combination of Theorem 3.9 and Proposition 3.11. $\qquad\square$

Remark 4.1.5. There is some redundancy in the definition as presented. Namely, β can be calculated from τ: an edge (u, v) is i-colored if and only if i lies in precisely one of $\tau(u)$ and $\tau(v)$, while $i + 1$ lies in precisely the other. Assaf needed a slightly more general definition,
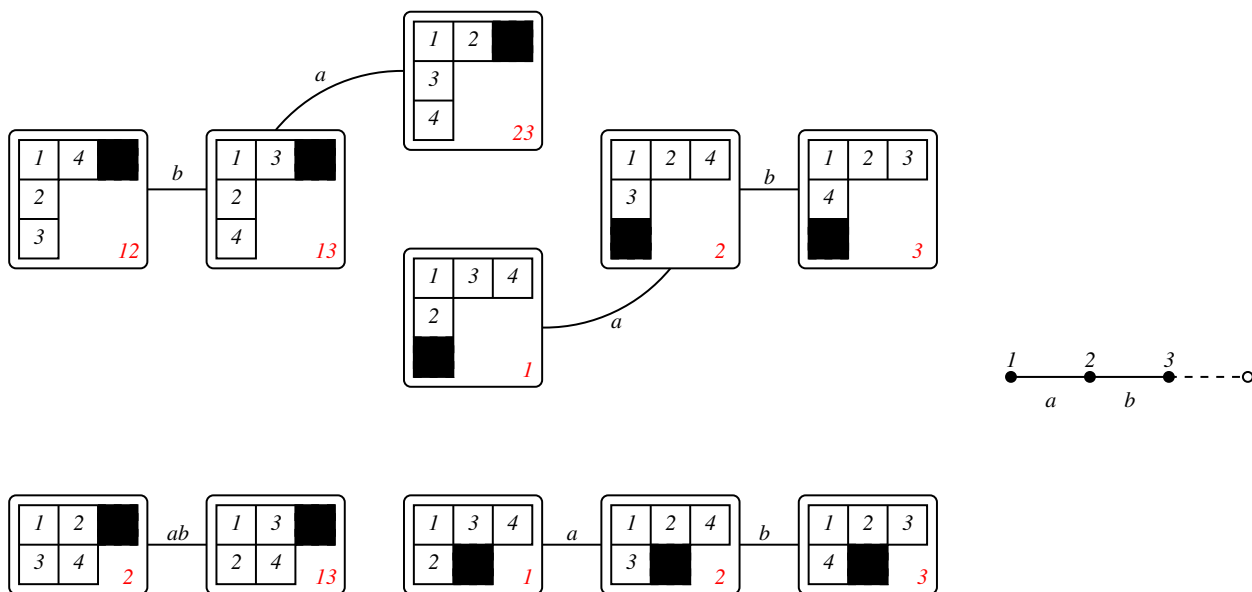
Figure 4.4: Examples of restriction for standard dual equivalence graphs.

so β was not redundant. We think of β as a piece of data about a dual equivalence graph, and keep it as part of the definition to be consistent with the original.

A *weak dual equivalence graph* is a signed colored graph satisfying conditions $1-5$ of the Definition 4.1.3.

4.1.3 Restriction

Suppose G is a signed colored graph of type $n+1$. For $0 \leqslant k < n+1$, a $(k+1)$-*restriction* of G consists of the same vertex set V, the τ function post-composed with intersection with $\{1, \ldots, k\}$, and the β function post-composed with restriction to $\{b_1, \ldots, b_{k-1}\}$. The $(k+1)$-restriction of G is a signed colored graph of type $k+1$. The property of being a (weak) dual equivalence graph is preserved by restriction. By a $(k+1)$-*component of G* we mean either the connected component of the restriction, or the induced subgraph of G on vertices corresponding to such connected component. It should be clear from the context which of these we are talking about.

The n-components of G_λ are obtained by fixing the position of $n+1$ in the tableau. Such a component is isomorphic to G_μ, where μ is formed from λ by erasing the outer corner which contained $n+1$. On the above examples this looks as shown in Figure 4.4.

The condition of being a weak dual equivalence graph is already quite powerful. The following lemma is relevant to us. It essentially says that a weak dual equivalence graph

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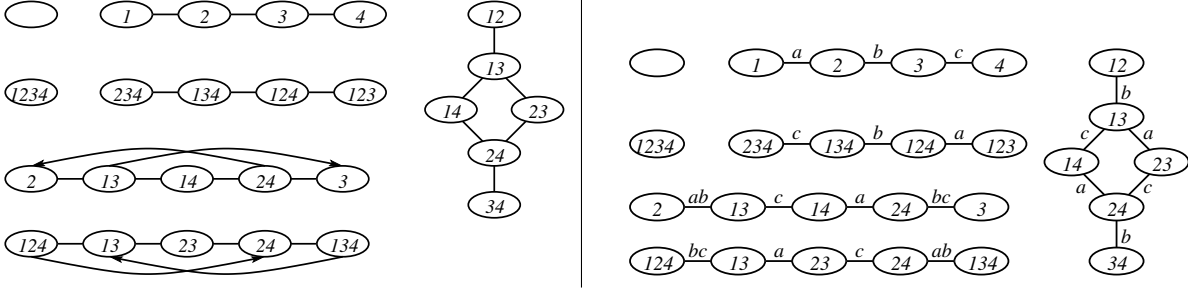
Figure 4.5: Molecules and dual equivalence graphs for type A_4.

with a nice restriction property is necessarily a cover of a dual equivalence graph.

Lemma 4.1.6. *Suppose G is a weak dual equivalence graph of type $n+1$. Suppose moreover that each n-component is a dual equivalence graph. Then there is a surjective morphism $\varphi : G \to G_\lambda$ for some partition λ of $n+1$, which restricts to an isomorphism on the n-components.*

Let $C \cong G_\mu$ be an n-component. Then for any partition $\nu \neq \mu$ of n with $\nu \subset \lambda$, there exists a unique n-component D with $\varphi(D) = G_\nu$ which is connected to C by an E_{n-1} edge. Also, two n-components which are isomorphic to G_μ are not connected by an E_{n-1} edge.

Proof. The references are again to [Ass08]. The existence of the morphism is shown in Theorem 3.14. Its surjectivity follows by Remark 3.8. The fact that it restricts to an isomorphism on the n-components follows from the proof of Theorem 3.14. The covering properties from the second paragraph are shown in Corollary 3.15, though the last one is not explicitly mentioned. □

4.1.4 Molecules and dual equivalence graphs

In this section we make more precise the relationship between molecules and dual equivalence graphs which may be guessed from Figure 4.5.

Proposition 4.1.7. *The simple part of an A_n-molecule, with the corresponding τ function and edges labeled by activated bonds, is a weak dual equivalence graph.*

Proof. Axioms (1), (2), (3) follow directly from SR, BR, and CR. Axiom (4) was demonstrated in Example 1.1.4 and Proposition 1.1.8. Axiom (5) is a weaker version of the Local Polygon Rule. □

Consider the graph G_λ. It is clear that (viewed as a directed graph with edge weights of 1) it is an admissible S-labeled graph for the A_n root system. It is well known that it forms the simple part of an A_n-molecule (the left Kazhdan-Lusztig cell) which we call $\overline{G_\lambda}$.

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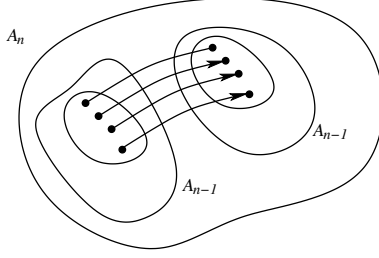
Figure 4.6: A schematic representation of a "cabling" of edges.

Definition 4.1.8. An A_n-molecule is *Kazhdan-Lusztig* if it is isomorphic to $\overline{G_\lambda}$, i.e. if its simple part is a dual equivalence graph.

Remark 4.1.9. We can explicitly describe the simple edges of any parabolic restriction of $\overline{G_\lambda}$. Let $J = \{j_1, \ldots, j_k\}$. Then the relevant tableau entries are $J' := \{j_1, j_1 + 1, j_2, j_2 + 1, \ldots, j_k, j_k + 1\}$. The simple edges of the W_J-restriction of G_λ are dual Knuth moves that exchange two entries of J' provided the "witness" between them is also in J'.

4.2   Classification of admissible A_n-molecules

In this section we show that any A_n-molecule is Kazhdan-Lusztig. The proof will proceed by induction on n, so the preliminary results will start with an A_n-molecule whose A_{n-1}-submolecules are Kazhdan-Lusztig.

The first of these results states that if two such A_{n-1}-submolecules are connected by a simple edge, then the connected A_{n-2}-submolecules are isomorphic and there is a "cabling" of edges (possibly arcs) of weight 1 between them (see Figure 4.6).

Lemma 4.2.1. *Let M be an A_n-molecule whose A_{n-1}-submolecules are Kazhdan-Lusztig. Suppose A and B are two such submolecules which are joined by a simple edge (in M), namely there exist $x \in A$, $y \in B$ such that the edge (x, y) is simple. Let A' (resp. B') be the A_{n-2}-submolecule of M containing x (resp. y). Then there is an isomorphism ψ between A' and B' such that $\psi(x) = y$. Moreover, if $n \in \tau(x)$ then $m(z, \psi(z)) = 1$ for all $z \in A'$.*

Proof. By Lemma 4.1.6 we know that there is a surjective morphism $\varphi : M \to \overline{G_\lambda}$ for some λ. Then $\varphi(A) \cong \overline{G_\mu}$ and $\varphi(B) \cong \overline{G_\nu}$, for some μ, ν which are formed from λ by erasing an outer corner (these outer corners must be different since no two molecules corresponding to the same shape may be connected; Lemma 4.1.6).

Let $T = \varphi(x), U = \varphi(y)$. Thus T has $n + 1$ in position $\lambda \setminus \mu$ and U has $n + 1$ in position $\lambda \setminus \nu$. Now there is a simple edge between T and U, i.e. one is obtained from the other

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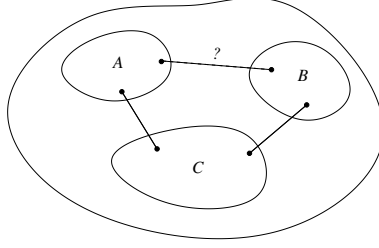
Figure 4.7: An illustration for the statement of Lemma 4.2.2.
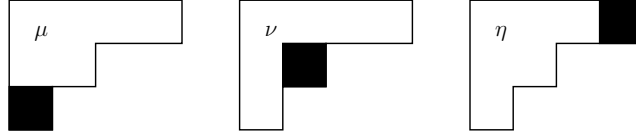
Figure 4.8: Partitions μ, ν and η from Lemma 4.2.2.

by a Knuth move. The only Knuth move in A_n which moves $n+1$ is one that exchanges n and $n+1$, in the presence of $n-1$ between them. Hence T has n in position $\lambda \setminus \nu$ and U has n in position $\lambda \setminus \mu$. Hence the A_{n-2}-molecule containing T has standard tableaux on $\lambda \setminus (\mu \cup \nu)$ as vertices, and Knuth moves between them as edges. The same is true for the A_{n-2} molecule containing U. Thus the two molecules are isomorphic and the isomorphism is given by switching n and $n+1$. Now we can use φ to lift it up to an isomorphism $\psi : A' \to B'$.

Suppose $n \in \tau(x)$. Then $n \notin \tau(y)$. So for any $x' \in A'$, we have $n \in \tau(x')$, and similarly for any $y' \in B'$, $n \notin \tau(y')$. Then repeated application of AT1 shows that the weight of the edge between $z \in A'$ and $\psi(z)$ is the same as between x and y, namely 1. $\qquad \square$

The second preliminary result shows that if, out of three A_{n-1}-submolecules, two pairs (satisfying some conditions) are connected by simple edges, then the third pair is also connected by a simple edge (see Figure 4.7). The conditions will later be removed to show that any two A_{n-1}-submolecules of an A_n-molecule are connected by a simple edge.

Lemma 4.2.2. *Let M be an A_n-molecule whose A_{n-1}-submolecules are Kazhdan-Lusztig. By Lemma 4.1.6, there is a surjective morphism $\varphi : M \to \overline{G_\lambda}$ for some partition λ of $n+1$. Let A, B, C be A_{n-1}-submolecules of M such that A and B are both connected to C by simple edges. Then $A \cong \overline{G_\mu}$, $B \cong \overline{G_\nu}$, $C \cong \overline{G_\eta}$, for some partitions formed by deleting outer corners of λ. The three partitions have to be different by Lemma 4.1.6. Suppose moreover that the deleted corner for η was the highest of the three (see Figure 4.8).*

Then A and B are connected by a simple edge.

Proof. Notice that the role of A and B is symmetric, so without loss of generality we may assume that the deleted corner for μ was the lowest of the three.
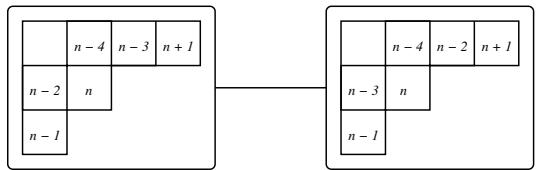
Figure 4.9: A simple edge in $\overline{G_\lambda}$.

To prove the existence of an edge between A and B, we will choose a simple edge of C and show, using arc transport rules, that its weight is equal to the weight of an edge between a vertex of A and a vertex of B whose τ invariants are incomparable. This will show that the edge in question is simple.

Consider a simple edge in $\overline{G_\lambda}$ (which happens to lie in the submolecule isomorphic to $\overline{G_\eta}$) of the form shown in Figure 4.9.

Let us describe precisely the kind of tableau we are looking for on the left. We want $n+1$ to occupy the cell $\lambda \setminus \eta$, n to occupy the cell $\lambda \setminus \nu$, and $n-1$ to occupy the cell $\lambda \setminus \mu$. There exists an outer corner of $\mu \cap \nu \cap \eta$ which now lies on a diagonal between n and $n+1$; place $n-3$ there. Place $n-2$ in the outer corner of $\mu \cap \nu \cap \eta$ between $n-1$ and n. Similarly, place $n-4$ between $n-2$ and $n-3$. Fill in the rest of the tableau in an arbitrary way. The two resulting tableaux differ by a Knuth move: one may flip $n-2$ and $n-3$ since $n-4$ is between them. So this is indeed a simple edge in $\overline{G_\lambda}$.

Now look at the A_4-molecules involved after restricting to the *rightmost* copy of A_4 in A_n. The restriction corresponds to allowing Knuth moves that exchange entries $\geqslant n-3$ provided the "witness" between them is also $\geqslant n-3$ (in particular, the original simple edge will become directed in the restriction). These are shown in Figure 4.10. The weight of the left blue (directed) edge is 1 since it was a simple edge before the restriction. It is equal to the weight of the right blue (dashed) edge by AT3.

In the original $\overline{G_\lambda}$, before restriction, we may then use the cabling of Lemma 4.2.1, to further transport this edge weight as in Figure 4.11. Thus we have shown that the weight of the right blue edge in this figure is 1.

In $\overline{G_\lambda}$ this is not very interesting since the two tableaux are seen to be related by a Knuth move; let us lift our sequence of moves up to M. Our original simple edge was located in the submolecule isomorphic to $\overline{G_\eta}$. The preimage under φ of that simple edge lies in C. Now consider the preimages of the two A_4-molecules. The preimage of the right end of the molecule on the top will lie in A since it is the only molecule isomorphic to $\overline{G_\mu}$ which is connected to C by a simple edge (Lemma 4.1.6). Similarly, the preimage of the right end of the molecule on the bottom is in B. So there is an edge of weight 1 from A to B.
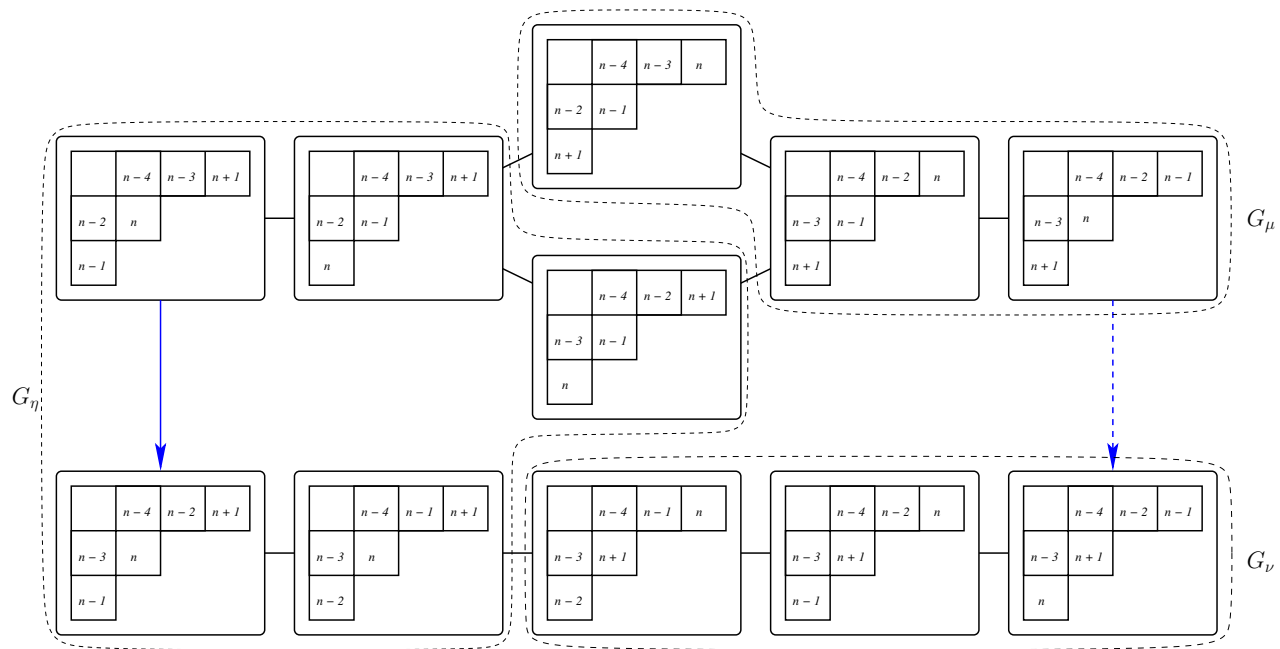
Figure 4.10: Transport along A_4 molecules in $\overline{G_\lambda}$.
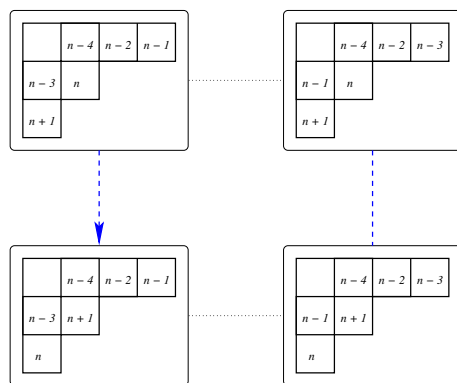
Figure 4.11: Transport along a cabling in $\overline{G_\lambda}$.

The transport along a cabling does not change the A_{n-1}-molecules involved, however the τ-invariants of the right blue edge in Figure 4.11 are manifestly incomparable (one has $n-1$ while the other has n). This produces a simple edge between A and B. $\square$

We can now finish the proof of the theorem.

Theorem 4.2.3. *Any A_n-molecule is Kazhdan-Lusztig.*

Proof. We know that the simple part of an A_n-molecule is a weak dual equivalence graph. It remains to show that it satisfies the axiom (6), namely that any two A_{n-1}-submolecules are connected by a simple edge.

Proceed by induction on n, the case $n = 1$ being trivial. Let M be an A_n-molecule. By inductive assumption, all A_{n-1}-molecules are Kazhdan-Lusztig. So, according to Lemma 4.1.6 there is a covering $M \to \overline{G_\lambda}$, for some partition λ of $n+1$.

Choose two of these A_{n-1}-submolecules of M, A and Z. Choose a path of simple edges between them which goes through the fewest number of submolecules. If it does not go through other submolecules, then we are done. Suppose that is not so. Let A, B, C be the first three submolecules on the path (it may happen that $Z = C$). The partitions μ, ν, η corresponding to A, B, and C are formed by removing outer corners of λ; they are all distinct by Lemma 4.1.6.

Consider the following string of submolecules connected by simple edges: $A-B-C-A'-B'$, with $A \cong A'$, $B \cong B'$, and some of these possibly equalities (this is possible by Lemma 4.1.6). Out of μ, ν, and η choose the partition which is formed by removing the highest box of λ. In the above string, choose a copy of the corresponding submolecule with submolecules attached on both sides (for example, if $\lambda \setminus \mu$ was highest of the three, then we should choose A'). Then the triple consisting of this submolecule and the two adjacent ones satisfies the condition of the Lemma 4.2.2 (in the example, it would be the triple $C-A'-B'$). Applying the lemma we get that $A' = A$, and $B' = B$. But then A is connected to C, contradicting our assumption that the path went through a minimal number of submolecules.

So any two A_{n-1}-submolecules are connected by an edge, finishing the proof. $\square$

Remark 4.2.4. In [Rob13], Roberts gives a revised version of Assaf's axiom 6 which is more suitable for computer calculations. Proving our theorem using this alternate axiomatization amounts to checking that all the A_5-molecules are Kazhdan-Lusztig. This provides a simple computerized proof of our result.

4.3 Cycles in the binding graph

In the introduction to this section we outlined a possible strategy for proving Conjecture 4.0.9. The second stage of that strategy is to show that no A_n-cell may contain multiple molecules. One way to achieve this would be to show that the binding graph (see section 1.1.6) is acyclic. As stated, this is known to be false. McLarnan and Warrington ([MW03]) have shown that the Kazhdan-Lusztig cell corresponding to the partition 5533 has an arc whose weight is bigger than 1. Since the Local Polygon Rule follows from arc transport, the self binding space of this molecule should be nontrivial. However the self-binding space of a molecule is an affine translate of the pairwise binding space of two copies of the molecule with the same parity. Hence the binding graph must have loops. A computation of Stembridge also found molecules where the pairwise binding space between a molecule and its copy with opposite parity is nonzero (for example, the molecule corresponding to the shape 4422).

Remark 4.3.1. The following simple observation about limitations of Kazhdan-Lusztig transport (and hence the Local Polygon Rule) will allow us to find more general cycles in the binding graph.

Suppose (W, S) is a Coxeter system with at most double bonds. For a subset J of S let grow(J) be the set $J \cup \{s \in S | s \sim t$ for some $t \in J\}$. Suppose $u \to v$ is a potential arc in an sBCS graph G and $\tau(u) \supseteq$ grow$(\tau(v))$. Then no instance of Kazhdan-Lusztig transport relates the weight of this arc to the weight of any other arc.

Example 4.3.2. We will look for a 2-cycle $G_1 \leftrightarrow G_2$ in the binding graph of type A, where G_2 is a hook-shape molecule and G_1 is a molecule of another shape (for us it will be a square). All we need for a possible cycle (as we shall see a little later) is a pair (J, L) of subsets of S which appear as τ-invariants in G_1 with the property that:

- There exist subsets K_1, K_2 with $|K_1| = |K_2|$, grow$(J) \subseteq K_1$ and grow$(K_2) \subseteq L$.

This can, for example, be found if the shape of G_1 is a 6x6 square. The τ-invariant of the row-superstandard tableau (shown in Figure 4.12 (a)) is $J = \{6, 12, 18, 24, 30\}$. The τ-invariant of the column-superstandard tableau (shown in Figure 4.12 (b)) is $L = [35] \setminus J$. Let

$$K_1 = \{5, 6, 7, 11, 12, 13, 17, 18, 19, 23, 24, 25, 29, 30, 31\}.$$

So $|K_1| = 15$. The largest set $\widetilde{K_2}$ such that grow$(\widetilde{K_2}) = L$ has size 20. Let K_2 be any 15-element subset of this set.

Now let us return to the concrete task. We take G_1 to be the molecule whose shape is a 6x6 square, and G_2 is a molecule whose shape is the hook $(21, 1^{15})$. Hence the τ-invariants

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

(a)

1	7	13	19	25	31
2	8	14	20	26	32
3	9	15	21	27	33
4	10	16	22	28	34
5	11	17	23	29	35
6	12	18	24	30	36

(b)

Figure 4.12: Superstandard tableaux.

of G_2 are all the 15-element subsets of S. Fix the parities of G_1 and G_2 so that the row-superstandard tableau in G_1 and the vertex with τ-invariant K_1 in G_2 have opposite parities. The freedom in the choice of K_2 allows us to pick K_2 so that the column-superstandard tableau in G_1 and the vertex with τ-invariant K_2 in G_2 have opposite parities. This gives a 2-cycle in the binding graph since by Remark 4.3.1 the dimension of the binding space between G_1 and G_2 with the given parities is at least 1 in either direction.

Example 4.3.3. A computation done by Stembridge found that the earliest 2-cycles obtained by means of Remark 4.3.1 occur in A_{13}:

$$(5,3,3,3) \leftrightarrow (4,4,3,3), \qquad (4,4,4,2) \leftrightarrow (4,4,4,1,1).$$

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APPENDIX A

Quasi-minuscule W-graphs for type E

We present the result of the calculation of the W-graphs of the quasi-minuscule quotients of types E_6, E_7, and E_8 below. The first column of each table gives the number of each vertex (in no particular order). The second column gives the coordinates of the corresponding root in terms of the simple roots. The third column gives the τ-invariant of the corresponding root, and the fourth lists all the vertices reached (possibly) from the given one.

The entries in black correspond to simple edges, while the entries in blue correspond to arcs of weight 1. Entries in red correspond to potential arcs whose weight was not determined by Kazhdan-Lusztig transport. All the weights of such arcs can be fixed by imposing the (full) Polygon Rule.

Table A.1: The W-graph of the quasi-minuscule quotient of E_6.

Vertex	Simple root coordinates	τ-invariant	Neighbors
1	$(1,2,2,3,2,1)$	$\{\}$	
2	$(1,1,2,3,2,1)$	$\{2\}$	3
3	$(1,1,2,2,2,1)$	$\{4\}$	$2,4,5$
4	$(1,1,2,2,1,1)$	$\{5\}$	$3,7$
5	$(1,1,1,2,2,1)$	$\{3\}$	$3,6$
6	$(0,1,1,2,2,1)$	$\{1\}$	5
7	$(1,1,2,2,1,0)$	$\{6\}$	4
8	$(1,1,1,2,1,1)$	$\{3,5\}$	$9,10,11,\textcolor{blue}{4,5,18,20}$
9	$(0,1,1,2,1,1)$	$\{1,5\}$	$8,12,13,\textcolor{blue}{6,35}$
10	$(1,1,1,2,1,0)$	$\{3,6\}$	$8,12,15,\textcolor{blue}{7,36}$
11	$(1,1,1,1,1,1)$	$\{4\}$	$8,14$
12	$(0,1,1,2,1,0)$	$\{1,6\}$	$9,10,\textcolor{blue}{41,42}$
13	$(0,1,1,1,1,1)$	$\{1,4\}$	$9,17,18,\textcolor{blue}{11,29}$
14	$(1,0,1,1,1,1)$	$\{2\}$	11
15	$(1,1,1,1,1,0)$	$\{4,6\}$	$10,19,20,\textcolor{blue}{11,30}$
16	$(0,1,1,1,1,0)$	$\{1,4,6\}$	$21,22,23,\textcolor{blue}{12,13,15,31,32,34}$
17	$(0,0,1,1,1,1)$	$\{1,2\}$	$13,24,\textcolor{blue}{14}$

(Continued on the next page.)

The W-graph of the quasi-minuscule quotient of E_6 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
18	$(0, 1, 0, 1, 1, 1)$	$\{3\}$	13
19	$(1, 0, 1, 1, 1, 0)$	$\{2, 6\}$	$15, 25, 14$
20	$(1, 1, 1, 1, 0, 0)$	$\{5\}$	15
21	$(0, 0, 1, 1, 1, 0)$	$\{1, 2, 6\}$	$16, 26, 28, 17, 19, 37, 61$
22	$(0, 1, 0, 1, 1, 0)$	$\{3, 6\}$	$16, 27, 18, 38$
23	$(0, 1, 1, 1, 0, 0)$	$\{1, 5\}$	$16, 27, 20, 40$
24	$(0, 0, 0, 1, 1, 1)$	$\{2, 3\}$	$17, 29, 18$
25	$(1, 0, 1, 1, 0, 0)$	$\{2, 5\}$	$19, 30, 20$
26	$(0, 0, 1, 1, 0, 0)$	$\{1, 2, 5\}$	$21, 32, 33, 23, 25, 56, 57$
27	$(0, 1, 0, 1, 0, 0)$	$\{3, 5\}$	$22, 23, 31$
28	$(0, 0, 0, 1, 1, 0)$	$\{2, 3, 6\}$	$21, 33, 34, 22, 24, 54, 55$
29	$(0, 0, 0, 0, 1, 1)$	$\{4\}$	$24, 35$
30	$(1, 0, 1, 0, 0, 0)$	$\{4\}$	$25, 36$
31	$(0, 1, 0, 0, 0, 0)$	$\{4\}$	$27, 37$
32	$(0, 0, 1, 0, 0, 0)$	$\{1, 4\}$	$26, 38, 30, 39, 42$
33	$(0, 0, 0, 1, 0, 0)$	$\{2, 3, 5\}$	$26, 28, 39, 27, 37, 38, 40, 48, 49, 50$
34	$(0, 0, 0, 0, 1, 0)$	$\{4, 6\}$	$28, 40, 29, 39, 41$
35	$(0, 0, 0, 0, 0, 1)$	$\{5\}$	$29, 41$
36	$(1, 0, 0, 0, 0, 0)$	$\{3\}$	$30, 42$
37	$(0, -1, 0, 0, 0, 0)$	$\{2\}$	31
38	$(0, 0, -1, 0, 0, 0)$	$\{3\}$	32
39	$(0, 0, 0, -1, 0, 0)$	$\{4\}$	33
40	$(0, 0, 0, 0, -1, 0)$	$\{5\}$	34
41	$(0, 0, 0, 0, 0, -1)$	$\{6\}$	35
42	$(-1, 0, 0, 0, 0, 0)$	$\{1\}$	36
43	$(0, -1, 0, -1, 0, 0)$	$\{2, 4\}$	$48, 49, 37, 39, 58, 72$
44	$(0, 0, -1, -1, 0, 0)$	$\{3, 4\}$	$48, 50, 52, 38, 39, 58, 66$
45	$(0, 0, 0, -1, -1, 0)$	$\{4, 5\}$	$49, 50, 51, 39, 40, 58, 65$
46	$(0, 0, 0, 0, -1, -1)$	$\{5, 6\}$	$51, 40, 41, 65$
47	$(-1, 0, -1, 0, 0, 0)$	$\{1, 3\}$	$52, 38, 42, 66$
48	$(0, -1, -1, -1, 0, 0)$	$\{2, 3\}$	$43, 44, 56$
49	$(0, -1, 0, -1, -1, 0)$	$\{2, 5\}$	$43, 45, 54$
50	$(0, 0, -1, -1, -1, 0)$	$\{3, 5\}$	$44, 45, 55, 57$
51	$(0, 0, 0, -1, -1, -1)$	$\{4, 6\}$	$45, 46, 54, 55$
52	$(-1, 0, -1, -1, 0, 0)$	$\{1, 4\}$	$44, 47, 56, 57$
53	$(0, -1, -1, -1, -1, 0)$	$\{2, 3, 5\}$	$58, 59, 60, 48, 49, 50, 65, 66, 70, 72$
54	$(0, -1, 0, -1, -1, -1)$	$\{2, 6\}$	$49, 51$
55	$(0, 0, -1, -1, -1, -1)$	$\{3, 6\}$	$50, 51, 61$
56	$(-1, -1, -1, -1, 0, 0)$	$\{1, 2\}$	$48, 52$
57	$(-1, 0, -1, -1, -1, 0)$	$\{1, 5\}$	$50, 52, 61$
58	$(0, -1, -1, -2, -1, 0)$	$\{4\}$	53

The W-graph of the quasi-minuscule quotient of E_6 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
59	$(0, -1, -1, -1, -1, -1)$	$\{2, 3, 6\}$	$53, 62, 64, 54, 55, 68$
60	$(-1, -1, -1, -1, -1, 0)$	$\{1, 2, 5\}$	$53, 63, 64, 56, 57, 69$
61	$(-1, 0, -1, -1, -1, -1)$	$\{1, 6\}$	$55, 57$
62	$(0, -1, -1, -2, -1, -1)$	$\{4, 6\}$	$59, 65, 58$
63	$(-1, -1, -1, -2, -1, 0)$	$\{1, 4\}$	$60, 66, 58$
64	$(-1, -1, -1, -1, -1, -1)$	$\{1, 2, 6\}$	$59, 60, 67, 61, 72$
65	$(0, -1, -1, -2, -2, -1)$	$\{5\}$	62
66	$(-1, -1, -2, -2, -1, 0)$	$\{3\}$	63
67	$(-1, -1, -1, -2, -1, -1)$	$\{1, 4, 6\}$	$64, 68, 69, 62, 63, 71$
68	$(-1, -1, -2, -2, -1, -1)$	$\{3, 6\}$	$67, 70, 66$
69	$(-1, -1, -1, -2, -2, -1)$	$\{1, 5\}$	$67, 70, 65$
70	$(-1, -1, -2, -2, -2, -1)$	$\{3, 5\}$	$68, 69, 71$
71	$(-1, -1, -2, -3, -2, -1)$	$\{4\}$	$70, 72$
72	$(-1, -2, -2, -3, -2, -1)$	$\{2\}$	71

Table A.2: The W-graph of the quasi-minuscule quotient of E_7.

Vertex	Simple root coordinates	τ-invariant	Neighbors
1	$(2, 2, 3, 4, 3, 2, 1)$	$\{\}$	
2	$(1, 2, 3, 4, 3, 2, 1)$	$\{1\}$	3
3	$(1, 2, 2, 4, 3, 2, 1)$	$\{3\}$	$2, 4$
4	$(1, 2, 2, 3, 3, 2, 1)$	$\{4\}$	$3, 5, 6$
5	$(1, 1, 2, 3, 3, 2, 1)$	$\{2\}$	4
6	$(1, 2, 2, 3, 2, 2, 1)$	$\{5\}$	$4, 8$
7	$(1, 1, 2, 3, 2, 2, 1)$	$\{2, 5\}$	$9, 10, 5, 6, 17$
8	$(1, 2, 2, 3, 2, 1, 1)$	$\{6\}$	$6, 11$
9	$(1, 1, 2, 3, 2, 1, 1)$	$\{2, 6\}$	$7, 12, 13, 8, 30$
10	$(1, 1, 2, 2, 2, 2, 1)$	$\{4\}$	$7, 14$
11	$(1, 2, 2, 3, 2, 1, 0)$	$\{7\}$	8
12	$(1, 1, 2, 3, 2, 1, 0)$	$\{2, 7\}$	$9, 16, 11$
13	$(1, 1, 2, 2, 2, 1, 1)$	$\{4, 6\}$	$9, 16, 17, 18, 10, 25, 27$
14	$(1, 1, 1, 2, 2, 2, 1)$	$\{3\}$	$10, 15$
15	$(0, 1, 1, 2, 2, 2, 1)$	$\{1\}$	14
16	$(1, 1, 2, 2, 2, 1, 0)$	$\{4, 7\}$	$12, 13, 20, 21$
17	$(1, 1, 2, 2, 1, 1, 1)$	$\{5\}$	13
18	$(1, 1, 1, 2, 2, 1, 1)$	$\{3, 6\}$	$13, 19, 21, 22, 14$
19	$(0, 1, 1, 2, 2, 1, 1)$	$\{1, 6\}$	$18, 23, 24, 15, 62$
20	$(1, 1, 2, 2, 1, 1, 0)$	$\{5, 7\}$	$16, 25, 17$
21	$(1, 1, 1, 2, 2, 1, 0)$	$\{3, 7\}$	$16, 18, 23$
22	$(1, 1, 1, 2, 1, 1, 1)$	$\{3, 5\}$	$18, 24, 27, 17, 36$
23	$(0, 1, 1, 2, 2, 1, 0)$	$\{1, 7\}$	$19, 21, 69$

(Continued on the next page.)

The W-graph of the quasi-minuscule quotient of E_7 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
24	$(0,1,1,2,1,1,1)$	$\{1,5\}$	$19, 22, 29, 55$
25	$(1,1,2,2,1,0,0)$	$\{6\}$	20
26	$(1,1,1,2,1,1,0)$	$\{3,5,7\}$	$28, 31, 32, 20, 21, 22, 40, 44, 52$
27	$(1,1,1,1,1,1,1)$	$\{4\}$	$22, 30$
28	$(0,1,1,2,1,1,0)$	$\{1,5,7\}$	$26, 33, 34, 23, 24, 48, 61$
29	$(0,1,1,1,1,1,1)$	$\{1,4\}$	$24, 35, 36, 27, 49$
30	$(1,0,1,1,1,1,1)$	$\{2\}$	27
31	$(1,1,1,2,1,0,0)$	$\{3,6\}$	$26, 34, 38, 25, 63$
32	$(1,1,1,1,1,1,0)$	$\{4,7\}$	$26, 37, 38, 27, 57$
33	$(0,1,1,1,1,1,0)$	$\{1,4,7\}$	$28, 39, 40, 41, 29, 32, 54$
34	$(0,1,1,2,1,0,0)$	$\{1,6\}$	$28, 31, 68, 70$
35	$(0,0,1,1,1,1,1)$	$\{1,2\}$	$29, 42, 30$
36	$(0,1,0,1,1,1,1)$	$\{3\}$	29
37	$(1,0,1,1,1,1,0)$	$\{2,7\}$	$32, 43, 30, 64$
38	$(1,1,1,1,1,0,0)$	$\{4,6\}$	$31, 32, 43, 44, 56$
39	$(0,0,1,1,1,1,0)$	$\{1,2,7\}$	$33, 45, 47, 35, 37, 99$
40	$(0,1,0,1,1,1,0)$	$\{3,7\}$	$33, 46, 36$
41	$(0,1,1,1,1,0,0)$	$\{1,4,6\}$	$33, 45, 46, 48, 34, 38, 57, 58, 60$
42	$(0,0,0,1,1,1,1)$	$\{2,3\}$	$35, 49, 36$
43	$(1,0,1,1,1,0,0)$	$\{2,6\}$	$37, 38, 50$
44	$(1,1,1,1,0,0,0)$	$\{5\}$	38
45	$(0,0,1,1,1,0,0)$	$\{1,2,6\}$	$39, 41, 51, 53, 43, 64, 94$
46	$(0,1,0,1,1,0,0)$	$\{3,6\}$	$40, 41, 52, 65$
47	$(0,0,0,1,1,1,0)$	$\{2,3,7\}$	$39, 53, 54, 40, 42, 91, 92$
48	$(0,1,1,1,0,0,0)$	$\{1,5\}$	$41, 52, 44, 67$
49	$(0,0,0,0,1,1,1)$	$\{4\}$	$42, 55$
50	$(1,0,1,1,0,0,0)$	$\{2,5\}$	$43, 56, 44$
51	$(0,0,1,1,0,0,0)$	$\{1,2,5\}$	$45, 58, 59, 48, 50, 87, 88$
52	$(0,1,0,1,0,0,0)$	$\{3,5\}$	$46, 48, 57$
53	$(0,0,0,1,1,0,0)$	$\{2,3,6\}$	$45, 47, 59, 60, 46, 84, 85$
54	$(0,0,0,0,1,1,0)$	$\{4,7\}$	$47, 60, 61, 49$
55	$(0,0,0,0,0,1,1)$	$\{5\}$	$49, 62$
56	$(1,0,1,0,0,0,0)$	$\{4\}$	$50, 63$
57	$(0,1,0,0,0,0,0)$	$\{4\}$	$52, 64$
58	$(0,0,1,0,0,0,0)$	$\{1,4\}$	$51, 65, 56, 66, 70$
59	$(0,0,0,1,0,0,0)$	$\{2,3,5\}$	$51, 53, 66, 52, 64, 65, 67, 77, 78, 79$
60	$(0,0,0,0,1,0,0)$	$\{4,6\}$	$53, 54, 67, 66, 68$
61	$(0,0,0,0,0,1,0)$	$\{5,7\}$	$54, 68, 55, 67, 69$
62	$(0,0,0,0,0,0,1)$	$\{6\}$	$55, 69$
63	$(1,0,0,0,0,0,0)$	$\{3\}$	$56, 70$
64	$(0,-1,0,0,0,0,0)$	$\{2\}$	57

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Vertex	Simple root coordinates	τ-invariant	Neighbors
65	$(0,0,-1,0,0,0,0)$	$\{3\}$	58
66	$(0,0,0,-1,0,0,0)$	$\{4\}$	59
67	$(0,0,0,0,-1,0,0)$	$\{5\}$	60
68	$(0,0,0,0,0,-1,0)$	$\{6\}$	61
69	$(0,0,0,0,0,0,-1)$	$\{7\}$	62
70	$(-1,0,0,0,0,0,0)$	$\{1\}$	63
71	$(0,-1,0,-1,0,0,0)$	$\{2,4\}$	$77,78,64,66,89,116$
72	$(0,0,-1,-1,0,0,0)$	$\{3,4\}$	$77,79,82,65,66,89,102$
73	$(0,0,0,-1,-1,0,0)$	$\{4,5\}$	$78,79,80,66,67,89,100$
74	$(0,0,0,0,-1,-1,0)$	$\{5,6\}$	$80,81,67,68,100,109$
75	$(0,0,0,0,0,-1,-1)$	$\{6,7\}$	$81,68,69,109$
76	$(-1,0,-1,0,0,0,0)$	$\{1,3\}$	$82,65,70,102,126$
77	$(0,-1,-1,-1,0,0,0)$	$\{2,3\}$	$71,72,87$
78	$(0,-1,0,-1,-1,0,0)$	$\{2,5\}$	$71,73,84$
79	$(0,0,-1,-1,-1,0,0)$	$\{3,5\}$	$72,73,85,88$
80	$(0,0,0,-1,-1,-1,0)$	$\{4,6\}$	$73,74,84,85,86$
81	$(0,0,0,0,-1,-1,-1)$	$\{5,7\}$	$74,75,86$
82	$(-1,0,-1,-1,0,0,0)$	$\{1,4\}$	$72,76,87,88$
83	$(0,-1,-1,-1,-1,0,0)$	$\{2,3,5\}$	$89,90,93,77,78,79,100,102,110,116$
84	$(0,-1,0,-1,-1,-1,0)$	$\{2,6\}$	$78,80,91$
85	$(0,0,-1,-1,-1,-1,0)$	$\{3,6\}$	$79,80,92,94$
86	$(0,0,0,-1,-1,-1,-1)$	$\{4,7\}$	$80,81,91,92$
87	$(-1,-1,-1,-1,0,0,0)$	$\{1,2\}$	$77,82$
88	$(-1,0,-1,-1,-1,0,0)$	$\{1,5\}$	$79,82,94$
89	$(0,-1,-1,-2,-1,0,0)$	$\{4\}$	83
90	$(0,-1,-1,-1,-1,-1,0)$	$\{2,3,6\}$	$83,95,96,98,84,85,106$
91	$(0,-1,0,-1,-1,-1,-1)$	$\{2,7\}$	$84,86$
92	$(0,0,-1,-1,-1,-1,-1)$	$\{3,7\}$	$85,86,99$
93	$(-1,-1,-1,-1,-1,0,0)$	$\{1,2,5\}$	$83,97,98,87,88,107,123$
94	$(-1,0,-1,-1,-1,-1,0)$	$\{1,6\}$	$85,88,99$
95	$(0,-1,-1,-2,-1,-1,0)$	$\{4,6\}$	$90,100,101,89,109$
96	$(0,-1,-1,-1,-1,-1,-1)$	$\{2,3,7\}$	$90,101,104,91,92,111$
97	$(-1,-1,-1,-2,-1,0,0)$	$\{1,4\}$	$93,102,89,124,126$
98	$(-1,-1,-1,-1,-1,-1,0)$	$\{1,2,6\}$	$90,93,103,104,94,116,121$
99	$(-1,0,-1,-1,-1,-1,-1)$	$\{1,7\}$	$92,94$
100	$(0,-1,-1,-2,-2,-1,0)$	$\{5\}$	95
101	$(0,-1,-1,-2,-1,-1,-1)$	$\{4,7\}$	$95,96,105$
102	$(-1,-1,-2,-2,-1,0,0)$	$\{3\}$	97
103	$(-1,-1,-1,-2,-1,-1,0)$	$\{1,4,6\}$	$98,106,107,108,95,97,113,115,120$
104	$(-1,-1,-1,-1,-1,-1,-1)$	$\{1,2,7\}$	$96,98,108,99,119$
105	$(0,-1,-1,-2,-2,-1,-1)$	$\{5,7\}$	$101,109,100$

The W-graph of the quasi-minuscule quotient of E_7 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
106	$(-1,-1,-2,-2,-1,-1,0)$	$\{3,6\}$	$103, 110, 111, 102$
107	$(-1,-1,-1,-2,-2,-1,0)$	$\{1,5\}$	$103, 110, 100, 122$
108	$(-1,-1,-1,-2,-1,-1,-1)$	$\{1,4,7\}$	$103, 104, 111, 112, 101, 118$
109	$(0,-1,-1,-2,-2,-2,-1)$	$\{6\}$	105
110	$(-1,-1,-2,-2,-2,-1,0)$	$\{3,5\}$	$106, 107, 113$
111	$(-1,-1,-2,-2,-1,-1,-1)$	$\{3,7\}$	$106, 108$
112	$(-1,-1,-1,-2,-2,-1,-1)$	$\{1,5,7\}$	$108, 114, 115, 105, 107$
113	$(-1,-1,-2,-3,-2,-1,0)$	$\{4\}$	$110, 116$
114	$(-1,-1,-2,-2,-2,-1,-1)$	$\{3,5,7\}$	$112, 117, 118, 110, 111, 122$
115	$(-1,-1,-1,-2,-2,-2,-1)$	$\{1,6\}$	$112, 117, 109, 126$
116	$(-1,-2,-2,-3,-2,-1,0)$	$\{2\}$	113
117	$(-1,-1,-2,-2,-2,-2,-1)$	$\{3,6\}$	$114, 115, 120, 125$
118	$(-1,-1,-2,-3,-2,-1,-1)$	$\{4,7\}$	$114, 119, 120, 113$
119	$(-1,-2,-2,-3,-2,-1,-1)$	$\{2,7\}$	$118, 121, 116$
120	$(-1,-1,-2,-3,-2,-2,-1)$	$\{4,6\}$	$117, 118, 121, 122, 124$
121	$(-1,-2,-2,-3,-2,-2,-1)$	$\{2,6\}$	$119, 120, 123$
122	$(-1,-1,-2,-3,-3,-2,-1)$	$\{5\}$	120
123	$(-1,-2,-2,-3,-3,-2,-1)$	$\{2,5\}$	$121, 124, 122$
124	$(-1,-2,-2,-4,-3,-2,-1)$	$\{4\}$	$123, 125$
125	$(-1,-2,-3,-4,-3,-2,-1)$	$\{3\}$	$124, 126$
126	$(-2,-2,-3,-4,-3,-2,-1)$	$\{1\}$	125

Table A.3: The W-graph of the quasi-minuscule quotient of E_8.

Vertex	Simple root coordinates	τ-invariant	Neighbors
1	$(2,3,4,6,5,4,3,2)$	$\{\}$	
2	$(2,3,4,6,5,4,3,1)$	$\{8\}$	3
3	$(2,3,4,6,5,4,2,1)$	$\{7\}$	$2,4$
4	$(2,3,4,6,5,3,2,1)$	$\{6\}$	$3,5$
5	$(2,3,4,6,4,3,2,1)$	$\{5\}$	$4,6$
6	$(2,3,4,5,4,3,2,1)$	$\{4\}$	$5,7,8$
7	$(2,2,4,5,4,3,2,1)$	$\{2\}$	6
8	$(2,3,3,5,4,3,2,1)$	$\{3\}$	$6,10$
9	$(2,2,3,5,4,3,2,1)$	$\{2,3\}$	$11, 12, 7, 8, 16$
10	$(1,3,3,5,4,3,2,1)$	$\{1\}$	8
11	$(2,2,3,4,4,3,2,1)$	$\{4\}$	$9,13$
12	$(1,2,3,5,4,3,2,1)$	$\{1,2\}$	$9, 14, 10, 26$
13	$(2,2,3,4,3,3,2,1)$	$\{5\}$	$11,15$
14	$(1,2,3,4,4,3,2,1)$	$\{1,4\}$	$12, 16, 17, 11, 24$
15	$(2,2,3,4,3,2,2,1)$	$\{6\}$	$13,18$
16	$(1,2,2,4,4,3,2,1)$	$\{3\}$	14

(Continued on the next page.)

The W-graph of the quasi-minuscule quotient of E_8 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
17	$(1, 2, 3, 4, 3, 3, 2, 1)$	$\{1, 5\}$	$14, 19, 20, 13$
18	$(2, 2, 3, 4, 3, 2, 1, 1)$	$\{7\}$	$15, 21$
19	$(1, 2, 2, 4, 3, 3, 2, 1)$	$\{3, 5\}$	$17, 22, 24, 16, 32$
20	$(1, 2, 3, 4, 3, 2, 2, 1)$	$\{1, 6\}$	$17, 22, 23, 15, 47$
21	$(2, 2, 3, 4, 3, 2, 1, 0)$	$\{8\}$	18
22	$(1, 2, 2, 4, 3, 2, 2, 1)$	$\{3, 6\}$	$19, 20, 27, 28, 45$
23	$(1, 2, 3, 4, 3, 2, 1, 1)$	$\{1, 7\}$	$20, 25, 27, 18$
24	$(1, 2, 2, 3, 3, 3, 2, 1)$	$\{4\}$	$19, 26$
25	$(1, 2, 3, 4, 3, 2, 1, 0)$	$\{1, 8\}$	$23, 29, 21$
26	$(1, 1, 2, 3, 3, 3, 2, 1)$	$\{2\}$	24
27	$(1, 2, 2, 4, 3, 2, 1, 1)$	$\{3, 7\}$	$22, 23, 29, 31$
28	$(1, 2, 2, 3, 3, 2, 2, 1)$	$\{4, 6\}$	$22, 30, 31, 32, 24, 40, 41$
29	$(1, 2, 2, 4, 3, 2, 1, 0)$	$\{3, 8\}$	$25, 27, 33$
30	$(1, 1, 2, 3, 3, 2, 2, 1)$	$\{2, 6\}$	$28, 34, 35, 26$
31	$(1, 2, 2, 3, 3, 2, 1, 1)$	$\{4, 7\}$	$27, 28, 33, 34, 36$
32	$(1, 2, 2, 3, 2, 2, 2, 1)$	$\{5\}$	28
33	$(1, 2, 2, 3, 3, 2, 1, 0)$	$\{4, 8\}$	$29, 31, 37, 38$
34	$(1, 1, 2, 3, 3, 2, 1, 1)$	$\{2, 7\}$	$30, 31, 37$
35	$(1, 1, 2, 3, 2, 2, 2, 1)$	$\{2, 5\}$	$30, 40, 32$
36	$(1, 2, 2, 3, 2, 2, 1, 1)$	$\{5, 7\}$	$31, 38, 41, 32, 50$
37	$(1, 1, 2, 3, 3, 2, 1, 0)$	$\{2, 8\}$	$33, 34$
38	$(1, 2, 2, 3, 2, 2, 1, 0)$	$\{5, 8\}$	$33, 36, 43$
39	$(1, 1, 2, 3, 2, 2, 1, 1)$	$\{2, 5, 7\}$	$42, 44, 46, 34, 35, 36, 54, 58, 67$
40	$(1, 1, 2, 2, 2, 2, 2, 1)$	$\{4\}$	$35, 45$
41	$(1, 2, 2, 3, 2, 1, 1, 1)$	$\{6\}$	36
42	$(1, 1, 2, 3, 2, 2, 1, 0)$	$\{2, 5, 8\}$	$39, 48, 49, 37, 38, 62, 105$
43	$(1, 2, 2, 3, 2, 1, 1, 0)$	$\{6, 8\}$	$38, 50, 41$
44	$(1, 1, 2, 3, 2, 1, 1, 1)$	$\{2, 6\}$	$39, 52, 41, 74, 77$
45	$(1, 1, 1, 2, 2, 2, 2, 1)$	$\{3\}$	$40, 47$
46	$(1, 1, 2, 2, 2, 2, 1, 1)$	$\{4, 7\}$	$39, 49, 51, 52, 40$
47	$(0, 1, 1, 2, 2, 2, 2, 1)$	$\{1\}$	45
48	$(1, 1, 2, 3, 2, 1, 1, 0)$	$\{2, 6, 8\}$	$42, 54, 55, 43, 44, 82, 97$
49	$(1, 1, 2, 2, 2, 2, 1, 0)$	$\{4, 8\}$	$42, 46, 56, 112$
50	$(1, 2, 2, 3, 2, 1, 0, 0)$	$\{7\}$	43
51	$(1, 1, 1, 2, 2, 2, 1, 1)$	$\{3, 7\}$	$46, 53, 56, 57, 45$
52	$(1, 1, 2, 2, 2, 1, 1, 1)$	$\{4, 6\}$	$44, 46, 57, 58, 70$
53	$(0, 1, 1, 2, 2, 2, 1, 1)$	$\{1, 7\}$	$51, 59, 60, 47, 119$
54	$(1, 1, 2, 3, 2, 1, 0, 0)$	$\{2, 7\}$	$48, 61, 50$
55	$(1, 1, 2, 2, 2, 1, 1, 0)$	$\{4, 6, 8\}$	$48, 61, 62, 63, 49, 52, 74, 76, 91$
56	$(1, 1, 1, 2, 2, 2, 1, 0)$	$\{3, 8\}$	$49, 51, 59, 120$
57	$(1, 1, 1, 2, 2, 1, 1, 1)$	$\{3, 6\}$	$51, 52, 60, 64$

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The W-graph of the quasi-minuscule quotient of E_8 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
58	$(1,1,2,2,1,1,1,1)$	$\{5\}$	52
59	$(0,1,1,2,2,2,1,0)$	$\{1,8\}$	$53, 56, 127, 128$
60	$(0,1,1,2,2,1,1,1)$	$\{1,6\}$	$53, 57, 66, 111$
61	$(1,1,2,2,2,1,0,0)$	$\{4,7\}$	$54, 55, 67, 68$
62	$(1,1,2,2,1,1,1,0)$	$\{5,8\}$	$55, 67, 58, 98$
63	$(1,1,1,2,2,1,1,0)$	$\{3,6,8\}$	$55, 65, 68, 69, 56, 57, 83$
64	$(1,1,1,2,1,1,1,1)$	$\{3,5\}$	$57, 66, 70, 58, 81$
65	$(0,1,1,2,2,1,1,0)$	$\{1,6,8\}$	$63, 71, 72, 59, 60, 86, 118$
66	$(0,1,1,2,1,1,1,1)$	$\{1,5\}$	$60, 64, 73, 104$
67	$(1,1,2,2,1,1,0,0)$	$\{5,7\}$	$61, 62, 74$
68	$(1,1,1,2,2,1,0,0)$	$\{3,7\}$	$61, 63, 71$
69	$(1,1,1,2,1,1,1,0)$	$\{3,5,8\}$	$63, 72, 75, 76, 62, 64, 88$
70	$(1,1,1,1,1,1,1,1)$	$\{4\}$	$64, 77$
71	$(0,1,1,2,2,1,0,0)$	$\{1,7\}$	$65, 68, 126$
72	$(0,1,1,2,1,1,1,0)$	$\{1,5,8\}$	$65, 69, 78, 79, 66, 110$
73	$(0,1,1,1,1,1,1,1)$	$\{1,4\}$	$66, 80, 81, 70, 96$
74	$(1,1,2,2,1,0,0,0)$	$\{6\}$	67
75	$(1,1,1,2,1,1,0,0)$	$\{3,5,7\}$	$69, 78, 83, 84, 67, 68, 93, 98, 107$
76	$(1,1,1,1,1,1,1,0)$	$\{4,8\}$	$69, 82, 84, 70$
77	$(1,0,1,1,1,1,1,1)$	$\{2\}$	70
78	$(0,1,1,2,1,1,0,0)$	$\{1,5,7\}$	$72, 75, 85, 86, 71, 102, 117$
79	$(0,1,1,1,1,1,1,0)$	$\{1,4,8\}$	$72, 85, 87, 88, 73, 76, 103$
80	$(0,0,1,1,1,1,1,1)$	$\{1,2\}$	$73, 89, 77$
81	$(0,1,0,1,1,1,1,1)$	$\{3\}$	73
82	$(1,0,1,1,1,1,1,0)$	$\{2,8\}$	$76, 90, 77$
83	$(1,1,1,2,1,0,0,0)$	$\{3,6\}$	$75, 86, 91, 74, 120$
84	$(1,1,1,1,1,1,0,0)$	$\{4,7\}$	$75, 76, 90, 91, 113$
85	$(0,1,1,1,1,1,0,0)$	$\{1,4,7\}$	$78, 79, 92, 93, 94, 84, 109$
86	$(0,1,1,2,1,0,0,0)$	$\{1,6\}$	$78, 83, 125, 128$
87	$(0,0,1,1,1,1,1,0)$	$\{1,2,8\}$	$79, 92, 95, 80, 82, 170$
88	$(0,1,0,1,1,1,1,0)$	$\{3,8\}$	$79, 93, 81$
89	$(0,0,0,1,1,1,1,1)$	$\{2,3\}$	$80, 96, 81$
90	$(1,0,1,1,1,1,0,0)$	$\{2,7\}$	$82, 84, 97, 121$
91	$(1,1,1,1,1,0,0,0)$	$\{4,6\}$	$83, 84, 97, 98, 112$
92	$(0,0,1,1,1,1,0,0)$	$\{1,2,7\}$	$85, 87, 99, 101, 90, 163$
93	$(0,1,0,1,1,1,0,0)$	$\{3,7\}$	$85, 88, 100$
94	$(0,1,1,1,1,0,0,0)$	$\{1,4,6\}$	$85, 99, 100, 102, 86, 91, 113, 114, 116$
95	$(0,0,0,1,1,1,1,0)$	$\{2,3,8\}$	$87, 101, 103, 88, 89, 159, 160$
96	$(0,0,0,0,1,1,1,1)$	$\{4\}$	$89, 104$
97	$(1,0,1,1,1,0,0,0)$	$\{2,6\}$	$90, 91, 105$
98	$(1,1,1,1,0,0,0,0)$	$\{5\}$	91

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Vertex	Simple root coordinates	τ-invariant	Neighbors
99	$(0,0,1,1,1,0,0,0)$	$\{1,2,6\}$	$92, 94, 106, 108, 97, 121, 156$
100	$(0,1,0,1,1,0,0,0)$	$\{3,6\}$	$93, 94, 107, 122$
101	$(0,0,0,1,1,1,0,0)$	$\{2,3,7\}$	$92, 95, 108, 109, 93, 152, 153$
102	$(0,1,1,1,0,0,0,0)$	$\{1,5\}$	$94, 107, 98, 124$
103	$(0,0,0,0,1,1,1,0)$	$\{4,8\}$	$95, 109, 110, 96$
104	$(0,0,0,0,0,1,1,1)$	$\{5\}$	$96, 111$
105	$(1,0,1,1,0,0,0,0)$	$\{2,5\}$	$97, 112, 98$
106	$(0,0,1,1,0,0,0,0)$	$\{1,2,5\}$	$99, 114, 115, 102, 105, 148, 149$
107	$(0,1,0,1,0,0,0,0)$	$\{3,5\}$	$100, 102, 113$
108	$(0,0,0,1,1,0,0,0)$	$\{2,3,6\}$	$99, 101, 115, 116, 100, 144, 145$
109	$(0,0,0,0,1,1,0,0)$	$\{4,7\}$	$101, 103, 116, 117$
110	$(0,0,0,0,0,1,1,0)$	$\{5,8\}$	$103, 117, 118, 104$
111	$(0,0,0,0,0,0,1,1)$	$\{6\}$	$104, 119$
112	$(1,0,1,0,0,0,0,0)$	$\{4\}$	$105, 120$
113	$(0,1,0,0,0,0,0,0)$	$\{4\}$	$107, 121$
114	$(0,0,1,0,0,0,0,0)$	$\{1,4\}$	$106, 122, 112, 123, 128$
115	$(0,0,0,1,0,0,0,0)$	$\{2,3,5\}$	$106, 108, 123, 107, 121, 122, 124, 136, 137, 138$
116	$(0,0,0,0,1,0,0,0)$	$\{4,6\}$	$108, 109, 124, 123, 125$
117	$(0,0,0,0,0,1,0,0)$	$\{5,7\}$	$109, 110, 125, 124, 126$
118	$(0,0,0,0,0,0,1,0)$	$\{6,8\}$	$110, 126, 111, 125, 127$
119	$(0,0,0,0,0,0,0,1)$	$\{7\}$	$111, 127$
120	$(1,0,0,0,0,0,0,0)$	$\{3\}$	$112, 128$
121	$(0,-1,0,0,0,0,0,0)$	$\{2\}$	113
122	$(0,0,-1,0,0,0,0,0)$	$\{3\}$	114
123	$(0,0,0,-1,0,0,0,0)$	$\{4\}$	115
124	$(0,0,0,0,-1,0,0,0)$	$\{5\}$	116
125	$(0,0,0,0,0,-1,0,0)$	$\{6\}$	117
126	$(0,0,0,0,0,0,-1,0)$	$\{7\}$	118
127	$(0,0,0,0,0,0,0,-1)$	$\{8\}$	119
128	$(-1,0,0,0,0,0,0,0)$	$\{1\}$	120
129	$(0,-1,0,-1,0,0,0,0)$	$\{2,4\}$	$136, 137, 121, 123, 150, 190$
130	$(0,0,-1,-1,0,0,0,0)$	$\{3,4\}$	$136, 138, 142, 122, 123, 150, 167$
131	$(0,0,0,-1,-1,0,0,0)$	$\{4,5\}$	$137, 138, 139, 123, 124, 150, 164$
132	$(0,0,0,0,-1,-1,0,0)$	$\{5,6\}$	$139, 140, 124, 125, 164, 177, 188$
133	$(0,0,0,0,0,-1,-1,0)$	$\{6,7\}$	$140, 141, 125, 126, 177, 189$
134	$(0,0,0,0,0,0,-1,-1)$	$\{7,8\}$	$141, 126, 127, 189, 240$
135	$(-1,0,-1,0,0,0,0,0)$	$\{1,3\}$	$142, 122, 128, 167, 217$
136	$(0,-1,-1,-1,0,0,0,0)$	$\{2,3\}$	$129, 130, 148$
137	$(0,-1,0,-1,-1,0,0,0)$	$\{2,5\}$	$129, 131, 144$
138	$(0,0,-1,-1,-1,0,0,0)$	$\{3,5\}$	$130, 131, 145, 149$
139	$(0,0,0,-1,-1,-1,0,0)$	$\{4,6\}$	$131, 132, 144, 145, 146$

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The W-graph of the quasi-minuscule quotient of E_8 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
140	$(0,0,0,0,-1,-1,-1,0)$	$\{5,7\}$	$132,133,146,147,\cancel{239}$
141	$(0,0,0,0,0,-1,-1,-1)$	$\{6,8\}$	$133,134,147$
142	$(-1,0,-1,-1,0,0,0,0)$	$\{1,4\}$	$130,135,148,149$
143	$(0,-1,-1,-1,-1,0,0,0)$	$\{2,3,5\}$	$150,151,155,136,137,138,164,167,179,190$
144	$(0,-1,0,-1,-1,-1,0,0)$	$\{2,6\}$	$137,139,152$
145	$(0,0,-1,-1,-1,-1,0,0)$	$\{3,6\}$	$138,139,153,156$
146	$(0,0,0,-1,-1,-1,-1,0)$	$\{4,7\}$	$139,140,152,153,154$
147	$(0,0,0,0,-1,-1,-1,-1)$	$\{5,8\}$	$140,141,154,\cancel{240}$
148	$(-1,-1,-1,-1,0,0,0,0)$	$\{1,2\}$	$136,142$
149	$(-1,0,-1,-1,-1,0,0,0)$	$\{1,5\}$	$138,142,156$
150	$(0,-1,-1,-2,-1,0,0,0)$	$\{4\}$	143
151	$(0,-1,-1,-1,-1,-1,0,0)$	$\{2,3,6\}$	$143,157,158,162,144,145,173$
152	$(0,-1,0,-1,-1,-1,-1,0)$	$\{2,7\}$	$144,146,159$
153	$(0,0,-1,-1,-1,-1,-1,0)$	$\{3,7\}$	$145,146,160,163$
154	$(0,0,0,-1,-1,-1,-1,-1)$	$\{4,8\}$	$146,147,159,160$
155	$(-1,-1,-1,-1,-1,0,0,0)$	$\{1,2,5\}$	$143,161,162,148,149,174,205$
156	$(-1,0,-1,-1,-1,-1,0,0)$	$\{1,6\}$	$145,149,163$
157	$(0,-1,-1,-2,-1,-1,0,0)$	$\{4,6\}$	$151,164,165,150,177$
158	$(0,-1,-1,-1,-1,-1,-1,0)$	$\{2,3,7\}$	$151,165,166,169,152,153,180$
159	$(0,-1,0,-1,-1,-1,-1,-1)$	$\{2,8\}$	$152,154$
160	$(0,0,-1,-1,-1,-1,-1,-1)$	$\{3,8\}$	$153,154,170$
161	$(-1,-1,-1,-2,-1,0,0,0)$	$\{1,4\}$	$155,167,150,209,217$
162	$(-1,-1,-1,-1,-1,-1,0,0)$	$\{1,2,6\}$	$151,155,168,169,156,190,200$
163	$(-1,0,-1,-1,-1,-1,-1,0)$	$\{1,7\}$	$153,156,170$
164	$(0,-1,-1,-2,-2,-1,0,0)$	$\{5\}$	157
165	$(0,-1,-1,-2,-1,-1,-1,0)$	$\{4,7\}$	$157,158,171,172$
166	$(0,-1,-1,-1,-1,-1,-1,-1)$	$\{2,3,8\}$	$158,172,176,159,160,187,235$
167	$(-1,-1,-2,-2,-1,0,0,0)$	$\{3\}$	161
168	$(-1,-1,-1,-2,-1,-1,0,0)$	$\{1,4,6\}$	$162,173,174,175,157,161,184,186,196$
169	$(-1,-1,-1,-1,-1,-1,-1,0)$	$\{1,2,7\}$	$158,162,175,176,163,195$
170	$(-1,0,-1,-1,-1,-1,-1,-1)$	$\{1,8\}$	$160,163$
171	$(0,-1,-1,-2,-2,-1,-1,0)$	$\{5,7\}$	$165,177,178,164,189$
172	$(0,-1,-1,-2,-1,-1,-1,-1)$	$\{4,8\}$	$165,166,178,236$
173	$(-1,-1,-2,-2,-1,-1,0,0)$	$\{3,6\}$	$168,179,180,167$
174	$(-1,-1,-1,-2,-2,-1,0,0)$	$\{1,5\}$	$168,179,164,201$
175	$(-1,-1,-1,-2,-1,-1,-1,0)$	$\{1,4,7\}$	$168,169,180,181,182,165,192$
176	$(-1,-1,-1,-1,-1,-1,-1,-1)$	$\{1,2,8\}$	$166,169,182,170,202,233$
177	$(0,-1,-1,-2,-2,-2,-1,0)$	$\{6\}$	171
178	$(0,-1,-1,-2,-2,-1,-1,-1)$	$\{5,8\}$	$171,172,183,237$
179	$(-1,-1,-2,-2,-2,-1,0,0)$	$\{3,5\}$	$173,174,184$

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The W-graph of the quasi-minuscule quotient of E_8 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
180	$(-1,-1,-2,-2,-1,-1,-1,0)$	$\{3,7\}$	173, 175, 187
181	$(-1,-1,-1,-2,-2,-1,-1,0)$	$\{1,5,7\}$	175, 185, 186, 188, 171, 174, 198
182	$(-1,-1,-1,-2,-1,-1,-1,-1)$	$\{1,4,8\}$	175, 176, 187, 188, 172, 199, 231
183	$(0,-1,-1,-2,-2,-2,-1,-1)$	$\{6,8\}$	178, 189, 177, 238, 240
184	$(-1,-1,-2,-3,-2,-1,0,0)$	$\{4\}$	179, 190
185	$(-1,-1,-2,-2,-2,-1,-1,0)$	$\{3,5,7\}$	181, 191, 192, 193, 179, 180, 201, 203, 212
186	$(-1,-1,-1,-2,-2,-2,-1,0)$	$\{1,6\}$	181, 191, 177, 217
187	$(-1,-1,-2,-2,-1,-1,-1,-1)$	$\{3,8\}$	180, 182, 234
188	$(-1,-1,-1,-2,-2,-1,-1,-1)$	$\{1,5,8\}$	181, 182, 193, 194, 178, 229
189	$(0,-1,-1,-2,-2,-2,-2,-1)$	$\{7\}$	183
190	$(-1,-2,-2,-3,-2,-1,0,0)$	$\{2\}$	184
191	$(-1,-1,-2,-2,-2,-2,-1,0)$	$\{3,6\}$	185, 186, 196, 213, 216
192	$(-1,-1,-2,-3,-2,-1,-1,0)$	$\{4,7\}$	185, 195, 196, 199, 184
193	$(-1,-1,-2,-2,-2,-1,-1,-1)$	$\{3,5,8\}$	185, 188, 197, 199, 187, 208, 228
194	$(-1,-1,-1,-2,-2,-2,-1,-1)$	$\{1,6,8\}$	188, 197, 198, 183, 186, 221, 227
195	$(-1,-2,-2,-3,-2,-1,-1,0)$	$\{2,7\}$	192, 200, 202, 190
196	$(-1,-1,-2,-3,-2,-2,-1,0)$	$\{4,6\}$	191, 192, 200, 201, 209
197	$(-1,-1,-2,-2,-2,-2,-1,-1)$	$\{3,6,8\}$	193, 194, 203, 204, 191, 219, 225
198	$(-1,-1,-1,-2,-2,-2,-2,-1)$	$\{1,7\}$	194, 203, 189
199	$(-1,-1,-2,-3,-2,-1,-1,-1)$	$\{4,8\}$	192, 193, 202, 230
200	$(-1,-2,-2,-3,-2,-2,-1,0)$	$\{2,6\}$	195, 196, 205
201	$(-1,-1,-2,-3,-3,-2,-1,0)$	$\{5\}$	196
202	$(-1,-2,-2,-3,-2,-1,-1,-1)$	$\{2,8\}$	195, 199, 232
203	$(-1,-1,-2,-2,-2,-2,-2,-1)$	$\{3,7\}$	197, 198, 207
204	$(-1,-1,-2,-3,-2,-2,-1,-1)$	$\{4,6,8\}$	197, 206, 207, 208, 196, 199, 215, 216, 223
205	$(-1,-2,-2,-3,-3,-2,-1,0)$	$\{2,5\}$	200, 209, 201
206	$(-1,-2,-2,-3,-2,-2,-1,-1)$	$\{2,6,8\}$	204, 210, 211, 200, 202, 218
207	$(-1,-1,-2,-3,-2,-2,-2,-1)$	$\{4,7\}$	203, 204, 210, 212
208	$(-1,-1,-2,-3,-3,-2,-1,-1)$	$\{5,8\}$	204, 212, 201, 226
209	$(-1,-2,-2,-4,-3,-2,-1,0)$	$\{4\}$	205, 213
210	$(-1,-2,-2,-3,-2,-2,-2,-1)$	$\{2,7\}$	206, 207
211	$(-1,-2,-2,-3,-3,-2,-1,-1)$	$\{2,5,8\}$	206, 214, 215, 205, 208
212	$(-1,-1,-2,-3,-3,-2,-2,-1)$	$\{5,7\}$	207, 208, 216
213	$(-1,-2,-3,-4,-3,-2,-1,0)$	$\{3\}$	209, 217
214	$(-1,-2,-2,-3,-3,-2,-2,-1)$	$\{2,5,7\}$	211, 218, 220, 210, 212, 226
215	$(-1,-2,-2,-4,-3,-2,-1,-1)$	$\{4,8\}$	211, 219, 220, 209
216	$(-1,-1,-2,-3,-3,-3,-2,-1)$	$\{6\}$	212
217	$(-2,-2,-3,-4,-3,-2,-1,0)$	$\{1\}$	213
218	$(-1,-2,-2,-3,-3,-3,-2,-1)$	$\{2,6\}$	214, 223, 216, 232
219	$(-1,-2,-3,-4,-3,-2,-1,-1)$	$\{3,8\}$	215, 221, 222, 213
220	$(-1,-2,-2,-4,-3,-2,-2,-1)$	$\{4,7\}$	214, 215, 222, 223

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The W-graph of the quasi-minuscule quotient of E_8 (ctd.)

Vertex	Simple root coordinates	τ-invariant	Neighbors
221	$(-2,-2,-3,-4,-3,-2,-1,-1)$	$\{1,8\}$	$219, 224, 217, 240$
222	$(-1,-2,-3,-4,-3,-2,-2,-1)$	$\{3,7\}$	$219, 220, 224, 225$
223	$(-1,-2,-2,-4,-3,-3,-2,-1)$	$\{4,6\}$	$218, 220, 225, 226, 230$
224	$(-2,-2,-3,-4,-3,-2,-2,-1)$	$\{1,7\}$	$221, 222, 227, 239$
225	$(-1,-2,-3,-4,-3,-3,-2,-1)$	$\{3,6\}$	$222, 223, 227, 228$
226	$(-1,-2,-2,-4,-4,-3,-2,-1)$	$\{5\}$	223
227	$(-2,-2,-3,-4,-3,-3,-2,-1)$	$\{1,6\}$	$224, 225, 229, 238$
228	$(-1,-2,-3,-4,-4,-3,-2,-1)$	$\{3,5\}$	$225, 229, 230, 226, 234$
229	$(-2,-2,-3,-4,-4,-3,-2,-1)$	$\{1,5\}$	$227, 228, 231, 237$
230	$(-1,-2,-3,-5,-4,-3,-2,-1)$	$\{4\}$	$228, 232$
231	$(-2,-2,-3,-5,-4,-3,-2,-1)$	$\{1,4\}$	$229, 233, 234, 230, 236$
232	$(-1,-3,-3,-5,-4,-3,-2,-1)$	$\{2\}$	230
233	$(-2,-3,-3,-5,-4,-3,-2,-1)$	$\{1,2\}$	$231, 235, 232$
234	$(-2,-2,-4,-5,-4,-3,-2,-1)$	$\{3\}$	231
235	$(-2,-3,-4,-5,-4,-3,-2,-1)$	$\{2,3\}$	$233, 236, 234$
236	$(-2,-3,-4,-6,-4,-3,-2,-1)$	$\{4\}$	$235, 237$
237	$(-2,-3,-4,-6,-5,-3,-2,-1)$	$\{5\}$	$236, 238$
238	$(-2,-3,-4,-6,-5,-4,-2,-1)$	$\{6\}$	$237, 239$
239	$(-2,-3,-4,-6,-5,-4,-3,-1)$	$\{7\}$	$238, 240$
240	$(-2,-3,-4,-6,-5,-4,-3,-2)$	$\{8\}$	239

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