On Existence and Properties of Rotating Star Solutions to the Euler-Poisson Equations

by

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To the Lord and my parents

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ABSTRACT

On Existence and Properties of Rotating Star Solutions to the Euler-Poisson Equations

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The Euler-Poisson equations are used in astrophysics to model rotating gaseous stars. Auchmuty and Beals in 1971 first found a family of rotating star solutions by solving a variational free boundary problem. A great many results followed to generalize the solutions to more diverse situations. Recent interests in extrasolar planet structures require extension of the picture to include a solid rocky core together with its gravitational potential. In this dissertation, we discuss various extensions of the classical rotating star results to incorporate a solid core. We also study the effect of a non-isentropic equation of state on the structure of the rotating star solutions.

CHAPTER I

Introduction

To seek a mathematical model for the internal structure of a star has always been a fascinating problem in astrophysics. Dating back to as early as Newton, there is a long history of investigation of the balancing shapes of rotating homogeneous incompressible fluid bodies. See Chandrasekhar [7], Kopal [24], Lamb [25], and Poincaré [37] for more details. Also see Chandrasekhar [8] for a historical account of equilibrium ellipsoids of homogeneous fluids. In comparison, the physical model of a star as self gravitating compressible gas in equilibrium had not capture significant attention until the late nineteenth century. Part of the reason for this delayed attention on the compressible nature of the model is that such a model requires a mature theory of thermodynamics to bring out a fruitful analysis, which was not really available at Newton's time. The first works on gaseous stellar structure models considered a spherically symmetric star in hydrostatic equilibrium. The well-known Lane-Emden equation is a key entity to summarize the properties of such a configuration. It allowed Lane [26] to calculate a theoretical estimation of the temperature of the sun. Chandrasekhar's classic [7] gives a detailed mathematical account of the analysis and implications of the Lane-Emden equation, and has long served as the starting point of many physical models on stellar structures. In contrast, a rigorous mathematical theory for rotating gaseous stars came much later. It turns out that in order to allow breaking of spherical symmetry, one needs to consider the system of partial differential equations known as the Euler-Poisson equations. This system is capable of describing the dynamical evolution of a star without any symmetry restriction. Auchmuty and Beals [3] initiated the search for rotating star solutions to the Euler-Poisson equations. As opposed to the spherically symmetric case where the solutions and their behaviors are more or less explicit, the rotating star solutions are provided by abstract existence theorems from the calculus of variations. Many works followed to provide existence theorems in diverse situations as well as to study the properties of the solutions whose existence are guaranteed by [3].

This dissertation is aiming to extend the work for rotating star solutions to the following new situations.

1. Studies in planet structures have revealed that a large number of giant planets are gaseous with a solid rocky core. Recent interests in the astrophysical community proposes the question of extending stellar structure models to include planets with solid cores. See Militzer et al. [34], Miller et al. [35], Burrows et al. [5], and Anderson and Adams [1]. I will prove numerous existence and non-existence results on a modified version of the Euler-Poisson equations. This allows one to characterize a generalized model for planet structures that include a solid rocky core.

2. Almost all of the previous works on Euler-Poisson equations assumed an isentropic equation of state for the gas.[1] Luo and Smoller [31] considered a non-isentropic equation of state for an ideal gas and proved some existence results

[1] Here isentropy means the pressure - density relation is given: $p = p(\rho)$, i.e. pressure is determined by density alone. We do not assume any particular form of this dependence. In the physics literature, this condition is more often called barotropy.

for simple prescribed entropies. I will prove several existence results related to the non-isentropic Euler-Poisson equations for more general entropy profiles.

The rest of this chapter will provide a background introduction to stellar structure models. In section 1.1, we derive and briefly study the Lane-Emden equation. In section 1.2, we describe the Euler-Poisson equations and give a precise formulation of the rotating star solutions. In section 1.3, we introduce the modified Euler-Poisson equations which describe a planet with a solid rocky core, and propose the problem of finding rotating planet solutions. In section 1.4, we introduce a non-isentropic equation of state, and present a preliminary study of their impact on the standard Euler-Poisson equations. In section 1.5, we give an overview of the entire thesis.

1.1 Stationary Stars and the Lane-Emden Equation

Let us consider a model for spherically symmetric stationary stars. Instead of building in the spherical symmetry from the very beginning, let us take a slightly more general approach. Throughout our presentation, integrals without limits or differentials mean volume integrals over space ($\mathbb{R}^3$). Let U be an arbitrary domain in the star. The gas in U is subject to two forces: the pressure from the surrounding gas on the boundary:

$$-\int_{\partial U} p\mathbf{n}\, dS, \tag{1.1}$$

and the gravitation of the star:

$$-\int_{U} \rho\nabla\phi\, dV. \tag{1.2}$$

Here $\mathbf{n}$ is the unit outward normal, p is pressure, ρ is gas density, and ϕ is the gravitational potential of the star. It is given by

$$-\phi(\mathbf{x}) = \rho * \frac{1}{|\mathbf{x}|} = \int \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}, \tag{1.3}$$

where we have chosen units so that Newton's gravitational constant G is equal to 1. When the star is in equilibrium, force must balance:

$$-\int_{\partial U} p\mathbf{n}\, dS - \int_U \rho\nabla\phi\, dV = 0. \tag{1.4}$$

Applying the divergence theorem on the first term, we get

$$-\int_U \nabla p\, dV - \int_U \rho\nabla\phi\, dV = 0. \tag{1.5}$$

Since U is arbitary, we conclude that

$$\nabla p = -\rho\nabla\phi. \tag{1.6}$$

Now let us impose spherical symmetry. Let r be distance to the origin, and $\mathbf{e}_r$ the unit vector in the outward radial direction, (1.6) becomes

$$\frac{dp}{dr}\mathbf{e}_r = -\rho\frac{d\phi}{dr}\mathbf{e}_r. \tag{1.7}$$

Furthermore, either by a direct calculation or by applying the divergence theorem and noticing that $\Delta\phi = 4\pi\rho$, one can get

$$-\frac{d\phi}{dr} = -\frac{1}{r^2}\int_0^r 4\pi s^2\rho(s)\, ds. \tag{1.8}$$

Hence

$$\frac{dp}{dr} = -\rho\frac{1}{r^2}\int_0^r 4\pi s^2\rho(s)\, ds. \tag{1.9}$$

This is a relation between p and ρ. One can impose an equation of state $p = p(\rho)$ to reduce it to an equation for ρ only. Following [7], let us use the polytropic equation of state

$$p = c\rho^\gamma, \tag{1.10}$$

then (1.9) becomes

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d(c\rho^\gamma)}{dr}\right) = -4\pi r^2\rho. \tag{1.11}$$

This is a second order ODE on ρ. We may try to solve the initial value problem for

$$\rho(0) = \rho_0, \quad \rho'(0) = 0. \tag{1.12}$$

The derivative condition on ρ is natural if we require ρ be smooth at the origin. The problem with this approach is that most of the time we do not know the value of ρ_0. What one can measure experimentally are some overall quantities of the star. For instance, one can measure the radius of the sun by astronomical observations; one can also measure the mass of the sun by knowing the gravitational acceleration it exerts on the earth and the distance between the earth and sun. We may ask the following question:

Question I.1. *Suppose we know the total mass M, and the radius R of a star. Furthermore, assume that we know the polytropic index γ of the gas. Can we estimate its interior density, pressure, temperature, etc?*

To illustrate a solution to this question, let us make the following transformation:

$$\rho = \rho_0\theta^q, \quad r = a\xi = \sqrt{\frac{(1+q)c}{4\pi}\rho_0^{\frac{1}{q}-1}}\,\xi, \tag{1.13}$$

where $q = \dfrac{1}{\gamma - 1}$. (1.11) then becomes dimensionless:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^q. \tag{1.14}$$

(1.14) is called the Lane-Emden equation of index q. If we want to solve the initial value problem (1.11), (1.12), the corresponding conditions for $\theta(\xi)$ will obviously be

$$\theta(0) = 1, \quad \theta'(0) = 0. \tag{1.15}$$

The advantage of this nondimensionalization procedure is that the resulting intial value problem of the Lane-Emden equation does not depend on the physical parameter ρ_0. The solution to (1.14) and (1.15) is called the Lane-Emden function of index q, and is denoted by θ_q. [7] provides a very nice account of the Lane-Emden functions. Some Lane-Emden functions have exact formulas. For example

$$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2, \tag{1.16}$$

$$\theta_1(\xi) = \frac{\sin(\xi)}{\xi}, \tag{1.17}$$

$$\theta_5(\xi) = \sqrt{\frac{1}{1+\frac{1}{3}\xi^2}}. \tag{1.18}$$

For general q, although one does not have exact formulas, the behavior of the solutions remain very nice due to the elliptic nature of the Lane-Emden equation. In fact, one has the following

Proposition I.2. *For $0 \leq q < 5$, $\theta_q(\xi)$ decreases monotonically to its first zero at some finite value ξ_q. For $q \geq 5$, $\theta_q(\xi)$ decreases monotonically but does not have a finite zero.*

Therefore for $q < 5$ $\left(\gamma > \dfrac{6}{5}\right)$, one has for the stellar radius R and total mass M:

$$R = a\xi_q = \sqrt{\frac{(1+q)c}{4\pi}\rho_0^{\frac{1}{q}-1}}\ \xi_q \tag{1.19}$$

$$
\begin{aligned}
M &= \int_0^R 4\pi r^2 \rho(r) dr \\
&= \int_0^{\xi_q} 4\pi a^3 \xi^2 \rho_0 \theta_q^q(\xi) d\xi \\
&= 4\pi a^3 \rho_0 \int_0^{\xi_q} \xi^2 \theta_q^q(\xi) d\xi \\
&= -4\pi a^3 \rho_0 \int_0^{\xi_q} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta_q}{d\xi}\right) d\xi \\
&= -4\pi a^3 \rho_0 \xi_q^2 \theta_q'(\xi_q) \\
&= -4\pi \sqrt{\frac{(1+q)c}{4\pi}}^{\,3} \rho_0^{\frac{3-q}{2q}} \xi_q^2 \theta_q'(\xi_q). \qquad (1.20)
\end{aligned}
$$

We observe that in (1.19) and (1.20), the only unknowns are c and ρ_0. Solving the two equations, we can determine the density function $\rho(r)$. By the equation of state, this will give us a complete characterization of the thermodynamic quantities in the interior of the star.

Although this is a pretty much oversimplified model of the structure of a star, the Lane-Emden equation does capture the dominant effect of gravitation on producing pressure and energy gradient in the stellar interior. Since its emergence in the late nineteenth century, it has become the standard equation for stellar models. The nondimensionality of the Lane-Emden functions makes their value tables powerful and easy to use, another great feature that adds to their popularity in practice.

1.2 Rotating Stars and the Compressible Euler-Poisson Equations

Relying on a simple force balance, the Lane-Emden equation is apparently insufficient if the dynamical evolution of a star must be taken into account. In this scenario, one needs the following compressible Euler-Poisson equations:

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla \phi
\end{cases} \qquad (1.21)
$$

Here ρ, p, ϕ are the same as before, whereas $\mathbf{v}$ is the velocity vector field of the motion of the gas under consideration. This system is closely related to the standard compressible Euler system in fluid dynamics. The first equation in (1.21) indicates mass conservation. The second equation indicates momentum conservation. The external forcing term $-\rho\nabla\phi$ signifies self coupling of the fluid via gravitation. The absence of second order derivatives of the velocity field $\mathbf{v}$ indicates that this is an invisid fluid. The derivation of the Euler equations is standard. See, for example, [10].

We observe immediately that (1.6) is none other than (1.21) with $\mathbf{v}$ and all t derivatives set to zero. Therefore the Euler-Poisson equations generalize the Lane-Emden equation. Like in the case of the Lane-Emden equation, we need to set an equation of state $p = p(\rho)$ to close the Euler-Poisson system.

Our goal in mind is to construct a model for rotating stars. For that purpose, we need to make the assumptions that the star is axisymmetric and that it is in dynamical equilibrium. One could apply these conditions at once and simplify the equations immediately. However, in order to better understand the effect of axisymmetry on the Euler-Poisson equations, let us take a first step in assuming axisymmetry only. A precise mathematical formulation is as follows:

Assumption I.3 (axisymmetry)**.** *Consider the Euler-Poisson equations in three spatial dimensions. Let* $\mathbf{x} = (x_1, x_2, x_3)$ *be spatial coordinates. We assume the following:*

- *Let* $Q \in SO(3)$ *be any rotation matrix which fixes the* x_3*-axis. Then all scalar functions* f *in the equations satisfy* $f(\mathbf{x}, t) = f(Q\mathbf{x}, t)$*. The velocity vector field* $\mathbf{v}$ *satisfies* $\mathbf{v}(\mathbf{x}, t) = Q^{-1}\mathbf{v}(Q\mathbf{x}, t)$*.*

The standard Euler equations have rotational symmetry. See [33]. A similar calculation shows that the same is true for the Euler-Poisson equations. In fact

Proposition I.4. *If $\rho(\mathbf{x},t)$, $p(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$ solve the Euler-Poisson equations, let $Q \in SO(3)$ be any rotation matrix, then $\rho(Q\mathbf{x},t)$, $p(Q\mathbf{x},t)$, $Q^{-1}\mathbf{v}(Q\mathbf{x},t)$ also solve the Euler-Poisson equations.*

This implies that a well-posed initial value problem for the Euler-Poisson equations will have an axisymmetric solution if the initial value is axisymmetric.

Let (r,θ,z) be the cylindrical coordinates in three spatial dimensions, and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ be the standard unit vector field in the cylindrical radial, angular and vertical directions. One sees easily that assumption I.3 is satisfied if and only if $f = f(r,z,t)$ and $\mathbf{v} = v_r(r,z,t)\mathbf{e}_r + v_\theta(r,z,t)\mathbf{e}_\theta + v_z(r,z,t)\mathbf{e}_z$. Here v_r, v_θ, v_z are the components of $\mathbf{v}$ in the $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ directions. An interesting quantity is conserved along particle trajectories in this axisymmetric setting.

Proposition I.5. *$j = rv_\theta$ is conserved along particle trajectories.*

Proof. Let us take the material derivative of $j = rv_\theta$,

$$
\begin{aligned}
&(rv_\theta)_t + \mathbf{v}\cdot\nabla(rv_\theta) \\
&= r(v_\theta)_t + (v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z)\cdot\left(\mathbf{e}_r v_\theta + r\left(\frac{\partial v_\theta}{\partial r}\mathbf{e}_r + \frac{\partial v_\theta}{\partial z}\mathbf{e}_z\right)\right) \\
&= r(v_\theta)_t + v_r\left(v_\theta + r\frac{\partial v_\theta}{\partial r}\right) + rv_z\frac{\partial v_\theta}{\partial z}.
\end{aligned}
\tag{1.22}
$$

The momentum balance equation in the Euler-Poisson system is equivalent to

$$
\rho\mathbf{v}_t + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = -\rho\nabla\phi. \tag{1.23}
$$

In an axisymmetric setting, ∇p and $\nabla\phi$ have zero $\mathbf{e}_\theta$ component. Let us project (1.23) onto the $\mathbf{e}_\theta$ direction:

$$
\rho(v_\theta)_t + \rho(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{e}_\theta = 0, \tag{1.24}
$$

or

$$(v_\theta)_t + (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{e}_\theta = 0, \tag{1.25}$$

if one looks only at the non-vacuum region. A direct calculation shows

$$(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{e}_\theta = \frac{v_r v_\theta}{r} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z}. \tag{1.26}$$

We may now observe that (1.22) is none other than r times the left hand side of (1.25). $\square$

Let us now give the assumptions that lead to rotating star solutions.

Assumption I.6 (rotation, dynamical equilibrium)**.** *Consider a solution to the Euler-Poisson equations that is axisymmetric in the sense of assumption I.3. Let* (r, θ, z) *and* $\mathbf{e}_r$*,* $\mathbf{e}_\theta$*,* $\mathbf{e}_z$ *be the cylindrical setup as before. We further assume*

- *The velocity field is given by* $\mathbf{v} = r\Omega(r, z)\mathbf{e}_\theta$*.*
- *None of the functions in the equations depends on* t*.*

Under assumption I.6, the first equation (mass conservation) in (1.21) is indentically satisfied, whereas the second equation (momentum conservation) is reduced to

$$\frac{\nabla p}{\rho} = -\nabla \phi + r\Omega^2 \mathbf{e}_r. \tag{1.27}$$

Notice that in (1.27) only the square of Ω appears. Therefore in principle Ω could change sign abruptly and not affect the solution. This is a consequence of our neglect of viscous effects in the fluid. In a realistic model Ω will always be smooth. Let us now present the rotating star problem.

Question I.7. *Does there exist a solution to* (1.27) *with given equation of state, angular velocity profile* Ω^2 *and total mass* $\int \rho = M$*?*

The answer is affirmative. The existence and properties of rotating star solutions to (1.27) were attained by Auchmuty and Beals [3], Auchmuty [2], Caffarelli and Friedman [6], Friedman and Turkington [15, 17, 16], Li [28], Chanillo and Li [9], Luo and Smoller [31], and Luo and Smoller [32].

Auchmuty and Beals [3] imposed some non-trival decay conditions on Ω^2 and got the first existence results for rotating stars. However, these conditions excluded constant Ω. Li [28] obtained existence results for small constant Ω and also a non-existence result for large constant Ω. In my work on a modified version of the Euler-Poisson equations, I will prove some parallel results for rotating planets with a solid core, and more.

Notice that if Ω^2 is prescribed, ρ is the only unknown in (1.27), since ϕ is the gravity potential of ρ, and p is determined by ρ from the equation of state $p = p(\rho)$. Different components of (1.27) seem to provide too many equations for the single unknown function ρ. This is a subtle point of the problem. In order to appreciate this subtlety, let us take the curl of (1.27),

$$\nabla \times \left(\frac{\nabla p}{\rho}\right) = \nabla \times (r\Omega^2 \mathbf{e}_r), \tag{1.28}$$

which simplifies to

$$\frac{\nabla p \times \nabla \rho}{\rho^2} = r\frac{\partial \Omega^2}{\partial z}\mathbf{e}_\theta. \tag{1.29}$$

(1.29) implies the following crucial proposition,

Proposition I.8. *Both sides of* (1.27) *are curl free if and only if* $\nabla p \times \nabla \rho = 0$ *if and only if* Ω^2 *depends only on* r.

For a given equation of state $p = p(\rho)$, one sees immediately the following

Corollary I.9. *If* p *is a given function of* ρ*, then* (1.27) *is curl free, and* Ω^2 *can depend only on* r.

This shows that in order for the equations to be consistent, we can only prescribe Ω^2 as functions of r. Notice that the vanishing of curl is a necessary condition for a vector field to be a gradient. That indeed is the case here. In fact, (1.27) can be rewritten as

$$\nabla\big(a(\rho)\big) = -\nabla\phi + \nabla J, \tag{1.30}$$

where

$$a(s) = \int_0^s \frac{p'(t)}{t}dt, \quad J(r) = \int_0^r s\Omega^2(s)ds. \tag{1.31}$$

Taking off the gradients, we get

$$a(\rho) = -\phi + J(r) + \lambda \tag{1.32}$$

for some constant λ. With ϕ given as the gravity potential of ρ, (1.32) appears a single equation for the unknown function ρ, although we still don't know the value of λ.

In the literature, there is a way of prescribing the angular velocity profile without giving Ω^2 directly. See Auchmuty and Beals [3], Friedman and Turkington [15, 17, 16], Caffarelli and Friedman [6], and Luo and Smoller [32]. What one does instead is to consider the function

$$m_\rho(r) = \int_{x_1^2+x_2^2\le r^2} \rho(\mathbf{x})d\mathbf{x} \tag{1.33}$$

representing the mass enclosed by a cylinder around the x_3-axis with radius r. One prescribes a function $j(s)$, and calculates the angular velocity by

$$r^2\Omega(r) = j(m_\rho(r)). \tag{1.34}$$

Under this setup, (1.27) can be written as

$$\frac{\nabla p}{\rho} = -\nabla\phi + \frac{j^2(m_\rho(r))}{r^3}\mathbf{e}_r, \tag{1.35}$$

and the equivalent for (1.32) is

$$a(\rho) = -\phi - \int_r^\infty \frac{j^2(m_\rho(s))}{s^3} ds + \lambda. \tag{1.36}$$

One can motivate this point of view as follows. First observe that $r^2\Omega$ is the same as rv_θ. Let us consider an axisymmetric dynamical solution to the Euler-Poisson equations in the sense of assumption I.3. Further assume that the v_r and the v_θ component of the velocity field depends only on r at any given time t. Let $X : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ be the particle trajectory mapping. Writing X in cylindrical coordinates, the assumptions can be summarized as

$$X(r, \theta, z, t) = \big(X_1(r,t), X_2(r,t), X_3(r,z,t)\big). \tag{1.37}$$

(1.37) is naturally satisfied by a configuration that has no z variation, but for general configurations is quite non-trivial. In any case, let us force this setup to get

$$m_{\rho(\cdot,t)}\big(X_1(r,t)\big) = m_{\rho(\cdot,0)}(r) \tag{1.38}$$

by mass conservation, and

$$X_1(r,t)v_\theta\big(X_1(r,t),t\big) = rv_\theta(r,0) \tag{1.39}$$

by proposition I.5. Since $X(\cdot,t)$ is a diffeomorphism, (1.38) and (1.39) shows that there is a fixed relation between rv_θ and $m_\rho(r)$ for any given time t. In other words, one could write

$$rv_\theta(r,t) = j(m_{\rho(\cdot,t)}(r)). \tag{1.40}$$

Therefore (1.34) can be thought of as an equilibrium version of (1.40). That being said, for the rest of this dissertation, we will focus mainly on the formulation with prescribed Ω, although a similar theory for prescribed j could also be developed.

The key idea to solve relation (1.32) is to regard it as the Euler-Lagrange equation of the following energy functional

$$E(\rho) = \int A(\rho) + \frac{1}{2}\rho B\rho - \rho J \tag{1.41}$$

subject to the constraint

$$\int \rho = M. \tag{1.42}$$

Here

$$A(s) = \int_0^s a(t)dt \tag{1.43}$$

and

$$B\rho = \rho * \frac{1}{|\mathbf{x}|} = \int \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}. \tag{1.44}$$

Under this formulation the unknown constant λ in (1.32) is naturally realized as a Lagrange multiplier. We will give a more detailed exposition of the existence theory by calculus of variations in chapter II.

1.3 Rotating Planets with a Solid Core

Recent observations on extrasolar giant planets have raised fundamental questions about their interior structure and origin. Many of the extrasolar planets possess unexpectedly small radii, suggesting high metallicity in their composition and possibly the existence of a solid core in the center (Anderson and Adams [1]). Efforts have been made to simulate the evolution of these planets, and evidence for the existence of a solid core has been found (Militzer et al. [34]). Models involving high metallicity and center core have been constructed and examined (Miller et al. [35], Burrows et al. [5]). As a first model from a mathematical perspective, one could modify the

Euler-Poisson equations for rotating stars to include a solid core and its gravitational potential. Let K be an axisymmetric bounded domain in $\mathbb{R}^3$, and ρ_K be a given axisymmetric non-negative function on K, indicating the density of the solid core. Let $\phi_K = -\rho_K * \frac{1}{|\mathbf{x}|}$ denote the gravitational potential of ρ_K. Then by the $-\phi_K$-modified Euler-Poisson system we mean the following

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla (\phi + \phi_K) \end{cases} \tag{1.45}$$

Here we only require the equations be satisfied on $\mathbb{R}^3 \setminus K$. As is in the case of rotating star solutions, we can again make assumption I.6, and reduce the equations to

$$\frac{\nabla p}{\rho} = -\nabla(\phi + \phi_K) + r\Omega^2 \mathbf{e}_r. \tag{1.46}$$

Since the new term ϕ_K appears inside the gradient, it does not contribute to the curl of the equation. Following the previous calculation, we again get corollary I.9. Therefore we can prescribe some Ω^2 which depends only on r, and ask the following

Question I.10. *Does there exist a solution to* (1.46) *with given equation of state, angular velocity profile* Ω^2 *and total mass* $\int \rho = M$*?*

At first sight, this question might seem easy if not trivial given all the previous work on rotating star solutions, for the following reasons. 1. From a mathematical standpoint, this is just one extra term in the equation, which is a given potential function. It should not add too much difficulty to the problem. 2. From a physical intuitive standpoint, the inclusion of a solid core potential should help pull the gas together and hence stabilize the star. An existence result is expected to hold "more true" in this circumstance. However, a further examination of the methodologies leading to the rotating star solutions reveals a different story. Indeed, in order to show that a solution to (1.32) exists, one needs to assume that J has small L^∞ norm. See [28]. The mathematical reason for this assumption cannot be easily explained without digging deep into the existence proof, but the physical intuition is that one should only expect a solution to exist if the rotation is slow. In fact, for sufficiently fast constant rotation, [28] provides a non-existence result. On the other hand, with the addition of the ϕ_K term, (1.32) gets modified to

$$a(\rho) = -\phi - \phi_K + J(r) + \lambda \tag{1.47}$$

Recall that $\phi_K = -\rho_K * \frac{1}{|\mathbf{x}|}$ is the physical potential of ρ_K, and is apparently negative. Therefore $-\phi_K$ and J in (1.47) are of the same sign! This seems to be in strong disagreement with the intuition that core gravity and centrifugal force from the rotation should somehow cancel each other. The reason for this strange phenomenon is, roughly speaking, that the centrifugal potential increases as one moves away from the center, while gravity potential decreases, hence, although they are in opposite directions in the force equation, on the potential level, they actually have the same sign. As we have pointed out, the previous work on rotating star solutions assumes smallness of J in light of slow rotation, but here in this planet structure model, there is no physical reason to assume $-\phi_K$ to be small. The core gravity potential need not be small for a slowly rotating planet. Therefore different methods are needed to treat this new case.

Moreover, coming back to the physical intuition of core gravity - centrifugal force balance, one could imagine a super heavy core pulling the surrounding gas very tightly around it, and spinning with a very fast angular velocity. This picture of heavy core and fast rotation suggests one to ask the following

Question I.11. *With given J, not necessarily small, given equation of state, and* *given total mass M, does there exist a solution to* (1.47) *when $-\phi_K$ is sufficiently large in the appropriate sense?*

Again, although the heuristic picture strongly suggests a positive answer, the current methodologies in rotating star solution theory does not provide a proof. If a regular sized $-\phi_K$ creates a problem in the arguments, a very large $-\phi_K$ is certainly going to make things worse.

On the other hand, one should still not expect a solution to exist if the core is of regular size while the rotation is too fast. Motivated by the non-existence result in [28], one could ask

Question I.12. *With given core potential $-\phi_K$, equation of state, and total mass M, can one be sure that there is no solution to* (1.47) *if J is given by a constant rotation that is too large?*

The argument provided in [28] is a subtle contradiction involving integral equalities derived from (1.47) and carefully chosen test functions. This subtle construction breaks down if one adds the ϕ_K term and change the domain from $\mathbb{R}^3$ to $\mathbb{R}^3 \setminus K$. As is the case for most non-existence proofs involving integral equalities, the arguments get quite sensitive to the equation and the domain and is usually not very flexible.

As we will show in this dissertation, all three questions raised in this section can be answered in the affirmative if suitable conditions are given. For question I.10, one has existence if J has small L^∞ norm or is given by small constant rotation. For question I.11, one has existence if the core potential is given by $\mu\phi_K$ for some given ϕ_K and large enough μ. For question I.12, one has non-existence for large enough constant rotation.

1.4 Non-isentropic Equation of State

As is manifested by corollary I.9, there is a direct connection between the isentropy of the equation of state and the fact that Ω^2 has only cylindrical radial dependence. Almost all previous works on rotating star solutions are based on an isentropic equation of state. If one attempts to study solutions that allow a general velocity profile that has non-trivial z dependence, a non-isentropic equation of state is inevitable:

$$p = p(\rho, s). \tag{1.48}$$

Here s is entropy. In standard thermodynamics, the state of a gas is determined by two state variables. Therefore (1.48) is sufficient to describe any general state changes in the gas. The introduction of the new variable s calls for another equation to close the system. Indeed, the full Euler-Poisson system has another equation for energy conservation, which we have been ignoring until now:

$$\left(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e\right)_t + \nabla \cdot \left((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e)\mathbf{v}\right) = -\rho\nabla\phi \cdot \mathbf{v} - \nabla \cdot (p\mathbf{v}). \tag{1.49}$$

Here e is specific internal energy. By the second law of thermodynamics, one has

$$de = T(\rho, s)ds + \frac{p(\rho, s)}{\rho^2}d\rho, \tag{1.50}$$

or

$$e(\rho, s) = \int_0^\rho \frac{p(\xi, s)}{\xi^2}d\xi. \tag{1.51}$$

Here we have assumed the relation

$$e(0, s) = 0.$$

(1.51) implies that the dependence of $e(\rho, s)$ and therefore $T(\rho, s)$ on ρ and s are determined if we pick an equation of state $p = p(\rho, s)$. (1.49) will then provide a new equation for s and ρ. Let us simplify (1.49) using (1.21). After a simple calculation, we get

$$\rho e_t + \rho \mathbf{v} \cdot \nabla e = -p \nabla \cdot \mathbf{v}. \tag{1.52}$$

By (1.50), this becomes

$$\begin{aligned}
\rho T s_t + \rho \frac{p}{\rho^2} \rho_t + \rho \mathbf{v} \cdot \left(T \nabla s + \frac{p}{\rho^2} \nabla \rho \right) &= -p \nabla \cdot \mathbf{v} \\
\rho T (s_t + \mathbf{v} \cdot \nabla s) + \frac{p}{\rho} (\rho_t + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}) &= 0 \\
\rho T (s_t + \mathbf{v} \cdot \nabla s) &= 0 \\
s_t + \mathbf{v} \cdot \nabla s &= 0. \qquad (1.53)
\end{aligned}$$

Notice that we have used the mass conservation equation to get the penultimate step. Let us summarize the equations as the full Euler-Poisson system

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla \phi \\ s_t + \mathbf{v} \cdot \nabla s = 0 \end{cases} \tag{1.54}$$

where the three equations stand for mass conservation, momentum conservation, and entropy transport, respectively. To look for rotating star solutions, let us again assume assumption I.6. We observe that mass conservation and entropy transport are automatically satisfied under these assumptions. The momentum balance equation is reduced to

$$\frac{\nabla p}{\rho} = -\nabla \phi + r \Omega^2 \mathbf{e}_r. \tag{1.55}$$

This equation looks deceptively similar to (1.27). The difference between the two will become apparent once we impose a non-isentropic equation of state. Let us introduce the equation of state of an ideal gas (Courant and Friedrichs [11])

$$p = \exp(s)\rho^\gamma \tag{1.56}$$

for some constant γ. (1.55) now becomes

$$\frac{\nabla(\exp(s)\rho^\gamma)}{\rho} = -\nabla\phi + r\Omega^2\mathbf{e}_r. \tag{1.57}$$

We wish that we could treat this vector equation as the gradient of a scalar equation like (1.32). To see that this is not possible, let us take the curl of (1.57),

$$\exp(s)\rho^{\gamma-2}\nabla s \times \nabla\rho = r\frac{\partial\Omega^2}{\partial z}\mathbf{e}_\theta. \tag{1.58}$$

Hence (1.57) is a gradient only if

$$\nabla s \; // \; \nabla\rho. \tag{1.59}$$

This is a very awkward relation between s and ρ. One way to make sure this condition be satisfied is to prescribe s as a function of ρ. But that also means we give up on non-isentropy. If we treat s and ρ as two independent functions, (1.59) is too strong a connection between the two and the resulting equation will still lack a variational structure, which is the very core of the method treating the classical rotating star problem.

In this dissertation we will prove two types of existence results pertaining to the non-isentropic Euler-Poisson equations. One is to consider the divegence of equation (1.57):

$$\nabla\cdot\left(\frac{\nabla(\exp(s)\rho^\gamma)}{\rho}\right) = -4\pi\rho + \nabla\cdot(r\Omega^2\mathbf{e}_r), \tag{1.60}$$

and to solve for ρ with s and Ω prescribed; the other is to consider equation (1.55) and to solve for p and Ω^2 with ρ prescribed.

To better see the structure of (1.60), let us make the change of variable:

$$w = \frac{\gamma}{\gamma - 1} \exp\left(\frac{\gamma - 1}{\gamma} s\right) \rho^{\gamma - 1}. \tag{1.61}$$

(1.60) now becomes

$$\nabla \cdot (\exp(\alpha s) \nabla w) + K \exp(-\alpha s) |w|^q - f = 0. \tag{1.62}$$

Here

$$q = \frac{1}{\gamma - 1}, \quad \alpha = \frac{1}{\gamma}, \quad K = 4\pi \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma - 1}}, \tag{1.63}$$

and

$$f = 2\Omega^2 + r\frac{\partial \Omega^2}{\partial r} = 2\Omega \frac{\partial}{\partial r}(r\Omega). \tag{1.64}$$

With s and Ω prescribed, Luo and Smoller [31] considered (1.62) and obtained some existence results when the entropy is assumed to be either constant or radially dependent, and a non-existence result when the entropy is non-constant. I will find some existence results for (1.62) with axisymmetric entropy. Standard elliptic theory (Gilbarg and Trudinger [19]) can solve the Dirichlet problem to (1.62) on bounded domains given suitable range of q, but in order to conclude positivity of w inside the domain, it is desirable that f be negative. Unfortunately for most physically interesting Ω, f is positive. For example, constant Ω will produce a positive f. Therefore the gist of the proofs is to show existence of positive solutions.

One could also consider (1.55) with prescribed ρ. Let us rewrite (1.55) in cylindrical coordinates.

$$\begin{cases} p_r = \rho(-\phi)_r + \rho r \Omega^2 \\ p_z = \rho(-\phi)_z \end{cases} \tag{1.65}$$

If ρ is given, the second equation in (1.65) will determine p, and consequently the first equation in (1.65) will give Ω^2. However, thus obtained solutions in general do not satisfy the following conditions to be considered physically significant: p and Ω^2 should be sufficiently regular (e.g. no blow ups), positive where ρ is positive, and vanish where ρ is zero. It turns out that it is possible to give sufficient conditions on ρ that would guarantee the existence of solutions with the above mentioned physical merits.

1.5 Outline of Thesis

In this dissertation, we will attempt to address the formerly proposed questions on rotating planets with a solid core and rotating stars with a non-isentropic equation of state. Chapter II serves as an introduction to the existence theory for rotating star solutions. We outline the ideas that yield existence of solutions for slow rotation and non-existence of solution for fast constant rotation. We also discuss a non-linear stability result whose proof relies on a stronger existence theorem. Chapter III deals with rotating planets with a solid core and answers the questions raised in section 1.3. Chapter IV treats a non-isentropic equation of state and answers the questions raised in section 1.4. Chapter V provides further inquiries about rotating stars, and indicates some possible directions for future research.

CHAPTER II

Existence of Rotating Star Solutions to the Isentropic Euler-Poisson Equations

In this chapter, we give an overview of the methods leading to existence theorems on rotating star solutions to the isentropic Euler-Poisson equations. Section 2.1 explains the variational formulation introduced in Auchmuty and Beals [3]. Section 2.2 and 2.3 outline the proofs of the existence and non-existence results in [3] and in Li [28]. Section 2.4 discusses a non-linear stability result in Luo and Smoller [32]. To avoid complicated technicalities, let us assume the equation of state used in this chapter to be[1]

$$p = c\rho^{\gamma}. \tag{2.1}$$

Here c and γ are constants. As we will see later, the value of c is insignificant to the analysis while the value of γ is important. We will freely choose c to make the equations look simpler. (2.1) is already sophisticated enough to capture the essential difficulty in results with more general equations of state.

[1] Gas satisfying this equation of state is often called a polytrope.

2.1 A Variational Free Boundary Problem

Recall that rotating stars are modeled by solutions to (1.32). Imposing the equation of state (2.1) and choosing $c = \gamma - 1$, this is

$$\gamma\rho^{\gamma-1} = B\rho + J(r) + \lambda, \tag{2.2}$$

where

$$B\rho(\mathbf{x}) = \rho * \frac{1}{|\mathbf{x}|} = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}. \tag{2.3}$$

We can view (2.2) as an Euler-Lagrange equation. In fact, let us consider the following energy functional

$$E(\rho) = \int_{\mathbb{R}^3} \left(\big(\rho(\mathbf{x})\big)^{\gamma} - \frac{1}{2}\rho(\mathbf{x})B\rho(\mathbf{x}) - \rho(\mathbf{x})J(r(\mathbf{x})) \right) d\mathbf{x} \tag{2.4}$$

on the space of admissible functions

(2.5)
$$W = \left\{ \rho : \mathbb{R}^3 \to \mathbb{R},\ \rho \text{ is axisymmetric},\ \rho \geq 0 \text{ a.e.},\ \rho \in L^{\gamma}(\mathbb{R}^3),\ \int_{\mathbb{R}^3} \rho\, d\mathbf{x} = M \right\}.$$

One has the following

Proposition II.1. *Assume J is smooth. Let ρ be a local minimum of E on W, then ρ is continuous on $\mathbb{R}^3$, smooth on its own positive set, and satisfies* (2.2) *there.*

To prove this proposition, one does a standard calculation to get the following variational inequality (See also [23]):

$$\gamma\rho^{\gamma-1} - B\rho - J \geq \lambda \quad \text{a.e.}, \tag{2.6}$$

$$\gamma\rho^{\gamma-1} - B\rho - J = \lambda \quad \text{a.e. where } \rho > 0. \tag{2.7}$$

From (2.6) and (2.7) one deduces that

$$\gamma\rho^{\gamma-1} = \max(B\rho + J + \lambda, 0). \tag{2.8}$$

The smoothing effect of B will now give the desired result.

Before we move on to discuss the existence of minimizers of E, let us present a few convolution inequalities which prove to be very useful in the rotating star existence theory. Their proofs can be found in [3].

Lemma II.2. *Suppose* $\rho \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, *and* $1 < p \leq \frac{3}{2}$. *Then* $B\rho \in L^r(\mathbb{R}^3)$ *for all* $3 < r < \frac{3p}{3-2p}$, *and*

$$\|B\rho\|_r \leq C(\|\rho\|_1^b \|\rho\|_p^{1-b} + \|\rho\|_1^c \|\rho\|_p^{1-c}) \tag{2.9}$$

for some constant C *and* $0 < b, c < 1$ *depending on* p *and* r. *If* $p > \frac{3}{2}$, *then* $B\rho$ *is bounded and continuous and satisfies* (2.9) *with* $r = \infty$.

Lemma II.3. *If* $\rho \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, *then*

$$\left| \int_{\mathbb{R}^3} \rho B\rho \, d\mathbf{x} \right| \leq C\Big(\int_{\mathbb{R}^3} |\rho|^{4/3} \, d\mathbf{x} \Big) \Big(\int_{\mathbb{R}^3} |\rho| \, d\mathbf{x} \Big)^{2/3}. \tag{2.10}$$

Lemma II.4. *If* $\rho \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ *for some* $p > 3$, *then* $B\rho$ *is continuously differentiable.*

To see that E is well-defined on W and is bounded away from $-\infty$, we apply lemma II.3 to the second term in (2.4).

$$\begin{aligned} &\int \frac{1}{2}\rho B\rho \\ \leq &C \int \rho^{4/3} \Big(\int \rho \Big)^{2/3} \\ \leq &C \int \rho^{4/3} M^{2/3} \end{aligned} \tag{2.11}$$

We wish to control (2.11) by the first term in (2.4). For that purpose, we need to make the crucial assumption

$$\gamma > \frac{4}{3}. \tag{2.12}$$

Now for any small $\epsilon > 0$, we can choose $s(\epsilon)$ sufficiently large, so that $\rho^{4/3} \le \epsilon \rho^\gamma$ for $\rho > s$, and

$$\begin{aligned}
& C \int \rho^{4/3} M^{2/3} \\
\le & C(M)\Big(\int_{\rho<1} \rho^{4/3} + \int_{1\le\rho\le s} \rho^{4/3} + \int_{\rho>s} \rho^{4/3} \Big) \\
\le & C(M)\Big(\int_{\rho<1} \rho + s^{1/3} \int_{1\le\rho\le s} \rho + \epsilon \int_{\rho>s} \rho^{\gamma} \Big) \\
\le & C(M, s(\epsilon)) + C(M)\ \epsilon \int \rho^\gamma. \qquad (2.13)
\end{aligned}$$

Choosing ϵ so small that $C(M)\ \epsilon < \dfrac{1}{2}$, we see that

$$\begin{aligned}
E(\rho) =& \int \Big(\rho^\gamma - \frac{1}{2}\rho B \rho - \rho J \Big)\ d\mathbf{x} \\
\ge & \frac{1}{2} \int \rho^\gamma \ d\mathbf{x} - C(M, s(\epsilon)) - M\|J\|_\infty. \qquad (2.14)
\end{aligned}$$

Therefore we have the following

Proposition II.5. *Suppose $\gamma > \dfrac{4}{3}$, and let $M > 0$ and $J \in L^\infty$ be given. Then there is a constant $C(M, J)$ such that*

$$E(\rho) \ge \frac{1}{2}\int \rho^\gamma \ d\mathbf{x} - C(M, J) \qquad (2.15)$$

for all $\rho \in W$.

Let $I = \inf_{\rho\in W} E(\rho)$. We see from proposition II.5 that $I > -\infty$. Pick a minimizing sequence $\{\rho_n\}$ of E in W. (2.15) implies that $\{\rho_n\}$ is bounded in L^γ, and therefore has a weakly convergent subsequence. Without loss of generality, we still denote that subsequence by $\{\rho_n\}$. In order for the limit to be a minimizer, we need to show that E is weakly lower semicontinuous on L^γ. The first term in (2.4) is weakly lower semicontinuous by a standard convexity argument. See, for example [29]. The third term in (2.4) is linear in ρ, hence is weakly continuous, at least if J has sufficient decay. To show convergence of the second term, it is desirable that $B\rho_n$ should converge in $L^{\gamma'}$ norm, where γ' is the conjugate exponent of γ. If the functions under consideration are restricted to S_R, a ball of radius R, then $B : L^\gamma(S_R) \to L^{\gamma'}(S_R)$ is compact when $\gamma > \dfrac{6}{5}$ by the Sobolev embedding theorem. Since we have already assumed $\gamma > \dfrac{4}{3}$, in that case, we do get the desired convergence. Unfortunately, we have no such knowledge a priori about the support of the functions in W. The unboundedness of the domain makes this problem difficult.

The prescription to circumvent loss of compactness is the following. We first minimize E on a restricted space of functions W_R, which comprises essentially those functions in W which are supported in S_R. This can be done exactly by the arguments given above. Such minimizers are denoted by ρ_R, and they will satisfy the Euler-Poisson equations in the interior of S_R, but there is no guarantee that they would vanish on the boundary. Such solutions are not sufficient to model rotating stars. However, we can try to prove bounds on the support of ρ_R, and hopefully, the size of the support would have some uniform bound for all large R. Once we let R increase past that bound, we will have found a compactly supported solution to the Euler-Poisson equations, since the support of ρ_R will be contained in a smaller ball inside S_R.

2.2 Existence of Solutions for Slow Rotation

In this section we outline the ideas in the proof of the following

Proposition II.6. *Given $M > 0$, $\gamma > \dfrac{4}{3}$, there is an $\epsilon > 0$ and $R_0 > 0$, such that the support of ρ_R is contained in S_{R_0} if $\|J\|_\infty < \epsilon$, and $R > R_0$. Here ρ_R is a minimizer of E on*

$$W_R = \left\{ \rho \in W \;\middle|\; \rho = 0 \text{ a.e. outside } S_R,\ \rho \le R \text{ a.e.} \right\}, \tag{2.16}$$

and S_R is the ball of radius R centered at the origin.

We first need an a priori bound on the L^∞ norm of ρ_R.

Lemma II.7. *There exists some constant C_1, depending on γ, M and $\|J\|_\infty$ such that*

$$\|\rho_R\|_\infty \le C_1 \quad \text{for all } R \ge 1. \tag{2.17}$$

This can be proven by a bootstrap argument. One starts with a uniform bound on $\|\rho_R\|_{4/3}$ given by (2.13). The fact that ρ_R minimizes E on W_R allows one to come up with estimates on the L^p norm of ρ_R for a slightly larger power p. After finitely many steps, p becomes greater than $\frac{3}{2}$. Lemma II.2 then gives the L^∞ bound.

Next we write down the Euler-Lagrange equations similar to (2.6) and (2.7), this time on S_R.

Lemma II.8. *For $R > C_1$, there is a unique constant λ_R such that*

$$\gamma\rho_R^{\gamma-1} - B\rho_R - J \ge \lambda_R \quad \text{a.e. on } S_R, \tag{2.18}$$

$$\gamma\rho_R^{\gamma-1} - B\rho_R - J = \lambda_R \quad \text{a.e. where } \rho_R > 0. \tag{2.19}$$

Our next goal is to show that λ_R is negative and bounded away from 0 for all R sufficiently large. This is a crucial step in the argument. The idea is to use (2.18). If we can find a point $\mathbf{x} \in S_R$ for which $\rho_R(\mathbf{x})$ is much smaller compared with $B\rho_R(\mathbf{x})$, λ_R will be negative. Hence we need some uniform *lower bound* on $\|B\rho_R\|_\infty$. This may not be true for a general family of functions having the same L^1 norm, because they could become more and more dilute and converge to zero as R increases.

Lemma II.9. *There exist $\delta > 0$, $\epsilon_1 > 0$, $R_1 > 0$ such that,*

$$\|B\rho_R\|_\infty \ge \delta \tag{2.20}$$

if $\|J\|_\infty < \epsilon_1$ and $R > R_1$.

The truthfulness of this lemma can be motivated as follows. Let us first assume that $J = 0$. By simple scaling considerations,

$$E(\rho_R) = \int \rho_R^\gamma - \frac{1}{2}\rho_R B \rho_R < 0 \tag{2.21}$$

for R large enough. We fix some R_1 large so that $E(\rho_{R_1})$ is negative. Since $\rho_{R_1} \in W_R$ for all $R > R_1$, we have

$$E(\rho_R) \leq E(\rho_{R_1}) < 0. \tag{2.22}$$

Therefore

$$0 < -E(\rho_{R_1}) \leq \frac{1}{2}\int \rho_R B \rho_R \leq \frac{M}{2}\|B\rho_R\|_\infty. \tag{2.23}$$

When J is non-zero but small, the same result holds true.

Once this lower bound on $\|B\rho_R\|_\infty$ is established, with some decent amount of work in the direction suggested above, we can get the following

Lemma II.10. *There exist $\delta > 0$, $\epsilon_1 > 0$, $R_1 > 0$ such that*

$$\lambda_R < -\delta \quad \textit{if } \|J\|_\infty < \epsilon_1, \ \textit{and } R > R_1. \tag{2.24}$$

The values of δ, ϵ_1 and R_1 may be different from the previous lemma. We use these letters to represent generic constants depending on γ and M.

Now, in view of (2.19), we have

$$\gamma(\rho_R(\mathbf{x}))^{\gamma-1} - B\rho_R(\mathbf{x}) - J(r(\mathbf{x})) < -\delta \tag{2.25}$$

whenever $\rho_R(\mathbf{x}) > 0$. If we furthermore assume that $\|J\|_\infty < \frac{\delta}{2}$, (2.25) implies

$$B\rho_R(\mathbf{x}) > \frac{\delta}{2} \quad \text{whenever } \rho_R(\mathbf{x}) > 0. \tag{2.26}$$

Therefore in order to show that $\rho_R(\mathbf{x})$ is zero outside some ball of uniformly bounded radius, one just need to estimate $B\rho_R(\mathbf{x})$ and show that it is smaller than $\frac{\delta}{2}$ there. In fact, one has the following

Lemma II.11. *There exist $\epsilon > 0$ and $R_0 > 0$ such that if $\|J\|_\infty < \epsilon$*

$$B\rho_R(\boldsymbol{x}) < \frac{\delta}{2} \quad \textit{for } |\boldsymbol{x}| > R_0. \tag{2.27}$$

To understand the main idea behind this lemma, let us provide a schematic argument for the case when $\mathbf{x}$ is far away from the x_3-axis. We split $B\rho_R$ into three parts:

$$\begin{aligned} B\rho_R(\mathbf{x}) &= \int \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &= \int_{|\mathbf{x}-\mathbf{y}|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{1<|\mathbf{x}-\mathbf{y}|<a} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|>a} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \end{aligned} \tag{2.28}$$

Denote by r the distance of $\mathbf{x}$ to the x_3-axis. The first term in (2.28) should be small, because by axisymmetry, the portion of ρ_R on $|\mathbf{x}-\mathbf{y}| < 1$ can be rotated around the x_3-axis to create a ring of radius r. Since the total mass M is given, the portion of ρ_R on $|\mathbf{x}-\mathbf{y}| < 1$ is $O\left(\frac{1}{r}\right)$, and will therefore be small when r is large. The last term in (2.28) will be small if we pick a large enough, since ρ_R is bounded in L^1. The integration domain of the second term can be covered by $O(a^3)$ balls of radius one. The contribution from each of these balls will be $O\left(\frac{1}{r-a}\right)$ by a similar argument as before. Hence the second term will be small if r is sufficiently large.

Lemma II.11 implies proposition II.6, which gives existence of compactly supported rotating star solutions for J having small L^∞ norm. Notice that for constant angular velocity Ω, $J(r) = \frac{1}{2}\Omega^2 r^2$ is not in L^∞. Knowing that the above mentioned solutions are supported in S_{R_0}, we replace J by a smooth cut-off function

$$\tilde{J}(r) = \begin{cases} \frac{1}{2}\Omega^2 r^2 & \text{when } r \le R_0 \\ \Omega^2 R_0^2 & \text{when } r > 2R_0 \end{cases}$$

where $\tilde{J}$ is smooth and increasing between R_0 and $2R_0$. The solution corresponding to $\tilde{J}$ will be supported in S_{R_0} as long as we pick Ω small enough to make sure that

$\|\tilde{J}\| < \epsilon$. Such solutions only see the effect of $\tilde{J}$ up to $r = R_0$, hence they will solve the Euler-Poisson equations with a uniform angular velocity profile.

2.3 Non-existence of Solution for Fast Constant Rotation

In this section we outline the ideas that lead to the following

Proposition II.12. *Given $\gamma > \dfrac{4}{3}$ and $M > 0$, there is an $\Omega_0 > 0$ such that if $\Omega(r) \equiv \Omega > \Omega_0$, there does not exist a continuous bounded function $\rho : \mathbb{R}^3 \to \mathbb{R}$ for which*

1. $\rho \geq 0$, $\displaystyle\int_{\mathbb{R}^3} \rho = M$.
2. *ρ satisfies the equilibrium relation*

$$\gamma\rho^{\gamma-1} - \frac{1}{2}\Omega^2 r^2 - B\rho = \lambda \tag{2.29}$$

where $\rho > 0$.

We first need a few direct estimates on the gravity potential $B\rho$. See [28] for their proofs.

Lemma II.13.

$$\|B\rho\|_\infty \leq CM^{2/3}\|\rho\|_\infty^{1/3}. \tag{2.30}$$

Compare lemma II.13 with (2.29). We see that

$$\frac{1}{2}\Omega^2 r^2 = \gamma\rho^{\gamma-1} - B\rho - \lambda \leq \gamma\|\rho\|_\infty^{\gamma-1} + CM^{2/3}\|\rho\|_\infty^{1/3} - \lambda. \tag{2.31}$$

Hence $d = \displaystyle\sup_{(r,\theta,z)\in \mathbf{Supp}\rho} r < \infty$ if ρ is bounded and $\Omega > 1$. Let us present some more estimates.

Lemma II.14. *There exist some constant $d_1(M, \|\rho\|_\infty)$ and $C > 0$ such that*

$$\|B\rho\|_\infty \leq C\|\rho\|_\infty d^2 \log\left(1 + \frac{M}{2\pi d^3\|\rho\|_\infty}\right) \tag{2.32}$$

for $d < d_1$.

Let d be defined as before and $\epsilon = \sup\{d - r \mid (r,\theta,z) \in \mathbf{Supp}\rho\}$. We have the following

Lemma II.15. *There exist some constant $d_2(M, \|\rho\|_\infty)$ and $C > 0$ such that*

$$\|B\rho\|_\infty \leq C\|\rho\|_\infty \epsilon d\left(\log\left(1 + \sqrt{\frac{d}{\epsilon}}\right) + \log\left(1 + \frac{M}{2\pi\epsilon d^2\|\rho\|_\infty}\right)\right) \tag{2.33}$$

for $d < d_2$ and $\epsilon < \dfrac{d}{2}$.

The logarithmic factors in lemma II.14 and lemma II.15 come naturally from calculating the gravity potential of a denisty function supported in a cylinder of radius d. Next, we observe that $\lambda \leq -\frac{1}{2}\Omega^2 d^2$ by evaluating (2.29) at a sequence of points $(r_n, \theta_n, z_n) \in \mathbf{Supp}\rho$ such that $r_n \to d$:

$$\begin{aligned}
\lambda &= \gamma(\rho(r_n,\theta_n,z_n))^{\gamma-1} - \frac{1}{2}\Omega^2 r_n^2 - B\rho(r_n,\theta_n,z_n) \\
&\leq \gamma(\rho(r_n,\theta_n,z_n))^{\gamma-1} - \frac{1}{2}\Omega^2 r_n^2 \to -\frac{1}{2}\Omega^2 d^2.
\end{aligned} \tag{2.34}$$

With this knowledge on λ, we conclude from (2.29) that

$$\gamma\rho^{\gamma-1} \leq B\rho. \tag{2.35}$$

Therefore

$$\|\rho\|_\infty \leq C\|B\rho\|_\infty^{\frac{1}{\gamma-1}}. \tag{2.36}$$

Combining (2.36) with lemma II.13 and using $\gamma > \frac{4}{3}$, we can get an a priori bound on the L^∞ norm of ρ.

Lemma II.16. *There is a constant $C_2(\gamma, M) > 0$ such that*

$$\|\rho\|_\infty \leq C_2. \tag{2.37}$$

On the other hand, combining (2.36) with lemma II.14 and lemma II.15, one can get rid of the logarithmic factors in these lemmas and even create an small extra power of d or ϵd.

Lemma II.17. *There exist some constant $d_3(M, \|\rho\|_\infty)$, $C_2 > 0$ and $\delta > 0$ such that*

$$\|B\rho\|_\infty \leq C_2 d^{2+\delta}, \tag{2.38}$$

and

$$\|B\rho\|_\infty \leq C_2(\epsilon d)^{1+\delta} \tag{2.39}$$

for $d < d_3$ and $\epsilon < \dfrac{d}{2}$.

Now let us prove proposition II.12 in the following two cases: first when $d < d_3$, and later when $d \geq d_3$.

Let us first assume $d < d_3$ so that lemma II.17 can be applied. The fact that $\lambda \leq -\dfrac{1}{2}\Omega^2 d^2$ can be further utilized with (2.29) to give

$$\begin{aligned} \frac{1}{2}\Omega^2 d^2 - \frac{1}{2}\Omega^2 r^2 &\leq B\rho, \\ d - r &\leq \frac{2\|B\rho\|_\infty}{\Omega^2 d}. \end{aligned} \tag{2.40}$$

where $\rho > 0$. Hence by (2.38)

$$\epsilon = \sup\{d - r \mid (r, \theta, z) \in \mathbf{Supp}\rho\} \leq \frac{2\|B\rho\|_\infty}{\Omega^2 d} \leq \frac{2C_2}{\Omega^2} d^{1+\delta} < \frac{d}{2} \tag{2.41}$$

if Ω is sufficiently large. (2.39) in lemma II.17 can now be applied to give

$$\Omega^2 \leq \frac{2\|B\rho\|_\infty}{\epsilon d} \leq 2C_2(\epsilon d)^\delta < 2C_2 d_3^{2\delta}. \tag{2.42}$$

This produces a contradiction when Ω is sufficiently large.

Having dealt with the $d < d_3$ case, let us now assume that $d \geq d_3$. Since $\|B\rho\|_\infty$ is uniformly bounded by lemma II.13 and lemma II.16, we have from (2.40) that

$$d - r \leq \frac{2C}{\Omega^2 d_3} \leq \frac{d_3}{2} \leq \frac{d}{2} \tag{2.43}$$

if Ω is sufficiently large. Therefore we get

$$r \geq \frac{d}{2} \tag{2.44}$$

whenever $\rho(r, \theta, z) > 0$.

Choose a smooth, non-negative, and compactly supported function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^3} g(z)\rho^{\gamma-1} > 0. \tag{2.45}$$

Multiply both sides of (2.29) by $g(z)\frac{1}{r}(r^2\rho^{\gamma-1})_r$ and integrate over $\mathbb{R}^3$. After a few steps of simple estimates and integration by parts, and using (2.44), one gets

$$\left(\frac{\Omega^2 d^2}{C} - C(1+d)\right) \int_{\mathbb{R}^3} g(z)\rho^{\gamma-1} \leq 0 \quad \text{for some constant } C, \tag{2.46}$$

which gives the desired contradiction when Ω is sufficiently large.

As one can see, this proof relies quite heavily on the domain being $\mathbb{R}^3$ (to facilitate integration by parts), and the equation being free of external potential term (to get smallness estimates like lemma II.14 and lemma II.15 for density function supported in a thin cylinder of radius d). In the case of rotating planets with a core potential, a different proof is needed.

2.4 Non-linear Stability

Luo and Smoller [32] obtained a non-linear stability result whose proof relies on a refined version of existence theorem. In this result they use the formulation in which one prescribes $r^2\Omega(r)$ as a function of $m_\rho(r)$, as is explained in section 1.2. In this case, the equilibrium relation is given by (1.36), and the corresponding energy functional is

$$E(\rho) = \int_{\mathbb{R}^3} \left(\big(\rho(\mathbf{x})\big)^\gamma - \frac{1}{2}\rho(\mathbf{x})B\rho(\mathbf{x}) + \frac{1}{2}\rho(\mathbf{x})\frac{j^2\big(m_\rho[r(\mathbf{x})]\big)}{r(\mathbf{x})^2} \right) d\mathbf{x}. \tag{2.47}$$

Compared with (2.4), (2.47) has a very desirable property: it coincides with the physical energy of the system. For a smooth dynamical solution, the physical energy is conserved. For a solution with shock waves, the physical energy can only decrease. This will prove to be crucial for the stability result. With the space of admissible functions W chosen, and a few physically reasonable conditions on the function j, they were able to show the following

Proposition II.18. *Let $\gamma > \dfrac{4}{3}$, $M > 0$ and j be given. If $\{\rho_n\} \in W$ is a minimizing sequence of E, then there exists a sequence of vertical shifts $a_n \boldsymbol{e}_3$, and a subsequence of $\{\rho_n\}$ (still labelled $\{\rho_n\}$), such that the weak limit of $T\rho_n := \rho_n(\boldsymbol{x}+a_n\boldsymbol{e}_3)$ in $L^\gamma(\mathbb{R}^3)$ is a minimizer of E in W. Furthermore $\{\nabla B(T\rho_n)\}$ converges to its limit in the $L^2(\mathbb{R}^3)$ norm.*

The reason we say this is a refined version of existence theorem is that it does not start from a special minimizing sequence as the other existence theorems do. Instead of using the minimizers on finite balls, this theorem asserts that any minimizing sequence contains a convergent subsequence. Its proof relies more explicitly on the concentration compactness principle (compare Lions [30]), in the sense that it does not assert the minimizing sequence to be compactly supported. It suffices to know that an arbitrarily large portion of the mass is contained in a bounded ball to conclude convergence of the energy functional and effective compactness of B.

To explain the ideas leading to the non-linear stability result, let us consider an axisymmetric dynamical solution to the Euler-Poisson equations in the sense of assumption I.3. As before we write the velocity field $\mathbf{v}$ as $\mathbf{v} = v_r(r,z,t)\mathbf{e}_r + v_\theta(r,z,t)\mathbf{e}_\theta + v_z(r,z,t)\mathbf{e}_z$. Let us further assume that the particle trajectory mapping satisfies (1.37), so that $rv_\theta(r,t)$ is given by (1.40) for some fixed function j. In this case the physical energy of the dynamical solution is

$$\begin{aligned} E_1(\rho(t),\mathbf{v}(t)) = \int_{\mathbb{R}^3} \bigg(\big(\rho(\mathbf{x},t)\big)^\gamma - \frac{1}{2}\rho(\mathbf{x},t)B\rho(\mathbf{x},t) \\ + \frac{1}{2}\rho(\mathbf{x},t)\frac{j^2\big(m_{\rho(\cdot,t)}[r(\mathbf{x})]\big)}{r(\mathbf{x})^2} + \frac{1}{2}\rho(v_r^2+v_z^2)(\mathbf{x},t)\bigg)\, d\mathbf{x}. \end{aligned} \tag{2.48}$$

Now consider the rotating star solution $\tilde{\rho}$ produced by proposition II.18 with the same mass M and angular velocity profile j as $\rho(0)$. Since such an equilibrium solution has no v_r and v_z component, its physical energy is the same $E(\tilde{\rho})$ given by (2.47). Because the physical energy is non-increasing, we have

$$E_1(\rho(t),\mathbf{v}(t)) - E(\tilde{\rho}) \le E_1(\rho(0),\mathbf{v}(0)) - E(\tilde{\rho}). \tag{2.49}$$

A careful analyis on $E_1(\rho(t),\mathbf{v}(t)) - E(\tilde{\rho})$ reveals that it has a very nice structure:

$$\begin{aligned} &E_1(\rho(t),\mathbf{v}(t)) - E(\tilde{\rho}) \\ &= d(\rho(t),\tilde{\rho}) + d_1(\rho(t),\tilde{\rho}) - \frac{1}{8\pi}\|\nabla B\rho(\cdot,t) - \nabla B\tilde{\rho}\|_2^2 \\ &\quad + \int_{\mathbb{R}^3} \frac{1}{2}\rho(v_r^2+v_z^2)(\mathbf{x},t)\, d\mathbf{x}. \end{aligned} \tag{2.50}$$

where d and d_1 are both *non-negative* expressions involving $\rho(t)$ and $\tilde{\rho}$. We even have

$$d(\rho(t),\tilde{\rho}) \ge C\|\rho(t)-\tilde{\rho}\|_2^2 \tag{2.51}$$

for $\gamma < 2$. Notice that in (2.50), if the sign in front of $\dfrac{1}{8\pi}$ was positive, we would have obtained a stability result already, because (2.49) says $E_1(\rho(t),\mathbf{v}(t)) - E(\tilde{\rho})$

should stay small if started small. Of course we cannot make this simple argument in reality, but let us take this as a good motivation for defining the following

$$\begin{aligned} &F(\rho(t),\mathbf{v}(t)) \\ &= d(\rho(t),\tilde{\rho}) + d_1(\rho(t),\tilde{\rho}) + \frac{1}{8\pi}\|\nabla B\rho(\cdot,t) - \nabla B\tilde{\rho}\|_2^2 \\ &\quad + \int_{\mathbb{R}^3} \frac{1}{2}\rho(v_r^2 + v_z^2)(\mathbf{x},t)\ d\mathbf{x}. \end{aligned} \tag{2.52}$$

$F(\rho(t),\mathbf{v}(t))$ may grow very large only if $\frac{1}{8\pi}\|\nabla B\rho(\cdot,t) - \nabla B\tilde{\rho}\|_2^2$ grows very large in time. Let us make a heuristic argument to see why $\|\nabla B\rho(\cdot,t) - \nabla B\tilde{\rho}\|_2$ must stay small. In fact, if $\rho(0)$ is very close to $\tilde{\rho}$, $F(\rho(0),\mathbf{v}(0))$ will be small. This implies that $E_1(\rho(0),\mathbf{v}(0)) - E(\tilde{\rho})$ is small, which entails the smallness of $E_1(\rho(t),\mathbf{v}(t)) - E(\tilde{\rho})$. Since by definition $E(\rho(t)) \leq E_1(\rho(t),\mathbf{v}(t))$, we conclude that $E(\rho(t)) - E(\tilde{\rho})$ is small. If we assume $\tilde{\rho}$ to be the unique minimizer of E, proposition II.18, roughly speaking, will imply that $\rho(t)$ is close to $\tilde{\rho}$ in the weak topology on $L^\gamma(\mathbb{R}^3)$ and that $\nabla B\rho(\cdot,t)$ and $\nabla B\tilde{\rho}$ are close in the $L^2(\mathbb{R}^3)$ norm. Therefore $F(\rho(t),\mathbf{v}(t))$ should stay small if started small. One can see that, in order for this line of reasoning to work, proposition II.18 must provide convergence for an arbitrary minimizing sequence, rather than a particularly chosen one as in the other existence proofs.

CHAPTER III

Existence of Rotating Planet Solutions to the Isentropic Euler-Poisson Equations with Core Potential

In this chapter, we establish a number of existence and non-existence theorems for the modified Euler-Poisson equations. Theorems III.1 and III.2 are proved in section 3.3. Theorems III.3 and III.4 are proved in section 3.4. Theorem III.5 is proved in section 3.5.

3.1 Statement of Results

Let us consider the following axisymmetric equilibrium Φ_K-modified Euler-Poisson equations in $\mathbb{R}^3 \setminus K$, for a bounded axisymmetric domain K:

$$\frac{\nabla p}{\rho} = \nabla\big(B\rho + J + \Phi_K\big), \tag{3.1}$$

which is the gradient of the following equilibrium relation

$$A'(\rho) - B\rho - J - \Phi_K = \lambda. \tag{3.2}$$

Here

$$B\rho(\mathbf{x}) = \int_{\mathbb{R}^3\setminus K} \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\ d\mathbf{y} \tag{3.3}$$

is the Newtonian potential of ρ,

$$J(r) = \int_0^r s\Omega^2(s)\ ds, \tag{3.4}$$

where $r = \sqrt{x_1^2 + x_2^2}$, is the potential of centrifugal force, and Φ_K is the potential generated by the core. We assume

$$\text{(3.5)} \qquad s\Omega^2(s) \text{ is a given non-negative function in } L^1[0,\infty) \cap C[0,\infty).$$

If gravity is the only effect of the core, Φ_K is given by

$$\Phi_K(\mathbf{x}) = B\rho_K(\mathbf{x}) = \int_K \frac{\rho_K(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} \tag{3.6}$$

where $\rho_K \in L^q(K)$ for some $q > 3$ is a given axisymmetric non-negative function on K. More generally, Φ_K is a function satisfying

$$\text{(3.7)} \qquad \Phi_K \in C^1(\mathbb{R}^3) \text{ is positive, axisymmetric, and } \lim_{\mathbf{x}\to\infty} \Phi_K(\mathbf{x}) = 0,$$

and

(3.8) there is a $z_0 > 0$ such that if $|x_3| > z_0$, Φ_K is non-increasing as $|x_3|$ increases.

The equation of state is given by $p = f(\rho)$, where f is a function satisfying

$$\text{(3.9)} \qquad f(s) \text{ is non-negative, continuous, and strictly increasing for } s > 0.$$

$$\lim_{s\to 0} f(s)s^{-\frac{4}{3}} = 0, \quad \lim_{s\to\infty} f(s)s^{-\frac{4}{3}} = \infty. \tag{3.10}$$

The internal energy potential A in (3.2) is related to f by

$$A(s) = s\int_0^s \frac{f(t)}{t^2} dt. \tag{3.11}$$

We then have the following

Theorem III.1. *Given $M > 0$, Φ_K satisfying (3.7), and f satisfying (3.9) and (3.10), there is an $\epsilon_1 > 0$, such that if $\|J\|_\infty < \epsilon_1$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus K \to [0,\infty)$, such that*

- *ρ is differentiable where it is positive, and satisfies the Φ_K-modified Euler-Poisson equation* (3.1) *there.*

- $\displaystyle\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x})\ d\mathbf{x} = M.$

Theorem III.2. *Given $M > 0$, Φ_K satisfying* (3.7)*, and f satisfying* (3.9) *and* (3.10)*, there is an $\epsilon_2 > 0$, such that if $\Omega(s) \equiv \Omega < \epsilon_2$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus K \to [0, \infty)$, such that*

- *ρ is differentiable where it is positive, and satisfies the Φ_K-modified Euler-Poisson equation* (3.1) *there.*

- $\displaystyle\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x})\ d\mathbf{x} = M.$

Theorem III.1 and theorem III.2 establish existence of rotating planet solutions with given mass and core potential for sufficiently small angular velocity profile. Furthermore, we have the following

Theorem III.3. *Given $M > 0$, J satisfying* (3.5)*, f satisfying* (3.9) *and* (3.10)*, and Φ_K satisfying* (3.7)*, there is a $\mu_0 > 0$, such that if $\mu > \mu_0$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus C \to [0, \infty)$, such that*

- *ρ is differentiable where it is positive, and satisfies the $\mu\Phi_K$-modified Euler-Poisson equations there.*

- $\displaystyle\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x})\ d\mathbf{x} = M.$

Theorem III.4. *Given $M > 0$, $\Omega(r) \equiv \Omega \geq 0$, f satisfying* (3.9) *and* (3.10)*, and Φ_K satisfying* (3.7)*, there is an $\mu_0 > 0$, such that if $\mu > \mu_0$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus C \to [0, \infty)$, such that*

- *ρ is differentiable where it is positive, and satisfies the $\mu\Phi_K$-modified Euler-Poisson equations there.*

- $\displaystyle\int_{\mathbb{R}^3\setminus K} \rho(\mathbf{x})\ d\mathbf{x} = M.$

Theorem III.3 and theorem III.4 establish existence of rotating planet solutions with given mass and angular velocity profile for sufficiently large core potential. Finally, in order to describe a non-existence theorem for fast constant rotation, we need some further assumptions on the equation of state f.

$$\liminf_{s\to\infty} f(s)s^{-\gamma} > 0, \text{ for some } \gamma > \frac{4}{3}. \tag{3.12}$$

$f(s)$ is continuously differentiable for $s > 0$ and

$$\liminf_{s\to 0} f'(s)s^{-\mu} > 0 \tag{3.13}$$

for some $\mu > 0$.

Theorem III.5. *Suppose f satisfies* (3.9), (3.10), (3.12) *and* (3.13). *Let Φ_K be given by* (3.6), *and let $M > 0$ be given. Also assume that K satisfies the "no trapping" condition:*

- *If $(x, y, z) \in \mathbb{R}^3 \setminus K$, then the half line $(x, y, z) + t(x, y, 0)$, ($t \geq 0$) also belongs to $\mathbb{R}^3 \setminus K$.*

Then there exists an $\Omega_0 > 0$ such that for $\Omega(r) \equiv \Omega > \Omega_0$, there does not exist a bounded continuous function $\rho : \mathbb{R}^3 \setminus C \to [0, \infty)$, such that

- *ρ satisfies* (3.2) *where positive.*

- $\displaystyle\int_{\mathbb{R}^3\setminus K} \rho(\mathbf{x})\ d\mathbf{x} = M.$

3.2 Variational Formulation

As [3] and [28], we will solve this problem via a variational approach. Let us consider the energy functional

$$E(\rho) = \int_{\mathbb{R}^3 \setminus K} \left(A(\rho)(\mathbf{x}) - \frac{1}{2}\rho(\mathbf{x}) B\rho(\mathbf{x}) - \rho(\mathbf{x}) J(\mathbf{x}) - \rho(\mathbf{x})\Phi_K(\mathbf{x}) \right) d\mathbf{x}, \tag{3.14}$$

where A is given by (3.11), on the space of admissible functions

$$W = \left\{ \rho : \mathbb{R}^3 \setminus K \to \mathbb{R},\ \rho \text{ is axisymmetric},\ \rho \geq 0 \text{ a.e.},\ \int_{\mathbb{R}^3 \setminus K} A(\rho) < \infty,\ \int_{\mathbb{R}^3 \setminus K} \rho = M \right\}.$$

We first verify that E is well-defined on W. From (3.10), it follows easily that

$$\lim_{s \to 0} A(s) s^{-\frac{4}{3}} = 0, \quad \lim_{s \to \infty} A(s) s^{-\frac{4}{3}} = \infty. \tag{3.15}$$

(3.9) and (3.15) imply the existence of a $c > 0$ such that

$$A(s) \geq c s^{4/3} \tag{3.16}$$

for $s > 1$. Hence

$$\begin{aligned} \int \rho^{4/3} &\leq \frac{1}{c} \int A(\rho) + \int_{\rho < 1} \rho^{4/3} \\ &\leq \frac{1}{c} \int A(\rho) + M. \end{aligned} \tag{3.17}$$

(3.17) and lemma II.3 give the finiteness of the second term in (3.14). The last two terms in (3.14) are finite because J and Φ_K are bounded functions. We have shown that E is well-defined on W.

The basic assertion is the following:

Proposition III.6. *If ρ is a local minimum for E in W, then ρ is continuous and is differentiable where it is positive, and satisfies* (3.1) *there.*

Proof. The proof is standard. See [3]. □

3.3 Existence of Solution for Slow Rotation and Fixed Core Density

In the following proof, we will come up with a number of bounds R_n on the size of the support of the density functions. Without further mentioning, we always assume that R_{n+1} is no less than R_n. All constants in the following may depend on M, f, $\|J\|_\infty$ and Φ_K. Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ and cylindrical coordinates (r, θ, z) are used interchangeably. To look for a minimizer of E in W, let us first show that E is bounded from below.

Proposition III.7. *There is a $C > 0$ such that $E(\rho) \geq -C$ for all $\rho \in W$.*

Proof. By lemma II.3, we have

$$E(\rho) \geq \int A(\rho)\, d\mathbf{x} - M\|J + \Phi_K\|_\infty - \frac{1}{2} c M^{2/3} \int \rho^{4/3}\, d\mathbf{x}. \tag{3.18}$$

By (3.15), there is an $s > 0$ such that for $\rho > s$, $A(\rho) > \frac{1}{2} c M^{2/3} \rho^{4/3}$. Therefore

$$\begin{aligned} E(\rho) &\geq \int_{\rho > s} A(\rho)\, d\mathbf{x} - M\|J + \Phi_K\|_\infty - \frac{1}{2} c M^{2/3} \int_{\rho > s} \rho^{4/3} - \frac{1}{2} c M^{2/3} s^{1/3} \int_{\rho < s} \rho\, d\mathbf{x} \\ &\geq -M\|J + \Phi_K\|_\infty - \frac{1}{2} c M^{5/3} s^{1/3}. \end{aligned}$$

□

Now that we know E is bounded from below, it makes sense to talk about the infimum of E. Let

$$I = \inf_{\rho \in W} E(\rho). \tag{3.19}$$

We will take a sequence of minimizers in bounded balls as a minimizing sequence for I. For that purpose, we need to define

$$W_R = \left\{ \rho \in W \;\middle|\; \mathbf{Supp}\rho \in S_R,\ 0 \leq \rho \leq R \text{ a.e.} \right\}. \tag{3.20}$$

Here S_R is the ball centered at the origin with radius $R > R_0$ so large that K is contained in S_R. As usual we will extend functions in W_R by zero values outside S_R, and treat them as functions defined on the whole space if necessary. The next assertion is the starting point of this method.

Proposition III.8. *There is an $R_0 > 0$ such that for $R > R_0$, there exists some $\rho_R \in W_R$ which minimizes E:*

$$I_R = E(\rho_R) = \inf_{\rho \in W_R} E(\rho). \tag{3.21}$$

Proof. The proof is standard. See [3] or [28]. □

As in [28], we can give a uniform L^∞ bound on ρ_R.

Lemma III.9. *There is a $C > 0$, such that*

$$\|\rho_R\|_\infty \leq C \tag{3.22}$$

for all $R \geq R_0$.

Proof. Notice that $\Phi_K \in L^\infty(\mathbb{R}^3)$. The proof in this case is basically the same as that in [28]. □

The L^∞ bound frees the restriction on ρ_R from above, and therefore implies a variational inequality in one direction:

Lemma III.10. *There is an $R_1 > 0$, such that for all $R > R_1$, there exists a λ_R such that*

$$A'(\rho_R) - B\rho_R - J - \Phi_K \geq \lambda_R, \quad \text{in } B_R, \tag{3.23}$$

$$A'(\rho_R) - B\rho_R - J - \Phi_K = \lambda_R, \quad \text{where } \rho_R > 0. \tag{3.24}$$

Proof. See [3]. □

Lemma III.11. *There is an $R_2 > 0$ and $e_1 > 0$, such that $I_R \leq -e_1$ for all $R > R_2$.*

Proof. Let

$$F(\rho) = \int_{\mathbb{R}^3 \setminus K} \left(A(\rho)(\mathbf{x}) - \frac{1}{2}\rho(\mathbf{x}) B\rho(\mathbf{x}) \right) d\mathbf{x}. \tag{3.25}$$

This is the corresponding energy functional for an Euler-Poisson system with no rotation and a zero density core. The method in [3] is fully applicable to this case. We therefore get a compactly supported minimizer $\sigma \in W$ of F. Let

$$e_1 = -F(\sigma) = -\inf_{\rho \in W} F(\rho). \tag{3.26}$$

e_1 is seen to be positive by the following scaling argument: pick a non zero $\rho \in W$ that is bounded and compactly supported in $\mathbb{R}^3 \setminus S_{\tilde{R}}$ for some $S_{\tilde{R}} \supset K$. Let

$$\rho_t(\mathbf{x}) = t^{-3} \rho(t^{-1}\mathbf{x}) \tag{3.27}$$

for $t > 1$. We see easily that ρ_t is supported in $\mathbb{R}^3 \setminus tB_{\tilde{R}}$, and therefore belongs to W.

$$\begin{aligned}
F(\rho_t) &= \int_{\mathbb{R}^3 \setminus tB_{\tilde{R}}} A(\rho_t) - \frac{1}{2}\rho_t B \rho_t \\
&= \int_{\mathbb{R}^3 \setminus B_{\tilde{R}}} (t^3 A(t^{-3}\rho) - \frac{1}{2} t^{-1} \rho B \rho) \\
&= \int_{\mathbf{Supp}\rho} o(t^{-4} \|\rho\|_\infty) t^3 - t^{-1} \frac{1}{2} \int \rho B \rho \\
&= o(t^{-1}) - \Theta(t^{-1}).
\end{aligned}$$

The penultimate step follows from (3.15). This shows that the minimum of F must be negative. Now let R_2 be large enough to contain the support of σ, then $\sigma \in W_R$

for $R > R_2$, and

$$\begin{aligned}E(\rho_R) &\le E(\sigma)\\ &= \int (A(\sigma) - \frac{1}{2}\sigma B\sigma - J\sigma - \Phi_K\sigma)\\ &\le \int (A(\sigma) - \frac{1}{2}\sigma B\sigma)\\ &= F(\sigma)\\ &= -e_1.\end{aligned}$$

□

Lemma III.12. *Suppose* $\|J\|_\infty < \dfrac{e_1}{2M}$. *There is an* $\epsilon_0 > 0$ *and an* $R_2 > 0$ *such that for all* $R > R_2$, $\epsilon_R := \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{|\mathbf{x}-\mathbf{y}|<1} \rho_R(\mathbf{y})d\mathbf{y} \ge \epsilon_0$.

Proof. Under the assumption on $\|J\|_\infty$

$$\begin{aligned}&\int \frac{1}{2}\rho_R B\rho_R + \rho_R\Phi_K\\ =& - E(\rho_R) + \int A(\rho_R) - \rho_R J\\ \ge&\quad e_1 - \|J\|_\infty M\\ \ge&\quad \frac{e_1}{2}.\end{aligned}$$

Therefore either

$$\int \frac{1}{2}\rho_R B\rho_R \ge \frac{e_1}{4}, \tag{3.28}$$

or

$$\int \rho_R\Phi_K \ge \frac{e_1}{4}. \tag{3.29}$$

If (3.28) happens, then

$$\frac{e_1}{2} \le \int \rho_R B\rho_R \le M\|B\rho_R\|_\infty. \tag{3.30}$$

Now

$$
\begin{aligned}
B\rho_R(\mathbf{x}) &= \int_{\mathbb{R}^3} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\
&= \int_{|\mathbf{y}-\mathbf{x}|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{1<|\mathbf{y}-\mathbf{x}|<\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{|\mathbf{y}-\mathbf{x}|>\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\
&:= B_1 + B_2 + B_3.
\end{aligned}
$$

By lemma III.9 and lemma II.2, we have

$$B_1 \leq C(\epsilon_R^b + \epsilon_R^c) \tag{3.31}$$

for some $0 < b, c < 1$. The annulus $1 < |\mathbf{y} - \mathbf{x}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius one, hence

$$B_2 \leq C\tilde{R}^3 \epsilon_R. \tag{3.32}$$

One clearly has

$$B_3 \leq \frac{M}{\tilde{R}}. \tag{3.33}$$

Hence

$$\|B\rho_R\|_\infty \leq C(\epsilon_R^b + \epsilon_R^c) + C\tilde{R}^3\epsilon_R + \frac{M}{\tilde{R}}. \tag{3.34}$$

Choosing $\tilde{R}$ sufficiently large and comparing (3.30) with (3.34), we see that there must be an $\epsilon_0 > 0$ such that $\epsilon_R > \epsilon_0$. Now let us assume that (3.29) happens. We have

$$
\begin{aligned}
&\int \rho_R \Phi_K \\
=&\int_{|\mathbf{x}|>\tilde{R}} \rho_R(\mathbf{x})\Phi_K(\mathbf{x}) d\mathbf{x} + \int_{|\mathbf{x}|<\tilde{R}} \rho_R(\mathbf{x})\Phi_K(\mathbf{x}) d\mathbf{x} \\
:=&B_1 + B_2.
\end{aligned}
$$

By (3.7), we can choose $\tilde{R}$ so large that $\Phi_K(\mathbf{x}) \leq \frac{e_1}{8M}$ when $|\mathbf{x}| > \tilde{R}$. Then

$$B_1 \leq \frac{e_1}{8}. \tag{3.35}$$

The ball $|\mathbf{x}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius one, hence

$$B_2 \leq C\tilde{R}^3\epsilon_R. \tag{3.36}$$

Therefore

$$\int \rho_R\Phi_K \leq \frac{e_1}{8} + C\tilde{R}^3\epsilon_R. \tag{3.37}$$

Comparing (3.29) with (3.37), we again see that such an ϵ_0 exists. □

Lemma III.13. *There is an $R_a > 0$ such that if*

$$\int_{|\mathbf{y}-\mathbf{x}|<1} \rho_R(\mathbf{y})d\mathbf{y} \geq \frac{\epsilon_0}{2}, \tag{3.38}$$

then $r(\mathbf{x}) \leq R_a$. Here $r(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$.

Proof. Assume $|r(\mathbf{x})| > \tilde{R} + 1$ where $S_{\tilde{R}} \supset K$. By the axisymmetry of ρ_R,

$$Cr(\mathbf{x})\frac{\epsilon_0}{2} \leq \int_T \rho_R \leq M,$$
$$r(\mathbf{x}) \leq \frac{2M}{C\epsilon_0}.$$

Here T is the torus obtained from rotating the the ball $|\mathbf{y} - \mathbf{x}| < 1$ around the z-axis. □

Lemma III.14. *Suppose $\|J\|_\infty \leq \frac{e_1}{2M}$. There is an $R_3 > R_a$ and an $e_2 > 0$ such that $\lambda_R \leq -e_2$ for all $R > R_2$.*

Proof. By lemma III.12, for $R > R_2$ there is an $\mathbf{x}_R$ such that

$$\int_{|\mathbf{y}-\mathbf{x}_R|<1} \rho_R(\mathbf{y})d\mathbf{y} \geq \frac{\epsilon_0}{2}. \tag{3.39}$$

By lemma III.13, $r(\mathbf{x}_R) < R_a$. Let $\mathbf{x}_0$ be on the z-axis such that $z(\mathbf{x}_0) = z(\mathbf{x}_R)$. Let $B(\mathbf{x}_0, R_3)$ be the ball centered at $\mathbf{x}_0$ with radius $R_3 > R_a$ to be determined. When $R > R_3$, the volume of the set $B(\mathbf{x}_0, R_3) \cap B_R$ is of order R_3^3. There must exist a point $\mathbf{x} \in B(\mathbf{x}_0, R_3) \cap B_R$ such that

$$\rho_R(\mathbf{x}) \leq \frac{CM}{R_3^3} \tag{3.40}$$

for some constant $C > 0$. Clearly

$$|\mathbf{x} - \mathbf{x}_R| \leq |\mathbf{x} - \mathbf{x}_0| + |\mathbf{x}_R - \mathbf{x}_0| \leq 2R_3. \tag{3.41}$$

Hence

$$B\rho_R(\mathbf{x}) \geq \int_{|\mathbf{y}-\mathbf{x}_R|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \geq \frac{1}{2R_3 + 1} \frac{\epsilon_0}{2}. \tag{3.42}$$

By (3.23),

$$\lambda_R \leq A'(\frac{CM}{R_3^3}) - \frac{1}{2R_3 + 1} \frac{\epsilon_0}{2} \tag{3.43}$$

Notice that (3.10) implies

$$\lim_{s \to 0} \frac{A'(s)}{s^{1/3}} = 0. \tag{3.44}$$

Hence (3.43) implies

$$\lambda_R \leq o(R_3^{-1}) - \Theta(R_3^{-1}). \tag{3.45}$$

Pick R_3 so large that the right hand side of (3.45) becomes negative, and call that $-e_2$. □

Lemma III.15. *Suppose* $\|J\|_\infty \leq \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$, *then*

$$B\rho_R + \Phi_K \geq \frac{e_2}{2} \quad \textit{where } \rho_R > 0 \tag{3.46}$$

for $R > R_3$.

Proof. By (3.24) and lemma III.14, we have

$$A'(\rho_R) - B\rho_R - J - \Phi_K = \lambda_R \le -e_2 \tag{3.47}$$

when $\rho_R > 0$. □

Lemma III.16. *Suppose* $\|J\|_\infty \le \min\left\{\dfrac{e_1}{2M}, \dfrac{e_2}{2}\right\}$. *There exists an* $R_4 > 0$ *such that* $\rho_R(\mathbf{x}) = 0$ *if* $R > r(\mathbf{x}) > R_4$.

Proof. We have

$$\begin{aligned} B\rho_R(\mathbf{x}) &= \int \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &= \int_{|\mathbf{x}-\mathbf{y}|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{1<|\mathbf{x}-\mathbf{y}|<\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|>\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &:= B_1 + B_2 + B_3. \end{aligned}$$

Clearly

$$B_3 \le \frac{M}{\tilde{R}}. \tag{3.48}$$

We choose $\tilde{R} > \dfrac{12M}{e_2}$, so that

$$B_3 < \frac{e_2}{12}. \tag{3.49}$$

By lemma II.2,

$$B_1 \le c_0\left(\left(\int_{|\mathbf{x}-\mathbf{y}|<1} \rho_R(\mathbf{y}) d\mathbf{y}\right)^b + \left(\int_{|\mathbf{x}-\mathbf{y}|<1} \rho_R(\mathbf{y}) d\mathbf{y}\right)^c\right) \tag{3.50}$$

for some $0 < b, c < 1$. By requiring $R > r(\mathbf{x}) > R_4$ to be large enough, we have

$$B_1 \le c_0\left(\left(\frac{CM}{R_4}\right)^b + \left(\frac{CM}{R_4}\right)^c\right) < \frac{e_2}{12} \tag{3.51}$$

by axisymmetry, just like in lemma III.13. The annulus $1 < |\mathbf{x}-\mathbf{y}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius 1. Again by axisymmetry, we have

$$B_2 \le \frac{C\tilde{R}^3 M}{R_4 - \tilde{R}} < \frac{e_2}{12}, \tag{3.52}$$

provided R_4 is chosen to be sufficiently large. Therefore

$$B\rho_R(\mathbf{x}) = B_1 + B_2 + B_3 < \frac{e_2}{4} \tag{3.53}$$

if $R > r(\mathbf{x}) > R_4$. Enlarge R_4 if necessary so that $\Phi_K(\mathbf{x}) < \frac{e_2}{4}$ when $r(\mathbf{x}) > R_4$. We get

$$B\rho_R(\mathbf{x}) + \Phi_K(\mathbf{x}) < \frac{e_2}{4} + \frac{e_2}{4} = \frac{e_2}{2} \tag{3.54}$$

when $R > r(\mathbf{x}) > R_4$. Comparing (3.54) with (3.46), we see that the assertion is true. $\square$

Lemma III.17. *Suppose* $\|J\|_\infty \le \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$. *There exist* $R_5 > 0$, $\delta > 0$ *and* $r > 0$ *such that if* $R > z(\mathbf{x}) > R_5$, *and if*

$$\int_{|z(\mathbf{x})-z_0|<r} \rho_R(\mathbf{x})d\mathbf{x} < \delta, \tag{3.55}$$

then $\rho(\mathbf{x}) = 0$ *for* $|z(\mathbf{x}) - z_0| < 1$.

Proof. Suppose $r > 2$. If $|z(\mathbf{x}) - z_0| < 1$, $\operatorname{dist}\big(\mathbf{x}, \{\mathbf{y} \mid |z(\mathbf{y}) - z_0| > r\}\big) > r - 1$. Just like in lemma III.16, we have

$$\begin{aligned} B\rho_R(\mathbf{x}) &= \int_{|z(\mathbf{y})-z_0|<r} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{|z(\mathbf{y})-z_0|>r} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &\le C(\delta^b + \delta^c) + \frac{M}{r-1} \\ &< \frac{e_2}{4} \end{aligned}$$

by choosing δ small and r large. Furthermore $\Phi_K(\mathbf{x}) < \frac{e_2}{4}$ if $z(\mathbf{x}) > R_5$ is sufficiently large. These imply

$$B\rho_R(\mathbf{x}) + \Phi_K(\mathbf{x}) < \frac{e_2}{2}. \tag{3.56}$$

The assertion follows again from a comparison with (3.46). $\square$

Lemma III.18. *Suppose* $\|J\|_\infty \le \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$. *There is an* $R_6 > 0$ *such that* $\rho_R(\mathbf{x}) = 0$ *if* $R > z(\mathbf{x}) > R_6$.

Proof. Let $Z_n = \{x : |z(x) - 2n| < 1\}$, $n = \pm([R_5]+1), \pm([R_5]+2), \dots,$ and let $Z_n' = \left\{x \mid |z(x) - 2n| < r\right\}$. By lemma III.17, if ρ_R is not identically zero on a Z_n, then $\int_{Z_n'} \rho_R \ge \delta$. Let m be the number of such n's. Since each point in $\mathbb{R}^3$ is covered by at most r different Z_n''s, $m\delta \le rM$. Also such Z_n's must be contiguous, if they lie in the region $|z| > z_0 + 2$ for z_0 given in (3.8). Otherwise there would be an "empty" Z_n below a "non-empty" half space. If one slides the whole "non-empty" half space down by two units to create a new ρ_R', $\int A(\rho_R) - J\rho_R = \int A(\rho_R') - J\rho_R'$, but $\int -\frac{1}{2}\rho_R B \rho_R - \rho_R \Phi_K > \int -\frac{1}{2}\rho_R' B \rho_R' - \rho_R' \Phi_K$. This implies $E(\rho_R) > E(\rho_R')$, but $\rho_R' \in W_R$, a contradiction. Now pick $R_6 > 2\left([R_5] + \frac{rM}{\delta}\right) + z_0 + 3$. The proof is complete. □

We are now in a position to prove theorem III.1 and theorem III.2.

Proof of theorem III.1. Let $\epsilon_1 = \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$. From lemma III.16 and lemma III.18, we see that $\rho_R = \rho_{R_7}$ when $R > R_7 := \sqrt{2}R_6$. Since $\Phi_K \in L^\infty(\mathbb{R}^3)$, a similar argument as in [3] shows that $\rho = \rho_{R_7}$ minimizes E in W. By proposition III.6, ρ solves (3.1) and has the stated properties. □

Proof of theorem III.2. Let $\epsilon_2 = \frac{\sqrt{\epsilon_1}}{R_7}$, and let $\tilde{J}(r) \in C^\infty(0,\infty)$ be an increasing function such that

$$\tilde{J}(r) = \begin{cases} \frac{1}{2}\Omega^2 r^2 & \text{if } r \le R_7, \\ \Omega^2 R_7^2 & \text{if } r \ge 2R_7. \end{cases} \tag{3.57}$$

If $\Omega < \epsilon_2$, we have $\|\tilde{J}\| < \epsilon_1$, hence by theorem III.1, there is a solution ρ to (3.1) where J is replaced by $\tilde{J}$, supported in S_{R_7}. Clearly such a ρ also solves (3.1) with the original J, and has the stated properties. $\square$

3.4 Existence of Solution for Fast Rotation and Heavy Core Density

In this section, we give proofs to theorem III.3 and theorem III.4. That corresponds to establishing existence of minimizer of

$$E_\mu(\rho) = \int_{\mathbb{R}^3\setminus C} \left(A(\rho)(\mathbf{x}) - \frac{1}{2}\rho(\mathbf{x})B\rho(\mathbf{x}) - \rho(\mathbf{x})J(\mathbf{x}) - \mu\rho(\mathbf{x})\Phi_K(\mathbf{x}) \right) d\mathbf{x} \tag{3.58}$$

for large enough μ. We will omit an argument in the proof if it runs parallel to the proof in the previous section.

As before, E_μ is bounded from below on W and has an infimum which we denote by I_μ. If we pick

$$W_R = \left\{ \rho \in W \;\middle|\; \mathbf{Supp}\rho \in S_R,\ \rho \geq 0 \text{ a.e.} \right\}. \tag{3.59}$$

E_μ will also attain its infimum $I_{\mu,R}$ on each W_R. We still denote the minimizers by ρ_R. It is understood that ρ_R implicitly depends on μ. Comparing (3.20) with (3.59), we see that the L^∞ bound on W_R (namely, the $\leq R$ constraint) is removed. This is to allow large ρ_R on B_R. As we will see later, the L^∞ bound of ρ_R depends on μ and J. For that purpose, we start by modifying the bound on $\|\rho\|_{4/3}$.

Lemma III.19. *Let ρ_R be a minimizer of E_μ in W_R, and assume that B_{R_0} contains the core K. There is a constant C depending only on f, M, J and Φ_K such that*

$$\int \rho_R^{\frac{4}{3}}\, d\mathbf{x} \leq C(1+\mu) \tag{3.60}$$

for all $R > R_0$.

Proof. Let ρ_0 be some fixed function in W_{R_0}. For $R > R_0$,

$$\begin{aligned}
&\int \left(A(\rho_0) - \rho_0 J - \frac{1}{2}\rho_0 B\rho_0 - \mu\rho_0\Phi_K \right) d\mathbf{x} \\
\geq &\int \left(A(\rho_R) - \rho_R J - \frac{1}{2}\rho_R B\rho_R - \mu\rho_R\Phi_K \right) d\mathbf{x} \\
\geq &\int \Big(A(\rho_R) - \rho_R (J + \mu\Phi_K) \Big) \, d\mathbf{x} - CM^{\frac{2}{3}} \int \rho_R^{\frac{4}{3}} \, d\mathbf{x}.
\end{aligned}$$

The last step follows from lemma II.3. By condition (3.10), there is an $s_1 > 0$ such that

$$A(s)s^{-\frac{4}{3}} > 2CM^{\frac{2}{3}} \tag{3.61}$$

for $s > s_1$. Therefore

$$\begin{aligned}
\tilde{C} = &\int \left(A(\rho_0) - \rho_0 J - \frac{1}{2}\rho_0 B\rho_0 - \mu\rho_0\Phi_K \right) d\mathbf{x} \\
\geq &\int_{\rho_R \leq s_1} A(\rho_R) \, d\mathbf{x} + \int_{\rho_R > s_1} A(\rho_R) \, d\mathbf{x} - M(\|J\|_\infty + \mu\|\Phi_K\|_\infty) \\
&\quad - CM^{\frac{2}{3}} s_1^{\frac{1}{3}} M - \int_{\rho_R > s_1} \frac{1}{2} A(\rho_R) \, d\mathbf{x} \\
\geq &\frac{1}{2} \int A(\rho_R) \, d\mathbf{x} - C'(1 + \mu).
\end{aligned}$$

Or,

$$\int A(\rho_R) \, d\mathbf{x} \leq C(M, s_1)(1 + \mu). \tag{3.62}$$

Notice that we have

$$\begin{aligned}
\int \rho_R^{\frac{4}{3}} \, d\mathbf{x} &= \int_{\rho_R \leq s_1} \rho_R^{\frac{4}{3}} \, d\mathbf{x} + \int_{\rho_R > s_1} \rho_R^{\frac{4}{3}} \, d\mathbf{x} \\
&\leq s_1^{\frac{1}{3}} M + \frac{1}{2CM^{\frac{2}{3}}} \int_{\rho_R > s_1} A(\rho_R) \, d\mathbf{x} \\
&\leq C(M, s_1)\left(1 + \int A(\rho_R) \, d\mathbf{x} \right).
\end{aligned}$$

The assertion is now apparent. $\square$

Now let us give an L^∞ bound on ρ_R. It is crucial to make the power of μ as low as possible.

Lemma III.20. *There is an $R_1 > 0$ and a constant C depending on f, M, J and Φ_K such that*

$$\|\rho_R\|_\infty \le C(1+\mu) \tag{3.63}$$

for $R > R_1$.

Proof. Let $E_R = \{\mathbf{x} \in \mathbb{R}^3 \setminus K \mid \rho_R(\mathbf{x}) > 10M\}$, $F_n = \{\mathbf{x} \in \mathbb{R}^3 \setminus K \mid 10M < \rho_R(\mathbf{x}) < n\}$ for n large. It is easy to see that the Lebesgue measure $|E_R| < \dfrac{1}{10}$. Choose $D \subset B_R \backslash E_R$ such that $|D| = 1$. This is possible if we choose some $R_1 > \max\{R_0, 10\}$. Now let $\gamma_1 = \dfrac{4}{3}$ and $\alpha_1 = \dfrac{5\gamma_1 - 6}{3} - \epsilon = \dfrac{2}{9} - \epsilon$ for some very small $\epsilon > 0$ to be determined later. Now define

$$v_1 = \begin{cases} -\rho_R^{1+\alpha_1} & \text{on } F_n \\ \displaystyle\int_{F_n} \rho_R^{1+\alpha_1} & \text{on } D \\ 0 & \text{otherwise} \end{cases} \tag{3.64}$$

One sees that $\rho_R + tv_1 \in W_R$ for $t > 0$ sufficiently small. Since ρ_R is a minimizer of E_μ in W_R, we have $\displaystyle\lim_{t \to 0^+} \frac{E_\mu(\rho_R + tv_1) - E_\mu(\rho_R)}{t} \ge 0$. Calculating the limit, we get

$$\int (A'(\rho_R) - J - B\rho_R - \mu\Phi_K)v_1 \ge 0, \tag{3.65}$$

from which it follows that

$$-\int_{F_n} v_1 A'(\rho_R) \le \int_D v_1 A'(\rho_R) - \int_{F_n} v_1(J + \mu\Phi_K) - \int_{F_n} v_1 B\rho_R. \tag{3.66}$$

Condition (3.10) on f implies that $A'(s) \ge C_1 \rho^{\frac{1}{3}}$ for $s > 10M$. Therefore

$$-\int_{F_n} v_1 A'(\rho_R) \ge \frac{1}{C_1} \int_{F_n} \rho_R^{\frac{4}{3}+\alpha_1}. \tag{3.67}$$

Furthermore,

$$
\begin{aligned}
\int_D v_1 A'(\rho_R) &\le A'(10M)\int_{F_n}\rho_R^{1+\alpha_1},\\
-\int_{F_n} v_1(J+\mu\Phi_K) &\le (\|J\|_\infty+\mu\|\Phi_K\|_\infty)\int_{F_n}\rho_R^{1+\alpha_1},\\
-\int_{F_n} v_1 B\rho_R &\le \|\rho_R^{1+\alpha_1}\|_{\frac{3\gamma_1}{5\gamma_1-3-3\epsilon}}\|B\rho_R\|_{(\frac{1}{\gamma_1}-\frac{2}{3}+\frac{\epsilon}{\gamma_1})^{-1}}\\
&= \|\rho_R\|^{1+\alpha_1}_{(1+\alpha_1)\frac{3\gamma_1}{5\gamma_1-3-3\epsilon}}\|B\rho_R\|_{(\frac{1}{\gamma_1}-\frac{2}{3}+\frac{\epsilon}{\gamma_1})^{-1}}\\
&= \|\rho_R\|^{1+\alpha_1}_{\gamma_1}\|B\rho_R\|_{(\frac{1}{\gamma_1}-\frac{2}{3}+\frac{\epsilon}{\gamma_1})^{-1}}\\
&\le C\|\rho_R\|^{2+\alpha_1}_{\gamma_1}.
\end{aligned}
$$

Here the last step follows from lemma II.2. Now

$$
\begin{aligned}
&\int_{F_n}\rho_R^{\frac{4}{3}+\alpha_1}\\
\le& C_1(A'(10M)+\|J\|_\infty+\mu\|\Phi_K\|_\infty)\int_{F_n}\rho_R^{1+\alpha_1}+C\|\rho_R\|^{2+\alpha_1}_{\gamma_1}\\
\le& C_2(1+\mu)\|\rho_R\|^{1+\alpha_1}_{1+\alpha_1}+C\|\rho_R\|^{2+\alpha_1}_{\gamma_1}.
\end{aligned}
$$

Since $1+\alpha_1<\gamma_1$, by the interpolation inequality for L^p spaces,

$$
\|\rho_R\|_{1+\alpha_1}\le C(M)\|\rho_R\|^{\frac{4\alpha_1}{1+\alpha_1}}_{\gamma_1}. \tag{3.68}
$$

Hence

$$
\begin{aligned}
&\int_{F_n}\rho_R^{\frac{4}{3}+\alpha_1}\\
\le& C_3(1+\mu)\|\rho_R\|^{4\alpha_1}_{\gamma_1}+C\|\rho_R\|^{2+\alpha_1}_{\gamma_1}\\
\le& C_3(1+\mu)\Big(\int\rho_R^{\frac{4}{3}}\Big)^{3\alpha_1}+C\Big(\int\rho_R^{\frac{4}{3}}\Big)^{\frac{3}{4}(2+\alpha_1)}\\
\le& C_4(1+\mu)^{1+3\alpha_1}+C_4(1+\mu)^{\frac{3}{4}(2+\alpha_1)}\\
\le& 2C_4(1+\mu)^{\frac{5}{3}}.
\end{aligned}
$$

Lemma III.19 is needed for the penultimate step, and the last step follows from the choice of α_1. Now let n tend to infinity. Since the F_n's increase to E_R, one gets

$$\int_{E_R} \rho_R^{\frac{4}{3}+\alpha_1} \leq 2C_4(1+\mu)^{\frac{5}{3}}. \tag{3.69}$$

$$\begin{aligned}\int \rho_R^{\frac{4}{3}+\alpha_1} &= \int_{E_R} \rho_R^{\frac{4}{3}+\alpha_1} + \int_{\rho_R \leq 10M} \rho_R^{\frac{4}{3}+\alpha_1} \\ &\leq 2C_4(1+\mu)^{\frac{5}{3}} + (10M)^{\frac{4}{3}+\alpha_1-1}M \\ &\leq C_5(1+\mu)^{\frac{5}{3}}.\end{aligned}$$

Or,

$$\|\rho_R\|_{\frac{4}{3}+\alpha_1} \leq C_5(1+\mu)^{\frac{5}{4+3\alpha_1}}. \tag{3.70}$$

Here we assumed that we had chosen ϵ so small that

$$\frac{4}{3}+\alpha_1 = \frac{14}{9} - \epsilon > \frac{3}{2}. \tag{3.71}$$

Let $b_1(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\chi_{S_1}(\mathbf{x})$ and $b_2(\mathbf{x}) = \frac{1}{|\mathbf{x}|} - b_1(\mathbf{x})$. We have $B\rho_R = \rho_R * b_1 + \rho_R * b_2$.

$$\|\rho_R * b_2\|_\infty \leq \|b_2\|_\infty \|\rho_R\|_1 \leq C. \tag{3.72}$$

Now let us pick some p between 1 and 2. Assume that we have chosen ϵ so small that the following is true

$$\frac{1}{1-\frac{p}{5}(1+3\alpha_1)} > \frac{3}{2}. \tag{3.73}$$

Notice that since $\alpha_1 = \frac{2}{9} - \epsilon < \frac{2}{9}$, $1-\frac{p}{5}(1+3\alpha_1) > 1-\frac{p}{3} > 0$. (3.73) is equivalent to $\alpha_1 > \frac{1}{3}(\frac{5}{3p}-1)$. Since the right hand side is less than $\frac{1}{3}(\frac{5}{3}-1) = \frac{2}{9}$, this is possible. Now choose q satisfying $q > \frac{3}{2}$, $q < \frac{4}{3}+\alpha_1$, $q < \frac{1}{1-\frac{p}{5}(1+3\alpha_1)}$. That this is possible follows from (3.71) and (3.73). Since $b_1 \in L^{q'}$ for $1 \leq q' < 3$,

$$\begin{aligned}\|\rho_R * b_1\|_\infty &\leq \|b_1\|_{q'} \|\rho_R\|_q \\ &\leq C(M)\|\rho_R\|^a_{\frac{4}{3}+\alpha_1}\end{aligned}$$

where $a = \dfrac{1-\frac{1}{q}}{1-\frac{1}{\frac{4}{3}+\alpha_1}}$, by the interpolation inequality for L^p spaces. Now it follows from (3.70) that

$$\|\rho_R * b_1\|_\infty \le C_6(1+\mu)^{\frac{5a}{4+3\alpha_1}}. \tag{3.74}$$

Combining this with (3.72), we get

$$\|B\rho_R\|_\infty \le C_7(1+\mu)^{\frac{5a}{4+3\alpha_1}}. \tag{3.75}$$

Let us calculate the exponent:

$$\begin{aligned}&\frac{5a}{4+3\alpha_1}\\ =&\frac{1-\frac{1}{q}}{1-\frac{1}{\frac{4}{3}+\alpha_1}}\frac{5}{4+3\alpha_1}\\ =&\frac{5(1-\frac{1}{q})}{1+3\alpha_1} < p\end{aligned}$$

by the choice of q. Therefore

$$\|B\rho_R\|_\infty \le C_7(1+\mu)^p. \tag{3.76}$$

Now if $p \ge 3$, the same inequality is obviously true since it is already true for smaller exponents. Now let $\alpha_{l+1} = \alpha_l + \dfrac{1}{3}$. Define

$$v_l = \begin{cases} -\rho_R^{1+\alpha_l} & \text{on } F_n \\ \displaystyle\int_{F_n} \rho_R^{1+\alpha_l} & \text{on } D \\ 0 & \text{otherwise} \end{cases} \tag{3.77}$$

and repeat the previous argment, only this time using the better estimate (3.76). That gives us

$$\int \rho_R^{\frac{4}{3}+\alpha_l} \le C_8(1+\mu)^p \int \rho_R^{1+\alpha_l}, \tag{3.78}$$

or,

$$\begin{aligned}\int \rho_R^{1+\alpha_{l+1}} &\leq C_8(1+\mu)^p \int \rho_R^{1+\alpha_l} \\ &\leq (C_8(1+\mu)^p)^l \int \rho_R^{1+\alpha_1} \\ &\leq (C_9(1+\mu)^p)^l \int \rho_R^{\frac{4}{3}} \\ &\leq (C_9(1+\mu)^p)^l C(1+\mu).\end{aligned}$$

Therefore

$$\begin{aligned}\|\rho_R\|_\infty &= \lim_{l\to\infty} \|\rho_R\|_{1+\alpha_{l+1}} \\ &\leq \lim_{l\to\infty} (C_9(1+\mu)^p)^{\frac{l}{l+1}} (C(1+\mu))^{\frac{1}{l+1}} \\ &\leq C_9(1+\mu)^p.\end{aligned}$$

We now use this better bound on ρ_R to estimate

$$\begin{aligned}\|\rho_R * b_1\|_\infty &\leq \|b_1\|_2 \|\rho_R\|_2 \\ &\leq C(M)\|\rho_R\|_\infty^{\frac{1}{2}} \\ &\leq C_{10}(1+\mu)^{\frac{p}{2}}.\end{aligned}$$

Hence

$$\|B\rho_R\|_\infty \leq C_{11}(1+\mu)^{\frac{p}{2}}. \tag{3.79}$$

Since we have chosen $p < 2$, this grows at most linearly in μ. We can now repeat the previous bootstrap argument with this better estimate on $\|B\rho_R\|_\infty$. One gets

$$\int \rho_R^{\frac{4}{3}+\alpha_l} \leq C_{12}(1+\mu) \int \rho_R^{1+\alpha_l}, \tag{3.80}$$

and the assertion of the lemma follows. □

ρ_R still satisfies variational equations like (3.23) and (3.24) for $R > R_1$. From here on, we will construct a series of bounds R_n on the support of ρ_R, and a series of lower bounds μ_n for μ. Let us emphasize from the beginning that although the μ_n's depend on f, M, Φ_K and J, the R_n's are independent of J and μ. Also we always take $R_{n+1} \geq R_n$ and $\mu_{n+1} \geq \mu_n$.

Lemma III.21. *There is an $R_2 > 0$ and a $\tilde{K} > 0$ such that $\lambda_R \leq 1 - \mu\tilde{K}$ for $R > R_2$.*

Proof. One first observes that if $R > R_2 > R_1$, there must be a point $\mathbf{x} \in S_{R_2}$ such that

$$\rho_R(\mathbf{x}) \leq \frac{M}{\frac{4}{3}\pi R_2^3}. \tag{3.81}$$

By (3.23),

$$\lambda_R \leq A'\left(\frac{M}{\frac{4}{3}\pi R_2^3}\right) - \mu\Phi_K(\mathbf{x}). \tag{3.82}$$

(3.10) implies that

$$\lim_{s\to 0} \frac{A'(s)}{s^{1/3}} = 0. \tag{3.83}$$

Hence

$$A'\left(\frac{M}{\frac{4}{3}\pi R_2^3}\right) = o(R_2^{-1}). \tag{3.84}$$

Pick R_2 large enough to make $A'\left(\frac{M}{\frac{4}{3}\pi R_2^3}\right) < 1$, and let $\tilde{K} = \inf_{B_{R_2}} \Phi_K > 0$. By (3.82),

$$\lambda_R \leq 1 - \mu\tilde{K}. \tag{3.85}$$

$\square$

Lemma III.22. *There is a $\mu_2 > 0$ such that if $\mu > \mu_2$ and $R > R_2$,*

$$B\rho_R + \mu\Phi_K \geq \frac{\mu\tilde{K}}{2} \quad \textit{where } \rho_R > 0. \tag{3.86}$$

Proof. By (3.24) and lemma III.21,

$$A'(\rho_R) - B\rho_R - J - \mu\Phi_K = \lambda_R \le 1 - \mu\tilde{K} \tag{3.87}$$

where $\rho_R > 0$. Hence

$$B\rho_R + \mu\Phi_K \ge \mu\tilde{K} - 1 - J. \tag{3.88}$$

Pick $\mu_2 > \dfrac{2(1+\|J\|_\infty)}{\tilde{K}}$ to get the result. □

Lemma III.23. *There is a* $\mu_3 > 0$ *and an* $R_3 > 0$ *such that* $\rho_R(\mathbf{x}) = 0$ *if* $R > |\mathbf{x}| > R_3$ *and* $\mu > \mu_3$.

Proof. We only need to prove $B\rho_R + \mu\Phi_K < \dfrac{\mu\tilde{K}}{2}$ in view of (3.86). By (3.79), $\|B\rho_R\|_\infty \le C(1+\mu)^a$ for some $0 < a < 1$. We may choose μ_3 so large that $\dfrac{C(1+\mu)^a}{\mu} < \dfrac{\tilde{K}}{4}$ when $\mu > \mu_3$, and R_3 so large that $\Phi_K(\mathbf{x}) < \dfrac{\tilde{K}}{4}$ when $|\mathbf{x}| > R_3$. The lemma then follows. □

Proof of theorem III.3 and theorem III.4. The argument goes exactly as before. For the constant angular velocity case just notice that the R_3 in lemma III.23 only depends on f, M, Φ_K and not on J and μ, so we can construct a smooth increasing function

$$J(r) = \begin{cases} \dfrac{1}{2}\Omega^2 r^2 & \text{if } r < R_3, \\ \Omega^2 R_3^2 & \text{if } r > 2R_3, \end{cases} \tag{3.89}$$

and find a μ_0 such that a solution exists and is supported in S_{R_3} if $\mu > \mu_0$. □

3.5 Non-existence of Solution for Fast Rotation and Fixed Core Density

We now show that a solution does not exist for large enough constant rotation if the core potential Φ_K is given by the gravity of a density function ρ_K. Let us start with a few estimates.

Lemma III.24. *Let $\rho \in L^\infty(\mathbb{R}^3)$ be a non-negative function such that $\displaystyle\int \rho = M$, then there is a $C > 0$ such that*

$$\|B\rho\|_\infty \le C M^{\frac{2}{3}} \|\rho\|_\infty^{\frac{1}{3}}. \tag{3.90}$$

Proof. See [14]. □

Lemma III.25. *Let $\rho \in L^\infty$ be a nonnegative function supported in the infinite cylinder $x_1^2 + x_2^2 \le d^2$. Then there is a $C > 0$, such that for $x_1^2 + x_2^2 \le d^2$,*

$$|(B\rho)_r(\mathbf{x})| \le C\|\rho\|_\infty \left(d + \sqrt{x_1^2 + x_2^2} \right). \tag{3.91}$$

Here the subscript r denotes directional derivative in the cylindrical radial direction, even if the function under consideration is not axisymmetric.

Proof. Without loss of generality, we may assume $x_1 \ge 0, x_2 = 0, x_3 = 0$.

$$\begin{aligned}
|(B\rho)_r(x_1, 0, 0)| &\le \left| \int_{\mathbf{supp}\rho} \frac{\rho(x_1', x_2', x_3')(x_1 - x_1')}{\sqrt{(x_1' - x_1)^2 + x_2'^2 + x_3'^2}^3} dx_1' dx_2' dx_3' \right| \\
&\le \int_{\mathbf{supp}\rho \cap \{x_1' < x_1\}} \frac{\|\rho\|_\infty (x_1 - x_1')}{\sqrt{(x_1' - x_1)^2 + x_2'^2 + x_3'^2}^3} dx_1' dx_2' dx_3' \\
&\le \|\rho\|_\infty \int_{-d < x_1' < x_1} \frac{x_1 - x_1'}{\sqrt{(x_1' - x_1)^2 + x_2'^2 + x_3'^2}^3} dx_1' dx_2' dx_3' \\
&= C\|\rho\|_\infty (d + x_1).
\end{aligned}$$

The last equality follows either from a direct calculation or a simple application of the divergence theorem. □

Lemma III.26. *Let $l = \sup\big\{|x_3| \;\big|\; (x_1, x_2, x_3) \in K\big\} + 1$, $Z = \big\{(x_1, x_2, x_3) \;\big|\; |x_3| \le l\big\}$. Then $\Phi_K|_Z = B\rho_K|_Z \in C^{1,\frac{3}{q}}(\bar{Z})$, $\Phi_K|_{\mathbb{R}^3 \setminus Z} = B\rho_c|_{\mathbb{R}^3 \setminus Z} \in C^{1,1}(\overline{\mathbb{R}^3 \setminus Z})$.*

Proof. We first estimate $\Phi_K|_{\mathbb{R}^3 \setminus Z}$:

$$\Phi_K(x_1, x_2, x_3) = \int_{\mathbf{Supp}\rho_K} \frac{\rho_K(x_1', x_2', x_3')}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}} dx_1' dx_2' dx_3'.$$

Since (x_1, x_2, x_3) is bounded away from $\mathbf{Supp}\rho_K$, we can differentiate under the integral sign and see that

$$\begin{aligned}|D\Phi_K(x,y,z)| &\leq C\int_{\mathbf{Supp}\rho_K}\frac{\rho_K(x_1', x_2', x_3')}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}^2}dx_1'dx_2'dx_3' \\ &\leq C\int_{\mathbf{Supp}\rho_K}\rho_K(x_1', x_2', x_3')dx_1'dx_2'dx_3' \\ &\leq C\|\rho_K\|_1 \\ &\leq \tilde{C}\|\rho_K\|_q^q.\end{aligned}$$

In the above inequalities, the second line is because $|x_3 - x_3'| \geq 1$, the last line is because $\mathbf{Supp}\rho_K$ is compact. We can give a similar estimate for $D^2\Phi_K$, therefore $\Phi_K|_{\mathbb{R}^3\setminus Z} \in C^{1,1}(\mathbb{R}^3 \setminus Z)$. As for $\Phi_K|_Z$, the Lipschitz continuity of the first derivative in a neighborhood of ∞ follows in the same way as above, whereas the Hölder continuity of the first derivative in a neighborhood of $\mathbf{Supp}\ \rho_K$ follows from the standard Calderon-Zygmund inequality and the Sobolev embedding theorem. □

From now on, we assume Ω is at least 1 and use cylindrical coordinates (r, θ, z). Let us suppose, contrary to the assertion of theorem III.5, that there is such a ρ satisfying all the properties stated.

Lemma III.27. *$d = \sup\{r \mid (r, \theta, z) \in \mathbf{Supp}\rho\} < \infty$.*

Proof. By (3.2),

$$\frac{1}{2}r^2 \leq \frac{1}{2}\Omega^2 r^2 \leq A'(\rho) - B\rho - \Phi_K - \lambda \leq A'(\rho) - \lambda. \tag{3.92}$$

We know that $A'(s) = \displaystyle\int_0^s \frac{f(t)}{t^2}dt + \frac{f(s)}{s}$. It follows from (3.9) and (3.10) that $A'(\rho) \in L^\infty$ if ρ is. □

By the expression of $A'(s)$ in the proof, we see that $A'(\rho) > 0$ iff $\rho > 0$.

Lemma III.28. $\lambda \leq -\frac{1}{2}\Omega^2 d^2$.

Proof. Pick a sequence (r_n, θ_n, z_n) such that $\rho(r_n, \theta_n, z_n) > 0$, and $r_n \to d$. We claim that $A'(\rho)(r_n, \theta_n, z_n) \to 0$. If not, a subsequence will be bounded away from zero. Without loss of generality, we still call that subsequence $A'(\rho)(r_n, \theta_n, z_n)$. By the no trapping condition, $A'(\rho)(r, \theta_n, z_n)$ is defined for all $r > r_n$, in particular we have $A'(\rho)(d, \theta_n, z_n) = 0$. By Rolle's theorem there is an r_n^* between r_n and d such that $A'(\rho)(r_n^*, \theta_n, z_n) > 0$ and $(A'(\rho))_r(r_n^*, \theta_n, z_n) \to -\infty$. By (3.2) and the smoothing effect of B, $A'(\rho)$ is differentiable when positive. Differentiating (3.2), we get

$$(A'(\rho))_r - \Omega^2 r - (B\rho)_r - (\Phi_c)_r = 0. \tag{3.93}$$

We see a contradiction if we evaluate this expression at (r_n^*, θ_n, z_n): the first term goes to $-\infty$ while the last three terms are bounded by lemma III.27, lemma III.25 and lemma III.26 respectively. Now evaluate (3.2) at (r_n, θ_n, z_n). By the limit of $A'(\rho)(r_n, \theta_n, z_n)$ and the positivity of $B\rho$ and Φ_K, we get the desired result. □

Lemma III.29. *There is a constant $C_1 > 0$, depending on Φ_K, f and M, such that $\|\rho\|_\infty \leq C_1$.*

Proof. By lemma III.28,

$$A'(\rho) \leq B\rho + \Phi_K. \tag{3.94}$$

By (3.12), there is a $C > 0$ such that if $s > C$,

$$Cs^{\gamma-1} \leq A'(s). \tag{3.95}$$

Hence either $\rho < C$ or $C\rho^{\gamma-1} \leq B\rho + \Phi_K$. Therefore

$$C\|\rho\|_\infty^{\gamma-1} \leq C^\gamma + \|\Phi_K\|_\infty + CM^{\frac{2}{3}}\|\rho\|_\infty^{\frac{1}{3}}. \tag{3.96}$$

The last term follows from lemma III.24. Here we have taken the liberty of using the same constant C. Now take $\epsilon > 0$ so small that $\epsilon M^{\frac{2}{3}} < \frac{1}{2}$. Since $\gamma - 1 > \frac{1}{3}$, we have

$$\|\rho\|_{\infty}^{\frac{1}{3}} \le \epsilon \|\rho\|_{\infty}^{\gamma-1} + C(\epsilon). \tag{3.97}$$

It follows that

$$\begin{aligned}
C\|\rho\|_{\infty}^{\gamma-1} &\le C^{\gamma} + \|\Phi_K\|_{\infty} + \frac{1}{2}C\|\rho\|_{\infty}^{\gamma-1} + C(M) \\
\frac{1}{2}C\|\rho\|_{\infty}^{\gamma-1} &\le C^{\gamma} + \|\Phi_K\|_{\infty} + C(M).
\end{aligned}$$

The assertion now follows from the fact that $\Phi_K \in L^{\infty}(\mathbb{R}^3)$. □

Lemma III.30. *There is an $\Omega_1 > 0$ and $0 < d_0 < \frac{1}{4}$ such that if $\Omega > \Omega_1$, then $d < d_0$.*

Proof. Pick an (r, θ, z) such that $\rho(r, \theta, z) > 0$, $r > \frac{d}{2}$. Then there is an r^* between r and d such that $(A'(\rho))_r(r^*, \theta, z) \le 0$. Evaluating (3.93) at this point, we have

$$\begin{aligned}
\Omega^2 \frac{d}{2} &\le \Omega^2 r^* \\
&\le (B\rho)_r(r^*, \theta, z) + (\Phi_K)_r(r^*, \theta, z).
\end{aligned}$$

The first term above is bounded by $2CC_1d$ by lemma III.25 and III.29. Noticing $(\Phi_K)_r(0, \theta, z) = 0$ by axisymmetry, the second term above is therefore bounded by $Cd^{\frac{3}{q}}$ by lemma III.26. Now we have

$$\begin{aligned}
\Omega^2 \frac{d}{2} &\le \tilde{C}(d + d^{\frac{3}{q}}) \\
\left(\frac{\Omega^2}{2} - \tilde{C}\right) d^{1-\frac{3}{q}} &\le \tilde{C},
\end{aligned} \tag{3.98}$$

and the assertion follows. □

Lemma III.31. *$\rho \in C^{0,\alpha}(\bar{S})$ for some $0 < \alpha < 1$, where S is any ball of radius $\frac{1}{2}$ whose center is on $(\mathbb{R}^3 \setminus Z) \cap x_3$-axis, and we have $\|\rho\|_{C^{0,\alpha}(\bar{S})} \le C_2$. Here C_2 is a constant depending on Φ_K f and M, and Z is the region given in lemma III.26.*

Proof. We first observe that since $A''(s) = \dfrac{f'(s)}{s}$, $A'(s)$ is strictly increasing. Also notice that $A'(0) = 0$. By (3.2), $A'(\rho)$ is uniformly Lipschitz continuous on $S \cap \{\rho > 0\}$ and continuous on S, hence is uniformly Lipschitz continuous on S. It is sufficient to prove

$$|\rho(\mathbf{x}) - \rho(\mathbf{y})| \leq \tilde{C}_2 |A'(\rho(\mathbf{x})) - A'(\rho(\mathbf{y}))|^{\alpha},$$

or

$$A'(t) - A'(s) \geq \tilde{C}_2 (t-s)^{\frac{1}{\alpha}} \tag{3.99}$$

for $0 \leq s < t \leq \|\rho\|_\infty \leq C_1$. By (3.13) there exists a $C > 0$ such that $f'(s) \geq Cs^{\mu}$ for $0 \leq s \leq \|\rho\|_\infty \leq C_1$. Now let $u = t - s$,

$$\begin{aligned}
&(A'(t) - A'(s))(t-s)^{-\frac{1}{\alpha}} \\
=&(A'(s+u) - A'(s))u^{-\frac{1}{\alpha}} \\
=&u^{-\frac{1}{\alpha}} \int_s^{s+u} A''(\xi) d\xi \\
=&u^{-\frac{1}{\alpha}} \int_s^{s+u} \frac{f'(\xi)}{\xi} d\xi \\
\geq&u^{-\frac{1}{\alpha}} \int_s^{s+u} \frac{C\xi^{\mu}}{\xi} d\xi \\
=&\tilde{C} u^{-\frac{1}{\alpha}} ((s+u)^{\mu} - s^{\mu})
\end{aligned} \tag{3.100}$$

If $\mu \geq 1$, (3.100) is equal to

$$\begin{aligned}
&\tilde{C} \left[\left(1 + \frac{s}{u}\right)^{\mu} - \left(\frac{s}{u}\right)^{\mu} \right] u^{\mu - \frac{1}{\alpha}} \\
\geq&\tilde{C} u^{\mu - \frac{1}{\alpha}} \\
\geq&\tilde{C} C_1^{\mu - \frac{1}{\alpha}} \geq \tilde{C}_2 > 0.
\end{aligned}$$

The last step is correct if we choose $\alpha < \dfrac{1}{\mu}$. On the other hand if $0 < \mu < 1$, (3.100) is equal to $\tilde{C}\mu\xi^{\mu-1}u^{1-\frac{1}{\alpha}}$, where ξ is between s and $s+u$. This in turn is greater than or equal to

$$\tilde{C}\mu C_1^{\mu-1}u^{1-\frac{1}{\alpha}} \geq \tilde{C}\mu C_1^{\mu-1}C_1^{1-\frac{1}{\alpha}} \geq \tilde{C}_2 > 0$$

if we choose an $\alpha < 1$. $\square$

Lemma III.32. *There is an $\Omega_2 > 0$ such that if $\Omega > \Omega_2$, $\rho_{\mathbb{R}^3\setminus Z} \equiv 0$. Here Z is the region given in lemma III.26.*

Proof. We first show that $\|B\rho\|_{C^{1,1}(\bar{S}_1)}$ is uniformly bounded, where S_1 is any ball of radius $\frac{1}{4}$ whose center is on $(\mathbb{R}^3 \setminus Z) \cap x_3$-axis. Let S be a ball concentric with S_1 of radius $\frac{1}{2}$, then $B\rho = B(\rho\chi_S) + B(\rho\chi_{\mathbb{R}^3\setminus S})$. The first term is bounded in $C^{2,\alpha}(\bar{S}_1)$ by lemma III.31 and elliptic Schauder estimates. The second term is bounded in $C^2(\bar{S}_1)$ by a direct differentiation under the integral sign argument since $\mathbb{R}^3 \setminus S$ is bounded away from S_1.

We first pick $\Omega > \Omega_1$ so that $d < d_0 < \frac{1}{4}$. Now suppose $\rho(r,\theta,z) > 0$ for some (r,θ,z) in $\mathbb{R}^3 \setminus Z$. Let us switch to Cartesian coordinates for the moment and, without loss of generality, denote this point $(x,0,z)$ with $x \geq 0$. Let $x^* = \sup\{x \mid \rho(x,0,z) > 0\}$. There must be a sequence $x_n \to x^*$ such that $\rho(x_n,0,z) > 0$ and $(A'(\rho))_x(x_n,0,z) \leq 0$, differentiating (3.2) with respect to x and evaluating at $(x_n,0,z)$, we have

$$\Omega^2 x_n \leq -(B\rho)_x(x_n,0,z) - (\Phi_K)_x(x_n,0,z). \tag{3.101}$$

Taking limit as $n \to \infty$, we get

$$\Omega^2 x^* \leq -(B\rho)_x(x^*,0,z) - (\Phi_K)_x(x^*,0,z). \tag{3.102}$$

If there was an $x_0 \in [0,x^*)$ such that $\rho(x_0,0,z) > 0$ and $(A'(\rho))_x(x_0,0,z) \geq 0$, we would have

$$\Omega^2 x_0 \geq -(B\rho)_x(x_0,0,z) - (\Phi_K)_x(x_0,0,z). \tag{3.103}$$

Subtracting (3.103) from (3.102), we get

$$\Omega^2(x^* - x_0) \leq \big((B\rho)_x(x_0, 0, z) - (B\rho)_x(x^*, 0, z)\big)$$

$$+\big((\Phi_K)_x(x_0, 0, z) - (\Phi_K)_x(x^*, 0, z)\big). \tag{3.104}$$

The first term on the right hand side is bounded by $C(x^* - x_0)$ because $B\rho$ is uniformly bounded in $C^{1,1}(\bar{S}_1)$ as indicated above, while the second term is bounded by $C(x^* - x_0)$ because of lemma III.26. Hence (3.104) becomes

$$\Omega^2(x^* - x_0) \leq 2C(x^* - x_0), \tag{3.105}$$

which is impossible if we choose $\Omega_2 > \max\{\Omega_1, 2C\}$. Therefore such an x_0 does not exist. This in particular implies that there is no $x \in [0, x^*)$ for which $\rho(x, 0, z) = 0$, which then implies that $\rho(0, 0, z) > 0$ and $(A'(\rho))_x(0, 0, z) < 0$. But exactly the same argument in the $-x$ direction would imply $(A'(\rho))_x(0, 0, z) > 0$. This contradiction indicates that there is no such (r, θ, z) in the first place, and the assertion is therefore true. □

We are now ready to give

Proof of theorem III.5. By lemma III.32, $\mathbf{Supp}\rho$ is uniformly bounded in the z direction. Recall from lemma III.26 that this bound is given by l,

$$\begin{aligned} M &= \int_{\mathbf{Supp}\rho} \rho \\ &= \int_{|z| \leq l, r \leq d} \rho \\ &\leq \|\rho\|_\infty 2\pi d^2 l \\ &\leq 2C_1 \pi d^2 l. \end{aligned} \tag{3.106}$$

Therefore

$$d \geq \sqrt{\frac{M}{2C_1 \pi l}}. \tag{3.107}$$

Compare this with (3.98), we get

$$(\frac{\Omega^2}{2} - \tilde{C})\sqrt{\frac{M}{2C_1\pi l}}^{1-\frac{3}{q}} \leq \tilde{C}, \tag{3.108}$$

which is clearly false if we choose $\Omega_0 > \Omega_2$ sufficiently large. This contradiction indicates that such a solution ρ does not exist. $\square$

CHAPTER IV

Existence of Rotating Star Solutions to the Non-isentropic Euler-Poisson Equations

In this chapter, we establish a number of existence theorems related to the non-isentropic Euler-Poisson equations. Theorems IV.1 is proved in section 4.2. Theorems IV.2 and IV.3 are proved in section 4.4. Theorems IV.5 and IV.7 are proved in section 4.5.

4.1 Statement of Results

Let us consider the following axisymmetric equilibrium non-isentropic Euler-Poisson equation in $\mathbb{R}^3$:

$$\frac{\nabla p}{\rho} = B\rho + r\Omega^2 \mathbf{e}_r \tag{4.1}$$

with equation of state

$$p = e^s \rho^\gamma. \tag{4.2}$$

Here γ is a constant called the adiabatic index. The divergence of (4.1) is

$$\nabla \cdot \left(\frac{\nabla(e^s \rho^\gamma)}{\rho} \right) = -4\pi\rho + \nabla \cdot (r\Omega^2 \mathbf{e}_r). \tag{4.3}$$

After the change of variable

$$w = \frac{\gamma}{\gamma - 1} e^{\frac{\gamma-1}{\gamma}s} \rho^{\gamma-1}, \tag{4.4}$$

(4.3) becomes

$$\nabla \cdot (e^{\alpha s} \nabla w) + K e^{-\alpha s} |w|^q - f = 0, \tag{4.5}$$

where

$$q = \frac{1}{\gamma - 1}, \quad \alpha = \frac{1}{\gamma}, \quad K = 4\pi \left(\frac{\gamma - 1}{\gamma} \right)^{\frac{1}{\gamma - 1}}, \tag{4.6}$$

and

$$f = 2\Omega^2 + r \frac{\partial \Omega^2}{\partial r} = 2\Omega \frac{\partial}{\partial r}(r\Omega). \tag{4.7}$$

We have the following

Theorem IV.1. *Let f and s be given axisymmetric smooth functions. If $0 < q < 1$ $(\gamma > 2)$, and $\mathbf{x} \cdot \nabla s \leq 0$, then there is a finite ball centered at the origin on which there exists an axisymmetric positive smooth solution to* (4.5) *with zero boundary value.*

The condition on entropy has the physical interpretation that it is decreasing in the radial direction, so that the star is thermally more active the further one goes down surface.

The $q > 1$ case is more difficult. Let us take a look at a simple ODE model $u'' + \lambda u^q = 0$. Suppose $q > 1$. In order for u to stay positive, symmetric about the origin, and be zero on the boundary of a given symmetric domain, $u(0)$ will be unbounded as λ gets close to 0. Therefore there is no a priori bound for Leray-Schauder type arguments. However, if one is allowed to rescale the velocity field, the equation can still be solved. The results are as follows:

Theorem IV.2. *Let f and s be given axisymmetric smooth bounded functions. Suppose there exists a $c > 0$ such that $f \geq c$, and suppose $1 < q < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$, then* *for any $R > 0$, and sufficiently large $P > 0$, there exists a non-negative axisymmetric function w in $H_0^1(S_R)$, and a $\lambda > 0$, such that w is smooth on its own positive set and satisfies*

$$\nabla \cdot (e^{\alpha s} \nabla w) + K e^{-\alpha s} w^q - \lambda f = 0, \tag{4.8}$$

and

$$\int_{S_R} f w \, d\mathbf{x} = P. \tag{4.9}$$

Here S_R is a ball of radius R centered at the origin. The positive set of w will turn out to be open, so there is no ambiguity in defining (4.8). Since $\lambda > 0$, (4.5) is solved with a rescaled the velocity field (compare (4.7)). Also

$$\begin{aligned} P &= \int_{S_R} f w \, d\mathbf{x} \\ &\leq \int_{S_R} C w \, d\mathbf{x} \\ &\leq \tilde{C} \int_{S_R} \rho^{\gamma-1} \, d\mathbf{x} \\ &\leq \tilde{\tilde{C}} \left(\int_{S_R} \rho \, d\mathbf{x} \right)^{\gamma-1}. \end{aligned}$$

Therefore the largeness of P implies the largeness of the total mass in this case.

The method for deriving this result is variational. It is possible to extend the variational method to allow functions defined on the entire $\mathbb{R}^3$ once we find a way to address the lost of compactness issue.

Theorem IV.3. *Let f and s be given axisymmetric smooth functions. Suppose s is bounded, $f \geq c > 0$, and $1 < q < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$, then for sufficiently large $P > 0$, there exists a non-negative axisymmetric function w in $H^1(\mathbb{R}^3)$, and a $\lambda > 0$, such* *that w is smooth on its own positive set, and satisfies* (4.8) *and*

$$\int_{\mathbb{R}^3} fw \, d\mathbf{x} = P. \tag{4.10}$$

Another way of investigating solutions to (4.1) is by prescribing ρ and solving for p and Ω^2. Apart from being suitably smooth, an obvious requirement for p and Ω^2 is that they should be positive where ρ is positive. Furthermore, p should be zero on the boundary of the positive set of ρ. It is possible to develop conditions on ρ that will guarantee the existence of such p and Ω^2. To find out what conditions on ρ are natural, we observe some features of the classical Auchmuty and Beals solutions with isentropic equation of state. In [6], Caffarelli and Friedman studied the shapes of the Auchmuty and Beals solutions. Some of their results can be summarized as follows:

Proposition IV.4. *Assume Ω^2 is analytic, and the equation of state is given by*

$$p = c\rho^\gamma \tag{4.11}$$

for some $\frac{4}{3} < \gamma < 2$ ($1 < q < 3$). Then the Auchmuty and Beals solution ρ to (4.1) *has the following properties:*

1. *Let $D = \left\{\mathbf{x} \in \mathbb{R}^3 \;\middle|\; \rho(\mathbf{x}) > 0\right\}$, then $\bar{D}$ is compact, ∂D is smooth and D is a finite union of sets of the form $\left\{(r,z) \;\middle|\; 0 \le a < r < b, |z| < \psi(r)\right\}$, where ψ is a function vanishing at the end points except if $a = 0$. $\rho \in C^{0,\beta}(\mathbb{R}^3) \cap C^\infty(D)$ for some $\beta > 0$.*
2. *$\rho(r,z) = \rho(r,-z)$.*
3. *$\rho_z(r,z) > 0$ for $(r,z) \in D$, $r > 0$, and $z < 0$.*
4. *$\rho_{zz}(r,0) < 0$ for $(r,0) \in D$.*

Motivated by this result, we will prove the following

Theorem IV.5. *Let ρ be an axisymmetric nonnegative function such that*

1. *$\rho \in C^k(\bar{D})$ $(k \geq 2)$, where D is a finite union of sets of the form $\{(r,z) \mid 0 \leq a < r < b, |z| < \psi(r)\}$, where ψ is a function vanishing at the end points except if $a = 0$. Also assume ∂D is smooth, $\rho > 0$ on D, $\rho = 0$ on ∂D.*
2. *$\rho(r,z) = \rho(r,-z)$.*
3. *$\rho_r(B\rho)_z - \rho_z(B\rho)_r \geq 0$ for $z < 0$.*
4. *$\rho_z > 0$ for $z < 0$.*

Also assume the following is satisfied:

(a) There is a $c > 0$, such that $\rho_{zz} < -c$ on $\{z = 0\} \cap \partial D$.

Then (4.1) *is solvable for a nonnegative angular velocity function $\Omega^2 \in C^{k-2}(D) \cap C^0(\bar{D})$ and a positive pressure $p \in C^k(\bar{D})$, such that $p = 0$ on ∂D.*

Remark IV.6. If $\nabla\rho$ and $\nabla(B\rho)$ point approximately to the center of the star, condition 3 in theorem IV.5 means that the gradient of ρ is more inclined with respect to the plane $\{z = 0\}$ than the gravity force. Simple calculations with ellipsoids suggest that shapes that are wider at the equator tend to satisfy condition 3.

It is desirable to relax the regularity conditions of ρ at the boundary, since for some γ, the Auchmuty and Beals solutions are only Hölder continuous at the boundary. A similar result with weaker boundary regularity needs more control on the derivatives when close to the boundary. Here is one way of formulating the conditions:

Theorem IV.7. *Let ρ be an axisymmetric nonnegative function such that*

1. *$\rho \in C^2(D) \cap C^{0,\beta}(\bar{D})$, for some $0 < \beta < 1$, where D is a finite union of sets of the form $\{(r,z) \mid 0 \leq a < r < b, |z| < \psi(r)\}$, where ψ is a function vanishing at the end points except if $a = 0$. Also assume that ∂D is smooth, convex at $(0, \pm\psi(0)) \in \partial D$ (if there are such points), i.e., the interior of the segment*

$(0,\psi(0)) - (r,\psi(r))$ lies in D for r sufficiently small. $\rho > 0$ on D, $\rho = 0$ on ∂D.

2. *$\rho(r,z) = \rho(r,-z)$.*

3. *$\rho_r(B\rho)_z - \rho_z(B\rho)_r \geq 0$ for $z < 0$.*

4. *$\rho_z > 0$ for $z < 0$.*

5. *$\forall \epsilon > 0$, $\exists C > 0$ such that on $D \cap \{|z| \geq \epsilon\}$: $|\rho_r| \leq C|\rho_z|$, $|\rho_{rr}| \leq C|\rho_z|$, $|\rho_{rz}| \leq C|\rho_z|$.*

Also assume that one of the following is satisfied:

(a) $\dfrac{\rho_{rz}}{\rho_{zz}}$ and $\dfrac{\rho_r}{\rho_{zz}}$ are bounded in a neighbourhood of $\{z = 0\} \cap \partial D$.

(a') $\rho_r \leq 0$ in a neighbourhood of $\{z = 0\} \cap \partial D$.

(a") $\dfrac{z\rho_r}{\rho_z}$ is bounded on $U \setminus \{z = 0\}$, where U is some neighbourhood of $\{z = 0\} \cap \partial D$.

Then (4.1) is solvable for a nonnegative angular velocity function $\Omega^2 \in C^0(D) \cap L^\infty(D)$ and a positive pressure $p \in C^1(D) \cap C^0(\bar{D})$, such that $p = 0$ on ∂D.

Remark IV.8. If D has only one connected component containing the origin, $\{z = 0\} \cap \partial D$ is the equator, and since ρ is zero on the equator and positive in the interior of D, the condition (a') is most likely satisfied in this case. Condition (a") is equivalently to $\dfrac{z}{r}\Big/\dfrac{\rho_z}{\rho_r}$ being bounded on $U \setminus \{z = 0\}$ and has the geometrical interpretation that the when $\mathbf{x}$ gets close to $\{z = 0\} \cap \partial D$, the inclination of $\mathbf{x}$ to the horizontal plane is bounded by the inclination of $\nabla\rho(\mathbf{x})$.

4.2 Existence of Solution for High Adiabatic Index

Without loss of generality, we may absorb α into s in (4.5) and work with

$$\nabla \cdot (e^s \nabla w) + K e^{-s} w^q - f = 0. \tag{4.12}$$

We first find a subsolution to this equation.

Lemma IV.9. *If $\mathbf{x} \cdot \nabla s \leq 0$, there is a ball of radius R, denoted by S_R, centered at the origin, on which there is a smooth spherically symmetric positive function $\underline{u}$ with zero boundary value satisfying*

$$\nabla \cdot (e^s \nabla \underline{u}) + K e^{-s} \underline{u}^q - f \geq 0. \tag{4.13}$$

Proof. Let A_1, A_2 be two positive constants such that $Ke^{-2s} \geq A_1$, $e^{-s} f \leq A_2$. We look for a positive function $\underline{u}$ on a ball which satisfies

$$\Delta \underline{u} + A_1 \underline{u}^q - A_2 \geq 0. \tag{4.14}$$

By lemma 3.1 in [41], we only need to check that the primitive of $g(t) = A_1 t^q - A_2$, which is $G(t) = \dfrac{A_1}{q+1} t^{q+1} - A_2 t$, satisfies $G(t) > 0$ for some $t > 0$. But this is certainly true for large enough t. It follows that there is a ball of radius R, and a spherically symmetric positive solution $\underline{u}$ of (4.14) on this ball with zero boundary value, which satisfies $\mathbf{x} \cdot \nabla \underline{u} < 0$. By the definition of A_1 and A_2, we have

$$\Delta \underline{u} + K e^{-2s} \underline{u}^q - e^{-s} f \geq \Delta \underline{u} + A_1 \underline{u}^q - A_2 \geq 0. \tag{4.15}$$

Furthermore, by

$$\nabla \underline{u} = -\frac{|\nabla \underline{u}|}{|\mathbf{x}|} \mathbf{x}, \tag{4.16}$$

we have

$$\nabla s \cdot \nabla \underline{u} = -(\mathbf{x} \cdot \nabla s) \frac{|\nabla \underline{u}|}{|\mathbf{x}|} \geq 0. \tag{4.17}$$

Therefore,

$$\Delta \underline{u} + \nabla s \cdot \nabla \underline{u} + K e^{-2s} \underline{u}^q - e^{-s} f \geq 0, \tag{4.18}$$

which differs from (4.13) only by a factor of e^{-s}. Hence the assertion is proved. $\square$

Having found a subsolution to (4.12), we now only need a supersolution to produce a genuine solution. That is given by

Lemma IV.10. *Suppose* $0 < q < 1$. *There is a smooth positive function* $\bar{u}$ *on* $\overline{S_R}$, *such that* $\bar{u} \geq \underline{u}$ *on* $\overline{S_R}$, *and satisfies*

$$\nabla \cdot (e^s \nabla \bar{u}) + K e^{-s} \bar{u}^q - f \leq 0. \tag{4.19}$$

Proof. Let $C = \|\underline{u}\|_{L^\infty(S_R)}$, and $M = \|f\|_{L^\infty(S_R)}$. Let $g(t) \geq 0$ be a smooth function on $\mathbb{R}$ such that

$$g(t) = \begin{cases} t^q & \text{if } t \geq C \\ \\ 0 & \text{if } t \leq 0 \end{cases} \tag{4.20}$$

and $0 \leq g'(t) \leq 2C^{q-1}$ when $0 < t < C$. We look for a solution to the equation:

$$\nabla \cdot (e^s \nabla u) + K e^{-s} g(u + C) + M = 0 \tag{4.21}$$

by the standard Leray-Schauder estimate. For that we define

$$A : H_0^1(S_R) \to H_0^1(S_R)$$
$$u \mapsto v$$

by

$$\begin{aligned} \nabla \cdot (e^s \nabla v) + K e^{-s} g(u + C) + M = 0 \quad &\text{on } S_R \\ v = 0 \quad &\text{on } \partial S_R \end{aligned}$$

By the definition of $g(t)$ we have

$$\begin{aligned} \big(g(u + C)\big)^2 &\leq C^{2q} + (u + C)^{2q}, \\ &\leq \tilde{C}(1 + u^2) \end{aligned}$$

where $\tilde{C}$ is a constant which will be enlarged appropriately in the following. Therefore $A(u) \in H^2(S_R)$, and

$$\|A(u)\|_{H^2(S_R)} \leq \tilde{C}(1 + \|u\|_{H_0^1(S_R)}). \tag{4.22}$$

It follows easily that A is continuous and compact. Furthermore if $u = tA(u)$, for $0 \leq t \leq 1$, we have

$$\nabla \cdot (e^s \nabla u) + t(Ke^{-s} g(u + C) + M) = 0 \tag{4.23}$$

weakly. Therefore for some $c > 0$

$$\begin{aligned}
&c \int_{S_R} |\nabla u|^2 \\
\leq &\int_{S_R} e^s |\nabla u|^2 \\
=&t \int_{S_R} Ke^{-s} g(u + C)u + Mu.
\end{aligned}$$

Notice that $g(u + C) \leq C^q + \tilde{C}(u^q + C^q)$,

$$\begin{aligned}
&c \int_{S_R} |\nabla u|^2 \\
\leq &\tilde{C}(1 + \int_{S_R} u^{q+1} + \tilde{C}Mu) \\
\leq &\tilde{C}(1 + C(\epsilon) + \epsilon \int_{S_R} u^2) \\
\leq &\tilde{C}(C(\epsilon) + \epsilon \|u\|^2_{H_0^1(S_R)}).
\end{aligned}$$

Here the constants $\tilde{C}$ and $C(\epsilon)$ are enlarged appropriately from line to line. Let us now choose ϵ so small that $\tilde{C}\epsilon < \frac{c}{2}$. It follows that $\{u \mid u = tA(u), 0 \leq t \leq 1\}$ is bounded in $H_0^1(S_R)$. Therefore there exists a u in $H_0^1(S_R)$ solving (4.21). By the

Sobolev imbedding theorem, $u \in H^2(S_R) \subset W^{1,6}(S_R) \subset C^{0,\frac{1}{2}}(\overline{S_R})$. Since

$$\begin{aligned}
&|g(u(x)+C) - g(u(y)+C)| \\
&\le |g'(\theta)||u(x)-u(y)| \\
&\le \max(2C^{q-1}, q(C+\|u\|_{C^0(\overline{S_R})})^{q-1})[u]_{0,\frac{1}{2};S_R}|x-y|^{\frac{1}{2}},
\end{aligned}$$

where θ is between $u(x)+C$ and $u(y)+C$, it follows that $g(u+C) \in C^{0,\frac{1}{2}}(\overline{S_R})$. Elliptic regularity estimates imply $u \in C^{2,\frac{1}{2}}(\overline{S_R})$, and an iteration of the regularity estimates imply that u is smooth. Now by the classical maximum principle, $u \ge 0$ on S_R, therefore u solves

$$\nabla \cdot (e^s \nabla u) + K e^{-s}(u+C)^q + M = 0. \tag{4.24}$$

Hence

$$\nabla \cdot (e^s \nabla (u+C)) + K e^{-s}(u+C)^q - f \le 0. \tag{4.25}$$

Let $\bar{u} = u + C$, the proof is complete. □

Proof of theorem IV.1. It follows from lemma IV.9, lemma IV.10, and a standard construction (see Smoller [40]) that a solution to (4.12) exists. The construction also guarantees the resulting solution to be axisymmetric if $\underline{u}$ is. □

4.3 Variational Formulation

The main purpose of this section is to show existence of minimizer of the following energy functional:

$$E(w) = \int_{\mathbb{R}^3} \left(\frac{e^s}{2}|\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-s} \right) d\mathbf{x} \tag{4.26}$$

subject to the constraint:

$$N(w) = \int_{\mathbb{R}^3} f w \; d\mathbf{x} = P. \tag{4.27}$$

where f is assumed to be locally bounded, and

$$f \geq c > 0. \tag{4.28}$$

We take the set W_P of admissible functions to be

(4.29)

$$H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \cap \big\{w : \mathbb{R}^3 \to \mathbb{R},\ w \geq 0 \text{ a.e.}, w \text{ is axisymmetric}, N(w) = P\big\}.$$

In fact, one has

Proposition IV.11. *If $1 < q < 3$, there exists a minimizer in W_P of the energy functional E for P sufficiently large.*

We will apply this proposition to construct solutions to (4.1) when $1 < q < 3$ and the domain is infinite.

The proof will need a bound of the L^{q+1} norm by the L^p norm and the L^2 norm of the derivative. We will only concern ourselves with the case in $\mathbb{R}^3$. This is given by the following inequality (see, for example, [36]).

Proposition IV.12 (Gagliaro-Nirenberg inequality)**.** *Let $1 \leq p < 6$, $p \leq q+1 \leq 6$. If $w \in L^p(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, then $\exists C > 0$, such that*

$$\|w\|_{L^{q+1}(\mathbb{R}^3)} \leq C \|\nabla w\|^a_{L^2(\mathbb{R}^3)} \|w\|^{1-a}_{L^p(\mathbb{R}^3)}.$$

If $w \in L^p(\mathbb{R}^3 \setminus S_R) \cap H^1(\mathbb{R}^3 \setminus S_R)$, where S_R is the ball centered at the origin with radius $R > R_0 > 0$, then $\exists C(R_0) > 0$, such that

$$\|w\|_{L^{q+1}(\mathbb{R}^3 \setminus S_R)} \leq C \|\nabla w\|^a_{L^2(\mathbb{R}^3 \setminus S_R)} \|w\|^{1-a}_{L^p(\mathbb{R}^3 \setminus S_R)}.$$

In both of these inequalities,

$$a = \frac{\frac{1}{p} - \frac{1}{q+1}}{\frac{1}{p} - \frac{1}{6}}.$$

Notice when $q \leq 5$, $0 < a \leq 1$. This is the useful range of exponents for us. With the Gagliardo-Nirenberg inequality, we can show that E is bounded from below on W_P.

Lemma IV.13. *Suppose $w \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, $N(w) = P$, and $q < 3$, then there exists a constant C depending only on P, such that*

$$E(w) \geq \frac{1}{2} \int_{\mathbb{R}^3} \frac{e^s}{2} |\nabla w|^2 \, d\mathbf{x} - C.$$

Proof. Since s is bounded,

$$\int w^{q+1} e^{-s} \, d\mathbf{x} \leq C \int w^{q+1} \, d\mathbf{x} = C\|w\|_{q+1}^{q+1}.$$

By the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} & C\|w\|_{q+1}^{q+1} \\ \leq & C\|\nabla w\|_{L^2}^{a(q+1)} \|w\|_{L^1}^{(1-a)(q+1)} \\ \leq & C(P)\|\nabla w\|_{L^2}^{a(q+1)}. \end{aligned}$$

The last inequality follows from the boundedness of s, (4.27), and (4.28).

Since $q < 3$, an easy calculation shows $a(q+1) < 2$. By an elementary inequality we have

$$\begin{aligned} & C(P)\|\nabla w\|_{L^2}^{a(q+1)} \\ \leq & \tilde{C}(P, \epsilon) + \epsilon \|\nabla w\|_{L^2}^2 \\ \leq & \tilde{C}(P, \epsilon) + \epsilon \int |\nabla w|^2 \, d\mathbf{x} \\ \leq & \tilde{C}(P, \epsilon) + C'\epsilon \int \frac{e^s}{2} |\nabla w|^2 \, d\mathbf{x}. \end{aligned}$$

Therefore,

$$E(w) \geq (1 - C'\epsilon) \int \frac{e^s}{2} |\nabla w|^2 \, d\mathbf{x} - \tilde{C}(P, \epsilon).$$

Choose ϵ so small that $(1 - C'\epsilon) > \dfrac{1}{2}$, the assertion is established. $\square$

Let us define:

$$I_P = \inf_{w \in W_P} \{E(w) | N(w) = P\}. \tag{4.30}$$

lemma IV.13 shows that $I_P > -\infty$. We can quickly find a few useful scaling inequalities on I_P.

Lemma IV.14. *Suppose $q > 1$. Given s and f, $I_P < 0$ for P sufficiently large. If $P' > P > 0$, then $I_{P'} \leq \left(\frac{P'}{P}\right)^{q+1} I_P$.*

Proof. Notice in (4.27) that $N(w)$ is linear in w. We have for $\theta > 1$,

$$\begin{aligned}
I_{\theta P} &= \inf \big\{E(w) \bigm| N(w) = \theta P\big\} \\
&= \inf \big\{E(\theta w) \bigm| N(w) = P\big\} \\
&= \inf \big\{ \int \frac{e^s}{2}\theta^2 |\nabla w|^2 - \frac{K}{q+1}\theta^{q+1} w^{q+1} e^{-s} | N(w) = P\big\}.
\end{aligned}$$

Now observe that

$$\int w^{q+1} e^{-s} > 0$$

and the term with the coefficient θ^{q+1} will dominate as θ increases, we can conclude that $I_{\theta P} < 0$ if θ is sufficiently large.

Following the same line of reasoning,

$$\begin{aligned}
I_{P'} &= \inf \big\{E(w) \bigm| N(w) = P'\big\} \\
&= \inf \left\{ E\left(\left(\frac{P'}{P}\right) w\right) \bigm| N(w) = P \right\} \\
&= \inf \left\{ \int \frac{e^s}{2} \left(\frac{P'}{P}\right)^2 |\nabla w|^2 - \frac{K}{q+1} \left(\frac{P'}{P}\right)^{q+1} w^{q+1} e^{-s} \bigm| N(w) = P \right\} \\
&= \left(\frac{P'}{P}\right)^{q+1} \inf \left\{ \int \frac{e^s}{2} \left(\frac{P'}{P}\right)^{1-q} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-s} \bigm| N(w) = P \right\} \\
&\leq \left(\frac{P'}{P}\right)^{q+1} \inf \left\{ \int \frac{e^s}{2} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-s} \bigm| N(w) = P \right\} \\
&= \left(\frac{P'}{P}\right)^{q+1} I_P.
\end{aligned}$$

We get the inequality because $P' > P$ and $q > 1$. $\square$

We are now ready to introduce a concentration compactness principle due to Lions [30]. This is the starting point of the existence argument.

Lemma IV.15. *Let $\{w_n\}$ be a sequence in $L^1(\mathbb{R}^3)$ such that $w_n \geq 0$ a.e. Suppose w_n's are axisymmetric, and $\displaystyle\int_{\mathbb{R}^3} f w_n \, d\mathbf{x} = P$. Then there exists a subsequence $\{w_{n_k}\}$ such that one of the following is true:*

1. $\exists \; \{a_k\} \in \mathbb{R}$ *such that* $\forall \epsilon > 0, \exists R > 0, K_0 > 0$ *such that* $\forall k > K_0$
$$P \geq \int_{a_k \mathbf{e}_3 + S_R} f w_{n_k} \, d\mathbf{x} \geq P - \epsilon.$$
2. $\forall R > 0$
$$\lim_{k \to \infty} \sup_{\mathbf{y} \in \mathbb{R}^3} \int_{\mathbf{y} + S_R} f w_{n_k} \, d\mathbf{x} = 0.$$
3. $\exists \lambda \in (0, P), \forall \epsilon > 0, \exists R_0 > 0, \{a_k\} \in \mathbb{R}, \forall R > R_0, \exists k_0 > 0, \forall k > k_0$*:*
$$\int_{a_k \mathbf{e}_3 + S_R} f w_{n_k} \, d\mathbf{x} > \lambda - \epsilon,$$
$$\int_{a_k \mathbf{e}_3 + S_{2R}} f w_{n_k} \, d\mathbf{x} < \lambda + \epsilon.$$

Proof. Denote $f w_n$ by ρ_n. Let $Q_n(t) = \displaystyle\sup_{\mathbf{y} \in \mathbb{R}^3} \int_{\mathbf{y} + S_t} \rho_n \, d\mathbf{x}$.

$Q_n(t)$ is a sequence of nondecreasing, nonnegative, uniformly bounded functions on $\mathbb{R}^+$, and $\displaystyle\lim_{t \to +\infty} Q_n(t) = P$. By the Helly selection theorem, there exists a subsequence $Q_{n_k}(t)$, and a function $Q(t)$, such that $Q_{n_k}(t) \to Q(t)$ pointwise on $\mathbb{R}^+$. $Q(t)$ is hence non-decreasing and non-negative.

Let $\lambda = \displaystyle\lim_{n \to \infty} Q(t) \in [0, P]$.

1. If $\lambda = P$, then $\forall \epsilon > 0$, $\exists R(\epsilon) > 0$ such that $Q(R) > P - \dfrac{\epsilon}{2}$.

Since $\lim_{k\to\infty} Q_{n_k}(R) = Q(R)$, $\exists K_0(\epsilon) > 0$, $\forall k > K_0(\epsilon)$: $Q_{n_k}(R) > P - \dfrac{\epsilon}{2}$.
Hence, $\exists \mathbf{y}_k(\epsilon) \in \mathbb{R}^3$ such that $\displaystyle\int_{\mathbf{y}_k(\epsilon)+S_R} \rho_{n_k}\, d\mathbf{x} > P - \frac{\epsilon}{2}$. Take $\mathbf{y}_k = \mathbf{y}_k\left(\dfrac{P}{2}\right)$.
We claim that $|\mathbf{y}_k(\epsilon) - \mathbf{y}_k| < R\left(\dfrac{P}{2}\right) + R(\epsilon)$ for ϵ small. If not,

$$\begin{aligned}\int_{\mathbb{R}^3} \rho_{n_k}\, d\mathbf{x} &\geq \int_{\mathbf{y}_k+R(\frac{P}{2})} \rho_{n_k}\, d\mathbf{x} + \int_{\mathbf{y}_k(\epsilon)+R(\epsilon)} \rho_{n_k}\, d\mathbf{x} \\ &> P - \frac{P}{2} + P - \frac{\epsilon}{2} \\ &= \frac{3P}{2} - \frac{\epsilon}{2} > P \quad \text{if } \epsilon \text{ is small.}\end{aligned}$$

Take $R'(\epsilon) = 2R(\epsilon) + R\left(\dfrac{P}{2}\right)$. By the previous inequality, we have

$$\mathbf{y}_k + R'(\epsilon) \supset \mathbf{y}_k(\epsilon) + R(\epsilon).$$

Therefore,

$$\int_{\mathbf{y}_k+B_{R'(\epsilon)}} \rho_{n_k}\, d\mathbf{x} > P - \frac{\epsilon}{2}.$$

Take $a_k = \mathbf{y}_k \cdot \mathbf{e}_3$, and let $r(\mathbf{y})$ be the distance of $\mathbf{y}$ to the $\mathbf{e}_3$ axis. There must exist an r_0 such that $r(\mathbf{y}_k) \leq r_0$. Otherwise the integral of ρ_{n_k} on the torus obtained from revolving $\mathbf{y}_k + S_{R(\frac{P}{2})}$ around the $\mathbf{e}_3$ axis will give

$$\int_{T_k} \rho_{n_k}\, d\mathbf{x} \geq C\left(P - \frac{P}{2}\right) r(\mathbf{y}_k)$$

for some constant C. The right hand side is bounded because the left hand side is.

Let $R''(\epsilon) = R'(\epsilon) + r_0$, then

$$\int_{a_k\mathbf{e}_3+B_{R''(\epsilon)}} \rho_{n_k}\, d\mathbf{x} > P - \epsilon.$$

2. If $\lambda = 0$, then $\lim_{R\to\infty} Q(R) = 0$, which implies $Q(R) \equiv 0$. The result follows immediately.

3. If $\lambda \in (0,P)$, since $\lim_{t\to\infty} Q(t) = \lambda$, $\lim_{k\to\infty} Q_{n_k}(t) = Q(t)$, we know:

   $\forall\epsilon > 0, \exists R(\epsilon) > 0, K_0 > 0, \forall k > K_0, R \geq R(\epsilon)$:

$$Q_n(R) = \sup_{\mathbf{y}\in\mathbb{R}^3} \int_{\mathbf{y}+S_R} \rho_{n_k}\ d\mathbf{x} > \lambda - \epsilon.$$

   Let $f_k(\mathbf{y}) = \displaystyle\int_{\mathbf{y}+B_{R(\epsilon)}} \rho_{n_k}\ d\mathbf{x}$. It is easy to verify that $f_k(\mathbf{y})$ is a continuous function. Consider the set $\{\mathbf{y} \mid f_k(\mathbf{y}) \geq \lambda - \epsilon\}$. This set is nonempty because $\sup_{\mathbf{y}\in\mathbb{R}^3} f_k(\mathbf{y}) > \lambda - \epsilon$, is closed by the continuity of f_k, and is bounded because the contrary will indicate that ρ_{n_k} has infinite mass. Therefore, there exists $\mathbf{y}_k \in \mathbb{R}^3$ such that

$$f_k(\mathbf{y}_k) = \int_{\mathbf{y}_k+S_{R(\epsilon)}} \rho_{n_k}\ d\mathbf{x} = \sup_{\mathbf{y}\in\mathbb{R}^3} \int_{\mathbf{y}+B_{R(\epsilon)}} \rho_{n_k}\ d\mathbf{x} > \lambda - \epsilon.$$

   Also for any $R \geq R(\epsilon)$, we have

$$\int_{\mathbf{y}_k+S_R} \rho_{n_k}\ d\mathbf{x} > \lambda - \epsilon.$$

   For the same reason as in case 1, there must be an $r_0 = r_0(\epsilon)$ such that $r(\mathbf{y}_k) \leq r_0$. Let $a_k = \mathbf{y}_k \cdot \mathbf{e}_3$, and $R_0 = R(\epsilon) + r_0$, $\forall R > R_0, k > K_0$,

$$\int_{a_k\mathbf{e}_3+S_R} \rho_{n_k}\ d\mathbf{x} > \lambda - \epsilon.$$

   On the other hand, because $\lim_{k\to\infty} Q_{n_k}(2R) \leq \lambda$, there must be a $k_0 > K_0$ such that $\forall k > k_0$:

$$Q_{n_k}(2R) < \lambda + \epsilon,$$

   which implies

$$\int_{a_k\mathbf{e}_3+S_{2R}} \rho_{n_k}\ d\mathbf{x} < \lambda + \epsilon.$$

This concludes the proof of the lemma. $\square$

Intuitively, lemma IV.15 says that if we have a sequence of densities with fixed total mass, then the densities will either concentrate in a ball of radius R, or vanish as n goes to infinity, or split up into at least two parts (with masses roughly λ and $M-\lambda$) that escape infinitely far from each other as n goes to infity. Our analysis in the following will show that case 2 and case 3 cannot happen, provided that the scaling inequalities hold. On the other hand, case 1 will force the existence of a minimizer.

Lemma IV.16. *Let $1<q<3$. If w_n is bounded in $L^1(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$, $w_n \geq 0$ a.e., and if*

$$\exists R>0,\ \lim_{n\to\infty}\sup_{\mathbf{y}\in\mathbb{R}^3}\int_{\mathbf{y}+S_R} w_n\ d\mathbf{x}\to 0,$$

Then $\displaystyle\int_{\mathbb{R}^3} w_n^{q+1}\ d\mathbf{x}\to 0.$

Proof. Fix $\alpha\in\left(\max\left\{\frac{3}{2},\frac{2(q+1)}{3}\right\},q+1\right)$, and let $\beta=\dfrac{q+1}{\alpha}$. We get $1<\beta<\dfrac{3}{2}$. For any $w\in L^1(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$, by Sobolev embedding $W^{1,1}\subset L^\beta$,

$$\begin{aligned}
&\int_{\mathbf{y}+S_R} w^{q+1}\ d\mathbf{x}\\
=&\int_{\mathbf{y}+S_R} w^{\alpha\beta}\ d\mathbf{x}\\
\leq& C(R)\Big(\int_{\mathbf{y}+S_R}(w^\alpha+\alpha w^{\alpha-1}|\nabla w|)\ d\mathbf{x}\Big)^\beta\\
\leq& C(R)\Big(\int_{\mathbf{y}+S_R} w^\alpha\ d\mathbf{x}+\alpha\big[\int_{\mathbf{y}+S_R} w^{2(\alpha-1)}\ d\mathbf{x}\big]^{\frac12}\big[\int_{\mathbf{y}+S_R}|\nabla w|^2\big]^{\frac12}\Big)^\beta\\
=& C(R)(\|w\|^\alpha_{L^\alpha(\mathbf{y}+S_R)}+\alpha\|\nabla w\|_{L^2(\mathbf{y}+S_R)}\cdot\|w\|^{\alpha-1}_{L^{2\alpha-2}(\mathbf{y}+S_R)})^\beta.
\end{aligned}\tag{4.31}$$

By the Gagliardo-Nirenberg inequality,

$$\|w\|_{L^\alpha(\mathbf{y}+S_R)}\leq C(R)\|\nabla w\|^a_{L^2(\mathbf{y}+S_R)}\|w\|^{1-a}_{L^1(\mathbf{y}+S_R)}$$

$$\|w\|_{L^{2\alpha-2}(\mathbf{y}+S_R)}\leq C(R)\|\nabla w\|^b_{L^2(\mathbf{y}+S_R)}\|w\|^{1-b}_{L^1(\mathbf{y}+S_R)},$$

where

$$a = \frac{1-\frac{1}{\alpha}}{1-\frac{1}{6}}$$
$$b = \frac{1-\frac{1}{2\alpha-2}}{1-\frac{1}{6}}.$$

One has $a, b \in (0,1)$ if $1 < \alpha < 6$, $1 < 2\alpha - 2 < 6$, or $\dfrac{3}{2} < \alpha < 4$, which is guaranteed by the choice of α. Hence given the hypotheses for w_n, we can easily get

$$\|w_n\|_{L^{\alpha}(\mathbf{y}+S_R)} \to 0,$$
$$\|w_n\|_{L^{2\alpha-2}(\mathbf{y}+S_R)} \to 0,$$

as $n \to \infty$. By (4.31)

$$\begin{aligned}
&\int_{\mathbf{y}+S_R} w_n^{q+1}\, d\mathbf{x} \\
\leq& C(R)\Big(\int_{\mathbf{y}+S_R} (w_n^{\alpha} + \alpha w_n^{\alpha-1}|\nabla w_n|)\, d\mathbf{x}\Big)^{\beta} \\
:=& C(R)\epsilon_n^{\beta} \\
\leq& C(R)\epsilon_n^{\beta-1}\int_{\mathbf{y}+S_R} (w_n^{\alpha} + \alpha w_n^{\alpha-1}|\nabla w_n|)\, d\mathbf{x},
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_n =& \int_{\mathbf{y}+S_R} (w_n^{\alpha} + \alpha w_n^{\alpha-1}|\nabla w_n|)\, d\mathbf{x} \\
\leq& \|w_n\|^{\alpha}_{L^{\alpha}(\mathbf{y}+S_R)} + \alpha\|\nabla w_n\|_{L^2(\mathbf{y}+S_R)} \cdot \|w_n\|^{\alpha-1}_{L^{2\alpha-2}(\mathbf{y}+S_R)} \\
\to& 0.
\end{aligned}$$

Cover $\mathbb{R}^3$ with these balls of radius R in such a way that each point in $\mathbb{R}^3$ is contained in an overlap of at most m balls. Then,

$$\int_{\mathbb{R}^3} w_n^{q+1}\, d\mathbf{x} \leq C(R)m\epsilon_n^{\beta-1}\int_{\mathbb{R}^3} (w_n^{\alpha} + \alpha w_n^{\alpha-1}|\nabla w|)\, d\mathbf{x}. \tag{4.32}$$

Just as in (4.31), we have

$$\int_{\mathbb{R}^3} \left(w_n^\alpha + \alpha w_n^{\alpha-1}|\nabla w|\right)\, d\mathbf{x}$$

$$\leq \|w_n\|^\alpha_{L^\alpha(\mathbb{R}^3)} + \alpha\|\nabla w_n\|_{L^2(\mathbb{R}^3)} \cdot \|w_n\|^{\alpha-1}_{L^{2\alpha-2}(\mathbb{R}^3)}.$$

Similarly by the Gagliardo-Nirenberg inequality,

$$\|w_n\|_{L^\alpha(\mathbb{R}^3)} \leq C\|\nabla w_n\|^a_{L^2(\mathbb{R}^3)}\|w_n\|^{1-a}_{L^1(\mathbb{R}^3)}$$

$$\|w_n\|_{L^{2\alpha-2}(\mathbb{R}^3)} \leq C\|\nabla w_n\|^b_{L^2(\mathbb{R}^3)}\|w_n\|^{1-b}_{L^1(\mathbb{R}^3)}.$$

By the boundedness of w_n in $L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, we conclude from (4.32) that

$$\int_{\mathbb{R}^3} w_n^{q+1}\, d\mathbf{x} \to 0$$

as $n \to \infty$. □

Corollary IV.17. *If $\{w_n\}$ is a minimizing sequence of E in W_P, and if $I_P < 0$, then case 2 in lemma IV.15 cannot happen.*

Proof. If case 2 in lemma IV.15 happens, there will be a subsequence $\{w_{n_k}\}$ such that $\forall R > 0$,

$$\lim_{k\to\infty} \sup_{\mathbf{y}\in\mathbb{R}^3} \int_{\mathbf{y}+S_R} f w_{n_k}\, d\mathbf{x} = 0$$

Since $f \geq c > 0$, this implies

$$\lim_{k\to\infty} \sup_{\mathbf{y}\in\mathbb{R}^3} \int_{\mathbf{y}+S_R} w_{n_k}\, d\mathbf{x} = 0$$

By lemma IV.16 and the boundedness of s this implies

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} \frac{K}{q+1} e^{-s} w_{n_k}^{q+1}\, d\mathbf{x} = 0,$$

which then implies $I_P \geq 0$. □

For the purpose of eliminating the possibility of case 3 in lemma IV.15, we need an elementary inequality.

Lemma IV.18. *If* $0 \le \lambda_1, \lambda_2 \le 1,\ \lambda_1 + \lambda_2 = 1,\ q > 1,$ *then*

$$1 - \lambda_1^{q+1} - \lambda_2^{q+1} \ge 2\lambda_1\lambda_2.$$

Proof. Since $q > 1$, $q + 1 > 2$. Hence

$$\begin{aligned} 1 - \lambda_1^{q+1} - \lambda_2^{q+1} &\ge 1 - \lambda_1^2 - \lambda_2^2 \\ &= (\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2 \\ &= 2\lambda_1\lambda_2. \end{aligned}$$

□

Now we are ready to eliminate case 3 in lemma IV.15.

Lemma IV.19. *Let* $\{w_n\}$ *be a minimizing sequence of* E *in* W_P*. Suppose* $I_P < 0$*, and* $\forall P_2 > P_1 > 0,\ I_{P_2} \le \left(\frac{P_2}{P_1}\right)^{q+1} I_{P_1}$*. Then case 3 in lemma IV.15 cannot happen.*

Proof. Assume the contrary. Then there exists a subsequence $\{w_{n_k}\}$ such that $\exists \lambda \in (0, P), \forall \epsilon > 0, \exists R_0 > 0, a_k \in \mathbb{R}, \forall R > R_0, \exists k_0 > 0, \forall k > k_0$:

$$\begin{aligned} &\int_{a_k \mathbf{e}_3 + S_R} f w_{n_k}\, d\mathbf{x} > \lambda - \epsilon, \\ &\int_{a_k \mathbf{e}_3 + S_{2R}} f w_{n_k}\, d\mathbf{x} < \lambda + \epsilon. \end{aligned} \tag{4.33}$$

Let $\varphi : \mathbb{R}^+ \to [0, 1]$ be a smooth cut off function, such that

$$\begin{aligned} \varphi(t) = 1 &\quad \text{when} \quad |t| \le 1, \\ \varphi(t) = 0 &\quad \text{when} \quad |t| \ge 2, \\ |\nabla\varphi(t)| \le 2 &\quad \text{for all} \quad t. \end{aligned}$$

Let us now define

$$\varphi_{k,1}(\mathbf{x}) = \varphi\left(\frac{|\mathbf{x} - a_k\mathbf{e}_3|}{R}\right),$$

$$\varphi_{k,2}(\mathbf{x}) = 1 - \varphi_{k,1}(\mathbf{x}),$$

$$w_{k,1}(\mathbf{x}) = \varphi_{k,1}(\mathbf{x})w_{n_k}(\mathbf{x}),$$

$$w_{k,2}(\mathbf{x}) = \varphi_{k,2}(\mathbf{x})w_{n_k}(\mathbf{x}),$$

$$P_{k,1} = \int_{\mathbb{R}^3} f w_{k,1}\, d\mathbf{x},$$

$$P_{k,2} = \int_{\mathbb{R}^3} f w_{k,2}\, d\mathbf{x}.$$

Obviously $w_{k,1} \in W_{P_{k,1}}, w_{k,2} \in W_{P_{k,2}}, |\nabla\varphi_{k,1}| \leq \dfrac{2}{R}, |\nabla\varphi_{k,2}| \leq \dfrac{2}{R}$, also $P = P_{k,1} + P_{k,2}$.

We now estimate

$$\begin{aligned}
E(w_{n_k}) &= \int_{\mathbb{R}^3} \left(\frac{e^s}{2}|\nabla w_{n_k}|^2 - \frac{K}{q+1}w_{n_k}^{q+1}e^{-s}\right) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \left(\frac{e^s}{2}|\nabla w_{k,1} + \nabla w_{k,2}|^2 - \frac{K}{q+1}(w_{n_k}^{q+1} - w_{k,1}^{q+1} - w_{k,2}^{q+1})e^{-s}\right. \\
&\qquad \left. - \frac{K}{q+1}(w_{k,1}^{q+1} + w_{k,2}^{q+1})e^{-s}\right) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \left(\frac{e^s}{2}|\nabla w_{k,1}|^2 - \frac{K}{q+1}w_{k,1}^{q+1}e^{-s}\right) d\mathbf{x} \\
&\quad + \int_{\mathbb{R}^3} \left(\frac{e^s}{2}|\nabla w_{k,2}|^2 - \frac{K}{q+1}w_{k,2}^{q+1}e^{-s}\right) d\mathbf{x} \\
&\quad + \int_{\mathbb{R}^3} \left(e^s\nabla w_{k,1}\cdot\nabla w_{k,2} - \frac{K}{q+1}w_{n_k}^{q+1}(1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1})e^{-s}\right) d\mathbf{x} \\
&\geq I_{P_{k,1}} + I_{P_{k,2}} + \int_{\mathbb{R}^3} \left(e^s\nabla w_{k,1}\cdot\nabla w_{k,2} - \frac{K}{q+1}w_{n_k}^{q+1}(1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1})e^{-s}\right) d\mathbf{x} \\
&\geq \left[\left(\frac{P_{k,1}}{P}\right)^{q+1} + \left(\frac{P_{k,2}}{P}\right)^{q+1}\right] I_P \\
&\quad + \int_{\mathbb{R}^3} \left(e^s\nabla w_{k,1}\cdot\nabla w_{k,2} - \frac{K}{q+1}w_{n_k}^{q+1}(1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1})e^{-s}\right) d\mathbf{x}.
\end{aligned}$$

The last inequality follows from the hypothesis in the lemma. If we denote

$$Re = \int_{\mathbb{R}^3} \left(e^s \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{n_k}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-s} \right) \, d\mathbf{x},$$

then the above estimate gives us

$$I_P - E(w_{n_k}) \leq \left[1 - \left(\frac{P_{k,1}}{P} \right)^{q+1} - \left(\frac{P_{k,2}}{P} \right)^{q+1} \right] I_P - Re.$$

Since $P = P_{k,1} + P_{k,2}$, and $I_P < 0$, by lemma IV.18, we get

$$\begin{aligned} I_P - E(w_{n_k}) &\leq \left[1 - \left(\frac{P_{k,1}}{P} \right)^{q+1} - \left(\frac{P_{k,2}}{P} \right)^{q+1} \right] I_P - Re \\ &\leq 2 \frac{P_{k,1} P_{k,2}}{P^2} I_P - Re, \end{aligned}$$

or

$$-\frac{2}{P^2} I_P P_{k,1} P_{k,2} \leq E(w_{n_k}) - I_P - Re. \tag{4.34}$$

Let us now estimate Re:

$$-Re = -\int_{\mathbb{R}^3} \left(e^s \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{n_k}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-s} \right) \, d\mathbf{x}.$$

By the definition of $\varphi_{k,1}$ and $\varphi_{k,2}$, we know $1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1} \in [0, 1]$, and is nonzero only when $R \leq |\mathbf{x} - a_k \mathbf{e}_3| \leq 2R$. Therefore

$$\begin{aligned} -Re &\leq -\int_{\mathbb{R}^3} e^s \nabla w_{k,1} \cdot \nabla w_{k,2} \, d\mathbf{x} + C(q, K, s) \int_{R \leq |\mathbf{x} - a_k \mathbf{e}_3| \leq 2R} w_{n_k}^{q+1} \, d\mathbf{x} \\ &= L_1 + L_2. \end{aligned}$$

We estimate L_1 and L_2 separately.

$$\begin{aligned}
L_1 &= -\int_{\mathbb{R}^3} e^s \nabla w_{k,1} \cdot \nabla w_{k,2}\ d\mathbf{x} \\
&= -\int_{\mathbb{R}^3} e^s \nabla (w_{n_k}\varphi_{k,1}) \cdot \nabla (w_{n_k}\varphi_{k,2})\ d\mathbf{x} \\
&= -\int_{\mathbb{R}^3} e^s \nabla \varphi_{k,1} \cdot \nabla \varphi_{k,2} |w_{n_k}|^2\ d\mathbf{x} - \int_{\mathbb{R}^3} e^s w_{n_k}\varphi_{k,2} \nabla \varphi_{k,1} \cdot \nabla w_{n_k}\ d\mathbf{x} \\
&\quad - \int_{\mathbb{R}^3} e^s w_{n_k}\varphi_{k,1} \nabla \varphi_{k,2} \cdot \nabla w_{n_k}\ d\mathbf{x} - \int_{\mathbb{R}^3} e^s \varphi_{k,1}\varphi_{k,2} |\nabla w_{n_k}|^2\ d\mathbf{x} \\
&\le -\int_{\mathbb{R}^3} e^s \nabla \varphi_{k,1} \cdot \nabla \varphi_{k,2} |w_{n_k}|^2\ d\mathbf{x} - \int_{\mathbb{R}^3} e^s w_{n_k}\varphi_{k,2} \nabla \varphi_{k,1} \cdot \nabla w_{n_k}\ d\mathbf{x} \\
&\quad - \int_{\mathbb{R}^3} e^s w_{n_k}\varphi_{k,1} \nabla \varphi_{k,2} \cdot \nabla w_{n_k}\ d\mathbf{x} \\
&\le \frac{C(s)}{R}.
\end{aligned}$$

The last inequality follows from $|\nabla \varphi_{k,1}| \le \dfrac{2}{R}$, $|\nabla \varphi_{k,2}| \le \dfrac{2}{R}$ and that $\{w_{n_k}\}$ is bounded in $H^1(\mathbb{R}^3)$. On the other hand, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
L_2 &\le C(q,K,s) \|w_{n_k}\|^{q+1}_{L^{q+1}(R \le |\mathbf{x}-a_k\mathbf{e}_3| \le 2R)} \\
&\le C(q,K,s) \|\nabla w_{n_k}\|^{a(q+1)}_{L^2(R \le |\mathbf{x}-a_k\mathbf{e}_3| \le 2R)} \|w_{n_k}\|^{(1-a)(q+1)}_{L^1(R \le |\mathbf{x}-a_k\mathbf{e}_3| \le 2R)} \\
&\le C(q,K,s)[(\lambda+\epsilon)-(\lambda-\epsilon)]^{(1-a)(q+1)}
\end{aligned}$$

The constant $C(q,K,s)$ is enlarged in different lines. The last inequality above follows from (4.33), and the fact that $\{w_{n_k}\}$ is bounded in $H^1(\mathbb{R}^3)$.

In summary, we have

$$-Re \le \frac{C(s)}{R} + C(q,K,s)(2\epsilon)^{(1-a)(q+1)}.$$

From the range of q, we deduce that $a \in (0,1)$. Choose $R > R_0$ so big that

$$-Re \le C(q,K,s)\epsilon^{(1-a)(q+1)}.$$

By the definition of $w_{k,1}$, we have

$$P_{k,1} \ge \int_{|\mathbf{x}-a_k\mathbf{e}_3| \le R} f w_{n_k}\ d\mathbf{x} > \lambda - \epsilon.$$

By (4.34), and the estimates on Re, we have

$$P_{k,2} \leq C(P, I_P, \lambda, q, K, s)(\epsilon + \epsilon^{(1-a)(q+1)}).$$

However,

$$\begin{aligned} P_{k,2} &= \int_{\mathbb{R}^3} f w_{k,2} \; d\mathbf{x} \\ &\geq \int_{\mathbb{R}^3 \backslash a_k \mathbf{e}_3 + S_{2R}} f w_{n_k} \; d\mathbf{x}. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^3 \backslash a_k \mathbf{e}_3 + S_{2R}} f w_{n_k} \; d\mathbf{x} \leq C(P, I_P, \lambda, q, K, s)(\epsilon + \epsilon^{(1-a)(q+1)}).$$

On the other hand,

$$\int_{a_k \mathbf{e}_3 + S_{2R}} f w_{n_k} \; d\mathbf{x} < \lambda + \epsilon.$$

This implies

$$\begin{aligned} P &= \int_{\mathbb{R}^3} f w_{n_k} \; d\mathbf{x} \\ &< \lambda + \epsilon + C(P, I_P, \lambda, q, K, s)(\epsilon + \epsilon^{(1-a)(q+1)}). \end{aligned}$$

If we have initially chosen ϵ so small that

$$\lambda + \epsilon + C(P, I_P, \lambda, q, K, s)(\epsilon + \epsilon^{(1-a)(q+1)}) < P.$$

a contradiction will be obtained. $\square$

With the preparation above, we are ready to prove the existence of a minimizer.

Proof of proposition IV.11. By lemma IV.14, the scaling inequalities are true in this q range, therefore lemma IV.15, lemma IV.16 and lemma IV.19 apply. For any minimizing sequence $\{w_n\}$, there exists a subsequence $\{w_{n_k}\}$ such that case 1 in lemma IV.15 is true. Without loss of generality, we assume $\{w_n\}$ is already shifted, and satisfies: $\forall\epsilon > 0$, $\exists R > 0, n_0 > 0$, $\forall n > n_0$:

$$P \geq \int_{S_R} f w_n \, d\mathbf{x} \geq P - \epsilon.$$

By lemma IV.13, $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$. The Banach-Alaoglu theorem implies that there exists a subsequence of $\{w_n\}$ which converges weakly in $H^1(\mathbb{R}^3)$ to $\tilde{w}$. Without loss of generality, we call this subsequence $\{w_n\}$ again. We claim that $\tilde{w}$ is a minimizer of $E(w)$ in W_M.

Let us first show $\tilde{w} \in W_M$. Obviously $\tilde{w} \in H^1(\mathbb{R}^3)$. Notice for any $R > 0$, we have $w_n \rightharpoonup w$ weakly in $H^1(S_R)$. By the Rellich-Kondrachov theorem, $H^1(S_R)$ is compactly embedded in $L^p(S_R)$ for $1 \leq p < 6$. This implies $\forall R > 0$, $w_n \to \tilde{w}$ in $L^q(S_R)$ for $1 \leq q < 6$. The conditions $w \geq 0$ a.e. and w axisymmetric are now easily established if we integrate the w_n's against positive smooth test functions with compact supports and take the limit. Let us now show $N(\tilde{w}) = P$. For that we observe $\forall\epsilon > 0$, $\exists R > 0, n_0 > 0$, $\forall n > n_0$:

$$\int_{S_R} f w_n \, d\mathbf{x} \geq P - \epsilon.$$

Since $w_n \to \tilde{w}$ in $L^q(S_R)$ for all $R > 0$, and f is locally bounded, we have

$$\int_{S_R} f \tilde{w} \, d\mathbf{x} \geq P - \epsilon.$$

Therefore for any $\epsilon > 0$

$$\int_{\mathbb{R}^3} f \tilde{w} \, d\mathbf{x} \geq P - \epsilon. \tag{4.35}$$

On the other hand, for any $R > 0$,

$$P \geq \int_{S_R} f w_n \, d\mathbf{x},$$

which implies

$$P \geq \int_{S_R} f\tilde{w}\ d\mathbf{x},$$

which implies

$$P \geq \int_{\mathbb{R}^3} f\tilde{w}\ d\mathbf{x}. \tag{4.36}$$

Combine (4.35) and (4.36), we get

$$\int_{\mathbb{R}^3} f\tilde{w}\ d\mathbf{x} = P.$$

This also shows $\tilde{w} \in L^1(\mathbb{R}^3)$. We have shown $\tilde{w} \in W_M$, it remains to establish the weak lower-semicontinuity of E. The first term in E can be treated by the standard method. Let us observe that

$$F_c = \left\{ w \;\middle|\; \int_{\mathbb{R}^3} \frac{e^s}{2}|\nabla w|^2\ d\mathbf{x} \leq c \right\}$$

is a convex norm closed set in $H^1(\mathbb{R}^3)$, therefore is weakly closed.

For the second term

$$-\int_{\mathbb{R}^3} \frac{K}{q+1} w^{q+1} e^{-s}\ d\mathbf{x}$$

we recall, $\forall \epsilon > 0$, $\exists R > 0, n_0 > 0$, $\forall n, n' > n_0$:

$$\int_{\mathbb{R}^3\setminus S_R} f w_n\ d\mathbf{x} \leq \epsilon$$
$$\int_{\mathbb{R}^3\setminus S_R} f w_{n'}\ d\mathbf{x} \leq \epsilon$$
$$\|w_n - w_{n'}\|_{L^{q+1}(S_R)} < \epsilon.$$

Therefore,

$$\begin{aligned}
\|w_n - w_{n'}\|_{L^{q+1}(\mathbb{R}^3)} &\leq \|w_n - w_{n'}\|_{L^{q+1}(S_R)} + \|w_n - w_{n'}\|_{L^{q+1}(\mathbb{R}^3\setminus S_R)} \\
&< \epsilon + C\big(\sup_n \|w_n\|_{H^1(\mathbb{R}^3)}\big)^a \big(\|w_n\|_{L^1(\mathbb{R}^3\setminus S_R)} + \|w_{n'}\|_{L^1(\mathbb{R}^3\setminus S_R)}\big)^{1-a} \\
&\leq \epsilon + C'\epsilon.
\end{aligned}$$

The second inequality above follows from the Gagliardo-Nirenberg inequality. Hence $\{w_n\}$ converges in $L^{q+1}(\mathbb{R}^3)$. But $w_n \to \tilde{w}$ in $L^{q+1}(S_R)$ for any $R > 0$. This implies $w_n \to \tilde{w}$ in $L^{q+1}(\mathbb{R}^3)$. Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} w_n^{q+1} e^{-s} \, d\mathbf{x} = \int_{\mathbb{R}^3} \tilde{w}^{q+1} e^{-s} \, d\mathbf{x}.$$

Combine the two terms in E. We have

$$\liminf_{n\to\infty} E(w_n) \geq E(\tilde{w}).$$

This shows that $\tilde{w}$ is a minimizer. □

One can establish a similar proposition for functions restricted to a finite ball. Since one has compact Sobolev embedding theorems on bounded balls, the corresponding proof will be a lot easier. In particular, if we let

$$E(w) = \int_{S_R} \left(\frac{e^s}{2} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-s} \right) d\mathbf{x}, \tag{4.37}$$

$$N(w) = \int_{S_R} f w \, d\mathbf{x}, \tag{4.38}$$

and let W_P be

$$H_0^1(S_R) \cap L^1(S_R) \cap \big\{ w : S_R \to \mathbb{R},\ w \geq 0 \text{ a.e.}, w \text{ is axisymmetric}, N(w) = P \big\},$$

then proposition IV.11 with these newly defined E and W_P remains true. The proof for that is standard.

4.4 Existence of Solution for Low Adiabatic Index

In this section, we give proofs to theorems IV.2 and IV.3. The argument is based on proposition IV.11. We will only lay out the demonstration for the compactly supported case, i.e. theorem IV.2. The whole space case is virtually identical.

We first study the Euler-Lagrange equation. Let W be

$$H_0^1(S_R)\cap L^1(S_R)\cap\big\{w:S_R\to\mathbb{R},\ w\geq 0 \text{ a.e.}, w \text{ is axisymmetric}, N(w)<\infty\big\},$$

one has

Lemma IV.20. $\exists\lambda\in\mathbb{R},\forall u\in W$:

$$\int_{S_R}\Big(e^s\nabla\tilde{w}\cdot\nabla(u-\tilde{w})-Ke^{-s}\tilde{w}^q(u-\tilde{w})\Big)\,d\mathbf{x}\geq-\lambda\int_{S_R}f(u-\tilde{w})\,d\mathbf{x}. \tag{4.39}$$

Proof. Given $u\in W$, when $t>0$ is small enough,

$$\tilde{w}+t\bigg[(u-\tilde{w})-\frac{N(u-\tilde{w})}{N(\tilde{w})}\tilde{w}\bigg]\in W_P,$$

therefore,

$$\frac{d}{dt}E\bigg(\tilde{w}+t\bigg[(u-\tilde{w})-\frac{N(u-\tilde{w})}{N(\tilde{w})}\tilde{w}\bigg]\bigg)\bigg|_{t=0+}\geq 0.$$

Denote $(u-\tilde{w})-\dfrac{N(u-\tilde{w})}{N(\tilde{w})}\tilde{w}$ by σ, we have

$$\begin{aligned}&\frac{E(\tilde{w}+t\sigma)-E(\tilde{w})}{t}\\ &=\int_{S_R}\bigg(e^s\nabla\tilde{w}\cdot\nabla\sigma-\frac{K}{q+1}e^{-s}(q+1)(\tilde{w}+\theta\sigma)^q\sigma\bigg)\,d\mathbf{x}+O(t),\end{aligned}$$

where θ is between 0 and t, and depends on $\mathbf{x}$. Take the limit as $t\to 0+$. By the dominated convergence theorem, we get

$$\lim_{t\to 0+}\frac{E(\tilde{w}+t\sigma)-E(\tilde{w})}{t}=\int_{S_R}\Big(e^s\nabla\tilde{w}\cdot\nabla\sigma-Ke^{-s}\tilde{w}^q\sigma\Big)\,d\mathbf{x}.$$

Denote this by $E'_{\tilde{w}}(\sigma)$, we have

$$\begin{aligned}0&\leq E'_{\tilde{w}}(\sigma)\\ &=E'_{\tilde{w}}(u-\tilde{w})-\frac{E'_{\tilde{w}}(\tilde{w})}{N(\tilde{w})}N(u-\tilde{w}).\end{aligned}$$

Let $-\lambda=\dfrac{E'_{\tilde{w}}(\tilde{w})}{N(\tilde{w})}$, the proof is complete. $\square$

Lemma IV.21. *If $I_P < 0, q > 1$, then $\lambda > 0$.*

Proof. Observe that $2\tilde{w} \in W$, therefore we may plug in $u = 2\tilde{w}$ to find

$$
\begin{aligned}
-\lambda P &= -\lambda \int_{S_R} f(2\tilde{w} - \tilde{w})\, d\mathbf{x} \\
&\leq \int_{S_R} (e^s |\nabla \tilde{w}|^2 - K e^{-s} \tilde{w}^{q+1})\, d\mathbf{x} \\
&= \int_{S_R} (\frac{e^s}{2} |\nabla \tilde{w}|^2 - \frac{K}{q+1} e^{-s} \tilde{w}^{q+1})\, d\mathbf{x} + \int_{S_R} (\frac{e^s}{2} |\nabla \tilde{w}|^2 - \frac{qK}{q+1} e^{-s} \tilde{w}^{q+1})\, d\mathbf{x} \\
&\leq 2 I_P \\
&< 0.
\end{aligned}
$$

□

For any $\varphi \in C_0^\infty(S_R)$, $\varphi \geq 0$, let $S(\varphi)$ be

$$S(\varphi)(r, \theta, z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r, \theta, z) d\theta. \tag{4.40}$$

Then $S(\varphi)$ is axisymmetric, $\tilde{w} + S(\varphi) \in W$, and

$$
\begin{aligned}
&\int_{S_R} \left(e^s \nabla \tilde{w} \cdot \nabla \varphi - K e^{-s} \tilde{w}^q \varphi + \lambda f \varphi \right) d\mathbf{x} \\
&= \int_{S_R} \left(S(e^s) \nabla S(\tilde{w}) \cdot \nabla \varphi - K S(e^{-s} \tilde{w}^q) \varphi + \lambda S(f) \varphi \right) d\mathbf{x} \\
&= \int_{S_R} \left(e^s \nabla \tilde{w} \cdot S(\varphi) - K e^{-s} \tilde{w}^q S(\varphi) + \lambda f S(\varphi) \right) d\mathbf{x}
\end{aligned}
$$

$$\geq 0. \tag{4.41}$$

One can pass from the second line to the third line by Fubini's theorem. The last line follows from (4.39). Therefore,

$$-\nabla \cdot (e^s \nabla \tilde{w}) - K e^{-s} \tilde{w}^q + \lambda f \tag{4.42}$$

is a positive distribution on S_R. By a theorem of Schwartz (see Schwartz [38]), it must be a positive Borel measure:

$$-\nabla \cdot (e^s \nabla \tilde{w}) - K e^{-s} \tilde{w}^q + \lambda f = d\mu. \tag{4.43}$$

Let us write $\tilde{w} = w_1 + w_2$, where $w_1 \in H_0^1(S_R)$ weakly solves

$$-\nabla \cdot (e^s \nabla w_1) - K e^{-s} \tilde{w}^q + \lambda f = 0, \tag{4.44}$$

and $w_2 \in H_0^1(S_R)$ weakly solves

$$-\nabla \cdot (e^s \nabla w_2) = d\mu. \tag{4.45}$$

From the range of q and the fact that $\tilde{w} \in H_0^1(S_R) \subset L^6(S_R)$, we have $\tilde{w}^q \in L^2(S_R)$. By standard elliptic regularity theory, w_1 is continuous. We next show that w_2 is lower semicontinuous, following Lewy and Stampacchia [27].

Lemma IV.22. *Let $\tilde{S}$ be any ball contained in S_R. Let $G(\mathbf{x}, \mathbf{y})$ be the Dirichlet Green's function of $\tilde{S}$ with respect to the operator $-\nabla \cdot (e^s \nabla)$, i.e.*

$$\begin{aligned} -\nabla_{\mathbf{x}}(e^s \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y})) = \delta_{\mathbf{y}} \quad & \textit{on } \tilde{S} \\ G(\mathbf{x}, \mathbf{y}) = 0 \quad & \textit{on } \partial\tilde{S} \end{aligned}$$

then

$$w_2(\mathbf{x}) = \int_{\tilde{S}} G(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) - \int_{\partial\tilde{S}} e^{s(\mathbf{y})} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} w_2(\mathbf{y}) d\sigma(\mathbf{y}) \tag{4.46}$$

in $\tilde{S}$, where σ is the standard surface measure on $\partial\tilde{S}$.

Proof. Pick any ball S contained in $\tilde{S}$. $\forall \varphi \in C_0^\infty(S)$, $\varphi \geq 0$, $\exists u$ solving

$$\begin{aligned} -\nabla \cdot (e^s \nabla u) = \varphi \quad & \text{on } \tilde{S} \\ u = 0 \quad & \text{on } \partial\tilde{S} \end{aligned}$$

It follows from the Green's theorem that

$$u(\mathbf{x}) = \int_S G(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) \, d\mathbf{y}. \tag{4.47}$$

Now

$$\begin{aligned}
&\int_S w_2(\mathbf{x})\varphi(\mathbf{x})\ d\mathbf{x}\\
&=\int_{\tilde S} w_2(\mathbf{x})\varphi(\mathbf{x})\ d\mathbf{x}\\
&=-\int_{\tilde S} w_2\nabla\cdot(e^s\nabla u)\ d\mathbf{x}\\
&=\int_{\tilde S} e^s\nabla w_2\cdot\nabla u\ d\mathbf{x}-\int_{\partial\tilde S} e^s w_2\frac{\partial u}{\partial n}d\sigma\\
&=\int_{\tilde S} u d\mu-\int_{\partial\tilde S} e^s w_2\frac{\partial u}{\partial n}d\sigma\\
&=\int_{\tilde S}\left(\int_S G(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\ d\mathbf{y}\right)d\mu(\mathbf{x})-\int_{\partial\tilde S} e^{s(\mathbf{x})}w_2(\mathbf{x})\left(\int_S\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\varphi(\mathbf{y})\ d\mathbf{y}\right)d\sigma(\mathbf{x})\\
&=\int_S\left(\int_{\tilde S} G(\mathbf{x},\mathbf{y})d\mu(\mathbf{x})\right)\varphi(\mathbf{y})\ d\mathbf{y}-\int_S\left(\int_{\tilde S} e^{s(\mathbf{x})}w_2(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}d\sigma(\mathbf{x})\right)\varphi(\mathbf{y})\ d\mathbf{y}.
\end{aligned}$$

The last equality follows from Fubini's theorem and the fact that $G(\mathbf{x},\mathbf{y})>0$ when $\mathbf{x}\neq\mathbf{y}$. □

Proof of theorem IV.2. Without loss of generality, we can assume $\alpha=1$ in (4.8). When $\mathbf{x}$ is in a compact subset of $\tilde S$, and $\mathbf{y}$ on $\partial\tilde S$, $\dfrac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}$ is a smooth function in $\mathbf{x}$ and $\mathbf{y}$. Hence first term in (4.46) is continuous in $\mathbf{x}$. Also notice that $G(\mathbf{x},\mathbf{y})$ is a pointwise limit of

$$G_a(\mathbf{x},\mathbf{y})=\begin{cases}G(\mathbf{x},\mathbf{y}) & \text{if } G(\mathbf{x},\mathbf{y})\le a\\ a & \text{if } G(\mathbf{x},\mathbf{y})>a\end{cases}$$

and that

$$\int_{\tilde S} G_a(\mathbf{x},\mathbf{y})d\mu(\mathbf{y}) \tag{4.48}$$

is continuous in $\mathbf{x}$ on $\tilde S$. By the monotone convergence theorem,

$$\int_{\tilde S} G(\mathbf{x},\mathbf{y})d\mu(\mathbf{y})$$

is an increasing pointwise limit of (4.48), and hence is lower semicontinuous. We can now conclude that w_2, and $\tilde{w}$ also, are lower semicontinuous. This implies that the set $U_+ = \{\mathbf{x} \in S_R \mid \tilde{w}(\mathbf{x}) > 0\}$ is open. If $\varphi \in C_0^\infty(U_+)$, then $\tilde{w} + tS(\varphi) \in W$ for $|t|$ sufficiently small. A similar calculation as (4.41) will show that

$$\int_{S_R} \left(e^s \nabla\tilde{w} \cdot \nabla\varphi - Ke^{-s}\tilde{w}^q\varphi + \lambda f\varphi \right) d\mathbf{x} = 0. \tag{4.49}$$

In other words, $\tilde{w}$ solves

$$\nabla \cdot (e^s \nabla w) + Ke^{-s}w^q - \lambda f = 0 \tag{4.50}$$

weakly on U_+. Regularity of the solution follows from standard elliptic regularity. □

4.5 Existence of Solution for Given Gas Density

We prove theorems IV.5 and IV.7 in this section. To avail ourselves in establishing regularity at $r = 0$, let us prove the following lemma.

Lemma IV.23. *Let $f : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbb{R}$ be such that $f(-r, z) = -f(r, z)$, and assume that $f \in C^k$, $k \geq 1$, then the function*

$$g(r, z) = \begin{cases} \dfrac{f(r, z)}{r} & r \neq 0 \\ f_r(0, z) & r = 0 \end{cases} \tag{4.51}$$

is in C^{k-1}.

Proof. Obviously $f(0,z)=0$, hence for $r\neq 0$,

$$\begin{aligned}
&g(r,z)\\
=&\frac{1}{r}(f(r,z)-f(0,z))\\
=&\frac{1}{r}\int_0^r f_s(s,z)ds\\
=&\frac{1}{r}\int_0^1 f_s(rs,z)rds\\
=&\int_0^1 f_s(rs,z)ds \qquad (4.52)
\end{aligned}$$

Apparently the same equation is true for $r=0$, and the assertion is clear from this formula. □

Proof of theorem IV.5. Let us write (4.1) in cylindrical coordinates:

$$\begin{cases} p_r=\rho(B\rho)_r+\rho r\Omega^2 \\ p_z=\rho(B\rho)_z \end{cases} \qquad (4.53)$$

From the definition of $B\rho$, we get

$$(B\rho)_z(\mathbf{x})=\int_D \frac{\rho_z(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}. \qquad (4.54)$$

Therefore $B\rho_z(r,-z)=-B\rho_z(r,z)$ and $B\rho_z(r,z)>0$ when $z<0$, by hypothesis 4 and the symmetry of ρ and D. Let

$$p(r,z)=\int_{-\psi(r)}^{z}\rho(r,\xi)B\rho_\xi(r,\xi)d\xi. \qquad (4.55)$$

From now on we allow r to take negative values by evenly extending all the relevant functions across $r=0$. It is easily seen that $p>0$ in D, $p=0$ on ∂D and that p satisfies the second equation in (4.53). Since $\rho\in C^k(\bar{D})$ and ∂D is smooth, we have $B\rho\in C^{k+1}(\bar{D})$. It is not difficult to see that $p\in C^k(\bar{D})$. Differentiate (4.55) under the integral sign, we get

$$p_r(r,z)=\int_{-\psi(r)}^{z}\left(\rho_r(r,\xi)B\rho_\xi(r,\xi)+\rho(r,\xi)B\rho_{r\xi}(r,\xi)\right)\,d\xi. \qquad (4.56)$$

By the first equation in (4.53), when $r > 0$, Ω^2 has to have the form:

$$
\begin{aligned}
\Omega^2 &= \frac{1}{r\rho}(p_r - \rho B\rho_r) \\
&= \frac{1}{r\rho}\left(\int_{-\psi(r)}^{z} \left(\rho_r B\rho_\xi + \rho B\rho_{r\xi}\right)\, d\xi - \rho B\rho_r\right) \\
&= \frac{1}{r\rho}\left(\int_{-\psi(r)}^{z} \left(\rho_r B\rho_\xi + \rho B\rho_{r\xi}\right)\, d\xi - \int_{-\psi(r)}^{z} \left(\rho B\rho_r\right)_\xi\, d\xi\right) \\
&= \frac{1}{r\rho}\int_{-\psi(r)}^{z} \left(\rho_r B\rho_\xi - \rho_\xi B\rho_r\right)\, d\xi. \qquad (4.57)
\end{aligned}
$$

(4.57) is non-negative on D by hypothesis 3. Define Ω^2 by (4.57), when $r > 0$, and if D contains points at $r = 0$, by

$$\Omega^2(0, z) = \frac{1}{\rho}(p_r - \rho B\rho_r)_r(0, z) \qquad (4.58)$$

Notice $p_r - \rho(B\rho)_r$ is odd in r. By lemma IV.23, $\rho\Omega^2 \in C^{k-2}(D)$, hence so is Ω^2. Such p and Ω^2 obviously satisfy (4.53). It remains to show that Ω^2 extends to a continuous function on $\bar{D}$. Let us consider the following three cases:

1. Let r_0 be a nonzero radius such that $(r_0, -\psi(r_0)) \in \partial D$ and $\psi(r_0) > 0$.

$$
\begin{aligned}
&\lim_{(r,z)\to(r_0,-\psi(r_0))} \Omega^2(r, z) \\
&= \lim_{(r,z)\to(r_0,-\psi(r_0))} \frac{1}{r\rho}\int_{-\psi(r)}^{z} \left(\rho_r B\rho_\xi - \rho_\xi B\rho_r\right)\, d\xi \\
&= \lim_{(r,z)\to(r_0,-\psi(r_0))} \frac{1}{r\rho_z}\left(\rho_r B\rho_z - \rho_z B\rho_r\right) \\
&= \frac{1}{r_0\rho_z(r_0, -\psi(r_0))}\left(\rho_r B\rho_z - \rho_z B\rho_r\right)(r_0, -\psi(r_0)). \qquad (4.59)
\end{aligned}
$$

Here we have used the differential mean value theorem and hypothesis 4.

2. If ∂D contains points at $r = 0$, since ∂D is smooth and symmetric about $z = 0$

and $r = 0$, we have $\psi(0) > 0$, $\psi'(0) = 0$. Hence

$$\begin{aligned}
&\lim_{\substack{(r,z)\to(0,-\psi(0))\\ r\neq 0}} \Omega^2(r,z)\\
=&\lim_{\substack{(r,z)\to(0,-\psi(0))\\ r\neq 0}} \frac{1}{r\rho}\int_{-\psi(r)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)\,d\xi\\
=&\lim_{\substack{(r,z)\to(0,-\psi(0))\\ r\neq 0}} \frac{1}{r\rho_z}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)\\
=&\frac{1}{\rho_z(0,-\psi(0))}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)_r(0,-\psi(0)), \qquad (4.60)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{z\to-\psi(0)} \Omega^2(0,z)\\
=&\lim_{z\to-\psi(0)} \frac{1}{\rho}(p_r - \rho B\rho_r)_r(0,z)\\
=&\lim_{z\to-\psi(0)} \frac{1}{\rho(0,z)}\Bigg(\int_{-\psi(0)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)_r d\xi\\
&\qquad + \psi'(0)\big(\rho_r B\rho_z - \rho_z B\rho_r\big)(0,-\psi(0))\Bigg)\\
=&\frac{1}{\rho_z(0,-\psi(0))}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)_r(0,-\psi(0)). \qquad (4.61)
\end{aligned}$$

3. Let r_0 be such that $\psi(r_0) = 0$. When (r,z) gets close to $(r_0, 0)$ and $z \le 0$, we observe by the differential mean value theorem that

$$\begin{aligned}
&\Omega^2(r,z)\\
=&\frac{1}{r\rho}\int_{-\psi(r)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)\,d\xi\\
=&\frac{1}{r\rho_z}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)(r,z')\\
=&\frac{1}{r\rho_{zz}}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)_z(r,z''), \qquad (4.62)
\end{aligned}$$

where z' is between $-\psi(r)$ and z, and z'' is between z' and 0. Therefore

$$\lim_{(r,z)\to(r_0,0)} \Omega^2(r,z) = \frac{1}{r\rho_{zz}}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)_z(r_0,0). \qquad (4.63)$$

We use hypothesis (a) to conclude that the limit is finite.

□

Remark IV.24. It is possible to establish higher regularity for Ω^2 at the first two types of boundary points. However, at the third type of boundary points, $\psi'(r_0) = \infty$, in order to get higher regularity, we need conditions on how fast $\psi'(r)$ grows at around r_0, which we do not employ ourselves doing here.

With relaxed regularity conditions at the boundary, the same computation works if further growth conditions are imposed on the derivatives of ρ when close to the boundary. Let us now give

Proof of theorem IV.7. As before, we define

$$p(r,z) = \int_{-\psi(r)}^{z} \rho(r,\xi) B\rho_\xi(r,\xi)\, d\xi. \tag{4.64}$$

It follows from hypothesis 1, 2, 4 that $p > 0$ in D, $p = 0$ on ∂D and that p satisfies the second equation in (4.53). Since $\rho \in C^\beta(\bar{D})$, we have $B\rho \in C^{2,\beta}(\bar{D})$, hence $p \in C^0(\bar{D})$. Now let us calculate the r partial derivative of p. In the following, let $b = \max\limits_{r \le s \le r+h} \big(-\psi(s)\big)$.

$$\begin{aligned}
&\frac{1}{h}\bigg(\int_{-\psi(r+h)}^{z} \rho B\rho_\xi(r+h,\xi)\, d\xi - \int_{-\psi(r)}^{z} \rho B\rho_\xi(r,\xi)\, d\xi \bigg) \\
=&\frac{1}{h}\bigg(\int_{-\psi(r+h)}^{b} \rho B\rho_\xi\, d\xi - \int_{-\psi(r)}^{b} \rho B\rho_\xi\, d\xi + \int_{b}^{z} \big(\rho B\rho_\xi(r+h,\xi) - \rho B\rho_\xi(r,\xi)\big)\, d\xi \bigg).
\end{aligned} \tag{4.65}$$

It is easily seen that the first two terms converge to 0 as h goes to 0. Let us focus on the last term:

$$\begin{aligned}
&\frac{1}{h}\int_{b}^{z} \big(\rho B\rho_\xi(r+h,\xi) - \rho B\rho_\xi(r,\xi)\big)\, d\xi \\
=&\int_{b}^{z} \big(\rho B\rho_\xi\big)_r (r',\xi)\, d\xi,
\end{aligned} \tag{4.66}$$

where r' is between r and $r+h$. We will use the dominated convergence theorem to compute the limit of (4.66). For that purpose we need an estimate on $\chi_{(b,z)}\left(\rho B\rho_\xi\right)_r(r',\xi)$. For the moment let us assume $z<0$. By hypothesis 5, there is a $C>0$ such that $|\rho_r|\le C|\rho_\xi|$, $|\rho_{rr}|\le C|\rho_\xi|$, $|\rho_{r\xi}|\le C|\rho_\xi|$ for $\xi<z$. Also $\rho_\xi>0$. Therefore

$$\begin{aligned}
&|\left(\rho B\rho_\xi\right)_r(r',\xi)|\\
\le&C_1|\rho_r(r',\xi)|+C_2|\rho(r',\xi)|\\
\le&C_1\left|\rho_r(r,\xi)+\int_r^{r'}\rho_{ss}(s,\xi)ds\right|+C_2\left|\rho(r,\xi)+\int_r^{r'}\rho_s(s,\xi)ds\right|\\
\le&(C_1+C_2)\left(|\rho_r(r,\xi)|+|\rho(r,\xi)|+C\int_{r-h_0}^{r+h_0}\rho_\xi(s,\xi)ds\right)\\
\le&\tilde{C}\left(\rho_\xi(r,\xi)+\rho(r,\xi)+\int_{r-h_0}^{r+h_0}\rho_\xi(s,\xi)ds\right),
\end{aligned}\tag{4.67}$$

for some fixed $h_0>h$. In the integral term, if (s,ξ) lies outside D, then extend ρ_ξ to be 0. The fact that the integral of (4.67) is finite is manifested by the following:

$$\int_{-\psi(r)}^{z}\rho_\xi(r,\xi)\;d\xi=\rho(r,z)\tag{4.68}$$

$$\int_{-\psi(r)}^{z}\rho(r,\xi)\;d\xi<\infty\tag{4.69}$$

$$\begin{aligned}
&\int_{-\psi(r)}^{z}\int_{r-h_0}^{r+h_0}\rho_\xi(s,\xi)\;dsd\xi\\
\le&\int_{r-h_0}^{r+h_0}\int_{-\psi(s)}^{z}\rho_\xi(s,\xi)\;d\xi ds\\
\le&\int_{r-h_0}^{r+h_0}\rho(s,z)ds\\
<&\infty.
\end{aligned}\tag{4.70}$$

Therefore, by the dominated convergence theorem,

$$p_r(r,z)=\int_{-\psi(r)}^{z}\left(\rho B\rho_\xi\right)_r\;d\xi.\tag{4.71}$$

Now if $z \geq 0$, the integral in (4.66) can be broken into two pieces: one from b to z' and the other from z' to z, for some $z' < 0$. Notice that the second piece lies completely inside D, where ρ is C^2, so the limit is the same as before. We have proved $p \in C^1(D)$. Now define Ω^2 by

$$\frac{1}{r\rho}\int_{-\psi(r)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)\, d\xi. \tag{4.72}$$

when $r > 0$, and if D contains points at $r = 0$, by

$$\Omega^2(0,z) = \frac{1}{\rho}\int_{-\psi(0)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)_r\, d\xi \tag{4.73}$$

The convergence of these integrals are guaranteed by hypothesis 5. It is easy to verify that such p and Ω^2 satisfy (4.53). Let us now show that Ω^2 is continuous on D. Since ∂D is smooth and convex at $(0, -\psi(0))$, $-\psi(r) = \max\limits_{0\leq s\leq r} \big(-\psi(s)\big)$ for r small enough. Therefore,

$$\begin{aligned}&\frac{1}{r}\int_{-\psi(r)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)\, d\xi\\ &= \int_{-\psi(r)}^{z} \big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)_r(r',\xi)\, d\xi,\end{aligned} \tag{4.74}$$

where r' is between 0 and r. As before we assume $z < 0$ and estimate the integrand,

$$\begin{aligned}&|\big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)_r(r',\xi)|\\ \leq& C_1(|\rho_{rr}(r',\xi)| + |\rho_{r\xi}(r',\xi)| + |\rho_r(r',\xi)| + |\rho_\xi(r',\xi)|)\\ \leq& \tilde{C}\rho_\xi(r',\xi)\\ \leq& \tilde{C}\left(\rho_\xi(0,\xi) + \int_0^{r'} \rho_{s\xi}(s,\xi)ds\right)\\ \leq& \tilde{C}\left(\rho_\xi(0,\xi) + \int_0^{r_0} \rho_\xi(s,\xi)ds\right),\end{aligned} \tag{4.75}$$

where $r_0 > r$ is small and fixed. As before, (4.75) has a finite ξ integral. By the dominated convergence theorem,

$$\lim_{\substack{(r,z)\to(0,z_0)\\ r\neq 0, z_0<0}} \Omega^2(r,z) = \Omega^2(0,z). \tag{4.76}$$

Again by splitting the integral (4.72) into boundary and interior parts, (4.76) continues to be true when $z_0 \geq 0$, and $\Omega^2(0, z)$ is evidently continuous in z. This establishes the continuity of Ω^2 at $D \cap \{r = 0\}$. The continuity of Ω^2 away from the z axis is obvious. It remains to show that $\Omega^2 \in L^\infty(D)$. Let us consider the following three cases:

1. Let r_0 be a nonzero radius such that $(r_0, -\psi(r_0)) \in \partial D$ and $\psi(r_0) > 0$. When $z < -\dfrac{1}{2}\psi(r_0)$,

$$\begin{aligned}
&\Omega^2(r_0, z)\\
=&\frac{1}{r_0\rho}\int_{-\psi(r_0)}^{z} \left(\rho_r B\rho_\xi - \rho_\xi B\rho_r\right)\, d\xi\\
=&\frac{1}{r_0\rho_z}\left(\rho_r B\rho_z - \rho_z B\rho_r\right)(r_0, z')\\
\leq&\frac{1}{r_0}(C|B\rho_z| + |B\rho_r|)\\
\leq&\frac{\tilde{C}}{r_0},
\end{aligned} \tag{4.77}$$

where C is given by hypothesis 5.

2. If ∂D contains points at $r = 0$, as (r, z) gets close to $(0, -\psi(0))$, $r \neq 0$,

$$\begin{aligned}
&\Omega^2(r, z)\\
=&\frac{1}{r\rho}\int_{-\psi(r)}^{z} \left(\rho_r B\rho_{xi} - \rho_\xi B\rho_r\right)\, d\xi\\
=&\frac{1}{r\rho_z}\left(\rho_r B\rho_z - \rho_z B\rho_r\right)(r, z')\\
=&\frac{\left(\rho_r B\rho_z - \rho_z B\rho_r\right)_r}{\rho_z + r\rho_{rz}}(r', z')\\
\leq&\frac{\tilde{C}}{1 + r'C}
\end{aligned} \tag{4.78}$$

where z' is between $-\psi(r)$ and z, and r' is between 0 and r. In this process we have used the mean value theorem several times, the justification being that the convexity of ∂D at $(0, -\psi(0))$ guarantees that all the relevant segments lie inside D. On the other hand if $r = 0$,

$$\begin{aligned}
&\Omega^2(0,z) \\
=&\frac{1}{\rho}\int_{-\psi(0)}^{z}\big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)_r\, d\xi \\
=&\frac{1}{\rho_z}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)_r(0,z') \\
\le&\tilde{C}. \qquad (4.79)
\end{aligned}$$

3. Let r_0 be such that $\psi(r_0) = 0$. When (r, z) gets close to $(r_0, 0)$ and $z \le 0$,

$$\begin{aligned}
&\Omega^2(r,z) \\
=&\frac{1}{r\rho}\int_{-\psi(r)}^{z}\big(\rho_r B\rho_\xi - \rho_\xi B\rho_r\big)\, d\xi \\
=&\frac{1}{r\rho_z}\big(\rho_r B\rho_z - \rho_z B\rho_r\big)(r,z') \qquad (4.80) \\
=&\big(\frac{\rho_r B\rho_z}{r\rho_z} - \frac{B\rho_r}{r}\big)(r,z') \qquad (4.81) \\
=&\frac{1}{r\rho_{zz}}\big(\rho_r B\rho_z\big)_z(r,z'') - \frac{B\rho_r}{r}(r,z'), \qquad (4.82)
\end{aligned}$$

where z' is between $-\psi(r)$ and z, and z'' is between z' and 0. If hypothesis (a) is satisfied, by (4.82), $\Omega^2(r, z)$ is bounded. If hypothesis (a') is satisfied, by (4.80) and the fact that $\rho_z > 0$, $B\rho_z > 0$ when $z < 0$,

$$0 \le \Omega^2(r,z) \le -\frac{1}{r}B\rho_r(r,z'). \qquad (4.83)$$

Therefore $\Omega^2(r, z)$ is bounded. If hypothesis (a") is satisfied, by (4.81) and the fact that $|B\rho_z| < C|z|$, we again get the boundedness of $\Omega^2(r, z)$.

$\square$

CHAPTER V

Further Extensions of the Physical Equations and Future Work

5.1 Solving for Entropy and Density Simultaneously

One differenet way of viewing equation (1.57) is to regard it as a system of equations for s and ρ with given Ω. That this gives the correct number of equations for the number of unknowns is manifested by its cylindrical form:

$$\begin{cases} (e^s\rho^\gamma)_r = \rho(B\rho)_r + \rho r\Omega^2 \\ (e^s\rho^\gamma)_z = \rho(B\rho)_z \end{cases} \tag{5.1}$$

Whether this system has a variational formulation is unknown. With $\Omega(r,z)$ having generic dependence on r as well as z, (5.1) poses a very different problem than the previously mentioned ones for isentropic Euler-Poisson equations. In chapter IV, we solved the divergence of (1.57) with prescribed s. In that process, s is essentially chosen arbitrarily since any axisymmetric stationary s would satisfy the entropy transport equation in the Euler-Poisson system, therefore is not excluded a priori. In comparison, (5.1) poses much stronger constraints on s to the extent that it becomes an unknown to solve for. Before arriving at the ultimate goal of solving (5.1), one may try to formulate questions that "interpolate" between these two views. Here is one possible question: for given total mass, what entropy configuration can give existence of a density function that solves the divergence equation?

5.2 Other Ways to Extend the Euler-Poisson Equations

Tassoul [42] provides a comprehensive exposition of the detailed physical principles employed in rotating star models. One can extend the physical equations to account for heat conduction, radiation pressure and nuclear fusion in the star. The resulting equation is conceivably much more complicated, but one could progressively include more terms into the analysis and come up with new results of increasing physical relevance to the state of the art stellar models used in astrophysics.

Another scheme of extending the Euler-Poisson equations is to incorporate electromagnetism. Federbush et al. [13] considered a model for stationary magnectic stars. They augmented the Euler-Poisson equations by the Maxwell equations (see also [12], [39] and [43]):

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla \phi + \dfrac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \\ \nabla \cdot \mathbf{B} = 0 \\ \Delta \phi = 4\pi \rho \end{cases} \tag{5.2}$$

and proved an existence result for axisymmetric non-rotating magnetic stars. The next possible step is to add rotation or perhaps even a solid core.

Another big problem related to the Euler-Poisson equations is the existence of rotating relativistic stars. See Friedman and Stergioulas [18] and Gourgoulhon [20] for an introduction. Heilig [22] provided solutions with small deviation from a Newtonian rotating star, assuming that the speed of light is sufficiently large. Hartle and

Sharp [21] and Bardeem [4] proposed variational formulations for rotating relativistic stars with given integral constraints in a flavor like the rotating star solutions to the Euler-Poisson equations, but the energy functionals are strongly indefinite and non-convex. Some fundamentally new ideas are needed to show existence in this setting.

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