Contributions to High-dimensional Data Analysis using Factor Models and Low Rank Approximations

by

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To my parents, Laiqi Zhang and Lijun Qiao,
to my beloved husband, Yingchuan Wang and our coming baby!

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ABSTRACT

Modern information technology has enabled collecting data of unprecedented size and complexity, but it also presents significant challenges to learn from these data. This thesis seeks to close some apparent gaps between the growing size of emerging datasets and the capabilities of existing approaches to statistical modeling and computing. Specifically, we focus on three problems that arise in learning from high-dimensional data and are of great use in practice. The first problem is to estimate high-dimensional covariance matrix for financial assets via the Barra model, which is one of the most widely used risk models in financial industry. We first study theoretical properties of the Barra model, which have not been investigated in the literature. A surprising conclusion is that as the sample size increases, the Barra approach is in fact not asymptotically consistent. To improve the estimation of the Barra approach, we re-interpret the Barra model via the framework of the random effects model and propose an EM-like method for estimating the Barra model, which is consistent and performs well when the number of assets is large. With the estimated covariance matrix for financial assets, the second problem we investigate is on selecting stable and sparse portfolios. The ℓ_1-norm regularized mean-variance portfolio analysis has the advantage of simultaneously controlling the estimation error and performing automatic portfolio selection. We propose an efficient algorithm that combines coordinate descent and augmented Lagrangian methods to solve the optimization problem. To further reduce the computational cost, we also propose a novel screening method for solving the ℓ_1-norm regularized optimization problem with an equality constraint. The innovated screening method is able to save substantial computational cost by quickly identifying and removing

the assets that are guaranteed to be zero weighted in the solution. The third problem we consider is to recover the underlying structure of corrupted low rank matrices. Specifically, we assume the observed data matrix is the summation of a low rank matrix, a sparse matrix and noise. We propose a series of spectral regularization algorithms, which are easy to implement and have less computational complexity comparing with existing algorithms. Convergence properties of the proposed algorithms have also been shown under certain conditions.

CHAPTER 1

Introduction

In recent decades, massive amounts of high-dimensional data with complex structures are produced from various domains, including biology, medicine, engineering, etc. The task of understanding and extracting useful knowledge from these massive data presents significant challenges for statistics and machine learning. For example, in the portfolio selection and allocation problem, investors need to estimate the covariance matrix of returns for thousands of assets in order to construct an optimal portfolio. In such a case, the data is high-dimensional but the number of useful samples is usually limited, as longer time series (larger n) increases modeling bias. Consider corrupted image data as another example. The goal here is to recover the underlying structure from the corrupted data. A variety of statistical models and optimization techniques have been proposed and developed to address the high-dimensional issue of these data. Among them, two most widely used ideas are factorization and regularization. In this thesis, we make use of these ideas to develop both flexible and computationally efficient methods to better understand and explore high dimensional and complex real datasets. Three problems are discussed in the following chapters.

In Chapter 2, we investigate the problem of estimating high-dimensional covariance matrix for financial assets using the Barra model. The Barra model is one of the most popular risk models in financial industry, and the Barra one-step and two-step approaches are widely used to implement the estimation. In Chapter 2, we first examine theoretical properties of the Barra model, which have not been studied in the literature. In particular, we investigate the impact of the sample size (i.e., the number of trading days) and the number of financial assets on the performance of the Barra model. We show that as the sample size increases, the Barra approach is in fact not asymptotically consistent. This result is a little surprising and has never been reported. On the other hand, when the sample size is fixed and the number of financial assets increases, which is more realistic in practice, we show that the Barra approach outperforms the sample covariance. To further improve the estimation of Barra approaches, we re-interpret the Barra model via the framework of the random effects model and propose an EM-like method to estimate the covariance. We show that under certain conditions, the new method is asymptotically consistent when the sample size increases, and when the sample size is fixed while the number of financial assets increases, the new method performs as well as the traditional Barra approach. Extensive simulation studies are used to support the theoretical results and compare the Barra approach, the new method and the sample covariance.

With the estimated covariance matrix for financial assets, the next problem we investigate in Chapter 3 is on selecting stable and sparse portfolios. Portfolio allocation with a gross-exposure constraint is an effective way to increase the efficiency and stability of portfolio selection among a vast pool of assets (Fan, Zhang and Yu, 2012). This problem can be formulated as a ℓ_1-norm regularized mean-variance portfolio criterion and it has the advantage of simultaneously controlling the estimation error, and performing automatic portfolio selection. We propose an efficient algorithm that combines coordinate descent with augmented Lagrangian methods to solve the optimization problem. To further reduce the computational cost, we also propose a novel screening algorithm for solving the ℓ_1-norm regularized mean-variance analysis via constraining its dual feasible set. The innovated screening method is able to quickly identify the assets that are guaranteed to be zero weighted in the solution, so that the assets can be safely removed from the optimization. To the best of our knowledge, this is the first safe screening algorithm that accommodates equality constraints in sparse learning problems. We demonstrate the effectiveness of the proposed algorithms using extensive simulation and empirical studies.

In Chapter 4, we consider the situation that outliers or corruptions exist in the observed high dimensional data. Specifically, we consider the scenario where the data matrix is the summation of a low rank matrix, a sparse matrix and noise. To recover the low rank matrix, various convex-optimization-based algorithms, including the well-recognized robust principle component analysis, have been proposed for the situation where noise is not assumed. However, there is a lack of systematic algorithms for the scenario where perturbations/noise are present. In Chapter 4, we propose a series of spectral regularization algorithms for denoising corrupted low rank matrices. Comparing with existing algorithms, the proposed algorithms are easy to implement and have less computational complexity. Convergence properties for the proposed algorithms have also been shown under certain conditions. Numerical results support the applicability of the proposed algorithms in practice.

CHAPTER 2

High Dimensional Covariance Matrix Estimation via the Barra Model

2.1 Introduction

2.1.1 Background

Covariance matrices and their inverses play a key role in portfolio allocation and risk management. For example, when determining a proper asset allocation, one aims at maximizing the expected return and at the same time minimizing the risk, which depends on the covariance matrix of the financial assets. The most classic estimate for the covariance matrix is the sample covariance

$$\widehat{\boldsymbol{\Sigma}}_{sam} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{r}_i - \bar{\mathbf{r}})(\mathbf{r}_i - \bar{\mathbf{r}})', \tag{2.1}$$

where $\mathbf{r}_i, i = 1, \ldots, n$ are n asset returns. More recently, Ledoit and Wolf (2004) introduced a shrinkage sample covariance estimate, which is an asymptotically optimal convex linear combination of the sample covariance matrix and the identity matrix. Optimality is in the sense with respect to a quadratic loss function. A main benefit of the shrinkage estimate is that it is invertible when the number of assets is larger than the sample size n, and the inversion does not amplify estimation error. Specifically, the shrinkage sample covariance can be expressed as

$$\widehat{\boldsymbol{\Sigma}}_{shk} = \rho_1 \mathbf{I}_p + \rho_2 \widehat{\boldsymbol{\Sigma}}_{sam}, \tag{2.2}$$

where $\mathbf{I}_p$ is the identity matrix, ρ_1 and ρ_2 are chosen such that $\widehat{\boldsymbol{\Sigma}}_{shr}$ minimizes a quadratic loss. In the context of financial risk management, both the sample covariance and the shrinkage sample covariance only use the returns of the assets, while not using other available information such as the risk attributes.

To address this issue, multi-factor models have been proposed to describe asset returns and their covariance matrix as functions of a limited number of risk attributes. Sharp (1963), Lintner (1965) and Mossin (1966) introduced the Capital Asset Pricing Model (CAPM) independently, which uses the stock beta to determine a theoretically appropriate required rate of return of an asset. This model only involves one relevant risk measure, which does not adequately explain the variation in stock returns. To further improve the CAPM, Ross (1976) posited the Arbitrage Pricing Theory (APT), which is a more general multi-factor structure for the return generating processes. This theory has been widely used. A specific example is Fama and French (1992), which identified the price to book value ratio and the market capitalization (in addition to the stock beta) to evaluate stock returns.

Another advantage of the multi-factor model arises when we deal with high dimensional assets. In practice, longer time series (larger n) increase modeling bias; thus, the number of stocks p is usually of the same order or even higher order compared to the sample size n. Under the multi-factor model, the high dimensional assets will be projected to low dimensional factors, resulting in a significant reduction on the number of parameters that we need to estimate.

In general, multi-factor models posit that the period returns of different assets are explained by common factors in a linear model. The asset returns are relevant to the factors according to the sensitivities of a specific asset to them. In addition, they are relevant to another component, the so-called specific returns, which are assumed to be independent of the factor returns. Specifically, a multi-factor model for a relevant asset can be expressed as

$$R_j = b_{j1}F_1 + \cdots + b_{jK}F_K + \epsilon_j, \qquad j = 1, \ldots, p, \tag{2.3}$$

where R_j is the return of the jth asset; p is the number of financial assets; b_{jk} is the sensitivity of asset j to factor k, which is also called factor loading; $F_1, \ldots, F_K$ are K factors; and ϵ_j is the specific return of the asset j, which plays the same role as noise in the linear model.

With some blurring at the boundaries, multi-factor models of asset returns can be divided into three types: the macroeconomic factor models, the statistical factor models and the fundamental factor models.

The macroeconomic factor models are the simplest and most intuitive type among multi-factor models. They use observable economic time series as measures of factors in asset returns. Simply speaking, the factors $F_1, \ldots, F_K$ can be observed at each time point, but the factor loadings are not observed. This kind of factor model has been studied in the literature for a long time. Sharp (1963) first analyzed the single factor model; Chen, Roll and Ross (1986) provided a description of commonly used macroeconomic factors for equity. A series of papers, such as Fan and Li (2006), Fan, Fan and Lv (2008), Fan and Lv (2010), and Fan, Liao and Mincheva (2011), which discussed the properties of the estimators of covariance matrix with a diverging number of parameters, are also based on this kind of macroeconomic factor models. In particular, Fan, Fan and Lv (2008) established asymptotic properties for both the estimator of the macroeconomic factor model and the traditional sample covariance estimator. However, a necessary assumption for the macroeconomic factor models is that the observations or measurements of factors are known, which is not always easy in practice.

The second common method for estimating factor models is the statistical factor analysis. In this type of approach, both the loadings and the factors are estimated simultaneously. Various maximum-likelihood and principal-components-based factor analysis procedures on cross-sectional/time-series samples of asset returns to identify the factors have been proposed to analyze the statistical factor models. See Anderson (1963), Priestley, Rao and Tong (1974), Brillinger (1981), Peña and Box (1987), Chamberlain and Rothschild (1983), Johnson and Wichern (2002), Bai (2003), Forni et al. (2000, 2004, 2005), Pan and Yao (2008) and Lam and Yao (2011), for references. This approach can on one hand, "optimally" explain the past, but on the other hand, can hardly be interpreted in finance.

The third kind of multi-factor models is the fundamental factor analysis. This kind of factor models rely on the empirical findings including company attributes such as firm size, dividend yield, book-to-market ratio, and industry classification which explain a substantial proportion of common returns. In contrast with macroeconomic factor models, all b_{ij} can be observed from the empirical data, but the $F_1, \ldots, F_K$ can be regarded as a kind of unobservable returns. The two most popular fundamental factor models are the Barra model and Fama-French model. The Barra model was proposed by Barr Rosenberg (Rosenberg, 1974), founder of Barra Inc., and the estimation procedure was named as the Barra approach by Grinold and Kahn (2000). Barra products, one of the star products of MSCI Barra, Inc., are powered by the Barra model. On the other hand, the Fama-French model has been studied more in the literature. The main idea of Fama-French model is that the factor realization F_k for a given fundamental is obtained by constructing some hedge portfolio based on observed fundamentals. This model is estimated by a two-stage cross-sectional regression method. Fama and MacBeth (1973) suggested conducting cross-sectional regression for each stage and then treating the estimates as independent samples of the estimated parameters. Jagannathan and Wang (1998) derived the asymptotic distribution of the estimators for the Fama-French model.

Connor (1995) and Sheikh (1996) made a comparison between the three kinds of multi-factor models. Connor (1995) pointed out that the fundamental factor model slightly outperforms the statistical factor model, and substantially outperforms the macroeconomic factor model in the sense of explanatory power. Sheikh (1996) showed that Barra's predicted betas are clearly better predictors of future betas compared to the rest of the models. However, a little surprising to us, there is not much literature directly discussing the properties of the Barra model, especially theoretical properties, which deserve attention.

Besides the multi-factor models we mentioned above, a variety of regularization methods, including banding, tapering, thresholding and penalization, have been introduced for estimating several classes of covariance and precision matrices with different structures. See, for example, Bickel and Levina (2008a), Bickel and Levina (2008b), Cai and Liu (2011), Cai et al. (2013), Lam and Fan (2009) and Ravikumar et al. (2011), among many others.

In this chapter, we focus on studying the Barra model. We re-interpret the Barra model in the framework of the linear random effects model, develop a new algorithm for estimating the Barra model, and also develop theoretical properties of the Barra estimates.

2.1.2 The Barra factor model

Let p be the number of financial assets, and $F_1, \ldots, F_K$ be the K factors. We rewrite the factor model in matrix form as

$$\mathbf{r} = \mathbf{B}\mathbf{f} + \boldsymbol{\epsilon}, \tag{2.4}$$

where $\mathbf{r} = (R_1, \ldots, R_p)'$, $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_p)'$ with $\mathbf{b}_j = (b_{j1}, \ldots, b_{jK})'$, $j = 1, \ldots, p$, $\mathbf{f} = (F_1, \ldots, F_K)'$, and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_p)'$. In the Barra model, we assume $\mathbf{B}$ is observed from the empirical findings regarding the company attributes, whereas factors $F_1, \ldots, F_K$ are not observed. We assume both $\mathbf{f}$ and $\boldsymbol{\epsilon}$ are random variables. Throughout we assume that $E(\boldsymbol{\epsilon}|\mathbf{f}) = \mathbf{0}$ and $\mathrm{cov}(\boldsymbol{\epsilon}|\mathbf{f}) = \boldsymbol{\Sigma}_\epsilon$ is diagonal but not necessarily homogeneous. In this work, we are interested in estimating the covariance matrix of returns, denoted by $\boldsymbol{\Sigma}$. Simple calculation shows that

$$\boldsymbol{\Sigma} = \mathrm{cov}(\mathbf{r}) = \mathbf{B}\mathrm{cov}(\mathbf{f})\mathbf{B}' + \boldsymbol{\Sigma}_\epsilon. \tag{2.5}$$

A natural idea for estimating $\boldsymbol{\Sigma}$ is to first estimate $\mathbf{f}_j$ and $\boldsymbol{\epsilon}_j$, $j = 1, \ldots, p$ using least squares regression, then estimate $\mathrm{cov}(\mathbf{f})$ and $\boldsymbol{\Sigma}_\epsilon$ using sample covariance. Therefore, we have a plugged-in estimate $\widehat{\boldsymbol{\Sigma}} = \mathbf{B}\widehat{\mathrm{cov}}(\mathbf{f})\mathbf{B}' + \widehat{\boldsymbol{\Sigma}}_\epsilon$.

Suppose we have observations $\mathbf{r}_1, \ldots, \mathbf{r}_n$ on n days. Let $(\mathbf{f}_1, \boldsymbol{\epsilon}_1), \ldots, (\mathbf{f}_n, \boldsymbol{\epsilon}_n)$ be independent and identically distributed (i.i.d.) samples of $(\mathbf{f}, \boldsymbol{\epsilon})$ and denote

$$\mathbf{F} = (\mathbf{f}_1, \ldots, \mathbf{f}_n), \mathbf{E} = (\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n) \text{ and } \mathbf{R} = (\mathbf{r}_1, \ldots, \mathbf{r}_n).$$

There are two methods to estimate the factors and specific returns. One simpler way is to use the ordinary least squares (OLS) estimation. While the covariance matrix of $\boldsymbol{\epsilon}$ is not homogeneous, one might suggest to apply the weighted least squares (WLS) to estimate the factors instead of OLS. Note that in the second method, OLS is also a necessary step. We refer the OLS method as the Barra one-step approach, and the WLS method as the Barra two-step approach. Specifically, the Barra one-step approach proceeds as follows

$$\widehat{\mathbf{F}}_o = (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{R}, \tag{2.6}$$

$$\widehat{\mathbf{E}}_o = \mathbf{R} - \mathbf{B}\widehat{\mathbf{F}}_o, \tag{2.7}$$

$$\widehat{\mathrm{cov}}(\mathbf{f})_o = \frac{1}{n-1}(\widehat{\mathbf{F}}_o\widehat{\mathbf{F}}'_o - \frac{1}{n}\widehat{\mathbf{F}}_o\mathbf{1}\mathbf{1}'\widehat{\mathbf{F}}'_o), \tag{2.8}$$

$$\widehat{\boldsymbol{\Sigma}}_{\epsilon,o} = \mathrm{diag}(\frac{1}{n}\widehat{\mathbf{E}}_o\widehat{\mathbf{E}}'_o), \tag{2.9}$$

$$\widehat{\boldsymbol{\Sigma}}_o = \mathbf{B}\widehat{\mathrm{cov}}(\mathbf{f})_o\mathbf{B}' + \widehat{\boldsymbol{\Sigma}}_{\epsilon,o}, \tag{2.10}$$

where the subscript o is used to denote the OLS estimate.

Given $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$, the Barra two-step approach estimates $\mathbf{F}$ in a slightly different way,

$$\widehat{\mathbf{F}}_w = (\mathbf{B}'\widehat{\boldsymbol{\Sigma}}^{-1}_{\epsilon,o}\mathbf{B})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}^{-1}_{\epsilon,o}\mathbf{R}, \tag{2.11}$$

$$\widehat{\mathbf{E}}_w = \mathbf{R} - \mathbf{B}\widehat{\mathbf{F}}_w, \tag{2.12}$$

$$\widehat{\mathrm{cov}}(\mathbf{f})_w = \frac{1}{n-1}(\widehat{\mathbf{F}}_w\widehat{\mathbf{F}}'_w - \frac{1}{n}\widehat{\mathbf{F}}_w\mathbf{1}\mathbf{1}'\widehat{\mathbf{F}}'_w), \tag{2.13}$$

$$\widehat{\boldsymbol{\Sigma}}_{\epsilon,w} = \mathrm{diag}(\frac{1}{n}\widehat{\mathbf{E}}_w\widehat{\mathbf{E}}'_w), \tag{2.14}$$

$$\widehat{\boldsymbol{\Sigma}}_w = \mathbf{B}\widehat{\mathrm{cov}}(\mathbf{f})_w\mathbf{B}' + \widehat{\boldsymbol{\Sigma}}_{\epsilon,w}, \tag{2.15}$$

where the subscript w is used to denote the WLS estimate.

Note that in practice, $\mathbf{B}$ may not be the same for different i, especially when a number of factors are involved. The estimates in the case with different $\mathbf{B}_i$'s are similar to the above. The details are given in the Appendix.

2.1.3 Random effects model

In this subsection, we point out a connection between the Barra model and the random effects model. Harville (1977) proposed the random effects model,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\mathbf{b} + \mathbf{e}, \tag{2.16}$$

where $\mathbf{y}$ is a $p \times 1$ vector, $\mathbf{X}$ and $\mathbf{Z}$ are $p \times r$ and $p \times s$ matrices respectively, $\boldsymbol{\alpha}$ is a $r \times 1$ vector of unobservable parameters, which are referred as fixed effects, $\mathbf{b}$ is a $s \times 1$ vector of unobservable random effects, and $\mathbf{e}$ is a $p \times 1$ vector of unobservable random errors. The model assumes $\mathrm{E}(\mathbf{b}) = \mathbf{0}$, $\mathrm{E}(\mathbf{e}) = \mathbf{0}$, and $\mathrm{cov}(\mathbf{b}, \mathbf{e}) = \mathbf{0}$. Let $\mathbf{V} = \mathrm{var}(\mathbf{y})$, then $\mathbf{V} = \mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{E}$, where $\mathbf{D} = \mathrm{var}(\mathbf{b})$, and $\mathbf{E} = \mathrm{var}(\mathbf{e})$. We see that if $\mathbf{X} = \mathbf{0}$, this model is similar to the Barra model. The main difference is the assumption for the error term. In the random effects model, $\mathbf{E} = \sigma^2\mathbf{I}$ is often assumed, while in the Barra model, the diagonal elements of $\boldsymbol{\Sigma}_\epsilon$ are not necessary the same. In terms of estimating the random effects model, the maximum likelihood approach and the restricted maximum likelihood approach are among the most popular ones, which have been discussed extensively in the literature such as Harville (1976, 1977), Dempster, Rubin and Tsutakawa (1981).

2.1.4 Major results

We first prove theoretical properties of the Barra approaches with diverging sample size and asset size for the purpose of covariance matrix estimation. Inspired by the connection between the Barra model and the random effects model, we then propose an EM approach to estimate the covariance matrix. The performances of the Barra approaches, the EM approach and the sample covariance estimation are compared, which lead to the following major results:

- The sample covariance estimate is consistent under the Barra model as the sample size tends to infinity, while both the Barra one-step estimator $\widehat{\boldsymbol{\Sigma}}_o$ and two-step estimator $\widehat{\boldsymbol{\Sigma}}_w$ are not consistent.

- The inverse of the covariance matrix estimator based on the Barra approaches cannot be consistent. However, this does not necessarily mean that the Barra model is not useful. When the number of assets p is close to the sample size n or of higher order than n, the performances of the Barra estimators are much better than the sample covariance estimator or the shrinkage sample covariance estimator.

- Under some regularity conditions, we show that the MLE of the Barra model is consistent. Hence, when an EM-like algorithm is used to approximate the MLE, we obtain a consistent estimator $\widehat{\mathbf{\Sigma}}_{EM}$. The new estimator performs well in both n is greater than p and p is greater than n scenarios.

2.1.5 Outline of the Chapter

In Section 2.2, we state the basic assumptions and present the theoretical properties of $\widehat{\mathbf{\Sigma}}_o$, $\widehat{\mathbf{\Sigma}}_w$ and $\widehat{\mathbf{\Sigma}}_{sam}$. Construction and properties of the MLE of the Barra model and the EM approach are discussed in Section 2.3. In Section 2.4, simulation studies are presented, which support the theoretical results. We conclude the results in Section 2.5. All proofs are given in the Appendix.

2.2 Theoretical results for the Barra approaches

Throughout the chapter, we use the Frobenius norm as our major measure for matrix norms, which is given by

$$||\mathbf{A}|| = \{\mathrm{tr}(\mathbf{A}\mathbf{A}')\}^{1/2}, \tag{2.17}$$

for any matrix $\mathbf{A} = (a_{ij})$. In particular, if $\mathbf{A}$ is a $q \times q$ symmetric matrix, $||\mathbf{A}|| = \{\sum_{i=1}^{q} \lambda_i(\mathbf{A})^2\}^{1/2}$, where $\lambda_1(\mathbf{A}), \ldots, \lambda_q(\mathbf{A})$ are the q eigenvalues of $\mathbf{A}$ in decreasing order. The Frobenius norm as well as many other matrix norms (Horn and Johnson, 1985) are intrinsically related to the eigenvalues or singular values of matrices.

2.2.1 Basic assumptions

Let $b = E||\mathbf{R}||^2$, $c = \max_{1 \le k \le K} E(F_k^4)$, and $d = \max_{1 \le j \le p} E(\epsilon_j^4)$.

(A) $\mathbf{f}_1, \ldots, \mathbf{f}_n$ are n i.i.d. samples of factors $\mathbf{f}$, $E(\boldsymbol{\epsilon}|\mathbf{f}) = \mathbf{0}$ and $\mathrm{cov}(\boldsymbol{\epsilon}|\mathbf{f}) = \mathbf{\Sigma}_\epsilon$ is diagonal, and for simplicity, we assume normality for $\boldsymbol{\epsilon}$, i.e. $\boldsymbol{\epsilon} \sim N(0, \mathbf{\Sigma}_\epsilon)$. Also, the distribution of $\mathbf{f}$ is continuous and $K \le p$.

The first and second parts are usual assumptions, and the normality of $\boldsymbol{\epsilon}$ is also common. The assumption that $\mathbf{f}$ has a continuous distribution is made to ensure the matrix $\mathbf{F}\mathbf{F}'$ is invertible with probability one when $p \ge K$.

(B) $b = O(p)$ and c and d are bounded.

$E||\mathbf{R}||^2 = \sum_{j=1}^{p} Er_j^2$, so $b = O(p)$ is a reasonable assumption. In addition, the assumptions of the boundedness of the fourth moments of $\mathbf{f}$ and $\boldsymbol{\epsilon}$ help facilitate the study of $\text{cov}(\mathbf{f})$ and $\boldsymbol{\Sigma}_\epsilon$.

(C) The number of assets p is of a polynomial order of the sample size n.

This assumption is fair because p is hard to exceed the polynomial order of n in practice, though p can be much larger than n.

(D) There exists a constant $\sigma_1 > 0$ such that $\lambda_K(\text{cov}(\mathbf{f})) > \sigma_1$.

(E) There exists a constant $\sigma_2 > 0$ such that $\lambda_p(\boldsymbol{\Sigma}_\epsilon) > \sigma_2$.

(F) If we consider multi-factor models with constant $\mathbf{B}$, there exists a constant $\delta > 0$ such that $|\lambda_K(\mathbf{B}'\mathbf{B})| > \delta$.

The above three assumptions are helpful to the study of the inverse of the covariance matrix, which are also commonly used in literature.

2.2.2 Main results

Theorem 2.2.1 (Lower bounds of covariance estimators for factors)**.** *Under conditions (A)-(F), asymptotically, we have*

$$||\widehat{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})||_F \geq c_1 u, \tag{2.18}$$
$$||\widehat{\text{cov}}(\mathbf{f})_w - \text{cov}(\mathbf{f})||_F \geq c_2 u, \tag{2.19}$$

where c_1 and c_2 are positive constants and $u = ||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||_F$.

Theorem 2.2.2 (Lower bounds of covariance estimators for specific returns)**.** *Under conditions (A)-(F), asymptotically, we have*

$$||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o} - \boldsymbol{\Sigma}_\epsilon||_F \geq c_3 v, \tag{2.20}$$
$$||\widehat{\boldsymbol{\Sigma}}_{\epsilon,w} - \boldsymbol{\Sigma}_\epsilon||_F \geq c_4 v, \tag{2.21}$$

where c_3 and c_4 are positive constants, and $v = ||\text{diag}(\mathbf{H}\boldsymbol{\Sigma}_\epsilon\mathbf{H}) - \text{diag}(\mathbf{H}\boldsymbol{\Sigma}_\epsilon + \boldsymbol{\Sigma}_\epsilon\mathbf{H})||_F$.

Theorem 2.2.3 (Lower bounds of covariance estimators for returns)**.** *Under conditions (A)-(F), we have*

$$||\widehat{\boldsymbol{\Sigma}}_o - \boldsymbol{\Sigma}||_F \geq c_5 w, \tag{2.22}$$
$$||\widehat{\boldsymbol{\Sigma}}_w - \boldsymbol{\Sigma}||_F \geq c_6 w, \tag{2.23}$$

where c_5 and c_6 are positive constants, and $w = ||\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H} + \text{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H}) - \text{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon + \mathbf{\Sigma}_\epsilon\mathbf{H})||_F$.

Theorem 2.2.1 and Theorem 2.2.2 indicate that the Frobenius errors of the covariance estimators for factors and specific returns based on the Barra one-step and two-step approaches are lower bounded. In general, u and v are greater than 0. To see it, we consider a simple example. Suppose $\mathbf{B}$ represents indicators for exclusive company attributes, that is, each row of $\mathbf{B}$ has only one 1, but all the rest are 0s. Denote $b_1, \cdots, b_K$ as the numbers of companies associated to the attributes. Further suppose $\mathbf{\Sigma}_\epsilon$ is the identity matrix. Then we can compute u and v, i.e. $u = (\sum_{k=1}^K 1/b_i^2)^{1/2}$ and $v = (\sum_{k=1}^K 1/b_i)^{1/2}$. This shows the inconsistency of the covariance estimators for factors and specific returns, and the lower bounds are determined by the factor loading $\mathbf{B}$ and $\mathbf{\Sigma}_\epsilon$. Furthermore, since the way we estimate $\mathbf{\Sigma}$ is by plugging in $\widehat{\text{cov}}(\mathbf{f})$ and $\widehat{\mathbf{\Sigma}}_\epsilon$, the Barra one-step and two-step estimators for $\mathbf{\Sigma}$ cannot be consistent either, which is shown in Theorem 2.2.3. Note that w can be also computed in the illustrative example above, which is greater than 0. It seems that this fact has never been reported, and it implies that when the sample size n is large, the Barra approaches may not be a good choice to estimate the covariance matrix.

Theorem 2.2.4 (Rates of covariance estimators for factors)**.** *Under conditions (A)-(F), we have*

$$||\widehat{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})||_F = O_p(n^{-1/2}p^{1/4}K^{3/4}) + O_p(p^{1/2}K^{1/2}), \tag{2.24}$$
$$||\widehat{\text{cov}}(\mathbf{f})_w - \text{cov}(\mathbf{f})||_F = O_p(n^{-1/2}p^{1/4}K^{3/4}) + O_p(p^{1/2}K^{1/2}). \tag{2.25}$$

Theorem 2.2.5 (Rates of covariance estimators for specific returns)**.** *Under conditions (A)-(F), we have*

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||_F = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}), \tag{2.26}$$
$$||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon||_F = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}). \tag{2.27}$$

The above two theorems indicate the upper bounds on the Frobenius norm error of covariance estimators for factors and specific returns based on the Barra one-step and two-step approaches. Note that all bounds contain one convergent term and one extra term $O_p(p^{1/2}K^{1/2})$, and the inconsistency is bounded by $O_p(p^{1/2}K^{1/2})$.

Theorem 2.2.6 (Rates of covariance estimators for returns)**.** *Under conditions (A)-(F), we* *have*

$$||\widehat{\mathbf{\Sigma}}_o - \mathbf{\Sigma}||_F = O_p(n^{-1/2}pK) + O_p(p^{1/2}K^{1/2}), \tag{2.28}$$

$$||\widehat{\mathbf{\Sigma}}_w - \mathbf{\Sigma}||_F = O_p(n^{-1/2}pK) + O_p(p^{1/2}K^{1/2}), \tag{2.29}$$

$$||\widehat{\mathbf{\Sigma}}_{sam} - \mathbf{\Sigma}||_F = O_p(n^{-1/2}pK). \tag{2.30}$$

Based on Theorems 2.2.1-2.2.5, it is not difficult to show Theorem 2.2.6. As discussed above, the Barra one-step and two-step estimators for covariance of returns are not consistent; further, the inconsistency is bounded by a term unrelated with n, i.e. $O_p(p^{1/2}K^{1/2})$. On the other hand, the sample covariance estimator is bounded by a convergent term, so that when n tends to infinity, the difference between the sample covariance estimator and the true covariance goes to zero. Therefore, in the sense of asymptotic properties of covariance matrix estimators, the sample covariance estimator is likely to be better than the Barra estimators. Note that the highest order of p in the above bounds are the same for all three estimators, which implies with fixed n, it is not necessary that the sample covariance estimator is better than the others. We will show more comparisons in the simulation section.

Theorem 2.2.7 (Rates of inverse-covariance estimators for returns)**.** *Under conditions (A)-(F), we have*

$$||\widehat{\mathbf{\Sigma}}_o^{-1} - \mathbf{\Sigma}^{-1}||_F = O_p(n^{-1/2}p^{5/4}K^{5/4}) + O_p(pK), \tag{2.31}$$

$$||\widehat{\mathbf{\Sigma}}_w^{-1} - \mathbf{\Sigma}^{-1}||_F = O_p(n^{-1/2}p^{5/4}K^{5/4}) + O_p(pK), \tag{2.32}$$

$$||\widehat{\mathbf{\Sigma}}_{sam}^{-1} - \mathbf{\Sigma}^{-1}||_F = O_p(n^{-1/2}p^2K). \tag{2.33}$$

Besides, the inverse of the covariance matrix is also quite important in financial risk management. One classic application is in determination of portfolio allocation. The inconsistency of the Barra estimators for the inverse of the covariance matrix inherits from that for the covariance matrix, whereas the inverse of sample covariance matrix estimator is again consistent. From this perspective, Barra estimators are not as good as the sample covariance matrix when n is large. On the other hand, if we do not require that n tends to infinity, but consider the case with increasing p, the order of p will dominate the performance of different estimators. In that case, $\widehat{\mathbf{\Sigma}}_o^{-1}$ and $\widehat{\mathbf{\Sigma}}_w^{-1}$ perform much better than $\widehat{\mathbf{\Sigma}}_{sam}^{-1}$, benefiting from the order of p. Since this is the more common situation in practice, the Barra approaches have an advantage over the sample covariance in real life, especially for estimating the inverse of the covariance matrix.

We have also conducted theoretical analysis in the case with changing $\mathbf{B}_i$'s. Similar conclusions can be drawn, and the details are given in the Appendix.

2.3 New estimation method based on the random effects model

2.3.1 The MLE of the random effects model

Because of the connection between the Barra model and the random effects models, we leverage theoretical results for the random effects models and apply them to the Barra model. Discussion on the random effects model in the literature often make use of the maximum likelihood estimation, and it is well-known that the MLE enjoys good theoretical properties such as consistency with fixed dimension p, but it is not obvious for the case with diverging p. Here, we study the properties of the MLE for the Barra model, with increasing n and p. We assume that the factors $\mathbf{f}_i$'s are from a multivariate normal distribution $N(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{f}})$, then the likelihood function of the Barra model can be written as

$$\mathbf{L}_n(\mathbf{\Sigma}_{\mathbf{f}}, \mathbf{\Sigma}_\epsilon|\mathbf{r}_i) = \prod_{i=1}^{n}(2\pi)^{-p/2}|\mathbf{B}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{B}' + \mathbf{\Sigma}_\epsilon|^{-1/2}\exp\{-\frac{1}{2}\mathbf{r}_i'(\mathbf{B}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{B}' + \mathbf{\Sigma}_\epsilon)^{-1}\mathbf{r}_i\}. \quad (2.34)$$

The next theorem illustrates the convergence rates of the maximum likelihood estimators.

Theorem 2.3.1 (Rates of convergence of MLE)**.** *Under regularity conditions Assumption 1 to Assumption 3, there exists a local maximizer of (2.34), such that*

$$||\widehat{\mathbf{\Sigma}}_{\mathbf{f}} - \mathbf{\Sigma}_{\mathbf{f}}^*||_F = O_p(n^{-1/2}K(\text{logK})^{1/2}) \textit{ and } ||\widehat{\mathbf{\Sigma}}_\epsilon - \mathbf{\Sigma}_\epsilon^*||_F = O_p(n^{-1/2}(p\text{log}p)^{1/2}).$$

Furthermore,

$$||\widehat{\mathbf{\Sigma}} - \mathbf{\Sigma}^*||_F = O_p(n^{-1/2}pK(\text{logK})^{1/2}), \quad (2.35)$$

$$||\widehat{\mathbf{\Sigma}}^{-1} - \mathbf{\Sigma}^{*-1}||_F = O_p(n^{-1/2}p\text{log}pK^{1/2}) + O_p(n^{-1/2}p^{1/2}K^{3/2}(\text{log}K)^{1/2}), \quad (2.36)$$

where $\mathbf{\Sigma}_{\mathbf{f}}^*$, $\mathbf{\Sigma}_\epsilon^*$ and $\mathbf{\Sigma}^*$ are the true values.

Theorem 2.3.1 implies that the MLE approach provides consistent estimators for $\mathbf{\Sigma}_{\mathbf{f}}$ and $\mathbf{\Sigma}_\epsilon$, and as a result, for $\mathbf{\Sigma}$ as well. Note that first, $\widehat{\mathbf{\Sigma}}$ works better than $\widehat{\mathbf{\Sigma}}_o$ and $\widehat{\mathbf{\Sigma}}_w$ when n is large because of the consistency. Second, $\widehat{\mathbf{\Sigma}}^{-1}$ enjoys a lower order of p than $\widehat{\mathbf{\Sigma}}_o^{-1}$ and $\widehat{\mathbf{\Sigma}}_w^{-1}$. This implies that the MLE out-performs the Barra estimators in both cases when sample size is large and when asset size is growing with the sample size. Compared to $\widehat{\mathbf{\Sigma}}_{sam}$, the convergence rate of covariance estimators are the same, in terms of n and p, but as for inverse-covariance estimators, $\widehat{\mathbf{\Sigma}}^{-1}$ performs much better than $\widehat{\mathbf{\Sigma}}_{sam}^{-1}$ when p increases. Overall, to some extent, the MLE overcomes the drawbacks of both the Barra estimators and the sample covariance.

2.3.2 EM approach

Much research have been done on exploring the MLE of the random effects model. For example, Laird and Ware (1982) discussed a unified approach based on a combination of empirical Bayes and maximum likelihood estimation of model parameters using the EM algorithm. Inspired by the EM algorithm, we construct a new estimation approach to approximate the maximum likelihood estimator.

Because of the normality and the transformation

$$\begin{pmatrix} \mathbf{f}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{f}_i \\ \boldsymbol{\epsilon}_i \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{f}_i \\ \boldsymbol{\epsilon}_i \end{pmatrix},$$

we obtain the joint distributions

$$\begin{pmatrix} \mathbf{f}_i \\ \mathbf{r}_i \end{pmatrix} \sim N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{\Sigma_f} & \mathbf{\Sigma_f B}' \\ \mathbf{B\Sigma_f} & \mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon \end{pmatrix} \right),$$

and

$$\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathbf{r}_i \end{pmatrix} \sim N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{\Sigma}_\epsilon & \mathbf{\Sigma}_\epsilon \\ \mathbf{\Sigma}_\epsilon & \mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon \end{pmatrix} \right).$$

Then the conditional distributions are given by

$$\begin{aligned} \mathbf{f}_i|\mathbf{r}_i &\sim N(\mathbf{\Sigma_f B}'(\mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon)^{-1}\mathbf{r}_i, \mathbf{\Sigma_f} - \mathbf{\Sigma_f B}'(\mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon)^{-1}\mathbf{B\Sigma_f}), \\ \boldsymbol{\epsilon}_i|\mathbf{r}_i &\sim N(\mathbf{\Sigma}_\epsilon(\mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon)^{-1}\mathbf{r}_i, \mathbf{\Sigma}_\epsilon - \mathbf{\Sigma}_\epsilon(\mathbf{B\Sigma_f B}' + \mathbf{\Sigma}_\epsilon)^{-1}\mathbf{\Sigma}_\epsilon). \end{aligned}$$

By Sherman-Morrison formula

$$(\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}\mathbf{B}' + \boldsymbol{\Sigma}_{\epsilon})^{-1} = \boldsymbol{\Sigma}_{\epsilon}^{-1} - \boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1},$$

we can simplify the above formulas as

$$\begin{aligned}
\mathrm{E}(\mathbf{f}_i|\mathbf{r}_i) &= (\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{r}_i, \\
\mathrm{Var}(\mathbf{f}_i|\mathbf{r}_i) &= (\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{f}}^{-1})^{-1}, \\
\mathrm{E}(\boldsymbol{\epsilon}_i|\mathbf{r}_i) &= \left[\mathbf{I} - \mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\right]\mathbf{r}_i, \\
\mathrm{Var}(\boldsymbol{\epsilon}_i|\mathbf{r}_i) &= \mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'.
\end{aligned}$$

Based on the conditional mean and conditional variance, the estimators for covariance matrices can be constructed through

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}} &= \widehat{\mathrm{Var}}(\mathrm{E}(\mathbf{f}_i|\mathbf{r}_i)) + \widehat{\mathrm{E}}(\mathrm{Var}(\mathbf{f}_i|\mathbf{r}_i)), \\
\widehat{\boldsymbol{\Sigma}}_{\epsilon} &= \widehat{\mathrm{Var}}(\mathrm{E}(\boldsymbol{\epsilon}_i|\mathbf{r}_i)) + \widehat{\mathrm{E}}(\mathrm{Var}(\boldsymbol{\epsilon}_i|\mathbf{r}_i)).
\end{aligned}$$

Therefore, by plugging in the conditional expectation and conditional variance, we propose another way to estimate the covariance matrix. The nested structure demands an iterative procedure as follows,

$$\widehat{\mathbf{f}}_i = \widehat{\mathrm{E}}(\mathbf{f}_i|\mathbf{r}_i) = (\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\mathbf{B} + \widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\mathbf{r}_i, \tag{2.37}$$

$$\widehat{\boldsymbol{\epsilon}}_i = \widehat{\mathrm{E}}(\boldsymbol{\epsilon}_i|\mathbf{r}_i) = \left[\mathbf{I} - \mathbf{B}(\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\mathbf{B} + \widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\right]\mathbf{r}_i, \tag{2.38}$$

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}} = \widehat{\mathrm{Var}}(\mathrm{E}(\mathbf{f}_i|\mathbf{r}_i)) + \widehat{\mathrm{E}}(\mathrm{Var}(\mathbf{f}_i|\mathbf{r}_i)) = \frac{1}{n}\sum_{i=1}^{n}\widehat{\mathbf{f}}_i\hat{\mathbf{f}}_i' + (\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\mathbf{B} + \widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1})^{-1}, \tag{2.39}$$

$$\widehat{\boldsymbol{\Sigma}}_{\epsilon} = \widehat{\mathrm{Var}}(\mathrm{E}(\boldsymbol{\epsilon}_i|\mathbf{r}_i)) + \widehat{\mathrm{E}}(\mathrm{Var}(\boldsymbol{\epsilon}_i|\mathbf{r}_i)) = \mathrm{diag}\left[\frac{1}{n}\sum_{i=1}^{n}\widehat{\boldsymbol{\epsilon}}_i\widehat{\boldsymbol{\epsilon}}_i' + \mathbf{B}(\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{-1}\mathbf{B} + \widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1})^{-1}\mathbf{B}'\right]. \tag{2.40}$$

We stop the iteration when the following two criteria are satisfied

$$||\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{(t+1)} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{(t)}||_F^2 < \delta \tag{2.41}$$

$$||\widehat{\boldsymbol{\Sigma}}_{\epsilon}^{(t+1)} - \widehat{\boldsymbol{\Sigma}}_{\epsilon}^{(t)}||_F^2 < \delta, \tag{2.42}$$

where t is the iteration index, and δ is a pre-specified small number.

Note that this iterative approach is intrinsically an EM-like algorithm. Specifically, let

Θ denote the unknown parameter, which includes $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2, \mathbf{\Sigma_f})$. If we observed $\mathbf{f}_i$ and $\boldsymbol{\epsilon}_i$ in addition to $\mathbf{r}_i$, we could easily find simple closed-form maximum likelihood estimates of the components of Θ, given by

$$\widehat{\sigma}_j^2 = \left[\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i'\right]_{jj}, \tag{2.43}$$

and

$$\widehat{\mathbf{\Sigma}}_{\mathbf{f}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{f}_i \mathbf{f}_i'. \tag{2.44}$$

We denote $\mathbf{t}_1 = \mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i')$ and $\mathbf{t}_2 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{f}_i \mathbf{f}_i'$. Hence, $\mathbf{t}_1$ and $\mathbf{t}_2$ are "sufficient statistics" for Θ. The maximum likelihood estimation based on observed sufficient statistics leads to the M-step of the EM algorithm.

If an estimate of Θ is available, we can use it to calculate "estimates" of the missing "sufficient statistics", by setting them equal to their expectations, conditional on the observed data vector $\mathbf{R}$. Let $\widehat{\Theta}$ denote the estimate of Θ, $\widehat{\mathbf{\Sigma}}_\epsilon(\Theta)$ and $\widehat{\mathbf{\Sigma}}_{\mathbf{f}}(\Theta)$ the corresponding estimates of $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}$ and $\mathbf{\Sigma_f}$, and $\widehat{\mathbf{t}}_1$ and $\widehat{\mathbf{t}}_2$ the "estimated sufficient statistics". Then, we have

$$\widehat{\mathbf{t}}_1 = \mathrm{E}\left[\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i')|\mathbf{r}_i, \widehat{\Theta}\right] = \mathrm{diag}\left[\frac{1}{n}\sum_{i=1}^{n} \widehat{\boldsymbol{\epsilon}}_i \widehat{\boldsymbol{\epsilon}}_i' + \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_\epsilon(\Theta)^{-1}\mathbf{B} + \widehat{\mathbf{\Sigma}}_{\mathbf{f}}(\Theta)^{-1})^{-1}\mathbf{B}'\right], \tag{2.45}$$

and

$$\widehat{\mathbf{t}}_2 = \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \mathbf{f}_i \mathbf{f}_i'|\mathbf{r}_i, \widehat{\Theta}\right] = \frac{1}{n}\sum_{i=1}^{n} \widehat{\mathbf{f}}_i \widehat{\mathbf{f}}_i' + (\mathbf{B}'\widehat{\mathbf{\Sigma}}_\epsilon(\Theta)^{-1}\mathbf{B} + \widehat{\mathbf{\Sigma}}_{\mathbf{f}}(\Theta)^{-1})^{-1}. \tag{2.46}$$

This is the E-step of the EM algorithm.

Note that we iterate between equation (2.43)-(2.44) and (2.45)-(2.46), it is equivalent to iterate between equation (2.37)-(2.38) and (2.39)-(2.40). Thus, we call our method the EM approach.

2.4 Simulation studies

In this section, we use simulation studies to illustrate our theoretical results and also to demonstrate finite-sample performance of the estimators for $\mathbf{\Sigma}$ as well as $\mathbf{\Sigma}^{-1}$.

We set the number of factors $K = 5$ in our simulation studies, leading to the model

$$R_j = b_{j1}F_1 + b_{j2}F_2 + b_{j3}F_3 + b_{j4}F_4 + b_{j5}F_5 + \epsilon_j, \quad j = 1, \ldots, p. \tag{2.47}$$

To make the simulation mimic real life, we generate the parameters based on a real dataset. Specifically, we collect the S&P 500 monthly data from 2000 to 2012. We set $\mathbf{B}$ to contain the companies' industry classification, and keep those companies that form the 5 largest industries. Then we fit the Barra model to obtain $\mathrm{cov}(\mathbf{f})$ and $\mathbf{\Sigma}_\epsilon$. The estimated $\mathrm{cov}(\mathbf{f})$ in Table 2.1. The median of the estimated variance of specific returns is 0.00693.

Table 2.1: $\mathrm{cov}(\mathbf{f}) \times 10^4$

9.827	3.406	-3.938	2.001	5.378
3.406	19.507	-0.860	2.510	-8.940
-3.938	-0.860	12.673	-2.151	-5.713
2.001	2.510	-2.151	6.073	-3.355
5.378	-8.940	-5.713	-3.355	46.348

The case with n increasing, while p and K being fixed

First, we explore the influence of the sample size n on different estimators. Fixing the number of assets p at 100, we increase n from 1000 to 2×10^5. We randomly choose 100 companies affiliated to 5 industries, and record the respective $\mathbf{B}$ and $\mathbf{\Sigma}_\epsilon$. Then for each replication, we do the following:

• Generate a random sample of $\mathbf{f} = (F_1, F_2, F_3, F_4, F_5)'$ with size n from the 5-variate normal distribution $N(\mathbf{0}, \mathrm{cov}(\mathbf{f}))$.

• Generate a random sample of $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_p)'$ with size n from the p-variate normal distribution $N(\mathbf{0}, \mathbf{\Sigma}_\epsilon)$.

• Using equation (2.47), we get a sample of $\mathbf{r} = (R_1, \ldots, R_p)'$ with size n.

• Using the Barra one-step approach and two-step approach, the EM approach, and sample covariance, we compute the estimates for the covariance matrix and the inverse of the covariance matrix, denoted by $\widehat{\mathbf{\Sigma}}_o$, $\widehat{\mathbf{\Sigma}}_o^{-1}$, $\widehat{\mathbf{\Sigma}}_w$, $\widehat{\mathbf{\Sigma}}_w^{-1}$, $\widehat{\mathbf{\Sigma}}_{EM}$, $\widehat{\mathbf{\Sigma}}_{EM}^{-1}$, $\widehat{\mathbf{\Sigma}}_{sam}$, and $\widehat{\mathbf{\Sigma}}_{sam}^{-1}$ respectively. Then we compute the difference between the estimate and the true covariance matrix with respect to the Frobenius norm.

We repeat the above simulation 50 times for each n and report the mean of the Frobenius error in Figure 2.1 (for covariance) and Figure 2.2 (for the inverse of the covarince).

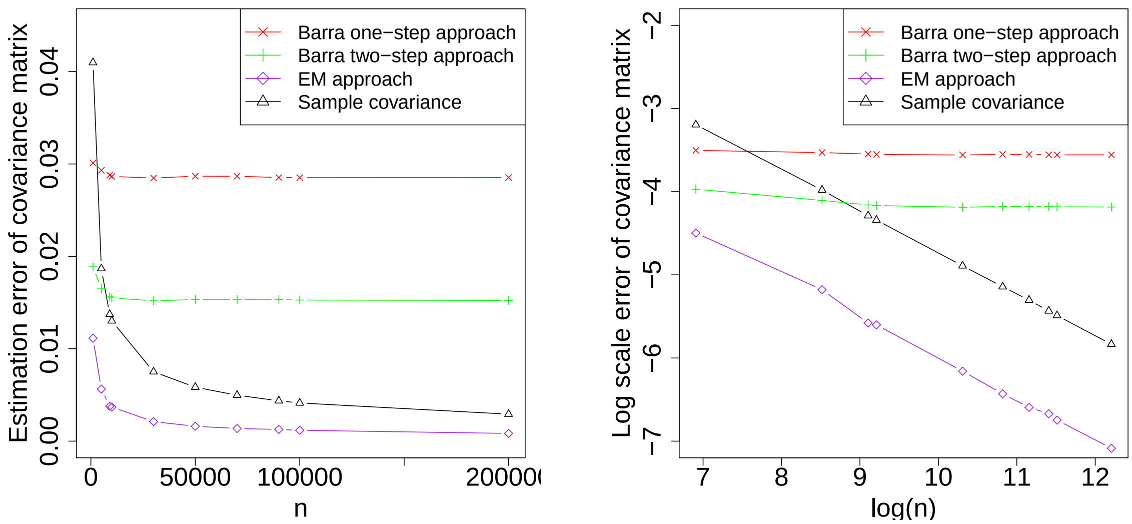


Figure 2.1: The number of assets p is fixed at 100. Left panel: The average error for estimating $\boldsymbol{\Sigma}$ over 50 replications. Right panel: The average error under logarithmic scale.

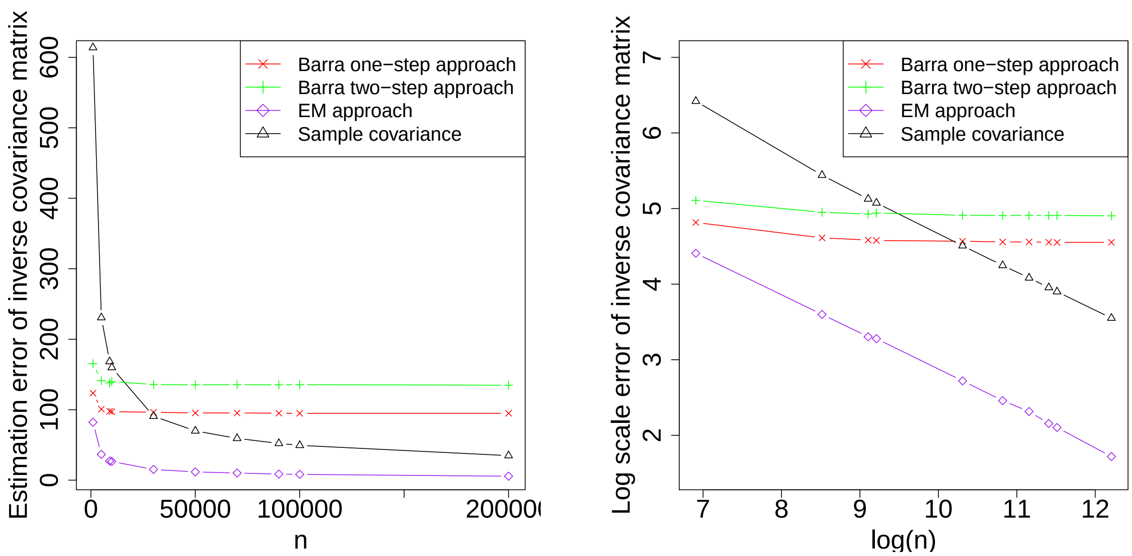


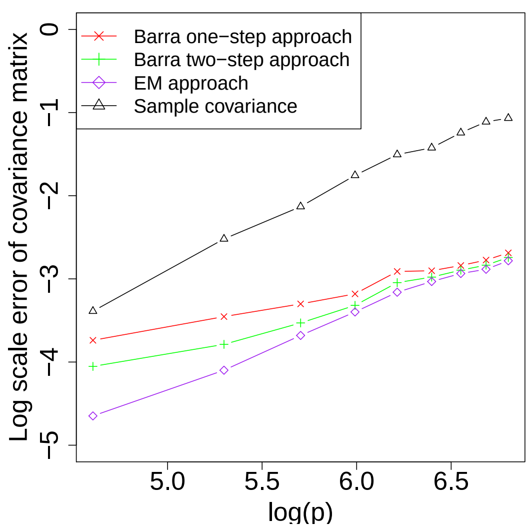
Figure 2.2: The number of assets p is fixed at 100. Left panel: The average error for estimating $\boldsymbol{\Sigma}^{-1}$ over 50 replications. Right panel: The average error under logarithmic scale.

We can see the errors of the two Barra estimates level off as n increases, indicating they are not consistent. This agrees with the result in Theorem 2.2.6. On the other hand, the errors of the EM estimate and the sample covariance decrease as n increases. The right panel at Figure 2.1 indicates that the slope for $\widehat{\Sigma}_{EM}$ is similar to that for $\widehat{\Sigma}_{sam}$, verifying the rate of the convergence results of Theorem 2.3.1.

Figure 2.2 demonstrates similar patterns, but now for estimates of the inverse of the covariance.

The case with p increasing, while n and K being fixed

In this section, we explore the influence of the number of assets p on different estimators. We consider two scenarios. In the first scenario, we study the case when p is less than n. Fixing the sample size n at 1000, we increase p from 100 to 900. In the second scenario, we study the case when p is larger than n, where we fix n at 100 and increase p from 100 to 1000. Note that when p is larger than n, the sample covariance is not invertible, and we use the shrinkage sample covariance $\widehat{\Sigma}_{shk}$ (Ledoit and Wolf, 2004) instead. The simulation procedure is similar to the previous one, but we need to generate different $\mathbf{B}$ and Σ_ϵ for different p, where we use sample with replacement. Again, we repeat the process 50 times for each p, and report the mean of the Frobenius norm error in Figure 2.3 (for covariance) and Figure 2.4 (for the inverse of the covariance).



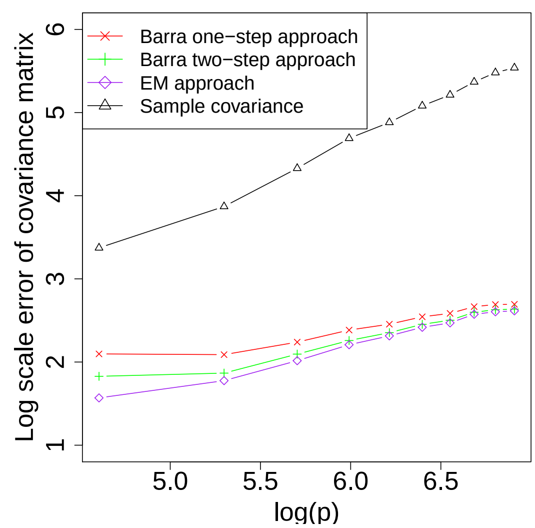


Figure 2.3: Left panel: The average error for estimating Σ with $n = 1000$. Right panel: The average error for estimating Σ with $n = 100$.

Figure 2.3 shows that when n is fixed and p increases, the performances of different estimates all degrade, with the sample covariance performing the worst and all estimates estimates degrading at about the same rate. The EM estimate performs the best, though the difference from the two Barra estimates is not large, and the gap gets smaller as p increases.

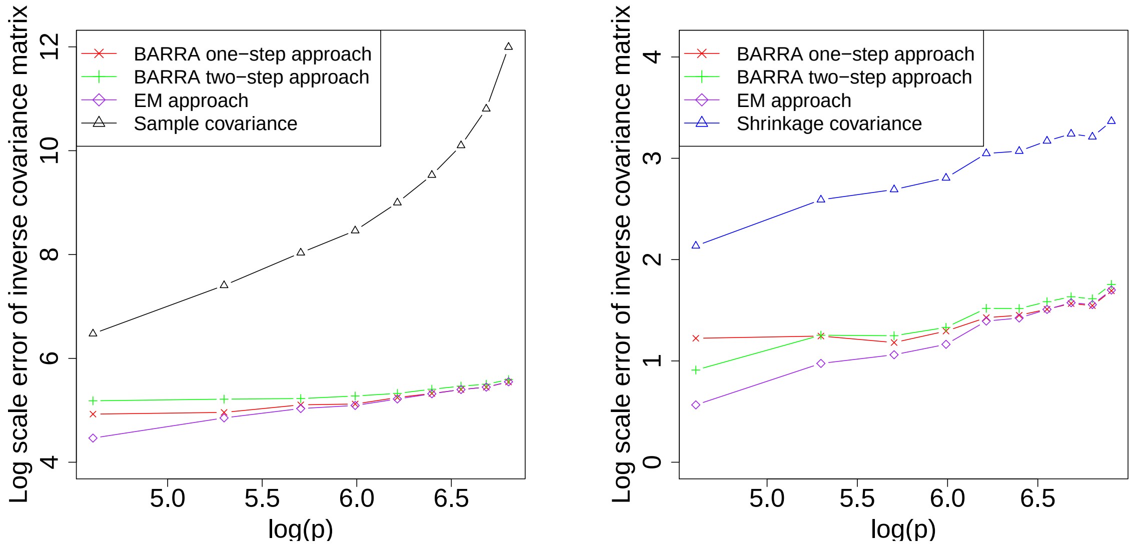


Figure 2.4: Left panel: The average error for estimating $\mathbf{\Sigma}^{-1}$ with $n = 1000$. Right panel: The average error for estimating $\mathbf{\Sigma}^{-1}$ with $n = 100$.

In Figure 2.4, the patterns for the inverse covariance estimates are similar, except that the sample covariance (when $n > p$) degrades the most quickly. The EM estimate still performs the best, though again the difference from the two Barra estimates is not large.

2.5 Conclusion

This chapter studies different approaches for estimating the covariance matrix of financial assets via the Barra model. Both the covariance matrix and the inverse of the covariance are fundamental in financial risk management. The Barra estimates are widely used in industry, but it turns out they are not consistent. We re-interpret the Barra model via the framework of random effects model, and propose an EM approach to estimate the Barra model. The EM estimate inherits good properties of the MLE of the random effects model, is consistent.

In practice, however, the sample size n is often not very large, whereas the number of financial assets p is usually close to n or even larger than n. In this scenario, in particular for estimating the inverse covariance matrix, our simulation studies indicate that the Barra estimates and the EM estimate are better than the sample covariance. Our theoretical properties also indicate that the EM estimate performs slightly better than the Barra estimates.

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CHAPTER 3

A Screening Method for Sparse and Stable Portfolio Selection

3.1 Introduction

Modern portfolio theory (MPT) addresses the issue on how risk-averse investors can construct portfolios to optimize or maximize the expected return based on a given level of risk. It is also called the mean-variance analysis, and has been a fundamental problem in finance ever since Markowitz (1952, 1959) laid down the groundbreaking work. Markowitz (Markowitz, 1952) posed the mean-variance analysis as solving a quadratic linear problem, i.e.

$$\begin{aligned} &\text{minimize} \quad \frac{\gamma}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} - \boldsymbol{\mu}^T\mathbf{w}, \\ &\text{subject to} \quad \mathbf{w}^T\mathbf{1} = 1, \end{aligned} \tag{3.1}$$

where $\mathbf{w} \in \mathbb{R}^p$ is the portfolio allocation vector, $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$ is the covariance matrix of the returns on the assets in the portfolio, $\boldsymbol{\mu} \in \mathbb{R}^p$ is a vector of expected asset returns, and γ is a coefficient for relative risk aversion. The optimization involves minimizing the risk $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$ and maximizing the expected return $\boldsymbol{\mu}^T\mathbf{w}$ on the portfolio. The work by Markowitz (1959) in the theory of financial economics leads to the celebrated capital asset pricing model (CAPM), developed by Sharpe (1964), Lintner (1965), and Black (1972). Thus, it is often referred as a milestone of modern finance.

On the other hand, there are also documented facts that the Markowitz portfolio is sensitive to errors in the estimates of the inputs, namely the expected return and the covariance matrix. The problem gets more severe when the portfolio size is large. To address this sensitivity problem, two kinds of efforts have been made. First, methods have been proposed to reduce the variation of estimates for the Markowitz portfolio input parameters. For example, Jorion (1986), Chopra and Ziemba (1993) proposed a Bayes-Stein estimate and James-Stein estimate for the expected return respectively. In terms of the covariance matrix, Ledoit and Wolf (2004) proposed to shrink the sample covariance matrix to achieve a more stable covariance estimate. Fan, Fan and Lv (2008) also developed the covariance matrix estimate based on macroeconomic factor model.

The other type of approaches try to modify the Markowitz mean-variance optimization problem, by imposing additional constraints on the portfolio weights, such that the resulting allocation depends less sensitively on the input parameters. For example, Jagannathan and Ma (2003) studied the no-short-sale constraint and found that such constraints improve the empirical performance of portfolios. Fan, Zhang and Yu (2012) proposed a gross-exposure constraint on the portfolio weights, i.e. $||\mathbf{w}||_1 \leq c$. They showed that the estimation error is bounded by a quadratic function of the ℓ_1 norm of portfolio weights, and thus constraining the portfolio norm can effectively constrain the estimation error. In solving the optimization problem, they developed an approximate solution path to the risk minimization problem taking advantage of the LARS-LASSO algorithm. However, the solution is not exact, and the quality of the approximation is not clear.

In this chapter, we follow the approach taken by Fan, Zhang and Yu (2012) and consider the following ℓ_1-norm regularized mean-variance analysis:

$$\begin{aligned} \text{minimize} \quad & \frac{\gamma}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} - \boldsymbol{\mu}^T\mathbf{w} + \lambda||\mathbf{w}||_1, \\ \text{subject to} \quad & \mathbf{w}^T\mathbf{1} = 1. \end{aligned} \tag{3.2}$$

Note that because of the regularization on the norm of $\mathbf{w}$, the gross exposure of the portfolio is controlled, such that extreme long and short positions can be avoided, and thus the portfolio allocation can be stabilized. Further, due to the ℓ_1 norm of $\mathbf{w}$, many elements of the solution to (3.2) will be exactly zero. This leads to automatic asset selection and has practical advantages for portfolio management. We make two contributions in this chapter. First, we propose an efficient algorithm that solves (3.2) exactly. Further, we develop a screening method that can identify zero elements of the solution before solving the optimization problem, and thus solving (3.2) in a much reduced space is even more efficient in terms of both computing time and memory storage.

The rest of this chapter is organized as follows. We develop an efficient algorithm for solving (3.2) in Section 3.2. In Section 3.3, we propose a screening method that effectively reduced the dimension of $\mathbf{w}$ by quickly identifying zero elements of the solution. The performance of proposed algorithms are illustrated by simulation studies in Section 3.4 and by real data in Section 3.5. In Section 3.6, we provide concluding remarks.

3.2 Regularized Mean-Variance Analysis and its Solution

The mean-variance analysis with the gross-exposure constraint ($||\mathbf{w}||_1 \leq c$) was introduced by Fan, Zhang and Yu (2012) for selecting optimal portfolio allocation. It has been proved that it helps with controlling the discrepancy between the empirical risk and the actual risk. Fan, Zhang and Yu (2012) proposed to approximate the solution by replacing $||\mathbf{w}||_1 \leq c$ with $|w_1| + \ldots |w_{p-1}| \leq d$ and ignoring the equality constraint, i.e. $\mathbf{w}^T\mathbf{1} = 1$, such that the optimization problem can be connected to the lasso regression and efficient LARS-LASSO algorithms can be used. However, there is no guarantee that the approximate solution is close to the exact solution. We consider the Lagrange format of the constrained optimization, i.e. (3.2). These two formulations are equivalent; in the sense that for a given $\lambda > 0$, there exists a $c > 0$ such that the two problems share the same solution and vice versa. Note that due to the equality constraint, coordinate decent methods for solving ℓ_1 regularized least-squares problems (Friedman et al., 2007) are not directly applicable. In this section, we propose to combine the coordinate descent method, which is efficient for solving ℓ_1-regularized problems, and augmented Lagrange method (Bertsekas, 2014), which is powerful for dealing with constraints for convex problems, and develop an efficient algorithm for solving (3.2). We refer our algorithm as the coordinate descent method with multipliers (CDM).

We first form the augmented Lagrangian for (3.2):

$$\mathbf{L}_\beta(\mathbf{w}, \alpha) = \frac{\gamma}{2}\mathbf{w}^T\mathbf{\Sigma}\mathbf{w} - \boldsymbol{\mu}^T\mathbf{w} + \lambda||\mathbf{w}||_1 + \alpha(1 - \sum_{j=1}^{p} w_j) + \frac{\beta}{2}(1 - \sum_{j=1}^{p} w_j)^2, \tag{3.3}$$

where α is the Lagrange multiplier and $\beta > 0$ is a penalty parameter. The augmented Lagrangian method for (3.2) consists of the following iterations:

$$\mathbf{w}^{k+1} \leftarrow \min \mathbf{L}_\beta(\mathbf{w}, \alpha^k), \tag{3.4}$$

$$\alpha^{k+1} \leftarrow \alpha^k + \beta(1 - \sum_{j=1}^{p} w_j^{k+1}). \tag{3.5}$$

To solve (3.4), we can separate the parameters and apply the coordinate descent method. Note that taking derivative with respective to w_j, we have

$$\gamma \sum_{i=1}^{p} \Sigma_{ji} w_k - \mu_j + \lambda v_j - \alpha - \beta(1 - \sum_{i=1}^{p} w_i) = 0, \tag{3.6}$$

where v_j is the derivative of $|w_j|$. With other components held fixed, the jth component of

$\mathbf{w}$ is updated by

$$w_j^{k+1} \leftarrow \frac{1}{\gamma\Sigma_{jj} + \beta} S_\lambda(\alpha^k + \beta(1 - \sum_{i \neq j} w_i^k) + \mu_j - \gamma \sum_{i \neq j} \Sigma_{ji} w_i^k), \tag{3.7}$$

where $S_\lambda(t) = \text{sign}(t)(|t| - \lambda)$ is the soft thresholding operator.

Combining (3.4), (3.5) and (3.7) yields Algorithm 1 for solving problem (3.2).

Algorithm 1 Coordinate descent method with multipliers (CDM)

Input: $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\lambda > 0$, $\beta > 0$

1: **Initialize**: $\mathbf{w}^0$ with $\mathbf{0}$ or a warm start, $\alpha^0 = 0$, and $k = 0$.
2: **while** not converged **do**
3: Update w_j^k by (3.7) until convergence
4: Update α^k by (3.5)
5: k = k+1
6: **end while**

Output: $\hat{\mathbf{w}} = \mathbf{w}^k$.

Note that the minimization of (3.4) need not be exact; it often suffices to adopt a stopping criterion such that the minimization is asymptotically exact in terms of the iterations. Convergence of Algorithm 1 and this relaxation can be proved by verifying the conditions in Theorem 4 of Rockafellar (1976).

In Section 3.4, we will compare the exact solution we propose here with the approximate one by Fan, Zhang and Yu (2012).

3.3 A screening method for regularized mean-variance analysis

Note that λ controls the number of nonzero weight assets in the optimal portfolio; thus it is often desirable to solve (3.2) for a series of λ values. In order to efficiently solve a series of ℓ_1-norm regularized optimization problems, the idea of screening has been shown to be useful. Screening aims at quickly identifying zero components in the solution and then removes them from the optimization problem. Thus, the number of parameters of the optimization problem is reduced, which often leads to substantial savings in both computational cost and memory usage.

Existing screening methods can be roughly divided into two categories: safe screening and heuristic screening. Safe screening methods guarantee that discarded features have zero coefficients in the solution. Ghaoui et al. (2012) laid the groundwork on safe screening methods. Wang et al. (2013) and Liu et al. (2013) improved the performance of safe screening by tools of convex optimization and extended the idea to more general ℓ_1-norm regularized sparse problems. Heuristic screening methods, on the other hand, may mistakenly discard features that have nonzero coefficients in the solution. Efficient methods in this category include the strong rule (Tibshirani et al., 2012) and sure independence screening (Fan and Lv, 2008).

Our goal here is to develop a safe screening rule for the ℓ_1-norm regularized mean-variance analysis. Note that (3.2) involves an equality constraint, i.e. the portfolio weights sum to one, which makes none of the existing screening methods applicable to this problem. In the following subsections, we develop a safe screening method that accommodates the equality constraint. Specifically, given a series of regularization parameters $\lambda_1 > \lambda_2 > \ldots > \lambda_m$, we wish the rule can be effective at identifying the zero weighted assets corresponding to each of the regularization parameter value and thus reduce the dimension of the corresponding optimization problem.

3.3.1 Overview of the proposed method

The main goal of the proposed safe screening for equality constrained optimization (SASECO), also most other existing screening methods, is to eliminate the inactive features in the solution before solving the optimization problem. To achieve that, it is often convenient to work with the dual problem. For example, consider the lasso problem: $\frac{1}{2}||\mathbf{y} - \mathbf{X}\boldsymbol{\beta}||^2 + \lambda||\boldsymbol{\beta}||_1$, Karush-Kuhn-Tucker (KKT) conditions indicate $\langle \mathbf{x}_j, \boldsymbol{\theta}^* \rangle < 1 \implies \beta_j^* = 0$, where $\boldsymbol{\theta}^*$ denotes the dual solution and $\boldsymbol{\beta}^*$ denotes the primal solution. The common idea of screening methods is to construct a feasible set for the dual solution and estimate an upper bound on the function of the dual solution in connection to the primal solution. The construction of a tight feasible set for the dual solution is the key to the success of screening techniques. Some prior knowledge is often required to accomplish the construction. For example, in our case, solving (3.2) with $\lambda_1 > \lambda_2 > \ldots > \lambda_m$ are of interests. Having computed the solution $\mathbf{w}^*_{(k-1)}$ at λ_{k-1}, the feasible set of dual solution corresponding to λ_k can be constructed by using variational inequalities, which provide sufficient and necessary optimality conditions for the dual problems with $\lambda = \lambda_{k-1}$ and λ_k. Therefore, in the following sections, we first introduce the dual problem for the constrained mean-variance problem, then a feasible set is constructed for the dual optimal solution, and an upper bound connected to the primal optimal solution over the feasible set is estimated; we present the main part of SASECO in Theorem 3.3.2.

3.3.2 The dual problem

To derive the dual problem of (3.2), let $\mathbf{X} = \sqrt{\gamma}\boldsymbol{\Sigma}^{\frac{1}{2}}$ and $\mathbf{y} = \frac{1}{\sqrt{\gamma}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}$. Then (3.2) can be re-written as

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}||\mathbf{y} - \mathbf{X}\mathbf{w}||^2 + \lambda||\mathbf{w}||_1, \\ \text{subject to} \quad & \mathbf{w}^T\mathbf{1} = 1. \end{aligned} \tag{3.8}$$

By introducing $\mathbf{Z} = \mathbf{y} - \mathbf{X}\mathbf{w}$, the dual variables $\boldsymbol{\eta} \in \mathbb{R}^p$, $\delta \in \mathbb{R}$, we obtain the Lagrangian function of (3.8):

$$L(\mathbf{w}, \mathbf{Z}, \boldsymbol{\eta}, \delta) = \frac{1}{2}||\mathbf{Z}||^2 + \lambda||\mathbf{w}||_1 + \boldsymbol{\eta}^T(\mathbf{y} - \mathbf{X}\mathbf{w} - \mathbf{Z}) + \delta(1 - \mathbf{w}^T\mathbf{1}). \tag{3.9}$$

Then the dual problem can be derived as follows

$$\min_{\mathbf{w},\mathbf{Z}} \max_{\boldsymbol{\eta},\delta} L(\mathbf{w}, \mathbf{Z}, \boldsymbol{\eta}, \delta) \tag{3.10}$$

$$= \max_{\boldsymbol{\eta},\delta} \min_{\mathbf{w}}(-\boldsymbol{\eta}^T\mathbf{X}\mathbf{w} + \lambda||\mathbf{w}||_1 - \delta\mathbf{w}^{\mathrm{T}}\mathbf{1}) + \min_{\mathbf{Z}}(\frac{1}{2}||\mathbf{Z}||^2 - \boldsymbol{\eta}^T\mathbf{Z}) + \boldsymbol{\eta}^T\mathbf{y} + \delta. \tag{3.11}$$

Denote the first part of the optimization problem as

$$f_1(\mathbf{w}) = -\boldsymbol{\eta}^T\mathbf{X}\mathbf{w} + \lambda||\mathbf{w}||_1 - \delta\mathbf{w}^T\mathbf{1}. \tag{3.12}$$

Consider its subgradient,

$$\partial f_1(\mathbf{w}) = -\boldsymbol{\eta}^T\mathbf{X} + \lambda\mathbf{v} - \delta\mathbf{1}, \tag{3.13}$$

in which $||\mathbf{v}||_\infty \leq 1$ and $\mathbf{v}^T\mathbf{w} = ||\mathbf{w}||_1$, i.e. $\mathbf{v}$ is the subgradient of $||\mathbf{w}||_1$. The necessary condition for f_1 to attain an optimum is

$$\exists\mathbf{w}', \quad \text{such that } 0 \in -\boldsymbol{\eta}^T\mathbf{X} + \lambda\mathbf{v}' - \delta\mathbf{1}, \tag{3.14}$$

where $\mathbf{v}' \in \partial||\mathbf{w}'||_1$. Thus, we have

$$\mathbf{v}' = \frac{\boldsymbol{\eta}^T\mathbf{X} + \delta\mathbf{1}}{\lambda} \quad \text{and} \quad ||\boldsymbol{\eta}^T\mathbf{X} + \delta\mathbf{1}||_\infty \leq \lambda. \tag{3.15}$$

Note that when plugging in $\mathbf{v}'$, $f_1(\mathbf{w}') = \min f_1(\mathbf{w}) = 0$.

Next, we consider the second part in (3.11). Denoting it as $f_2(\mathbf{Z})$. It is not difficult to see that

$$\min f_2(\mathbf{Z}) = f_2(\boldsymbol{\eta}) = -\frac{1}{2}||\boldsymbol{\eta}||^2. \tag{3.16}$$

Combining the two parts, the dual problem yields

$$\begin{aligned} \max_{\boldsymbol{\eta},\delta} \quad & -\frac{1}{2}||\boldsymbol{\eta}||^2 + \boldsymbol{\eta}^T\mathbf{y} + \delta, \\ \text{subject to} \quad & ||\boldsymbol{\eta}^T\mathbf{X} + \delta\mathbf{1}||_\infty \leq \lambda. \end{aligned} \tag{3.17}$$

By a simple re-scaling of the dual variables $\boldsymbol{\eta}$, δ, i.e., denote $\boldsymbol{\xi} = \frac{\boldsymbol{\eta}}{\lambda}$ and $\zeta = \frac{\delta}{\lambda}$, (3.17) becomes

$$\begin{aligned} \min_{\boldsymbol{\xi},\zeta} \quad & \frac{\lambda}{2}||\boldsymbol{\xi}||^2 - \boldsymbol{\xi}^T\mathbf{y} - \zeta, \\ \text{subject to} \quad & ||\boldsymbol{\xi}^T\mathbf{X} + \zeta\mathbf{1}||_\infty \leq 1. \end{aligned} \tag{3.18}$$

Note that (3.8) is convex and the constraints are all affine. By Slater's condition, we will have strong duality as long as (3.8) is feasible. Let $\mathbf{w}^*, \mathbf{Z}^*, \boldsymbol{\xi}^*, \zeta^*$ be the optimal primal and dual variables. We have the following relationship between the primal variables and the dual variables:

$$\mathbf{y} - \mathbf{X}\mathbf{w}^* = \mathbf{Z}^* = \lambda\boldsymbol{\xi}^* \tag{3.19}$$

$$\mathbf{x}_j^T\boldsymbol{\xi}^* + \zeta^* \in \begin{cases} \text{sign}(w_j^*) & \text{if } w_j^* \neq 0, \\ [-1, 1] & \text{if } w_j^* = 0. \end{cases} \tag{3.20}$$

Note (3.20) implies that if $|\mathbf{x}_j^T\boldsymbol{\xi}^* + \zeta^*| < 1$, then w_j^* must be 0. This property provides us with the direction to construct the screening rule.

3.3.3 Feasible set construction

Let $\boldsymbol{\xi}_1^*$ and ζ_1^* denote the solution at λ_1 and $\boldsymbol{\xi}_2^*$ and ζ_2^* denote the solution at λ_2, where $\lambda_1 > \lambda_2$. Note that it is impossible to achieve the exact value of $|\mathbf{x}_j^T\boldsymbol{\xi}^* + \zeta^*|$ without solving the optimization problem, however, we are able to estimate an upper bound of $|\mathbf{x}_j^T\boldsymbol{\xi}_2^* + \zeta_2^*|$ at λ_2 taking advantage of $\boldsymbol{\xi}_1^*$ and ζ_1^*, which can be obtained from the primal optimal solution $\mathbf{w}_{(1)}^*$ at λ_1. In the following, we will make use of the variational inequality as in Lemma (3.3.1) to construct a feasible set for the dual optimal solution $(\boldsymbol{\xi}_2^*, \zeta_2^*)$.

Lemma 3.3.1. *[Nesterov (2007)] For the constrained convex optimization problem:*

$$\min_{G} \quad f(\mathbf{x}), \tag{3.21}$$

with G being convex and closed and $f(\cdot)$ being convex and differentiable, $x^* \in G$ is an optimal solution of (3.21) if and only if

$$\langle f'(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \quad \forall \mathbf{x} \in G. \tag{3.22}$$

Since $(\boldsymbol{\xi}_1^*, \zeta_1^*)$ and $(\boldsymbol{\xi}_2^*, \zeta_2^*)$ are optimal solutions to (3.18) at λ_1 and λ_2 respectively, we can apply Lemma (3.3.1) to (3.18) and obtain

$$\left\langle \begin{pmatrix} \lambda_1 \boldsymbol{\xi}_1^* - \mathbf{y} \\ -1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\xi} - \boldsymbol{\xi}_1^* \\ \zeta - \zeta_1^* \end{pmatrix} \right\rangle \geq 0, \quad \forall (\boldsymbol{\xi}, \zeta) : ||\boldsymbol{\xi}^T \mathbf{X} + \zeta \mathbf{1}||_\infty \leq 1, \tag{3.23}$$

$$\left\langle \begin{pmatrix} \lambda_2 \boldsymbol{\xi}_2^* - \mathbf{y} \\ -1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\xi} - \boldsymbol{\xi}_2^* \\ \zeta - \zeta_2^* \end{pmatrix} \right\rangle \geq 0, \quad \forall (\boldsymbol{\xi}, \zeta) : ||\boldsymbol{\xi}^T \mathbf{X} + \zeta \mathbf{1}||_\infty \leq 1. \tag{3.24}$$

Plugging $(\boldsymbol{\xi}_2^*, \zeta_2^*)$ and $(\boldsymbol{\xi}_1^*, \zeta_1^*)$ into (3.23) and (3.24) respectively, we have

$$\langle \lambda_1 \boldsymbol{\xi}_1^* - \mathbf{y}, \boldsymbol{\xi}_2^* - \boldsymbol{\xi}_1^* \rangle \geq \zeta_2^* - \zeta_1^*, \tag{3.25}$$

$$\langle \lambda_2 \boldsymbol{\xi}_2^* - \mathbf{y}, \boldsymbol{\xi}_1^* - \boldsymbol{\xi}_2^* \rangle \geq \zeta_1^* - \zeta_2^*. \tag{3.26}$$

As a result, (3.25) and (3.26) provide us with the following feasible set for $(\boldsymbol{\xi}_2^*, \zeta_2^*)$:

$$\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*) = \{(\boldsymbol{\xi}, \zeta) : \langle \lambda_1 \boldsymbol{\xi}_1^* - \mathbf{y}, \boldsymbol{\xi} - \boldsymbol{\xi}_1^* \rangle \geq \zeta - \zeta_1^*, \langle \lambda_2 \boldsymbol{\xi} - \mathbf{y}, \boldsymbol{\xi}_\mathbf{1}^* - \boldsymbol{\xi} \rangle \geq \zeta_1^* - \zeta \}. \tag{3.27}$$

3.3.4 Upper bound estimation

Given the feasible set $\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)$, we seek to obtain an the upper bound of $|\mathbf{x}_j^T \boldsymbol{\xi}_2^* + \zeta_2^*|$ via the optimization problem

$$\max_{\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)} |\mathbf{x}_j^T \boldsymbol{\xi} + \zeta|. \tag{3.28}$$

With the upper bound, KKT condition (3.20) yields

$$\max_{\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)} |\mathbf{x}_j^T \boldsymbol{\xi} + \zeta| < 1 \implies w_j^* = 0. \tag{3.29}$$

In the following of this subsection, we show how to solve (3.28). For convenience, we introduce the following variables:

$$\mathbf{a} = \frac{\mathbf{y}}{\lambda_1} - \boldsymbol{\xi}_1^* = \frac{\mathbf{X}\mathbf{w}_1^*}{\lambda_1}, \tag{3.30}$$

$$\mathbf{b} = \frac{\mathbf{y}}{\lambda_2} - \boldsymbol{\xi}_1^* = \mathbf{a} + (\frac{\mathbf{y}}{\lambda_2} - \frac{\mathbf{y}}{\lambda_1}), \tag{3.31}$$

$$\mathbf{r} = 2\boldsymbol{\xi} - (\boldsymbol{\xi}_1^* + \frac{\mathbf{y}}{\lambda_2}), \tag{3.32}$$

$$d = \zeta_1^* - \zeta. \tag{3.33}$$

where $\mathbf{a}$ denotes the "prediction" based on $\mathbf{w}_1^*$ scaled by $\frac{1}{\lambda_1}$, and $\mathbf{b}$ differs $\mathbf{a}$ by a term capturing the change of inputs to the dual problem from λ_1 and λ_2, and d measures the difference between optimal ζ_1^* and ζ.

With the above variables, the inequalities in $\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)$ can be reformulated as follows:

$$\langle \frac{\mathbf{y}}{\lambda_1} - \boldsymbol{\xi}_1^*, 2\boldsymbol{\xi} - 2\boldsymbol{\xi}_1^* \rangle = \langle \mathbf{a}, \mathbf{r} + \mathbf{b} \rangle \leq \frac{2}{\lambda_1} d, \tag{3.34}$$

$$\langle 2\boldsymbol{\xi} - 2\frac{\mathbf{y}}{\lambda_2}, 2\boldsymbol{\xi} - 2\boldsymbol{\xi}_1^* \rangle = \langle \mathbf{r} - \mathbf{b}, \mathbf{r} + \mathbf{b} \rangle \leq -\frac{4}{\lambda_2} d. \tag{3.35}$$

Furthermore, the objective function in (3.28) can be written as

$$|\mathbf{x}_j^T \boldsymbol{\xi} + \zeta| = \frac{1}{2} |\langle \mathbf{x}_j, \boldsymbol{\xi}_1^* + \frac{\mathbf{y}}{\lambda_2} \rangle + 2\zeta_1^* + \langle \mathbf{x}_j, \mathbf{r} \rangle - 2d|. \tag{3.36}$$

To solve (3.28), we only need to compute the unknown part $\max |\langle \mathbf{x}_j, \mathbf{r} \rangle - 2d|$ over the feasible set. Note that the maximization problem can be transformed to minimizing the inverse, i.e. $\min_{\mathbf{r},d} \Big(\langle \mathbf{x}_j, \mathbf{r} \rangle - 2d, -\langle \mathbf{x}_j, \mathbf{r} \rangle + 2d \Big)$. We focus on the first part of the objective function in the following discussion, and the second part will be similar. It follows the reformulated optimization problem that

$$\begin{aligned} \min_{\mathbf{r},d} \quad & \langle \mathbf{x}_j, \mathbf{r} \rangle - 2d, \\ \text{subject to} \quad & \langle \mathbf{a}, \mathbf{r} + \mathbf{b} \rangle \leq \frac{2}{\lambda_1} d, ||\mathbf{r}||^2 - ||\mathbf{b}||^2 \leq -\frac{4}{\lambda_2} d. \end{aligned} \tag{3.37}$$

Since the problem is convex and the Slater's condition holds, we have strong duality that the optimal primal value and the optimal dual value are the same. By introducing the Lagrange multipliers μ_1 and μ_2, the dual problem of the first part of (3.37) can be derived as:

$$
\begin{aligned}
&\min_{\mathbf{r},d} \max_{\mu_1 \geq 0, \mu_2 \geq 0} \langle \mathbf{x}_j, \mathbf{r} \rangle - 2d + \mu_1(\langle \mathbf{a}, \mathbf{r} + \mathbf{b} \rangle - \frac{2}{\lambda_1} d) + \\
&\quad \frac{\mu_2}{2}(||\mathbf{r}||^2 - ||\mathbf{b}||^2 + \frac{4}{\lambda_2} d), \qquad (3.38)\\
= &\max_{\mu_1 \geq 0, \mu_2 \geq 0} \min_{\mathbf{r}} (\langle \mathbf{x}_j, \mathbf{r} \rangle + \mu_1 \langle \mathbf{a}, \mathbf{r} \rangle + \frac{\mu_2}{2} ||\mathbf{r}||^2) + \\
&\quad \min_{d} (\frac{2\mu_2}{\lambda_2} - \frac{2\mu_1}{\lambda_1} - 2) d + \mu_1 \langle \mathbf{a}, \mathbf{b} \rangle - \frac{\mu_2}{2} ||\mathbf{b}||^2. \qquad (3.39)
\end{aligned}
$$

Note the optimal primal and dual solutions $\mathbf{r}^*$, d^*, μ_1^* and μ_2^* satisfy the following equations by KKT conditions:

$$
\mathbf{x}_j + \mu_1^* \mathbf{a} + \mu_2^* \mathbf{r}^* = \mathbf{0}, \qquad (3.40)
$$

$$
(\frac{\mu_2^*}{\lambda_2} - \frac{\mu_1^*}{\lambda_1} - 1) d^* = 0. \qquad (3.41)
$$

In order for (3.41) to be satisfied, there are two possibilities:

$$
\frac{\mu_2}{\lambda_2} - \frac{\mu_1}{\lambda_1} - 1 = 0 \qquad (3.42)
$$

or

$$
d = 0. \qquad (3.43)
$$

In the first case, (3.42) offers a relationship between μ_1 and μ_2. Thus, $\mu_2 = \max(\lambda_2(\frac{\mu_1}{\lambda_1} + 1), 0)$ can be plugged in (3.39). On the other hand, taking derivative of (3.39) with respective to $\mathbf{r}$ leads us to a relationship between $\mathbf{r}$ and μ_1, μ_2, which can be also applied to (3.39). Through simple algebra, we can obtain the optimal μ_1 in (3.39):

$$
\mu_1 = \max(\frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1 \boldsymbol{\xi}_1^*||_2}{\lambda_2 ||(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) \boldsymbol{\xi}_1^*||_2} - \lambda_1, 0), \qquad (3.44)
$$

and the corresponding optimal μ_2 and $\mathbf{r}$:

$$
\mu_2 = \max(\frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1 \boldsymbol{\xi}_1^*||_2}{\lambda_1 ||(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) \boldsymbol{\xi}_1^*||_2}, \lambda_2), \qquad (3.45)
$$

$$
\mathbf{r} = \frac{-\mathbf{x}_j - \mu_1 \mathbf{a}}{\mu_2}. \qquad (3.46)
$$

Recall the constraints in (3.37). Then the following inequalities should be satisfied

$$\frac{\lambda_1}{2}\langle \mathbf{a}, \mathbf{r}+\mathbf{b}\rangle \leq d \leq -\frac{\lambda_2}{4}(||\mathbf{r}||^2 - ||\mathbf{b}||^2). \tag{3.47}$$

If $\frac{\lambda_1}{2}\langle \mathbf{a}, \mathbf{r}+\mathbf{b}\rangle \leq -\frac{\lambda_2}{4}(||\mathbf{r}||^2 - ||\mathbf{b}||^2)$, the optimal d to (3.37) exists; by plugging in (3.46) and (3.47), the optimal value of (3.37) can be obtained. Otherwise, the KKT condition (3.41) cannot be achieved by (3.42).

On the other hand, suppose (3.43) holds. We see that (3.37) can be simplified to the following:

$$\begin{aligned} \min_{\mathbf{r},d} \quad & \langle \mathbf{x}_j, \mathbf{r}\rangle, \\ \text{subject to} \quad & \langle \mathbf{a}, \mathbf{r}+\mathbf{b}\rangle \leq 0, ||\mathbf{r}||^2 - ||\mathbf{b}||^2 \leq 0, \end{aligned} \tag{3.48}$$

which can be solved in a similar fashion (details skipped).

Overall, (3.37) can be solved using above discussions. Similar calculations can be done when changing the objective function of (3.37) to $\min_{r,d}(-\langle \mathbf{x}_j, \mathbf{r}\rangle + 2d)$, thus finishing computing the unknown part of (3.36).

Next, we present an upper bound of $|\mathbf{x}_j^T\boldsymbol{\xi}_2^* + \zeta_2^*|$ in the following theorem.

Theorem 3.3.2. *Let* $\mathbf{y} \neq \mathbf{0}$*, and* $||\mathbf{X}^T\mathbf{y}||_\infty \geq \lambda_1 > \lambda_2 > 0$*. Denote*

$$u_j^+(\lambda_2) = \max_{\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)} \mathbf{x}_j^T\boldsymbol{\xi} + \zeta, \tag{3.49}$$

$$u_j^-(\lambda_2) = \max_{\Omega(\boldsymbol{\xi}_2^*, \zeta_2^*)} -\mathbf{x}_j^T\boldsymbol{\xi} - \zeta. \tag{3.50}$$

Then $u_j^+(\lambda_2)$ *and* $u_j^-(\lambda_2)$ *can be computed using the following steps.*
Case 1: $\mathbf{a} \neq \mathbf{0}$*, we compute*

Step 1:

1. *If* $\frac{\lambda_1}{2}\langle \mathbf{a}, \mathbf{r}+\mathbf{b}\rangle \leq -\frac{\lambda_2}{4}(||\mathbf{r}||^2 - ||\mathbf{b}||^2)$*, we compute*

$$u_j^+(\lambda_2)^{(1)} = \langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{1}{2}\langle \mathbf{x}_j - \lambda_1\mathbf{a}, \mathbf{r}^+ + \mathbf{b}\rangle + \zeta_1^*, \tag{3.51}$$

where $\mathbf{r}^+ = \frac{\mathbf{x}_j - \mu_1^+\mathbf{a}}{\mu_2^+}$*,* $\mu_1^+ = \frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1\boldsymbol{\xi}_1^*||_2}{\lambda_2||(\frac{1}{\lambda_1} + \frac{1}{\lambda_1})\boldsymbol{\xi}_1^*||_2} - \lambda_1$*, and* $\mu_2^+ = \frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1\boldsymbol{\xi}_1^*||_2}{\lambda_1||(\frac{1}{\lambda_1})\boldsymbol{\xi}_1^*||_2}$.

$$u_j^-(\lambda_2)^{(1)} = -\langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle - \frac{1}{2}\langle \mathbf{x}_j + \frac{\lambda_2}{2}(\mathbf{r}^- - \mathbf{b}), \mathbf{r}^- + \mathbf{b}\rangle - \zeta_1^*, \tag{3.52}$$

where $\mathbf{r}^- = \frac{-\mathbf{x}_j - \mu_1^-\mathbf{a}}{\mu_2^-}$*,* $\mu_1^- = max(\frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1\boldsymbol{\xi}_1^*||_2}{\lambda_2||(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})\boldsymbol{\xi}_1^*||_2} - \lambda_1, 0)$*, and* $\mu_2^- = max(\frac{||\mathbf{x}_j - \mathbf{y} + \lambda_1\boldsymbol{\xi}_1^*||_2}{\lambda_1||(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})\boldsymbol{\xi}_1^*||_2}, \lambda_2)$.

2. *If* $\frac{\lambda_1}{2}\langle \mathbf{a}, \mathbf{r}+\mathbf{b}\rangle > -\frac{\lambda_2}{4}(||\mathbf{r}||^2 - ||\mathbf{b}||^2)$*, then* $u_j^+(\lambda_2)^{(1)} = -\infty$ *and* $u_j^-(\lambda_2)^{(1)} = -\infty$.

Step 2:

1. *If* $\frac{\langle \mathbf{b},\mathbf{a}\rangle}{||\mathbf{b}||_2||\mathbf{a}||_2} > \frac{|\langle \mathbf{x}_j,\mathbf{a}\rangle|}{||\mathbf{x}_j||_2||\mathbf{a}||_2}$*, we compute*

$$u_j^+(\lambda_2)^{(2)} = \langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}{2}[||\mathbf{x}_j^\perp||_2||\mathbf{y}^\perp||_2 + \langle \mathbf{x}_j^\perp, \mathbf{y}^\perp\rangle] + \zeta_1^*, \tag{3.53}$$

$$u_j^-(\lambda_2)^{(2)} = -\langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}{2}[||\mathbf{x}_j^\perp||_2||\mathbf{y}^\perp||_2 - \langle \mathbf{x}_j^\perp, \mathbf{y}^\perp\rangle] - \zeta_1^*, \tag{3.54}$$

where $\mathbf{x}_j^\perp = \mathbf{x}_j - \mathbf{a}\langle \mathbf{x}_j, \mathbf{a}\rangle/||\mathbf{a}||_2^2$ *and* $\mathbf{y}^\perp = \mathbf{y} - \mathbf{a}\langle \mathbf{y}, \mathbf{a}\rangle/||\mathbf{a}||_2^2$.

2. *If* $\langle \mathbf{x}_j, \mathbf{a}\rangle > 0$ *and* $\frac{\langle \mathbf{b},\mathbf{a}\rangle}{||\mathbf{b}||_2||\mathbf{a}||_2} \le \frac{|\langle \mathbf{x}_j,\mathbf{a}\rangle|}{||\mathbf{x}_j||_2||\mathbf{a}||_2}$*, then* $u_j^+(\lambda_2)^{(2)}$ *satisfies (3.53), and*

$$u_j^-(\lambda_2)^{(2)} = -\langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{1}{2}[||\mathbf{x}_j||_2||\mathbf{b}||_2 - \langle \mathbf{x}_j, \mathbf{b}\rangle]. \tag{3.55}$$

3. *If* $\langle \mathbf{x}_j, \mathbf{a}\rangle < 0$ *and* $\frac{\langle \mathbf{b},\mathbf{a}\rangle}{||\mathbf{b}||_2||\mathbf{a}||_2} \le \frac{|\langle -\mathbf{x}_j,\mathbf{a}\rangle|}{||\mathbf{x}_j||_2||\mathbf{a}||_2}$*, then* $u_j^-(\lambda_2)^{(2)}$ *satisfies (3.54), and*

$$u_j^+(\lambda_2)^{(2)} = \langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{1}{2}[||\mathbf{x}_j||_2||\mathbf{b}||_2 + \langle \mathbf{x}_j, \mathbf{b}\rangle]. \tag{3.56}$$

Step 3:

$$u_j^+(\lambda_2) = \max(u_j^+(\lambda_2)^{(1)}, u_j^+(\lambda_2)^{(2)}), \tag{3.57}$$

$$u_j^-(\lambda_2) = \max(u_j^-(\lambda_2)^{(1)}, u_j^-(\lambda_2)^{(2)}). \tag{3.58}$$

Case 2: $\mathbf{a} = \mathbf{0}$*, we compute*

$$u_j^+(\lambda_2) = \langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{1}{2}[||\mathbf{x}_j||_2||\mathbf{b}||_2 + \langle \mathbf{x}_j, \mathbf{b}\rangle], \tag{3.59}$$

$$u_j^-(\lambda_2) = -\langle \mathbf{x}_j, \boldsymbol{\xi}_1^*\rangle + \frac{1}{2}[||\mathbf{x}_j||_2||\mathbf{b}||_2 - \langle \mathbf{x}_j, \mathbf{b}\rangle]. \tag{3.60}$$

Note that it follows from (3.29), if $\max(u_j^+(\lambda_2), u_j^-(\lambda_2)) < 1$, then the jth asset can be safely eliminated for the computation of $\mathbf{w}^*_{(2)}$.

3.3.5 Choice of λ_1

Note that when λ is large enough, combining with the constraint $\mathbf{w}^T\mathbf{1} = 1$, the ℓ_1-norm regularization leads to $||\mathbf{w}||_1 = 1$, and the solution corresponds to the optimal no-short-sale portfolio. One can solve the no-short-sale portfolio solution using either a quadratic programming or a variation of our CDM algorithm. In terms of the start point of the regularization parameter λ_1, a natural choice is thus the smallest λ_0 corresponding to $||\mathbf{w}||_1 = 1$.

Denote the optimal no-short-sale portfolio as $\mathbf{w}^*_{(0)}$. It follows from the KKT conditions (3.19) and (3.20) that

$$\lambda_0 \mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\mathbf{w}^*_{(0)}) + \zeta^*_{(0)} \in \begin{cases} 1 & \text{if } w^*_{(0)j} \neq 0, \\ [-1, 1] & \text{if } w^*_{(0)j} = 0. \end{cases} \tag{3.61}$$

Note there are two unknown variables λ_0 and $\zeta^*_{(0)}$, but only one equality when $w^*_{(0)j} \neq 0$ and a series of inequalities $\lambda_0 \mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\mathbf{w}^*_{(0)}) + \zeta^*_{(0)} \geq -1$. The exact λ_0 cannot be solved by (3.61), but a range of λ_0 can be obtained by simple calculations. We choose the lower bound of the range as λ_1, and in our numerical studies, we found λ_1 and λ_0 are almost the same.

3.4 Simulation studies

In this section, we evaluate the performance of our proposed coordinate descent method with multipliers (CDM) and safe screening for equality constrained optimization (SASECO) using simulation studies. Specifically, we first demonstrate that the optimal portfolio obtained by CDM performs substantially better than the approximate solution proposed in Fan, Zhang and Yu (2012) (referred as LARS). Second, we show that SASECO can significantly save the computational cost of CDM.

3.4.1 Multi-factor model

Let R_j be the excessive return over the risk-free interest rate of the jth asset. A K-dimensional multi-factor model for a relevant asset can be expressed as

$$R_j = b_{j1}F_1 + \cdots + b_{jK}F_K + \epsilon_j, \qquad j = 1, \ldots, p, \tag{3.62}$$

where b_{jk} is the factor loading on the jth stock on the factor F_k; and ϵ_j is the specific return of the asset j, which plays the same role as noise in a linear model. Many popular risk models in financial industry, such as the Barra (Rosenberg, 1974) model and the Fama and French (Fama and French, 1992) model, can be classified as multi-factor models. Throughout the simulation, we assume that $\mathrm{E}(\boldsymbol{\epsilon}|\mathbf{f}) = \mathbf{0}$ and $\mathrm{cov}(\boldsymbol{\epsilon}|\mathbf{f}) = \boldsymbol{\Sigma}_\epsilon$ is diagonal but not necessarily homogeneous. We rewrite the multi-factor model in a matrix form as

$$\mathbf{R} = \mathbf{B}\mathbf{f} + \boldsymbol{\epsilon}. \tag{3.63}$$

Then (3.63) implies that $\mathbf{R}$ has mean and variance as follows

$$\boldsymbol{\mu} = \mathbf{B} \cdot \mathrm{E}(\mathbf{f}), \tag{3.64}$$

$$\boldsymbol{\Sigma} = \mathbf{B}\mathrm{cov}(\mathbf{f})\mathbf{B}^T + \boldsymbol{\Sigma}_\epsilon = \mathbf{B}\mathrm{cov}(\mathbf{f})\mathbf{B}^T + \mathrm{diag}(\sigma_1^2, \ldots, \sigma_p^2). \tag{3.65}$$

We evaluate the proposed algorithms in two sets of simulations. To generate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, our first attempt is to generate the data matrix of interests from Normal or Uniform distribution, and the second attempt is to generate the parameters from a fit of the Barra model based on a real dataset. We illustrate the procedure of the simulations in the next two subsections.

3.4.2 Simulation based on artificial settings

Entries of the factor loading $\mathbf{B}$ and the mean vector of the factors $\mathrm{E}(\mathbf{f})$ are generated from $N(0, 1)$; variances of the specific returns $\sigma_1^2, \ldots, \sigma_p^2$ are generated from a uniform distribution with minimum 0.5. We set the diagonals of $\mathrm{cov}(\mathbf{f})$ to be 1 and off diagonals to be 0.2. We consider two scenarios for the number of factor $k = 3$ and $k = 30$ and also two possibilities for the number of assets $p = 1000$ and $p = 3000$. The variances of the specific returns are adjusted with the signals.

In each setting, we first apply the CDM algorithm (without screening) on a pre-specified grid of 100 λ values, where $\lambda_1 > \lambda_2 > \ldots > \lambda_{100}$. Specifically, λ_1 is chosen using the strategy described in section 3.3.5, and 20 equally-spaced λ values are chosen such that λ/λ_1 ranges from 0.5 to 1, and 80 equally-spaced λ values are chosen such that λ/λ_1 ranges from 0.05 to 0.5. The reason we use a coarser grid when λ is large and a finer grid when λ is small is because when λ is large, the number of selected assets changes slowly, while when λ is small, the number of selected assets changes more rapidly. In each setting, we also apply the LARS approximation proposed in Fan, Zhang and Yu (2012). The results are shown in the left column of Fig. 3.1 and Fig. 3.2. Note the vertical axis is the value of the mean-variance objective function, and the horizontal axis is the ℓ_1-norm of $\mathbf{w}$, i.e. c. First, we note that the value of the objective function based on the exact (CDM) solution is much lower than that of based on the approximate solution, indicating that the discrepancy between the exact solution and the approximate solution can be large. Second, we note that as c increases, the value of the objective function based on the exact

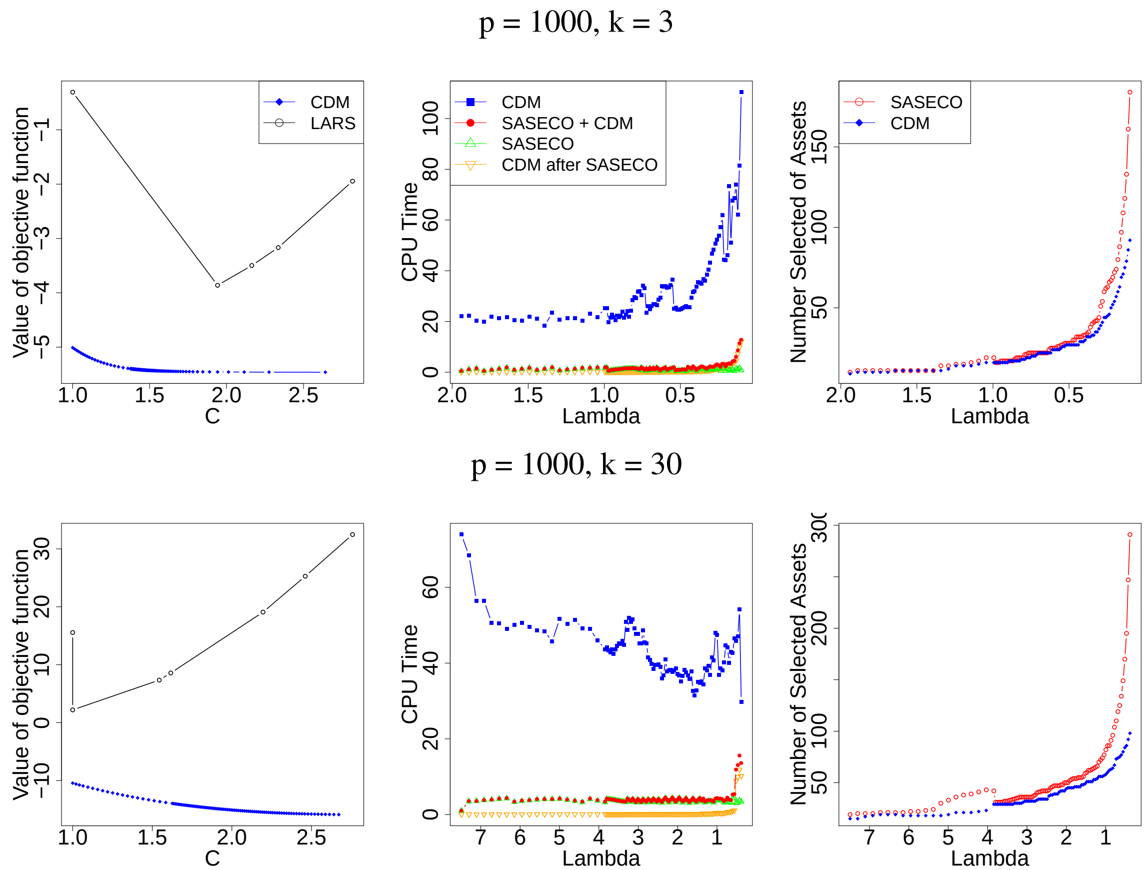

Figure 3.1: Left panels compare the value of the objective function based on the exact solution with that based on the approximate solution, as a function of the exposure parameter c. Middle panels compare the computational cost of the CDM algorithm without screening to that with screening. Right panels illustrate the number of remaining assets after screening and the number of assets in the final exact solution.

solution decreases, which is expected, while the value of the objective function based on the approximate solution first decreases then increases, which indicates that the quality of the approximate solution becomes worse when c either very small or very large.

Then in each setting, we apply the screening algorithm SASECO to first remove those assets that we are able to identify as having zero weights in the final solution, and then apply the CDM algorithm to the remaining assets to obtain the final exact solutions. The results are shown in the second and third columns of Fig. 3.1 and Fig. 3.2. As we can see, through the second column of Fig. 3.1 and Fig. 3.2, the computational cost of CDM without screening is much higher than that with screening. Further, we can also see, through the third columns of Fig. 3.1 and Fig. 3.2, that SASECO is very effective at removing zero-weighted assets especially when λ is relatively large, i.e. the number of remaining assets after screening is very close to the number of assets in the final exact solution. As λ

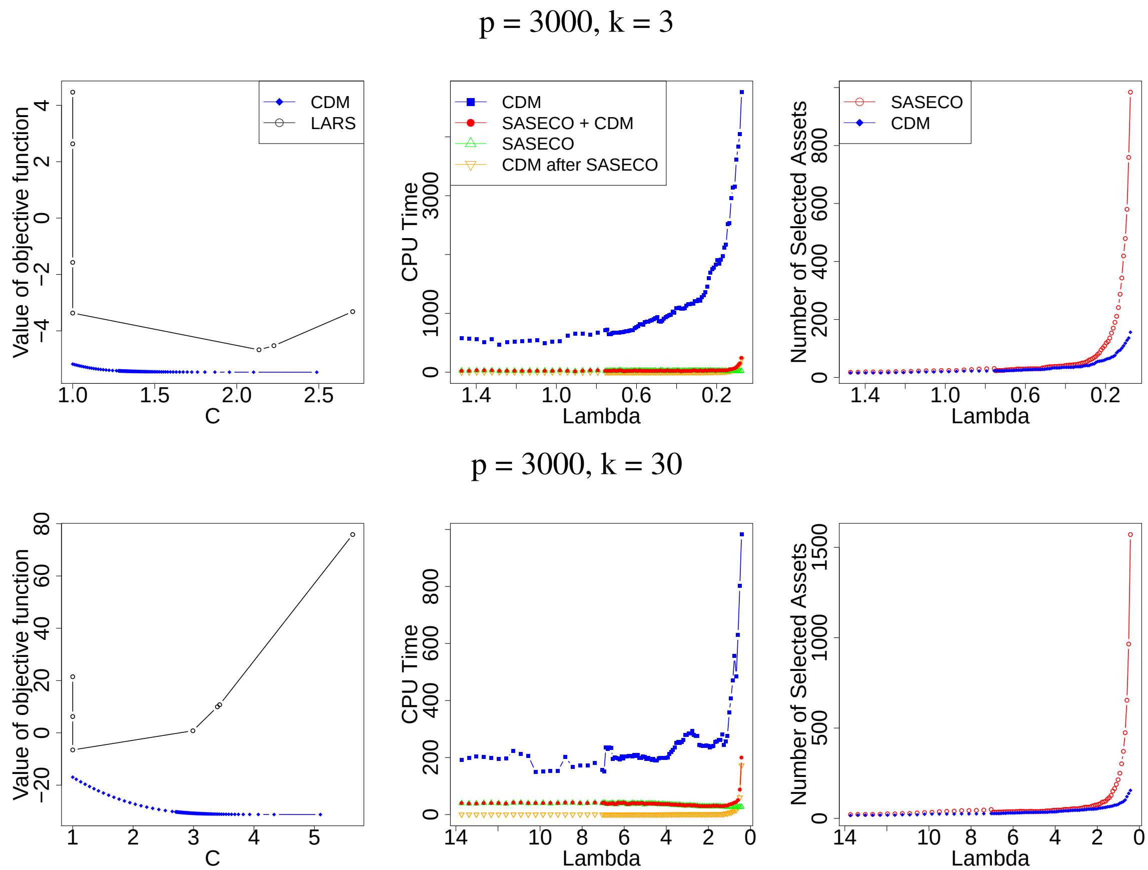

Figure 3.2: Left panels compare the value of the objective function based on the exact solution with that based on the approximate solution, as a function of the exposure parameter c. Middle panels compare the computational cost of the CDM algorithm without screening to that with screening. Right panels illustrate the number of remaining assets after screening and the number of assets in the final exact solution.

decreases, the effectiveness of screening degrades, but still, it is able to remove a significant amount of zero-weighted assets.

3.4.3 Simulation based on real data

In this section, we set $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ based on real data. Specifically, we collect S&P 500 monthly data from 2000 to 2012, and apply the Barra model (see Chapter 2) to obtain estimates for $\mathrm{E}(\mathbf{f})$, $\mathrm{cov}(\mathbf{f})$ and $\boldsymbol{\Sigma}_\epsilon$. Then we generate n-period returns of p assets as follows. We generate the factors returns over n-periods from the normal distribution $N(\mathrm{E}(\mathbf{f}), \mathrm{cov}(\mathbf{f}))$. See Table (3.1) for $\mathrm{E}(\mathbf{f})$ and $\mathrm{cov}(\mathbf{f})$. Further, we generate the entries of the factor loadings matrix $\mathbf{B}$ from the uniform distribution $U(0, 1)$. The specific returns are generated from a gamma distribution with the shape parameter 1.6322 and the scale parameter 0.006663, conditioned on the noise level of at least 0.001.

Table 3.1: Parameters for factor returns

$E(\mathbf{f}) \times 10^4$	$\text{cov}(\mathbf{f}) \times 10^4$				
-39.950	9.827	3.406	-3.938	2.001	5.3780
317.758	3.406	19.507	-0.860	2.510	-8.940
-468.772	-3.938	-0.860	12.673	-2.151	-5.713
-344.863	2.001	2.510	-2.151	6.073	-3.355
1091.531	5.378	-8.940	-5.713	-3.355	46.348

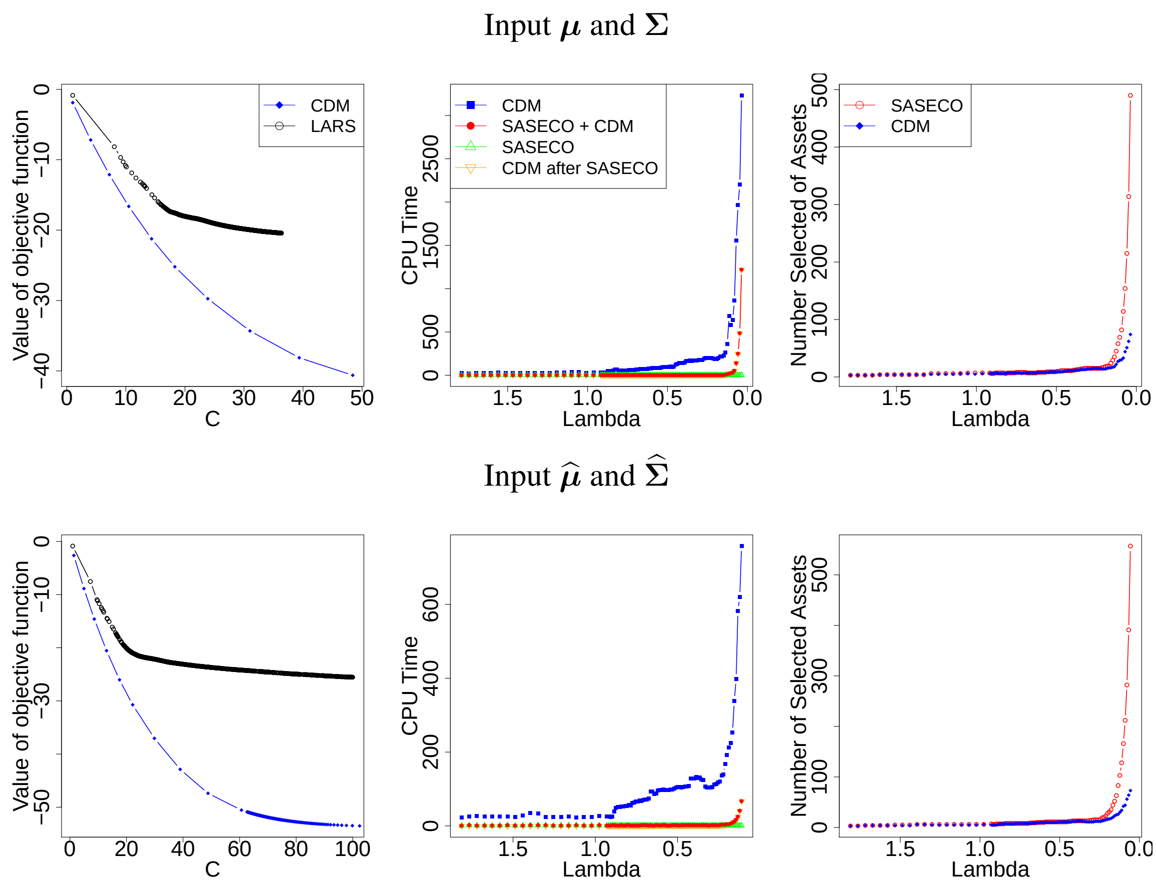

Figure 3.3: Left panels compare the value of the objective function based on the exact solution with that based on the approximate solution, as a function of the exposure parameter c. Middle panels compare the computational cost of the CDM algorithm without screening to that with screening. Right panels illustrate the number of remaining assets after screening and the number of assets in the final exact solution.

We set $n = 252$ and $p = 1000$ in the simulation. Given the simulated data, we first apply the Barra model (see Chapter 2) to obtain estimates for $\boldsymbol{\mu}$, Σ, denoted as $\widehat{\boldsymbol{\mu}}$ and $\widehat{\Sigma}$, and then use them as inputs for (3.2) and apply both CDM and SASECO + CDM algorithms. As a comparison, we also use $\boldsymbol{\mu}$ and Σ, obtained via fitting the original S&P 500 data, as inputs for (3.2) and apply both CDM and SASECO + CDM algorithms. The results are shown in

Fig. 3.3. The patterns are similar to those found in Fig. 3.1 and Fig. 3.2.

3.5 Empirical study

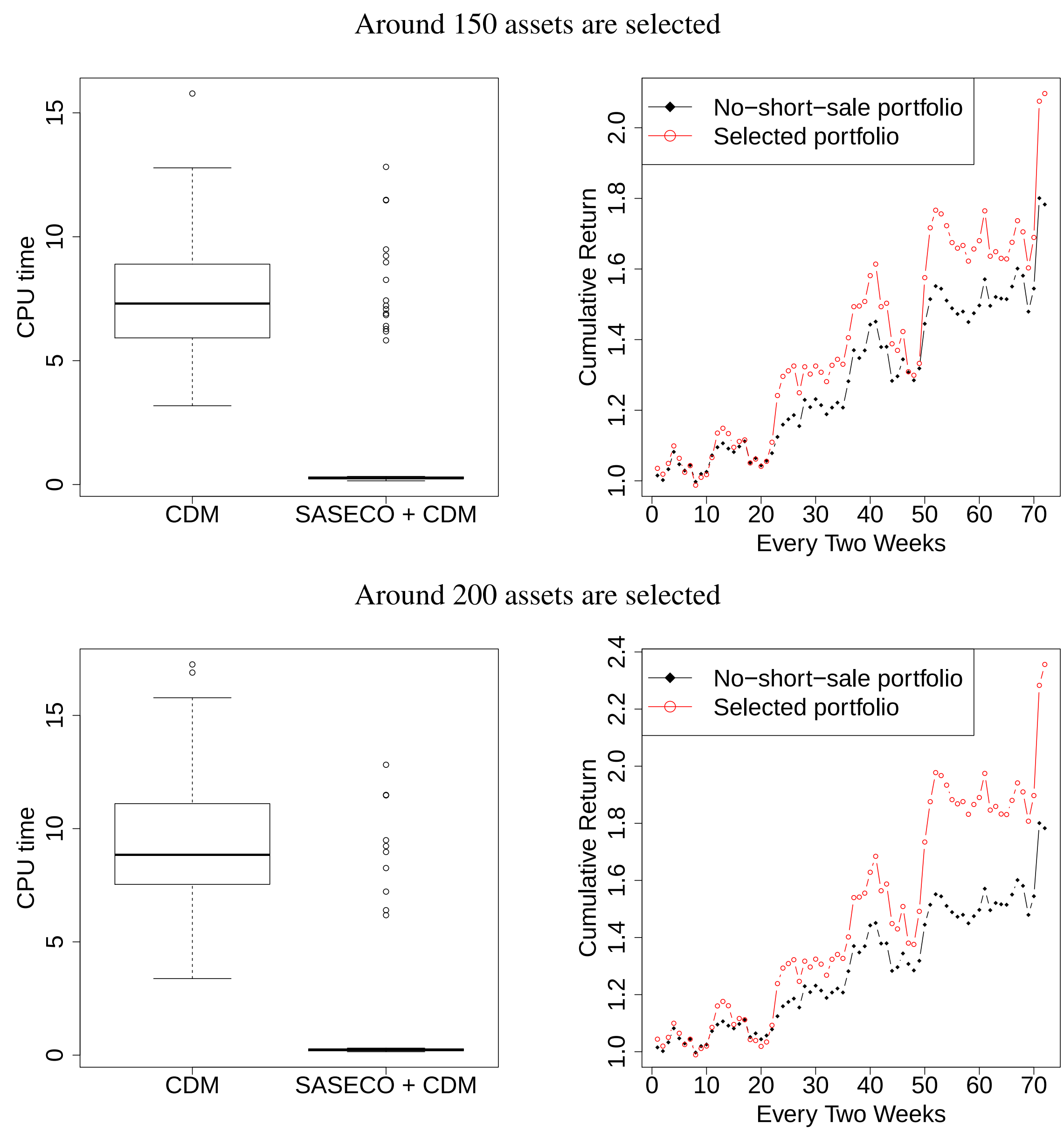

Figure 3.4: The left panels show the boxplots of time costs for obtaining the optimal portfolios by plain CDM and CDM with SASECO. The right panels describe the accumulative returns of the no-short-sale portfolios and the optimal portfolios with size around 150 and 200 (first row and second row) as a function of regularization parameters.

In this section, we apply our algorithms to the Chinese stock market. The dataset consists of daily stock returns for over 1600 companies and their associated industry classifications from Jan 4, 2011 to Dec 31, 2014. The daily returns have been adjusted by dividend, right of exclusion and other events. Daily HS300 returns as market returns are also available. Companies that have been suspended for an extended period of time or have too many missing values have been removed, and in the end 1140 assets are kept.

We fit the Barra model to the 1140 assets every two weeks. The factor loading matrix consists of 29 industry categories; the covariance of the factors and the covariance of specific returns are estimated using returns on the past 120 days with exponential decay (half-life of the decay is set to 60 days); the mean of the factors is estimated using returns on the past 30 days, again with exponential decay (half-life of the decay is set to 10 days). We then plug in the estimated return and covariance in (3.2) to construct the optimal portfolio. The λ value is chosen such that about 150 (or 200) assets are selected in the final solution. Once again, the optimal portfolio gets updated every two weeks.

We record the computational time and also the cumulative return of constructed portfolios. The results are shown in Fig. 3.4. From the left panels, we can see, similar to what we observe in Fig. 3.1 - 3.3, the computational cost of SASECO + CDM is much lower than that of CDM without screening. The outliers in the boxplots for SASECO + CDM correspond to the computational cost incurred at the starting value of λ, i.e. λ_1, or the no-short-sale solution. In the right panels of Fig. 3.4, we compare the cumulative returns of the optimal portfolio with that of the no-short-sale portfolio. We can see that the optimal portfolio, allowing negative weights, tends to perform better than the no-short-sale portfolio. The optimal portfolio that consists of 200 assets also seems to perform slightly better than that consisting of 150 assets.

3.6 Conclusion

The ℓ_1 regularized mean-variance analysis is an efficient tool to select and allocate stable and sparse portfolios, especially when the number of assets is large. To solve the corresponding optimization problem exactly and efficiently, we have developed a coordinate descent with multipliers (CDM) algorithm. We have demonstrated that the exact solution can be much better than the approximate solution in the literature. Further, we have also developed a screening rule (SASECO) that can effectively identify the zero-weighted assets before solving the optimization problem. To our knowledge, SASECO is the first screening rule that accommodates equality constraint in sparse learning algorithms. Our simulation studies indicate that it can significantly reduce the computational cost comparing to without screening.

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CHAPTER 4

Spectral Regularization Algorithms for Learning Corrupted Low-Rank Matrices

4.1 Introduction

4.1.1 Motivation

In this chapter, we consider the problem of recovering a low rank matrix from an observed data matrix. The principle component analysis (PCA) is a classical example, and one of the best known techniques in multivariate analysis. However, it is also well known that PCA is sensitive to outliers or corrupted observations. To address this problem, Wright et al. (2009) and Candès and Recht (2009) proposed the Robust PCA. They assume that a data matrix $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$ can be decomposed as $\mathbf{M} = \mathbf{L}_0 + \mathbf{S}_0$, where $\mathbf{L}_0$ is a low rank matrix and $\mathbf{S}_0$ is a sparse matrix.

Note that in the above formulation, besides $\mathbf{S}_0$, no other noise component is considered. However, in real-world applications, small noise and outliers often exist together. For example, in the image denoising problem, the grouped patches usually have almost identical structural content, which leads to low-rank structure. Since the quality of the image depends on many factors such as capturing technology, lighting conditions, and transmission errors, small perturbations are un-avoidable. Furthermore, it is also likely that a few pixels are largely corrupted by movement, change of conditions and so on. Therefore, it is more reasonable to decompose the observed data into a low rank component, a sparse component and a noise component, rather than just a low rank component and a sparse component.

Another example can be drawn from applications in finance. Suppose we have price data of hundreds of companies over a period of time, and usually a few principle components, corresponding to the overall economic status and industry trend, can explain well the variance in the data. However, it is also likely that there are a few companies whose prices do not follow the overall economic trend, thus making themselves outliers for the

PCA analysis. In this case, unlike in the image denoising problem where only a few pixels are corrupted, entire data points (companies) are corrupted. Still, one may assume that the observed data matrix can be decomposed into a low rank matrix that explains most of the variance in the data, a sparse matrix that represents outliers, and a noise matrix.

4.1.2 Background and related work

Specifically, we consider the following problem: suppose we are given a data matrix $\mathbf{M}$, where rows correspond to data points and columns correspond to variables, and it can be decomposed as

$$\mathbf{M} = \mathbf{L}_0 + \mathbf{S}_0 + \mathbf{N}_0, \tag{4.1}$$

where $\mathbf{L}_0$ is a low rank matrix, $\mathbf{S}_0$ is a sparse matrix with either randomly distributed non-zero entries or non-zero rows, and $\mathbf{N}_0$ is a noise matrix.

4.1.2.1 The element-wise sparse case

We first focus on the case when $\mathbf{S}_0$ is a randomly distributed sparse matrix. If $\mathbf{N}_0 = \mathbf{0}$, the problem can be addressed by robust PCA, i.e.

$$\begin{aligned} &\text{minimize} \quad && ||\mathbf{L}||_* + \lambda ||\mathbf{S}||_1, \\ &\text{subject to} \quad && \mathbf{L} + \mathbf{S} = \mathbf{M}, \end{aligned} \tag{4.2}$$

where $||\mathbf{L}||_*$ is the nuclear norm, or the sum of the singular values of $\mathbf{L}$, and $||\mathbf{S}||_1$ is the ℓ_1-norm of $\mathbf{S}$ with $\mathbf{S}$ seen as a long vector. Many theoretical results, including Candès and Recht (2009) and Candès et al. (2011) have been derived for identifying the low rank matrix and the sparse matrix based on (4.2). They showed that under certain assumptions, the true underlying low rank matrix can be recovered with high accuracy.

Methods for solving the optimization problem have also been extensively discussed. The two most popular ones are the proximal gradient algorithm (PGA) (Lin et al., 2009) and the augmented Lagrange multiplier (ALM) algorithm (Lin, Chen and Ma, 2010). Experiments show the ALM algorithm is one of the most efficient algorithms for robust PCA. Details of the algorithm were presented by Lin, Chen and Ma (2010). Specifically, they considered on the augmented Lagrange multiplier:

$$f(\mathbf{L}, \mathbf{S}, \mathbf{Y}, \mu) = ||\mathbf{L}||_* + \lambda ||\mathbf{S}||_1 + < \mathbf{Y}, \mathbf{M} - \mathbf{L} - \mathbf{S} > + \frac{\mu}{2} ||\mathbf{M} - \mathbf{L} - \mathbf{S}||_{\mathrm{F}}^2, \tag{4.3}$$

and iteratively updated $\mathbf{L}, \mathbf{S}, \mathbf{Y}$ to reach convergence.

Note that in (4.2) the noise component $\mathbf{N}_0$ is assumed $\mathbf{0}$. This is often not reasonable in practice. Zhou et al. (2010) considered a relaxation of (4.2) to allow the noise component, i.e.

$$\begin{aligned} &\text{minimize} \quad ||\mathbf{L}||_* + \lambda||\mathbf{S}||_1, \\ &\text{subject to} \quad ||\mathbf{M} - \mathbf{L} - \mathbf{S}||_F \leq \delta, \end{aligned} \tag{4.4}$$

where δ controls the deviation of $\mathbf{L} + \mathbf{S}$ from the observed data matrix $\mathbf{M}$. Further, Zhou et al. (2010) derived an upper bound for the error of estimation on $\mathbf{L}$ and $\mathbf{S}$ under similar conditions as in the robust PCA.

Algorithms for solving (4.4) are not discussed as extensively as those for (4.2). In scope of the ALM, Tao and Yuan (2011) developed the alternative splitting argumented Lagrange method (ASALM) and its variant (VASALM). It turns out that in order to achieve convergence, the ASALM, though a direct modification of ALM, needs to satisfy certain conditions that are complicated and difficult to check; while the VASALM algorithm is more complicated.

4.1.2.2 The row sparse case

The second case we consider is when entire data points are corrupted. This corresponds to the case when a small fraction of rows (rather than a few elements) of $\mathbf{S}$ are non-zero. A major change here is suppose that the data matrix $\mathbf{M}$ can be decomposed by a low rank matrix $\mathbf{L}_0$ and a row sparse matrix $\mathbf{S}_0$ with nonzero rows indexed by $\mathcal{I}$. Then if $\mathbf{L}_{0,\mathcal{I}}$ does not lie in the space $\mathcal{L}$ generated by the rows of $\mathbf{L}_{0,\mathcal{I}^c}$, the existence of $\mathbf{S}_0$ makes the detection of $\mathbf{L}_{0,\mathcal{I}}$ impossible. On the other hand, if $\mathbf{L}_{0,\mathcal{I}}$ does lie in the space $\mathcal{L}$, then $\mathbf{L}_{0,\mathcal{I}}$ could be $\mathbf{0}$, be several rows from $\mathbf{L}_{0,\mathcal{I}^c}$, or be any matrix lying in $\mathcal{L}$, and the nonzero entries of the corresponding $\mathbf{S}_0$ can be $(\mathbf{M}-\mathbf{L}_0-\mathbf{N}_0)_{\mathcal{I}}$. Thus more constraints are needed for identifying $\mathbf{L}_{0,\mathcal{I}}$ and $\mathbf{S}_{0,\mathcal{I}}$.

Xu, Caramanis and Sanghavi (2010) are he first that studied the row corruption problem and they proposed the following criteria:

(1) For the noiseless case:

$$\begin{aligned} &\text{minimize} \quad ||\mathbf{L}||_* + \lambda||\mathbf{S}||_{1,2} \\ &\text{subject to} \quad \mathbf{L} + \mathbf{S} = \mathbf{M} \end{aligned} \tag{4.5}$$

(2) For the noisy case:

$$\begin{aligned} &\text{minimize} \quad ||\mathbf{L}||_* + \lambda||\mathbf{S}||_{1,2} \\ &\text{subject to} \quad ||\mathbf{M} - \mathbf{L} - \mathbf{S}||_F \leq \delta \end{aligned} \tag{4.6}$$

where $||\mathbf{S}||_{1,2}$ is the sum of the ℓ_2 norm of rows in $\mathbf{S}$. They also pointed out that exact recovery of $\mathbf{L}$ and $\mathbf{S}$ does not make sense in this setting. Instead, they proved that in the noiseless case, under certain conditions, the solution to problem (4.5) will exactly recover the row space of $\mathbf{L}_0$ and exactly identify the indices of rows corresponding to outliers; in the noisy case, under similar conditions, there exists $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{S}}$ such that $\mathbf{M}_0 = \tilde{\mathbf{L}} + \tilde{\mathbf{S}}$, such that $\tilde{\mathbf{L}}$ contains the correct row space and $\tilde{\mathbf{S}}$ contains the correct row support, and the difference between the solution to (4.6) and $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{S}}$ is upper bounded. It turns out both theory and algorithms are scarce in dealing with the noisy case. See Agarwal, Negahban and Wainwright (2012) for another piece of theoretical work for the noisy case. In terms of methods for solving (4.5) and (4.6), we were only able to find two algorithms for (4.5) (Chen et al., 2011, Xu, Caramanis and Sanghavi, 2010), but none for (4.6).

For the rest of the chapter, we will focus on the noisy case, considering both element-wise sparse and row sparse scenarios, with the goal of developing efficient and well-performed algorithms, for recovering the underlying low rank matrices.

The rest of the chapter is organized as follows. In Section 4.2, we propose our criteria for the noisy case and present theoretical properties. In Section 4.3, we develop two natural algorithms for optimizing the proposed criteria. In Section 4.4, we develop two alternative algorithms for optimizing the proposed criteria, and we also establish their convergence properties. In Section 4.5, we extend the results to the matrix completion problem. Simulation studies and real data applications are presented in Section 4.6 and Section 4.7 respectively. In Section 4.8, we conclude the chapter.

4.2 Proposed criteria and theoretical properties

Different from previous work, we propose the following criteria.

(1) For the element-wise corruption case, we consider

$$\text{minimize} \qquad f_{\lambda,\tau}(\mathbf{L},\mathbf{S}) = \frac{1}{2}||\mathbf{M}-\mathbf{L}-\mathbf{S}||_{\mathrm{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}||_1, \tag{4.7}$$

where λ and τ are tuning parameters and $||\mathbf{S}||_1$ is the ℓ_1 norm of $\mathbf{S}$.

(2) For data point corruption case, we consider

$$\text{minimize} \qquad f_{\lambda,\tau}(\mathbf{L},\mathbf{S}) = \frac{1}{2}||\mathbf{M}-\mathbf{L}-\mathbf{S}||_{\mathrm{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}||_{1,2}, \tag{4.8}$$

where λ, τ are again tuning parameters, and $||\mathbf{S}||_{1,2}$ is the sum of ℓ_2 norm of rows of $\mathbf{S}$.

Interpretations of (4.7) and (4.8) are straightforward. The first parts of both (4.7) and (4.8) measure the loss between observations and estimations; the second parts control the nuclear norm of $\mathbf{L}$, a relaxation of the rank of $\mathbf{L}$; the third parts control the sparsity of $\mathbf{S}$.

Next, we show theoretical properties of the solution to the proposed criteria. For notational simplicity, we let $n_1 = n_2 = n$ in the following results.

Theorem 4.2.1. *Suppose we observe* $\mathbf{M} = \mathbf{L}_0^{(1)} + \mathbf{S}_0^{(1)} + \mathbf{N}_0^{(1)}$*, where* $\mathbf{L}_0^{(1)}$ *and* $\mathbf{S}_0^{(1)}$ *satisfy identifiability conditions in Candès et al. (2011) with sufficiently small numerical constants, with high probability (over the choice of the support of* $\mathbf{S}_0^{(1)}$*), the solution* $(\hat{\mathbf{L}}^{(1)}, \hat{\mathbf{S}}^{(1)})$ *to (4.7) with* τ/λ *in a specific range satisfies*

$$||\hat{\mathbf{L}}^{(1)} - \mathbf{L}_0^{(1)}||_{\mathrm{F}}^2 + ||\hat{\mathbf{S}}^{(1)} - \mathbf{S}_0^{(1)}||_{\mathrm{F}}^2 \leq \frac{C_1}{\tau^2}(\lambda\sqrt{n} + \tau n + \delta_1)^4, \tag{4.9}$$

where $\delta_1 = ||\mathbf{N}_0^{(1)}||_{\mathrm{F}}$*, and* C_1 *is a numerical constant.*

The above theorem provides an upper error bound for recovering the low rank matrix and sparse matrix in the presence of noise. Note that δ_1 is generally of order $O(n)$. When τ is $O(1)$, the upper bound is basically of order $O(\delta_1^2 n^2)$, which is the most common case. In Zhou et al. (2010), they provided an upper error bound for convex program (4.4), that is $||\hat{\mathbf{L}}^{(1)} - \mathbf{L}_0^{(1)}||_{\mathrm{F}}^2 + ||\hat{\mathbf{S}}^{(1)} - \mathbf{S}_0^{(1)}||_{\mathrm{F}}^2 \leq C\delta^2 n^2$. However, their δ is not the exact Frobenius norm of noise matrix, but a pre-determined upper bound. One can imagine that if every entry-wise noise is over estimated by a numerical constant, δ can be $O(n)$ and larger than δ_1. With the true δ_1, our result is more precise to some extent.

A similar property can also be shown for the row sparse case.

Theorem 4.2.2. *Suppose we observe* $\mathbf{M} = \mathbf{L}_0^{(2)} + \mathbf{S}_0^{(2)} + \mathbf{N}_0^{(2)}$*, where* $\mathbf{L}_0^{(2)}$ *and* $\mathbf{S}_0^{(2)}$ *satisfy identifiability conditions in Xu, Caramanis and Sanghavi (2010) with sufficiently small numerical constants. Let the solution to (4.8) be* $(\hat{\mathbf{L}}^{(2)}, \hat{\mathbf{S}}^{(2)})$*. Then with* τ/λ *in a specific range, there exists* $(\tilde{\mathbf{L}}^{(2)}, \tilde{\mathbf{S}}^{(2)})$ *such that* $\mathbf{M}_0^{(2)} = \tilde{\mathbf{L}}^{(2)} + \tilde{\mathbf{S}}^{(2)}$*,* $\tilde{\mathbf{L}}^{(2)}$ *has the correct row space, and* $\tilde{\mathbf{S}}^{(2)}$ *has the correct row support, and*

$$||\hat{\mathbf{L}}^{(2)} - \tilde{\mathbf{L}}^{(2)}||_{\mathrm{F}} + ||\hat{\mathbf{S}}^{(2)} - \tilde{\mathbf{S}}^{(2)}||_{\mathrm{F}} \leq \frac{C_2}{\tau}(\tau\sqrt{n} + \delta_2)^2, \tag{4.10}$$

where $\delta_2 = ||\mathbf{N}_0^{(2)}||_{\mathrm{F}}$*, and* C_2 *is a numerical constant.*

As we mentioned in introduction, without additional structure, the problem to recover the true low rank matrix and corrupted data point is not identifiable. Thus, we are interested in recovering the row space and the outlier support. From the proof, we find that for the un-corrupted part, $\tilde{\mathbf{L}}^{(2)}$ is exactly $\mathbf{L}_0^{(2)}$, while for the corrupted part, $\tilde{\mathbf{L}}^{(2)}$ is mixed with corrupted data. Since exact recovery is not demanded, it makes sense the error bound in Theorem 4.2.2 is smaller than that in Theorem 4.2.1. The upper error bound in Theorem 4.2.2 mainly depends on the order of τ and δ_2. In practice, τ is controlling the ℓ_2 norm of a row vector, contributing to order of $\sqrt{n}$, and δ_2 is of order $O(n)$. Hence, the order of upper error bound will probably be $\delta_2\sqrt{n}$. On the other hand, the upper error bound of convex program (4.6) is provided by Xu, Caramanis and Sanghavi (2010), which is $C\delta\sqrt{n}$. Again, δ can be much larger than δ_2, resulting that the bound is possibly not as tight as the one in Theorem 4.2.2.

4.3 Two natural algorithms

In this section, we propose two algorithm for solving (4.7) and (4.8) respectively. We first introduce some lemma that are useful for the proposed algorithms.

Lemma 4.3.1. *Suppose the rank of matrix $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$ is r, and the singular value decomposition (SVD) of $\mathbf{M}$ is given by $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ with $\mathbf{D} = \text{diag}[d_1, \ldots, d_r]$. Then the solution to the optimization problem*

$$\text{minimize} \quad f(\mathbf{L}) = \frac{1}{2}||\mathbf{M} - \mathbf{L}||_F^2 + \lambda||\mathbf{L}||_*, \tag{4.11}$$

is given by $\hat{\mathbf{L}} = D_\lambda(\mathbf{M})$, where $D_\lambda(\mathbf{M}) = \mathbf{U}\mathbf{D}_\lambda\mathbf{V}^T$ with $\mathbf{D}_\lambda = \text{diag}[(d_1 - \lambda)_+, \ldots, (d_r - \lambda)_+]$, and $t_+ = \max(t, 0)$.

Lemma 4.3.2. *Suppose the matrix $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$. Then the solution to the optimization problem*

$$\text{minimize} \quad g(\mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{S}||_{\mathrm{F}}^2 + \tau||\mathbf{S}||_1, \tag{4.12}$$

is given by $\hat{S}_{i,j} = H_\tau(M_{i,j})$ where $H_\tau(M_{i,j}) = \text{sign}(M_{i,j})(|M_{i,j}| - \tau)_+$.

Lemma 4.3.3. *Suppose the matrix $M \in \mathbb{R}^{n_1 \times n_2}$. Then the solution to the optimization problem*

$$\text{minimize} \quad g^r(\mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{S}||_{\mathrm{F}}^2 + \tau||\mathbf{S}||_{1,2}, \tag{4.13}$$

is given by $\hat{S}_{i,j} = H_\tau^r(M_{i,j})$ where $H_\tau^r(M_{i,j}) = \text{sign}(M_{i,j})(|M_{i,j}| - \tau \times \frac{|M_{i,j}|}{||M_{i,\cdot}||_2})_+$.

Lemma 4.3.1 is a crucial result for the nuclear norm regularization, which was proved by Cai, Candès and Shen (2010) and Keshavan et al. (2009) using various techniques. Lemma 4.3.2 and Lemma 4.3.3 play similar important roles in the ℓ_1 norm regularization and in the $\ell_{1,2}$ norm regularization respectively. The proofs are straightforward; we omit them here.

4.3.1 Algorithms

To solve for (4.7) and (4.8), we use a block-wise coordinate descent approach. Consider (4.7) first. Note that if $\mathbf{S}$ is given, the solution for $\mathbf{L}$ can be calculated by Lemma 4.3.1, and if $\mathbf{L}$ is given, the solution for $\mathbf{S}$ has a closed form by Lemma 4.3.2. We iterate between the two steps until convergence. See Algorithm 2 for details.

The rationale for solving (4.8) is similar. Given $\mathbf{L}$, the solution for $\mathbf{S}$ can be obtained by Lemma 4.3.3. See Algorithm 3 for details.

The stopping criterion can be set in many ways. We use the relative change in the value of the objective function and the Frobenius norm of the low rank matrix as the stopping criterion, that is,

$$\max\Big(\frac{|f_{\lambda,\tau}(\mathbf{L}^{k+1},\mathbf{S}^{k+1}) - f_{\lambda,\tau}(\mathbf{L}^{k},\mathbf{S}^{k})|}{|f_{\lambda,\tau}(\mathbf{L}^{k},\mathbf{S}^{k})|}, \frac{||\mathbf{L}^{k+1}-\mathbf{L}^{k}||_{\mathrm{F}}^2}{||\mathbf{L}^{k}||_{\mathrm{F}}^2}\Big) \leq \epsilon. \tag{4.14}$$

Algorithm 2

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ
1: **Initialize**: $\mathbf{L}^0 = \mathbf{0}, \mathbf{S}^0 = \mathbf{0}, k = 0$
2: **while** the stopping criterion is not satisfied **do**
3: $\quad \mathbf{L}^{k+1} = D_\lambda(\mathbf{M} - \mathbf{S}^k)$,
4: $\quad \mathbf{S}^{k+1} = H_\tau(\mathbf{M} - \mathbf{L}^{k+1})$
5: $\quad k \leftarrow k+1$
6: **end while**
Output: $\mathbf{L}^k$ and $\mathbf{S}^k$

Algorithm 3

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ
1: **Initialize**: $\mathbf{L}^0 = \mathbf{0}, \mathbf{S}^0 = \mathbf{0}, k = 0$
2: **while** the stopping criterion is not satisfied **do**
3: $\quad \mathbf{L}^{k+1} = D_\lambda(\mathbf{M} - \mathbf{S}^k)$,
4: $\quad \mathbf{S}^{k+1} = H_\tau^r(\mathbf{M} - \mathbf{L}^{k+1})$
5: $\quad k \leftarrow k+1$
6: **end while**
Output: $\mathbf{L}^k$ and $\mathbf{S}^k$

4.3.2 Convergence analysis

In this section, we study convergence properties of the proposed algorithms. Since the two algorithms are similar, we only investigate convergence properties of Algorithm 2. All properties we state below are also applicable to Algorithm 3.

First, it is straightforward to see that the objective function never increases its value over the iterations.

Lemma 4.3.4. *Let $\{\mathbf{L}^k, \mathbf{S}^k\}$ be the iterates generated by Algorithm 2. Then the value of the objective function is monotonically decreasing, i.e.*

$$f_{\lambda,\tau}(\mathbf{L}^k, \mathbf{S}^k) \geq f_{\lambda,\tau}(\mathbf{L}^{k+1}, \mathbf{S}^k) \geq f_{\lambda,\tau}(\mathbf{L}^{k+1}, \mathbf{S}^{k+1}). \tag{4.15}$$

Since $f_{\lambda,\tau}(\mathbf{L}^k, \mathbf{S}^k)$ is nonnegative, Lemma 4.3.4 implies convergence of the value of objective function. Note that our objective function is convex, and the nondifferentiable part is separable in terms of coordinate blocks. Tseng (2001) established the convergence theorem of the parameters $\mathbf{L}^k$ and $\mathbf{S}^k$.

Note that though the natural algorithms are easy to implement, SVD computation is needed in every iteration, which can be very time consuming. This is also a drawback for all the other previous algorithms to recover the low rank matrices, including PGA, ALM and their extensions.

4.4 Two alternative algorithms

In this section, we develop two alternative algorithms for solving (4.7) and (4.8), where SVD computation is not required in every iteration. The idea relies on a remarkable connection between the nuclear norm of a matrix and the sum of two Frobenius norms arising from a decomposition of the matrix.

Lemma 4.4.1. *Let $\mathbf{L} \in \mathbb{R}^{n_1 \times n_2}$, $\mathbf{A} \in \mathbb{R}^{n_1 \times r}$, $\mathbf{B} \in \mathbb{R}^{n_2 \times r}$, with $0 < r \leq \min(n_1, n_2)$. Then*

$$||\mathbf{L}||_* = \min_{\mathbf{A},\mathbf{B}:\mathbf{L}=\mathbf{A}\mathbf{B}^T} \frac{1}{2}(||\mathbf{A}||_{\mathrm{F}}^2 + ||\mathbf{B}||_{\mathrm{F}}^2). \tag{4.16}$$

Based on this fact, one can derive the following result.

Lemma 4.4.2. *Let $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$. Then for any given r such that $0 < r \leq \min(n_1, n_2)$, the solution to the optimization problem*

$$\text{minimize}_{\mathbf{L}:\text{rank}(\mathbf{L})\leq r} \quad f(\mathbf{L}) = \frac{1}{2}||\mathbf{M} - \mathbf{L}||_F^2 + \lambda||\mathbf{L}||_*, \tag{4.17}$$

is given by $\hat{\mathbf{L}} = D_\lambda(\mathbf{M}_r)$*, where* $D_\lambda(\mathbf{M}_r) = \mathbf{U}_r\mathbf{D}_{r,\lambda}\mathbf{V}_r^T$ *with* $\mathbf{U}_r\mathbf{D}_r\mathbf{V}_r^T$ *is the rank-*r *SVD of* $\mathbf{M}$*, and* $\mathbf{D}_{r,\lambda} = \text{diag}[(d_1 - \lambda)_+, \ldots, (d_r - \lambda)_+]$.

Let $\mathbf{A} \in \mathbb{R}^{n_1 \times r}$ *and* $\mathbf{B} \in \mathbb{R}^{n_1 \times r}$*. Then for the same* r*, the solution to the optimization problem*

$$\text{minimize} \qquad F_\lambda(\mathbf{A}, \mathbf{B}) = \frac{1}{2}||\mathbf{M} - \mathbf{A}\mathbf{B}^T||_{\text{F}}^2 + \frac{\lambda}{2}(||\mathbf{A}||_{\text{F}}^2 + ||\mathbf{B}||_{\text{F}}^2), \tag{4.18}$$

is given by $\hat{\mathbf{A}} = \mathbf{U}_r(\mathbf{D}_{r,\lambda})^{\frac{1}{2}}$ *and* $\hat{\mathbf{B}} = \mathbf{V}_r(\mathbf{D}_{r,\lambda})^{\frac{1}{2}}$*, and all solutions satisfy* $\hat{\mathbf{A}}\hat{\mathbf{B}}^T = \hat{\mathbf{L}}$*, where* $\hat{\mathbf{L}}$ *is as given in (4.17).*

Motivated by this result, we now consider the following criterion:

(1) For the element-wise sparse case

$$\text{minimize} \qquad F_{\lambda,\tau}(\mathbf{A}, \mathbf{B}, \mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}(||\mathbf{A}||_{\text{F}}^2 + ||\mathbf{B}||_{\text{F}}^2) + \tau||\mathbf{S}||_1 \tag{4.19}$$

(2) For the row sparse case

$$\text{minimize} \qquad F_{\lambda,\tau}(\mathbf{A}, \mathbf{B}, \mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}(||\mathbf{A}||_{\text{F}}^2 + ||\mathbf{B}||_{\text{F}}^2) + \tau||\mathbf{S}||_{1,2} \tag{4.20}$$

where the dimension of $\mathbf{A}$ is $n_1 \times r$ and the dimension of $\mathbf{B}$ is $n_2 \times r$.

4.4.1 Algorithms

To solve (4.19) and (4.20), we can again use the idea of block descent. When $\mathbf{B}$ and $\mathbf{S}$ are fixed, it is easy to see that this optimization decouples into n_1 separate ridge regressions. Similarly, when $\mathbf{A}$ and $\mathbf{S}$ are fixed, solving for $\mathbf{B}$ is equivalently to n_2 separate ridge regressions. When $\mathbf{A}$ and $\mathbf{B}$ are fixed, we can solve for $\mathbf{S}$ using Lemma 4.3.2 and Lemma 4.3.3. See Algorithm 4 and Algorithm 5 for details. In terms of the stopping criterion, we use something comparable to (4.14), i.e.

$$\max\left(\frac{|F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1}) - F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)|}{|F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)|}, \frac{||\mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T - \mathbf{A}^k(\mathbf{B}^k)^T||_{\text{F}}^2}{||\mathbf{A}^k(\mathbf{B}^k)^T||_{\text{F}}^2}\right) \leq \epsilon. \tag{4.21}$$

Remarks:

1 Note that all previous algorithms, including PGA, ALM, and Algorithms 2 and 3, need to compute SVD in every iteration, and this can be time consuming. However, in Algorithm 4 and 5, SVD is only needed in initialization. Instead, matrix inversion is needed in the new algorithms, but the dimension of the matrix that needs to be inverted is only $r \times r$, where r is often much smaller than n_1 and n_2.

2 In theory, r should be chosen such that it is larger than the rank of $\mathbf{L}_0$. In practice, the choice of r can be guided by the SVD of $\mathbf{M}$.

3 Since the alternating ridge regression might not exactly reveal the rank of the solution, we add an eigenvalue thresholding step at the end of the algorithm, that is, to estimate $\mathbf{L}$ by $D_\lambda(\mathbf{A}\mathbf{B}^T)$.

Algorithm 4

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ, r

Initialize: $\mathbf{A} = \mathbf{U}\mathbf{D}, \mathbf{B} = \mathbf{V}\mathbf{D}$ and $\mathbf{S} = \mathbf{0}$

while the stopping criterion is not satisfied **do**

  Given $\mathbf{B}$ and $\mathbf{S}$, solve for $\mathbf{A}$: $\min_{\mathbf{A}} ||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}||\mathbf{A}||_{\text{F}}^2$

  $\implies \mathbf{A} = (\mathbf{M} - \mathbf{S})\mathbf{B}(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{I})^{-1}$

  Given $\mathbf{A}$ and $\mathbf{S}$, solve for $\mathbf{B}$: $\min_{\mathbf{B}} ||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}||\mathbf{B}||_{\text{F}}^2$

  $\implies \mathbf{B} = (\mathbf{M} - \mathbf{S})^T\mathbf{A}(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}$

  Given $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{S} = H_\tau(\mathbf{M} - \mathbf{A}\mathbf{B}^T)$

end while

Output: $\mathbf{A}, \mathbf{B}$ and $\mathbf{S}$

Algorithm 5

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ, r

Initialize: $\mathbf{A} = \mathbf{U}\mathbf{D}, \mathbf{B} = \mathbf{V}\mathbf{D}$ and $\mathbf{S} = \mathbf{0}$

while the stopping criterion is not satisfied **do**

  Given $\mathbf{B}$ and $\mathbf{S}$, solve for $\mathbf{A}$: $\min_{\mathbf{A}} ||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}||\mathbf{A}||_{\text{F}}^2$

  $\implies \mathbf{A} = (\mathbf{M} - \mathbf{S})\mathbf{B}(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{I})^{-1}$

  Given $\mathbf{A}$ and $\mathbf{S}$, solve for $\mathbf{B}$: $\min_{\mathbf{B}} ||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\text{F}}^2 + \frac{\lambda}{2}||\mathbf{B}||_{\text{F}}^2$

  $\implies \mathbf{B} = (\mathbf{M} - \mathbf{S})^T\mathbf{A}(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}$

  Given $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{S} = H_\tau^r(\mathbf{M} - \mathbf{A}\mathbf{B}^T)$

end while

Output: $\mathbf{A}, \mathbf{B}$ and $\mathbf{S}$

4.4.2 Convergence analysis

In this section, we investigate convergence properties of the Algorithm 4 and 5 in the context of (4.19)-(4.20) and (4.7)-(4.8). First, we show the value of objective function is monotone decreasing.

Theorem 4.4.3. *Let* $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ *be the iterates generated by Algorithm 4. The values of the objective function (4.19) are monotone decreasing, i.e.*

$$F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) \geq F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) \geq F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^k) \geq F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1}).$$

A similar result holds for (4.20). Again, since $F_{\lambda,\tau}(\mathbf{A}, \mathbf{B}, \mathbf{S})$ is positive, convergence of the value of the objective function is guaranteed.

4.4.2.1 Properties of convergent sequences

In addition to the convergence of the value of the objective function, properties of the limit points of the sequence $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ are also of interests. Since both (4.19) and (4.20) are non-convex, it is difficult to guarantee a unique limit point for either problem. Instead, we will discuss stationary points of the sequence $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$. The stationary point here is in the sense of first order stationarity, that is,

$$\partial_A F_{\lambda,\tau}(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = 0, \quad \partial_B F_{\lambda,\tau}(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = 0, \quad \partial_S F_{\lambda,\tau}(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = 0,$$

where $\partial_x f(x,y)$ is partial derivative of $f(x,y)$ with respect to x. Thus, $(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*)$ will be a fixed point of the updates.

Before we show the results on stationary points, we first introduce the following lemma.

Lemma 4.4.4. *Let* $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ *be the estimates at iterate the* k*th iteration. Then the following inequality holds:*

$$\begin{aligned} F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1}) \geq\; & \frac{1}{2}(||(\mathbf{A}^k - \mathbf{A}^{k+1})(\mathbf{B}^k)^T||_{\mathrm{F}}^2 + ||\mathbf{A}^{k+1}(\mathbf{B}^k - \mathbf{B}^{k+1})^T||_{\mathrm{F}}^2) \\ & + \frac{\lambda}{2}(||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2 + ||\mathbf{B}^k - \mathbf{B}^{k+1}||_{\mathrm{F}}^2) \\ & + (F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}_k) - F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1})) \\ & \doteq \eta^k. \end{aligned} \tag{4.22}$$

Further, we have

$$\mathbf{A}^k - \mathbf{A}^{k+1} \to \mathbf{0}, \quad \mathbf{B}^k - \mathbf{B}^{k+1} \to \mathbf{0}, \quad \mathbf{S}^k - \mathbf{S}^{k+1} \to \mathbf{0}, \quad as\ k \to \infty. \tag{4.23}$$

The above lemma shows that the difference between successive iterates goes to zero. This is a necessary condition for the following theorem.

Theorem 4.4.5. *Let $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ be the iterates generated by Algorithm 4 (or Algorithm 5). Then for any $\lambda > 0$ and $\tau > 0$, we have:*

(a) every limit point of $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ is a stationary point;

(b) the associated (sub)sequence $\{(\mathbf{A}^{n_k}, \mathbf{B}^{n_k}, \mathbf{S}^{n_k})\}$ have the same convergence. That is to say, if a sequence $\mathbf{A}^{n_k}$ converges to a limit point $\mathbf{A}^*$, then the sequences $\mathbf{B}^{n_k}$ and $\mathbf{S}^{n_k}$ also converge. Similarly, if $\mathbf{B}^{n_k}$ is known to converge to $\mathbf{B}^*$, the sequences $\mathbf{A}^{n_k}$ and $\mathbf{S}^{n_k}$ also converge.

Theorem 4.4.5 implies that if we observe a convergent subsequence of $\{\mathbf{A}^k\}$ or $\{\mathbf{B}^k\}$, the corresponding subsequence of $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ converges to a stationary point. In the next theorem, we explore the rate at which Algorithm 4 (and Algorithm 5) reaches to a stationary point.

Theorem 4.4.6. *Let $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ be the iterates generated by Algorithm 4 (or Algorithm 5) and f^∞ be the limit point of $F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$. We have $\eta^k \to 0$, and finite convergence rate*

$$\min_{1 \le k \le K} \eta^k \le \frac{F_{\lambda,\tau}(\mathbf{A}^1, \mathbf{B}^1, \mathbf{S}^1) - f^\infty}{K}, \quad \forall K > 0. \tag{4.24}$$

Further, the difference between successive iterates will satisfy

$$\min_{1 \le k \le K} (||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2 + ||\mathbf{B}^k - \mathbf{B}^{k+1}||_{\mathrm{F}}^2) \le \frac{2}{\lambda}\Big(\frac{F_{\lambda,\tau}(\mathbf{A}^1, \mathbf{B}^1, \mathbf{S}^1) - f^\infty}{K}\Big). \tag{4.25}$$

Theorem 4.4.6 implies that for any $\epsilon > 0$, we need at most $K = O(1/\epsilon)$ steps to reach a point $(\mathbf{A}^{k^*}, \mathbf{B}^{k^*}, \mathbf{S}^{k^*})$, such that $\eta^{k^*} \le \epsilon$, where $1 \le k^* \le K$. The sum of successive differences for $\mathbf{A}^{k^*}$ and $\mathbf{B}^{k^*}$ will be smaller than $2\epsilon/\lambda$. A $O(1/K)$ convergence rates for Algorithm 4 and Algorithm 5 are thus established.

4.4.2.2 Connections to the two formulations

In this section, we study the connections between (4.19)-(4.20) and (4.7)-(4.8). We first focus on the value of the objective functions. Note that the objective functions in (4.7) and

(4.8) can be written as

$$f(\mathbf{A}\mathbf{B}^T, \mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\mathrm{F}}^2 + \lambda||\mathbf{A}\mathbf{B}^T||_* + \tau||\mathbf{S}||_1, \tag{4.26}$$

$$f(\mathbf{A}\mathbf{B}^T, \mathbf{S}) = \frac{1}{2}||\mathbf{M} - \mathbf{A}\mathbf{B}^T - \mathbf{S}||_{\mathrm{F}}^2 + \lambda||\mathbf{A}\mathbf{B}^T||_* + \tau||\mathbf{S}||_{1,2}. \tag{4.27}$$

Without confusion, we omit the subscript λ and τ in objective functions. Using the fact $||\mathbf{A}\mathbf{B}^T||_* \leq \frac{1}{2}(||\mathbf{A}||_{\mathrm{F}}^2 + ||\mathbf{B}||_{\mathrm{F}}^2)$, we have $f(\mathbf{A}^k(\mathbf{B}^k)^T, \mathbf{S}^k) \leq F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$ for both element-wise sparse and row sparse penalization. We also note that if $\mathbf{A}^k$ and $\mathbf{B}^k$ are stored in the "SVD format", that is, $\mathbf{A}^k = \mathbf{U}^k(\mathbf{D}^k)^{1/2}$ and $\mathbf{B}^k = \mathbf{V}^k(\mathbf{D}^k)^{1/2}$, it follows that $f(\mathbf{A}^k(\mathbf{B}^k)^T, \mathbf{S}^k) = F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$ by Lemma 4.4.1. Remind that $F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$ converges to f^∞. Under the assumption that $\mathbf{A}^k$ and $\mathbf{B}^k$ are stored in the "SVD format", $f(\mathbf{A}^k(\mathbf{B}^k)^T, \mathbf{S}^k)$ is a decreasing sequence, and the sequence also converges to f^∞. However, f^∞ is not necessary the minimum of $f(\mathbf{A}\mathbf{B}^T, \mathbf{S})$.

Next, we focus on comparing the stationary points of (4.19)-(4.20) to (4.7)-(4.8). First we consider the element-wise sparse senario, i.e. (4.19) and (4.7). For $(\mathbf{L}^*, \mathbf{S}^*)$ to be a stationary point of (4.7), the first order sub-gradient condition should be satisfied, i.e.

$$\partial_L f(\mathbf{L}^*, \mathbf{S}^*) = (\mathbf{L}^* + \mathbf{S}^* - \mathbf{M}) + \lambda\partial||\mathbf{L}^*||_* = 0 \implies (\mathbf{L}^* + \mathbf{S}^* - \mathbf{M}) + \lambda\mathbf{U}^*\partial(\mathbf{D}^*)\mathbf{V}^{*T} = 0, \tag{4.28}$$

$$\partial_S f(\mathbf{L}^*, \mathbf{S}^*) = (\mathbf{L}^* + \mathbf{S}^* - \mathbf{M}) + \tau\partial||\mathbf{S}^*||_1 = 0 \implies (\mathbf{L}^* + \mathbf{S}^* - \mathbf{M}) + \tau\mathrm{sgn}(\mathbf{S}^*) = 0, \tag{4.29}$$

where $\mathbf{U}^*\mathbf{D}^*\mathbf{V}^*$ is the SVD of $\mathbf{L}^*$, $\partial(\mathbf{D}^*)$ is a subgradient of the nuclear norm $||\mathbf{D}^*||$, and $\mathrm{sgn}(\mathbf{S}^*)$ is sign function of $\mathbf{S}^*$. On the other hand, the conditions for $(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*)$ to be a stationary point of (4.19) are as follows:

$$\partial_A F(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = (\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M})\mathbf{B}^* + \lambda\mathbf{A}^* = 0 \implies (\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M})V^* + \lambda\mathbf{U}^* = 0, \tag{4.30}$$

$$\partial_B F(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = (\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M})^T\mathbf{A}^* + \lambda\mathbf{B}^* = 0 \implies U^{*T}(\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M}) + \lambda\mathbf{V}^{*T} = 0, \tag{4.31}$$

$$\partial_S F(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*) = (\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M}) + \tau\partial||\mathbf{S}^*||_1 = 0 \implies (\mathbf{A}^*\mathbf{B}^{*T} + \mathbf{S}^* - \mathbf{M}) + \tau\mathrm{sgn}(\mathbf{S}^*) = 0. \tag{4.32}$$

We can see two things from the above conditions. First, a stationary point of the convex problem about $f(\mathbf{L}, \mathbf{S})$ corresponds to a stationary point of the optimization problem $F(\mathbf{A}, \mathbf{B}, \mathbf{S})$. Second, a stationary point of the optimization problem about $F(\mathbf{A}, \mathbf{B}, \mathbf{S})$

is not necessary a stationary point of the convex problem about $f(\mathbf{L}, \mathbf{S})$. The first statement can be verified by left multiplying a matrix $\mathbf{U}^{*T}$ and right multiplying a matrix $\mathbf{V}^*$ to (4.28). In this case, one representative is $\mathbf{U}_F^* = \mathbf{U}_f^* \partial(\mathbf{D}_f^*)$, $\mathbf{V}_F^* = \mathbf{V}_f^* \partial(\mathbf{D}_f^*)$, $(\mathbf{D}_F^*)_{ii} = [(\mathbf{D}_f^*)_{ii}/(\partial(\mathbf{D}_f^*)_{ii})^2]$ and $\mathbf{S}_F^* = \mathbf{S}_f^*$, where the subscript represents the function it belongs to. In terms of the second statement, it makes sense the non-convex problem about $F(\mathbf{A}, \mathbf{B}, \mathbf{S})$ needs extra conditions to match the stationary points with the convex problem about $f(\mathbf{L}, \mathbf{S})$. Recall Lemma 4.4.2 that the rank of $\mathbf{A}^*$ and $\mathbf{B}^*$ should be greater than the rank of $\mathbf{L}^*$. Under this condition, if there exists a minimizer for function $F(\mathbf{A}, \mathbf{B}, \mathbf{S})$, the minimizer will correspond to the solution of the convex problem about $f(\mathbf{L}, \mathbf{S})$ with $\mathbf{L}_f^* = \mathbf{A}_F^* \mathbf{B}_F^{*T}$ and $\mathbf{S}_f^* = \mathbf{S}_F^*$ by Lemma 4.4.2.

Analyses for row sparse scenario are similar. Overall, we have the following conclusions.

Theorem 4.4.7. *Let* $\mathbf{A}^k \in \mathbb{R}^{n_1 \times r}$*,* $\mathbf{B}^k \in \mathbb{R}^{n_2 \times r}$*,* $\mathbf{S}^k \in \mathbb{R}^{n_1 \times n_2}$ *be the sequence generated by Algorithm 4 (or Algorithm 5) and let* $(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*)$ *denote a limit point of the sequence. Suppose the problem* $\min f(\mathbf{L}, \mathbf{S})$ *has a minimizer* L^* *with rank at most* r*. If* $(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*)$ *is the global minimizer of* $F(\mathbf{A}, \mathbf{B}, \mathbf{S})$*, then* $\mathbf{L}^* = \mathbf{A}^* \mathbf{B}^{*T}$ *and* $\mathbf{S}^*$ *is a solution to the convex problem of* $\min f(\mathbf{L}, \mathbf{S})$.

4.5 Extension to the matrix completion problem

In general, the matrix completion problem is underdetermined without additional information, since filling in the missing entries with any values will complete the matrix. In many instances, however, the target matrix is known to be structured in the sense that it is low-rank or approximately low-rank. Many practical cases fall in this general category. One popular example is the Netflix problem. Each row of the matrix consists of a user, each column corresponds to a movie, and each entry in the matrix is the rating that a user gives to a movie. The missing entries of this matrix are ratings that users have not yet rated and the goal is to predict these missing values in order to provide movie recommendations to the users. In this case, the low-rank or approximately low-rank structure of the matrix is assumed, as it is commonly believed that only a few factors contribute to a user's tastes or preferences.

Similar to the fully observed matrix, small noise and outliers or corrupted entries may exist in the incomplete matrix. Under similar assumptions, we extend our methods to estimate the low rank matrix and corrupted entries to the matrix completion problem. We also consider two scenarios, the element-wise sparse case and the row sparse case. Let

M be the observed matrix (containing missing values), $Sigma$ be the set of indices of the observed entries. Let $P_\Omega(\mathbf{M})$ denote the projection of the incomplete matrix to the observed entries, i.e.

$$P_\Omega(\mathbf{M})_{i,j} = \begin{cases} M_{i,j} & \text{if } (i,j) \in \Omega \\ 0 & \text{if } (i,j) \notin \Omega. \end{cases} \tag{4.33}$$

Taking advantage of the projection notation, we consider the following criteria for matrix completion:

$$\text{minimize} \quad f_{\lambda,\tau}(L,S) = \frac{1}{2}||P_\Omega(\mathbf{M} - \mathbf{L} - \mathbf{S})||_{\mathrm{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}||_1 \tag{4.34}$$

$$\text{minimize} \quad f_{\lambda,\tau}(L,S) = \frac{1}{2}||P_\Omega(\mathbf{M} - \mathbf{L} - \mathbf{S})||_{\mathrm{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}||_{1,2} \tag{4.35}$$

Note that Algorithm 2 and 3 can be modified to accommodate the incomplete matrix. See Algorithm 6 for details. The only difference is in updating the low rank matrix. Instead of only using the observed entries of the matrix, we also use information on the incomplete part from the most recent iteration. We found through empirical studies that this approach works well in terms of both convergence efficiently and estimation accuracy.

Algorithm 6

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ
1: **Initialize**: $\mathbf{L}^0 = \mathbf{0}, \mathbf{S}^0 = \mathbf{0}, k = 0$
2: **while** not converged **do**
3: $\quad \mathbf{L}^{k+1} = D_\lambda(P_\Omega(\mathbf{M} - \mathbf{S}^k) + P_{\Omega^\perp}(\mathbf{L}^k))$,
4: $\quad \mathbf{S}^{k+1} = H_\tau(\mathbf{M} - \mathbf{L}^{k+1})$ (for the elment-wise sparse case)
$\quad$ or $\mathbf{S}^{k+1} = H_\tau^r(\mathbf{M} - \mathbf{L}^{k+1})$ (for the row sparse case)
5: $\quad k \leftarrow k + 1$
6: **end while**
Output: $\mathbf{L}^k$ and $\mathbf{S}^k$

Similarly, Algorithms 4 and 5 can also be modified to accommodate the incomplete matrix. See Algorithm 7 for details.

Note that all previous convergence analysis can be carried over and done in a similar fashion for Algorithm 6 and 7. We omit the details here. Further, one can combine Algorithm 6 and 7 with cross-validation to select tuning parameter λ and τ in (4.7)-(4.8) and (4.19)-(4.20).

Algorithm 7

Input: $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$, λ, τ

1: **Initialize**: $\mathbf{A}^0 = \mathbf{UD}, \mathbf{B}^0 = \mathbf{VD}$ and $\mathbf{S}^0 = \mathbf{0}, k = 0$

2: **while** not converged **do**

3: Given $\mathbf{B}^k, \mathbf{S}^k$, solve for $\mathbf{A}^{k+1}$: $\min_A ||P_\Omega(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T - \mathbf{S}^k)||_{\mathrm{F}}^2 + \frac{\lambda}{2}||\mathbf{A}^k||_{\mathrm{F}}^2$
$\implies \mathbf{A}^{k+1} = \mathbf{M}^*\mathbf{B}^k((\mathbf{B}^k)^T\mathbf{B}^k + \lambda I)^{-1}$,
where $\mathbf{M}^* = P_\Omega(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T - \mathbf{S}^k) + \mathbf{A}^k(\mathbf{B}^k)^T$

4: Given $\mathbf{A}^{k+1}, \mathbf{S}^k$, solve for $\mathbf{B}^{k+1}$: $\min_B ||P_\Omega(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^k)^T - \mathbf{S}^k||_{\mathrm{F}}^2 + \frac{\lambda}{2}||\mathbf{B}^k||_{\mathrm{F}}^2$
$\implies \mathbf{B}^{k+1} = \mathbf{M}^{*T}\mathbf{A}^{k+1}((\mathbf{A}^{k+1})^T\mathbf{A}^{k+1} + \lambda I)^{-1}$,
where $\mathbf{M}^* = P_\Omega(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^k)^T - \mathbf{S}^k) + \mathbf{A}^{k+1}(\mathbf{B}^k)^T$

5: Given $\mathbf{A}^{k+1}, \mathbf{B}^{k+1}$, $\mathbf{S}^{k+1} = H_\tau(P_\Omega(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T))$
(for the element-wise sparse case)
or $\mathbf{S}^{k+1} = H_\tau^r(P_\Omega(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T)$ (for the row sparse case)

6: $k \leftarrow k + 1$

7: **end while**

Output: $\mathbf{A}^k, \mathbf{B}^k$ and $\mathbf{S}^k$

4.6 Simulation studies

In this section, we evaluate the performances of Algorithm 2-5. Following (4.1), we use the model $\mathbf{M}_{n_1 \times n_2} = \mathbf{U}_{n_1 \times r_0}\mathbf{V}_{r_0 \times n_2}^T + \mathbf{S}_0 + \sigma\mathbf{N}_0$, where n_1 and n_2 are fixed at 400, $\mathbf{U}$ and $\mathbf{V}$ are random matrices with $Normal(0, 2)$ entries, and the entries of $\mathbf{N}_0$ are i.i.d. from $Normal(0, 1)$.

First, we consider the case where the low rank matrix is corrupted by an element-wise sparse matrix. Specifically, we randomly assign 5% entries to $\mathbf{S}_0$ to be non-zero, and these non-zero entries are generated i.i.d. from $Uniform(1, 5)$. We change the rank r_0 from 5 to 30 and also range the noise level σ from 0.01 to 0.5. In Algorithm 4, we set the rank parameter r to 50 so that $r \geq r_0$ is always satisfied. We compare Algorithm 2 and Algorithm 4 with the ASALM algorithm of Tao and Yuan (2011), one of the most efficient and accurate algorithms in the literature that assumes the same model as our algorithms. To evaluate the performance, we use the relative error of the low rank matrix, i.e.

$$\text{Relative Error} = \frac{||\hat{\mathbf{L}} - \mathbf{U}\mathbf{V}^T||_{\mathrm{F}}^2}{||\mathbf{U}\mathbf{V}^T||_{\mathrm{F}}^2}.$$

Table 4.1 shows the results. As one can see, as the rank r_0 increases, the performance of each algorithm is pretty stable. Algorithm 4 works the best, Algorithm 2 performs slightly worse than Algorithm 4, but both work better than ASALM. On the other hand, as the noise level increases, performances of all three algorithms degrade. When the noise level is low, the difference between the three algorithms is not obvious, while when the noise level

Table 4.1: Mean(standard deviation) $\times 10^3$ of relative error over 10 replicants in the element-wise corruption case

$\sigma = 0.1$				
r_0	5	10	20	30
ASALM	3.75(0.0707)	3.73(0.0422)	3.37(0.125)	3.31(0.0316)
Algorithm 2	3.18(0.0632)	3.16(0.0843)	3.14(0.0843)	3.12(0.0422)
Algorithm 4	2.95(0.0707)	2.93(0.0675)	2.84(0.0843)	2.66(0.0843)
$r_0 = 20$				
σ	0.01	0.05	0.1	0.5
ASALM	0.032(0.0632)	1.18(0.0789)	3.37(0.125)	17.14(0.190)
Algorithm 2	0.064(0.0516)	1.62(0.0422)	3.14(0.0843)	15.83(0.356)
Algorithm 4	0.052(0.0632)	1.42(0.0422)	2.84(0.0843)	15.23(0.283)

is large, the advantage of Algorithm 2 and 4 over ASALM becomes more obvious. This might be due to the reason that in ASALM, one needs to pre-specify the noise level, and if the noise level is not set appropriately, the algorithm may not work well.

Table 4.2: CPU time per iteration in the element-wise corruption case

$\sigma = 0.1$				
r_0	5	10	20	30
ASALM	1.34	2.14	1.79	1.55
Algorithm 2	16.2	17.3	18.9	21.7
Algorithm 4	0.21	0.22	0.22	0.22
$r_0 = 20$				
σ	0.01	0.05	0.1	0.5
ASALM	1.82	1.78	1.79	1.93
Algorithm 2	20.9	20.3	18.9	18.9
Algorithm 4	0.21	0.22	0.22	0.22

Table 4.2 records the computational cost per iteration for each algorithm. Note that Algorithm 4 is significantly more efficient than the other two algorithms, because unlike the other two algorithms, it does not compute the SVD in every iteration. On the other hand, Algorithm 2 is the least efficient algorithm; this is probably due to the fact that it needs to estimate the parameter iteratively in every step of the algorithm.

We also consider the case where the low rank matrix is corrupted by a row sparse matrix. Keeping all other setting the same as before, we randomly assign 5% of the rows of $\mathbf{S}_0$ as non-zero rows, and their entries are generated i.i.d. from $Uniform(1, 5)$. We are not aware of any algorithms in the literature that deal specifically with this case. Thus, we only compare Algorithm 3 and Algorithm 5. Due to the identifiability issue, we use a slightly different measure to evaluate the performance,

$$\text{Relative Error} = \frac{||\hat{\mathbf{L}}_{\mathcal{I}_0^c} - (\mathbf{U}\mathbf{V}^T)_{\mathcal{I}_0^c}||_\text{F}^2 + ||P_{\mathbf{U}\mathbf{V}^{T\perp}}(\hat{\mathbf{L}}_{\mathcal{I}_0})||_\text{F}^2}{||\mathbf{U}\mathbf{V}^T||_\text{F}^2},$$

where $\mathcal{I}_0$ is the indices of the corrupted rows, $P_{\mathbf{U}\mathbf{V}^{T\perp}}(X)$ is the projection of X on the orthogonal space generated by $\mathbf{U}\mathbf{V}^T$.

Table 4.3: Mean(standard deviation) $\times 10^3$ of relative error over 10 replicants in the row corruption case

$\sigma = 0.1$				
r_0	5	10	20	30
Algorithm 3	3.83(0.0675)	4.13(0.0483)	4.50(0.115)	4.65(0.135)
Algorithm 5	3.41(0.0568)	3.51(0.0738)	3.64(0.0699)	3.78(0.0919)
$r_0 = 20$				
σ	0.01	0.05	0.1	0.5
Algorithm 3	1.49(0.11)	2.63(0.11)	4.50(0.115)	18.63(0.309)
Algorithm 5	0.42(0.0632)	1.79(0.0738)	3.64(0.0699)	16.16(0.381)

Table 4.4: CPU time per iteration in the row corruption case

$\sigma = 0.1$				
r_0	5	10	20	30
Algorithm 3	1.28	1.31	1.40	1.77
Algorithm 5	0.28	0.28	0.28	0.28
$r_0 = 20$				
σ	0.01	0.05	0.1	0.5
Algorithm 3	1.34	1.41	1.40	4.57
Algorithm 5	0.28	0.28	0.28	0.66

The results are shown in Table 4.3 and 4.4. The patterns are similar to those in the element-wise corruption case. Algorithm 5 works better than Algorithm 3 in terms of both estimation accuracy and computational cost.

4.7 Data applications

In this section, we apply the proposed methods to two data examples.

4.7.1 Background modeling from surveillance video

Detecting the background variations in a scene is one of the basic tasks in video surveillance. Because of the correlation between frames, it could be regarded as a low-rank modeling problem. Candès et al. (2011) applied robust principal component analysis to solve the problem. They model the background variations as low rank, and foreground objects as sparse errors. However, we investigated the differences between frames, finding that not only the foreground objects are different, but also the parts that look the same have small differences. In this case, treating the frames as the combination of a low rank component, an element-wise sparse component and small noise component is a more reasonable choice.

We consider the first example in Candès et al. (2011). The video is a sequence of 200 grayscale frames taken at an airport. The resolution for each frame is 176 $\times$ 144, and all frames can be stacked to be rows of a matrix. Thus, the dimension of data matrix $\mathbf{M}$ is $200 \times 25,344$. Note that the video has a relatively static background, but significant foreground variations. It is thus natural to assume that the rank of the low rank matrix is 1. We apply Algorithm 4 here with $r = 1$, λ being $0.01 \times \lambda_{\max}(\mathbf{M})$, and τ being 5 times of the noise magnitude. The left column of Figure 4.1 shows three frames taken from the original video, while the middle column and the right column show the recovered low-rank and sparse components, respectively. Note that the low-rank component correctly identifies the background with the person appearing throughout the video, while the sparse component correctly identifies the moving pedestrians.

Algorithm 4 enjoys several advantages to the Principal Component Pursuit (PCP) in Candès et al. (2011). First, the background model could be measured more accurately by imposing a rank in Algorithm 4. Note that $r = 1$ restricts all low-rank images to be the same, whereas it is hard to tell the difference in the low rank images through PCP. Second, Algorithm 4 can choose to strengthen or weaken the sparse component by adjusting the tuning parameter τ. For example, if one wishes see most of the anomaly in the frames, such as the characters seen in the top right subfigure of Figure 4.1, τ can be relatively small, while if only large objects are of interests, τ can be adjusted to be large. PCP, on the other hand, can only obtain one kind of sparse component, treating large corruption and small noise as the same. The third benefit of Algorithm 4 is in computational cost. On a desktop PC with a 2.13 GHz Core4 Duo processor, it takes Algorithm 4 76 iterations, roughly 1 minute to converge, while for PCP, it takes 806 iterations and 43 minutes.

Figure 4.1: The panels describe background modeling from a 200 frame sequence video taken in an airport. The first column shows three original frames. The second and third columns display low-rank $\hat{\mathbf{L}}$ and sparse $\hat{\mathbf{S}}$ obtained by Algorithm 4.

4.7.2 Face recognition

Face recognition is another domain that has been investigated using low-rank models. It is commonly assumed that images taken under distant illumination lie in a low dimensional linear subspace, Candès et al. (2011) for instance. Various methods have been proposed to remove shadows and peculiarities from the face images. We consider a more complicated situation here: images are contaminated. In other words, not only the shadows and peculiarities but also the corrupted images would disturb the recovery of low-rank model.

We take a sequence of face images from the Yale B face database as an example. There are 64 grayscale images of the same subject; the resolution of each image is 480×640. Stacking each image as a row, we obtain our data matrix $\mathbf{M} \in \mathbb{R}^{64\times 307{,}200}$. Of the 64 images, one has been contaminated by random noise generated from $Uniform(1, 100)$

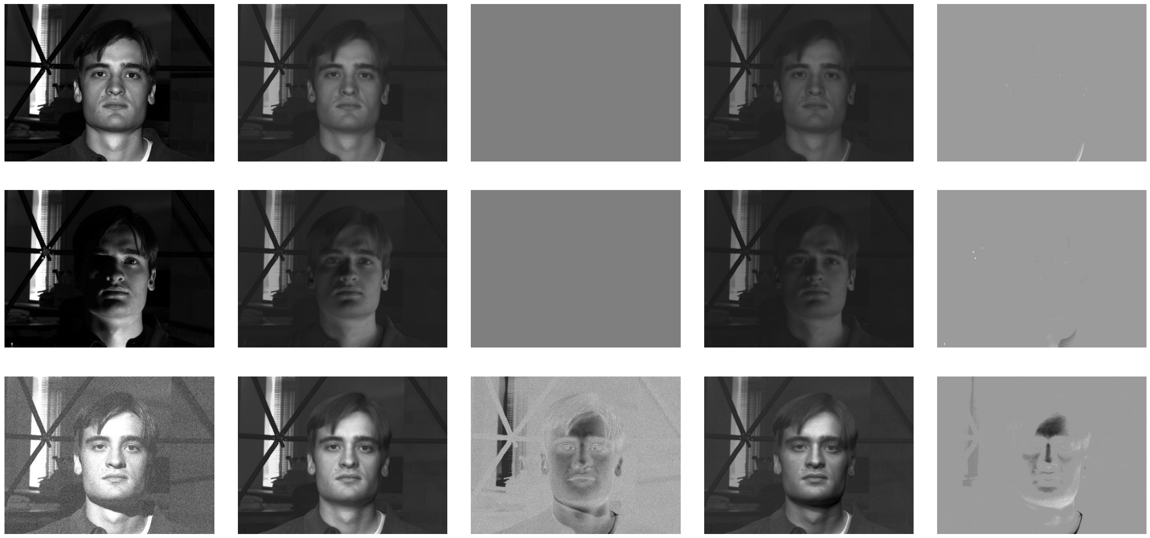

Figure 4.2: The panels describe face recognition problem with a person's face images from Yale B face dataset. There are 64 images under different illuminations. In the first column, the top two are original images, and the bottom one is a contaminated image. The second and third columns display low-rank $\hat{\mathbf{L}}_1$ and sparse $\hat{\mathbf{S}}_1$, which are estimated by Algorithm 5. The fourth and fifth columns are low-rank $\hat{\mathbf{L}}_2$ and sparse $\hat{\mathbf{S}}_2$, estimated by Algorithm 4.

on each pixel. We apply Algorithm 5 here. We set $r = 4$, $\lambda = 0.01 \times \lambda_{\max}(\mathbf{M})$ and $\tau = 16000$ so that the face and light direction information will be kept while the shadow will be removed. The first column of Figure 4.2 shows three original images with the third one being the contaminated image. The second and third columns show the estimated low-rank and sparse components by Algorithm 5, respectively. As a comparison, we also apply Algorithm 4 here assuming shadow and corruption as element-wise sparse component. The fourth and fifth columns in Figure 4.2 show the corresponding recovered low-rank and sparse components, respectively.

As one can see, Algorithm 5 is very effective at recovering the low rank component of these images. Both the shadow and contamination have been largely removed. The sparse component in the third column of Figure 4.2 clearly illustrate which image is contaminated. The results given by Algorithm 4 are inferior. The low rank components in the fourth column are not as clear as the corresponding changes in the second column, for example, the right face in the second image and the chin in the third image. Further, the third image in the fourth column indicates that the contamination was not fully removed.

4.8 Conclusion

This chapter considers two low rank recovery problems, one for the element-wise corrupted and noisy low-rank matrix, and the other for the row corrupted and noisy low-rank matrix. We proposed new criteria to address the problems and we show upper error bounds for the recovered low-rank and sparse components through our criteria. The obtained upper bounds are comparable to related literature. More importantly, we developed a series of spectral regularization algorithms that are easy to implement and efficient in solving the optimization problems. In particular, we have demonstrated that Algorithm 4 and Algorithm 5 have advantages in both computational efficiency and estimation accuracy.

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CHAPTER 5

Future Work

Problems of high-dimensional data analysis continue to challenge statisticians. The size of the data keeps growing and the scope of the analyses is no longer limited to finding significant relations, but also to develop efficient algorithms. In this thesis, we propose an EM-like method to estimate the covariance matrix for financial assets via the Barra model (Chapter 2), the coordinate descent with multipliers (CDM) algorithm and a screening rule (SASECO) to select stable and sparse portfolios (Chapter 3), and a series of spectral regularization algorithms to recover the low rank component from corrupted data matrices (Chapter 4). Good properties of these methods have been demonstrated both theoretically and numerically.

Looking into future, we list some of potential directions that one may be continue to pursue. In Chapter 2, we assume the factors and the specific returns are independent and identically distributed, but it does not necessarily hold in practice. A time series effects on factors or on specific returns across days may be taken into consideration. In this case, we can also apply the idea of EM algorithm to deal with the dependence, but the derivation of the theoretical properties of the MLE will be more challenging. On the other hand, a relaxation of the assumption for the covariance matrix of specific returns $\mathbf{\Sigma}_\epsilon$ is also possible. For example, $\mathbf{\Sigma}_\epsilon$ could be a sparse matrix instead of a diagonal matrix. Extra structure or information is probably needed, as well as optimization techniques such as adaptive thresholding.

In Chapter 3, the portfolio selection problem may involve more constraints on individual assets in real life. A possible extension of our methods would be to take into account extra constraints, for example, constraints related to percentage of allocations on each sector or industry, in the form of more linear equality constraints. Both the CDM and the SASECO can be extented to other linear equality constraints, as well as certain linear inequality constraints. Such an extension is likely to be beneficial to a variety of personalized investment strategies.

In Chapter 4, we have studied to recover the low rank matrix from both element-wise corrupted data matrices and row corrupted data matrices. It is possible that these two kinds of corruption exist simultaneously in noisy high-dimensional data. In this case, one more regularization term needs to be added to the objective function, which makes both theoretical property derivation, as well as algorithm development more challenging. Further, the format of our objective function can be adopted to broader domains. For example, replacing the Frobenius loss with the loss or the likelihood of regression with matrix covariates will lead to the so-called regularized matrix regression. Our algorithms can be modified to accommodate this scenario.

APPENDIX A

Proofs of the Main Results in Chapter 2

A.1 Proof of Theorem 2.2.1

Proof. (1) First, we prove the lower bound for the Barra one-step estimator $\widehat{\mathrm{cov}}(\mathbf{f})_o$ under Frobenius norm. We see that

$$\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f}) = \frac{1}{n-1}(\widehat{\mathbf{F}}_o\widehat{\mathbf{F}}_o' - \frac{1}{n}\widehat{\mathbf{F}}_o\mathbf{1}\mathbf{1}'\widehat{\mathbf{F}}_o') - \mathrm{cov}(\mathbf{f}). \tag{A.1}$$

On the other hand, we can decompose $\widehat{\mathbf{F}}_o$ with respect to $\mathbf{F}$, that is

$$\widehat{\mathbf{F}}_o = \mathbf{F} + (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}.$$

Plugging it in (A.1) and arranging the order, we get

$$\begin{aligned}\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f}) = & \frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f}) \\ & + (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')] \\ & + [\frac{1}{n-1}(\mathbf{F}\mathbf{E}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1} \\ & + (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}.\end{aligned} \tag{A.2}$$

This shows that the difference is a four-term perturbation of the population covariance matrix, and this presentation is our key technical tool. Now we analyze the four terms one by one.

Before going further, let us bound $||\mathbf{B}||$. From assumption (C), we know that $\mathrm{cov}(\mathbf{f}) \geq \sigma_1\mathbf{I}_K$. Here, the inequality $A_1 \geq A_2$ means $A_1 - A_2$ is positive semidefinite, for any symmetric positive semidefnite matrices A_1 and A_2. Then, we get $\sigma_1\mathbf{B}\mathbf{B}' = \mathbf{B}(\sigma_1\mathbf{I}_K)\mathbf{B}' \leq$

$\mathbf{B}\mathrm{cov}(\mathbf{f})\mathbf{B}' \leq \mathbf{\Sigma}$. With assumption (B), it follows that

$$||\mathbf{B}||^2 = \mathrm{tr}(\mathbf{B}\mathbf{B}') \leq \mathrm{tr}(\mathbf{\Sigma})/\sigma_1 \leq b/\sigma_1 = O(p),$$

i.e.

$$||\mathbf{B}|| = O(p^{1/2}). \tag{A.3}$$

Now we consider the first term $\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})$. From $c = O(1)$ in assumption (B), we see that the fourth moments of $\mathbf{f}$ are bounded across n, thus a routine calculation shows that

$$\mathrm{E}(||\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})||^2) = O(n^{-1}K^2),$$

which implies $||\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})||^2 = O_p(n^{-1}K^2)$.

As for the second term, we can decompose it into two parts

$$\begin{aligned}
&\mathrm{E}||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]||^2 \\
\leq &2(\mathrm{E}||\frac{1}{n-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{F}'||^2 + \mathrm{E}||\frac{1}{n(n-1)}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'||^2).
\end{aligned}$$

We see that

$$\begin{aligned}
&\mathrm{E}||\frac{1}{n-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{F}'||^2 \\
=&\frac{1}{(n-1)^2}\mathrm{E}\mathrm{tr}[(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{F}'\mathbf{F}\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}] \\
=&\frac{1}{(n-1)^2}\mathrm{E}\mathrm{tr}(\mathbf{F}'\mathbf{F}\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}) \\
=&\frac{1}{(n-1)^2}\mathrm{E}\mathrm{tr}[\mathbf{F}'\mathbf{F}E(\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}|\mathbf{F})] \\
=&\frac{1}{(n-1)^2}\mathrm{E}\mathrm{tr}(\mathbf{F}'\mathbf{F})\mathrm{tr}[(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}] \\
=&\frac{1}{n}\mathrm{E}(\mathbf{f}\mathbf{f}')\mathrm{tr}[(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}] \\
\leq&\frac{1}{n}\mathrm{E}(\mathbf{f}\mathbf{f}')||\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'|| \cdot ||\mathbf{\Sigma}_\epsilon|| \\
=&\frac{1}{n}\mathrm{E}(\mathbf{f}\mathbf{f}')[\mathrm{tr}(\mathbf{B}'\mathbf{B})^{-2}])^{1/2}[\mathrm{tr}(\mathbf{\Sigma}_\epsilon^2)]^{1/2} \\
=&O(n^{-1}p^{1/2}K^{3/2}).
\end{aligned} \tag{A.4}$$

The last equation is because of our assumption (B), (E) and (F). On the other hand,

$$\begin{aligned}
&\mathrm{E}||\frac{1}{n(n-1)}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'||^2\\
=&\frac{1}{n^2(n-1)^2}\mathrm{Etr}[(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}]\\
=&\frac{1}{n^2(n-1)^2}\mathrm{Etr}(\mathbf{F}'\mathbf{F})\mathrm{E}(\mathbf{1}\mathbf{1}'\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}\mathbf{1}\mathbf{1}'|\mathbf{F})\\
\le&\frac{1}{n^2(n-1)^2}\mathrm{Etr}(\mathbf{F}'\mathbf{F})||\mathbf{1}\mathbf{1}'\mathbf{1}\mathbf{1}'||\mathrm{E}(\mathbf{E}'\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E}|\mathbf{F})\\
=&O(n^{-1}p^{1/2}K^{3/2}).
\end{aligned}\tag{A.5}$$

The last equation can be easily followed by (A.4).

So summing up (A.4) and (A.5), we get the order for second term in (A.2)

$$\mathrm{E}||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{F}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]||^2=O(n^{-1}p^{1/2}K^{3/2}),\tag{A.6}$$

which results in $||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{F}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]||^2=O_p(n^{-1}p^{1/2}K^{3/2})$.

The third term is just the transpose of the second term, so they have the same result.

At last, we analyze the last term in (A.2). By large sample theory, we know that

$$\frac{1}{n-1}(\mathbf{E}\mathbf{E}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')=\mathbf{\Sigma}_\epsilon+O(n^{-1/2})\mathbf{1}_{p\times p},$$

So

$$\begin{aligned}
&||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{E}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||\\
=&||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'(\mathbf{\Sigma}_\epsilon+O(n^{-1/2})\mathbf{1}_{p\times p})\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||.
\end{aligned}\tag{A.7}$$

Hence, going back to $\widehat{\mathrm{cov}}(\mathbf{f})_o-\mathrm{cov}(\mathbf{f})$, we see that when n goes to infinity, only one term won't vanish, that is $(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}$. Thus, we have the lower bound of the difference,

$$||\widehat{\mathrm{cov}}(\mathbf{f})_o-\mathrm{cov}(\mathbf{f})||\ge c_1||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||,\tag{A.8}$$

which is the first part in Theorem 2.2.1.

We further consider the order of the left term. First, if all diagonal elements of $\mathbf{\Sigma}_\epsilon$ are identical, say σ^2, then

$$
\begin{aligned}
&||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}|| \\
=&||\sigma^2(\mathbf{B}'\mathbf{B})^{-1}|| = \sigma^2(\sum_{i=1}^{k}\frac{1}{\lambda_i(\mathbf{B}'\mathbf{B})^2})^{1/2}.
\end{aligned} \tag{A.9}
$$

By assumption (F), we know all $\lambda_i(\mathbf{B}'\mathbf{B})$ are bounded. As a result, $\sigma^2(\sum_{i=1}^{k} 1/\lambda_i(\mathbf{B}'\mathbf{B})^2)^{1/2}$ can be denoted as $c_1^* K^{1/2}$, where c_1 is a positive constant. For general case, i.e. the diagonal elements of $\boldsymbol{\Sigma}_\epsilon$ are not necessarily identical. Then we can bound $\boldsymbol{\Sigma}_\epsilon$ as follows

$$
\boldsymbol{\Sigma}_\epsilon \geq \sigma_{(1)}^2\mathbf{I},
$$

where $\sigma_{(1)}^2$ is the smallest element in diagonal $\boldsymbol{\Sigma}_\epsilon$. This inequality is in the sense of positive definite. So we can further have

$$
||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}|| \geq ||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\cdot\sigma_{(1)}^2\mathbf{I}\cdot\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}|| = c_1^* K^{1/2}. \tag{A.10}
$$

According to the above analysis, the lower bound in Theorem 2.2.1 must be greater than 0.

(2) Second, we prove the lower bound for the Barra two-step estimator $\widehat{\text{cov}}(\mathbf{f})_w$ under Frobenius norm.

Note that $\widehat{\text{cov}}(\mathbf{f})_w$ is quite similar to $\widehat{\text{cov}}(\mathbf{f})_o$, so we can also decompose $\widehat{\text{cov}}(\mathbf{f})_w - \text{cov}(\mathbf{f})$ into four parts. Replacing $\widehat{\mathbf{F}}_o, \widehat{\mathbf{E}}_o$ by

$$
\begin{aligned}
\widehat{\mathbf{F}}_w &= \mathbf{F} + (\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{E}, \\
\widehat{\mathbf{E}}_w &= [\mathbf{I} - \mathbf{B}(\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}]\mathbf{E},
\end{aligned}
$$

similarly to (A.2), we have

$$\widehat{\mathrm{cov}}(\mathbf{f})_w - \mathrm{cov}(\mathbf{f}) = \frac{1}{n-1}(\mathbf{FF}' - \frac{1}{n}\mathbf{F11}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})$$
$$+ (\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}[\frac{1}{n-1}(\mathbf{EF}' - \frac{1}{n}\mathbf{E11}'\mathbf{F}')]$$
$$+ [\frac{1}{n-1}(\mathbf{FE}' - \frac{1}{n}\mathbf{F11}'\mathbf{E}')]\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}$$
$$+ (\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}[\frac{1}{n-1}(\mathbf{EE}' - \frac{1}{n}\mathbf{E11}'\mathbf{E}')]\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}. \tag{A.11}$$

The first term is the same as that in part (1). When we bound the rest of the three terms as in part (1), similar deduction can be done. So the difference here is to replace $\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}'$ with $\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-2}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}$. When we obtain the OLS estimator for $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$, we get p diagonal elements $\hat{\sigma}_{i,o}^2, i = 1, \ldots, p$. Arranging them from the smallest to the largest, we get $\hat{\sigma}_{(1),o}^2, \ldots, \hat{\sigma}_{(p),o}^2$. Then, following calculation can be done:

$$\hat{\sigma}_{(1),o}^2\mathbf{I} \le \widehat{\mathbf{\Sigma}}_{\epsilon,o} \le \hat{\sigma}_{(p),o}^2\mathbf{I},$$
$$\hat{\sigma}_{(p),o}^{-2}\mathbf{I} \le \widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} \le \hat{\sigma}_{(1),o}^{-2}\mathbf{I},$$
$$\hat{\sigma}_{(p),o}^{-2}\mathbf{B}'\mathbf{B} \le \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B} \le \hat{\sigma}_{(1),o}^{-2}\mathbf{B}'\mathbf{B},$$
$$(\hat{\sigma}_{(1),o}^{-2}\mathbf{B}'\mathbf{B})^{-1} \le (\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1} \le (\hat{\sigma}_{(p),o}^{-2}\mathbf{B}'\mathbf{B})^{-1},$$
$$\hat{\sigma}_{(1),o}^{4}\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}' \le \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-2}\mathbf{B}' \le \hat{\sigma}_{(p),o}^{4}\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}',$$
$$\frac{\hat{\sigma}_{(1),o}^4}{\hat{\sigma}_{(p),o}^4}\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}' \le \widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-2}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} \le \frac{\hat{\sigma}_{(p),o}^4}{\hat{\sigma}_{(1),o}^4}\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}'.$$

By Lemma 1, we see that $\hat{\sigma}_{(1),o}^2$ and $\hat{\sigma}_{(p),o}^2$ are bounded away from 0 in probability one. So the last three terms in (A.11) can be bounded by corresponding terms in (A.2). Hence, we will get the same result as part (1). We denote a new positive constant C_2, and achieve the second part in Theorem 2.2.1, $||\widehat{\mathrm{cov}}(\mathbf{f})_w - \mathrm{cov}(\mathbf{f})|| \ge c_2||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||$. □

A.2 Proof of Theorem 2.2.2

Proof. (1) First, we prove the lower bound for the Barra one-step estimator $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ under Frobenius norm. We see that

$$\begin{aligned}\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon &= \text{diag}(\frac{1}{n}\widehat{\mathbf{E}}_o\widehat{\mathbf{E}}_o') - \mathbf{\Sigma}_\epsilon \\ &= \text{diag}(\frac{1}{n}[\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}']\mathbf{E}\mathbf{E}'[\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}']) - \mathbf{\Sigma}_\epsilon \\ &\equiv \text{diag}(\frac{1}{n}[(\mathbf{I} - \mathbf{H})\mathbf{E}\mathbf{E}'(\mathbf{I} - \mathbf{H})]) - \mathbf{\Sigma}_\epsilon \qquad \text{where } \mathbf{H} = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}' \\ &= \text{diag}(\frac{1}{n}\mathbf{E}\mathbf{E}') - \mathbf{\Sigma}_\epsilon - \text{diag}[\frac{1}{n}(\mathbf{H}\mathbf{E}\mathbf{E}' + \mathbf{E}\mathbf{E}'\mathbf{H})] + \text{diag}(\frac{1}{n}\mathbf{H}\mathbf{E}\mathbf{E}'\mathbf{H}) \quad \text{(A.12)}\end{aligned}$$

Considering the difference under Frobenius norm, we have

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon|| \geq ||\text{diag}(\frac{1}{n}\mathbf{H}\mathbf{E}\mathbf{E}'\mathbf{H}) - \text{diag}[\frac{1}{n}(\mathbf{H}\mathbf{E}\mathbf{E}' + \mathbf{E}\mathbf{E}'\mathbf{H})]|| - ||\text{diag}(\frac{1}{n}\mathbf{E}\mathbf{E}') - \mathbf{\Sigma}_\epsilon|| \quad \text{(A.13)}$$

By large sample theorem, it is easy to see

$$||\text{diag}(\frac{1}{n}\mathbf{E}\mathbf{E}') - \mathbf{\Sigma}_\epsilon|| = O_p(n^{-1/2}p^{1/2}). \quad \text{(A.14)}$$

So the last term in (A.13) will vanish when n goes to infinity. At the same time, we can write $\frac{1}{n}\mathbf{E}\mathbf{E}'$ in (A.13) as $\mathbf{\Sigma}_\epsilon + O(n^{-1/2})\mathbf{1}_{p\times p}$. So asymptotically, we will get

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon|| \geq c_3||\text{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H}) - \text{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon + \mathbf{\Sigma}_\epsilon\mathbf{H})||,$$

which is the first part in Theorem 2.2.2.

(2) Second, we prove the lower bound for the Barra two-step estimator $\widehat{\mathbf{\Sigma}}_{\epsilon,w}$ under Frobenius norm.

Replacing $\widehat{\mathbf{E}}_o$ by $\widehat{\mathbf{E}}_w = [\mathbf{I} - \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}]\mathbf{E}$ in (A.12), we get

$$\begin{aligned}\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon = \ &\text{diag}(\frac{1}{n}\mathbf{E}\mathbf{E}') - \mathbf{\Sigma}_\epsilon \\ &- \text{diag}[\frac{1}{n}(\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{E}\mathbf{E}' + \mathbf{E}\mathbf{E}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}')] \\ &+ \text{diag}[\frac{1}{n}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{E}\mathbf{E}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}']. \quad \text{(A.15)}\end{aligned}$$

To get the asymptotic lower bound for it, we use our boundedness strategy again. Similar calculation can be done to get

$$\mathrm{tr}[(\frac{\hat{\sigma}^2_{(1),o}}{\hat{\sigma}^2_{(p),o}})^2\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H}] \leq \mathrm{diag}[\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}^{-1}_{\epsilon,o}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}^{-1}_{\epsilon,o}\mathbf{\Sigma}_\epsilon\widehat{\mathbf{\Sigma}}^{-1}_{\epsilon,o}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}^{-1}_{\epsilon,o}\mathbf{B})^{-1}\mathbf{B}'] \leq \mathrm{tr}[(\frac{\hat{\sigma}^2_{(p),o}}{\hat{\sigma}^2_{(1),o}})^2\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H}]$$

Since the boundedness of $\hat{\sigma}^2_{(1),o}$ and $\hat{\sigma}^2_{(p),o}$ by Lemma 1, we conclude that $c||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon|| \leq ||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon|| \leq c'||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||$ with probability one, where c and c' are some positive constants. Hence, $||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon|| \geq c_4||\mathrm{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon\mathbf{H}) - \mathrm{diag}(\mathbf{H}\mathbf{\Sigma}_\epsilon + \mathbf{\Sigma}_\epsilon\mathbf{H})||$ asymptotically. □

A.3 Proof of Theorem 2.2.3

Proof. Since $||\widehat{\mathbf{\Sigma}}_o - \mathbf{\Sigma}|| = ||\mathbf{B}[\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})]\mathbf{B}' + \widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||$, we can easily get the lower bound by plugging in the two separate parts' lower bounds, and similar to the Barra two-step estimator. □

A.4 Proof of Theorem 2.2.4

Proof. (1) First, we show the upper bound for the Barra one-step estimator $\widehat{\mathrm{cov}}(\mathbf{f})_o$ under Frobenius norm.

Similar to Theorem 2.2.1, we decompose $\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})$ into four parts, see $(A.2)$. Further, by Cauchy-Schwarz inequality,

$$\begin{aligned}||\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})||^2 \leq 4(&||\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})||^2 \\ &+ ||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]||^2 \\ &+ ||[\frac{1}{n-1}(\mathbf{F}\mathbf{E}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||^2 \\ &+ ||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||^2). \quad (A.16)\end{aligned}$$

We have analyzed the first three terms in the proof of Theorem 2.2.1. So what we need to focus on is the order of the last term. We see that

$$\begin{aligned}
&\mathrm{E}||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{E}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||^2\\
\leq&||\mathbf{B}(\mathbf{B}'\mathbf{B})^{-2}\mathbf{B}'||^2\mathrm{E}||\frac{1}{n-1}(\mathbf{E}\mathbf{E}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')||^2\\
\leq&O(K)\mathrm{Etr}[(\epsilon_i-\bar{\epsilon})(\epsilon_i-\bar{\epsilon})'(\epsilon_i-\bar{\epsilon})(\epsilon_i-\bar{\epsilon})']\\
=&O(pK). \qquad\qquad (A.17)
\end{aligned}$$

The last equation is based on d is bounded from assumption (B). The above calculation leads to $||(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'[\frac{1}{n-1}(\mathbf{E}\mathbf{E}'-\frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}||^2=O_p(pK)$.

Combining (A.4), (A.6) and (A.17), we obtain part (1) of Theorem 2.2.3,

$$||\widehat{\mathrm{cov}}(\mathbf{f})_o-\mathrm{cov}(\mathbf{f})||=O_p(n^{-1/2}p^{1/4}K^{3/4})+O_p(p^{1/2}K^{1/2}).$$

(2) As for the Barra two-step estimator $\widehat{\mathrm{cov}}(\mathbf{f})_w$, the upper bound is the same as Barra one-step estimator. Because similarly to what we have shown in the proof of part (2) of Theorem 2.2.1, we see that the difference between Barra two-step estimator and the true value can be bounded by the difference between Barra one-step estimator and the true value. It implies that the upper bound of $||\widehat{\mathrm{cov}}(\mathbf{f})_w-\mathrm{cov}(\mathbf{f})||$ is also bounded by the upper bound of $||\widehat{\mathrm{cov}}(\mathbf{f})_o-\mathrm{cov}(\mathbf{f})||$. So the second part of Theorem 2.2.3 is followed

$$||\widehat{\mathrm{cov}}(\mathbf{f})_w-\mathrm{cov}(\mathbf{f})||=O_p(n^{-1/2}p^{1/4}K^{3/4})+O_p(p^{1/2}K^{1/2}).$$

□

A.5 Proof of Theorem 2.2.5

Proof. (1) First, we show the upper bound for the Barra one-step estimator $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ under Frobenius norm.

Similar to Theorem 2.2.2, we decompose $\widehat{\mathbf{\Sigma}}_{\epsilon,o}-\mathbf{\Sigma}_\epsilon$ into three parts,

$$\begin{aligned}
&||\widehat{\mathbf{\Sigma}}_{\epsilon,o}-\mathbf{\Sigma}_\epsilon||^2\\
=&||\frac{1}{n}\mathrm{diag}(\mathbf{E}\mathbf{E}')-\mathbf{\Sigma}_\epsilon+\frac{1}{n}\mathrm{diag}(\mathbf{H}\mathbf{E}\mathbf{E}'\mathbf{H})-[\frac{1}{n}\mathrm{diag}(\mathbf{E}\mathbf{E}'\mathbf{H}+\mathbf{H}\mathbf{E}\mathbf{E}')]||^2\\
\leq&3(||\frac{1}{n}\mathrm{diag}(\mathbf{E}\mathbf{E}')-\mathbf{\Sigma}_\epsilon||^2+||\frac{1}{n}\mathrm{diag}(\mathbf{H}\mathbf{E}\mathbf{E}'\mathbf{H})||^2+||\frac{1}{n}\mathrm{diag}(\mathbf{E}\mathbf{E}'\mathbf{H}+\mathbf{H}\mathbf{E}\mathbf{E}')||^2).
\end{aligned}$$
(A.18)

(A.14) reveals the order of the first term, now we analyze the second and third term. For the second term, we can relieve the diagonal condition so that

$$||\frac{1}{n}\text{diag}(\mathbf{HEE'H})||^2 \le ||\frac{1}{n}\mathbf{HEE'H}||^2 \le ||\mathbf{H}||^2||\frac{1}{n}\mathbf{EE'}||^2 = O_p(pK). \tag{A.19}$$

The last equation is because $E||\frac{1}{n}\mathbf{EE'}||^2 = E[\text{tr}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}'\boldsymbol{\epsilon}\boldsymbol{\epsilon}')] = O(p)$ under assumption d is bounded.

For the third term, similarly to the second one, we have

$$||\frac{1}{n}\text{diag}(\mathbf{EE'H} + \mathbf{HEE'})||^2 \le 2||\frac{1}{n}\text{diag}(\mathbf{EE'H})||^2 \le 2||\mathbf{H}||^2||\frac{1}{n}\mathbf{EE'}||^2 = O_p(pK). \tag{A.20}$$

So applying above results into (A.18), we get the first part of Theorem 2.2.4, that is,

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon|| = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}).$$

(2) Turn to the Barra two-step estimator $\widehat{\mathbf{\Sigma}}_{\epsilon,w}$. Similar to the proof of part (2) in Theorem 2.2.2, we can bound $||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon||$ by $||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||$, so that the upper bound for $||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon||$ will inherit the upper bound of $||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||$. So it follows the second part of Theorem 2.2.4,

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon|| = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}).$$

□

A.6 Proof of Theorem 2.2.6

Proof. (1) First, we prove the asymptotic result for the Barra one-step estimator $\widehat{\mathbf{\Sigma}}_o$ under Frobenius norm. We see that

$$\begin{aligned}\widehat{\mathbf{\Sigma}}_o - \mathbf{\Sigma} &= \mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})]\mathbf{B}' + \widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon \qquad\qquad (\text{A.21})\\ &= \mathbf{B}[\frac{1}{n-1}(\widehat{\mathbf{F}}_o\widehat{\mathbf{F}}_o' - \frac{1}{n}\widehat{\mathbf{F}}_o\mathbf{1}\mathbf{1}'\widehat{\mathbf{F}}_o') - \text{cov}(\mathbf{f})]\mathbf{B}' + \text{diag}(\frac{1}{n}\widehat{\mathbf{E}}_o\widehat{\mathbf{E}}_o') - \mathbf{\Sigma}_\epsilon.\end{aligned}$$

On the other hand, we can decompose $\widehat{\mathbf{F}}_o$ and $\widehat{\mathbf{E}}_o$ with respect to $\mathbf{F}$ and $\mathbf{E}$, that is,

$$\begin{aligned}\widehat{\mathbf{F}}_o &= \mathbf{F} + (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{E},\\ \widehat{\mathbf{E}}_o &= [\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}']\mathbf{E} \equiv (\mathbf{I} - \mathbf{H})\mathbf{E},\end{aligned}$$

where we define projection matrix $\mathbf{H} = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$. Plugging them in (A.21) and arrange the order, we get

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_o - \boldsymbol{\Sigma} = & \mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})]\mathbf{B}' \\
& + \mathbf{H}[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' \\
& + \mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{E}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{H} \\
& + \mathbf{H}[\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{H} \\
& + \mathrm{diag}((\mathbf{I} - \mathbf{H})\frac{1}{n}\mathbf{E}\mathbf{E}'(\mathbf{I} - \mathbf{H})) - \boldsymbol{\Sigma}_\epsilon. \qquad (A.22)
\end{aligned}$$

This shows that the difference is a four-term perturbation of the population covariance matrix, and this presentation is our key technical tool. By the Cauchy-Schwarz inequality, it follows from (A.22) that

$$\begin{aligned}
||\widehat{\boldsymbol{\Sigma}}_o - \boldsymbol{\Sigma}||^2 \leq & 4\{||\mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})]\mathbf{B}'||^2 \\
& + ||\mathbf{H}[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' + \mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{E}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{H}||^2 \\
& + ||\mathbf{H}[\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{H}||^2 \\
& + ||\mathrm{diag}((\mathbf{I} - \mathbf{H})\frac{1}{n}\mathbf{E}\mathbf{E}'(\mathbf{I} - \mathbf{H})) - \boldsymbol{\Sigma}_\epsilon||^2\} \qquad (A.23) \\
\equiv & 4(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4). \qquad (A.24)
\end{aligned}$$

First, we consider the term $\mathcal{L}_1$. Combining (A.3) and (A.4), we can get

$$\begin{aligned}
\mathcal{L}_1 &= \mathrm{tr}\{\mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})]\mathbf{B}'\}^2 \\
&\leq ||\mathbf{B}'\mathbf{B}||^2(||\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \mathrm{cov}(\mathbf{f})||^2) \\
&= O_p(n^{-1}p^2K^2). \qquad (A.25)
\end{aligned}$$

Then let us consider the second term $\mathcal{L}_2$. It shows that

$$\begin{aligned}
\mathcal{L}_2 &= ||\frac{1}{n-1}[(\mathbf{H}\mathbf{E}\mathbf{F}'\mathbf{B}' + \mathbf{B}\mathbf{F}\mathbf{E}'\mathbf{H}) - \frac{1}{n}(\mathbf{H}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'\mathbf{B}' + \mathbf{B}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}'\mathbf{H})]||^2 \\
&\leq \frac{8}{(n-1)^2}(||\mathbf{H}\mathbf{E}\mathbf{F}'\mathbf{B}'||^2 + \frac{1}{n^2}||\mathbf{H}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'\mathbf{B}'||^2). \qquad (A.26)
\end{aligned}$$

Similarly to (A.4) and (A.5), the two terms in (A.26) conditioning on $\mathbf{F}$ results in

$$\begin{aligned}
\mathrm{E}||\mathbf{HEF'B'}||^2 &\leq ||\mathbf{H}||^2 E||\mathbf{EF'B'}||^2 = K \cdot \mathrm{E}[\mathrm{tr}(\mathbf{EF'B'BFE'})] \\
&= K \cdot \mathrm{E}[\mathrm{tr}(\mathbf{F'B'BF}E(\mathbf{E'E}|\mathbf{F}))] = K \cdot \mathrm{E}[\mathrm{tr}(\mathbf{F'B'BF})] \cdot \mathrm{tr}(\boldsymbol{\Sigma}_\epsilon) \\
&= nK\mathrm{tr}(\boldsymbol{\Sigma}_\epsilon) \cdot \mathrm{tr}[\mathbf{B'B}\mathrm{E}(\mathbf{ff'})] \leq nK\mathrm{tr}(\boldsymbol{\Sigma}_\epsilon)||\mathbf{B'B}||||\mathrm{E}(\mathbf{ff'})|| \\
&= O(np^2K^2),
\end{aligned} \tag{A.27}$$

and similarly,

$$\begin{aligned}
\mathrm{E}||\mathbf{HE11'F'B'}||^2 &\leq ||\mathbf{H}||^2 \mathrm{E}||\mathbf{E11'F'B'}||^2 = K \cdot \mathrm{E}[\mathrm{tr}(\mathbf{E11'F'B'BF11'E'})] \\
&= K \cdot \mathrm{E}[\mathbf{F'B'BF11'}\mathrm{E}(\mathbf{E'E}|\mathbf{F})\mathbf{11'})] = K \cdot \mathrm{tr}(\boldsymbol{\Sigma}_\epsilon)\mathrm{E}[\mathbf{F'B'BF11'11'})] \\
&\leq nK\mathrm{tr}(\boldsymbol{\Sigma}_\epsilon)||\mathbf{B'B}||||\mathrm{E}(\mathbf{ff'})||||\mathbf{11'11'}|| = n^3K^2\mathrm{tr}(\boldsymbol{\Sigma}_\epsilon)||\mathbf{B'B}|| \\
&= O(n^3p^2K^2).
\end{aligned} \tag{A.28}$$

(A.27) and (A.28) conclude the result for (A.26), that is

$$\mathcal{L}_2 = O_p(n^{-1}p^2K^2). \tag{A.29}$$

As for the term $\mathcal{L}_3$, similarly to (A.17), we have

$$\begin{aligned}
\mathcal{L}_3 &\leq ||\mathbf{H}||^2||\frac{1}{n-1}(\mathbf{EE'} - \frac{1}{n}\mathbf{E11'E'})||^2 \\
&= K \cdot ||\frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})'||^2 \\
&\leq K \cdot \mathrm{tr}(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})'(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})(\boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}})' = O_p(pK).
\end{aligned} \tag{A.30}$$

The finial result $O(pK)$ does not contain n, that is because the last inequality cancelled out the effects of $1/n$, which can not be released. So $\mathcal{L}_3$ becomes a key term in the proof.

Last, from the proof of Theorem 2.2.4, we see

$$\mathcal{L}_4 = O_p(n^{-1}p) + O_p(pK). \tag{A.31}$$

Therefore, combining the above results for $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ together gives $\widehat{\boldsymbol{\Sigma}}_o$,

$$||\widehat{\boldsymbol{\Sigma}}_o - \boldsymbol{\Sigma}||^2 = O_p(n^{-1}p^2K^2) + O_p(pK),$$

which is the result for the Barra one-step estimator.

(2) Second, we prove the asymptotic result for the Barra two-step estimator $\widehat{\mathbf{\Sigma}}_w$. Notice that $\widehat{\mathbf{\Sigma}}_w$ is quite similar to $\widehat{\mathbf{\Sigma}}_o$, so we can also decompose $\widehat{\mathbf{\Sigma}}_w - \mathbf{\Sigma}$ into four parts. Replacing $\widehat{\mathbf{F}}_o, \widehat{\mathbf{E}}_o$ by

$$\widehat{\mathbf{F}}_w = \mathbf{F} + (\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{E},$$
$$\widehat{\mathbf{E}}_w = [\mathbf{I} - \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}]\mathbf{E},$$

and with some simple calculation as (A.21) and (A.22), we can get the decomposition for $\widehat{\mathbf{\Sigma}}_w$, denoting

$$||\widehat{\mathbf{\Sigma}}_w - \mathbf{\Sigma}||^2 \leq 4(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4), \tag{A.32}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= ||\mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}') - \text{cov}(\mathbf{f})]\mathbf{B}'||^2,\\
\mathcal{A}_2 &= ||\mathbf{S}[\frac{1}{n-1}(\mathbf{E}\mathbf{F}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' + \mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{E}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{S}'||^2,\\
\mathcal{A}_3 &= ||\mathbf{S}[\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}')]\mathbf{S}'||^2,\\
\mathcal{A}_4 &= ||\text{diag}((\mathbf{I}-\mathbf{S})\frac{1}{n}\mathbf{E}\mathbf{E}'(\mathbf{I}-\mathbf{S}')) - \mathbf{\Sigma}_\epsilon||^2,
\end{aligned} \tag{A.33}$$

with $\mathbf{S} = \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}$. So we see that the difference between $\widehat{\Sigma}_w$ and $\widehat{\Sigma}_o$ is just replacing $\mathbf{H}$ with $\mathbf{S}$. What we intend to do next is to connect the two terms to get the properties for $\widehat{\Sigma}_w$.

Before going further, let us consider $\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'$ first. Similar to part (2) of Theorem 2.2.1,

$$\begin{aligned}
&\hat{\sigma}_{(1),o}^2\mathbf{I} \leq \widehat{\mathbf{\Sigma}}_{\epsilon,o} \leq \hat{\sigma}_{(p),o}^2\mathbf{I},\\
&\hat{\sigma}_{(p),o}^{-2}\mathbf{I} \leq \widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} \leq \hat{\sigma}_{(1),o}^{-2}\mathbf{I},\\
&\hat{\sigma}_{(p),o}^{-2}\mathbf{B}'\mathbf{B} \leq \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B} \leq \hat{\sigma}_{(1),o}^{-2}\mathbf{B}'\mathbf{B},\\
&(\hat{\sigma}_{(1),o}^{-2}\mathbf{B}'\mathbf{B})^{-1} \leq (\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1} \leq (\hat{\sigma}_{(p),o}^{-2}\mathbf{B}'\mathbf{B})^{-1},\\
&\hat{\sigma}_{(1),o}^2\mathbf{H} \leq \mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}' \leq \hat{\sigma}_{(p),o}^2\mathbf{H}.
\end{aligned}$$

Since $||\mathbf{S}||^2 = \text{tr}[\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-2}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}']$, we can bound it with

$$\text{tr}[(\hat{\sigma}_{(1),o}^2)^2\mathbf{H}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-2}\mathbf{H}] \leq \text{tr}[\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-2}\mathbf{B}(\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B})^{-1}\mathbf{B}'] \leq \text{tr}[(\hat{\sigma}_{(p),o}^2)^2\mathbf{H}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-2}\mathbf{H}],$$

then we have

$$\mathrm{tr}[(\frac{\hat{\sigma}^2_{(1),o}}{\hat{\sigma}^2_{(p),o}})^2\mathbf{H}] \leq ||\mathbf{S}||^2 \leq \mathrm{tr}[(\frac{\hat{\sigma}^2_{(p),o}}{\hat{\sigma}^2_{(1),o}})^2\mathbf{H}].$$

By assumption (C) and Lemma, we know that $||\mathbf{S}||^2 = O(1)\mathrm{tr}(\mathbf{H}) = O(K)$.

Moreover, we will need $||\mathbf{S}'\mathbf{S}||^2$ later. The bound can be written as

$$\mathrm{tr}[(\frac{\hat{\sigma}^2_{(1),o}}{\hat{\sigma}^2_{(p),o}})^4\mathbf{H}] \leq ||\mathbf{S}'\mathbf{S}||^2 \leq \mathrm{tr}[(\frac{\hat{\sigma}^2_{(p),o}}{\hat{\sigma}^2_{(1),o}})^4\mathbf{H}],$$

and similar result can be gotten that $||\mathbf{S}'\mathbf{S}||^2 = O(1)\mathrm{tr}(\mathbf{H}) = O(K)$.

Now let us go back to (A.32). Term $\mathcal{A}_1$ has been discussed in the proof of part (1). Then as for term $\mathcal{A}_2$, it can be similarly decomposed as

$$\mathcal{A}_2 \leq \frac{8}{(n-1)^2}(||\mathbf{SEF}'\mathbf{B}'||^2 + \frac{1}{n^2}||\mathbf{SE11}'\mathbf{F}'\mathbf{B}'||^2). \tag{A.34}$$

Similarly to (A.27) and (A.28), it follows that

$$||\mathbf{SEF}'\mathbf{B}'||^2 \leq ||\mathbf{S}||^2||\mathbf{EF}'\mathbf{B}'||^2 = O(K)||\mathbf{EF}'\mathbf{B}'||^2 = O_p(np^2K^2),$$
$$||\mathbf{SE11}'\mathbf{F}'\mathbf{B}'||^2 \leq ||\mathbf{S}||^2||\mathbf{E11}'\mathbf{F}'\mathbf{B}'||^2 = O(K)||\mathbf{E11}'\mathbf{F}'\mathbf{B}'||^2 = O_p(n^3p^2K^2),$$

which leads to the same result as $\mathcal{L}_2$, that is $\mathcal{A}_2 = O_p(n^{-1}p^2K^2)$.

Next, it turns to analyze term $\mathcal{A}_3$. Based on the calculation for $\mathcal{L}_3$ and the result for $||\mathbf{S}'\mathbf{S}||^2$, we see that

$$\mathcal{A}_3 \leq ||\mathbf{S}'\mathbf{S}||^2||\frac{1}{n-1}(\mathbf{EE}' - \frac{1}{n}\mathbf{E11}'\mathbf{E}')||^2 = O_p(pK). \tag{A.35}$$

At last, the same argument as $\mathcal{L}_4$ can be done for $\mathcal{A}_4$. We can show that

$$\mathcal{A}_4 = O_p(n^{-1}p) + O_p(pK). \tag{A.36}$$

Therefore, it follows from (A.32), (A.33), (A.25), and (A.34)-(A.36) that Barra two-step estimator has the same result as Barra one-step estimator,

$$||\widehat{\mathbf{\Sigma}}_w - \mathbf{\Sigma}||^2 = O_p(n^{-1}p^2K^2) + O_p(pK).$$

(3) Finally, we discuss the asymptotic properties for $\widehat{\mathbf{\Sigma}}_{sam}$. We see that

$$\begin{aligned}
\widehat{\mathbf{\Sigma}}_{sam} &= \frac{1}{n-1}(\mathbf{R}\mathbf{R}' - \frac{1}{n}\mathbf{R}\mathbf{1}\mathbf{1}'\mathbf{R}') \\
&= \frac{1}{n-1}[(\mathbf{B}\mathbf{F}+\mathbf{E})(\mathbf{B}\mathbf{F}+\mathbf{E})' - \frac{1}{n}(\mathbf{B}\mathbf{F}+\mathbf{E})\mathbf{1}\mathbf{1}'(\mathbf{B}\mathbf{F}+\mathbf{E})'] \\
&= \mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' + \frac{1}{n-1}[(\mathbf{B}\mathbf{F}\mathbf{E}' + \mathbf{E}\mathbf{F}'\mathbf{B}') \\
&\quad - \frac{1}{n}(\mathbf{B}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}' + \mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{F}'\mathbf{B}')] + \frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}'). \qquad \text{(A.37)}
\end{aligned}$$

This shows that $\widehat{\mathbf{\Sigma}}_{sam}$ is also a four-term perturbation of the population covariance matrix. By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
||\widehat{\mathbf{\Sigma}}_{sam} - \mathbf{\Sigma}||^2 \leq 4[||\mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' - \mathrm{cov}(\mathbf{f}))||^2 \\
+ \frac{2}{(n-1)^2}||\mathbf{B}\mathbf{F}\mathbf{E}'||^2 + \frac{2}{n^2(n-1)^2}||\mathbf{B}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}'||^2 \\
+ ||\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}') - \mathbf{\Sigma}_\epsilon||^2]. \qquad \text{(A.38)}
\end{aligned}$$

We have shown that

$$\begin{aligned}
&||\mathbf{B}[\frac{1}{n-1}(\mathbf{F}\mathbf{F}' - \frac{1}{n}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{F}')]\mathbf{B}' - \mathrm{cov}(\mathbf{f}))||^2 = O_p(n^{-1}p^2K^2), \\
&||\mathbf{B}\mathbf{F}\mathbf{E}'||^2 = O_p(np^2K), \\
&||\mathbf{B}\mathbf{F}\mathbf{1}\mathbf{1}'\mathbf{E}'||^2 = O_p(n^3p^2K), \\
&||\frac{1}{n-1}(\mathbf{E}\mathbf{E}' - \frac{1}{n}\mathbf{E}\mathbf{1}\mathbf{1}'\mathbf{E}') - \mathbf{\Sigma}_\epsilon||^2 = O_p(n^{-1}p^2K^2). \qquad \text{(A.39)}
\end{aligned}$$

So applying the above results to (A.38) yields

$$||\widehat{\mathbf{\Sigma}}_{sam} - \mathbf{\Sigma}||^2 = O_p(n^{-1}p^2K^2).$$

□

A.7 Proof of Theorem 2.2.7

Proof. (1) First, we show the asymptotic result for the Barra one-step estimator $\widehat{\mathbf{\Sigma}}_o^{-1}$ under Frobenius norm. From the structure of Barra approach, we know that $\widehat{\mathbf{\Sigma}}_o = \mathbf{B}\widehat{\mathrm{cov}}(\mathbf{f})_o\mathbf{B}' +$

$\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$, which along with the Sherman-Morrison-Woodbury formula shows that

$$\widehat{\boldsymbol{\Sigma}}_o^{-1} = \widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}. \tag{A.40}$$

Inverse of the true covariance matrix $\boldsymbol{\Sigma}^{-1}$ has this format as well, that is

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_{\epsilon}^{-1} - \boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}[\text{cov}(\mathbf{f})^{-1} + \mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}. \tag{A.41}$$

Taking the difference and decomposing the term, we get

$$\begin{aligned}
||\widehat{\boldsymbol{\Sigma}}_o^{-1} - \boldsymbol{\Sigma}^{-1}||^2 \leq{}& ||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1} - \boldsymbol{\Sigma}_{\epsilon}^{-1}||^2 + ||(\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1} - \boldsymbol{\Sigma}_{\epsilon}^{-1})\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}||^2 \\
&+ ||\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'(\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1} - \boldsymbol{\Sigma}_{\epsilon}^{-1})||^2 \\
&+ ||\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}\{[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1} - [\text{cov}(\mathbf{f})^{-1} + \mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}^{-1}||^2 \\
\equiv{}& \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4.
\end{aligned} \tag{A.42}$$

To study $||\widehat{\boldsymbol{\Sigma}}_o^{-1} - \boldsymbol{\Sigma}^{-1}||^2$, we need to examine each of the above four terms $\mathcal{K}_1, \ldots, \mathcal{K}_4$ separately. First, note that from the proof of Theorem 2.2.1 and Theorem 2.2.2, we know that $\widehat{\text{cov}}(\mathbf{f})_o$ and $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ are not consistent to $\text{cov}(\mathbf{f})$ and $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$. So it is hard to say $\widehat{\text{cov}}(\mathbf{f})_o^{-1}$ and $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}$ are consistent. Then, what we can do is to separate the consistent part and the inconsistent part. Let

$$\widetilde{\text{cov}}(\mathbf{f})_o = \widehat{\text{cov}}(\mathbf{f})_o - (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon}\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1} \equiv \widehat{\text{cov}}(\mathbf{f})_o - \mathbf{C}\boldsymbol{\Sigma}_{\epsilon}\mathbf{C}' \tag{A.43}$$

$$\widetilde{\boldsymbol{\Sigma}}_{\epsilon} = \widehat{\boldsymbol{\Sigma}}_{\epsilon} - \text{diag}(\mathbf{H}\boldsymbol{\Sigma}_{\epsilon}\mathbf{H}) - \text{diag}(\mathbf{H}\boldsymbol{\Sigma}_{\epsilon} + \boldsymbol{\Sigma}_{\epsilon}\mathbf{H}) \equiv \widehat{\boldsymbol{\Sigma}}_{\epsilon} - \mathbf{D} \tag{A.44}$$

Based on a basic fact in matrix theory, we have

$$||\widetilde{\text{cov}}(\mathbf{f})_o^{-1} - \text{cov}(\mathbf{f})_o^{-1}|| \leq \frac{||\text{cov}(\mathbf{f})_o^{-1}||^2||\widetilde{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})_o||}{1 - ||\text{cov}(\mathbf{f})_o^{-1}||||\widetilde{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})_o||},$$

whenever $||\text{cov}(\mathbf{f})_o^{-1}||||\widetilde{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})_o|| < 1$. From Theorem 2.2.4, we know that

$$||\widetilde{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})_o|| = O_p(n^{-1/2}p^{1/4}K^{3/4}).$$

And by Assumption (D), we have $||\text{cov}(\mathbf{f})_o^{-1}|| = O(K^{1/2})$. Thus,

$$||\widetilde{\text{cov}}(\mathbf{f})_o^{-1} - \text{cov}(\mathbf{f})_o^{-1}|| = O_p(n^{-1/2}p^{1/4}K^{7/4}). \tag{A.45}$$

On the other hand, since $\widetilde{\mathbf{\Sigma}}_\epsilon$ and $\mathbf{\Sigma}_\epsilon$ are diagonal, and each entry of them are bounded with probability one, we get the following by Theorem 2.2.5,

$$||\widetilde{\mathbf{\Sigma}}_\epsilon^{-1} - \mathbf{\Sigma}_\epsilon^{-1}|| \leq c^{(0)}||\widetilde{\mathbf{\Sigma}}_\epsilon - \mathbf{\Sigma}_\epsilon|| = O_p(n^{-1/2}p^{1/2}), \tag{A.46}$$

where $c^{(0)}$ is a constant.

Now we will analyze the four decomposition terms one by one. For the first term,

$$\begin{aligned} \mathcal{K}_1 = &||[\widetilde{\mathbf{\Sigma}}_\epsilon + \mathbf{D}]^{-1} - \mathbf{\Sigma}^{-1}||^2 \leq c^{(1)}||\widetilde{\mathbf{\Sigma}}_\epsilon + \mathbf{D} - \mathbf{\Sigma}||^2 \\ \leq& 2c^{(1)}(||\widetilde{\mathbf{\Sigma}}_\epsilon - \mathbf{\Sigma}||^2 + ||\mathbf{D}||^2) = O_p(n^{-1}p) + O_p(p), \end{aligned} \tag{A.47}$$

where $c^{(1)}$ is a constant.

Next, we consider the second term $\mathcal{K}_2$. By the properties for Frobenius norm, we have

$$\mathcal{K}_2 \leq ||(\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} - \mathbf{\Sigma}_\epsilon^{-1})\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2}||^2||\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}||^2||\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}||^2, \tag{A.48}$$

so we need to explore the above three terms. Since both $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ and $\mathbf{\Sigma}_\epsilon$ are diagonal, and the expectation of diagonal entries are bounded, we can show that with a constant $c^{(2)}$,

$$||(\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} - \mathbf{\Sigma}_\epsilon^{-1})\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2}||^2 \leq c^{(2)}||\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1} - \mathbf{\Sigma}_\epsilon^{-1}||^2 = O_p(n^{-1}p) + O_p(p), \tag{A.49}$$

and

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}||^2 = O_p(p). \tag{A.50}$$

Moreover, we notice that $\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}$ is symmetric positive semidefinite with rank at most K and $\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2} \geq 0$. Thus it follows from (A.40) that

$$\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2} = \mathbf{I}_p - \widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{1/2} \leq \mathbf{I}_p,$$

which implies that $\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1} + \mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}$ has at most K positive eigenvalues and all of them are bounded by one. So $E||\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1/2}||^2 = O(K)$, which along with (A.49) and (A.50) gives

$$\mathcal{K}_2 = O_p(n^{-1}p^2K) + O_p(p^2K). \tag{A.51}$$

Similarly, we can show that

$$\mathcal{K}_3 \leq ||\boldsymbol{\Sigma}_\epsilon^{-1}\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{1/2}||^2||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1/2}\mathbf{B}[\widehat{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1/2}||^2||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{1/2}(\widehat{\Sigma}_\epsilon^{-1}-\boldsymbol{\Sigma}_\epsilon^{-1})||^2$$
$$= O_p(p)\cdot O_p(K)\cdot[O_P(n^{-1}p)+O_p(p)] = O_p(n^{-1}p^2K)+O_p(p^2K). \quad \text{(A.52)}$$

Finally, we consider term $\mathcal{K}_4$.

$$||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}\{[\widehat{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}\mathbf{B}]^{-1}-[\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$
$$= ||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}\{[(\widetilde{\text{cov}}(\mathbf{f})_o+\mathbf{C}\boldsymbol{\Sigma}_\epsilon\mathbf{C}')^{-1}+\mathbf{B}'(\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}+\mathbf{D})^{-1}\mathbf{B}]^{-1}-[\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$
$$= ||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}\{[\widetilde{\text{cov}}(\mathbf{f})_o^{-1}-\widetilde{\text{cov}}(\mathbf{f})_o^{-1}\mathbf{C}(\boldsymbol{\Sigma}_\epsilon^{-1}+\mathbf{C}'\widetilde{\text{cov}}(\mathbf{f})_o^{-1}\mathbf{C})^{-1}\mathbf{C}'\widetilde{\text{cov}}(\mathbf{f})_o^{-1}$$
$$+\mathbf{B}'[\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}-\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}(\mathbf{D}^{-1}+\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1})\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}]^{-1}\mathbf{B}]^{-1}-[\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$
$$= ||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}\{[\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B}-\mathbf{X}]^{-1}-[\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$

where $X=\widetilde{\text{cov}}(\mathbf{f})_o^{-1}\mathbf{C}(\boldsymbol{\Sigma}_\epsilon^{-1}+\mathbf{C}'\widetilde{\text{cov}}(\mathbf{f})_o^{-1}\mathbf{C})^{-1}\mathbf{C}'\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}(\mathbf{D}^{-1}+\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1})^{-1}\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B}$

$$= ||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}\{(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}-(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}[(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}$$
$$-\mathbf{X}^{-1}]^{-1}(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}-[\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}]^{-1}\}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$
$$\leq 2||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}[(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}-(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}]\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2+2||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}$$
$$+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}[(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}-\mathbf{X}^{-1}]^{-1}(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2$$
$$\equiv 2\mathcal{K}_{4,1}+2\mathcal{K}_{4,2} \quad \text{(A.53)}$$

So we split term $\mathcal{K}_4$ into 2 parts, the convergent part and the non-convergent part. For the convergent part, we can use the basic fact in matrix theory.

$$
\begin{aligned}
\mathcal{K}_{4,1} \leq & ||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}[-\text{cov}(\mathbf{f})^{-1}(\widetilde{\text{cov}}(\mathbf{f})_o-\text{cov}(\mathbf{f}))\text{cov}(\mathbf{f})^{-1} \\
& -\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}(\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}-\boldsymbol{\Sigma}_\epsilon)\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}](\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2 \\
\leq & 2||\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\text{cov}(\mathbf{f})^{-1}(\widetilde{\text{cov}}(\mathbf{f})_o-\text{cov}(\mathbf{f}))\text{cov}(\mathbf{f})^{-1}\cdot \\
& (\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}||^2 \\
& +2||\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}\mathbf{B}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}(\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}-\boldsymbol{\Sigma}_\epsilon)\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}\cdot \\
& \mathbf{B}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}||^2||\boldsymbol{\Sigma}_\epsilon^{-1}||^2 \\
\leq & ||\widetilde{\text{cov}}(\mathbf{f})_o-\text{cov}(\mathbf{f})||^2||\text{cov}(\mathbf{f})^{-1}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-2}\text{cov}(\mathbf{f})^{-1}||^2||\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-2}\mathbf{B}|| \\
& +||\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}(\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}-\boldsymbol{\Sigma}_\epsilon)\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}||^2||[\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}\mathbf{B}(\text{cov}(\mathbf{f})^{-1}+\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-\frac{1}{2}}]^2||^2||\boldsymbol{\Sigma}_\epsilon^{-1}||^2 \\
= & O_p(n^{-1}p^{1/2}K^{3/2})\cdot O_p(K)\cdot O_p(p^2)+O_p(n^{-1}p)\cdot O_p(K)\cdot O_p(p) \\
= & O_p(n^{-1}p^{5/2}K^{5/2}). \qquad \text{(A.54)}
\end{aligned}
$$

In the above proof, we have used the property of Frobenius norm that if $\mathbf{A} \leq \mathbf{B}$ in the sense of positive definite, then $||\mathbf{A}|| \leq ||\mathbf{B}||$.

On the other hand, for the non-convergent part $\mathcal{K}_{4,2}$, we see that

$$
\begin{aligned}
\mathcal{K}_{4,2} \leq & ||\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-2}\mathbf{B}||^2||(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}[(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}-\mathbf{X}^{-1}]^{-1} \\
& (\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}||^2 \\
\leq & ||\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{-2}\mathbf{B}||^2||(\widetilde{\text{cov}}(\mathbf{f})_o^{-1}+\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon},o}^{-1}\mathbf{B})^{-1}||^2 \\
= & O_p(p^2K^2). \qquad \text{(A.55)}
\end{aligned}
$$

Combining equation (A.54) and equation (A.55), we will get the rate of equation A.53, which is $O_p(n^{-1}p^{5/2}K^{5/2})+O_p(p^2K^2)$.

Therefore, it follows from (A.47), (A.48), (A.52) and (A.53) that

$$
||\widehat{\boldsymbol{\Sigma}}_o^{-1}-\boldsymbol{\Sigma}^{-1}||^2 = O_p(n^{-1}p^{5/2}K^{9/2})+O_p(p^2K^2), \qquad \text{(A.56)}
$$

which is the asymptotic result for the Barra one-step estimator for the inverse of the covariance matrix.

(2) Now we briefly describe the proof for the Barra two-step estimator. From the proof in last part, we notice that the key point is the boundedness of diagonal entries of $\boldsymbol{\Sigma}_\epsilon$ and $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$. So here we need to demonstrate the boundedness of diagonal entries of $\widehat{\boldsymbol{\Sigma}}_{\epsilon,w}$. From

(12), we have $\widehat{\mathbf{\Sigma}}_{\epsilon,w} = \text{diag}((\mathbf{I}-\mathbf{S})\frac{1}{n}\mathbf{E}\mathbf{E}'(\mathbf{I}-\mathbf{S}'))$, and from Lemma 1, we can show that with probability one, every entry of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ is bounded. Then, it is easy to show that every diagonal entry of $(\mathbf{I}-\mathbf{S})\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{S}')$ is bounded. So same strategies can be applied to $\widehat{\mathbf{\Sigma}}_{\epsilon,w}$ as in part (1). Therefore, it follows the asymptotic result for Barra two-step estimator for the inverse of covariance matrix that

$$||\widehat{\mathbf{\Sigma}}_w^{-1} - \mathbf{\Sigma}^{-1}||^2 = O_p(n^{-1}p^{5/2}K^{9/2}) + O_p(p^2K^2). \tag{A.57}$$

(3) At last, we prove the asymptotic result for inverse of the sample covariance matrix $\widehat{\mathbf{\Sigma}}_{sam}^{-1}$ under Frobenius norm. Here we define $\mathbf{D} = \widehat{\mathbf{\Sigma}}_{sam} - \mathbf{\Sigma}$ for simplicity. It is a basic fact in matrix theory that

$$||\widehat{\mathbf{\Sigma}}_{sam}^{-1} - \mathbf{\Sigma}^{-1}|| \leq ||\mathbf{\Sigma}^{-1}||\frac{||\mathbf{\Sigma}^{-1}\mathbf{D}||}{1-||\mathbf{\Sigma}^{-1}\mathbf{D}||} \leq \frac{||\mathbf{\Sigma}^{-1}||^2||\mathbf{D}||}{1-||\mathbf{\Sigma}^{-1}||||\mathbf{D}||}, \tag{A.58}$$

whenever $||\mathbf{\Sigma}^{-1}||||\mathbf{D}|| < 1$. From Theorem 2.2.1, we know that

$$||\mathbf{D}|| = O_p(n^{-1/2}pK).$$

And it is easy to get that $||\mathbf{\Sigma}^{-1}|| = O(p^{1/2})$. Therefore, it follows that

$$||\widehat{\mathbf{\Sigma}}_{sam}^{-1} - \mathbf{\Sigma}^{-1}|| = O_p(n^{-1/2}p^2K).$$

□

A.8 Proof of Theorem 2.3.1

The following technical regularity conditions are assumed throughout the proofs:

Assumption 1 Denote $\mathbf{u}_i = \mathbf{B}'\mathbf{\Sigma}^{-1}\mathbf{r}_i$, $\tilde{\mathbf{u}}_i = \mathbf{\Sigma}^{-1}\mathbf{r}_i$, $\mathbf{W}_{i,ab} = (\mathbf{u}_i)_a(\mathbf{u}_i)_b - (\mathbf{B}'\mathbf{\Sigma}^{-1}\mathbf{B})_{ab}$ and $\widetilde{\mathbf{W}}_{i,ab} = (\tilde{\mathbf{u}}_i)_a(\tilde{\mathbf{u}}_i)_b - (\mathbf{\Sigma}^{-1})_{ab}$. The random variables $\mathbf{W}_{i,ab}$ and $\widetilde{\mathbf{W}}_{i,ab}$ for $i = 1,\ldots,n$ and $a, b = 1,\ldots,K$, satisfy the uniformly sub-Gaussian condition:

$$\max_{i=1,\ldots,n} L^2\{\text{E}\exp(|\mathbf{W}_{i,ab}|^2/L^2 - 1)\} < C^2, \quad \max_{i=1,\ldots,n} L^2\{\text{E}\exp(|\widetilde{\mathbf{W}}_{i,ab}|^2/L^2 - 1)\} < C^2,$$

for any a and b, where L and C are finite constants.

Assumption 2 The eigenvalues of $\mathbf{B}'\mathbf{\Sigma}^{-1}\mathbf{B} \otimes \mathbf{B}'\mathbf{\Sigma}^{-1}\mathbf{B}$ and $\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}$ are positive and bounded with probability one. Furthermore, each element of $\mathbf{\Sigma}^{-1}\mathbf{B}$ is bounded.

Assumption 3 For any $||\boldsymbol{\delta}|| \le O(n^{-1/2}K\sqrt{\log K})$, assume that

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(\boldsymbol{\delta}'\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}\otimes\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{r}_i\mathbf{r}_i'\boldsymbol{\Sigma}^{-1}\mathbf{B}\boldsymbol{\delta}) < \infty.$$

And for any $||\widetilde{\boldsymbol{\delta}}|| \le O(n^{-1/2}p^{1/2})$, assume that

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(\widetilde{\boldsymbol{\delta}}'\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\mathbf{r}_i\mathbf{r}_i'\boldsymbol{\Sigma}^{-1}\widetilde{\boldsymbol{\delta}}) < \infty.$$

Proof. The main idea of the proof follows Li et al. (2013). First, we define $-2\log[\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon|\mathbf{r}_i)]$ as our target function $\mathbf{L}_n$, that is,

$$\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon) = -2\log(\mathbf{L}_n) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i'(\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}\mathbf{B}'+\boldsymbol{\Sigma}_\epsilon)^{-1}\mathbf{r}_i + \log(|\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}\mathbf{B}'+\boldsymbol{\Sigma}_\epsilon|) + p\log(2\pi).$$

We will divide the proof in three parts. In the first part, we prove $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon^*) \ge \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*,\boldsymbol{\Sigma}_\epsilon^*)$ for $||\boldsymbol{\Sigma}_{\boldsymbol{f}}-\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*||_F \ge O_p(n^{-1/2}K\sqrt{\log K})$. In the second part we show that $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon) \ge \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon^*)$ for $||\boldsymbol{\Sigma}_\epsilon-\boldsymbol{\Sigma}_\epsilon^*||_F \ge O_p(n^{-1/2}(p\log p)^{1/2})$. Combining the two parts, we can draw conclusions on the convergence rate of the covariance estimator. In the third part, we show the convergence rate of $\widehat{\boldsymbol{\Sigma}}^{-1}$.

Let us consider $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon^*)$ first. We see

$$\begin{aligned}
&\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon^*) - \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*,\boldsymbol{\Sigma}_\epsilon^*)\\
=&\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i'[\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}^*\mathbf{B}'+\boldsymbol{\Sigma}_\epsilon^*+\mathbf{B}(\boldsymbol{\Sigma}_{\mathbf{f}}-\boldsymbol{\Sigma}_{\mathbf{f}}^*)\mathbf{B}']^{-1}\mathbf{r}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i'(\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}^*\mathbf{B}')^{-1}\mathbf{r}_i\\
&+\log(|\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}^*\mathbf{B}'+\boldsymbol{\Sigma}_\epsilon^*+\mathbf{B}(\boldsymbol{\Sigma}_{\mathbf{f}}-\boldsymbol{\Sigma}_{\mathbf{f}}^*)\mathbf{B}'|) - \log(|\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{f}}^*\mathbf{B}'|).
\end{aligned}$$

Using Taylor expansion of log and determinant functions, we have $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}},\boldsymbol{\Sigma}_\epsilon^*)-\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*,\boldsymbol{\Sigma}_\epsilon^*) = K_1+K_2+o_p(1)$, where

$$\begin{aligned}
K_1 &= \mathrm{tr}\{[\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}-\mathbf{B}'\boldsymbol{\Sigma}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{-1}\mathbf{B}]\Delta_{\mathbf{f}}\},\\
K_2 &= \mathrm{tr}\{\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}\Delta_{\mathbf{f}}[\mathbf{B}'\boldsymbol{\Sigma}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{-1}\mathbf{B}-\frac{1}{2}\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}]\Delta_{\mathbf{f}}\},
\end{aligned}$$

with $\boldsymbol{\Sigma} = \mathbf{B}'\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*\mathbf{B}+\boldsymbol{\Sigma}_\epsilon^*$ and $\Delta_{\mathbf{f}} = \boldsymbol{\Sigma}_{\boldsymbol{f}}-\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^*$.

If the sub-Gaussian condition for $\mathbf{W}_{i,ab}(i=1,\dots,n)$ in Assumption 1 is satisfied, similar to Lemma A.2 and Lemma A.3 in Bickel and Levina (2008a), we have

$$\max_{a,b}|[\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B} - \mathbf{B}'\boldsymbol{\Sigma}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{-1}\mathbf{B}]_{ab}| \leq O_p((\log K/n)^{1/2}).$$

Consequently, we have

$$\begin{aligned}|K_1| &= \max_{a,b}|[\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B} - \mathbf{B}'\boldsymbol{\Sigma}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{-1}\mathbf{B}]_{ab}| \cdot |\text{vec}(\Delta_{\mathbf{f}})|_1 \\ &\leq O_p((\log K/n)^{1/2}) \cdot \sqrt{K^2}||\text{vec}(\Delta_{\mathbf{f}})||_F = O_p((K^2\log K/n)^{1/2})||\text{vec}(\Delta_{\mathbf{f}})||_F.\end{aligned} \tag{A.59}$$

As for K_2, by the property of the Kronecker product, we know that

$$K_2 = \text{vec}(\Delta_{\mathbf{f}})'\{(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}) \otimes [\mathbf{B}'\boldsymbol{\Sigma}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{-1}\mathbf{B} - \frac{1}{2}\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}]\}\text{vec}(\Delta_{\mathbf{f}}).$$

By Assumption 3 and Kolmogorov's three series theorem, we have

$$\begin{aligned}K_2 &= \text{vec}(\Delta_{\mathbf{f}})'(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}) \otimes (\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})\text{vec}(\Delta_{\mathbf{f}}) + o(1) \\ &\geq \lambda_1^2||\text{vec}(\Delta_{\mathbf{f}})||_F^2,\end{aligned} \tag{A.60}$$

where λ_1^2 is the smallest eigenvalue of $\frac{1}{2}(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}) \otimes (\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})$ and by Assumption 2, $\lambda_1^2 > 0$. Note that if $K_1 + K_2 + o_p(1) \geq K_2 - |K_1| + o_p(1) \geq o_p(1)$, we can obtain $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}, \boldsymbol{\Sigma}_\epsilon^*) \geq \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}^*, \boldsymbol{\Sigma}_\epsilon^*)$. The condition can be derived from equation (A.59) and equation (A.60), which implies $||\text{vec}(\Delta_{\mathbf{f}})||_F \geq O_p((K^2\log K/n)^{1/2})$. Thus, we complete the first part of the proof.

Now we start to show the second part of the proof. Similarly to the first part, we take advantage of the taylor expansions and make transformations. We see $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}, \boldsymbol{\Sigma}_\epsilon) - \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}, \boldsymbol{\Sigma}_\epsilon^*) = \widetilde{K_1} + \widetilde{K_2} + o(1)$, where

$$\widetilde{K}_1 = \text{tr}\{[\widetilde{\boldsymbol{\Sigma}}^{-1} - \widetilde{\boldsymbol{\Sigma}}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\widetilde{\boldsymbol{\Sigma}}^{-1}]\Delta_\epsilon\}, \tag{A.61}$$

$$\widetilde{K}_2 = \text{tr}\{\widetilde{\boldsymbol{\Sigma}}^{-1}\Delta_\epsilon[\widetilde{\boldsymbol{\Sigma}}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\widetilde{\boldsymbol{\Sigma}}^{-1} - \frac{1}{2}\widetilde{\boldsymbol{\Sigma}}^{-1}]\Delta_\epsilon\}, \tag{A.62}$$

with $\widetilde{\boldsymbol{\Sigma}} = \mathbf{B}\boldsymbol{\Sigma}_{\boldsymbol{f}}\mathbf{B}' + \boldsymbol{\Sigma}_\epsilon^*$, and $\Delta_\epsilon = \boldsymbol{\Sigma}_\epsilon - \boldsymbol{\Sigma}_\epsilon^*$.

Taking $\widetilde{K}_1$ into consideration, we need to analyze $\widetilde{\boldsymbol{\Sigma}}$ first. We notice that

$$\widetilde{\boldsymbol{\Sigma}}^{-1} = (\boldsymbol{\Sigma}^* + \mathbf{B}\Delta_{\mathbf{f}}\mathbf{B}')^{-1} = \boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1}.$$

Consequently, we have

$$\begin{aligned}
&\widetilde{\boldsymbol{\Sigma}}^{-1} - \widetilde{\boldsymbol{\Sigma}}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\widetilde{\boldsymbol{\Sigma}}^{-1} \\
=&\boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}^{*-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1} \\
&+ \boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{*-1} \\
&+ \boldsymbol{\Sigma}^{*-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1} \\
&- \boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1}.
\end{aligned}$$

Since $(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1} \leq \Delta_{\mathbf{f}}$, we know that $\boldsymbol{\Sigma}^{*-1}\mathbf{B}(\Delta_{\mathbf{f}}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{*-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{*-1} \leq \boldsymbol{\Sigma}^{*-1}\mathbf{B}\Delta_{\mathbf{f}}\mathbf{B}'\boldsymbol{\Sigma}^{*-1}$ with each element is $o(p^{-1})$. From Assumption 1 and similar argument in the first part, we have

$$\max_{a,b}|[\boldsymbol{\Sigma}^{*-1} - \boldsymbol{\Sigma}^{*-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\boldsymbol{\Sigma}^{*-1}]_{ab}| \leq O_p((\log p/n)^{1/2})$$

Thus,

$$\begin{aligned}
|\widetilde{K}_1| &\leq [O_p((\log p/n)^{1/2}) + o(p^{-1})] \cdot |\text{vec}(\Delta_\epsilon)|_1 \leq [O_p((\log p/n)^{1/2}) + o(p^{-1})] \cdot \sqrt{p}||\text{vec}(\Delta_\epsilon)||_F \\
&= O_p(n^{-1/2}(p\log p)^{1/2})||\text{vec}(\Delta_\epsilon)||_F. \qquad \text{(A.63)}
\end{aligned}$$

In terms of $\widetilde{K}_2$, combine the technique for K_2 and in dealing with $\widetilde{\boldsymbol{\Sigma}}^{-1}$, we see

$$\begin{aligned}
\widetilde{K}_2 &= \text{vec}(\Delta_\epsilon)'\{\widetilde{\boldsymbol{\Sigma}}^{-1} \otimes [\widetilde{\boldsymbol{\Sigma}}^{-1}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{r}_i\mathbf{r}_i')\widetilde{\boldsymbol{\Sigma}}^{-1} - \frac{1}{2}\widetilde{\boldsymbol{\Sigma}}^{-1}]\}\text{vec}(\Delta_\epsilon) \\
&= \text{vec}(\Delta_\epsilon)'(\boldsymbol{\Sigma}^{*-1} \otimes \frac{1}{2}\boldsymbol{\Sigma}^{*-1})\text{vec}(\Delta_\epsilon)' + o(1) \\
&\geq \widetilde{\lambda}_1^2||\text{vec}(\Delta_\epsilon)||_F^2, \qquad \text{(A.64)}
\end{aligned}$$

where $\widetilde{\lambda}_1^2$ is the smallest eigenvalue of $\boldsymbol{\Sigma}^{*-1} \otimes \boldsymbol{\Sigma}^{*-1}/2$ and by Assumption 2, $\widetilde{\lambda}_1^2 > 0$.

Similarly to part one, within $||\text{vec}(\Delta_\epsilon)||_F \geq O_p(n^{-1/2}(p\text{log}p)^{1/2})$, $\widetilde{K}_1$ is dominated by $\widetilde{K}_2$, we can achieve $\mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}, \boldsymbol{\Sigma}_\epsilon) \geq \mathbf{L}_n(\boldsymbol{\Sigma}_{\boldsymbol{f}}, \boldsymbol{\Sigma}_\epsilon^*)$. Therefore, we complete the second part of the proof.

Combining the above two parts, we know if we obtain a local maximizer $\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}$ and $\widehat{\boldsymbol{\Sigma}}_\epsilon$ of the likelihood function, they must satisfy the convergence rates. As a result, the MLE $\widehat{\boldsymbol{\Sigma}}$ must satisfy $||\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^*||_F = O_p(n^{-1/2}pK(\text{logK})^{1/2})$.

In the last part, we will discuss the properties of $\widehat{\boldsymbol{\Sigma}}^{-1}$. Similar techniques to the proof of Theorem 2.2.7 can be applied, so we will split $\widehat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{\Sigma}^{*-1}$ into four parts and analyze them one by one.

$$\begin{aligned}
||\widehat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{\Sigma}^{*-1}||^2 \leq\ & ||\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1} - \boldsymbol{\Sigma}_\epsilon^{*-1}||^2 + ||(\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1} - \boldsymbol{\Sigma}_\epsilon^{*-1})\mathbf{B}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1}||^2 \\
& + ||\boldsymbol{\Sigma}_\epsilon^{*-1}\mathbf{B}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'(\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1} - \boldsymbol{\Sigma}_\epsilon^{*-1})||^2 \\
& + ||\boldsymbol{\Sigma}_\epsilon^{*-1}\mathbf{B}[(\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1}\mathbf{B})^{-1} - (\boldsymbol{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}]\mathbf{B}'\boldsymbol{\Sigma}_\epsilon^{*-1}||^2 \\
\equiv\ & \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 \qquad \text{(A.65)}
\end{aligned}$$

Since $\widehat{\boldsymbol{\Sigma}}_\epsilon$ and $\boldsymbol{\Sigma}_\epsilon^*$ are diagonal matrices and all the diagonal elements are bounded, denoting the lower bound as $c^{(1)}$. It is easy to get

$$\mathcal{J}_1 \leq \frac{1}{c^{(1)}}||\widehat{\boldsymbol{\Sigma}}_\epsilon - \boldsymbol{\Sigma}_\epsilon^*||^2 = O_p(n^{-1}p\text{log}p), \qquad \text{(A.66)}$$

Next, for the second term, we have

$$\begin{aligned}
\mathcal{J}_2 &\leq ||(\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1} - \boldsymbol{\Sigma}_\epsilon^{*-1})\widehat{\boldsymbol{\Sigma}}_\epsilon^{1/2}||^2||\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1/2}\mathbf{B}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1} + \mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1/2}||^2||\widehat{\boldsymbol{\Sigma}}_\epsilon^{-1/2}||^2 \\
&= O_p(n^{-1}p\text{log}p) \cdot O_p(K) \cdot O_p(p\text{log}p) = O_p(n^{-1}(p\text{log}p)^2K) \qquad \text{(A.67)}
\end{aligned}$$

Similarly, $\mathcal{J}_3 = O_p(n^{-1}(p\text{log}p)^2K)$. Taylor expansion can be took advantage of in the fourth term.

$$
\begin{aligned}
\mathcal{J}_4 \leq & ||\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{Z}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}||^2 \\
& \text{where } \mathbf{Z} = -\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1}\Delta_{\mathbf{f}}\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} - \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\Delta_\epsilon\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B} \\
\leq & 2||\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1}\Delta_{\mathbf{f}}\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}||^2 \\
& + 2||\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\mathbf{B}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\Delta_\epsilon\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\cdot \\
& (\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}||^2||\mathbf{\Sigma}_\epsilon^{*-1}||^2 \\
\leq & 2||\Delta_{\mathbf{f}}||^2||\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-2}\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1}||^2||\mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-2}\mathbf{B}||^2 \\
& + 2||\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\Delta_\epsilon\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}||^2||[\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}\mathbf{B}(\mathbf{\Sigma}_{\boldsymbol{f}}{}^{*-1} + \mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Sigma}_\epsilon^{*-\frac{1}{2}}]^2||||\mathbf{\Sigma}_\epsilon^{*-1}||^2 \\
= & O_p(K^2\text{log}K/n)\cdot O_p(K)\cdot O_p(p) + O_p(n^{-1}p\text{log}p)\cdot O_p(K)\cdot O_p(p) \\
= & O_p(n^{-1}pK^3\text{log}K) + O_p(n^{-1}p^2\text{log}pK)
\end{aligned} \tag{A.68}
$$

Since p is usually much greater than K, it follows from (A.66), (A.67) and (A.68) that

$$||\widehat{\mathbf{\Sigma}}^{-1} - \mathbf{\Sigma}^{*-1}||^2 = O_p(n^{-1}(p\text{log}p)^2K) + O_p(n^{-1}pK^3\text{log}K). \tag{A.69}$$

□

A.9 Consider cases with changing B_i's across n

If we consider the models with different $\mathbf{B}_i$'s across n, we need to add further assumption to Assumption (F).

There exist constants δ_i's such that $|\lambda_K(\mathbf{B}_i'\mathbf{B}_i)| > \delta_i$ for all n. Besides,

$$\frac{\max_i||\mathbf{t}'(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'||_2^2}{\sum_{i=1}^n||\mathbf{t}'(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'||_2^2} \to 0,$$

for all $\mathbf{t} \in \mathbb{R}^K$, but $\mathbf{t} \neq \mathbf{0}$.

This assumption puts some constrains on $\mathbf{B}_i$'s, which implies that every element in $\mathbf{B}_i$'s should not differ much. Intuitively, if $\mathbf{B}_i$'s differ a lot, the distributions of $\mathbf{r}_i$'s will be quite different, so it makes no sense to pursue the covariance.

Theorem A.9.1 (Rates on the bias of covariance estimators for factors under Frobenius norm). *Under conditions (A)-(F), we have*

$$||\widehat{\text{cov}}(\mathbf{f})_o - \text{cov}(\mathbf{f})||_F = O_p(n^{-1/2}p^{1/4}K^{3/4}) + O_p(p^{1/2}K^{1/2}), \tag{A.70}$$

$$||\widehat{\text{cov}}(\mathbf{f})_w - \text{cov}(\mathbf{f})||_F = O_p(n^{-1/2}p^{1/4}K^{3/4}) + O_p(p^{1/2}K^{1/2}). \tag{A.71}$$

Theorem A.9.2 (Rates on the bias of covariance estimators for errors under Frobenius norm). *Under conditions (A)-(F), we have*

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,o} - \mathbf{\Sigma}_\epsilon||_F = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}), \tag{A.72}$$

$$||\widehat{\mathbf{\Sigma}}_{\epsilon,w} - \mathbf{\Sigma}_\epsilon||_F = O_p(n^{-1/2}p^{1/2}) + O_p(p^{1/2}K^{1/2}). \tag{A.73}$$

Theorem A.9.3 (Rates on the bias of inverse of covariance estimators for returns under Frobenius norm). *Under conditions (A)-(F), we have*

$$||\widehat{\mathbf{\Sigma}}_o^{-1} - \mathbf{\Sigma}^{-1}||_F = O_p(pK), \tag{A.74}$$

$$||\widehat{\mathbf{\Sigma}}_w^{-1} - \mathbf{\Sigma}^{-1}||_F = O_p(pK). \tag{A.75}$$

We see that all of the above theorems have the same results as the case with constant $\mathbf{B}$, so that similar conclusions can be drawn for this setting of Barra model. But because it is hard to measure the exact difference of $\mathbf{B}$ and previous $\mathbf{B}_i$'s, we can only get very loose bound for the rates on the bias of covariance matrix estimators. Here, we do not illustrate theorems for them. In the following analysis, we restrict $\mathbf{B}$ as constant.

Proof of Theorem A.9.1

Proof. Here, we just show the asymptotic result for the Barra one-step estimator $\widehat{\text{cov}}(\mathbf{f})_o$ for different $\mathbf{B}_i$'s across n under Frobenius norm. As for the result for the Barra two-step estimator $\widehat{\text{cov}}(\mathbf{f})_w$, it can be bounded by $\widehat{\text{cov}}(\mathbf{f})_o$ as before, so we omit the proof here. We can rewrite $\widehat{\text{cov}}(\mathbf{f})_o$ as

$$\begin{aligned}
\widehat{\text{cov}}(\mathbf{f})_o = & \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{f}_i-\bar{\mathbf{f}})(\mathbf{f}_i-\bar{\mathbf{f}})' \\
& + \frac{1}{n-1}\sum_{i=1}^{n}[(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i](\mathbf{f}_i-\bar{\mathbf{f}})' \\
& + \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{f}_i-\bar{\mathbf{f}})[(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i]' \\
& + \frac{1}{n-1}[(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i][(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\epsilon}_i]'.
\end{aligned} \tag{A.76}$$

To simplify the notation, we define $\mathbf{W}_i=(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'$. Before going further, we discuss the normality of $\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i)$. By Cramér-Wold device, it is equivalent to consider for any $\mathbf{t}\in\mathbb{R}^K$, the normality of $\mathbf{t}'\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i)$. We know that

$$\text{var}(\mathbf{t}'\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i))=\frac{1}{n}\sum_{i=1}^{n}\mathbf{t}'\mathbf{W}_i\boldsymbol{\Sigma}_\epsilon\mathbf{W}_i'\mathbf{t}.$$

Hence, if it satisfies that $\frac{\max_i||\mathbf{t}'(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'||_2^2}{\sum_{i=1}^{n}\mathbf{t}'(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'\boldsymbol{\Sigma}_\epsilon\mathbf{B}_i(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{t}}\to 0$, for all $\mathbf{t}\in\mathbb{R}^K$, but $\mathbf{t}\neq\mathbf{0}$, then we will obtain

$$\frac{\sqrt{n}\mathbf{t}'\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\mathbf{t}'\mathbf{W}_i\boldsymbol{\Sigma}_\epsilon\mathbf{W}_i'\mathbf{t}}}\Rightarrow N(0,1),$$

which implies the normality of $\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i)$, that is

$$(\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\Sigma}_\epsilon\mathbf{W}_i')^{-1/2}\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i-\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i)\Rightarrow N(0,I_p).$$

This is guaranteed by assumption (F).

We then consider the difference $\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})$. It follows from (A.76) that

$$\begin{aligned}||\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})||^2 \leq 4[||\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{f}_i - \bar{\mathbf{f}})(\mathbf{f}_i - \bar{\mathbf{f}})' - \mathrm{cov}(\mathbf{f})||^2 \\ + ||\frac{1}{n-1}\sum_{i=1}^{n}[\mathbf{W}_i\boldsymbol{\epsilon}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i](\mathbf{f}_i - \bar{\mathbf{f}})'||^2 \\ + ||\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{f}_i - \bar{\mathbf{f}})[\mathbf{W}_i\boldsymbol{\epsilon}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i]'||^2 \\ + ||\frac{1}{n-1}\sum_{i=1}^{n}[\mathbf{W}_i\boldsymbol{\epsilon}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i][\mathbf{W}_i\boldsymbol{\epsilon}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i]'||^2].\end{aligned} \tag{A.77}$$

The decomposition is similar to that in Theorem 2.2.3. Here, we just analyze $||\frac{1}{n-1}\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i\mathbf{f}_i'||^2$ and $||\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i)(\mathbf{W}_i\boldsymbol{\epsilon}_i)'||^2$, since they are the dominated part in second term and fourth term. The rest of the proof is the same as that in Theorem 2.2.3. We denote $\mathbf{S} = (\mathbf{W}_1\boldsymbol{\epsilon}_1, \ldots, \mathbf{W}_n\boldsymbol{\epsilon}_n)$, and as before $\mathbf{F} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)$, then $\sum_{i=1}^{n}\mathbf{W}_i\boldsymbol{\epsilon}_i\mathbf{f}_i' = \mathbf{S}\mathbf{F}'$. So if we take expectation, it follows

$$\begin{aligned}\mathrm{E}||\mathbf{S}\mathbf{F}'||^2 &= \mathrm{E}[\mathrm{tr}(\mathbf{S}\mathbf{F}'\mathbf{F}\mathbf{S}')] = \mathrm{E}[\mathrm{tr}(\mathbf{F}'\mathbf{F}\mathbf{S}'\mathbf{S})] \\ &= \mathrm{E}[\mathrm{tr}(\mathbf{F}'\mathbf{F}E(\mathbf{S}'\mathbf{S}|\mathbf{F}))] = \mathrm{E}[\mathbf{f}_1'\mathbf{f}_1\mathrm{tr}(\mathbf{W}_1\boldsymbol{\Sigma}_\epsilon\mathbf{W}_1') + \cdots + \mathbf{f}_n'\mathbf{f}_n\mathrm{tr}(\mathbf{W}_n\boldsymbol{\Sigma}_\epsilon\mathbf{W}_n')] \\ &= [\mathrm{tr}(\mathbf{W}_1\boldsymbol{\Sigma}_\epsilon\mathbf{W}_1') + \cdots + \mathrm{tr}(\mathbf{W}_n\boldsymbol{\Sigma}_\epsilon\mathbf{W}_n')]\mathrm{E}(\mathbf{f}'\mathbf{f}).\end{aligned}$$

We see that

$$\begin{aligned}\mathrm{tr}(\mathbf{W}_1\boldsymbol{\Sigma}_\epsilon\mathbf{W}_1') &= \mathrm{tr}[(\mathbf{B}_1'\mathbf{B}_1)^{-1}\mathbf{B}_1'\boldsymbol{\Sigma}_\epsilon\mathbf{B}_1(\mathbf{B}_1'\mathbf{B}_1)^{-1}] \leq ||\mathbf{B}_1(\mathbf{B}_1'\mathbf{B}_1)^{-1}(\mathbf{B}_1'\mathbf{B}_1)^{-1}\mathbf{B}_1'||||\boldsymbol{\Sigma}_\epsilon|| \\ &= (\mathrm{tr}[(\mathbf{B}_1'\mathbf{B}_1)^{-2}])^{1/2}[\mathrm{tr}(\boldsymbol{\Sigma}_\epsilon^2)]^{1/2} = O(p^{1/2}K^{1/2}).\end{aligned}$$

And according to assumption (F), it should be valid for $i = 1, \ldots, n$. So $||\frac{1}{n-1}\mathbf{S}\mathbf{F}'||^2 = O_p(n^{-1}p^{1/2}K^{3/2})$, the same order as the second term in Theorem 2.2.3. Furthermore, we see

$$\begin{aligned}
\mathrm{E}||\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{W}_i\boldsymbol{\epsilon}_i)(\mathbf{W}_i\boldsymbol{\epsilon}_i)'||^2 &\leq \mathbf{max}_i\{\mathrm{E}||(\mathbf{W}_i\boldsymbol{\epsilon}_i)(\mathbf{W}_i\boldsymbol{\epsilon}_i)'||^2\} \\
&= \mathbf{max}_i\mathrm{Etr}(\mathbf{W}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{W}_i'\mathbf{W}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{W}_i') \\
&\leq \mathbf{max}_i\mathrm{E}||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{W}_i'\mathbf{W}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||||\mathbf{W}_i'\mathbf{W}_i|| \\
&\leq \mathbf{max}_i\mathrm{E}||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||||\mathbf{W}_i'\mathbf{W}_i||^2 \leq \mathbf{max}_i\mathrm{E}||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||^2||\mathbf{W}_i'\mathbf{W}_i||^2 \\
&= \mathbf{max}_i\mathrm{E}||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||^2||\mathbf{B}_i(\mathbf{B}_i'\mathbf{B}_i)^{-2}\mathbf{B}_i'||^2 = O(pK).
\end{aligned}$$

The last equation is because of assumption $|\lambda_K(\mathbf{B}_i'\mathbf{B}_i)| \geq \delta_i$ for $i = 1, \ldots, n$, in assumption (F). Based on this result, we conclude that the order of the fourth term of (A.77) is also the same as that in Theorem 2.2.3. Hence, same result will be followed as

$$||\widehat{\mathrm{cov}}(\mathbf{f})_o - \mathrm{cov}(\mathbf{f})||^2 = O_p(n^{-1}p^{1/2}K^{3/2}) + O_p(pK).$$

□

Proof of Theorem A.9.2

Proof. We omit the proof for the Barra two-step estimator $\widehat{\boldsymbol{\Sigma}}_{\epsilon,w}$ here, because similar to the proof in Theorem 2.2.4, we can bound its result by corresponding $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$. So we only show the proof for asymptotic result of $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ with different $\mathbf{B}_i$'s across n under Frobenius norm. Note that similar to (A.12), the difference between $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ and $\boldsymbol{\Sigma}_\epsilon$ can be written as

$$\begin{aligned}
||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o} - \boldsymbol{\Sigma}_\epsilon||^2 \leq 4[||\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i') - \boldsymbol{\Sigma}_\epsilon||^2 + ||\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i')||^2 \\
+ ||\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{H}_i)||^2 + ||\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{H}_i)||^2], \qquad \text{(A.78)}
\end{aligned}$$

where $\mathbf{H}_i = \mathbf{B}_i(\mathbf{B}_i'\mathbf{B}_i)^{-1}\mathbf{B}_i'$. Now we analyze the second term and the fourth term in (A.78). The main idea is to bound the mean by its largest term. We see that

$$\begin{aligned}
||\mathrm{diag}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i')||^2 \leq ||\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||^2 \leq \mathrm{max}_i||\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||^2 \\
\leq \mathrm{max}_i||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||||\mathbf{H}_i|| = O_p(pK^{1/2}),
\end{aligned}$$

and

$$||\text{diag}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{H}_i)||^2 \leq ||\frac{1}{n}\sum_{i=1}^{n}\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{H}_i||^2 \leq \max_i||\mathbf{H}_i\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'\mathbf{H}_i||^2$$
$$\leq \max_i||\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'||^2||\mathbf{H}_i||^2 = O_p(pK).$$

So the four terms in (A.78) again have the same order as those in Theorem 2.2.4. So we conclude that $||\widehat{\boldsymbol{\Sigma}}_{\epsilon,o} - \boldsymbol{\Sigma}_\epsilon||^2 = O_p(n^{-1}p) + O_p(pK)$. □

Proof of Theorem A.9.3

Proof. This theorem is similar to Theorem 2.2.6. As it is shown in part (2), $\widehat{\boldsymbol{\Sigma}}_{\epsilon,w}$ can be bounded by $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$, so similarly we can get the same rate for the difference between $\widehat{\boldsymbol{\Sigma}}_{\epsilon,w}$ and true $\boldsymbol{\Sigma}_\epsilon$ as the difference between $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ and true $\boldsymbol{\Sigma}_\epsilon$. Here we just discuss the asymptotic result for Barra one-step estimator $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ with different B_i's across n.

Compared with the proof of Theorem 2.2.6, we know that the main difference in this theorem is $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$. As long as the order of each entry of $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ and $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}$ are the same as the ones with constant $\mathbf{B}$, we can use the same proof as Theorem 2.2.6. So our main task here is to show the boundedness of the diagonal entries of $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}^{-1}$ as well as the boundedness of the largest diagonal entry $\lambda_p(\widehat{\boldsymbol{\Sigma}}_{\epsilon,o})$.

Since $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o} = \text{diag}[\frac{1}{n}\sum_{i=1}^{n}(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'(\mathbf{I}-\mathbf{H}_i)]$, it is better to analyze $\widehat{\boldsymbol{\Sigma}}_{\epsilon,o}$ entry by entry. From assumption (D) about the normality of $\boldsymbol{\epsilon}$, we have

$$(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i \sim N(0, (\mathbf{I}-\mathbf{H}_i)\boldsymbol{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)).$$

So for the jth diagonal entry, we can get

$$\frac{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i]_j}{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj}^{1/2}} \sim N(0,1),$$

furthermore,

$$\frac{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'(\mathbf{I}-\mathbf{H}_i)]_{jj}}{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj}} \sim \chi_1^2.$$

When we sum i from 1 to n, we have

$$\sum_{i=1}^{n}\frac{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'(\mathbf{I}-\mathbf{H}_i)]_{jj}}{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj}} \sim \chi_n^2,$$

denoting it as R_j. Since we know that R_j^{-1} follows inverse-chi-square distribution, which implies that $\mathrm{E}(R_j^{-2}) = \frac{1}{(n-2)(n-4)}$, we can bound the first diagonal entry by

$$\mathrm{E}[(\hat{\sigma}_j^2)^{-2}] \leq \frac{n^2 \mathrm{E}(R_j^{-2})}{(\min_i [(\mathbf{I} - \mathbf{H}_i)\mathbf{\Sigma}_\epsilon(\mathbf{I} - \mathbf{H}_i)]_{jj})^2},$$

which is bounded accordingly. The above equation can be applied to j from 1 to n. So all the diagonal entries of $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1})^2$ are bounded.

On the other hand, to show the boundedness of all diagonal entries of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$, we can follow the proof of Lemma. Note that

$$\hat{\sigma}_{j,o}^2 = \frac{1}{n} \sum_{i=1}^{n} [(\mathbf{I} - \mathbf{H}_i)\boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i' (\mathbf{I} - \mathbf{H}_i)]_{jj},$$

and after defining $U_{j,n} = \frac{1}{n} \sum_{i=1}^{n} \frac{[(\mathbf{I}-\mathbf{H}_i)\boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i'(\mathbf{I}-\mathbf{H}_i)]_{jj}}{[(\mathbf{I}-\mathbf{H}_i)\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj}}$,

$$U_{j,n} \sim \frac{1}{n} \chi_n^2.$$

To bound $\hat{\sigma}_{j,o}^2$ by $U_{j,n}$, we see $\min_i [(\mathbf{I}-\mathbf{H}_i)\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj} U_{j,n} \leq \hat{\sigma}_{j,o}^2 \leq \max_i [(\mathbf{I}-\mathbf{H}_i)\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H}_i)]_{jj} U_{j,n}$. We know the boundedness of $U_{j,n}$ from Lemma, as a result, the boundedness of $\hat{\sigma}_{j,o}^2$ can be obtained.

In sum, the boundedness of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ and $\widehat{\mathbf{\Sigma}}_{\epsilon,o}^{-1}$ is the same as that in Theorem 2.2.6. Then, following the same proof process, we get the result

$$||\widehat{\Sigma}_o^{-1} - \Sigma^{-1}||^2 = O_p(p^2 K^2).$$

□

A.10 Lemma

Lemma 1. *When p is in polynomial order of n, all diagonal entries of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$, i.e. $\hat{\sigma}_{1,o}^2, \ldots, \hat{\sigma}_{p,o}^2$ are bounded away from 0 in probability.*

Proof. Since $\widehat{\mathbf{\Sigma}}_{\epsilon,o} = \mathrm{diag}[\frac{1}{n}(\mathbf{I} - \mathbf{H})\mathbf{E}\mathbf{E}'(\mathbf{I} - \mathbf{H})]$, for each diagonal entry, we have

$$\hat{\sigma}_{i,o}^2 = \frac{1}{n} [(\mathbf{I} - \mathbf{H})\mathbf{E}\mathbf{E}'(\mathbf{I} - \mathbf{H})]_{ii}.$$

Based on assumption (A), we have $\frac{[(\mathbf{I}-\mathbf{H})\mathbf{E}\mathbf{E}'(\mathbf{I}-\mathbf{H})]_{ii}}{[(\mathbf{I}-\mathbf{H})\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H})]_{ii}} \sim \chi_n^2$, which implies that

$$\frac{\hat{\sigma}_{i,o}^2}{[(\mathbf{I}-\mathbf{H})\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H})]_{ii}} \sim \frac{1}{n}\chi_n^2.$$

We define $X_{i,n} = \frac{\hat{\sigma}_{i,o}^2}{[(\mathbf{I}-\mathbf{H})\mathbf{\Sigma}_\epsilon(\mathbf{I}-\mathbf{H})]_{ii}}$ for convenience. By law of large numbers, we know that

$$X_{i,n} \xrightarrow{p} 1.$$

Moreover, by Markov's inequality, we have for every ϵ,

$$P(|X_{i,n} - 1| > \epsilon) \leq \frac{1}{\epsilon^k}\mathrm{E}(|X_{i,n} - 1|^k).$$

If we assign $k = 4$, it is not hard to get that $\mathrm{E}(|X_{i,n} - 1|^4) = c/n^2$, where c is a constant. Thus, it follows that

$$P(|X_{i,n} - 1| > \epsilon) \leq \frac{c}{\epsilon^4 n^2}. \tag{A.79}$$

Now we consider the boundedness of all $X_{i,n}$, $i = 1, \ldots, p$. We know that

$$P(|X_{i,n} - 1| \geq \epsilon, \exists i) \leq p \cdot \frac{c}{\epsilon^4 n^2}, \tag{A.80}$$

so the complement entails

$$P(|X_{i,n} - 1| \leq \epsilon, \forall i) \geq 1 - \frac{cp}{\epsilon^4 n^2}. \tag{A.81}$$

Since the boundedness of $X_{i,n}$ implies the boundedness of $\hat{\sigma}_{i,o}^2$ for all $i = 1, \ldots, p$, we conclude that with probability greater than $1 - cp/\epsilon^4 n^2$, all diagonal entries of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ are bounded.

If we choose a different k, similar deduction will lead to a corresponding probability $1 - cp/\epsilon^k n^{k/2}$. So when p is in polynomial order of n, we can choose k so that $cp/\epsilon^k n^{k/2}$ goes to zero, which implies all the entries of $\widehat{\mathbf{\Sigma}}_{\epsilon,o}$ are bounded away from 0 asymptotically.

$\square$

APPENDIX B

Proofs of the Main Results in Chapter 4

B.1 Proof of Theorem 4.2.1

Proof. Recall that

$$(\hat{\mathbf{L}}^{(1)}, \hat{\mathbf{S}}^{(1)}) = \text{argmin}_{L,S} \frac{1}{2}||\mathbf{M} - \mathbf{L} - \mathbf{S}||_{\text{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}||_1, \tag{B.1}$$

where $\mathbf{M} = \mathbf{M}_0^{(1)} + \mathbf{N}_0^{(1)}$, and with high probability Candès et al. (2011),

$$(\mathbf{L}_0^{(1)}, \mathbf{S}_0^{(1)}) = \text{argmin}_{L,S} ||\mathbf{L}||_* + \frac{\tau}{\lambda}||\mathbf{S}||_1, \text{ s.t. } \mathbf{L} + \mathbf{S} = \mathbf{M}. \tag{B.2}$$

Our proof mainly uses the properties of the above two equations and some analysis tools from Zhou et. al. (2010). Firstly, we introduce some notations measuring the differences of the two pairs. Let $\mathbf{H}_L^{(1)} = \hat{\mathbf{L}}^{(1)} - \mathbf{L}_0^{(1)}$, $\mathbf{H}_S^{(1)} = \hat{\mathbf{S}}^{(1)} - \mathbf{S}_0^{(1)}$, and $\mathbf{H} = (\mathbf{H}_L^{(1)}, \mathbf{H}_S^{(1)})$.

From equation (B.1) and equation (B.2), we have that

$$\begin{aligned}
&\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\text{F}}^2 + \lambda||\hat{\mathbf{L}}^{(1)}||_* + \tau||\hat{\mathbf{S}}^{(1)}||_1 \\
\leq &\frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\text{F}}^2 + \lambda||\mathbf{L}_0^{(1)}||_* + \tau||\mathbf{S}_0^{(1)}||_1 \\
\leq &\frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\text{F}}^2 + \lambda(||\hat{\mathbf{L}}^{(1)} - \mathbf{H}_L^{(1)} - \mathbf{H}_S^{(1)}||_* + \frac{\tau}{\lambda}||\hat{\mathbf{S}}^{(1)}||_1).
\end{aligned}$$

The second inequality is because $\hat{\mathbf{L}}^{(1)} - \mathbf{H}_L^{(1)} - \mathbf{H}_S^{(1)} + \hat{\mathbf{S}}^{(1)} = \mathbf{M}_0$. Then, it follows that

$$\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\text{F}}^2 - \frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\text{F}}^2 \leq \lambda||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_*.$$

Furthermore, the inequality between nuclear norm and Frobenius norm leads us that

$$\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(1)}-\mathbf{S}_0^{(1)}||_{\mathrm{F}}^2-\langle \mathbf{M}-\mathbf{L}_0^{(1)}-\mathbf{S}_0^{(1)}, \mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}\rangle+\frac{1}{2}||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}}^2-\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(1)}-\mathbf{S}_0^{(1)}||_{\mathrm{F}}^2$$
$$\leq \sqrt{n}\lambda||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}}.$$

Thus, we get

$$\begin{aligned}\frac{1}{2}||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}}^2 &\leq \lambda\sqrt{n}||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}}+\langle \mathbf{M}-\mathbf{L}_0^{(1)}-\mathbf{S}_0^{(1)}, \mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}\rangle \\ &\leq (\lambda\sqrt{n}+\delta_1)||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}}.\end{aligned}$$

A key fact follows that

$$||\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}||_{\mathrm{F}} \leq 2(\lambda\sqrt{n}+\delta_1), \tag{B.3}$$

which reveals the upper bound for the sum of the differences $\mathbf{H}_L^{(1)}$ and $\mathbf{H}_S^{(1)}$.

We need to borrow two important lemmas in Zhou et al. (2010) for further analysis. Before we make use of the lemmas, we briefly restate the notation in their paper. Let Ω denote the support of $\mathbf{S}_0^{(1)}$, and let $\mathcal{P}_\Omega$ be the projection operator onto the space of matrices supported on Ω. Let T denote the subspace generated by matrices with the same column space or row space as $\mathbf{L}_0^{(1)}$, and let $\mathcal{P}_T$ be the projection operator onto this subspace. For any pair $\mathbf{X}=(\mathbf{L},\mathbf{S})$, let $||\mathbf{X}||_{\mathrm{F}}=(||\mathbf{L}||_{\mathrm{F}}^2+||\mathbf{S}||_{\mathrm{F}}^2)^{1/2}$, and define the projection operator $\mathcal{P}_\Omega\times\mathcal{P}_T$: $(\mathbf{L},\mathbf{S})\rightarrow(\mathcal{P}_T(\mathbf{L}),\mathcal{P}_\Omega(\mathbf{S}))$. Define the subspace $\Gamma=\{\mathbf{Q},\mathbf{Q}|\mathbf{Q}\in\mathbb{R}^{n\times n}\}$ and $\Gamma^\perp=\{\mathbf{Q},-\mathbf{Q}|\mathbf{Q}\in\mathbb{R}^{n\times n}\}$, and let $\mathcal{P}_\Gamma$ and $\mathcal{P}_{\Gamma^\perp}$ denote their respective projection operators. Define $||\mathbf{X}||_\diamond=||\mathbf{L}||_*+\frac{\tau}{\lambda}||\mathbf{S}||_1$, and $\mathbf{X}_0^{(1)}=(\mathbf{L}_0^{(1)},\mathbf{S}_0^{(1)})$. Lemma 5 in Zhou et. al. (2010) proved that, for any $\mathbf{H}^{(1)}=(\mathbf{H}_L^{(1)},\mathbf{H}_S^{(1)})$ obeying $\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}=0$, assuming $||\mathcal{P}_\Omega\mathcal{P}_T||\leq\frac{1}{2}$ and $\frac{\tau}{\lambda}\leq\frac{1}{2}$,

$$||\mathbf{X}_0^{(1)}+\mathbf{H}^{(1)}||_\diamond \leq ||\mathbf{X}_0^{(1)}||_\diamond+(\frac{3}{4}-||\mathcal{P}_{T^\perp}(\mathbf{\Lambda})||)||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)})||_*+(\frac{3\tau}{4\lambda}-||\mathcal{P}_{\Omega^\perp}(\mathbf{\Lambda})||_\infty)||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)})||_1, \tag{B.4}$$

where $\mathbf{\Lambda}=\mathbf{U}\mathbf{V}^T+\mathbf{W}$, and $\mathbf{W}$ is the dual certificate. Candès et al. (2011) showd W exists with high probability. Furthermore, Lemma 6 in Zhou et al. (2010) stated that, assuming $||\mathcal{P}_T\mathcal{P}_\Omega||\leq\frac{1}{2}$, for any pair $\mathbf{X}=(\mathbf{L},\mathbf{S})$,

$$||\mathcal{P}_\Gamma(\mathcal{P}_T\times\mathcal{P}_\Omega)(\mathbf{X}))||_{\mathrm{F}}^2 \leq \frac{1}{4}||(\mathcal{P}_T\times\mathcal{P}_\Omega)(\mathbf{X})||_{\mathrm{F}}^2. \tag{B.5}$$

Consider $||\mathbf{H}^{(1)}||_{\mathrm{F}}$ now. Write $\mathbf{H}^{(1)\Gamma} = \mathcal{P}_{\Gamma}(\mathbf{H}^{(1)}) = \left(\frac{\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}}{2}, \frac{\mathbf{H}_L^{(1)}+\mathbf{H}_S^{(1)}}{2}\right)$, and $\mathbf{H}^{(1)\Gamma^\perp} = \mathcal{P}_{\Gamma^\perp}(\mathbf{H}^{(1)}) = \left(\frac{\mathbf{H}_L^{(1)}-\mathbf{H}_S^{(1)}}{2}, \frac{\mathbf{H}_S^{(1)}-\mathbf{H}_L^{(1)}}{2}\right)$. It follows that

$$\begin{aligned}
||\mathbf{H}^{(1)}||_{\mathrm{F}}^2 &= ||\mathbf{H}^{(1)\Gamma}||_{\mathrm{F}}^2 + ||\mathbf{H}^{(1)\Gamma^\perp}||_{\mathrm{F}}^2 \\
&= ||\mathbf{H}^{(1)\Gamma}||_{\mathrm{F}}^2 + ||(\mathcal{P}_T \times \mathcal{P}_\Omega)(\mathbf{H}^{(1)\Gamma^\perp})||_{\mathrm{F}}^2 + ||(\mathcal{P}_{T^\perp} \times \mathcal{P}_{\Omega^\perp})(\mathbf{H}^{(1)\Gamma^\perp})||_{\mathrm{F}}^2 \\
&\equiv \mathrm{I}^2 + \mathrm{II}^2 + \mathrm{III}^2. \qquad \text{(B.6)}
\end{aligned}$$

We need to find the upper bound for (B.6). Considering the first part I^2, we have $\mathrm{I}^2 \leq 4(\sqrt{n}\lambda + \delta)^2$ by (B.3). Then we turn to the third part III^2. Notice that

$$\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\mathrm{F}}^2 + \lambda||\mathbf{X}_0^{(1)} + \mathbf{H}^{(1)}||_\diamond \leq \frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\mathrm{F}}^2 + \lambda||\mathbf{X}_0^{(1)}||_\diamond \quad \text{(B.7)}$$

and

$$\begin{aligned}
&\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\mathrm{F}}^2 + \lambda||\mathbf{X}_0^{(1)} + \mathbf{H}^{(1)}||_\diamond \\
\geq&\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\mathrm{F}}^2 + \lambda||\mathbf{X}_0^{(1)} + \mathbf{H}^{(1)\Gamma^\perp}||_\diamond - \lambda||\mathbf{H}^{(1)\Gamma}||_\diamond \\
\geq&\frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\mathrm{F}}^2 + \lambda\Big(||\mathbf{X}_0^{(1)}||_\diamond + \frac{1}{4}||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)\Gamma^\perp})||_* + \frac{\tau}{4\lambda}||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)\Gamma^\perp})||_1\Big) - \lambda||\mathbf{H}^{(1)\Gamma}||_\diamond,
\end{aligned} \quad \text{(B.8)}$$

where the second inequality comes from (B.4). Combining (B.7) and (B.8), we can get

$$\begin{aligned}
&\frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\mathrm{F}}^2 - \frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\mathrm{F}}^2 + \lambda||\mathbf{H}^{(1)\Gamma}||_\diamond \\
\geq&\frac{\lambda}{4}||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)\Gamma^\perp})||_* + \frac{\tau}{4}||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)\Gamma^\perp})||_1. \qquad \text{(B.9)}
\end{aligned}$$

On the other hand, based on the inequalities between different norms, we know that

$$\mathrm{III} \leq ||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)\Gamma^\perp})||_{\mathrm{F}} + ||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)\Gamma^\perp})||_{\mathrm{F}} \leq ||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)\Gamma^\perp})||_* + ||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)\Gamma^\perp})||_1. \quad \text{(B.10)}$$

With the assumption $\tau \leq \lambda$, we use the inequality in (B.9) to get

$$\begin{aligned}\text{III} &\leq \frac{1}{\tau}(\lambda||\mathcal{P}_{T^\perp}(\mathbf{H}_L^{(1)\Gamma^\perp})||_* + \tau||\mathcal{P}_{\Omega^\perp}(\mathbf{H}_S^{(1)\Gamma^\perp})||_1) \\ &\leq \frac{4}{\tau}(\frac{1}{2}||\mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}||_{\text{F}}^2 - \frac{1}{2}||\mathbf{M} - \hat{\mathbf{L}}^{(1)} - \hat{\mathbf{S}}^{(1)}||_{\text{F}}^2 + \lambda||\mathbf{H}^{(1)\Gamma}||_\diamond) \\ &\leq \frac{2}{\tau}(-||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_{\text{F}}^2 + 2\langle \mathbf{M} - \mathbf{L}_0^{(1)} - \mathbf{S}_0^{(1)}, \mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}\rangle + 2\lambda||\mathbf{H}^{(1)\Gamma}||_\diamond). \quad \text{(B.11)}\end{aligned}$$

The only unknown part in (B.11) is $||\mathbf{H}^{(1)\Gamma}||_\diamond$ at this moment. Note that

$$\begin{aligned}||\mathbf{H}^{(1)\Gamma}||_\diamond = ||\mathbf{H}_L^{(1)\Gamma}||_* + \frac{\tau}{\lambda}||\mathbf{H}_S^{(1)\Gamma}||_1 &\leq \sqrt{n}||\mathbf{H}_L^{(1)\Gamma}||_{\text{F}} + \frac{\tau n}{\lambda}||\mathbf{H}_S^{(1)\Gamma}||_{\text{F}} \\ &\leq (\sqrt{n} + \frac{\tau n}{\lambda})||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_{\text{F}}. \quad \text{(B.12)}\end{aligned}$$

We can have an upper bound for III, when combining (B.11), (B.12) and (B.3)

$$\begin{aligned}\text{III} &\leq \frac{2}{\tau}[(2\lambda\sqrt{n} + 2\tau n)||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_{\text{F}} + 2\delta_1||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_{\text{F}} - ||\mathbf{H}_L^{(1)} + \mathbf{H}_S^{(1)}||_{\text{F}}^2] \\ &\leq \frac{2}{\tau}(\lambda\sqrt{n} + \tau n + \delta_1)^2 \quad \text{(B.13)}\end{aligned}$$

In terms of the second part II in (B.6), we have II $\leq 2 \times$ III from Zhou et al. (2010). Therefore, (B.13) together with the properties of I and II, gives us the desired result,

$$||\mathbf{H}^{(1)}||_{\text{F}}^2 \leq \frac{C_1}{\tau^2}(\lambda\sqrt{n} + \tau n + \delta_1)^4. \quad \text{(B.14)}$$

□

B.2 Proof of Theorem 4.2.2

Proof. Let $(\hat{\mathbf{L}}^{(2)}, \hat{\mathbf{S}}^{(2)})$ be an optimal solution of (4.8). Then, define $\mathbf{H}_L^{(2)} = \hat{\mathbf{L}}^{(2)} - \mathbf{L}_0^{(2)}$ and $\mathbf{H}_S^{(2)} = \hat{\mathbf{S}}^{(2)} - \mathbf{S}_0^{(2)}$. We would use some results in Xu, Caramanis and Sanghavi (2010) to prove the theorem. Notice that their conclusions are for the column corruption case, but all the results should also apply for the row corruption case, by simply transferring the target matrix.

Before we display the details of our proof, we first restate some useful definitions in Xu, Caramanis and Sanghavi (2010). For any $(\mathbf{L}'^{(2)}, \mathbf{S}'^{(2)})$ satisfying $\mathbf{L}'^{(2)} + \mathbf{S}'^{(2)} = \mathbf{M}$,

$\mathcal{P}_{V_0}(\mathbf{L}'^{(2)}) = \mathbf{L}'^{(2)}$, and $\mathcal{P}_{\mathcal{I}_0}(\mathbf{S}'^{(2)}) = \mathbf{S}'^{(2)}$, define

$$\mathcal{N}(\mathbf{L}'^{(2)}) \equiv \mathbf{U}'^{(2)}\mathbf{V}'^{(2)T};$$

$$\mathcal{B}(\mathbf{S}'^{(2)}) \equiv \{\mathbf{H} \in \mathbb{R}^{n_1 \times n_2} | \mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}) = 0; \forall i \in \mathcal{I}' : \mathbf{H}_i = \frac{\mathbf{S}_i'^{(2)}}{||\mathbf{S}_i'^{(2)}||_2}; \forall i \in \mathcal{I}_0 \cap (\mathcal{I}')^c : ||\mathbf{H}_i||_2 \leq 1\},$$

where the SVD of $\mathbf{L}'^{(2)}$ is $\mathbf{L}'^{(2)} = \mathbf{U}'^{(2)}\mathbf{D}'^{(2)}\mathbf{V}'^{(2)T}$, and the row support of $\mathbf{S}'^{(2)}$ is $\mathcal{I}'$. Furthermore, define the operator $\mathcal{P}_{T(\mathbf{L}')}(\cdot)$: $\mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ as

$$\mathcal{P}_{T(\mathbf{L}')}(\mathbf{X}) = \mathcal{P}_{U'^{(2)}}(\mathbf{X}) + \mathcal{P}_{V'^{(2)}}(\mathbf{X}) - \mathcal{P}_{\mathbf{U}'^{(2)}}\mathcal{P}_{\mathbf{V}'^{(2)}}(\mathbf{X}).$$

By Theorem 2 and Theorem 5 in Xu, Caramanis and Sanghavi (2010), if certain conditions are satisfied, there exists a $\mathbf{Q}$ such that

$$\mathcal{P}_{T(\mathbf{L}')}(\mathbf{Q}) \in \mathcal{N}(\mathbf{L}'^{(2)}); \;\; ||\mathcal{P}_{T(\mathbf{L}')^\perp}(\mathbf{Q})|| \leq \frac{1}{2}; \;\; \mathcal{P}_{\mathcal{I}_0}(\mathbf{Q})/(\frac{\tau}{\lambda}) \in \mathcal{B}(\mathbf{S}'^{(2)}); \;\; ||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{Q})||_{\infty,2} \leq \frac{\tau}{2\lambda}.$$

Furthermore, in the proof of Theorem 3 in Xu, Caramanis and Sanghavi (2010), they showed if the above Q exists, for any fixed $\triangle \neq 0$, $(\mathbf{L}'^{(2)} + \triangle, \mathbf{S}'^{(2)} - \triangle)$ is strictly worse than $(\mathbf{L}'^{(2)}, \mathbf{S}'^{(2)})$ unless $\triangle \in \mathcal{P}_{V_0} \cap \mathcal{P}_{\mathcal{I}_0}$, we can find a $\mathbf{W}$ and a $\mathbf{F}$ such that

$$||\mathbf{W}|| = 1, \;\; \langle \mathbf{W}, \mathcal{P}_{T(\mathbf{L}')^\perp}(\triangle) \rangle = ||\mathcal{P}_{T(\mathbf{L}')^\perp}(\triangle)||_*, \;\; \mathcal{P}_{T(\mathbf{L}')^\perp}(\mathbf{W}) = 0,$$

and

$$\mathbf{F}_i = \begin{cases} \frac{-\triangle_i}{||\triangle_i||_2} & \text{if } i \notin \mathcal{I}_0 \text{ and } \triangle_i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathcal{P}_{T(\mathbf{L}')}(\mathbf{Q}) + \mathbf{W}$ is a subgradient of $||\mathbf{L}'^{(2)}||_*$, and $\mathcal{P}_{\mathcal{I}_0}(\mathbf{Q})/(\frac{\tau}{\lambda}) + \mathbf{F}$ is a subgradient of $||\mathbf{S}'^{(2)}||_{1,2}$.

Choosing the above $\mathbf{W}$ and $\mathbf{F}$, our target function has the following properties

$$
\begin{aligned}
&\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2+\lambda||\mathbf{L}_0^{(2)}||_*+\tau||\mathbf{S}_0^{(2)}||_{1,2}\\
\geq&\frac{1}{2}||\mathbf{M}-\hat{\mathbf{L}}^{(2)}-\hat{\mathbf{S}}^{(2)}||_{\mathrm{F}}^2+\lambda||\hat{\mathbf{L}}^{(2)}||_*+\tau||\hat{\mathbf{S}}^{(2)}||_{1,2}\\
\geq&\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2-\langle\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)},\hat{\mathbf{L}}^{(2)}-\mathbf{L}_0^{(2)}+\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)}\rangle+\frac{1}{2}||\hat{\mathbf{L}}^{(2)}-\mathbf{L}_0^{(2)}+\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2\\
&+\lambda||\mathbf{L}_0^{(2)}||_*+\lambda\langle\mathcal{P}_{T(\mathbf{L}_0)}(\mathbf{Q})+\mathbf{W},\hat{\mathbf{L}}^{(2)}-\mathbf{L}_0^{(2)}\rangle+\tau||\mathbf{S}_0^{(2)}||_{1,2}+\tau\langle\mathcal{P}_{\mathcal{I}_0}(\mathbf{Q})/(\frac{\tau}{\lambda})+\mathbf{F},\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)}\rangle\\
=&\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2-\langle\mathbf{N}_0^{(2)},\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}\rangle+\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2+\lambda||\mathbf{L}_0^{(2)}||_*+\tau||\mathbf{S}_0^{(2)}||_{1,2}\\
&+\lambda\langle\mathcal{P}_{T(\mathbf{L}_0)}(\mathbf{Q})+\mathbf{W},\mathbf{H}_L^{(2)}\rangle+\tau\langle\mathcal{P}_{\mathcal{I}_0}(\mathbf{Q})/(\frac{\tau}{\lambda})+\mathbf{F},\mathbf{H}_S^{(2)}\rangle\\
=&\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2+\lambda||\mathbf{L}_0^{(2)}||_*+\tau||\mathbf{S}_0^{(2)}||_{1,2}-\langle\mathbf{N}_0^{(2)},\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}\rangle+\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2\\
&+\lambda||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{H}_L^{(2)})||_*+\lambda\langle\mathbf{Q}-\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{Q}),\mathbf{H}_L^{(2)}\rangle+\tau||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{1,2}+\lambda\langle\mathbf{Q}-\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{Q}),\mathbf{H}_S^{(2)}\rangle\\
\geq&\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2+\lambda||\mathbf{L}_0^{(2)}||_*+\tau||\mathbf{S}_0^{(2)}||_{1,2}+\lambda(1-||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{Q})||)||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{H}_L^{(2)})||_*\\
&+(\tau-\lambda||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{Q})||_{\infty,2})||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{1,2}+\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2+\langle\lambda\mathbf{Q}-\mathbf{N}_0^{(2)},\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}\rangle
\end{aligned}
$$

Since $||\mathbf{Q}||_{\infty,2}\leq\frac{\tau}{\lambda}$, we have $||\mathbf{Q}||_{\mathrm{F}}\leq\frac{\tau}{\lambda}\sqrt{n}$. Thus, it follows that

$$
\begin{aligned}
&\lambda(1-||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{Q}))||)||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{H}_L^{(2)})||_*+(\tau-\lambda||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{Q})||_{\infty,2})||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{1,2}\\
\leq&\langle\mathbf{N}_0^{(2)}-\lambda\mathbf{Q},\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}\rangle-\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2\\
\leq&(||\mathbf{N}_0^{(2)}||_{\mathrm{F}}+\lambda||\mathbf{Q}||_{\mathrm{F}})||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}-\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2\\
\leq&(\delta_2+\tau\sqrt{n})||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}-\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2.
\end{aligned}
$$

Therefore, we can see

$$
||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{H}_L^{(2)})||_{\mathrm{F}}\leq||\mathcal{P}_{T(\mathbf{L}_0)^\perp}(\mathbf{H}_L^{(2)})||_*\leq\frac{2}{\lambda}[(\delta_2+\tau\sqrt{n})||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}-\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2], \tag{B.15}
$$

$$
||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{\mathrm{F}}\leq||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{1,2}\leq\frac{2}{\tau}[(\delta_2+\tau\sqrt{n})||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}-\frac{1}{2}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}^2]. \tag{B.16}
$$

Note that an upper bound for $||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}$ could be achieved by similar procedure as in proof of Theorem 4.2.1. We only list the key step here:

$$\frac{1}{2}||\mathbf{M}-\hat{\mathbf{L}}^{(2)}-\hat{\mathbf{S}}^{(2)}||_{\mathrm{F}}^2-\frac{1}{2}||\mathbf{M}-\mathbf{L}_0^{(2)}-\mathbf{S}_0^{(2)}||_{\mathrm{F}}^2\leq\tau||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{1,2}\leq\tau\sqrt{n}||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}.$$

Then, we get

$$||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}\leq 2(\tau\sqrt{n}+\delta_2). \tag{B.17}$$

Next, inspired by Xu, Caramanis and Sanghavi (2010), we introduce several new terms. Let $\mathbf{H}_L^{(2)+}=\mathbf{H}_L^{(2)}-\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0}(\mathbf{H}_L^{(2)})$ and $\mathbf{H}_S^{(2)+}=\mathbf{H}_S^{(2)}-\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0}(\mathbf{H}_S^{(2)})$. Intuitively, the rows of $\mathbf{H}_L^{(2)+}$ and $\mathbf{H}_S^{(2)+}$ within $\mathcal{I}_0$ are orthogonal to V_0. In the proof of Theorem 5 in Xu, Caramanis and Sanghavi (2010) showed that

$$||\mathcal{P}_{\mathcal{I}_0}(\mathbf{H}_S^{(2)+})||_{\mathrm{F}}\leq||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}+||\mathcal{P}_{T(\mathbf{L}_0)^{\perp}}(\mathbf{H}_L^{(2)})||_{\mathrm{F}}+||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{\mathrm{F}}+\varphi||\mathcal{P}_{\mathcal{I}_0}(\mathbf{H}_S^{(2)+})||_{\mathrm{F}}, \tag{B.18}$$

where $\varphi<1/4$. Now using the fact that $\tau\leq\lambda$ (Xu, Caramanis and Sanghavi, 2010) and the inequalities (B.15), (B.16) and (B.17), we have

$$||\mathcal{P}_{\mathcal{I}_0}(\mathbf{H}_S^{(2)+})||_{\mathrm{F}}\leq[\frac{2}{\tau}(\tau\sqrt{n}+\delta_2)^2+2(\tau\sqrt{n}+\delta_2)]/(1-\varphi)\leq\frac{16}{3\tau}(\tau\sqrt{n}+\delta_2)^2.$$

Furthermore, we can get

$$||\mathbf{H}_S^{(2)+}||_{\mathrm{F}}\leq||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})+\mathcal{P}_{\mathcal{I}_0}(\mathbf{H}_S^{(2)+})||_{\mathrm{F}}\leq||\mathcal{P}_{\mathcal{I}_0^c}(\mathbf{H}_S^{(2)})||_{\mathrm{F}}+||\mathcal{P}_{\mathcal{I}_0}(\mathbf{H}_S^{(2)+})||_{\mathrm{F}}\leq\frac{19}{3\tau}(\tau\sqrt{n}+\delta_2)^2$$

Note that $\mathbf{H}_S^{(2)+}=(\mathcal{I}-\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0})(\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)})=\hat{\mathbf{S}}^{(2)}-[\mathbf{S}_0^{(2)}+\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0}(\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)})]$. Letting $\tilde{\mathbf{S}}^{(2)}=\mathbf{S}_0^{(2)}+\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0}(\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)})$, we have $\tilde{\mathbf{S}}^{(2)}\in\mathcal{P}_{\mathcal{I}_0}$ and $||\hat{\mathbf{S}}^{(2)}-\tilde{\mathbf{S}}^{(2)}||_{\mathrm{F}}\leq\frac{19}{3\tau}(\tau\sqrt{n}+\delta_2)^2$. On the other hand, we let $\tilde{\mathbf{L}}^{(2)}=\mathbf{L}_0^{(2)}-\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{V_0}(\hat{\mathbf{S}}^{(2)}-\mathbf{S}_0^{(2)})$, leading to

$$||\hat{\mathbf{L}}^{(2)}-\tilde{\mathbf{L}}^{(2)}||_{\mathrm{F}}\leq||\mathbf{H}_L^{(2)}+\mathbf{H}_S^{(2)}||_{\mathrm{F}}+||\hat{\mathbf{S}}^{(2)}-\tilde{\mathbf{S}}^{(2)}||_{\mathrm{F}}\leq\frac{25}{3\tau}(\tau\sqrt{n}+\delta_2)^2.$$

Theorem 4.2.2 is thus established. □

B.3 Proof of Lemma 4.3.4

Proof. We consider $(\mathbf{L}^k, \mathbf{S}^k)$ to get $(\mathbf{L}^{k+1}, \mathbf{S}^{k+1})$. The two steps in iteration to achieve $\mathbf{L}^{k+1}$ and $\mathbf{S}^{k+1}$ implies that

$$\begin{aligned} f_{\lambda,\tau}(\mathbf{L}^k, \mathbf{S}^k) &= \frac{1}{2}||M - \mathbf{L}^k - \mathbf{S}^k||_{\mathrm{F}}^2 + \lambda||\mathbf{L}^k||_* + \tau||\mathbf{S}^k||_1 \\ &\geq \min_L \frac{1}{2}||\mathbf{M} - \mathbf{L} - \mathbf{S}^k||_{\mathrm{F}}^2 + \lambda||\mathbf{L}||_* + \tau||\mathbf{S}^k||_1 \\ &= \frac{1}{2}||\mathbf{M} - \mathbf{L}^{k+1} - \mathbf{S}^k||_{\mathrm{F}}^2 + \lambda||\mathbf{L}^{k+1}||_* + \tau||\mathbf{S}^k||_1 = f_{\lambda,\tau}(\mathbf{L}^{k+1}, \mathbf{S}^k) \\ &\geq \min_S \frac{1}{2}||\mathbf{M} - \mathbf{L}^{k+1} - \mathbf{S}||_{\mathrm{F}}^2 + \lambda||\mathbf{L}^{k+1}||_* + \tau||\mathbf{S}||_1 \\ &= \frac{1}{2}||\mathbf{M} - \mathbf{L}^{k+1} - \mathbf{S}^{k+1}||_{\mathrm{F}}^2 + \lambda||\mathbf{L}^{k+1}||_* + \tau||\mathbf{S}^{k+1}||_1 = f_{\lambda,\tau}(\mathbf{L}^{k+1}, \mathbf{S}^{k+1}). \end{aligned}$$

Therefore, the inequalities in lemma have been deduced. □

B.4 Proof of Lemma 4.4.3

Proof. It is simply based on the arguments we mentioned about the surrogate functions and it is similar to the proof of Lemma (4.3.4), so we omit it here. □

B.5 Proof of Lemma 4.4.4

Proof. In the proof, we omit the subscribe λ and τ if no confusion causes. Firstly, note that

$$\begin{aligned} F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1}) \geq\ & F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) \\ &+ F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^k) \\ &+ F(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1}). \end{aligned} \tag{B.19}$$

We observe that

$$F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) \geq F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k).$$

To further study the difference of the two terms in (**??**), we notice a fact that the ridge regression problem

$$H(\boldsymbol{\beta}) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\boldsymbol{\beta}||_2^2 + \frac{\lambda}{2}||\boldsymbol{\beta}||_2^2$$

has the following property

$$H(\boldsymbol{\beta}) - H(\boldsymbol{\beta}^*) = \frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{\beta}^*)^T(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})(\boldsymbol{\beta} - \boldsymbol{\beta}^*) = \frac{1}{2}||X(\boldsymbol{\beta} - \boldsymbol{\beta}^*)||_2^2 + \frac{\lambda}{2}||\boldsymbol{\beta} - \boldsymbol{\beta}^*||_2^2, \tag{B.20}$$

where $\boldsymbol{\beta}^* \in \text{argmin} H(\boldsymbol{\beta})$. Therefore, we have

$$F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) \geq \frac{1}{2}||(\mathbf{A}^k - \mathbf{A}^{k+1})(\mathbf{B}^k)^T||_{\mathrm{F}}^2 + \frac{\lambda}{2}||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2. \tag{B.21}$$

Combining (**??**) and (B.21), we have

$$F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) \geq \frac{1}{2}||(\mathbf{A}^k - \mathbf{A}^{k+1})(\mathbf{B}^k)^T||_{\mathrm{F}}^2 + \frac{\lambda}{2}||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2. \tag{B.22}$$

Similarly, the second part of (B.19) has the inequality

$$F(\mathbf{A}^{k+1}, \mathbf{B}^k, \mathbf{S}^k) - F(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^k) \geq \frac{1}{2}||\mathbf{A}^{k+1}(\mathbf{B}^k - \mathbf{B}^{k+1})^T||_{\mathrm{F}}^2 + \frac{\lambda}{2}||\mathbf{B}^k - \mathbf{B}^{k+1}||_{\mathrm{F}}^2. \tag{B.23}$$

Therefore, (B.22) and (B.23) give us (4.22).

On the other hand, because the function values $F(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$ are monotonically decreasing and lower bounded by 0, the left hand side of (4.22) goes to 0 as $k \to \infty$. Then, it follows that

$$\begin{aligned}
&\frac{\lambda}{2}||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2 \leq \eta^k \to 0, \\
&\frac{\lambda}{2}||\mathbf{B}^k - \mathbf{B}^{k+1}||_{\mathrm{F}}^2 \leq \eta^k \to 0, \\
&\frac{1}{2}||(\mathbf{A}^k - \mathbf{A}^{k+1})(\mathbf{B}^k)^T||_{\mathrm{F}}^2 \leq \eta^k \to 0, \\
&\frac{1}{2}||\mathbf{A}^{k+1}(\mathbf{B}^k - \mathbf{B}^{k+1})^T||_{\mathrm{F}}^2 \leq \eta^k \to 0.
\end{aligned}$$

Thus, we have

$$\mathbf{A}^k - \mathbf{A}^{k+1} \to \mathbf{0}, \quad \mathbf{B}^k - \mathbf{B}^{k+1} \to \mathbf{0}, \quad (\mathbf{A}^k - \mathbf{A}^{k+1})(\mathbf{B}^k)^T \to \mathbf{0}, \quad \mathbf{A}^{k+1}(\mathbf{B}^k - \mathbf{B}^{k+1})^T \to \mathbf{0}.$$

In the meanwhile, adding the last two equations above will result in $\mathbf{A}^k(\mathbf{B}^k)^T - \mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T \to \mathbf{0}$. Remind that the $\mathbf{S}^k$ can be written as functions of $\mathbf{A}^k\mathbf{B}^k$, that is,

$$\mathbf{S}^k = H_\tau(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T), \quad \text{if element-wise sparse case is considered;}$$
$$\mathbf{S}^k = H_\tau^r(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T), \quad \text{if row sparse case is considered.}$$

Hence, $\mathbf{S}^k - \mathbf{S}^{k+1} = H_\tau(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T) - H_\tau(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T)$ or $\mathbf{S}^k - \mathbf{S}^{k+1} = H_\tau^r(\mathbf{M} - \mathbf{A}^k(\mathbf{B}^k)^T) - H_\tau^r(\mathbf{M} - \mathbf{A}^{k+1}(\mathbf{B}^{k+1})^T)$. It can be easily proved that both function $H_\tau(X)$ and function $H_\tau^r(X)$ are continuous functions. Therefore, we also have $\mathbf{S}^k - \mathbf{S}^{k+1} \to 0$ as $k \to \infty$, completing the proof of the lemma. □

B.6 Proof of Theorem 4.4.5

Proof. Part (a): Note that if $\lambda > 0$, the sequence $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$ is bounded and thus has a limit point. Let $(\mathbf{A}^*, \mathbf{B}^*, \mathbf{S}^*)$ be any limit point of the sequence $\{(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)\}$. There exists a subsequence $\{n_k\} \subset \{1, 2, \ldots\}$ such that $\mathbf{A}^{n_k} \to \mathbf{A}^*, \mathbf{B}^{n_k} \to \mathbf{B}^*, \mathbf{S}^{n_k} \to \mathbf{S}^*$ as $k \to \infty$.

The properties of surrogate functions lead us to

$$\partial_A Q^A(\mathbf{A}^{n_k} | \mathbf{A}^{n_k-1}, \mathbf{B}^{n_k-1}, \mathbf{S}^{n_k-1}) \ni \mathbf{0},$$
$$\partial_B Q^A(\mathbf{B}^{n_k} | \mathbf{A}^{n_k}, \mathbf{B}^{n_k-1}, \mathbf{S}^{n_k-1}) \ni \mathbf{0},$$
$$\partial_S R^S(\mathbf{S}^{n_k} | \mathbf{A}^{n_k}, \mathbf{B}^{n_k}) \ni \mathbf{0}.$$

Notice that all the above subgradient functions are continuous functions. Furthermore, we have

$$\mathbf{A}^{n_k} - \mathbf{A}^{n_k-1} \to \mathbf{0}, \quad \mathbf{B}^{n_k} - \mathbf{B}^{n_k-1} \to \mathbf{0}, \quad \mathbf{S}^{n_k} - \mathbf{S}^{n_k-1} \to \mathbf{0}, \text{ as } k \to \infty,$$

following from Lemma 4.4.4. Then, it concludes that

$$\begin{aligned}
&\partial_A Q^A(\mathbf{A}^*|\mathbf{A}^*,\mathbf{B}^*,\mathbf{S}^*) \ni \mathbf{0},\\
&\partial_B Q^A(\mathbf{B}^*|\mathbf{A}^*,\mathbf{B}^*,\mathbf{S}^*) \ni \mathbf{0},\\
&\partial_S R^S(\mathbf{S}^*|\mathbf{A}^*,\mathbf{B}^*) \ni \mathbf{0},
\end{aligned}$$

indicating $(\mathbf{A}^*,\mathbf{B}^*,\mathbf{S}^*)$ a first order stationary point.
Part (b): Suppose $\mathbf{A}^{n_k} \to \mathbf{A}^*$, the statement is to show $\mathbf{B}^{n_k}$ and $\mathbf{S}^{n_k}$ has a unique limit point. To prove it, we firstly suppose there are two limit points for $\mathbf{B}^{n_k}$, i.e. $\mathbf{B}^{n_{k1}} \to \mathbf{B}_1$, $\mathbf{B}^{n_{k2}} \to \mathbf{B}_2$, and $\mathbf{B}_1 \neq \mathbf{B}_2$.

The derivation of inequality (B.23) tells us

$$F(\mathbf{A}^{k+1},\mathbf{B},\mathbf{S}^k) - F(\mathbf{A}^{k+1},\mathbf{B}^{k+1},\mathbf{S}^k) \geq \frac{\lambda}{2}||\mathbf{B}-\mathbf{B}^{k+1}||_{\mathrm{F}}^2.$$

Then, we have

$$\begin{aligned}
&F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k1}},\mathbf{S}^{n_{k1}-1})\\
=&F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k1}-1}) + F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k1}-1}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k1}},\mathbf{S}^{n_{k1}-1})\\
\geq&F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k1}-1}) + \frac{\lambda}{2}||\mathbf{B}^{n_{k2}}-\mathbf{B}^{n_{k1}}||_{\mathrm{F}}^2. \qquad \text{(B.24)}
\end{aligned}$$

Since $F(\mathbf{A}^k,\mathbf{B}^k,\mathbf{S}^k) \to f^\infty$, and $F(\mathbf{A}^k,\mathbf{B}^k,\mathbf{S}^k) - F(\mathbf{A}^k,\mathbf{B}^{k-1},\mathbf{S}^{k-1}) \to 0$, it follows that

$$F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k1}-1},\mathbf{S}^{n_{k1}-1}) \to f^\infty - f^\infty = 0.$$

Whereas, the right hand side of inequality (B.24) has limiting properties

$$\begin{aligned}
&F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) \to F(\mathbf{A}^*,\mathbf{B}_2,H_\tau(\mathbf{M}-\mathbf{A}^*(\mathbf{B}_2)^T)),\\
&F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k1}-1}) \to F(\mathbf{A}^*,\mathbf{B}_2,H_\tau(\mathbf{M}-\mathbf{A}^*(\mathbf{B}_2)^T)),\\
&||\mathbf{B}^{n_{k2}}-\mathbf{B}^{n_{k1}}||_{\mathrm{F}}^2 \to ||\mathbf{B}_2-\mathbf{B}_1||_{\mathrm{F}}^2,
\end{aligned}$$

giving rise to $F(\mathbf{A}^{n_{k2}+1},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k2}}) - F(\mathbf{A}^{n_{k1}},\mathbf{B}^{n_{k2}},\mathbf{S}^{n_{k1}-1}) + \frac{\lambda}{2}||\mathbf{B}^{n_{k2}}-\mathbf{B}^{n_{k1}}||_{\mathrm{F}}^2 \not\to 0$. The same results will apply when replacing $H_\tau(\mathbf{X})$ to $H_\tau^r(\mathbf{X})$ for row sparse case. This contradiction implies $\mathbf{B}^{n_k}$ has only one limit point. As a result, $\mathbf{S}^{n_k}$ has one corresponding limit point.

Exactly the same argument holds true for the sequence $\mathbf{B}^k$, leading to the conclusion of the other part of Part (b). □

B.7 Proof of Theorem 4.4.6

Proof. We make use of (4.22) and add both sides of the inequality over $k = 1, \dots, K$, which leads to:

$$\sum_{k=1}^{K}(F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1})) \geq \sum_{k=1}^{K} \eta^k \geq K(\min_{1 \leq k \leq K} \eta^k). \tag{B.25}$$

Since $F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k)$ is a decreasing sequence and converge to f^∞, we then get

$$\sum_{k=1}^{K}(F_{\lambda,\tau}(\mathbf{A}^k, \mathbf{B}^k, \mathbf{S}^k) - F_{\lambda,\tau}(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}, \mathbf{S}^{k+1})) \leq F_{\lambda,\tau}(\mathbf{A}^1, \mathbf{B}^1, \mathbf{S}^1) - f^\infty. \tag{B.26}$$

Combining (B.25) and (B.26), we obtain (4.24).

Concluding the second part can be owe to the fact from (4.22) that

$$||\mathbf{A}^k - \mathbf{A}^{k+1}||_{\mathrm{F}}^2 + ||\mathbf{B}^k - \mathbf{B}^{k+1}||_{\mathrm{F}}^2 \leq \frac{2}{\lambda}\eta^k.$$

□