Uniform Symbolic Topologies in Non-Regular Rings

by

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ABSTRACT

When does a Noetherian commutative ring R have uniform symbolic topologies (USTP) on primes – read, when does there exist an integer $D > 0$ such that the symbolic power $P^{(Dr)} \subseteq P^r$ for all prime ideals $P \subseteq R$ and all $r > 0$? Groundbreaking work of Ein – Lazarsfeld – Smith, as extended by Hochster and Huneke, and by Ma and Schwede in turn, provides a beautiful answer in the setting of finite-dimensional excellent regular rings. Their work shows that there exists a D depending only on the Krull dimension: in other words, the exact same D works for all regular rings as stated of a fixed dimension.

Referring to this last observation, we say in the thesis that the class of excellent regular rings enjoys class solidarity relative to the uniform symbolic topology property (USTP class solidarity), a strong form of uniformity. In contrast, this thesis shows that for certain classes of non-regular rings including rational surface singularities and select normal toric rings, a uniform bound D does exist but depends on the ring, not just its dimension. In particular, for rational double point surface singularities over $\mathbb{C}$, we show that USTP solidarity is plainly impossible.

It is natural to sleuth for analogues of the Improved Ein – Lazarsfeld – Smith Theorem where the ring R is non-regular, or where the above ideal containments can be improved using a linear function whose growth rate is slower. This thesis lies in the overlap of these research directions, working with Noetherian domains.

CHAPTER I

Introduction

1.1 An Invitational Sojourn to My Mathematical Playground

I will begin with two quotations. The first is attributed to the late mathematician Sophie Germain: *Algebra is nothing more than geometry in words; geometry is nothing more than algebra in pictures.* Indeed, many people regard algebra and geometry to be at once antipodal yet interconnected – they often come together like siblings in a sort of yin and yang relationship. Now onto the second quote which pairs well with the first. While reading a MAA Book Review of Gregor Kemper's *A Course in Commutative Algebra* [45], the reviewer attributed the following perspective to the late algebraic geometer George Kempf: *Algebraic Geometry is a seesaw balancing between two Mediterranean traditions of mathematical inquiry: the Arabic algebraic tradition on the one hand, and the Greek geometric tradition on the other hand.*

Algebraic varieties, common zero sets of systems of polynomial equations, are the central objects of study in algebraic geometry. For instance, given a collection $A = \{F_k\}_{k\in K}$ – possibly infinite, uncountable – of polynomials in n unknowns, either real variables or complex variables, we might write

$$\mathbb{V}(A)_{\mathbb{R}^n} = \{p \in \mathbb{R}^n\colon F(p) = 0, \forall F \in A\},$$

$$\mathbb{V}(A)_{\mathbb{C}^n} = \{p \in \mathbb{C}^n\colon F(p) = 0, \forall F \in A\}$$

for the real- and complex zero sets, respectively. In case a single nonconstant equation suffices, we call the variety an **algebraic hypersurface**. For instance, from Day One in undergraduate complex analysis we know that

$$\mathbb{V}(x^2+1)_{\mathbb{R}} = \varnothing, \text{ while } \mathbb{V}(x^2+1)_{\mathbb{C}} = \{i, -i\},$$

where $i = \sqrt{-1}$ is the imaginary unit, relative to which $\mathbb{C} = \{a + ib \colon a, b \in \mathbb{R}\}$. Indeed, any complex algebraic hypersurface must be a non-empty set, courtesy of the Fundamental Theorem of Algebra. The latter result extends to a result David Hilbert proved for complex varieties – Hilbert's Nullstellensatz, a famed theorem from classical algebraic geometry over algebraically closed fields like $\mathbb{C}$.

That said, real algebraic hypersurfaces are often non-empty – indeed, infinite sets when $n > 1$ – as illustrated by the following gallery of figures.

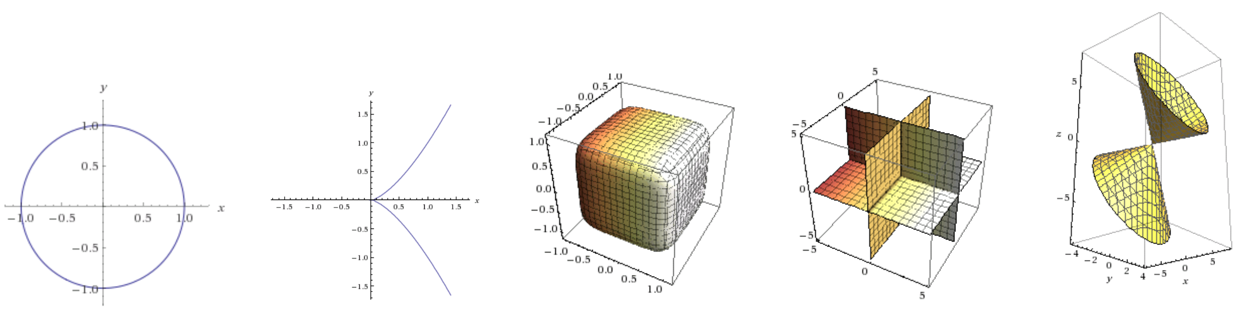

Figure 1.1: A gallery of real algebraic curves and surfaces.

Figure 1.1 features two **curves** (one-dimensional objects), and three **surfaces** (two-dimensional). Herwig Hauser's online Gallery of Algebraic Surfaces provides even more variety in the profile pictures one can study and admire.

I work primarily in commutative algebra. One facet of commutative algebra is the formal study of rings of polynomial functions on algebraic varieties, and their associated modules and algebras. As exposited by David Eisenbud [17, Ch. 1], the formal development of commutative algebra started in the 1800s, an outgrowth of ongoing activity in algebraic geometry, algebraic number theory, and invariant theory.

How does commutative algebra interact with and correspond to algebraic geometry? There are *several echelons* to answering this question, and the remainder of this section offers one brush stroke answer, in part – closing with remarks on the Affine Nullstellensatz Correspondence. We will work with real algebraic varieties for now, abruptly incepting complex varieties later. Let $R = \mathbb{R}[X_1, \ldots, X_N]$ denote the ring of real polynomials in N variables, viewed as functions $\mathbb{R}^N \to \mathbb{R}$. Given an algebraic variety $V \subseteq \mathbb{R}^N$, each $f \in R$ can be restricted to a function $V \to \mathbb{R}$. The **(affine) coordinate ring** $\mathbb{R}[V]$ of V consists of all functions $V \to \mathbb{R}$ obtained by restricting polynomials. For instance, the subsequent figure showcases a curve $\mathcal{C} = \mathbb{V}(F)_{\mathbb{R}^2}$ and a surface $\mathcal{S} = \mathbb{V}(G)_{\mathbb{R}^3}$, the zero sets of the respective polynomials below:

$$\begin{aligned} F(x,y) &= (x^2 + (y-1.5)^2 - 1/9) \cdot (x^2 + (y+1.5)^2 - 1/9) \\ &\quad \cdot (x^2 + y^2 - 9) \cdot (x + 0.1(y^3 - 9y)); \\ G(x,y,z) &= x^6 + y^6 + z^6 - 1. \end{aligned}$$

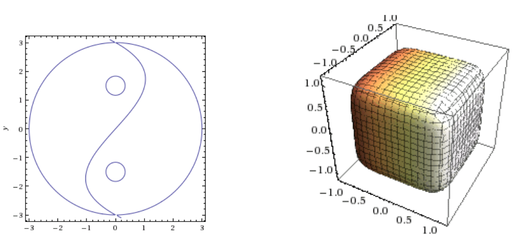

Figure 1.2: The curve $\mathcal{C} = \{F(x,y) = 0\}$, and the surface $\mathcal{S} = \{G(x,y,z) = 0\}$.

The polynomial F above defines a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ which vanishes on $\mathcal{C}$, that is, $F\colon \mathcal{C} \to \mathbb{R}$ is the zero function, $F(p) = 0$ for all points p in $\mathcal{C}$.

Given V as above, this notion of vanishing functions translates to an ideal in $\mathbb{R}[V]$. There is a natural surjective ring homomorphism (*Restriction to* V):

$$\phi\colon \mathbb{R}[x_1, \ldots, x_N] \twoheadrightarrow \mathbb{R}[V], \quad \phi(f) = f|_V \colon V \to \mathbb{R}.$$

Setting $\mathbb{I}(V) = \ker\phi = \{f\colon f(p) = 0, \forall p \in V\}$, we have a ring isomorphism

$$\mathbb{R}[V] \cong \frac{\mathbb{R}[x_1, \ldots, x_N]}{\mathbb{I}(V)}.$$

For instance, $\mathbb{R}[\mathcal{C}] \cong \mathbb{R}[x, y]/(F)$ and $\mathbb{R}[\mathcal{S}] \cong \mathbb{R}[x, y, z]/(G)$. Since $\mathbb{R}[V]$ is isomorphic to a quotient of a polynomial ring in finitely-many real variables, $\mathbb{R}[V]$ is an **affine $\mathbb{R}$-algebra**, or an algebra **of finite type over** $\mathbb{R}$. The **defining ideal** $\mathbb{I}(V)$ of V is always a *radical ideal*: that is, if $f^E \in \mathbb{I}(V)$ for some $E > 0$, then in fact $f \in \mathbb{I}(V)$. In words, the only nilpotent function in $\mathbb{R}[V]$ is the zero function, where by nilpotent function $f \in \mathbb{R}[V]$, we mean that f^E is the zero function on V for some $E > 0$. Equivalently, the ring $\mathbb{R}[V]$ is **reduced**.

Rather than sojourning into category theory to define what an *equivalence of categories* is, we simply note that Nullstellensatz Correspondence is a name given to several such equivalences which rigorously formalize the content of both Germain's and Kempf's quotations from earlier. In particular, results from the general theory of Noetherian commutative rings may be rendered to fruitful effect in geometric parlance pertaining to algebraic varieties. The Nullstellensatz Correspondence over $\mathbb{C}$ (or any algebraically closed field) implies, among other things, a series of bijective correspondences translating between the geometry of varieties V and the algebra of their coordinate rings $\mathbb{C}[V]$ (e.g., between $\mathbb{C}^N$ and $\mathbb{C}[x_1, \ldots, x_N]$):

- Isomorphism classes of algebraic varieties over $\mathbb{C}$ are in bijection with isomorphism classes of reduced, affine $\mathbb{C}$-algebras. Fixing one such variety or algebra, algebraic subvarieties of a variety V are in order-reversing bijection with radical ideals in $\mathbb{C}[V]$. In turn, *irreducible* algebraic subvarieties of V are in bijection with *prime* ideals. In particular, the points p in V correspond to maximal ideals $\mathfrak{m}_p = \mathbb{I}(p) = \{F \in \mathbb{C}[V]\colon F(p) = 0\}$.

- One can assign a value in $\mathbb{Z}_{\geq -1} \cup \{\infty\}$ to any commutative ring called its **(Krull) dimension**, which turns out be a *finite* numerical invariant for (isomorphism classes of) reduced, affine $\mathbb{C}$-algebras. In turn, one defines a notion of dimension for (isomorphism classes of) affine algebraic varieties over $\mathbb{C}$ in order that the dimension of a variety V coincides rigorously with the Krull dimension of $\mathbb{C}[V]$.

To strike a contrast, there are pairs of reduced, affine $\mathbb{R}$-algebras whose dimensions disagree (hence are non-isomorphic) that correspond to the same variety up to isomorphism. For instance, if $V = \mathbb{V}(x^2+1)_{\mathbb{R}}$ and $W = \mathbb{V}(1)_{\mathbb{R}}$ then $V = W = \varnothing$, but $\mathbb{R}[V] \cong \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$ is zero dimensional while the zero ring $\mathbb{R}[W] \cong \mathbb{R}[x]/(1)$ has dimension -1. Pointedly, this *never* happens over an algebraically closed field. We have reached the end of the invitational sojourn, the remaining sections read more like a research seminar talk.

1.2 A Highlight Reel Backdrop to the Dissertation Problem

In this chapter, **all** rings are nonzero Noetherian commutative with identity.

This thesis is focused on comparing the asymptotic growth of symbolic powers of ideals in Noetherian commutative rings, relative to regular powers. Echoing Sarah Mayes [51, Introduction]: *The asymptotic behavior of collections of algebraic objects has been a fruitful research trend for several decades now, motivated by the philosophy that there is often a uniformity or stability achieved in the limit that is hidden when studying individual objects.* Working with Noetherian commutative rings, this dissertation falls under said trend; see Huneke [39] and the survey by Huneke – Raicu [43]. We investigate two collections of ideals, namely, the regular and symbolic powers of a fixed ideal, invoking geometric, combinatorial, or algebraic considerations.

To clarify, suppose we fix an ideal I in a *Noetherian* commutative ring R, say $I =$

$(f_1, \ldots, f_t)R$ consists of all R-linear combinations of $f_1, \ldots, f_t \in R$. For each positive integer N, the **N-th regular power** I^N of I consists of all R-linear combinations of N-fold products of $f_1, \ldots, f_t$. For instance, in the polynomial ring $R = \mathbb{R}[x, y]$ in two real variables, if $I = (x, y)R$, then $I^2 = (x^2, xy, y^2)R$. Meanwhile, the *symbolic powers* of I are a family of ideals $\{I^{(N)}\}$ in R indexed by positive (or nonnegative) integers N such that $I^N \subseteq I^{(N)}$ for all N.

We wind up to a general definition in stages. If P is a prime ideal in a Noetherian commutative ring R, its **N-th ($N \in \mathbb{Z}_{>0}$) symbolic power**

$$P^{(N)} = P^N R_P \cap R := \{r \in R \colon ur \in P^N \text{ for some } u \in R - P\} \supseteq P^N$$

is the unique P-primary component in any minimal primary decomposition of P^N. Indeed, $P^{(N)}$ is the smallest P-primary ideal containing P^N. More generally, if $I = P_1 \cap \cdots \cap P_c$ is any radical ideal of R, expressed as a finite intersection of its minimal primes, then the symbolic power $I^{(N)} := P_1^{(N)} \cap \cdots \cap P_c^{(N)}$.

Definition I.1 (Cf., [37, Introduction])**.** When I is any proper ideal of a Noetherian commutative ring R, and $\mathrm{Ass}_R(R/I)$ is the set of associated primes of I, its **N-th symbolic power** ($N \geq 0$ an integer) is the following ideal:

$$I^{(N)} := I^N W^{-1} R \cap R, \text{ where } W = R - \bigcup \{P \colon P \in \mathrm{Ass}_R(R/I)\}.$$

In particular,

$$I^{(N)} := \left\{ f \in R \colon sf \in I^N \text{ for some } s \notin \bigcup_{P \in \mathrm{Ass}_R(R/I)} P \right\}.$$

By convention, we set $I^{(0)} = I^0 = R$ to be the unit ideal.

Remark I.2. Note that $I^{(1)} = I$ since $\mathrm{zdiv}_R(R/I) = \bigcup_{P \in \mathrm{Ass}_R(R/I)} P$ is the set of all zerodivisors modulo I. It is always true that $I^m \subseteq I^{(m)} \subseteq I^{(r)}$ when $m \geq r \geq 1$. These ideal containments are strict in general, as we now illustrate.

Example I.3. Consider the prime ideal $P = (x, y)R$ in $R = \mathbb{C}[x, y, z]/(y^2 - xz)$ which is standard graded. Then

$$P^3 = (x^3, x^2y, x^2z, xyz)R \subsetneq P^{(3)} = (x^2, xy)R \subsetneq P^{(2)} = (x)R.$$

The third equality can be checked directly. The second equality holds by applying the main result of Chapter II, Lemma II.1, on divisor class groups with bounded torsion: the divisor class group of R is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The reader should not infer from the above example that computation of symbolic powers is easy. Indeed, symbolic powers are difficult to understand algebraically – it is generally hard to find generating sets for them. They are more readily intuitive from the perspective of classical algebraic geometry in characteristic zero, intimately tied to order of vanishing of functions via the Zariski – Nagata theorem. Working over an algebraically closed field $\mathbb{F}$ of characteristic zero, suppose that $S = \mathbb{F}[X]$ is an **affine regular ring**, that is, the coordinate ring of a smooth affine algebraic variety X. For $x \in X$, let $\mathfrak{m}_x \subseteq S$ be the maximal ideal of regular functions that vanish at x. If $f \in S$ is a nonzero regular function vanishing at x, there is a unique $e \in \mathbb{Z}_{>0}$ such that $f \in \mathfrak{m}_x^e \setminus \mathfrak{m}_x^{e+1}$; we say f **vanishes at** x **to order** e, and let $\operatorname{ord}_x(f) := e$ denote the **order of vanishing at** f **at** x. By convention, $\operatorname{ord}_x(0) = \infty$.

Given $S = \mathbb{F}[X]$ as above, we now fix a radical ideal I in S, and $Z = \operatorname{Zeros}(I) \subseteq X$ the zero locus of I in X. The **Zariski – Nagata theorem** says the symbolic- and differential powers of I coincide (see [17, Thm. 3.14], [18], and [59, Cor. 2.9]):

$$I^{(N)} = I^{\langle N \rangle} := \{f \in S \colon f \in \mathfrak{m}_x^N \text{ for all } x \in Z\} \tag{1.1}$$

consists of the regular functions on X that vanish to order at least N at all points of the zero locus of I. Working in a polynomial ring $S = \mathbb{F}[x_1, \ldots, x_n]$ in characteristic zero, with standard monomial notation $x^A = x_1^{a_1} \cdots x_n^{a_n}$ with $A \in (\mathbb{Z}_{\geq 0})^n$ of degree $|A| = \sum_{i=1}^n a_i$, it is often convenient to re-express identity (1.1) in terms of partial derivatives up to a fixed order: for all $N > 0$,

$$(1.2)\quad I^{(N)} = \left\{ F \in S \colon \text{all partials } \tfrac{\partial^{|A|} F}{\partial x^A} \in I \text{ for all } A \in (\mathbb{Z}_{\geq 0})^n \text{ with } |A| \leq N-1 \right\}.$$

Analogues of this Zariski – Nagata theorem have been uncovered recently, for polynomial rings containing a perfect field, and for smooth $\mathbb{Z}$-algebras in mixed characteristic – see [13, Prop. 2.14, Exer. 2.15] and [62] for precise statements.

Now we motivate the problem that guides the thesis: Problem I.9 below. First,

The Ideal Containment Problem I.4. *Given an ideal I in a Noetherian commutative ring R, study the set of pairs $\{(N, r) \in (\mathbb{Z}_{>0})^2 \colon I^{(N)} \subseteq I^r\}$. In particular, is it the case that for each integer $r > 0$, there is an integer $N > 0$ such that $I^{(N)} \subseteq I^r$?*

Problem I.4 is plainly hard, but Irena Swanson's theorem on linear equivalence of ideal topologies provides a more aesthetic, *linearized* rendering of Problem I.4:

Theorem I.5 (cf., [63, Main Theorem 3.3])**.** *Given an ideal I in a Noetherian commutative ring R, Problem I.4 has an affirmative answer for I if and only if there is an integer $M := M(I)$ such that $I^{(Mr)} \subseteq I^r$ for all integers $r > 0$.*

The Ideal Containment Problem I.6. *Under the setup of Problem I.4, when is there an integer $M > 0$ such that $I^{(Mr)} \subseteq I^r$ for all integers $r > 0$?*

Thus when Problem I.4 – or equivalently, Problem I.6 – has an affirmative answer, the topologies on R induced by the I-adic filtration and I-symbolic filtration, respectively, are said to be **(linearly) equivalent or cofinal**; see Eisenbud [17, Ch. 5].

On the one hand, Problem I.4 can have a negative answer by example.

Example I.7. Set $R = \frac{\mathbb{C}[x,y,z]}{(xy,xz,yz)}$ and $P = (x,y)R$. Then for all $N > 0$, $P^{(N)} = P \not\subseteq P^r$ for all integers $r \geq 2$. As R is standard graded, the minimal degree of a generator of P^N is N, so $P \not\subseteq P^r$ for any $r \geq 2$. By definition, $P^{(N)} \subseteq P$ for all $N > 0$. Conversely, $P \subseteq P^{(N)}$ for all $N > 0$ as $xz = yz = 0$ in R and $z \in R - P$.

On the other hand, Problem I.6 can have an affirmative answer for a family of ideals.

Example I.8 ([41, Prop. 2.4])**.** Problems I.4 and I.6 have an affirmative answer for:

1. All prime ideals in a complete local domain [41, Proof of Prop. 2.4].

2. All radical ideals in a normal domain of finite type over a field [41, Proof of Prop. 2.4, first three sentences]. In particular, this covers the coordinate ring of any normal affine algebraic variety in arbitrary characteristic.

This thesis is guided by pursuit of *uniform* affirmative answers for ideal families.

The Uniform Containment Problem I.9 ([13, Ch. 3])**.** *Given a Noetherian commutative ring R, and a family $\mathcal{F}$ of ideals in R satisfying Problem I.6, is there an integer $M := M(R, \mathcal{F}) > 0$ such that $I^{(Mr)} \subseteq I^r$ for all integers $r > 0$ and all $I \in \mathcal{F}$?*

When Problem I.9 has an affirmative answer, we say that R has **uniform symbolic topologies (USTP) on $\mathcal{F}$** [13, Ch. 3]. Here, *uniform* indicates that the exact same multiplier M works for all members of the specified family of ideals: in Problem I.6, the multiplier depends *a priori* on the ideal. Frequently, one expects $\mathcal{F}$ to consist of all prime ideals in R or all radical ideals in R. We call a USTP result *constructive* (or *effective*) if an explicit multiplier D is given, and *nonconstructive* otherwise.

Example I.10. A special case of Problem I.6 asks when $I^{(N)} = I^N$ for all N. The answer is affirmative in the following cases ([13, Ch. 4] points to some cases we omit):

1. I is a maximal ideal. For any prime ideal P, $P^{(N)} = P^N$ if and only if the ideal P^N is P-primary. It thus suffices to recall that any ideal whose radical is a maximal ideal is primary [4, Lecture 18].

2. If R is Artinian, then $I^{(N)} = I^N$ for all N and any radical ideal I. Indeed, all prime ideals are maximal, and one can show that (symbolic) powers of distinct maximal ideals are comaximal, so ideal intersections and ideal products coincide. The radical ideal I is a finite product of maximal ideals $I = m_1 \cdots m_\ell$, and so
$$I^{(N)} = \bigcap_{i=1}^{\ell} m_i^N = \prod_{i=1}^{\ell} m_i^N = I^N.$$

3. If R is a one-dimensional domain, then $I^{(N)} = I^N$ for all N and any radical ideal I. We may assume I is nonzero: the argument for case (2) is adaptable.

4. I is generated by a regular sequence in a Cohen-Macaulay ring; see Zariski-Samuel [74, Lem. 5, Appendix 6]. The same can be said when I is generated locally by a regular sequence. In particular, this holds when an ideal $I \subseteq \mathbb{F}[V]$ defines a smooth subvariety of a smooth affine algebraic variety V.

5. If R is a two-dimensional unique factorization domain (UFD), then $Q^{(N)} = Q^N$ for all $N > 0$ and all radical ideals Q in R. The key observation to make is that given comaximal ideals I, J in any Noetherian commutative ring R, the symbolic powers $I^{(a)}$ and $J^{(b)}$ are comaximal for all $a, b \in \mathbb{Z}_{\geq 0}$, and hence $I^{(a)} \cap J^{(b)} = I^{(a)} J^{(b)}$; to piece this together, one can follow the exercises on comaximal ideals preceding the ring-theoretic Chinese Remainder Theorem in [4, Lecture 1]. We note that $Q^{(N)} = Q^N$ for all N when all associated primes of Q have the same height. The case $Q = (0)$ is easy, the height two case follows as in case (2) above. The case where all associated primes of Q have height one

follows from Corollary II.13 of Lemma II.1. If $Q = Q_1 \cap Q_2$ in terms of radical ideals, where all associated primes of Q_i have height i, one can use comaximality to prove that $Q^{(N)} = Q_1^{(N)} Q_2^{(N)} = (Q_1 Q_2)^N = Q^N$ for all N.

6. When a squarefree monomial ideal arises as the edge ideal $I = I(G)$ of a simple finite graph G, $I^{(N)} = I^N$ for all N if and only if G is bipartite [23].

Remark I.11. We now indicate two senses in which an affirmative answer to Problem I.9 could be uniform for a class of rings relative to a specified family of ideals. For instance, we can talk about the class of all excellent regular rings, or the class of Cohen-Macaulay, normal domains of finite type over $\mathbb{C}$. We could further curate the class by appending one or several common numerical invariants. For instance, we can talk about all excellent regular rings of Krull dimension 3264, or all Cohen-Macaulay, normal domains of finite type over $\mathbb{C}$ with Krull dimension two and Hilbert-Samuel multiplicity two relative to an isolated singularity. Working with a class of rings curated in said way which is an affirmative case of Problem I.9, relative to the same family $\mathcal{F}$ of ideals in each ring in the class, e.g., all prime ideals:

1. The first, strong sense of uniformity holds – a **USTP panacea (or class solidarity)** relative to the ideal family $\mathcal{F}$ and the specified invariants – if the exact same multiplier M works for every ring of the class, and M depends only on select numerical invariants among those indicated, and not on an individual ring's designated class or structure in some more nuanced, intimate way.

2. The second category of uniformity is when a USTP panacea is plainly unknown, or decisively unattainable in deference to a clear and present *obstruction* to such a multiplier M existing for all rings in the class simultaneously.

In particular, all finite-dimensional excellent regular rings enjoy a USTP panacea relative to radical ideals and the Krull dimension – see Theorem I.18. In stark contrast, almost every class of non-regular rings known to satisfy some version of Problem I.9 in the literature currently falls short of a USTP panacea; see Corollary I.27 for a noteworthy exception. This observation on non-regular rings inspires Question I.34 below, which we confront in Chapter II of the thesis.

Before proceeding, we record a key concept going forward.

Definition I.12. We say that a sequence $\{J_i\}_{i\in\mathbb{Z}_{>0}}$ of ideals in a Noetherian commutative ring R is **graded** if $J_a \cdot J_b \subseteq J_{a+b}$ for all $a, b \geq 1$.

If I is an ideal in a Noetherian commutative ring R, its regular powers $\{I^N\}$ and its symbolic powers $\{I^{(N)}\}$ each form a graded sequence of ideals. As discussed in papers such as [13, 14, 16, 37, 40, 41, 47, 50, 51], we articulate a heuristic:

Observation I.13. *Asymptotic stability or uniformity in the comparative anatomy and behaviors of two graded sequences, like $\{I^N\}$ and $\{I^{(N)}\}$, may be influenced by – and may carry – algebraic, combinatorial, or geometric information. The same may often be said for auxiliary invariant objects associated to a graded sequence of ideals.*

The cases recorded under Example I.10 are already indicative of this heuristic.

1.2.1 From Ein – Lazarsfeld – Smith to Non-Regular Rings and Finite Extensions

We now illustrate Observation I.13 with weightier affirmative results on Problem I.9. The most celebrated answer in this vein is the Ein – Lazarsfeld – Smith Theorem. We now rehearse a few iterations in formulating this vexillary result.

We call an ideal I in a Noetherian commutative ring R **unmixed** if it has no embedded primes, and moreover **pure of height** h if all associated primes have height h. Ein, Lazarsfeld, and Smith proved that for any unmixed ideal I in a $\mathbb{C}$-affine regular ring $\mathbb{C}[V]$ – the coordinate ring of a smooth affine variety V over $\mathbb{C}$,

$$I^{(hr)} \subseteq I^r \text{ for all } r > 0, \tag{1.3}$$

where h is the maximal height of an associated prime of I, cf., [16, Thm. 2.2]. To prove this, they developed an asymptotic multiplier ideal theory for graded systems of ideals in R, availing tools of the trade in modern birational algebraic geometry such as log resolution morphisms and vanishing theorems. They leverage properties of this nascent theory – such as subadditivity – relative to the graded systems $\{I^N\}$ and $\{I^{(N)}\}$ in case I is radical, extending to unmixed ideals as a brief follow-up discussion. See also Hara [29]. This groundbreaking result was quite striking back in 2000 (see [16, Introduction]): the simplicity and stability of the linear bound in (1.3) ran contrary to what algebraic geometers would expect heuristically as the singularities of the zero locus of I become nastier.

In 2001, Hochster and Huneke deduced (1.3) for all ideals I in a regular ring R containing a field [37, Thm. 1.1], the first conclusion stated in the following theorem:

Theorem I.14 ([37, Thm. 1.1])**.** *Let R be a Noetherian ring containing a field. Let I be an ideal of R and h be the largest height of an associated prime of I.*

1. *If R is regular, then $I^{(hr+kr)} \subseteq (I^{(k+1)})^r$ for all integers $k \geq 0$ and $r \geq 1$.*
2. *If I has finite projective dimension, then $I^{(hr)} \subseteq (I^r)^*$ for all $r \geq 1$, where J^* is the tight closure of an ideal J in R.*

They first prove their result in positive characteristic using tight closure methods and recover the characteristic zero version by reduction to positive characteristic.

Remark I.15. In Theorem I.14(2), suppose the ring R is of prime characteristic and **weakly F-regular**, meaning $J^* = J$ for all ideals J in R. Then we get (1.3) for all ideals I in R of finite projective dimension. It follows that finite-dimensional weakly F-regular rings enjoy USTP solidarity as in Remark I.11(1), relative to ideals of finite projective dimension and Krull dimension.

Remark I.16. The papers Hara [29], Hochster – Huneke [38], Takagi – Yoshida [65], and Johnson [44] follow in the footsteps of [37], extending what is known about the behavior of symbolic powers in Noetherian regular rings containing a field.

In 2017, Ma and Schwede deduced (1.3) for all radical ideals I in any excellent regular ring R even in mixed characteristic:

Theorem I.17 ([48, Thm. 7.4])**.** *If R is any Noetherian regular ring with reduced fibers (e.g., any excellent regular ring), I is a radical ideal in R, and h is the maximal height of an associated prime of I, then $I^{(hr)} \subseteq I^r$ for all $r > 0$.*

As summarized in [48, Introduction], Ma and Schwede emulate the line of attack first appearing in [16] and in [29], which we noted earlier. In passing, we record the following version of the improved Ein – Lazarsfeld – Smith Theorem, a vexillary USTP panacea:

Theorem I.18 ([16, Thm. 2.2], [37, Thm. 1.1], [48, Thm. 7.4])**.** *If R is a d-dimensional excellent regular ring and $D = \max\{1, d-1\}$, then $Q^{(Dr)} \subseteq Q^r$ for all radical ideals $Q \subseteq R$ and all $r > 0$.*

Remark I.19. The improved Ein – Lazarsfeld – Smith Theorem I.18 gives a bound D depending only on the Krull dimension. Thus the same D works for all excellent regular rings of dimension 3264, say, namely $D = 3263$. It turns out that it is too much to hope for in general that a uniform bound depending only on the Krull dimension might hold. We show in Chapter II, for example, that there are nice classes of two-dimensional normal domains which require an arbitrarily large multiplier D.

The Ein – Lazarsfeld – Smith theorem, as extended to Theorem I.14 by Hochster and Huneke, inspired two directions of follow-up work. One direction as spearheaded by Huneke – Katz – Validashti [40, 41] considers non-regular rings and finite extensions of domains. The other direction as spearheaded by Brian Harbourne concerns when given homogeneous ideals in a standard graded polynomial ring we can improve the containments $I^{(Dr)} \subseteq I^r$ via a linear function whose growth rate is slower. We consider the first direction for now, and backtrack to discuss the second direction after posing Questions I.34 and I.35 below.

We now cover six USTP results by other mathematicians, representing the state-of-the-art for non-regular rings. In 2009, Huneke – Katz – Validashti [40] deduced the following USTP theorem via non-constructive methods for a large class of isolated singularities (see also [40, Thm. 1.2, Thm. 3.5]).

Theorem I.20 ([40, Cor. 3.10]). *Let R be an equicharacteristic Noetherian local domain such that R is an isolated singularity. Assume that either:*

1. *The ring R is essentially of finite type over a field of characteristic zero; or*

2. *The ring R has positive characteristic, is F-finite and analytically irreducible.*

Then there exists an $E := E(R) \geq 1$ depending on R such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all prime ideals P in R.

The above theorem is a far cry from a USTP panacea – indeed, for good reason.

Remark I.21. Suppose that R is the coordinate ring of an affine variety over any perfect field $\mathbb{F}$, whose singular locus is zero dimensional. When paired with Theorem I.14, Theorem I.20 yields a uniform bound E for all primes in R. In particular, this covers rings such as $\mathbb{F}[x, y, z]/(y^2 - xz)$, $\mathbb{F}[x, y, z, w]/(xy - zw)$, and more generally when R corresponds to the affine cone over any smooth projective variety.

In 2013, Hochster's doctoral student Ajinkya A. More obtained the following result, where the USTP property *descends* along a finite extension of domains.

Theorem I.22 ([53, Thm. 4.4, Cor. 4.5])**.** *Suppose $R \subseteq S$ is a finite extension of equicharacteristic normal domains such that:* (1) *S is a regular ring generated as an R-module by n elements, and $n!$ is invertible in S; and* (2) *R is either essentially of finite type over an excellent Noetherian local ring (or over $\mathbb{Z}$), or is of prime characteristic $p > 0$ and F-finite. Then there exists an $E := E(R) \geq 1$ depending on the ring R such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all prime ideals P in R.*

The next theorem, due to Huneke – Katz – Validashti [41] in 2015, improves upon More's USTP descent theorem. But first, in step with [13, Ch. 3] we record a

Hypothesis I.23 ([13, Hypothesis 3.1])**.** *We consider a Noetherian reduced ring A satisfying one of the following conditions:*

1. *A is essentially of finite type over $\mathbb{Z}$ or over an excellent ring containing a field;*
2. *A is of positive characteristic and is F-finite;*
3. *A is an excellent Noetherian ring which is the homomorphic image of a regular ring of finite Krull dimension such that for all primes P, A/P has a resolution of singularities obtained by blowing up an ideal.*

Theorem I.24 ([13, Thm. 3.25], [41, Cor. 3.4])**.** *Let $R \subseteq S$ be a finite extension of domains, with R normal, such that both R and S satisfy Hypothesis I.23. If S has USTP on primes, then so does R.*

Remark I.25. As noted in [41, Introduction], Huneke – Katz – Validashti ask whether a complete local domain has uniform symbolic topologies on primes. Any complete local domain is a finite extension of a regular local ring, and thus they were actually interested in the extent to which the USTP property on prime ideals *ascends* along finite extensions of domains. In particular, an ascent version of the previous theorem would imply that complete local domains have USTP on primes. However, USTP ascent is trickier; see [41, Sec. 4] for the only known ascent results.

In 2017, Dao – De Stefani – Grifo – Huneke – Núñez-Betancourt proved the following two results, special cases of Theorem I.24 which give effective bounds.

Theorem I.26 ([13, Thm. 3.29])**.** *Fix a polynomial ring $S = K[x_1, \ldots, x_s]$ over a field K, $R \subseteq S$ a direct summand. Suppose that S is finitely generated as an R-module. Let $P \subseteq R$ be a prime ideal of height h. Let $e = [S : R]$, i.e., the degree of the fraction field of S over the fraction field of R. If $e!$ is invertible in R, then*

$$P^{(her)} \subseteq P^r \textit{ for all } r \geq 1.$$

Corollary I.27 ([13, Cor. 3.30])**.** *With notation as in Theorem I.26, fix G a finite group that acts on S. Let $R = S^G$ be the ring of invariants. Let $P \subseteq R$ be a prime ideal of height h. Set $e = [S : R] = \#G$. If $e!$ is invertible in K, then*

$$P^{(her)} \subseteq P^r \textit{ for all } r \geq 1.$$

In 2018 joint work, Carvajal-Rojas – Smolkin [11] have specified a class of F-regular rings called **diagonally F-regular rings** that are engineered to satisfy USTP emulating the proof strategy of [16] and Hara [29]. They deduce:

Theorem I.28 (Carvajal-Rojas – Smolkin [11, cf. Thm. 4.1])**.** *Fix an arbitrary field $\mathbb{F}$ of positive characteristic. If R is a diagonally F-regular $\mathbb{F}$-algebra essentially of finite type, then R satisfies USTP on primes using the Ein – Lazarsfeld – Smith multiplier $D = \max\{1, \dim R - 1\}$.*

Theorem I.29 (Carvajal-Rojas – Smolkin [11, cf. Thm. 5.6, Cor. 5.7])**.** *Fix a perfect field $\mathbb{F}$ of positive characteristic. Given positive integers r and s, the affine cone over $\mathbb{P}^r_{\mathbb{F}} \times \mathbb{P}^s_{\mathbb{F}}$ satisfies USTP on primes using the Ein – Lazarsfeld – Smith multiplier $D = r + s$.*

Theorem I.30 (Carvajal-Rojas – Smolkin [11, cf. Prop. 5.5])**.** *Suppose R and S are diagonally F-regular algebras over an arbitrary field $\mathbb{F}$ of positive characteristic. Then $R \otimes_{\mathbb{F}} S$ is also diagonally F-regular.*

Remark I.31*.* As a consequence of Theorem I.29 we get that the hypersurface ring $R = \mathbb{F}[\mathbb{P}^1 \times \mathbb{P}^1] = \mathbb{F}[x, y, z, w]/(xy - zw)$ has USTP on primes with $D = 2$. This addresses a decade's worth of fixation in the wake of Theorem I.20 – and frustration.

Remark I.32*.* As [11, Sec. 6] and [26, Sec. 5] illustrate, one heuristic advantage of working in positive characteristic is that a result one deduces frequently admits a nigh-verbatim analogue in characteristic zero, and the latter follows immediately through standard techniques grouped under *reduction to positive characteristic*. In particular, Theorem I.28 admits a characteristic zero analogue for algebras **of dense diagonally F-regular type** [11, Thm. 6.1].

Remark I.33*.* More recently, Page – Smolkin – Tucker [56] build upon the work in [11] in the setting of Hibi Rings, a class of normal toric rings associated to finite posets. More precisely, they determine a subclass of Hibi Rings that in fact have diagonally F-regular singularities relative to a stipulation on the underlying poset structure – a subclass under which the Segre products in Theorem I.29 fall as members vacuously.

We have covered six USTP results on non-regular rings. Prior to this thesis work, Remarks I.21 and I.31 are indicative that almost all examples and constructions applying these six results in the literature pertain to rings with isolated singularities.

In this thesis, we offer results and examples that represent first steps towards addressing the following questions. The first question is a follow-up to Remarks I.11 and I.19. The second is addressed both below and recently by [11, Cor. 5.9] and [56].

Question I.34. *Suppose that R belongs to a class of non-regular rings of dimension d with uniform symbolic topologies on primes. Is there a bound $E := E(d)$ depending only on d such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all primes $P \subseteq R$?*

Question I.35. *What can be said in the direction of witnessing uniform symbolic topologies in rings with non-isolated singularities?*

1.2.2 Harbourne – Huneke Symbolic Indices

Referring back to the Ein – Lazarsfeld – Smith Theorem, we fix a polynomial algebra $R = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, \ldots, x_N]$ standard graded over a field $\mathbb{F}$. Then

$$I^{(Nr)} \subseteq I^r$$

for all $r > 0$ and all graded ideals $I \subseteq R$. Bocci – Harbourne showed that these containments cannot be improved asymptotically, replacing N in the symbolic power with a uniform $0 < c < N$ that works for all $r \gg 0$ ([9, Thm. 2.4.3], [64, Prop. 3.2]). In this sense, the multiplier N is understood to be asymptotically optimal.

When $N = r = 2$, we obtain the containment $I^{(4)} \subseteq I^2$ for all graded ideals I in $\mathbb{F}[\mathbb{P}^2]$. This observation spurred Craig Huneke to ask if the containment improves to $I^{(3)} \subseteq I^2$ in case I is a radical ideal defining a finite set of points in $\mathbb{P}^2$. More generally, one could ask the following

Question I.36 (Dropping the symbolic power by a constant)**.** *When $N \geq 2$, is $I^{(Nr-c)} \subseteq I^r$ for all $r > 0$, all graded $I \subseteq \mathbb{F}[\mathbb{P}^N]$, and some uniform constant $c > 0$?*

In particular, Harbourne asked whether $c = N - 1$ will work:

Question I.37 (Harbourne [6, Conj. 8.4.2], [30, Conj. 4.1])**.** *When $N \geq 2$, is*

$$I^{(N(r-1)+1)} \subseteq I^r$$

for all $r > 0$ and all graded $I \subseteq \mathbb{F}[\mathbb{P}^N]$?

An affirmative answer to the latter question would imply an affirmative answer to Huneke's question on the $I^{(3)} \subseteq I^2$ containment. We call the index $N(r-1)+1$ a **Harbourne – Huneke bound**. All known affirmative results in the direction of uncovering these bounds are consolidated in the ideal containment problem survey preprint [64, Thm. 3.8] by Szemberg – Szpond. In particular, over any field $\mathbb{F}$,

$$I^{(N(r-1)+1)} \subseteq I^r$$

for all $r > 0$ and all monomial ideals $I \subseteq \mathbb{F}[\mathbb{P}^N]$ [6, Ex. 8.4.5]. In the case of radical monomial ideals, recent work of Grífo – Huneke [26] generalizes this result along with work by Takagi – Yoshida [65]. See also Grífo [27] for exciting recent developments.

However, negative results are known in both zero and odd positive characteristic, and we invite the reader to look into [2, 3, 15, 31, 64] for details. In passing, we record the following example, which in fact answers Huneke's question negatively.

Example I.38 (Dumnicki – Szemberg – Tutaj-Gasińska [15, Thm. 2.2])**.** The height two radical ideal below satisfies $I^{(3)} \not\subseteq I^2$:

$$I = (x_0(x_1^3 - x_2^3), x_1(x_0^3 - x_2^3), x_2(x_0^3 - x_1^3)) \subseteq R = \mathbb{C}[\mathbb{P}^2] = \mathbb{C}[x_0, x_1, x_2].$$

1.3 Thesis Outline: Main Results, Applications, General Conventions

This thesis consolidates results from four papers ([68, 69, 70, 71]), results that support Huneke's philosophy in [39] that there are uniform bounds lurking in Noetherian rings. First, we deduce in Chapter II the following result:

Theorem I.39 (Cf., Theorem II.1)**.** *Let R be a Noetherian normal domain whose global divisor class group* $\mathrm{Cl}(R) := \mathrm{Cl}(\mathrm{Spec}(R))$ *is annihilated by an integer $D > 0$. Then*

$$\mathfrak{q}^{(D(r-1)+s)} = (\mathfrak{q}^{(D)})^{r-1}\mathfrak{q}^{(s)}, \text{ and } \mathfrak{q}^{(D(r-1)+1)} \subseteq \mathfrak{q}^r,$$

for all prime ideals $\mathfrak{q} \subseteq R$ of height one, all $r > 0$, and all $0 \leq s < D$.

At the end of Chapter II, we apply Theorem I.39 to conclude that the answer to Question I.34 is **no**. Indeed, all rational surface singularities have Krull dimension two and finite divisor class group. Applying Theorem I.39, all rational surface singularities have uniform symbolic topologies on primes. One might then ask whether there exists a common bound B in terms of the Krull dimension – a USTP panacea. However, we show that the optimal bound B for obtaining the containments $P^{(Br)} \subseteq P^r$ does depend on the ring, and in particular, can be arbitrarily large; see Remark II.17. In fact, we identify a sharp effective B by inspecting the divisor class group – this is the content of the final column of Table 2.1 in Subsection 2.4.

As for Question I.35, I provide *quid pro quo* USTP results that are essentially constructive: I record a formula for D at the cost of focusing on a prescribed family of prime ideals inside of a class of rings with a prescribed structure. Theorem I.39 illustrates this theme, pairing well with key results in each of Chapters III and IV.

In Chapter III, we focus on the coordinate rings of normal affine toric varieties. Such algebras are combinatorially-defined, finitely generated, Cohen-Macaulay, normal, and have rational singularities when working in characteristic zero. To illustrate the main results of Chapter III, we record Theorems I.40 and I.41 below.

Adopting notation in toric algebra [12], we fix a convex rational polyhedral cone $C \subseteq \mathbb{R}^n$ which is the $\mathbb{R}_{\geq 0}$-linear span of a finite set $G \subseteq \mathbb{Z}^n$; without loss of generality, we may assume that each $v \in G$ is primitive, so the coordinates of v have gcd one.

The convex polyhedral cone **dual to** C is also rational

$$C^\vee := \{w \in \mathbb{R}^n \colon w \bullet v \geq 0 \text{ for all } v \in C\} \text{ (where } \bullet \text{ denotes the dot product).}$$

We further assume that C is both full and pointed so, respectively, the $\mathbb{R}$-span of C is $\mathbb{R}^n$ and C contains no line through the origin. In this case, there are uniquely-determined sets of irredundant primitive generators for C and for $C^\vee$. The lattice points of $C^\vee$ form a semigroup under vector addition, $S_C := C^\vee \cap \mathbb{Z}^n$, in which every vector in S_C is a $\mathbb{Z}_{\geq 0}$-linear combination over some finite subset $\mathcal{A} \subseteq S_C$. Working over an arbitrary field $\mathbb{F}$, the semigroup ring $R_\mathbb{F} = \mathbb{F}[C^\vee \cap \mathbb{Z}^n]$ is called the **toric $\mathbb{F}$-algebra associated to** C; it is always a normal domain of finite type over $\mathbb{F}$ [12, Thm. 1.3.5]. The algebra $R_\mathbb{F}$ has an $\mathbb{F}$-basis of monomials $\{\chi^M \colon M \in C^\vee \cap \mathbb{Z}^n\}$, and an ideal of $R_\mathbb{F}$ is **monomial (or torus-invariant)** if it is generated by monomials.

The following results originally appeared as [70, Thm. 1.1, Thm. 1.2]. First,

Theorem I.40 (Cf., Theorem III.1)**.** *Suppose $C \subseteq \mathbb{R}^n$ is a full-dimensional pointed polyhedral cone as above. Set $D := \max_{w \in \mathcal{B}} (w \bullet v_G) \in \mathbb{Z}_{>0}$, where $\mathcal{B}$ is a generating set for the semigroup $C^\vee \cap \mathbb{Z}^n$ and $v_G \in \mathbb{Z}^n$ is the sum of any (finite) set G of vectors in $\mathbb{Z}^n$ generating C. Then*

$$P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^r,$$

for all $r > 0$ and all monomial prime ideals P in the toric ring $R_\mathbb{F} = \mathbb{F}[C^\vee \cap \mathbb{Z}^n]$.

We get the best result in Theorem I.40 by taking G to consist of the unique set of primitive generators for C, in which case we write v_C in place of v_G; see Section 3.3.

The next result covers select non-monomial primes in a **simplicial** toric algebra.

Theorem I.41 (Cf., Theorem III.2)**.** *With notation as in Theorem I.40, assume moreover that the divisor class group $\mathrm{Cl}(R_\mathbb{F})$ is finite; that is, $R_\mathbb{F}$ is a simplical toric* *algebra. Set* $U := \operatorname{lcm}\{\max_{w \in \mathcal{B}}(w \bullet v_C), \#\operatorname{Cl}(R_{\mathbb{F}})\} \in \mathbb{Z}_{>0}$*, where* $\mathcal{B}$ *and* v_C *are as in Theorem I.40. Then* $P^{(U(r-1)+1)} \subseteq P^r$ *for all* $r > 0$*, all monomial primes in* $R_{\mathbb{F}}$*, and all height one primes in* $R_{\mathbb{F}}$*.*

We draw connections between the multiplier U and the so-called F-signature of $R_{\mathbb{F}}$ in Section 3.4. We then discuss the extent to which the multipliers D and U in Theorems I.40 and I.41 are sharp, deferring to examples consolidated in Section 3.5.

Along the way to deducing Theorems I.40 and I.41, we confronted various issues related to studying Problem I.6 even in the non-toric case. One such issue concerns how the USTP property interacts with faithfully flat ring extensions, analogous to the Huneke – Katz – Validashti USTP Descent Theorem I.24.

Remark I.42. The following is presented in Grifo – Huneke [26, Rem. 6]. Suppose that $\varphi\colon A \to B$ is a faithfully flat map of Noetherian rings. Suppose that P is a prime ideal in A, and that the ideal $Q = PB := \langle \varphi(P) \rangle B$. Then a containment

$$Q^{(a)} \subseteq Q^b \Longrightarrow P^{(a)} \subseteq P^b.$$

By flatness, $Q \cap A = P$, and nonzerodivisors in A/P remain nonzerodivisors in $B/Q = A/P \otimes_A B$. Thus $P^{(a)} \subseteq Q^{(a)} \cap A \subseteq Q^b \cap A = P^b$. So again we see that in efforts to preserve uniform symbolic topologies along a ring extension – whether finite or faithfully flat, *going down is easier than going up*; Melvin Hochster usually indicates the opposite when discussing the Going Up and Going Down Theorems for Integral Extensions in the Math 614 Commutative Algebra class at UM-Ann Arbor.

We deduce a result, Proposition IV.6, which implies that if the ideal Q in Remark I.42 is prime, then the implication above improves to an if-and-only-if statement. This proves to be handy in deducing the main result of Chapter IV, the following theorem (Cf., Theorem IV.1).

Theorem I.43. *Let $\mathbb{F}$ be an algebraically closed field. Let $R_1, \ldots, R_n$ $(n \geq 2)$ be domains, finitely generated as $\mathbb{F}$-algebras. Suppose that for each $1 \leq i \leq n$, there exists a positive integer D_i such that for all prime ideals P in R_i, either:*

1. *$P^{(D_i r)} \subseteq P^r$ for all $r > 0$ and for all i; or, even stronger,*
2. *$P^{(D_i(r-1)+1)} \subseteq P^r$ for all $r > 0$ and for all i.*

Fix any n prime ideals P_i in R_i, and consider the expanded ideals $P_i' = P_i T$ in the domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$, along with their sum $Q = \sum_{i=1}^n P_i'$ in T. Then:

(a) When (1) holds, $Q^{(Dr)} \subseteq Q^r$ for all $r > 0$, where $D = D_1 + \cdots + D_n$.

(b) When (2) holds, this improves to $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where instead $D = \max\{D_1, \ldots, D_n\}$.

The proof of this theorem leverages a multinomial formula for the symbolic powers of the prime ideal Q in T (Theorem IV.11). Hà – Nguyen – Trung – Trung announced in 2017 a binomial theorem for symbolic powers of ideal sums [28, Thm. 3.4], generalizing [8, Thm. 7.8], where one takes two arbitrary ideals $I \subseteq A, J \subseteq B$ inside of two Noetherian commutative algebras over a common field k, whose tensor product $R = A \otimes_k B$ is Noetherian; see Remark IV.17 for details. However, we give a proof of the Multinomial Theorem IV.11 which is more elementary and self-contained.

What is the intended mode of application for Theorem I.43? Referring back to Remark I.21, note that the class of rings R to which Theorems I.20, I.22, I.24, and I.26 apply is large. Applying Theorem IV.1 to any collection of two or more rings under Remark I.21, Remark IV.16 says that we can create an infinite set of primes – namely, the set $\mathcal{Q}_{ED}(T)$ below – as a vantage point for data suggestive of uniform symbolic topologies in the corresponding tensor product domain T. Pointedly, since the domain we create has non-isolated singularities, prior to [11, Cor. 5.9] **no** theorem in the literature affirmed that the domain has uniform symbolic topologies on all primes. We illustrate how these matters occur together in Example I.44 below.

But first, we fix an algebraically closed field $\mathbb{F}$. If R is a domain, finitely generated as an $\mathbb{F}$-algebra, we use the tensor power notation $T = R^{\otimes N} = (\bigotimes_{\mathbb{F}})_{i=1}^{N} R_i$ to denote the domain obtained by tensoring together N copies of R over $\mathbb{F}$. Leasing notation from Remark IV.16, we record the following set of *multinomial* prime ideals in T:

$$\mathcal{Q}_{ED}(T) := \left\{ Q = \sum_{i=1}^{N} P_i T \in \operatorname{Spec}(T) \colon \text{ each } P_i \in \operatorname{Spec}(R_i) \right\}.$$

Example I.44. *Fix an algebraically closed field $\mathbb{F}$. Given integers a and d both at least two, consider an affine hypersurface domain $R = \mathbb{F}[z_1, \ldots, z_a]/(F_d)$ where F_d is an irreducible homogeneous polynomial of degree d, with isolated singularity at the origin. Consider the varieties $V_R = \operatorname{Spec}(R) \subseteq \mathbb{F}^a$ and $V = \operatorname{Spec}(T) \subseteq \mathbb{F}^{aN}$ where*

$$T = R^{\otimes N} = \frac{\mathbb{F}[z_{i,1}, \ldots, z_{i,a} \colon 1 \leq i \leq N]}{(F_d(z_{i,1}, \ldots, z_{i,a}) \colon 1 \leq i \leq N)}.$$

Per Remark I.21, Theorem IV.1 implies that $Q^{(NE\cdot r)} \subseteq Q^r$ for all $r > 0$ and all primes $Q \in \mathcal{Q}_{ED}(T)$. In terms of n-factor Cartesian products, the singular locus

$$\operatorname{Sing}(V) = (\{0\} \times V_R \times \cdots \times V_R) \cup (V_R \times \{0\} \times V_R \times \cdots \times V_R) \cup \cdots \cup (V_R \times \cdots \times V_R \times \{0\})$$

is equidimensional of dimension $(a-1)(N-1)$. In particular, while T is not an isolated singularity when $N \geq 2$, the set $\mathcal{Q}_{ED}(T)$ is infinite by Remark IV.16 and provides a vantage point for witnessing uniform linear multipliers.

Chapter V closes the thesis, consolidating a few prospects for follow-up work. In particular, we focus on some immediate – and frustratingly open – questions that linger in the wake of our results in the toric setting. We close by mentioning one avenue for formal inquiry into extending our results on uniform symbolic topologies to cover all prime ideals in a simplicial toric algebra. This concludes our outline.

Before proceeding to Chapter II, we record ongoing conventions for the thesis, consolidating those from the four papers ([68, 69, 70, 71]) in a list below.

Conventions I.45. *All rings considered in this thesis are commutative with identity. Except when stated otherwise, all of our rings R will be Noetherian domains.*

1. *In Chapters II and III, R will typically be a normal domain as well. In these two chapters, when we say R is* ($\mathbb{N}$-)***graded***, *we mean that $R = \bigoplus_{d \geq 0} R_d$ is graded by the set $\mathbb{N} = \mathbb{Z}_{\geq 0}$ of nonnegative integers, with R_0 being a field, and $\mathfrak{m} = \bigoplus_{d>0} R_d$ the unique homogeneous maximal ideal.*

2. *In Chapter IV, our rings will typically be* ***affine*** *$\mathbb{F}$-algebras, that is, of finite type over a fixed field $\mathbb{F}$ of arbitrary characteristic. By* ***algebraic variety over*** *$\mathbb{F}$, we will mean an integral scheme of finite type over the field $\mathbb{F}$.*

3. *Throughout, $\mathbb{F}$ denotes by default an arbitrary ground field of arbitrary characteristic.*

CHAPTER II

Sharp Bounds for Domains with Finite Divisor Class Groups

This chapter consolidates the relevant material in our papers [68] and [69]. The following theorem, the main result of this chapter, allows us to produce effective bounds for uniform symbolic topologies:

Theorem II.1. *Let R be a Noetherian normal domain whose global divisor class group* $\mathrm{Cl}(R) := \mathrm{Cl}(\mathrm{Spec}(R))$ *is annihilated by an integer $D > 0$. Then*

$$\mathfrak{q}^{(D(r-1)+s)} = (\mathfrak{q}^{(D)})^{r-1}\mathfrak{q}^{(s)}, \textit{ and } \mathfrak{q}^{(D(r-1)+1)} \subseteq \mathfrak{q}^{r},$$

for all ideals $\mathfrak{q} \subseteq R$ of pure height one, all $r > 0$, and all $0 \leq s < D$.

In particular, when R in Theorem II.1 has Krull dimension two, it satisfies the uniform symbolic topology property on prime ideals, since height one primes are the only nontrivial class to check in this instance.

In Section 2.4, our endgame is to apply Theorem II.1 to answer Question I.34 in the negative: even if a class of d-dimensional non-regular rings R satisfies uniform symbolic topologies on primes, there need not be a multiplier $M := M(d)$ depending only on the Krull dimension d such that $P^{(Mr)} \subseteq P^r$ for all $r > 0$ and all primes P in R. We show that the answer is no, even restricted to rationally singular surfaces, or better still, rings of invariants of finite groups acting on $\mathbb{C}^2$.

The reader is invited to revisit the list of Conventions I.45 as appropriate.

2.1 Preliminaries on Divisor Class Groups

Our main references are Fossum [20], Hartshorne [32, II.6], Hochster [36], and Matsumura [49, Ch. 11]. However, we opt to state mathematical definitions and results from these sources only for Noetherian normal domains, rather than for Krull domains in general as is done in [20] and [49].

Throughout, R will denote a Noetherian normal domain. Let $\mathcal{P}$ denote the set of height-one primes in R. As noted in Matsumura's chapter on Krull rings ([49, Corollary of Thm. 12.3]), when $f \in R$ is a nonzero nonunit, and ν_P is the discrete valuation on the DVR R_P (for $P \in \mathcal{P}$), we have a unique primary decomposition

$$(f)R = \bigcap_{P \in \mathcal{P}} P^{(N_P)}, \text{ where } N_P := \nu_P(f) = 0 \text{ for all but finitely many } P.$$

We define the **Weil divisor of** f to be $\operatorname{div}(f) := \sum_{P \in \mathcal{P}} N_P \cdot P$. Additionally, we define the *trivial* effective Weil divisor $\operatorname{div}(\langle 1 \rangle R) = \operatorname{div}(R) = [R] := 0$ of the unit ideal to have identically zero $\mathbb{Z}$-coefficients.

Definition II.2. The **divisor class group** of a Noetherian normal domain R,

$$\operatorname{Cl}(R) = \operatorname{Cl}(\operatorname{Spec}(R)),$$

is the free abelian group on the set $\mathcal{P}$ of height one prime ideals of R modulo relations

$$a_1 P_1 + \cdots + a_r P_r = 0,$$

whenever the ideal $P_1^{(a_1)} \cap \ldots \cap P_r^{(a_r)}$ is principal.

In particular, $\operatorname{Cl}(R)$ is trivial if and only if R is a UFD [32, II.6]. Both conditions mean that every height one prime ideal in R is principal. By the next lemma, this latter assertion is equivalent to all symbolic powers of a height one prime ideal $P \subseteq R$ being principal, so $P^{(a)} = P^a$ for all $a > 0$ and all height one primes P in a UFD.

Lemma II.3. *If S is an arbitrary Noetherian ring, and $P = (f)$ is prime with f being a nonzerodivisor in S, then $P^{(a)} = (f^a)$ for all $a > 0$.*

Proof. Induce on a with base case $a = 1$. Assuming the statement for some $a \geq 1$, take $x \in P^{(a+1)}$. Since $x \in P^{(a+1)} \subseteq P^{(a)} = (f^a)$, $x = f^a y$ for some y. By the choice of x, there is $s \notin P$ with $sx = sf^a y \in P^{a+1} = (f^{a+1})$. Since f is a nonzerodivisor, $sy \in (f) = P$, which is prime. Therefore, $y \in P = (f)$, and $x \in (f^{a+1})$. □

We now record three theorems without formal proof, consolidating some results from Ch.II, Sections 7, 8, and 10 of Fossum [20]. The first result consolidates some immediate consequences of a fact called Nagata's theorem [20, Thm. 7.1].

Theorem II.4 (cf., Fossum [20, Cor. 7.2, Cor. 7.3])**.** *Let S be a multiplicatively closed subset of a Noetherian normal domain A. Then:*

1. *The natural map $\mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$ is a surjection of abelian groups. The kernel is generated by the classes of the height one prime ideals which meet S.*
2. *If S is generated by prime elements of A, then $\mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$ is an isomorphism of abelian groups.*

The next two results will allow us to streamline class group computations in Chapter III to a particularly nice setup where we have an incisive handle on computing class groups up to isomorphism.

Theorem II.5 (cf., Fossum [20, Thm. 8.1, Cor. 8.2])**.** *Working with polynomial extensions of a Noetherian normal domain A, we have isomorphisms for any $n \in \mathbb{Z}_{>0}$:*

$$\mathrm{Cl}(A) \cong \mathrm{Cl}(A[X_1, \ldots, X_n]) \cong \mathrm{Cl}(A[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]).$$

Proof Sketch. One can induce on n with base case $n = 1$. Assuming $n = 1$, the left-hand isomorphism is the content of Fossum [20, Thm. 8.1]. For the right-hand isomorphism, apply Theorem II.4(2) to the polynomial ring $B = A[X]$ and the multiplicatively closed set $S \subseteq B$ generated by the prime element $X \in B$, so $S^{-1}B = A[X^{\pm 1}]$ is a Laurent polynomial ring in one variable over A. $\square$

Theorem II.6 (cf., Fossum [20, Cor. 10.3, Cor. 10.7])**.** *Suppose $A = \oplus_{n=0}^{\infty} A_i$ is an $\mathbb{N}$-graded Noetherian normal domain where $A_0 = \mathbb{F}$ is a field, with homogeneous maximal ideal $\mathfrak{m} = \oplus_{n=1}^{\infty} A_i$. Then:*

1. *We have an isomorphism $\mathrm{Cl}(A) \cong \mathrm{Cl}(A_{\mathfrak{m}})$.*

2. *Suppose that $\mathbb{F}'$ is any field extension of $A_0 = \mathbb{F}$, and that $A' := A \otimes_{\mathbb{F}} \mathbb{F}'$ is a Noetherian normal domain. Then A' is faithfully flat over A and the induced homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(A')$ is injective.*

2.2 Annihilation of Divisor Class Groups

We now assume that R is a Noetherian normal ring, meaning that the local ring $R_{\mathfrak{p}}$ is a normal domain for all primes $\mathfrak{p}$ in R. In particular, R is reduced.

Definition II.7. Given a Noetherian normal ring R, the affine scheme $X = \mathrm{Spec}(R)$ is **locally factorial** if $R_{\mathfrak{p}}$ is a UFD for all prime ideals $\mathfrak{p} \subseteq R$. Equivalently, the local class group $\mathrm{Cl}(R_{\mathfrak{p}}) = 0$ is trivial for all primes $\mathfrak{p} \subseteq R$.

Definition II.8. A Noetherian normal ring R is **(locally) uniformly annihilated** if there exists an integer (multiplier) $D > 0$ such that any of the following equivalent conditions will hold:

1. $D \cdot \mathrm{Cl}(R_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p} \subseteq R$. More precisely, $P^{(D)}R_{\mathfrak{p}} = (PR_{\mathfrak{p}})^{(D)}$ is principal for all height one primes $P \subseteq \mathfrak{p}$.

2. $D \cdot \mathrm{Cl}(R_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$.

3. The annihilator ideal $\operatorname{Ann}_{\mathbb{Z}}(\operatorname{Cl}(R_{\mathfrak{p}})) \supseteq D\mathbb{Z}$ for all prime ideals $\mathfrak{p} \subseteq R$.

Notice that (2) implies (1) since principal ideals remain principal when we extend along the ring map $R_{\mathfrak{m}} \to R_{\mathfrak{p}}$. Note that since one can take $D = 1$ at locally factorial points of $\operatorname{Spec}(R)$, it suffices to compute D at maximal ideals $\mathfrak{m} \subseteq R$ such that $R_{\mathfrak{m}}$ is not a UFD. In language more familiar to algebraic geometers, D annihilates the **local class group** $\operatorname{Cl}_{loc}(X) := \operatorname{Cl}(X)/\operatorname{Pic}(X)$ of $X = \operatorname{Spec}(R)$.

Remark II.9. A Noetherian normal domain R is uniformly annihilated when the annihilator ideal $\operatorname{Ann}_{\mathbb{Z}}(\operatorname{Cl}(R)) \neq 0$ (e.g., if $\operatorname{Cl}(R)$ is finite). However, the smallest local uniform multiplier need not annihilate $\operatorname{Cl}(R)$ globally. For example, if the Dedekind domain $R = \mathbb{Z}[\sqrt{-5}]$, then $\operatorname{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z}$. However, any Dedekind domain R is locally factorial, so $D = 1$ is the optimal local uniform multiplier.

Remark II.10. If R as in Remark II.9 is local, then the smallest uniform multiplier would generate $\operatorname{Ann}_{\mathbb{Z}}(\operatorname{Cl}(R))$. When R is $\mathbb{N}$-graded over a field with unique graded maximal ideal $\mathfrak{m}$, $\operatorname{Cl}(R) \cong \operatorname{Cl}(R_{\mathfrak{m}})$ by Theorem II.6 and we again consider a generator of $\operatorname{Ann}_{\mathbb{Z}}(\operatorname{Cl}(R))$. In each case, this optimal multiplier D is the class group's order if and only if the group is finite cyclic; we fill in Table 2.1's final column via this fact.

Recall from Chapter I that a proper ideal I in a Noetherian ring has **pure height** h if all of its associated primes have height h, in particular, none are embedded.

Proposition II.11. *Let R be a Noetherian normal domain, and $\mathfrak{q}$ any ideal of pure height one with associated primes $P_1, \ldots, P_c$. Then:*

(a) There exist positive integers $b_1, \ldots, b_c$, uniquely determined by $\mathfrak{q}$, such that the symbolic power $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \ldots \cap P_r^{(Eb_c)}$ for all $E > 0$.

(b) If either (1) $D \cdot \operatorname{Cl}(R) = 0$*, or simply* (2) $\mathfrak{q}^{(D)}$ *is principal, then for all integers $r > 0$, $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r$ is principal and $\mathfrak{q}^{(Dr)} \subseteq \mathfrak{q}^r$.*

Proof. First, we prove (a). For each i, the local ring $S_i = R_{P_i}$ is a discrete valuation ring, and we let $t_i \in S_i$ be a local uniformizing parameter. Then S_i is a PID, so an ideal $J \subseteq S_i$ is $P_iS_i = (t_i)S_i$-primary if and only if $J = (P_iS_i)^n = P_i^nS_i = (t_i^n)S_i$ for some $n > 0$. In particular, $\mathfrak{q}S_i$ is P_iS_i-primary, say $\mathfrak{q}S_i = (t_i^{b_i})S_i$. Then the P_i-primary component of $\mathfrak{q}$ is $P_i^{(b_i)}$. Thus $\mathfrak{q} = P_1^{(b_1)} \cap \ldots \cap P_r^{(b_r)}$ and clearly the b_i are uniquely determined by $\mathfrak{q}$. Similarly, $\mathfrak{q}^ES_i = (t_i^{Eb_i})S_i$ for $E > 0$, so $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \ldots \cap P_r^{(Eb_r)}$ for all $E > 0$.

For (b), first note that (1) implies (2): indeed, since $\mathfrak{q} = P_1^{(b_1)} \cap \cdots \cap P_r^{(b_c)}$, it yields an element $[\mathfrak{q}] := b_1[P_1] + \cdots + b_c[P_c] \in \mathrm{Cl}(R)$, and since $0 = D[\mathfrak{q}] = [\mathfrak{q}^{(D)}] \in \mathrm{Cl}(R)$, we conclude that $\mathfrak{q}^{(D)}$ is principal. So we proceed assuming $\mathfrak{q}^{(D)}$ is principal. Since $\mathfrak{q}^{(D)} \subseteq \mathfrak{q}^{(1)} = \mathfrak{q}$, by taking r-th powers, part (b) follows in full once we explain how $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r$ for all integers $r > 0$. Indeed, using the notation in the proof of (a), $\mathfrak{q}^{(Dr)}S_i = (\mathfrak{q}^{(D)})^rS_i = (t^{Drb_i})S_i$ for all i, and we simply contract back to R. $\square$

Per Proposition II.11(a), we may define Weil divisors

$$\mathrm{div}[\mathfrak{q}] := b_1 \cdot P_1 + \cdots + b_c \cdot P_c, \quad \mathrm{div}[\mathfrak{q}^{(E)}] := E \cdot \mathrm{div}[\mathfrak{q}] = Eb_1 \cdot P_1 + \cdots + Eb_c \cdot P_c,$$

where $E > 0$. In particular, $\mathrm{div}[\mathfrak{q}^{(A+B)}] = \mathrm{div}[\mathfrak{q}^{(A)}] + \mathrm{div}[\mathfrak{q}^{(B)}]$ for all pairs of non-negative integers A and B.

2.3 Proving the Main Result, Two Immediate Corollaries

Proof of Theorem II.1. Our proof of the first claim replaces $r-1$ with $r \geq 0$. Per Proposition II.11(b), suppose $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r = (f^r)$ is principal for all $r \geq 0$ and some nonzero $f \in R$. Now set $I = \mathfrak{q}^{(s)}$. Following the first proof in Hochster's notes [36], we have a short exact sequence

$$0 \to \frac{(f^r)R}{(f^r)I} \to \frac{R}{(f^r)I} \to \frac{R}{(f^r)R} \to 0$$

and $\frac{(f^r)R}{(f^r)I} \cong R/I$ as R-modules via the R-linear map $\phi\colon R \twoheadrightarrow \frac{(f^r)R}{(f^r)I}$ with $\phi(g) = \overline{gf^r}$. Thus per our exact sequence (cf., Thm. 6.3 of Matsumura [49]),

$$\varnothing \neq \operatorname{Ass}_R(R/(f^r)I) \subseteq \operatorname{Ass}_R(R/I) \cup \operatorname{Ass}_R(R/(f^r)R)$$

and so $\operatorname{Ass}_R(R/(f^r)I)$ contains only height one primes since the latter two sets do. Finally, comparing Weil divisors of pure height one ideals

$$\operatorname{div}[(f^r)I = (\mathfrak{q}^{(D)})^r\mathfrak{q}^{(s)}] \overset{(*)}{=} \operatorname{div}[(f^r)R] + \operatorname{div}[I] = \operatorname{div}[\mathfrak{q}^{(Dr)}] + \operatorname{div}[\mathfrak{q}^{(s)}] = \operatorname{div}[\mathfrak{q}^{(Dr+s)}].$$

As Hochster notes, one can check identity (*) after first localizing at each height one prime Q; in this case, the identity is obvious in a DVR. Per (*), the ideals $\mathfrak{q}^{(Dr+s)}$ and $(\mathfrak{q}^{(D)})^r\mathfrak{q}^{(s)}$ have the exact same primary decomposition and hence are equal. Since $\mathfrak{q}^{(D)} \subseteq \mathfrak{q}^{(1)} = \mathfrak{q}$, setting $s = 1$ yields $\mathfrak{q}^{(D(r-1)+1)} = (\mathfrak{q}^{(D)})^{r-1}\mathfrak{q}^{(1)} \subseteq \mathfrak{q}^{r-1+1} = \mathfrak{q}^r$. $\square$

We now record two immediate corollaries of Theorem II.1:

Corollary II.12. *Let R be a Noetherian normal ring. Suppose D annihilates $\operatorname{Cl}(R_\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ in R. Then the symbolic power*

$$\mathfrak{q}^{(D(r-1)+s)} = (\mathfrak{q}^{(D)})^{r-1}\mathfrak{q}^{(s)}, \text{ and } \mathfrak{q}^{(D(r-1)+1)} \subseteq \mathfrak{q}^r,$$

for all ideals $\mathfrak{q} \subseteq R$ of pure height one, all $r > 0$, and all $0 \leq s < D$.

Proof. Reduce to the local case: given two ideals I, J in R, the inclusion $I \subseteq J$ holds (that is, the R-module $\frac{J+I}{J} = 0$) if and only if $IR_\mathfrak{m} \subseteq JR_\mathfrak{m}$ (that is, the $R_\mathfrak{m}$-module $\frac{(J+I)R_\mathfrak{m}}{JR_\mathfrak{m}} = \frac{JR_\mathfrak{m}+IR_\mathfrak{m}}{JR_\mathfrak{m}} = 0$) for all maximal ideals $\mathfrak{m} \subseteq R$. So we may assume R is a normal Noetherian local domain and D annihilates $\operatorname{Cl}(R)$ – invoke Theorem II.1. $\square$

Corollary II.13. *For a Noetherian normal domain R, the following assertions are equivalent to R being a UFD (i.e., every height one prime ideal is principal):*

1. *The divisor class group* $\mathrm{Cl}(R) = \mathrm{Cl}(\mathrm{Spec}(R)) = 0$ *is trivial.*

2. *Every ideal in* R *of pure height one is principal.*

3. *All symbolic powers of any ideal in* R *of pure height one are principal.*

In this case, $\mathfrak{q}^{(r)} = \mathfrak{q}^r$ *for all* $r > 0$ *and all ideals* $\mathfrak{q} \subsetneq R$ *of pure height one.*

2.4 Applications to Rational Surface Singularities

We emphasize two classes of rings that satisfy the hypotheses of Theorem II.1. We consider rationally singular surfaces now, and postpone a discussion of simplicial toric rings until Chapter III.

Definition II.14 ([46])**.** A two-dimensional, normal Noetherian local domain $(R, \mathfrak{m})$ to have **rational singularities** if there is a proper, birational map $f \colon X \to \mathrm{Spec}(R)$ from a regular scheme X such that $H^1(X, \mathcal{O}_X) = 0$.

In particular, two-dimensional regular local rings have rational singularities.

We record the following result due to Joe Lipman.

Theorem II.15 (Lipman [46, Prop 17.1])**.** *Let* $(R, \mathfrak{m})$ *be a normal Noetherian local domain of dimension two. If* R *has a rational singularity, then* $\mathrm{Cl}(R)$ *is finite.*

By applying Theorem II.1 to any rationally singular surface, we conclude that

Theorem II.16. *All two-dimensional rational singularities* $(R, \mathfrak{m})$ *satisfy uniform symbolic topologies on primes.*

In step with the improved Ein – Lazarsfeld – Smith Theorem I.18, applying Corollary II.12 yields a separate proof that all two-dimensional regular rings R satisfy uniform symbolic topology on primes and $P^{(r)} = P^r$ for all $r > 0$ and all primes P in R.

To obtain additional explicit, effective multipliers, we turn to the case of complete, normal Noetherian local domains S in equal characteristic zero with du Val (ADE) isolated singularity and algebraically closed residue field; for simplicity, we work with $\mathbb{C}$. In [46, Sec. 24], Lipman computes the class group isomorphism type (as a $\mathbb{Z}$-module) of each du Val singularity. The du Val (ADE) singularities, also known as *rational double points*, are the most basic isolated surface singularities. Their minimal resolutions can be understood and classified by the simply-laced Dynkin diagrams of types A, D, and E. We can express S as above as the quotient of the power series $\mathbb{C}[[x, y, z]]$ by a single local equation. We now situate a succinct data table, where the last column's entries are the optimal uniform multipliers from Remark II.10:

du Val Singularity type	Local Equation	Class group (isomorphism type)	$D_{\min}(S)$
A_n $(n \geq 1)$	$xz - y^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$
D_n $(n \geq 4)$	$x^2 + yz^2 - z^{n-1}$	$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & n \text{ odd.} \end{cases}$	$\begin{cases} 2 & n \text{ even} \\ 4 & n \text{ odd.} \end{cases}$
E_n $(n = 6, 7, 8)$	$\begin{cases} x^4 + y^3 + z^2 & \text{if } n = 6 \\ x^3y + y^3 + z^2 & \text{if } n = 7 \\ x^5 + y^3 + z^2 & \text{if } n = 8 \end{cases}$	$\begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } n = 6 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 7 \\ 0 & \text{if } n = 8 \end{cases}$	$\begin{cases} 3 & \text{if } n = 6 \\ 2 & \text{if } n = 7 \\ 1 & \text{if } n = 8 \end{cases}$

Table 2.1: Pertinent data for each Complex du Val surface singularity type.

An analogous data table can be drafted for affine du Val singularity hypersurfaces in $\mathbb{C}^3$ (affine, $\mathbb{N}$-graded case), which arise as the quotients $\mathbb{C}^2/G$ by the action of a finite subgroup $G \subseteq \mathrm{SU}_2(\mathbb{C})$ of the special unitary group.

To close, we swiftly settle Question I.34 from Chapter I in the negative.

Remark II.17. When a Noetherian ring R satisfies the uniform symbolic topology property (USTP) on prime ideals, experts might initially expect that the optimal multiplier $D = D_{\min}(R)$ should depend only on simple numerical invariants of R, such as Krull dimension, or the multiplicity of R at an isolated singularity. However,

A_n-singularities and E_8-singularities have multiplicity two, being rational double points. At one extreme, $D_{\min}(A_n) = n+1$ is optimal, grows arbitrarily large with n, and does **not** depend on any such numerical invariants of A_n. At the other extreme, $D_{\min}(E_8) = 1$ is lower than both the Krull dimension and the multiplicity. Therefore, a (sharp) uniform bound depending only on such numerical invariants need not exist, even if we restrict to rings of invariants of actions of finite groups on $\mathbb{C}^2$.

We have now fulfilled the endgame promised at the start of the chapter.

Before vaulting into toric algebra jargon in Chapter III, we pause to rehearse an iteration of Problem I.9 and the motivational questions at stake in this thesis.

Question II.18. *Given a Noetherian commutative ring R, when is there an integer D, depending only on R, such that the symbolic power $P^{(Dr)} \subseteq P^r$ for all prime ideals $P \subseteq R$ and all positive integers r? In short, when does R have* ***uniform symbolic topologies*** *on primes [41]? Moreover, can we effectively compute the multiplier D in terms of simple data about R?*

We will bear Question II.18 in mind as we vault into Chapters III and IV below – the main results of these chapters concern prime ideals only. The first page of each chapter will record the main result(s) for the benefit of readers already familiar with the relevant jargon and notation. The remainder of each chapter is curated to get all other readers on the same page – as far as necessary background and any follow-up considerations taken up after deducing our main results.

CHAPTER III

Uniform Symbolic Topologies in Normal Toric Domains

This chapter consolidates material from our papers [68], [69], and [70]. We answer Question II.18 for torus-invariant primes in a normal toric (or monomial, or semigroup) algebra – the coordinate rings of normal affine toric varieties, hence also Cohen-Macaulay and combinatorially-defined. We now state our main results for readers accustomed to conventions in Cox – Little – Schenck [12] and Fulton [21].

Theorem III.1. *Let $C \subseteq N_{\mathbb{R}}$ be a full pointed rational polyhedral cone. Let $R_{\mathbb{F}} = \mathbb{F}[C^{\vee} \cap M]$ be the associated toric algebra over a field $\mathbb{F}$. Set $D := \max_{m \in \mathcal{B}} \langle m, v_C \rangle$, where $\mathcal{B}$ is the minimal generating set for $C^{\vee} \cap M$ and $v_C \in N$ is the sum of the primitive generators for C. Then*

$$P^{(D(r-1)+1)} \subseteq P^r$$

for all $r > 0$, and all monomial primes P in $R_{\mathbb{F}}$.

Corollary III.2. *With notation as in Theorem III.1, we assume further that C is simplicial. Define $T := \max \{\max_{m \in \mathcal{B}} \langle m, v_C \rangle, D\}$, where D is any positive integer such that $D \cdot \mathrm{Cl}(R_{\mathbb{F}}) = 0$. Then*

$$P^{(T(r-1)+1)} \subseteq P^r$$

for all $r > 0$, all monomial primes, and all height one primes in $R_{\mathbb{F}}$.

The reader is invited to revisit the list of Conventions I.45 as appropriate.

3.1 Tapas of Toric Algebra for Full-Dimensional Cones

As in Cox – Little – Schenck [12, Ch.1,3,4] and Fulton [21, Ch.1,3], a lattice is a free abelian group of finite rank. We fix a perfect bilinear pairing $\langle\cdot,\cdot\rangle\colon M \times N \to \mathbb{Z}$ between two lattices M and N; this identifies M with $\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ and N with $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$. Our pairing extends to a perfect pairing of finite-dimensional vector spaces $\langle\cdot,\cdot\rangle\colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$, where $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Going forward, we fix a full pointed N-**rational** polyhedral cone $C \subseteq N_{\mathbb{R}}$ and its M-rational dual: respectively, for some **finite** subset $G \subseteq N - \{0\}$ these are closed, convex sets of the form

$$C = \mathrm{Cone}(G) := \left\{ \sum_{v \in G} a_v \cdot v\colon \text{ each } a_v \in \mathbb{R}_{\geq 0} \right\} \subseteq N_{\mathbb{R}}, \text{ and}$$

$$C^\vee := \{w \in M_{\mathbb{R}}\colon \langle w, v\rangle \geq 0 \text{ for all } v \in C\} = \{w \in M_{\mathbb{R}}\colon \langle w, v\rangle \geq 0 \text{ for all } v \in G\}.$$

By definition, the **dimension** of a cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is the dimension of the real vector subspace it spans; a cone is **full(-dimensional)** if it spans the full ambient space. A cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is **pointed (or strongly convex)** if it contains no line through the origin. A pointed full-dimensional cone C is said to be **simplicial** if it can be generated by exactly $d = \dim_{\mathbb{R}}(N_{\mathbb{R}})$ elements in N.

There is a uniquely-determined *minimal* finite generating set $\mathcal{B}$ for the semigroup $C^\vee \cap M$, its **Hilbert basis**. This basis consists of the **irreducible** vectors $m \in C^\vee \cap M - \{0\}$, read, nonzero vectors that cannot be expressed as a sum of two nonzero vectors in $C^\vee \cap M$ [12, Prop. 1.2.17, Prop. 1.2.23].

Fix an arbitrary ground field $\mathbb{F}$. The semigroup ring $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ is the **toric** $\mathbb{F}$**-algebra associated to** C. This ring $R_{\mathbb{F}}$ is a normal domain of finite type over

$\mathbb{F}$ [12, Thm. 1.3.5]. Note that $R_{\mathbb{F}}$ has an $\mathbb{F}$-basis $\{\chi^m : m \in C^\vee \cap M\}$ of monomials, giving $R_{\mathbb{F}}$ an M-grading, where $\deg(\chi^m) := m$. A **monomial ideal (also called an M-homogeneous or torus-invariant ideal)** in $R_{\mathbb{F}}$ is an ideal generated by a subset of these monomials.

3.2 Proof of Main Results and Example Computations

Definition III.3. Fix a lattice $\mathcal{L}$ and a convex polyhedral cone $\mathcal{C} \subseteq \mathcal{L}_{\mathbb{R}}$. A **face** of $\mathcal{C}$ is a convex polyhedral cone F in $\mathcal{L}_{\mathbb{R}}$ obtained by intersecting $\mathcal{C}$ with a hyperplane which is the kernel of a linear functional $m \in \mathcal{C}^\vee$; F is **proper** if $F \neq \mathcal{C}$.

There is a bijective inclusion-reversing correspondence between faces F of C and faces F^* of $C^\vee$, where $F^* = \{w \in C^\vee : \langle w, v\rangle = 0 \text{ for all } v \in F\}$ is the face of $C^\vee$ **dual to** F [21, Sec. 1.2]. Under this correspondence, it turns out that $C^\vee$ is pointed if and only if C is full – and vice versa, and

$$\dim(F) + \dim(F^*) = \dim(N_{\mathbb{R}}) = \dim(M_{\mathbb{R}}). \tag{3.1}$$

Proof of Theorem III.1. We may fix a face $F \neq \{0\}$ of the full pointed rational cone C, and $P = P_F$ the corresponding monomial prime in $R = R_{\mathbb{F}}$. First, we note F has a uniquely-determined set G_F of primitive generators – by definition, a vector $v \in N$ is **primitive** if $\frac{1}{k} \cdot v \notin N$ for all $k \in \mathbb{Z}_{>1}$. Fulton [21, p.53] records a surjective M-graded ring map between integral domains in terms of the face F^* of $C^\vee$ **dual** to F (cf., the discussion following Definition III.3 below), even when C is not full:

$$\phi_F : R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M] \twoheadrightarrow \mathbb{F}[F^* \cap M], \quad \phi_F(\chi^m) = \begin{cases} \chi^m & \text{if } \langle m, v\rangle = 0 \text{ for all } v \in F \\ 0 & \text{if } \langle m, v\rangle > 0 \text{ for some } v \in F. \end{cases}$$

The **monomial prime ideal of** F, $P_F := \ker(\phi_F)$, has height equal to $\dim(F)$.

Lemma III.4. *Fix a face F of a pointed rational cone C, and the monomial prime $P_F \subseteq R_{\mathbb{F}}$ above. Let G_F be the set of primitive generators of F, and set $v_F := \sum_{v \in G_F} v \in F \cap N$. Then*

$$P_F = (\{\chi^m \colon m \in C^\vee \cap M \text{ and the integer } \langle m, v_F \rangle > 0\})R_{\mathbb{F}}. \tag{3.2}$$

Proof. First, in defining $\phi_F(\chi^m)$ above, notice we can work with $v \in G_F$ without loss of generality. Now, fix $m \in C^\vee \cap M$. Then $\langle m, v \rangle \in \mathbb{Z}_{\geq 0}$ for all $v \in C \cap N$. As $\langle \cdot, \cdot \rangle$ is bilinear, (3.2) follows since a sum of nonnegative integers is positive if and only if one of the summands is positive. □

Lemma III.5. *For each integer $E \geq 1$, we have $P_F^{(E)} \subseteq I_F(E) \subseteq P_F^{\lceil E/D' \rceil} \subseteq P_F^{\lceil E/D \rceil}$ where*

$$I_F(E) := (\chi^m \colon \langle m, v_F \rangle \geq E)\, R, \quad D := \max_{m \in \mathcal{B}} \langle m, v_C \rangle\,, \text{ and } D' := \max_{m \in \mathcal{B}} \langle m, v_F \rangle \leq D.$$

Proof. First, $I_F(E)$ is P_F-primary for all $E \geq 1$, i.e., if $sf \in I_F(E)$ for some $s \in R - P_F$, then $f \in I_F(E)$. As $I_F(E)$ is monomial, we may test this by fixing $\chi^m \in I_F(E)R_{P_F} \cap R$ and $\chi^q \in R - P_F$ such that $\chi^m \cdot \chi^q = \chi^{m+q} \in I_F(E)$: $\langle q, v_F \rangle = 0$, while $E \leq \langle m + q, v_F \rangle = \langle m, v_F \rangle + \langle q, v_F \rangle = \langle m, v_F \rangle$, so $\chi^m \in I_F(E)$. Thus all $I_F(E)$ are P_F-primary, and certainly $P_F^E \subseteq I_F(E)$. Thus $P_F^{(E)} \subseteq I_F(E)$, being the smallest P_F-primary ideal containing P_F^E.

Now fix any monomial $\chi^\ell \in I_F(E)$, say $\ell = \sum_{m \in \mathcal{B}} a_m \cdot m$ with $a_m \in \mathbb{Z}_{\geq 0}$. Let $S \subseteq \mathcal{B}$ consist of those $m \in \mathcal{B}$ such that the monomials χ^m form a minimal generating set for P. By linearity of $\langle \bullet, v_F \rangle$,

$$E \leq \langle \ell, v_F \rangle = \sum_{m \in \mathcal{B}} a_m \langle m, v_F \rangle = \sum_{m \in S} a_m \langle m, v_F \rangle \leq \sum_{m \in S} a_m \cdot D' \Longrightarrow \sum_{m \in S} a_m \geq \lceil E/D' \rceil.$$

Thus $\chi^\ell \in P_F^{\sum_{m \in S} a_m} \subseteq P_F^{\lceil E/D' \rceil}$, ergo $I_F(E) \subseteq P_F^{\lceil E/D' \rceil} \subseteq P_F^{\lceil E/D \rceil}$, proving the lemma. □

To finish the proof of Theorem III.1, set $E = D(r-1)+1$ in the lemma. Thus $P_F^{(D(r-1)+1)} \subseteq P_F^r$ for all $r > 0$, as desired. □

Remark III.6. In passing, we invite the reader to compare the ideals $I_F(\bullet)$ in Lemma III.5 with Bruns and Gubeladze's terminology and description [10, Ch. 4, p. 149] for the symbolic powers of the height one monomial primes in terms of a full pointed cone. Lemma III.5 works in any height, and the Bruns – Gubeladze description may be adapted to this general height case too.

Proof of Corollary III.2. Since C is simplicial, $\#\,\mathrm{Cl}(R_{\mathbb{F}})$ is finite by Theorem III.23 and Lemma III.27 below. Now we simply combine Theorem III.1 with Lemma II.1, and take the maximum of the values. □

Remark III.7. When the cone C in Corollary III.2 is smooth – read, C is generated by a $\mathbb{Z}$-basis for N, then $T = 1$ and $P^{(r)} = P^r$ for all $r > 0$, all monomial primes, and all height one primes in $R_{\mathbb{F}}$. As C is smooth, C and $C^{\vee}$ are generated by a $\mathbb{Z}$-basis for N and the dual basis for M, respectively. Also, $\#\,\mathrm{Cl}(R_{\mathbb{F}}) = \#\,\mathrm{Cl}(R_{\overline{\mathbb{F}}}) = 1$. Note that in general, this means our multiplier T will not confirm uniform symbolic topologies for *all* primes P in a toric algebra. For example, even in a polynomial ring of dimension three, there are height two primes for which $P^{(r)} \neq P^r$ for some $r \geq 2$; [13, page two of Introduction] gives an example.

Remark III.8. Two-dimensional toric algebras are always simplicial with cyclic divisor class group. In this case, the conclusion of Corollary III.2 holds using the multiplier $\#\,\mathrm{Cl}(R_{\mathbb{F}})$. This multiplier is sharp by Proposition III.32.

Remark III.9. Theorem III.1 and its corollary can be adapted to the non-full case by replacing $R_{\mathbb{F}}$ with $R'_{\mathbb{F}}$ as in Proposition III.22, and applying both Theorem II.5 and Proposition IV.9 to the faithfully flat map φ from Proposition III.22.

Example III.10. Fix a field $\mathbb{F}$ and integers $n \geq 2$ and $E \geq 2$. Let

$$R = \frac{\mathbb{F}[x_1, \ldots, x_n, z]}{(z^E - x_1 \cdots x_n)}.$$

Then Theorem III.1 and its corollary ensure that $P^{(T(r-1)+1)} \subseteq P^r$ for all $r > 0$, all monomial primes, and all height one primes in R, where $T = \max\{n, E\}$. Indeed, R is a toric algebra arising from the simplical full pointed cone $C \subseteq \mathbb{R}^n$ spanned by $\{e_n,\ E \cdot e_i + e_n \colon i = 1, \ldots, n-1\} \subseteq \mathbb{Z}^n$, where $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. We can compute that $\mathrm{Cl}(R) \cong (\mathbb{Z}/E\mathbb{Z})^{n-1}$ so $E \cdot \mathrm{Cl}(R) = 0$; see Example III.46 below for details. Meanwhile, in the notation of Theorem III.1 $\mathcal{B} = \{e_1, \ldots, e_{n-1}, e_n, E \cdot e_n - e_1 - \cdots - e_{n-1}\} \subseteq \mathbb{Z}^n$ – see Lemma III.41, and the vector $v_C = n \cdot e_n + E \cdot (e_1 + \cdots + e_{n-1}) \in \mathbb{Z}^n$, so we compute that $\max_{m \in \mathcal{B}} \langle m, v_C \rangle = n$.

3.2.1 Closing Example Computation: Segre – Veronese algebras

In what follows, $\mathbb{F}$ is a fixed arbitrary field. For more on Segre products, see [24]. Segre – Veronese algebras are a well-known class of normal toric rings.

Definition III.11. Fix a family $A_1, \ldots, A_k$ of k standard graded algebras of finite type over $\mathbb{F}$, with $A_i = \mathbb{F}[a_{i,1}, \ldots, a_{i,b_i}]$ in terms of algebra generators. Their **Segre product** over $\mathbb{F}$ is the ring $S = (\#_{\mathbb{F}})_{i=1}^k A_i$ generated up to isomorphism as an $\mathbb{F}$-algebra by all k-fold products of the $a_{i,j}$.

Definition III.12. We fix integers $E \geq 1$ and $m \geq 2$. Suppose $A = \mathbb{F}[x_1, \ldots, x_m]$ is a standard graded polynomial ring in m variables over a field $\mathbb{F}$. Let $V_{E,m} \subseteq A$ denote the E-th **Veronese subring** of A, the standard graded $\mathbb{F}$-subalgebra generated by all monomials of degree E in the x_i. There are $\binom{m-1+E}{E}$ such monomials; this number is the **embedding dimension** of $V_{E,m}$.

Definition III.13. Fix k-tuples $\overline{E} = (E_1, \ldots, E_k) \in (\mathbb{Z}_{\geq 1})^k$ and $\overline{m} = (m_1, \ldots, m_k) \in (\mathbb{Z}_{\geq 2})^k$ of integers, with $k \geq 1$. Furthermore, we set $d(j) = \left(\sum_{i=1}^{j} m_i\right) - (j-1)$ for each $1 \leq j \leq k$: $d(k)$ is the Krull dimension of the Segre product $SV\left(\overline{E}, \overline{m}\right) = (\#_{\mathbb{F}})_{i=1}^{k} V_{E_i, m_i}$ of k Veronese rings in $m_1, \ldots, m_k$ variables, respectively; we call this algebra a **Segre – Veronese algebra with degree sequence** $\overline{E}$**.**

Theorem III.14. *Suppose* $A = SV\left(\overline{E}, \overline{m}\right)$ *is a Segre – Veronese algebra over* $\mathbb{F}$ *with* $\overline{E} = (E_1, \ldots, E_k)$*. Let* $D := \sum_{i=1}^{k} E_i$*. Then* $P^{(D(r-1)+1)} \subseteq P^r$ *for all* $r > 0$ *and all monomial primes* P *in* A*.*

Proof. Given a lattice $N \cong \mathbb{Z}^d$ we will use $e_1, \ldots, e_d \in N$ to denote a choice of basis for N will dual basis $e_1^*, \ldots, e_d^*$ for M. In the setup of Theorem III.1, the cardinality of the minimal generating set $\mathcal{B}$ of $C^\vee \cap M$ is the **embedding dimension** of $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ – see [12, Sec. 1.0, Proof of Thm. 1.3.10].

We now provide an explicit cone $C \subseteq N_{\mathbb{R}}$ and an explicit Hilbert basis $\mathcal{B}$ to feed into Theorem III.1. Fix k-tuples $\overline{E} \in (\mathbb{Z}_{\geq 1})^k$ and $\overline{m} \in (\mathbb{Z}_{\geq 2})^k$. Set $d(j) = \left(\sum_{i=1}^{j} m_i\right) - (j-1)$ for $1 \leq j \leq k$, while $d(0) = 0$. Given $SV\left(\overline{E}, \overline{m}\right) = (\#_{\mathbb{F}})_{i=1}^{k} V_{E_i, m_i}$, we fix a lattice $N \cong \mathbb{Z}^{d(k)}$ and record a cone $C = C\left(\overline{E}, \overline{m}\right) \subseteq N_{\mathbb{R}} \cong \mathbb{R}^{d(k)}$ as stipulated with $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M] \cong SV\left(\overline{E}, \overline{m}\right)$. Specifically, consider the cone $C \subseteq N_{\mathbb{R}}$ generated by the following irredundant collection of primitive vectors:

$$\mathcal{A} = \bigcup_{1 \leq j \leq k} A_j, \text{ where } A_1 = \{e_1, \ldots, e_{m_1 - 1}, -e_1 - \cdots - e_{m_1 - 1} + E_1 \cdot e_{m_1}\},$$

$$\text{and for each } 2 \leq j \leq k, \quad A_j = \left\{ e_h,\ E_j \cdot e_{m_1} - \sum_{h=d(j-1)+1}^{d(j)} e_h \colon d(j-1)+1 \leq h \leq d(j) \right\}.$$

The semigroup $C^\vee \cap M$ is generated by the following set of irreducible vectors:

$$\mathcal{B} = \left\{ e_{m_1}^* + \sum_{j=1}^{k} \sum_{\ell=1}^{m_j - 1} a_{j,\ell} \cdot e_{d(j-1)+\ell}^* \colon 0 \leq \sum_{\ell=1}^{m_j - 1} a_{j,\ell} \leq E_j \text{ for } 1 \leq j \leq k \right\}.$$

Indeed, $\#\mathcal{B} = \prod_{j=1}^{k} \binom{m_j-1+E_j}{E_j}$, the embedding dimension of $SV(\overline{E}, \overline{m})$. Finally, one can record a bijection between the monomial generators of $R_{\mathbb{F}}$ and those typically used to present $SV(\overline{E}, \overline{m})$; cf., Lemma III.43 below for how the bijection would look in the coordinates $a_{j,\ell}$ for each j. Feeding $v_C = \sum_{u \in \mathcal{A}} u = (\sum_{j=1}^{k} E_j) \cdot e_{m_1}$ and $\mathcal{B}$ into Theorem III.1 yields the multiplier $D = \sum_{j=1}^{k} E_j$. $\square$

Over any perfect field $\mathbb{K}$, a Segre – Veronese algebra has uniform symbolic topologies on all primes, per [16, Thm. 2.2], [37, Thm. 1.1], and [40, Cor. 3.10]. However, no explicit multiplier is provided by these cited results; indeed, the cases $k = 1$ and $k = 2$ (with $\overline{E} = (1,1)$) were only addressed recently via [13, Cor. 3.30] and [11, Thm. 5.6] [56], respectively. Meanwhile, the multiplier in Theorem III.14 covers the torus-invariant primes in Segre – Veronese algebras over an arbitrary field.

3.3 Tapas of Toric Algebra for Arbitrary Cones

Throughout, $C \subseteq N_{\mathbb{R}}$ will be an arbitrary rational convex polyhedral cone. In particular, although C need not be full-dimensional or pointed:

Remark III.15. In forming the toric algebra $\mathbb{F}[C^\vee \cap M]$, there is no loss of generality in assuming C is pointed in $N_{\mathbb{R}}$. Indeed, because $C^\vee \cap M = C^\vee \cap M'$ where $M' = M \cap \{\mathbb{R}\text{-span of } C^\vee \text{ in } M_{\mathbb{R}}\}$, we may replace M by M' to assume $C^\vee$ is full in $(M')_{\mathbb{R}}$. Now, replacing N and C by the duals of M' and $C^\vee$, we may assume that C is pointed in $N' = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Z})$. Compare with [12, Proof of Thm. 1.3.5] for details.

Remark III.16. When $C^\vee$ is pointed, $R_{\mathbb{F}}$ also has a non-canonical $\mathbb{N}$-grading obtained by fixing any group homomorphism $M \to \mathbb{Z}$ taking positive values $C^\vee \cap M - \{0\}$. The set $\{\chi^m : m \in C^\vee \cap M - \{0\}\}$ generates the unique homogeneous maximal ideal $\mathfrak{m}$ under this $\mathbb{N}$-grading. Our proofs of Lemma III.27 and Proposition III.32 use this.

We now clarify Remark III.9 from above, working towards Proposition III.22.

Proposition III.17 (Minkowski sum – Ideal sum)**.** *Suppose $C \subseteq N_{\mathbb{R}}$ is a pointed rational polyhedral cone, and $R_{\mathbb{F}} = \mathbb{F}[C^{\vee} \cap M]$ is the corresponding toric $\mathbb{F}$-algebra. When a face $F = Cone(G_F) = \rho_1 + \ldots + \rho_\ell$ as a Minkowski sum of rays,*

$$P_F = \sum_{j=1}^{\ell} P_{\rho_j} \tag{3.3}$$

as a sum of ideals.

Proof. Let $G_F = \{u_{\rho_j} \colon 1 \leq j \leq \ell\}$ consist of the primitive ray generators. Any $v \in F$ satisfies

$$v = \sum_{j=1}^{\ell} a_j u_{\rho_j}, \text{ for some } a_1, \ldots, a_\ell \in \mathbb{R}_{\geq 0}.$$

Given any $w \in C^{\vee}$, $\langle w, v\rangle \geq 0$ for all $v \in C$. Thus for $v \in F$ as above,

$$0 \leq \langle w, v\rangle = \sum_{j=1}^{\ell} a_j \langle w, u_{\rho_j}\rangle, \text{ for some } a_1, \ldots, a_\ell \in \mathbb{R}_{\geq 0},$$

and so $\langle w, v\rangle$ is positive if and only if $\langle w, u_{\rho_j}\rangle > 0$ for some $1 \leq j \leq \ell$. We infer that P_F and $\sum_{j=1}^{\ell} P_{\rho_j}$ have a generating set in common, and hence coincide. □

Definition III.18. With notation as in Proposition III.17, we call (3.3) a **Minkowski sum – ideal sum decomposition** for P_F.

Remark III.19*.* Adapting the proof of Proposition III.17 accordingly, we could use any decomposition of F as a Minkowski sum of faces, the latter need not be rays.

Our next goalpost is Lemma III.21 on decomposing monomial primes in tensor products of normal toric rings. Fix two pointed rational polyhedral cones $C_i = \text{Cone}(S_i) \subset (N_i)_{\mathbb{R}}$ $(i = 1, 2)$, where each S_i consists of the primitive ray generators. Define lattices $N = N_1 \times N_2, M = M_1 \times M_2$ per the standing conventions. Let $\langle , \rangle_i \colon M_i \times N_i \to \mathbb{Z}$ and $\langle , \rangle \colon M \times N \to \mathbb{Z}$ indicate our three designated bilinear pairings.

Remark III.20. While tedious, we could pedantically write down compatibility conditions to the effect that the output values of these pairings will agree relative to the obvious $\mathbb{Z}$-linear embeddings $N_i \hookrightarrow N$ and $M_i \hookrightarrow M$, e.g., $N_1 \cong N_1 \times \{0\}$. In particular, in a slight abuse of notation, going forward we identify

$$\langle , \rangle = \langle , \rangle_1 + \langle , \rangle_2.$$

This generalizes the usual dot product setup naturally, $\mathbb{Z}^E \subseteq \mathbb{R}^E$, where $E = m + n$ as a sum of positive integers.

The **product cone** $C = C_1 \times C_2$ in $N_{\mathbb{R}}$ is a pointed rational polyhedral cone. In terms of ray generators, C is generated as

$$C = (C_1 \times \{0\}) + (\{0\} \times C_2) = \mathrm{Cone}[(S_1 \times \{0\}) \cup (\{0\} \times S_2)] \subseteq N_{\mathbb{R}}.$$

Note that

$$C^\vee = (C_1 \times \{0\})^\vee \cap (\{0\} \times C_2)^\vee = C_1^\vee \times C_2^\vee.$$

For the right-hand equality, we defer to Remark III.20.

Lemma III.21. *For $n \geq 2$, let $R_1, \ldots, R_n$ be normal toric rings over a field $\mathbb{F}$, built from pointed rational polyhedral cones $C_i \subseteq (N_i)_{\mathbb{R}}$, respectively. Consider the normal toric ring $R \cong R_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$. Every monomial prime ideal Q in R is a sum $Q = \sum_{i=1}^n (P_i R)$ of expanded ideals, where each ideal $P_i \subseteq R_i$ is a monomial prime.*

Proof. Induce on n with base case $n = 2$; we focus on the base case for the remainder of the proof. Suppose $R_i = (R_i)_{\mathbb{F}} = \mathbb{F}[C_i^\vee \cap M_i]$, and

$$R = R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M] \cong R_1 \otimes_{\mathbb{F}} R_2.$$

Any monomial prime in R corresponds bijectively with a face of C. All faces of C are of the form $F = F_1 \times F_2$ where F_i is a face of C_i. Given F as stated, with $Q_F \subseteq R$ the corresponding monomial prime, the base case follows from proving that

(1) $Q_{F_1\times F_2} = Q_{F_1\times\{0\}} + Q_{\{0\}\times F_2}$; and

(2) As expansions of monomial ideals, $Q_{F_1\times\{0\}} = P_{F_1}R$, $Q_{\{0\}\times F_2} = P_{F_2}R$.

The **Minkowski sum – ideal sum decomposition** (3.3) suffices to verify both claims. First, to see **(1)**, notice $F_1 \times F_2 = (F_1 \times \{0\}) + (\{0\} \times F_2)$ as a Minkowski sum of faces. As for **(2)**, (3.3) allows us to reduce verification to the case where the F_i are rays. We do so explicitly for $Q_{\rho\times\{0\}}$ where ρ is a ray of C_1. We will use notations χ^a, ϕ^b, ψ^c for characters in R, R_1, R_2 respectively. We express an arbitrary

$$w = (w_1, w_2) \in C^\vee \cap M = (C_1^\vee \cap M_1) \times (C_2^\vee \cap M_2), \tag{3.4}$$

where $w_i \in C_i^\vee \cap M_i$. For w as in (3.4), the three characters $\chi^w, \chi^{(w_1,0)} = \phi^{w_1}, \chi^{(0,w_2)} = \psi^{w_2}$ all lie in R. Indeed, given any $v = (v_1, v_2) \in C$ with $v_i \in C_i$, and w as in (3.4), all dot product terms below are *nonnegative*: deferring to Remark III.20,

$$\langle w, v\rangle = \langle w_1, v_1\rangle + \langle w_2, v_2\rangle$$
$$\langle (w_1, 0), v\rangle = \langle w_1, v_1\rangle \geq 0, \quad \langle (0, w_2), v\rangle = \langle w_2, v_2\rangle \geq 0.$$

In particular, since $v \in C$ was arbitrary both $(w_1, 0)$ and $(0, w_2)$ lie in $C^\vee \cap M$.

Now suppose $\chi^w = \chi^{(w_1,0)}\chi^{(0,w_2)} = \phi^{w_1}\psi^{w_2} \in Q_{\rho\times\{0\}}$, i.e., $\langle w, v\rangle > 0$ for some vector $v = (v_1, v_2) \in \rho \times \{0\}$. Since $v_2 = 0$ here, equivalently $\langle w, v\rangle = \langle w_1, v_1\rangle > 0$ for some $v_1 \in \rho$, i.e., the character $\chi^{(w_1,0)} = \phi^{w_1} \in P_\rho R$. Since $\chi^{(0,w_2)} = \psi^{w_2} \in R$, $\chi^w = \phi^{w_1}\psi^{w_2} \in P_\rho R$. Thus $Q_{\rho\times\{0\}} \subseteq P_\rho R$. For the other inclusion: the characters $\chi^{(w_1,0)} = \phi^{w_1}$ as above generate $P_\rho R$, and each such generator lies in $Q_{\rho\times\{0\}}$ since we already indicated above that $\chi^w \in Q_{\rho\times\{0\}}$ if and only if $\chi^{(w_1,0)} = \phi^{w_1} \in P_\rho R$. $\square$

In case the pointed cone C is not full, the next proposition is handy.

Proposition III.22. *Let $N'_{\mathbb{R}}$ be the $\mathbb{R}$-span of a pointed cone $C \subseteq N_{\mathbb{R}}$. Set $N' = N'_{\mathbb{R}} \cap N$, and consider C as a full-dimensional cone in $N'_{\mathbb{R}}$ (relabeled as C'). Let*

$M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$ be the dual lattice. Then working over an arbitrary ground field $\mathbb{F}$, the toric ring $R_{\mathbb{F}} := \mathbb{F}[C^\vee \cap M]$ is isomorphic to $R'_{\mathbb{F}} \otimes_{\mathbb{F}} L$ where the toric ring $R'_{\mathbb{F}} := \mathbb{F}[(C')^\vee \cap M']$ and L is a Laurent polynomial ring over $\mathbb{F}$. In particular, there is a bijective correspondence between the monomial primes of $R'_{\mathbb{F}}$ and $R_{\mathbb{F}}$ given by expansion and contraction of ideals along the faithfully flat ring map $\varphi\colon R'_{\mathbb{F}} \hookrightarrow R'_{\mathbb{F}} \otimes L = R_{\mathbb{F}}$. Moreover, the divisor class groups of $R_{\mathbb{F}}$ and $R'_{\mathbb{F}}$ are isomorphic.

Proof. While Cox – Little – Schenck [12, Proof of Prop. 3.3.9] yields the first assertion, Lemma III.21 yields the second since a Laurent polynomial ring has no nonzero monomial primes. As for the class group assertion, $R_{\mathbb{F}}$ is a Laurent polynomial ring over $R'_{\mathbb{F}}$ after base change, so we may simply apply Theorem II.5. □

We now recall how to compute divisor class groups up to isomorphism when working over algebraically closed fields. Working over an algebraically closed field $\mathbb{F}$, fix a pointed cone C as in Remark III.15 and the pair of rings $R_{\mathbb{F}}$ and $R'_{\mathbb{F}}$ as in Proposition III.22. When $C \neq \{0\}$, each $\rho \in \Sigma(1)$, the collection of **rational rays (one-dimensional faces)** of C, yields a unique primitive generator $u_\rho \in \rho \cap N$ for C and a torus-invariant height one prime ideal P_ρ in $R'_{\mathbb{F}}$; cf., [12, Thm. 3.2.6]. The torus-invariant height one primes generate a free abelian group $\bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}P_\rho$ which maps surjectively onto the divisor class group of $R'_{\mathbb{F}}$. More precisely, we record the following well-known theorem; see [12, Ch. 4].

Theorem III.23. *With notation as in Proposition III.22, let $C \subseteq N_{\mathbb{R}}$ be a pointed cone with primitive generators $\Sigma(1)$ as described above. Then there is a short exact sequence of abelian groups*

$$0 \to M' \xrightarrow{\phi} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}P_\rho \to \mathrm{Cl}(R'_{\mathbb{F}}) \to 0, \tag{3.5}$$

where $\phi(m) = \mathrm{div}(\chi^m) = \sum_{\rho\in\Sigma(1)}\langle m, u_\rho\rangle P_\rho$*. Furthermore,* $\mathrm{Cl}(R_{\mathbb{F}})$ *and* $\mathrm{Cl}(R'_{\mathbb{F}})$ *are isomorphic,* $\mathrm{Cl}(R_{\mathbb{F}})$ *is finite abelian if and only if* C *is simplicial, and trivial if and only if* C *is smooth.*

Remark III.24*.* This above result follows from [12, Prop. 3.3.9, Prop. 4.1.1-4.1.2, Thm. 4.1.3, Exer. 4.1.1-4.1.2, Prop. 4.2.2, Prop. 4.2.6, and Prop. 4.2.7], essentially consolidating what facts we need to bear in mind going forward in the manuscript.

Definition III.25. The cone $C \subseteq N_{\mathbb{R}}$ is **simplicial (respectively, smooth)** if $C = \{0\}$ or the primitive ray generators form part of an $\mathbb{R}$-basis for $N_{\mathbb{R}}$ (resp., a $\mathbb{Z}$-basis for N). We also apply the adjectives simplicial and smooth to the corresponding toric algebra $R_{\mathbb{F}}$ and the toric $\mathbb{F}$-variety $\mathrm{Spec}(R_{\mathbb{F}})$.

Remark III.26*.* In algebro-geometric language, if C as in Theorem III.23 is simplicial, then all Weil divisors on $\mathrm{Spec}(R_{\mathbb{F}})$ are $\mathbb{Q}$-Cartier of index at most the order of $\mathrm{Cl}(R_{\mathbb{F}})$.

The next lemma says we can reduce all toric divisor class group computations to the case where $\mathbb{F}$ is algebraically closed, to leverage Theorem III.23.

Lemma III.27. *With notation as in Proposition III.22, the divisor class groups* $\mathrm{Cl}(R_{\mathbb{F}}) \cong \mathrm{Cl}(R_{\overline{\mathbb{F}}})$ *are isomorphic.*

Proof. By now it is clear we can reduce to the case where C is a full pointed cone in $N_{\mathbb{R}}$. The algebra $R_{\mathbb{F}}$ admits an $\mathbb{N}$-grading with its zeroth graded piece being $\mathbb{F}$; see the passage above Remark III.15. We may then cite Theorem II.6(2) to conclude that up to isomorphism, $\mathrm{Cl}(R_{\mathbb{F}}) \subseteq \mathrm{Cl}(R_{\overline{\mathbb{F}}})$ as a subgroup. This improves to an equality for normal toric rings because the divisor classes of height one monomial primes belong to both groups and generate the latter by Theorem III.23. $\square$

3.4 Drawing Connections to Convex Polytopes and Their Volumes

3.4.1 Improved Uniform Symbolic Topologies for Simplicial Toric Rings

Aside from our results on rational surface singularities in Chapter II, we have:

Theorem III.28. *Let $C \subseteq N_{\mathbb{R}}$ be a simplicial full pointed rational polyhedral cone, $\mathbb{F}$ an arbitrary field, and let $R_{\mathbb{F}} = \mathbb{F}[C^{\vee} \cap M]$ be the corresponding toric $\mathbb{F}$-algebra. Then $R_{\mathbb{F}}$ satisfies uniform symbolic topologies on all ideals of pure height one with multiplier D, where D is the volume of the parallelotope spanned by the primitive generators of C, where the volume form* Vol *on $N_{\mathbb{R}}$ is chosen so that a hypercube spanned by primitive generators of N has volume one.*

Proof. Assuming without loss of generality that $\mathbb{F}$ is algebraically closed, the short exact sequence (3.5) makes it easy to compute the divisor class group $\mathrm{Cl}(R_{\mathbb{F}})$ up to isomorphism. In practice, we pick a basis $e_1, \ldots, e_n$ of N with dual basis $e_1^*, \ldots, e_n^*$ for M, so that both N and M are isomorphic to $\mathbb{Z}^n$. Then the pairing $\langle \cdot, \cdot \rangle$ becomes dot product, and our volume form Vol agrees with Lebesgue measure on $\mathbb{R}^n$.

The collection $\Sigma(1) = \{\rho_1, \ldots, \rho_n\}$ of rays of C has n elements as C is full and simplicial. Let $u_{\rho_1}, \ldots, u_{\rho_r} \in N$ be the primitive ray generators. In terms of the isomorphism $N \cong \mathbb{Z}^n$, we express u_{ρ_i} as the column vector $(\langle e_1^*, u_{\rho_i} \rangle, \ldots, \langle e_n^*, u_{\rho_i} \rangle)^T$ where T denotes transpose. The map ϕ in Theorem III.23 can be treated, up to isomorphism, as a map $\mathbb{Z}^n \to \mathbb{Z}^n$ given by the matrix $A_C := (u_{\rho_1}, \ldots, u_{\rho_n})^T$ in terms of transpose: the i-th row of A_C is given by the coordinates of u_{ρ_i}. Theorem III.23 says $\mathrm{Cl}(R_{\mathbb{F}})$ is the cokernel of ϕ, and hence can be computed up to isomorphism by first finding the Smith normal form of A_C.

Note that since the alternating sum of the ranks in the exact sequence (3.5) vanishes, $\mathrm{Cl}(R_{\mathbb{F}})$ has rank zero and hence is finite abeilan. As A_C defines the action of ϕ, the Smith normal form indicates that $D := |\det(A_C)|$ is the order of $\mathrm{Cl}(R_{\mathbb{F}})$. In the Lebesgue measure on $\mathbb{R}^n$, this D is the volume of the parallelotope spanned by the primitive generators of C. To finish, we simply invoke Lemma II.1. $\square$

3.4.2 Teasing a Connection with Von Korff's Toric F-Signature Formula

Now fix a perfect field $\mathbb{K}$ of positive characteristic p. Given an F-finite $\mathbb{N}$-graded domain R of finite type over $\mathbb{K}$, for each integer $e \geq 0$, we have an R-module isomorphism $R^{1/p^e} \cong R^{a_e} \oplus M$ where M has no free summand, and the integer $a_e \leq p^{ed}$ where $d = \dim R$. By definition, the **F-signature** of R is (see [42], [66])

$$s(R) := \limsup_{e\to\infty} \frac{a_e}{p^{ed}} = \lim_{e\to\infty} \frac{a_e}{p^{ed}}, \quad 0 \leq s(R) \leq 1.$$

The F-signature has ties to measuring F-singularities: for instance, $s(R)$ is positive if and only if R is strongly F-regular [1], and $s(R) = 1$ if and only if R is regular [66, Thm. 4.16]; see also [33] and [57].

Over the perfect field $\mathbb{K}$, any normal toric ring is strongly F-regular and its F-signature is rational [60]. We now state Von Korff's result [67, Thm. 3.2.3]; see also Watanabe – Yoshida [72, Thm. 5.1] and Yao [73, Rem. 2.3(4)]:

Theorem III.29 (cf., Von Korff [67, Thm. 3.2.3])**.** *With notation as in Proposition III.22, we define a convex polytope,*

$$P_{C'} := \{w \in M'_{\mathbb{R}} \colon 0 \leq \langle w, v\rangle < 1, \forall v \in G\} \subsetneq (C')^{\vee},$$

where G is the set of primitive generators of $C' \neq \{0\}$. Then over any perfect field $\mathbb{K}$ of positive characteristic, the F-signature $s(R_{\mathbb{K}}) = s(R'_{\mathbb{K}}) = \mathrm{Vol}(P_{C'}) \in \mathbb{Q}_{>0}$, where the volume form Vol on $M'_{\mathbb{R}}$ is chosen so that a hypercube spanned by primitive generators of M' has volume one.

Corollary III.30. *When the cone C' in Theorem III.29 is simplicial, $s(R'_{\mathbb{K}}) = 1/D$ where the integer $D = \#\operatorname{Cl}(R'_{\mathbb{K}})$.*

Proof. We may reduce to the case where C is full and $\mathbb{K}$ is algebraically closed via Theorem III.23, Lemma III.27, and Theorem III.29. We then pick a basis $e_1, \ldots, e_n$ for the lattice N and the dual basis $e_1^*, \ldots, e_n^*$ for M as in the proof of Theorem III.28, so the volume form Vol agrees with the Lebesgue measure on $\mathbb{R}^n$. Then one can confirm the claim that $s(R_{\mathbb{K}}) \cdot \#\operatorname{Cl}(R_{\mathbb{K}}) = 1$ either via linear algebra or by applying the change-of-variables formula from multivariate calculus: either way, invoking the proof of Theorem III.28 is key.

Let $u_1, \ldots, u_n$ be the primitive generators for C, indexed so that the matrix $A = A_C$ from the proof of Theorem III.28 is positive-definite. Defining new coordinates $\ell_i := \ell_i(m) = \langle m, u_i \rangle$, for all $m = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set

$$\Cube_n := \{(\ell_1, \ldots, \ell_n) \in \mathbb{R}^n \colon 0 \leq \ell_i < 1\},$$

the unit n-cube in $\mathbb{R}^n$ in the coordinates $\ell_1, \ldots, \ell_n$ defined above. Let

$$L_A \colon \mathbb{R}^n_{(x_1,\ldots,x_n)} \xrightarrow{\cong} \mathbb{R}^n_{(\ell_1,\ldots,\ell_n)}$$

denote the $\mathbb{R}$-linear change-of-coordinates map defined by A. As the polytope $P_C = L_A^{-1}(\Cube_n)$, basic linear algebra indicates that

$$1 = \operatorname{Vol}(\Cube_n) = \det(A) \cdot \operatorname{Vol}(P_C).$$

The corollary then follows by the proof of Theorem III.28. $\square$

Remark III.31. Given Theorem III.29, we could opt to replace the invariant T from Corollary III.2 with the possibly larger invariant featured in Theorem I.41:

$$U = \operatorname{lcm}\{\max_{m \in \mathcal{B}} \langle m, v_C \rangle, \#\operatorname{Cl}(R_{\mathbb{K}})\} \in \mathbb{Z}_{>0}.$$

By Corollary III.30, U is an integer multiple of the reciprocal $1/s(R_{\mathbb{K}}) = \#\operatorname{Cl}(R_{\mathbb{K}})$ of the F-signature of $R_{\mathbb{K}}$ that controls the asymptotic growth of symbolic powers.

We briefly revisit this connection in our closing Chapter V.

3.5 (Non-)Sharp Multipliers, Segre – Veronese algebras revisited

This section is complementary in part to Section 2.4. To start, we deduce a result that occasionally provides first examples of sharp multipliers in the toric setting.

Proposition III.32. *With notation as in Corollary III.2, we assume C is a simplicial full pointed rational polyhedral cone. We now set $B := \max_{w\in\mathcal{PG}} \langle w, v_C\rangle$ where $\mathcal{PG} \subseteq \mathcal{B}$ consists of the primitive generators of $C^\vee$. There exists a monomial prime P in $R = R_{\mathbb{F}}$ of height one such that:*

1. *$P^{(B(r-1))} \not\subseteq P^r$ for some $r \geq 2$;*
2. *There is no positive integer $D' < B$ such that $P^{(D'(r-1)+1)} \subseteq P^r$ for all $r > 0$.*

Proof. Let $v_1, \ldots, v_n \in N$ and $w_1, \ldots, w_n \in M$ denote the primitive generators for C and for $C^\vee$, respectively. We index these generators so that the nonnegative integer $\langle w_j, v_i\rangle$ is positive if and only if $i = j$: we may do this citing the notion of facet normals [12, after Prop. 1.2.8]. In deference to Lemma III.4(3.2), let P_j $(1 \leq j \leq n)$ be the height one monomial prime in $R_{\mathbb{F}}$ such that a monomial $\chi^m \in P_j$ if and only if $\langle m, v_j\rangle > 0$. In particular, $\chi^{w_j} \in P_j$ for each j.

Next we show that for each $1 \leq j \leq n$, $\langle w_j, v_j\rangle$ is the order of the element in $\operatorname{Cl}(R_{\mathbb{F}})$ corresponding to P_j. We may leverage exact sequence (3.5) from Theorem III.23, since Lemma III.27 allows us reduce to the case where $\mathbb{F}$ is algebraically closed. For $1 \leq j \leq n$, we have $0 = [\operatorname{div}(\chi^{w_j})] = \langle w_j, v_j\rangle[D_{\rho_j}]$, where ρ_j is the rational ray of C generated by v_j. Thus $P_j^{(\langle w_j, v_j\rangle)} = (\chi^{w_j})R$. Since the order of $[D_{\rho_j}]$ is the smallest

$E_j > 0$ such that $P_j^{(E_j)} = (\chi^{m_j})R$ is principal for some $m_j \in C^\vee \cap M - \{0\}$, E_j divides $\langle w_j, v_j\rangle$, and $P_j^{(\langle w_j,v_j\rangle)} = (P_j^{(E_j)})^L$ where $L = \langle w_j, v_j\rangle/E_j$. As $C^\vee$ is pointed, $\chi^{w_j} = \chi^{L\cdot m_j}$ and $L = 1$, since $w_j \in \mathcal{B}$ is an irreducible vector in $C^\vee \cap M$.

To prove (1), notice $B = \langle w_{j_0}, v_{j_0}\rangle$ for some $1 \le j_0 \le n$. Then

$$(*): \quad (\chi^{w_{j_0}})R = P_{j_0}^{(B)} \not\subseteq P_{j_0}^2.$$

By Remark III.16, when the cone C is full-dimensional in $N_{\mathbb{R}}$, the semigroup algebra $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ can be $\mathbb{N}$-graded. Thus any minimal generator f of a homogeneous ideal I satisfies $f \in I - I^2$ by Nakayama's lemma. In our situation, $I = P_{j_0}$ and $f = \chi^{w_{j_0}}$. Observation (*) also gives part (2), arguing by contradiction and using Lemma IV.12 accordingly. $\square$

We offer several examples to show that establishing sharpness of our bilinear multipliers is a delicate matter meriting further study.

Example III.33. We fix integers $n \ge 2$ and $E \ge 2$, and an arbitrary field $\mathbb{F}$. Let $V_{E,n}$ be the E-th Veronese subalgebra of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$, that is, the $\mathbb{F}$-algebra generated by all monomials of degree E in $x_1, \ldots, x_n$. Then $P^{(E(r-1)+1)} \subseteq P^r$ for all $r > 0$, all monomial primes, and all primes of height one by Corollary III.2; see Lemma III.43 for details. However, for any $E' < E$, the proof of Proposition III.32 guarantees that we can find a prime $P \subseteq V_{E,n}$ (monomial, height one) such that $P^{(E'(r-1)+1)} \not\subseteq P^r$ for some $r \ge 2$, namely, for $r = 2$. In fact, this last observation holds for all monomial primes in $V_{E,n}$, aside from the zero ideal and the maximal monomial ideal for which $E' = 1$ will do; see the proof of Theorem III.44 below.

Despite Example III.33, Corollary III.2 does not give sharp multipliers in general.

Example III.34. For any $n > 2$, let $R = \mathbb{F}[Z, X_1, \ldots, X_n]/(Z^2 - X_1 \cdots X_n)$ as in

Example III.10. Citing the proof of Theorem III.42, when $P \subseteq R$ is any monomial prime of height at least 2:

- $P^{(r)} = P^r$ for all $r > 0$; however,
- The multiplier D' corresponding to P in Lemma III.5 always satisfies $D' \geq 2$.

Theorem III.1 gives a uniform multiplier D that works for all monomial primes. Even when this multiplier is sharp across all monomial primes, it need not be best possible for all monomial primes of a given height, contrasting with Example III.33.

Example III.35. Let $R = \mathbb{F}[C^\vee \cap \mathbb{Z}^3] = \mathbb{F}[x,y,z,w]/(xy - zw)$, for the non-simplicial cone $C \subseteq \mathbb{R}^3$ with $e_1, e_2, e_1 + e_3, e_2 + e_3$ as primitive generators. Theorem III.1 says $P^{(2r-1)} \subseteq P^r$ for all $r > 0$ and all monomial primes in R, observing that $C^\vee \cap \mathbb{Z}^3$ is minimally generated by $\mathcal{B} = \{e_1, e_2, e_3, (1,1,-1)\}$. Given any height two monomial prime P in R, these containments cannot be improved to $P^{(r)} = P^r$ for all $r \geq 2$. For instance, if $P = (x,y,z)R$, then for any $s \geq 1$, $z^s \in P^{(2s)} - P^{2s}$ and $z^{s+1} \in P^{(2s+1)} - P^{2s+1}$: indeed, $w^s \in R - P$ and R can be standard graded, so the least degree of a homogeneous element of P^r is r. By contrast, $P^{(r)} = P^r$ for all r and for any height one monomial prime P in R: the invariant $D' = 1$ in Lemma III.5 via direct computation.

Remark III.36. To reiterate, Carvajal-Rojas and Smolkin show $D = 2$ works for all primes in $R = \mathbb{F}[x,y,z,w]/(xy - zw)$ over a perfect field $\mathbb{F}$ of positive characteristic [11], translating to a verbatim analogue in characteristic zero. See also [56].

We do not currently see how to extend Theorems III.1 or III.14 to a form covering all prime ideals. However, we now deduce results that make a bit of progress in this direction. We work with select primes which are homogeneous with respect to the standard grading on a Segre – Veronese algebra. We start with the case of Segre products of polynomial rings, where the proof is simpler to record.

Theorem III.37. *Working over an algebraically closed field $\mathbb{F}$, let R be the homogeneous coordinate ring of the image $\Sigma \subseteq \mathbb{P}^A$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_T}$ under the Segre embedding map. Let P be any homogeneous prime ideal in R of height* $\dim(R) - 1 = n_1 + \cdots + n_T$. *Then*

$$P^{(Tr)} \subseteq P^{(T(r-1)+1)} \subseteq P^r \tag{3.6}$$

for all $r > 0$.

Proof. Consider the transitive action of $G = \prod_{i=1}^{T} \mathrm{PGL}(n_i + 1, \mathbb{F})$ on $\prod_{i=1}^{T} \mathbb{P}^{n_i}$ and the action of G on R via ring isomorphisms. Any prime P as stated corresponds to a unique point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_T}$. By transitivity, there exists some $g \in G$ inducing a ring isomorphism $g \colon R \to R$ such that $g(P) = \widetilde{P}$ where $\widetilde{P}$ is the monomial prime ideal corresponding to the torus-invariant point

$$([0\colon \cdots : 0\colon 1], [0\colon \cdots : 0\colon 1], \ldots, [0\colon \cdots : 0\colon 1]) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_T}.$$

Since (3.6) holds for $\widetilde{P}$ by Theorem III.14, by applying g^{-1} we conclude that (3.6) holds for P as well. □

Theorem III.38. *Working over an algebraically closed field $\mathbb{F}$, let R be the homogeneous coordinate ring of the image $\Sigma \subseteq \mathbb{P}^A$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_T}$ under the Segre – Veronese embedding map determined by a line bundle $\mathcal{O}(D_1, \ldots, D_T)$ corresponding to the degree sequence $(D_1, \ldots, D_T) \in (\mathbb{Z}_{\geq 1})^T$. Let P be any homogeneous prime ideal in R of height* $\dim(R) - 1 = n_1 + \cdots + n_T$. *Then*

$$P^{(Er)} \subseteq P^{(E(r-1)+1)} \subseteq P^r \tag{3.7}$$

for all $r > 0$, where the multiplier $E := \sum_{i=1}^{T} D_i$.

Proof. Let $G = \mathrm{PGL}(n_1+1, \mathbb{F}) \times \cdots \times \mathrm{PGL}(n_T+1, \mathbb{F})$. We note that G acts transitively on the Segre – Veronese variety Σ. For R as stated, note that

$$R \subseteq S = \mathbb{F}[u_{ij} \colon 1 \leq i \leq T, 0 \leq j \leq n_i]$$

is the $\mathbb{N}^T$-graded subalgebra generated by monomials of degree $(D_1, \ldots, D_T)$–degree D_i in the u_{ij}'s for a given i, each of which has degree $D_i \cdot e_i$ in terms of standard basis vectors. We note that G acts on S and takes elements of degree $(D_1, \ldots, D_T)$ to elements of degree $(D_1, \ldots, D_T)$. Thus G acts on R.

A prime $P = P_x$ in R of height $\dim R - 1$ corresponds to a point x in Σ. By transitivity, there exists $g \in G$ sending x to a torus-invariant point $y \in \Sigma$. The corresponding ring isomorphism $g \colon R \to R$ sends P_x to the monomial prime ideal $g(P_x) = P_y$ in R. Since (3.7) holds for P_y by Theorem III.14 for E as stated, by applying g^{-1} we conclude that (3.7) holds for P_x as well. $\square$

Remark III.39. In Theorems III.37 and III.38, if $\mathbb{F}$ is not algebraically closed, the statement still holds for homogeneous primes that remain prime after base changing to the algebraic closure $\overline{\mathbb{F}}$; see Proposition IV.9 to aid in digesting this point.

3.6 Wrap Up: Further Example Computations

Several of the examples above refer to work going into proving the following theorem in [69, Sec. 4]. We rehearse this work below for convenience.

Theorem III.40. *Let $S = \mathbb{F}[x_1, \ldots, x_n]$ $(n \geq 1)$ be a polynomial ring over an arbitrary field $\mathbb{F}$. Consider the finite extensions of normal toric rings $V_D \subseteq S \subseteq H_D$, where*

1. *$V_D \subseteq S$ is the D-th Veronese subring with its standard $\mathbb{N}$-grading, and*

2. *$H_D = \mathbb{F}[z, x_1, \ldots, x_n]/(z^D - x_1 \cdots x_n)$ is a hypersurface ring.*

Then $P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all monomial prime ideals P in each ring.

Theorem III.40 is easy if $n = 1$ or $D = 1$: all rings in sight are polynomial rings and monomial primes are complete intersections. Thus going forward, **we will assume that $n \geq 2$ and $D \geq 2$.** We will give presentations of our rings as subrings of the domain of Laurent polynomials $L = \mathbb{F}[s_1^{\pm 1}, \ldots, s_{n-1}^{\pm 1}, u^{\pm 1}]$ in n indeterminates over the field $\mathbb{F}$. The proof will proceed in cases, starting with the ring $H_D = \frac{\mathbb{F}[x_1,\ldots,x_n,z]}{(z^D - x_1 \cdots x_n)}$.

In practice, we pick a basis $e_1, \ldots, e_n$ of N with dual basis $e_1^*, \ldots, e_n^*$ for M, so that both N and M are isomorphic to $\mathbb{Z}^n$. Then the pairing $\langle \cdot, \cdot \rangle$ becomes dot product.

In the proofs of Theorems III.42 and III.44 below, we implicitly use the fact that for any prime ideal P in a Noetherian ring R, $P^{(E)} = P^E :_R (s)^\infty := \bigcup_{j \geq 0} (P^E :_R s^j)$ for any $s \notin P$ belonging to all associated primes of P^E. See [13, Ch. 3] for details.

The Hypersurface Case:

We first observe that H_D is a toric ring, up to isomorphism:

Lemma III.41. *Consider the full-dimensional simplicial pointed rational polyhedral cone $\sigma_D^{(n)} \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ whose ray generators are $De_i + e_n \in N$ for $1 \leq i < n$ and $e_n \in N$ in terms of the selected basis for N.*

1. *The Hilbert basis of the semigroup $(\sigma_D^{(n)})^\vee \cap M$ consists of $n+1$ vectors: the n dual basis vectors $e_1^*, \ldots, e_n^*$, together with the vector $-e_1^* \cdots - e_{n-1}^* + De_n^* \in M$.*

2. *The toric ring $\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M] \cong \frac{\mathbb{F}[x_1,\ldots,x_n,z]}{(z^D - x_1 \cdots x_n)} = H_D$.*

Proof. The reader can use the hilbertBasis algorithm implemented in the `Polyhedra` package in Macaulay2 [25] to check (1). For (2), recall that to each $m = \sum_{i=1}^n m_i e_i^* \in (\sigma_D^{(n)})^\vee \cap M$ we assign a Laurent monomial $\chi^m = s_1^{m_1} \cdots s_{n-1}^{m_{n-1}} u^{m_n}$ in the semigroup ring $\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]$. Given (1), in terms of $\mathbb{F}$-algebra generators we have

$$\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M] = \mathbb{F}\left[s_1, \ldots, s_{n-1}, \frac{u^D}{(s_1 \cdots s_{n-1})}, u\right] \subseteq \mathbb{F}[s_1^{\pm 1}, \ldots, s_{n-1}^{\pm 1}, u^{\pm 1}].$$

Given a polynomial ring $R = \mathbb{F}[x_1, \ldots, x_{n-1}, x_n, z]$ in $n+1$ variables, consider the surjective algebra map $\phi\colon R = \mathbb{F}[x_1, \ldots, x_{n-1}, x_n, z] \twoheadrightarrow \mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]$ under which $x_i \mapsto s_i$ for each $1 \leq i \leq n-1$, $x_n \mapsto \frac{u^D}{(s_1 \cdots s_{n-1})}$, and $z \mapsto u$. Since $\dim(R) = \dim(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]) + 1$, we conclude that the kernel of ϕ is a height one prime in the UFD R, and hence is principal. Now $F = z^D - x_1 \cdots x_n \in R$ is irreducible by Eisenstein's Criterion and belongs to the kernel of ϕ, so $\ker \phi = (F)$, and the isomorphism claim follows. $\square$

We now deduce the following refinement of Theorem III.40 for H_D:

Theorem III.42. *Take the ring $H_D = \mathbb{F}[x_1, \ldots, x_n, z]/(z^D - x_1 \cdots x_n)$, and P one of the monomial prime ideals of H_D (i.e., M-graded /torus-invariant); assume P is nonzero and nonmaximal. When $D \leq \operatorname{ht}(P)$ (the **height** of P), $P^{(E)} = P^E$ for all $E > 0$. If $D \geq \operatorname{ht}(P)$ and $E \equiv 1 \pmod{D}$, then*

$$P^{(E)} \subseteq P^{\operatorname{ht}(P)\left(\frac{E-1}{D}\right)+1}.$$

In particular, $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^{\operatorname{ht}(P)(r-1)+1} \subseteq P^r$ for all $r > 0$.

Proof. To start, the height j prime ideal $P_j := (z, x_1, \ldots, x_j)H_D$, for $1 \leq j \leq n-1$, equals P_τ for the j-dimensional face τ of $\sigma_D^{(n)}$ generated by $De_i + e_n$ for $1 \leq i \leq j$. As a saturation, $P_j^{(E)} = P_j^E :_{H_D} (\prod_{i=j+1}^n x_i)^\infty$. Since $P_j^{(E)}$ is monomial, in chasing down inclusions below it suffices to discern which monomial classes $g = (z^\ell x_1^{a_1} \cdots x_j^{a_j})(x_{j+1}^{a_{j+1}} \cdots x_n^{a_n}) \in H_D$ multiply a power of $m = \prod_{i=j+1}^n x_i$ into P_j^E. For g

as above, by definition $g \in P_j^{(E)}$ if and only if for all $T \gg 0$,

$$\begin{aligned} P_j^E \ni m^T g &= z^\ell \left(\prod_{i=j+1}^{n} x_i^{a_i+T}\right)\left(\prod_{i=1}^{j} x_i^{a_i}\right) \\ &= z^\ell \left(\prod_{i=1}^{n} x_i\right)^{T'} \left(\prod_{i=j+1}^{n} x_i^{a_i+T-T'}\right)\left(\prod_{i=1}^{j} x_i^{a_i-T'}\right) \\ &= \left(z^{D\cdot T'+\ell}\prod_{i=1}^{j} x_i^{a_i-T'}\right)\left(\prod_{i=j+1}^{n} x_i^{a_i+T-T'}\right) \end{aligned}$$

where $T' = T'(T) := \min(a_1, \ldots, a_j, a_{j+1}+T, \ldots, a_n+T) = \min(a_1, \ldots, a_j)$ for all $T \gg 0$. We conclude that $z^{D\cdot T'+\ell}\left(\prod_{i=1}^{j} x_i^{a_i-T'}\right) \in P_j^E$, and infer the inequality

$$(D-j)T' + \left(\sum_{i=1}^{j} a_i\right) + \ell \geq E. \tag{3.8}$$

Before proceeding, notice that since $T' \geq 0$, when $D \leq j$ so that the number $(D-j)T'$ is nonpositive, (3.8) implies that $\left(\sum_{i=1}^{j} a_i\right) + \ell \geq E$, so $(z^\ell x_1^{a_1} \cdots x_j^{a_j}) \in P_j^E$ and hence $g \in P_j^E$ already. Thus $P_j^{(E)} = P_j^E$ for all $E > 0$ when $D \leq j$, since both are generated by monomial classes. Thus in the remainder of the proof **we will assume that** $D \geq j = \mathrm{ht}(P_j)$, i.e., $D - j \geq 0$.

In this case, assuming $E \equiv 1 \pmod{D}$, we now show that $P_j^{(E)} \subseteq P_j^{1+j\left(\frac{E-1}{D}\right)}$. Fix a monomial

$$g = \left(z^\ell \prod_{i=1}^{j} x_i^{a_i}\right)\left(\prod_{i=j+1}^{n} x_i^{a_i}\right) \in P_j^{(E)},$$

and $T' = \min(a_1, \ldots, a_j)$ exactly as before. Now $g \in P_j^G$ where $G := \ell + \sum_{i=1}^{j} a_i$. The more involved case for us is when (**) $T' \leq (E-1)/D$: otherwise

$$G \geq a_1 + \cdots + a_j \geq jT' \geq j(E-1)/D + 1,$$

whence one easily infers that $g \in P_j^{j\left(\frac{E-1}{D}\right)+1}$. Assuming (**), we now show that $G \geq j\left(\frac{E-1}{D}\right) + 1$. Suppose to the contrary that $G \leq j\left(\frac{E-1}{D}\right)$. Since $g \in P_j^{(E)}$, inequality (3.8) above says

$$(D-j)T' + G = (D-j)T' + \left(\sum_{i=1}^{j} a_i\right) + \ell \geq E \Longrightarrow G \geq E - (D-j)T'.$$

Then since $E - 1 - DT' \geq 0$ by (**), and $D - j \geq 0$, we see that

$$\begin{aligned} j\,(E-1) = Dj\left(\frac{E-1}{D}\right) \geq DG \geq DE - D(D-j)T' & \\ &= D(E-1) + D - D(D-j)T' \\ &= j(E-1) + D + (D-j)(E-1-DT') \\ &\geq j(E-1) + D + (D-j)(0) \\ &= j(E-1) + D \end{aligned}$$

a contradiction. Thus $G \geq j\left(\frac{E-1}{D}\right) + 1$, so $g \in P_j^{1+j\left(\frac{E-1}{D}\right)}$. In particular, when $E = D(r-1)+1$, we have $P_j^{(D(r-1)+1)} \subseteq P_j^{1+j(r-1)}$. Finally, applying coordinate changes according to every permutation of $x_{[n]} := \{x_1, \ldots, x_n\}$, any (nonzero, nonmaximal) monomial prime ideal in H_D can be obtained from the P_j running through all indices $1 \leq j \leq n-1$, along with obtaining the desired containments. $\square$

The Veronese Case:

Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denote the set of **nonnegative integers**. To start,

Lemma III.43. *Consider the full-dimensional simplicial pointed rational polyhedral cone* $\eta_D^{(n)} \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ *whose ray generators are* e_i *for* $1 \leq i < n$ *along with the vector* $-e_1 - \ldots - e_{n-1} + De_n$ *in terms of the basis selected for* N*.*

1. *The Hilbert basis of the semigroup* $(\eta_D^{(n)})^\vee \cap M$ *is the set of vectors*

$$\left\{ e_n^* + \sum_{i=1}^{n-1} a_i e_i^* \in M \colon \text{all } a_i \geq 0 \text{ and } 0 \leq \sum_{i=1}^{n-1} a_i \leq D \right\}.$$

2. *The toric ring* $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] \cong V_D$*, the* D*-th Veronese subring of the polynomial ring* $\mathbb{F}[s_1, \ldots, s_{n-1}, u]$ *in the* n *indeterminates* $s_1, \ldots, s_{n-1}, u$*.*

Proof. The reader can use the hilbertBasis algorithm implemented in the `Polyhedra` package in Macaulay2 [25] to check (1). Given (1), as an algebra over $\mathbb{F}$, we have

$$\begin{aligned}\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] &= \mathbb{F}\left[s_1^{a_1}\cdots s_{n-1}^{a_{n-1}}u \colon \text{each } a_i \geq 0,\, 0 \leq \sum_{i=1}^{n-1} a_i \leq D\right] \\ &\cong \frac{\mathbb{F}[x_{(a_1,\ldots,a_{n-1})} \colon \text{each } a_i \geq 0,\, 0 \leq \sum_{i=1}^{n-1} a_i \leq D]}{(x_e x_f - x_g x_h \colon e+f = g+h \in \mathbb{N}^{n-1})}.\end{aligned}$$

Within the polynomial ring $\mathbb{F}[s_1,\ldots,s_{n-1},u]$, applying the correspondence

$$s_1^{a_1}\cdots s_{n-1}^{a_{n-1}}u \longleftrightarrow s_1^{a_1}\cdots s_{n-1}^{a_{n-1}}u^{D-a_1-\cdots-a_{n-1}}$$

takes the generators in the presentation of $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M]$ and recovers the usual presentation of V_D in terms of degree D monomials in n variables. Therefore, (2) holds: $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] \cong V_D$. $\square$

We use the toric presentation of V_D to deduce the following refinement of Theorem (III.40) for V_D:

Theorem III.44. *Over an arbitrary field $\mathbb{F}$, take the D-th Veronese subring $V_D \subseteq \mathbb{F}[s_1,\ldots,s_{n-1},u]$ and P one of the monomial prime ideals of V_D. When P is nonzero and nonmaximal, $P^{(E)} \subseteq P^r$ if and only if $r \leq \lceil E/D \rceil$. In particular, $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and the right-hand containment is sharp.*

Proof. For all $1 \leq j \leq n-1$, define height one primes

$$P_j = P_{e_j} = \left(s_1^{a_1}\cdots s_{n-1}^{a_{n-1}}u \colon a_j > 0, \text{ and } 1 \leq \sum_{b=1}^{n-1} a_b \leq D\right)V_D.$$

Then by the **Minkowski sum-ideal sum decomposition** (3.3) $P_{j_1<\cdots<j_k} := P_{j_1} + \cdots + P_{j_k}$ is a prime of height $1 \leq k \leq n-1$ for each size-k subset $j_1 < \ldots < j_k$ of $[n-1] = \{1,\ldots,n-1\}$. In particular, we focus on $P_{1<\cdots<k} = (s^{\bar{a}}u \colon \bar{a} \in T_k)V_D$, where

$$T_k := \left\{\bar{a} = (a_1,\ldots,a_{n-1}) \in \mathbb{N}^{n-1} \colon 1 \leq \sum_{b=1}^{k} a_b \leq \sum_{b=1}^{n-1} a_b \leq D\right\}.$$

Any monomial g in $P^{(E)}_{1<\cdots<k} \subseteq P_{1<\cdots<k} \subseteq P_{1<\cdots<n-1}$ belongs to $P_{1<\cdots<k}$ and so decomposes (for some $B \geq 0$) as

$$g = u^B \prod_{\bar{a}\in T_{n-1}} (s^{\bar{a}}u)^{i_{\bar{a}}} = \prod_{\bar{a}\in T_k} (s^{\bar{a}}u)^{i_{\bar{a}}} \left(u^B \prod_{\bar{a}\in T_{n-1}-T_k} (s^{\bar{a}}u)^{i_{\bar{a}}} \right) \in P^{\sum_{\bar{a}\in T_k} i_{\bar{a}}}_{1<\cdots<k}.$$

Note that this factorization of g into two monomial pieces (T_k versus $T_{n-1} - T_k$) is unique up to applying the Veronese relations $s^{\bar{e}}u \cdot s^{\bar{f}}u = s^{\bar{g}}u \cdot s^{\bar{h}}u \quad (\bar{e} + \bar{f} = \bar{g} + \bar{h})$. Setting the monomial $m := u \cdot \prod_{\bar{a}\in T_{n-1}-T_k} s^{\bar{a}}u \in V_D$ to be the **product** of the monomials $s_1^{a_1} \cdots s_{n-1}^{a_{n-1}}u$ with $a_j = 0$ for all $1 \leq j \leq k (\leq n-1)$, we have $P^{(E)}_{1<\cdots<k} = P^E_{1<\cdots<k} :_{V_D} (m)^\infty$, and the monomial g is in $P^{(E)}_{1<\cdots<k}$ precisely when for all $T \gg 0$,

$$g \cdot m^T = \left(u^{B+T} \prod_{\bar{a}\in T_k} (s^{\bar{a}}u)^{i_{\bar{a}}} \right) \prod_{\bar{a}\in T_{n-1}-T_k} (s^{\bar{a}}u)^{i_{\bar{a}}+T} \in P^E_{1<\cdots<k}.$$

In particular, the monomial in parentheses is in $P^E_{1<\cdots<k}$ so it is a multiple of some E-fold product of generators of $P_{1<\cdots<k} = (s^{\bar{a}}u \colon \bar{a} \in T_k)V_D$. Thus we infer that two inequalities must hold, signifying we have enough u's and s_j's ($1 \leq j \leq k$) at our disposal, respectively, to feasibly form such a E-fold product. These inequalities are **(1)** $\sum_{\bar{a}\in T_k} i_{\bar{a}} + B + T \geq E$, and **(2)** the sum

$$\sum_{\bar{a}\in T_k} i_{\bar{a}}(a_1 + \cdots + a_k) = \sum_{j=1}^{D} \ell_j \cdot j \geq E,$$

where $\ell_j := \sum_{\bar{a}\in T_{k,j}} i_{\bar{a}}$, $T_{k,j} := \{\bar{a} \in T_k \colon$ the partition $a_1 + \cdots + a_k = j\}$. Indeed,

$$E \leq \sum_{j=1}^{D} \ell_j \cdot j \leq D \left(\sum_{j=1}^{D} \ell_j \right) \Longrightarrow \sum_{j=1}^{D} \ell_j \geq \lceil E/D \rceil,$$

so **(2)** implies that **(3)** $\sum_{\bar{a}\in T_k} i_{\bar{a}} = \sum_{j=1}^{D} \ell_j \geq \lceil E/D \rceil$.[1] For any monomial $g \in P^{(E)}_{1<\cdots<k}$, **(3)** implies that $g \in P^{\lceil E/D \rceil}_{1<\cdots<k}$. Thus $P^{(E)}_{1<\cdots<k} \subseteq P^{\lceil E/D \rceil}_{1<\cdots<k}$ for all $E > 0$.

[1] Together, inequalities **(1)** and **(3)** are equivalent to

$$\sum_{\bar{a}\in T_k} i_{\bar{a}} = \sum_{j=1}^{D} \ell_j \geq \max\{\lceil E/D \rceil, E - (B+T)\} \equiv \lceil E/D \rceil \text{ for all } T \geq E.$$

Additionally if we consider R with its **standard $\mathbb{N}$-grading**, then the minimal degree of a monomial (e.g., a monomial generator) in $P^r_{1<\cdots<k}$ is r. Noticing that for $1 \leq j \leq k$, the degree $\lceil E/D \rceil$ monomial $(s_j^D u)^{\lceil E/D \rceil} \in P^E_{1<\cdots<k} : (u^{(E+1)-\lceil E/D \rceil}) \subseteq P^E_{1<\cdots<k} : (m^{(E+1)-\lceil E/D \rceil}) \subseteq P^{(E)}_{1<\cdots<k}$, we obtain the only-if part of: for each $1 \leq k \leq n$, $P^{(E)}_{1<\cdots<k} \subseteq P^r_{1<\cdots<k}$ if and only if $r \leq \lceil E/D \rceil$.

Setting $E = Dr - (D-1) = D(r-1)+1$, we have $\lceil E/D = (r-1) + 1/D \rceil = r$, so that $P^{(Dr-(D-1))}_{1<\cdots<k} \subseteq P^r_{1<\cdots<k}$ for all $r > 0$ and this containment is sharp.

In review, our argument **does not** depend crucially on which size-k index subset $j_1 < \ldots < j_k$ of $[n] = \{1, 2, \ldots, n\}$ we worked with; going with $1 < 2 < \ldots < k$ merely simplifies notation. In other words, in applying suitable permutations of the algebra generators for V_D, one obtains the above characterization of ideal containment for all of the monomial prime ideals in the ring **having one of the P_j as an ideal summand.** To handle monomial primes having the height one prime

$$P_{(-1,\ldots,-1,D)} = \left(s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \colon 0 \leq \sum_{i=1}^{n-1} a_i \leq D-1 \right)$$

as a summand, we use the $\mathbb{F}$-algebra isomorphisms $\phi_j \colon V_D \to V_D$ $(1 \leq j \leq n-1)$ under which a monomial algebra generator $g = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ with $0 \leq A := \sum_{i=1}^{n-1} a_i \leq D$ is sent to

$$\phi_j(g) = \begin{cases} s_1^{a_1} \cdots s_j^{D-A} \cdots s_{n-1}^{a_{n-1}} u & \text{if } A \leq D-1 \text{ and } a_j = 0 \\ s_1^{a_1} \cdots s_j^{0} \cdots s_{n-1}^{a_{n-1}} u & \text{if } A = D \text{ and } a_j > 0 \\ g & \text{if } A \leq D-1 \text{ and } a_j > 0 \\ g & \text{if } A = D \text{ and } a_j = 0. \end{cases}$$

We note that $\phi_j^2 = \phi_j \circ \phi_j$ is the identity, and the height one prime $\phi_j(P_{(-1,\ldots,-1,D)}) = P_j$: indeed, when $h = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ is a generator of P_j, $a_j > 0$; when $A \leq$

$D-1$, $h=\phi_j(h)$, or else $D-A=0$, $a_j = D-\left(\sum_{1\le i\ne j\le n-1} a_i\right) > 0$, and $h = \phi_j(g)$ where $g = s_1^{a_1}\cdots s_j^0 \cdots s_{n-1}^{a_{n-1}} u \in P_{(-1,\ldots,-1,D)}$. Moreover, we conclude that a (sharp) containment $Q^{(m)} \subset Q^r$ for any monomial prime Q with P_j as a summand translates under ϕ_j to a (sharp) containment $(Q')^{(m)} \subset (Q')^r$ for a monomial prime Q' of the same height as Q, with $P_{(-1,\ldots,-1,D)}$ replacing P_j as an ideal summand. Having analyzed ideals with one of the P_j as a summand quite thoroughly, this final observation completes the proof. □

To close the chapter, we now account for divisor class group computations cited in select examples recorded earlier.

Remark III.45. With notation as in Theorem III.23 and in the first paragraph of the proof of Theorem III.28, we note that if $C \subseteq N_{\mathbb{R}}$ is a full pointed rational polyhedral cone, then we have the following presentation for the divisor class group:

$$\mathrm{Cl}(\mathbb{F}[C^\vee \cap M]) \cong \frac{\bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot [D_\rho]}{\langle \sum_{\rho\in\Sigma(1)} \langle e_i^*, u_\rho\rangle [D_\rho] = 0 \colon 1\le i\le n\rangle},$$

where the $e_i^* \in M$ form the dual basis to the basis $e_1,\ldots,e_n \in N$ chosen in N.

Example III.46. We work with the polyhedral cones in the proof of Theorem III.40, showing that $\mathrm{Cl}(H_D) \cong (\mathbb{Z}/D\mathbb{Z})^{n-1}$ and $\mathrm{Cl}(V_D) \cong \mathbb{Z}/D\mathbb{Z}$. Although these class group facts are well known in certain circles and can be deduced by other means (see e.g., [61]), for completeness of exposition we include succinct computations.

1. The cone $\sigma_D^{(n)} \subseteq N_{\mathbb{R}}$ has ray generators $f_i = De_i + e_n$ for $1 \leq i < n$ and e_n, and

$$\begin{aligned}
\mathrm{Cl}(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n]) &\cong \frac{\mathbb{Z} \cdot [\mathbf{D_{e_n}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot [D_{f_i}]}{\langle D[D_{f_i}] = 0 \ (1 \leq i < n), [\mathbf{D_{e_n}}] = -[\mathbf{D_{f_1}}] - \cdots - [\mathbf{D_{f_{n-1}}}] \rangle} \\
&\cong \frac{\mathbb{Z} \cdot -[\mathbf{D_{f_1}}] - \cdots - [\mathbf{D_{f_{n-1}}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot [D_{f_i}]}{\langle D[D_{f_1}] = 0, \ldots, D[D_{f_{n-1}}] = 0 \rangle} \\
&= \frac{\bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot [D_{f_i}]}{\langle D[D_{f_1}] = 0, \ldots, D[D_{f_{n-1}}] = 0 \rangle} \\
&\cong \boxed{(\mathbb{Z}/D\mathbb{Z})^{n-1}.}
\end{aligned}$$

2. The cone $\eta_D^{(n)} \subseteq N_{\mathbb{R}}$ has ray generators e_i for $1 \leq i < n$ and $f_n = De_n - \sum_{i=1}^{n-1} e_i$, and

$$\begin{aligned}
\mathrm{Cl}(\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n]) &\cong \frac{\mathbb{Z} \cdot [\mathbf{D_{f_n}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot [D_{e_i}]}{\langle [\mathbf{D_{e_i}}] - [\mathbf{D_{f_n}}] = \mathbf{0} \ (\mathbf{1} \leq \mathbf{i} < \mathbf{n}),\ D[D_{f_n}] = 0 \rangle} \\
&\cong \frac{\mathbb{Z} \cdot [D_{f_n}]}{\langle D[D_{f_n}] = 0 \rangle} \\
&\cong \boxed{(\mathbb{Z}/D\mathbb{Z}).}
\end{aligned}$$

CHAPTER IV

Uniform Symbolic Topologies via Multinomial Expansions

This chapter consolidates the relevant material in our paper [71]. The following theorem is the main result of this chapter – a ready and cost-effective half-measure to partially address Question II.18 from the close of Chapter II.

Theorem IV.1. *Let $\mathbb{F}$ be an algebraically closed field. Let $R_1, \dots, R_n$ $(n \geq 2)$ be affine commutative $\mathbb{F}$-algebras which are domains. Suppose that for each $1 \leq i \leq n$, there exists a positive integer D_i such that for all prime ideals P in R_i, either:*

1. *$P^{(D_i r)} \subseteq P^r$ for all $r > 0$ and for all i; or, even stronger,*
2. *$P^{(D_i(r-1)+1)} \subseteq P^r$ for all $r > 0$ and for all i.*

Fix any n prime ideals P_i in R_i, and consider the expanded ideals $P_i' = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$, along with their sum $Q = \sum_{i=1}^n P_i'$ in T. Then:

(a) When (1) holds, $Q^{(Dr)} \subseteq Q^r$ for all $r > 0$, where $D = D_1 + \cdots + D_n$.

(b) When (2) holds, this improves to $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where instead $D = \max\{D_1, \dots, D_n\}$.

We previously discussed and illustrated the intended mode of application of this theorem in Chapter I, and thus will not repeat ourselves later in the chapter.

We first record the relevant preliminaries to prove this result, culminating in a proof of the Symbolic Power Multinomial Theorem IV.11 which inspires the title of the chapter. Our main theorem is then deduced as a swift corollary of Theorem IV.11. We close the chapter with a few remarks, in particular, tying our work to that of Há – Nguyen – Trung – Trung in [28].

The reader is invited to revisit the list of Conventions I.45 as appropriate.

4.1 Preliminaries, the Symbolic Power Multinomial Theorem

Torsion free modules over Noetherian Domains

A module M over a domain R is **torsion free** if whenever $rx = 0$ for some $x \in M$ and $r \in R$, then either $r = 0$ or $x = 0$. We first record a lemma on torsion free modules to be used both here and in the next subsection (cf., Lemmas 15.6.7-8 from the Stacks Project page [5] on torsion free modules):

Lemma IV.2. *Let R be a Noetherian domain. Let M be a nonzero finitely generated R-module. Then the following assertions are equivalent:*

1. *M is torsion free;*
2. *M is a submodule of a finitely generated free module;*
3. (0) *is the only associated prime of M, i.e.,* $\mathrm{Ass}_R(M) = \{(0)\}$.

Working over an arbitrary field $\mathbb{F}$, we fix two affine $\mathbb{F}$-algebras R and S which are domains. The tensor product $T = R \otimes_{\mathbb{F}} S$ will be an affine $\mathbb{F}$-algebra. The $\mathbb{F}$-algebra T is a domain when $\mathbb{F}$ is algebraically closed (Milne [52, Prop. 4.15]). We note that when R and S are duly nice (e.g., polynomial, or normal toric rings more generally), T is a domain over any field. We now record two additional lemmas.

Lemma IV.3. *Suppose that all three of R, S, and $T = R \otimes_{\mathbb{F}} S$ are affine domains over a field $\mathbb{F}$. If M and N are finitely generated torsion free modules over R and S, respectively, then $M \otimes_{\mathbb{F}} N$ is a finitely generated torsion free T-module.*

Proof. Viewed as vector spaces, $M \otimes_{\mathbb{F}} N = 0$ if and only if $M = 0$ or $N = 0$, in which case torsion freeness is vacuous. So we will assume all three of M, N, and $M \otimes_{\mathbb{F}} N$ are nonzero. Per Lemma IV.2, suppose we have embeddings $M \subseteq R^a$ and $N \subseteq S^b$. Apply the functor $\bullet \otimes_{\mathbb{F}} N$ to the first inclusion to get $M \otimes N \subseteq R^a \otimes N$, which in turn is contained in $R^a \otimes S^b$ by tensoring the inclusion $N \subseteq S^b$ with R^a. Thus $M \otimes N \subseteq R^a \otimes S^b \cong (R \otimes S)^{ab} = T^{ab}$, where the isomorphism is easily checked in the category of $\mathbb{F}$-vector spaces since direct sum commutes with tensor product. Of course, this inclusion holds in the category of T-modules, and all T-submodules of T^{ab} are finitely generated since T is Noetherian, so we are done by invoking Lemma IV.2 again. $\square$

Lemma IV.4. *For any prime P in any Noetherian ring A, the finitely generated module $P^{(a)}/P^{(a+1)}$ is torsion free as an A/P-module for all integers $a \geq 0$.*

Proof. Say $\overline{x} \in (P^{(a)}/P^{(a+1)})$ is killed by $\overline{r} \in A/P$. This means, lifting to A, that $x \in P^{(a)}$ and $rx \in P^{(a+1)}$. Localize at P. Then $rx \in P^{(a+1)}A_P = P^{a+1}A_P$. If $r \notin P$, this means $x \in P^{a+1}A_P \cap A = P^{(a+1)}$. That is, either $\overline{r} = 0$ in A/P or otherwise, $\overline{x} = 0$ in $(P^{(a)}/P^{(a+1)})$. Ergo by definition, $(P^{(a)}/P^{(a+1)})$ is a torsion-free A/P-module. $\square$

Finally, we record a consequence of Lemma IV.3 that will be important in the next subsection. The following proposition follows immediately from Lemmas IV.3 and IV.4

Proposition IV.5. *Suppose that all three of R, S, and $T = R \otimes_{\mathbb{F}} S$ are affine domains over a field $\mathbb{F}$. Fix two prime ideals P and Q in R and S respectively, such that the affine $\mathbb{F}$-algebra $T' = (R/P) \otimes_{\mathbb{F}} (S/Q)$ is a domain. Then $(P^{(a)}/P^{(a+1)}) \otimes_{\mathbb{F}} (Q^{(b)}/Q^{(b+1)})$ is finitely generated and torsion free over T' for any pair of nonnegative integers a and b.*

USTP Preservation along Faithfully Flat Maps

Picking up from Remark I.42, we state a result first proved in [69, Prop. 2.1]: it provides a convenient setup for preserving containments of the type $I^{(N)} \subseteq I^r$ along a faithfully flat ring extension. Consider a flat map $\phi\colon A \to B$ of Noetherian rings. In what follows, the ideal $JB := \langle \phi(J) \rangle B$ for any ideal J in A, and $J^r B = (JB)^r$ for all $r \geq 0$ since the two ideals share a generating set. For any A-module E, the proof of Theorem 23.2 (ii) in Matsumura [49] shows that

$$\mathrm{Ass}_B(E \otimes_A B) = \bigcup_{P \in \mathrm{Ass}_A(E)} \mathrm{Ass}_B(B/PB). \tag{4.1}$$

We define a set

$$\mathcal{I}(A) = \{\text{proper ideals } I \subseteq A\colon\ \mathrm{Ass}_B(B/IB) = \{PB\colon P \in \mathrm{Ass}_A(A/I)\}\}.$$

Setting the module $E = A/I$ in (4.1), we observe that $I \in \mathcal{I}(A)$ if and only if the extended ideal PB is prime for all $P \in \mathrm{Ass}_A(A/I)$. That PB is prime is not automatic, per the example following our proof of

Proposition IV.6. *Suppose $\phi\colon A \to B$ is a faithfully flat map of Noetherian rings. Then for each $I \in \mathcal{I}(A)$ and all integer pairs $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, we have*

$$I^{(N)} B = (IB)^{(N)}, \tag{4.2}$$

and $I^{(N)} \subseteq I^r$ if and only if $(IB)^{(N)} = I^{(N)} B \subseteq I^r B = (IB)^r$.

Proof. First, $I^{(N)}B \subseteq (IB)^{(N)}$: indeed, if $f \in I^{(N)}$, then $sf \in I^N$ for some $s \in A$ such that

$$s \notin \bigcup_{P \in \operatorname{Ass}_A(A/I)} P \overset{(\star)}{=} \bigcup_{P \in \operatorname{Ass}_A(A/I)} (PB \cap A) = \left(\bigcup_{P \in \operatorname{Ass}_A(A/I)} PB \right) \cap A$$

where $(\star)$ holds by faithful flatness. Thus $s \notin \bigcup_{P \in \operatorname{Ass}_A(A/I)} PB = \bigcup_{Q \in \operatorname{Ass}_B(B/IB)} Q$, where equality holds since $I \in \mathcal{I}(A)$ by hypothesis. We thus conclude that $f \in (IB)^{(N)}$.

By definition, $(IB)^{(N)}B_W = (IB)^N B_W = I^N B_W$ since all three ideals contract to $(IB)^{(N)}$, where B_W is the localization of B at the multiplicative system

$$W = B - \left(\bigcup_{Q \in \operatorname{Ass}_B(B/IB)} Q \right) = B - \left(\bigcup_{P \in \operatorname{Ass}_A(A/I)} PB \right).$$

Notice that since $I^{(N)}B \subseteq (IB)^{(N)}$, the right-hand containment holds in

$$I^N B_W \subseteq I^{(N)} B_W = (I^{(N)}B)B_W \subseteq (IB)^{(N)} B_W = I^N B_W.$$

Thus $I^{(N)}B$ and $(IB)^{(N)}$ localize to the same ideal $I^N B_W$; contracting back to B, we conclude that (4.2) holds for all $N \geq 0$. Finally, (4.2) gives both implications of the second part of the proposition, adducing faithful flatness once more to contract an ideal containment to A. $\square$

Remark IV.7. Two remarks in passing: on the one hand, when B is a polynomial ring in finitely many variables over a Noetherian ring A, the set $\mathcal{I}(A)$ consists of all proper ideals in A; on the other hand, $\mathcal{I}(A)$ may miss some proper ideals relative to an arbitrary faithfully flat ring extension.

Example IV.8. The extension

$$R = \frac{\mathbb{R}[x]}{(x^2+1)} \hookrightarrow \mathbb{C} \otimes_{\mathbb{R}} R \cong \frac{\mathbb{C}[x]}{(x^2+1)}$$

is faithfully flat over R, since R is free and hence faithfully flat over $\mathbb{R}$; the zero ideal in R is maximal, hence prime, but it fails to extend to a primary ideal of $\frac{\mathbb{C}[x]}{(x^2+1)}$ as $(x^2+1) = (x+i) \cap (x-i)$ in $\mathbb{C}[x]$.

We now define the set $\mathcal{P}(A) = \{\text{prime ideals } P \subseteq A \colon PB \text{ is prime}\}$ to consist of prime ideals that extend along ϕ to prime ideals of B. A special case of the above proposition to be used below is the following

Proposition IV.9. *Suppose $\phi\colon A \to B$ is a faithfully flat map of Noetherian rings. Then for each prime ideal $P \in \mathcal{P}(A)$ and all integer pairs $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, we have*

$$P^{(N)}B = (PB)^{(N)}, \tag{4.3}$$

and $P^{(N)} \subseteq P^r$ if and only if $(PB)^{(N)} = P^{(N)}B \subseteq P^rB = (PB)^r$.

When B is a polynomial ring in finitely many variables over A and ϕ is inclusion, $\mathcal{P}(A) = \mathrm{Spec}(A)$. It is possible that $\mathcal{P}(A) \neq \mathrm{Spec}(A)$ in Proposition IV.9, per Example IV.8 above. Working over a field $\mathbb{F}$, we use Proposition IV.9 when $B = A \otimes_{\mathbb{F}} C$ for two affine $\mathbb{F}$-algebras, so B is an affine $\mathbb{F}$-algebra; when A and C are domains and $\mathbb{F}$ is algebraically closed, B is a domain, $\mathcal{P}(A) = \mathrm{Spec}(A)$ and $\mathcal{P}(C) = \mathrm{Spec}(C)$.

Proving the Multinomial Theorem

Working over an algebraically closed field $\mathbb{F}$, we fix two affine $\mathbb{F}$-algebras R and S that are domains, and two prime ideals $P \subseteq R$, $Q \subseteq S$. Let

$$T = R \otimes S \supseteq P \otimes S + R \otimes Q =: PT + QT, \quad T' = (R/P) \otimes (S/Q) \cong T/(PT + QT),$$

where all tensor products are over $\mathbb{F}$. Both T and T' are affine domains over $\mathbb{F}$. Because $\mathbb{F}$ is algebraically closed, the extended ideals PT, QT are both prime, along with their sum $PT + QT$. We cannot relax the assumption that $\mathbb{F}$ is algebraically closed to its merely being perfect. For instance, $\mathbb{R}$ is perfect (being of characteristic zero), and along the ring extension

$$S := \frac{\mathbb{R}[x]}{(x^2+1)} \cong \mathbb{C} \hookrightarrow T := \frac{\mathbb{C}[x]}{(x^2+1)} \cong \mathbb{C} \otimes_{\mathbb{R}} S \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$

the zero ideal of S (which is maximal) extends to a radical ideal which is not prime.

We now prove a binomial theorem for the symbolic powers of $PT + QT$.

Theorem IV.10. *For all $n \geq 1$, the symbolic power*

$$(PT+QT)^{(n)} = \sum_{a+b=n} (PT)^{(a)}(QT)^{(b)}.$$

Proof. We'll drop the T's from the notation, and we will assume that both P, Q are nonzero to justify the effort. For $0 \leq c \leq n$, set $J_c = \sum_{t=0}^{c} P^{(c-t)}Q^{(t)}$, so $J_c \subseteq J_{c-1}$ for all $1 \leq c \leq n$, since $P^{(c-t)} \subseteq P^{(c-1-t)}$ for $t \leq c-1$ and for $t = c$, $Q^{(c)} \subseteq Q^{(c-1)}$. Note that

$$(P+Q)^n = \sum_{a+b=n} P^a Q^b \subseteq J_n = \sum_{a+b=n} P^{(a)}Q^{(b)} \overset{(!)}{\subseteq} (P+Q)^{(n)},$$

and **(!)** is easy to verify term-by-term for each $P^{(a)}Q^{(b)}$. Indeed, $P^{(a)}Q^{(b)}$ is generated by elements of the form fg with $f \in P^{(a)} \subset R$ and $g \in Q^{(b)} \subset S$ (viewing them as elements of T). We need $fg \in (P+Q)^{(a+b)}$. Per Proposition IV.9, there exist $u \in R - P$ and $v \in S - Q$ such that $uf \in P^a$ and $vg \in Q^b$. Viewing u and v as elements of the overring T, we have $uv \notin (P+Q)$. Indeed, since $P+Q$ is prime, if $uv \in P+Q$, then either u or v is in $P+Q$, but $(P+Q)T \cap R = P$ and $(P+Q)T \cap S = Q$, contradicting that $u \notin P$ and $v \notin Q$. Therefore, in T, $(uf)(vg) = (uv)(fg) \in P^aQ^b \subset (P+Q)^{a+b}$, which means $fg \in (P+Q)^{(a+b)}$. Thus **(!)** holds, and notably J_n is a proper ideal–read, $J_n \subsetneqq T$.

Since J_n contains $(P+Q)^n$, and $(P+Q)^{(n)}$ is the smallest $(P+Q)$-primary ideal containing $(P+Q)^n$, the opposite inclusion to (!) will follow once we show that J_n

is $(P+Q)$-primary, i.e., that the set of associated primes $\mathrm{Ass}_T(T/J_n) = \{P+Q\}$. We have short exact sequences of T-modules

$$0 \to J_{c-1}/J_c \to T/J_c \to T/J_{c-1} \to 0, \quad \text{for all } 1 \le c \le n.$$

Thus $\mathrm{Ass}_T(J_{c-1}/J_c) \subseteq \mathrm{Ass}_T(T/J_c) \subseteq \mathrm{Ass}_T(J_{c-1}/J_c) \cup \mathrm{Ass}_T(T/J_{c-1})$ for all $1 \le c \le n$, using the fact that given an inclusion of modules $N \subseteq M$,

$$\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N).$$

Thus by iterative unwinding and using that $J_0 = T$, i.e., $\mathrm{Ass}_T(T/J_0) = \varnothing$, we conclude that

$$\varnothing \neq \mathrm{Ass}_T(T/J_n) \subseteq \bigcup_{c=1}^{n} \mathrm{Ass}_T(J_{c-1}/J_c). \tag{4.4}$$

Taking all direct sums and tensor products over $\mathbb{F}$, we have a series of vector space isomorphisms

$$J_{c-1}/J_c \cong \bigoplus_{a+b=c-1} [P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}], \quad 1 \le c \le n. \tag{4.5}$$

We prove this first, considering two chains of symbolic powers, where each ideal is expressed as a direct sum of $\mathbb{F}$-vector spaces:

$$P^{(c)} = V_0 \subseteq P^{(c-1)} = V_0 \oplus V_1 \subseteq \ldots \subseteq P^{(0)} = R = V_0 \oplus \cdots \oplus V_c,$$
$$Q^{(c)} = W_0 \subseteq Q^{(c-1)} = W_0 \oplus W_1 \subseteq \ldots \subseteq Q^{(0)} = S = W_0 \oplus \cdots \oplus W_c.$$

In particular, for all pairs $0 \le a, b \le c-1$,

$$P^{(a)} = \bigoplus_{i=0}^{c-a} V_i, \quad P^{(a+1)} = \bigoplus_{i=0}^{c-a-1} V_i, \quad Q^{(b)} = \bigoplus_{j=0}^{c-b} W_j, \quad Q^{(b+1)} = \bigoplus_{j=0}^{c-b-1} W_j.$$

For any pair a, b as above with $a+b = c-1$, $c-b = a+1$, and so

$$\bigoplus_{a+b=c-1} \frac{P^{(a)}}{P^{(a+1)}} \otimes_{\mathbb{F}} \frac{Q^{(b)}}{Q^{(b+1)}} \cong \bigoplus_{a+b=c-1} V_{c-a} \otimes W_{c-b} = \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1}.$$

We now prove (4.5) by killing off a common vector space. First,

$$J_{c-1} = \sum_{a+b=c-1} P^{(a)}Q^{(b)} = \bigoplus_{\substack{0\le a\le c-1\\ 0\le i\le c-a, 0\le j\le a+1}} V_i \otimes W_j$$

$$= \boxed{\bigoplus_{\substack{0\le a\le c-1\\ 0\le i<c-a \text{ or } 0\le j<a+1}} (V_i \otimes W_j)} \oplus \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1},$$

while $$J_c = \sum_{a+b=c} P^{(a)}Q^{(b)} = \boxed{\bigoplus_{\substack{0\le a\le c\\ 0\le i\le c-a, 0\le j\le a}} V_i \otimes W_j.}$$

Identifying repeated copies of a $V_i \otimes V_j$ term with $i + j \le c$ (we can do this since we are working with vector subspaces of the ring T), it is straightforward to check that the boxed sums are equal. Thus for each $1 \le c \le n$, we have canonical isomorphisms of $\mathbb{F}$-vector spaces:

$$J_{c-1}/J_c \cong \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1} \cong \bigoplus_{a+b=c-1} \frac{P^{(a)}}{P^{(a+1)}} \otimes_{\mathbb{F}} \frac{Q^{(b)}}{Q^{(b+1)}}.$$

Therefore, since for each $1 \le c \le n$ there is a natural surjective T-module map (hence $\mathbb{F}$-linear)

$$\bigoplus_{a+b=c-1} [P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}] \to J_{c-1}/J_c,$$

this map must be injective per isomorphism (4.5). Thus for all $1 \le c \le n$,

$$\mathrm{Ass}_T(J_{c-1}/J_c) = \bigcup_{a+b=c-1} \mathrm{Ass}_T[P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}].$$

For any $1 \le c \le n$ such that $J_{c-1}/J_c \ne 0$, i.e., $\mathrm{Ass}_T(J_{c-1}/J_c) \ne \varnothing$, in turn the above identity implies that one of the modules $P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}$ is nonzero, in which case

$$\mathrm{Ass}_T(J_{c-1}/J_c) = \bigcup_{a+b=c-1} \mathrm{Ass}_T[P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}] = \{P + Q\}. \tag{4.6}$$

To explain the right-hand equality: for any pair $(a, b) \in (\mathbb{Z}_{\geq 0})^2$, Proposition IV.5 says that

$$M_{a,b} := P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}$$

is a finitely generated torsion-free module over $T' = (R/P) \otimes (S/Q) \cong T/(P+Q)$; thus when $M_{a,b} \neq 0$, we have $\mathrm{Ass}_{T/(P+Q)}(M_{a,b}) = \{(0)\}$ by Lemma IV.2: that is, $\mathrm{Ass}_T(M_{a,b}) = \{P+Q\}$.

Finally, combining (4.6) with the inclusion (4.4) for $\mathrm{Ass}_T(T/J_n) \neq \varnothing$–recall, J_n is a proper ideal, we conclude that $\mathrm{Ass}_T(T/J_n) = \bigcup_{c=1}^{n} \mathrm{Ass}_T(J_{c-1}/J_c) = \{P+Q\}$, that is, the ideal J_n is $(P+Q)$-primary as was to be shown. Thus $J_n \supseteq (P+Q)^{(n)}$, and indeed this is an equality. □

We now deduce a multinomial theorem by induction on the number of tensor factors:

Theorem IV.11. *Let $\mathbb{F}$ be an algebraically closed field. Let $R_1, \ldots, R_n$ $(n \geq 2)$ be affine commutative $\mathbb{F}$-algebras which are domains. Fix any n prime ideals P_i in R_i, and consider the expanded ideals $P_i' = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^{n} R_i$. Then the symbolic power*

$$\left(\sum_{i=1}^{n} P_i'\right)^{(N)} = \sum_{A_1+\cdots+A_n=N} \prod_{i=1}^{n} (P_i')^{(A_i)} \text{ for any } N \geq 0. \tag{4.7}$$

Proof. Induce on the number n of tensor factors with base case $n = 2$ being Theorem IV.10. Now suppose $n \geq 3$, and assume the result for tensoring up to $n-1$ factors. Suppose that $R = R_1$ and $S = R_2 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$, and that we have an expansion result in S of the form

$$\left(\sum_{i=2}^{n} P_i\right)^{(N)} = \sum_{A_2+\ldots+A_n=N} \prod_{i=2}^{n} P_i^{(A_i)} \quad \text{for all nonnegative integers } N \tag{4.8}$$

for $n-1$ primes $P_i \subseteq R_i$ $(2 \leq i \leq n)$. The sum $Q := \sum_{i=2}^n P_i$ is prime along with all extensions of the P_i to S. Given a prime $P = P_1$ in R, the sum $P + Q$ is prime in $T = R \otimes_{\mathbb{F}} S$, together with all extensions P_iT and QT being prime. The first equality below holds by Theorem IV.10, and applying Proposition IV.9 to the extension $\phi\colon S \hookrightarrow T$, the second equality holds by (4.8):

$$(P+Q)^{(N)} = \sum_{A_1+B=N} P^{(A_1)}Q^{(B)} = \sum_{A_1=0}^{N} P_1^{(A_1)} \left(\sum_{A_2+\ldots+A_n=N-A_1} \prod_{i=2}^{n} P_i^{(A_i)} \right)$$
$$\subseteq \sum_{A_1+A_2+\ldots+A_n=N} \prod_{i=1}^{n} P_i^{(A_i)},$$

using the fact that $I(J+K) \subseteq IJ + IK$ whenever I, J, K are ideals in a commutative ring. This proves the n-fold version of the hard inclusion in the proof of Theorem IV.10; deducing the opposite inclusion is about as easy as before, hence the above inclusion is an equality. $\square$

4.2 Proving the Main Theorem, Closing Remarks

Before proceeding, we record the following handy result, an asymptotic conversion lemma [69, Lem. 3.3]:

Lemma IV.12. *Given any proper ideal I in a Noetherian ring S, and $E \in \mathbb{Z}_{\geq 0}$,*

$$(1)\ I^{(N)} \subseteq I^{\lceil N/E \rceil} \text{ for all } N \geq 0 \iff (2)\ I^{(E(r-1)+1)} \subseteq I^r \text{ for all } r > 0.$$

Proof. The case $N = 0$ is trivial (the unit ideal is contained in itself), so we show equivalence when $N > 0$. Given $r > 0$, setting $N = E(r-1)+1$ in (1) gives (2). That (2) implies (1) follows from noticing that for any two positive integers N, r, we have $r = \lceil N/E \rceil$ if and only if $N = E(r-1)+j$ for some $1 \leq j \leq E$, and $I^{(m)} \subseteq I^{(n)}$ when $m \geq n$. $\square$

We now use the Multinomial Theorem IV.11 to deduce a corollary. Note that Theorem IV.1 is the version of this corollary where all tensor factors are assumed to satisfy uniform symbolic topologies on primes.

Corollary IV.13. *Let $\mathbb{F}$ be an algebraically closed field. Let $R_1, \ldots, R_n$ $(n \geq 2)$ be affine commutative $\mathbb{F}$-algebras which are domains. Fix n primes $P_i \subseteq R_i$, and consider the expanded ideals $P_i' = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$; set $Q = \sum_{i=1}^n P_i'$. Suppose that for each $1 \leq i \leq n$, there exists a positive integer D_i such that either:*

1. *$P_i^{(D_i r)} \subseteq P_i^r$ for all $r > 0$ and for all i; or, even stronger,*
2. *$P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ for all $r > 0$ and for all i.*

When (1) *holds, $Q^{(Dr)} \subseteq Q^r$ for all $r > 0$, where $D = D_1 + \cdots + D_n$. When* (2) *holds, this improves to $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where $D = \max\{D_1, \ldots, D_n\}$.*

Proof. Assume (1) holds. Per Theorem IV.11 note that for $D = D_1 + D_2 + \cdots + D_n$,

$$Q^{(Dr)} = \sum_{A_1 + A_2 + \cdots + A_n = D_1 r + D_2 r + \cdots + D_n r} \prod_{i=1}^n (P_i')^{(A_i)}.$$

In each n-tuple of indices $(A_1, \ldots, A_n)$, we must have that $A_j \geq D_j r$ for some j, otherwise $\sum_{i=1}^n A_i < \sum_{i=1}^n D_i r$, a contradiction. Thus each summand $\prod_{i=1}^n (P_i')^{(A_i)}$ will lie in some $(P_j')^r$ applying (1) and Proposition IV.9, and hence also in Q^r. Since $r > 0$ was arbitrary, we win.

If (2) holds, then $P_i^{(D(r-1)+1)} \subseteq P_i^r$ for all $r > 0$ and all i, where $D = \max_{1 \leq i \leq n} D_i$, so equivalently per Lemma IV.12 and Proposition IV.9, for all n-tuples $(A_1, \ldots, A_n) \in (\mathbb{Z}_{\geq 0})^n$, we have containments $(P_i')^{(A_i)} \subseteq (P_i')^{\lceil A_i/D \rceil} \subseteq Q^{\lceil A_i/D \rceil}$. For all nonnegative integers N, per Theorem IV.11

$$Q^{(N)} = \sum_{A_1+\cdots+A_n=N} \prod_{i=1}^{n} (P_i')^{(A_i)} \subseteq \sum_{A_1+\cdots+A_n=N} \prod_{i=1}^{n} (P_i')^{\lceil A_i/D \rceil}$$
$$\subseteq \sum_{A_1+\cdots+A_n=N} \prod_{i=1}^{n} Q^{\lceil A_i/D \rceil} \subseteq Q^{\lceil N/D \rceil},$$

since the integer $\sum_{i=1}^{n} \lceil A_i/D \rceil \geq \lceil (\sum_{i=1}^{n} A_i)/D \rceil = \lceil N/D \rceil$ for all n-tuples $(A_1, \ldots, A_n) \in (\mathbb{Z}_{\geq 0})^n$ with $\sum_{i=1}^{n} A_i = N$. Thus equivalently, $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$ by Lemma IV.12. $\square$

Remark IV.14. We get a much stronger conclusion in Corollary IV.13 when (2) holds. This is because we can then give a proof using Lemma IV.12 as a workaround. It is less clear what the strongest conclusion to shoot for is when (1) holds. We note that if (1) holds under Corollary IV.13, then setting $D = \max D_i$, one can alternatively prove by contradiction that

$$Q^{(n(Dr-1)+1)} \subseteq Q^r \text{ for all } r > 0.$$

In part (1) of the proof above, simply adjust the claim "$A_j \geq D_j r$ for some j" to "$A_j \geq Dr$ for some j." Otherwise, some tuple satisfies $n(Dr-1)+1 = \sum_{i=1}^{n} A_i \leq n(Dr-1)$, a contradiction.

Remark IV.15. One can state variants of Theorem IV.1 and Corollary IV.13 when $\mathbb{F}$ is not algebraically closed. One instead assumes that after base changing to the algebraic closure, the tensor product T is a domain along with $T \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, and the primes P_i remain prime when expanded to $R_i \otimes \overline{\mathbb{F}}$. See Proposition IV.9.

Remark IV.16. When all hypotheses are satisfied, Corollary IV.13 typically applies to an infinite set of prime ideals in the tensor product T. If R is a Noetherian ring of dimension at least two, or a Noetherian ring of dimension one which has infinitely many maximal ideals, then $\operatorname{Spec}(R)$ is infinite; see [4, Exercises 21.11-21.12]. Now suppose $R_1, \ldots, R_n$ are $\mathbb{F}$-affine domains, with $n \geq 2$ and $\mathbb{F}$ algebraically closed, at least one of which is of dimension one or more. Then in the domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$, the following set $\mathcal{Q}_{ED}(T) := \{Q = \sum_{i=1}^n P_i T \in \operatorname{Spec}(T) \colon \forall 1 \leq i \leq n, P_i \in \operatorname{Spec}(R_i)\}$ is infinite.

Remark IV.17. Let $R = \mathbb{F}[x_1, \ldots, x_m]$, $S = \mathbb{F}[y_1, \ldots, y_n]$, and $T = R \otimes_{\mathbb{F}} S \cong \mathbb{F}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be polynomial rings over a field $\mathbb{F}$. Our original inspiration for Theorem IV.11 was the following

Theorem (Thm. 7.8 of Bocci et.al [8]). Let $I \subseteq R$ and $J \subseteq S$ be squarefree monomial ideals in the polynomial rings R and S, respectively. Let $I' = IT$ and $J' = JT$ be their expansions to T. Then for any $N \geq 0$, the symbolic power

$$(I' + J')^{(N)} = \sum_{i=0}^{N} (I')^{(N-i)} (J')^{(i)} = \sum_{A+B=N} (I')^{(A)} (J')^{(B)}. \tag{4.9}$$

Há – Nguyen – Trung – Trung [28, Thm. 3.4] recently extended the above theorem to the case of two nonzero ideals $I \subseteq R, J \subseteq S$ in two Noetherian commutative $\mathbb{F}$-algebras such that $T = R \otimes_{\mathbb{F}} S$ is also Noetherian. A general multinomial theorem then follows by adapting the proof of Theorem IV.11, where one containment would require the n-fold version of [28, Lem. 2.1(i)]. Combining this multinomial expansion with Proposition IV.6 and Lemma IV.12, one can extend Corollary IV.13 to a form allowing, for instance, any proper ideals $I_i \subseteq R_i$. As a final note in passing, the proof of Proposition IV.6 still works up to a tweak of multiplicative system, for those who opt to define symbolic powers of proper ideals using only minimal associated primes as in [28], rather than using all associated primes as in [69].

Before proceeding to the final chapter, we remind the reader that our intended mode of application for Theorem IV.1 was expounded upon and illustrated in the introductory chapter to the thesis. Thus we opt not to repeat ourselves here.

CHAPTER V

Uniform Symbolic Topologies: A Few Avenues for Follow-Up Work

To summarize, we have deduced several *quid pro quo* results on uniform symbolic topologies for select families of ideals across a wide range of Noetherian domains. These results are also stated in a manner that incentivizes the search for Harbourne – Huneke symbolic indices in non-regular domains.

In Chapter II, we deduced criteria – Lemma II.1 and Corollary II.12 – to uncover uniform symbolic topologies on ideals of pure height one in Noetherian normal rings. In Chapter III, we deduced a USTP result for monomial primes in normal toric rings which adapts to cover primes of height one for simplicial toric rings – Theorems III.1 and III.2. Between Chapters II and III, we also demonstrated the utility of these criteria relative to familiar classes of local- or graded Cohen-Macaulay domains with rational singularities [35, 34]. We now list two natural lines for further investigation.

1. Can Lemma II.1 be strengthened to cover all (non-prime) ideals of height one?

2. Can we identify a candidate mechanism (e.g., group-theoretic) to verify the uniform symbolic topology property and/or Harbourne – Huneke bounds for: all prime ideals of height one in non-simplicial toric rings; or all prime ideals of height two or more, even in the case of simplicial toric rings of dimension at

least three?

In Chapter IV, we deduced a powerful criterion for proliferating uniform linear bounds on the growth of symbolic powers of prime ideals (e.g., Harbourne – Huneke bounds) – Theorem IV.1. In the setting of domains of finite type over algebraically closed fields, this criterion contributes further evidence for Huneke's philosophy in [39] about uniform bounds lurking throughout commutative algebra. A goalpost question that exceeds our grasp at present is:

Question V.1. *Do analogues of Theorem IV.1 exist for other product constructions in commutative algebra, such as Segre products of* $\mathbb{N}$*-graded rings, or fiber products of toric rings?*

We now focus on the toric setting. We expound upon the following natural problems lingering in the wake of our work in Chapter III. Namely:

Question V.2. *Working over an arbitrary field* $\mathbb{F}$*, can one identify an effective multiplier relative to which a normal toric* $\mathbb{F}$*-algebra satisfies uniform symbolic topologies on all radical monomial ideals?*

Question V.3. *Working over an arbitrary field* $\mathbb{F}$*, does a normal toric* $\mathbb{F}$*-algebra satisfy uniform symbolic topologies on all prime ideals? Can one identify an effective multiplier?*

Since one-dimensional normal toric algebras are regular, the above questions are only potentially harrowing in Krull dimension two or higher. That being said, the two-dimensional case is also as well-behaved as one could dare to hope for:

Remark V.4. Recall that over any field, all two-dimensional normal toric algebras are simplicial. We note that both parts of both questions have an affirmative answer for all radical ideals in this case, using the order of the divisor class group. The latter matches the reciprocal of the F-signature when working over perfect fields of positive characteristic; see Corollary III.30. If the reader is curious how to prove the USTP on all radical ideals, one can adapt the comaximality-based argument from Chapter I showing that all symbolic and ordinary powers coincide for any radical ideal in a two-dimensional UFD: see under Example I.10(5).

Given the connection between divisor class group order and the F-signature of simplicial toric rings in Corollary III.30, experts in positive characteristic commutative algebra might be tempted by the following generalization of Question V.3.

Question V.5. *Suppose R is of prime characteristic $p > 0$ and R is F-finite(=finitely generated as a module over the subring R^p of p-th powers). Can we specify suitable hypothesis relative to which a strongly F-regular ring R must satisfy uniform symbolic topologies on prime ideals, with an explicit, effective multiplier involving the reciprocal $1/s(R)$ of the F-signature?*

Personally, I have yet to find this prospectus fruitful, and not simply because I currently lack an incisive handle on positive characteristic techniques, and potentially relevant techniques in related fields like singularity theory in algebraic geometry.

Moving on, in deference to Question V.2, there are two immediate hurdles to extending Theorems III.1 and III.2 to the case of radical ideals. The first is conceptual, and frustratingly unassailable despite banging one's head against a chalkboard: suppose we know that there exists a bound $E \geq 1$ such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all minimal primes of a radical ideal I. How would it then follow that

$$I^{(Fr)} := \bigcap_{P \in \mathrm{Min}_R(R/I)} P^{(Fr)} \subseteq I^r$$

for all $r > 0$ and some value $F \geq 1$? All of the work that culminated in the Improved

Ein-Lazarsfeld-Smith Theorem I.18 used techniques that provided seamless–though not necessarily elementary–workarounds to this conceptual hurdle.

To speak to our second hurdle, a potential workaround that would appeal to my taste, and which is currently lacking, is a description of the radical monomial ideals on par with Lemma III.4 for the monomial primes. The latter is what eventually allowed us to deduce Theorem III.1 using the bilinearity of the pairing $\langle , \rangle$. That being said, this potential workaround might ultimately be a non-starter. Even if we had such a description of radical monomial ideals, as the latter form an infinite family of ideals, there need not be a sensible analogue of the multiplier constructed in Theorem III.1–the latter multiplier is the maximum among a *finite* set of integers.

Transitioning, we briefly discuss where we stand on approaching Question V.3. Alas, the situation is not great. As of right now, select affirmative results on the latter part of Question V.3 are known when attention is restricted to Segre-Veronese algebras over perfect fields. For instance, Corollary I.27 applies to Veronese rings. That being said, examples in Chapter III involving Segre-Veronese algebras are indicative that even if the answer to Question V.3 turns out to be affirmative in a prodigious range of cases, it is pointless to sleuth moreover for an affirmation of USTP class solidarity relative to prime ideals in the toric setting.

5.1 USTP formulas for rationally singular combinatorial algebras

In future work, I would aim to adjust the invariant D in Theorem III.1 to a multiplier for all primes in R, based on the following

Problem Rubric V.6. *Identify hypotheses on a Noetherian ring R such that $P^{(Ehr)} \subseteq P^r$ for all $r > 0$ and all primes P in R, where h is the height of P and the integer E depends only on R.*

I would propose to fill in Rubric V.6 for simplicial toric rings first. My proposed line of attack is to flesh out a correspondence between annihilation of Chow groups in the toric case and symbolic topologies on primes. To any Noetherian normal domain R, one can associate a family $\{A^h(R)\}_{h\in\mathbb{Z}_{>0}}$ of abelian groups called Chow groups, where $A^h(R)$ is presented as the free abelian group on the set of height h prime ideals (or codimension h-cycles) in R modulo some relations [58]. While these groups are notoriously difficult to compute in general, the prospects improve for toric rings. For example, the first Chow group $A^1(R) = \mathrm{Cl}(R)$ is the divisor class group.

In the toric setting, Chow groups are finitely-generated abelian groups [21, Ch. 5] [22]. For finitely-generated abelian groups, the condition of being annihilated by an integer $D > 0$ as in Lemma II.1 is equivalent to being of finite order. I propose to show that for a simplicial toric ring R we can fill in Rubric V.6 one height at a time, proving an analogue of Lemma II.1 for each Chow group $A^h(R)$. Indeed, I aim to show that there is some choice of common annihilator D in terms of intersection numbers involving the canonical divisor K_V of the toric variety $V = \mathrm{Spec}(R)$.

An outstanding question left open by my thesis – a dream deferred – is:

Question V.7. *Can we characterize which normal toric algebras satisfy USTP – if not all of them?*

All normal toric algebras of dimension one or two are simplicial and thus satisfy USTP. Thus issues can occur only in dimension three or higher. Ultimately, given a toric algebra R we have access to several invariants handed to us from possibly orthogonal theories and methodology. Ideally, I want to scaffold connections between the invariants, taking first steps towards a more streamlined and unified theory to attack the above question.

In a separate direction, cluster algebras are combinatorially defined algebras defined by Sergey Fomin and Andrei Zelevinski as a unifying framework for studying total positivity in a wide range of algebraic and geometric contexts. Cluster algebra structures have been uncovered in corners of math and physics such as: quiver representations, Teichmüller theory, discrete integrable systems, knot theory, Poisson geometry, statistical physics and mirror symmetry. Locally acyclic cluster algebras as defined by Greg Muller [54], prior to being one of my adviser's postdoctoral advisees, are a particularly well-behaved and prodigious subclass. Locally acyclic cluster algebras are known to have rational singularities over fields of characteristic zero [7] and divisor class groups free of finite rank [19]. That said, there remains considerable room beyond the entries [7, 19] to explore interactions between cluster algebra structures and matters at the heart of commutative algebra. I would seek to:

Project V.8. *Identify locally acyclic cluster algebras satisfying uniform symbolic topologies on primes.*

I would propose the Plücker homogeneous coordinate ring $R = \mathbb{F}[\mathrm{Gr}(k,n)]$ of Grassmann varieties as a starting place for Project V.8, which is a locally acyclic cluster algebra as deduced by Muller – Speyer [55]. Working with $\mathrm{Gr}(2,4)$ first, we get $R \cong \mathbb{F}[A,B,C,D,E,F]/(AF - BE + CD)$ which is a cluster algebra of rank five. I propose to investigate this hypersurface ring first as part of post-dissertation work.

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