Quantum AdS/CFT: Black Holes and Wilson Loops

by

Vimal Rathee

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Physics)
in The University of Michigan
2019

Doctoral Committee:

Professor Leopoldo A. Pando Zayas, Chair
Professor Gordon L. Kane
Professor James T. Liu
Professor Jennifer Ogilvie
Professor Yongbin Ruan

Vimal Rathee

vimalr@umich.edu

ORCID iD: 0000-0002-6596-1051

For my grandparents,

Karan Singh and Vedvati Rathee,

Chanda Singh and Ram Rati Pehal

And my love, Siva.

For believing in me.

ACKNOWLEDGEMENTS

First I would like to thank my parents, Suresh Kumar and Parveen Rathee for making me the person I am today and teaching me the value of hard-work in life. Special appreciation to my brother, Rishi and my aunt, Naveen for their unwavering faith and love towards me.

Next I would like to thank my advisor, Leopoldo A. Pando Zayas for his continued guidance and enthusiastic attitude in regards to research. You have been a tremendous mentor for me. This accomplishment would not have been possible without you. I would also like to extend my gratitude to his family and friends for making me feel like a part of their family and hosting all the wonderful gatherings.

I would like to express my sincere thanks to Alberto, Alejandro, Guillermo, Ioannis, James, Jeremías, Uri, Wenli and Wolfgang, for many fruitful and impeccable collaborations. Special thanks to Alberto and Guillermo for being a part of my journey and creating priceless memories together.

I am very much thankful to all my colleagues at University of Michigan, Anthony, Arash, Gino, Pedro, Sebastian, Sam, Midhat, Brian, Callum, Marios, Shruti, Josh and Marina for insightful discussions and fun Oscar Klein room reminiscences. I am also grateful to Ratindranath Akhoury for teaching me much needed QFT courses for my research.

At last, I would like to thank all my friends, Animesh, Siva, Mohit, Tanvi, Prachi, Manu, Dhruv, Surojit and my undergraduate friends Amit, Garima and Tulika for their emotional support and continuous encouragement during my PhD. Specially, Animesh and Siva for being my best friends and sharing all crazy adventures together. You both mean a lot to me. Finally, my golden retriever, Snow, for his unconditional love and one of the best decisions of my life.

This work has been partially supported by the US Department of Energy under Grant No. DE-SC0017808 "Topics in the AdS/CFT Correspondence: Precision tests with Wilson loops, quantum black holes and dualities" and Grant No. DE-SC0007859 and multiple Student Instructor positions in the Department of Physics. I also thank ICTP for providing hospitality at various stages.

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ABSTRACT

One of the important successes of string theory has been the AdS/CFT correspondence which conjectures a mathematical equivalence between string theories (containing gravity) and field theories. The main focus of this thesis is to understand AdS/CFT correspondence more deeply, at the quantum level, in the context of Black Hole Entropy and Holographic Wilson loops.

It has been recently shown that the topologically twisted index for 3d supersymmetric Chern-Simons-matter theory (known as ABJM theory) in a certain limit reproduces the Bekenstein-Hawking entropy of magnetically charged asymptotically AdS_4 black holes. In the first part of thesis, we investigate sub-leading logarithmic corrections in the large N limit to the topologically twisted index in ABJM theory and black hole entropy in the dual one-loop quantum supergravity, focusing on both the near horizon geometry and the full AdS_4 black hole background. We explicitly focus on understanding the quantum aspects of microstate counting of the black hole entropy, which provides an invaluable benchmark for quantum gravity theories.

Another aspect of this thesis is precision holography with supersymmetric Wilson loops. The main idea of precision holography is to better understand string perturbation theory in curved spaces beyond the semi-classical approximation, given exact results from localization. The expectation value of Wilson loop operators can be computed exactly via supersymmetric localization. Holographically, these operators are mapped to string configurations in the gravity dual. In the large N limit, the on-shell string action reproduces the large coupling limit of the gauge theory expectation value. There should be a precise match between the sub-leading corrections to these limits as guided by AdS/CFT correspondence. Such precision tests have been done in the literature in the context of $\mathcal{N} = 4$ SYM theory revealing various subtleties in the choice of regularization scheme for one-loop computations.

In the second part of thesis, we perform a test of this match at next-to-leading order in string theory by computing the ratio between one-loop determinants of the quadratic fluctuations around the classical string configurations dual to BPS latitude and circular Wilson loops in both $\mathcal{N} = 4$ SYM and ABJM theory. We find a match for sub-leading corrections in the limit of small latitude angle, using zeta function regularization scheme. Another crucial result of this calculation is that the string partition function is determined entirely by some special modes, which points to a potential bulk localization.

CHAPTER I

Introduction

Gravity is described by Einstein's theory of general relativity and other interactions in nature are described by gauge field theories. One of the most intriguing questions in the history of science has been the quest to find a unified theory of the fundamental forces of nature. String Theory has proved to be a promising candidate towards realizing this dream of grand unification. It first originated in late 1960's in order to understand the strong interactions [1],[2]. After the discovery of QCD as the theory of hadrons, the focus of string research shifted to the Plank scale domain of quantum gravity [3].

The most interesting questions in particle physics concern the numerous proposals for new physics beyond the Standard Model that rely on strongly coupled dynamics. However, the physics of strong interactions is extremely challenging and requires non-perturbative techniques, which are poorly understood. Several plausible directions have been explored in recent years, but only a small number of calculations are tractable in practice. The gauge/gravity correspondence has been immensely useful in studying these strongly coupled theories by identifying their dual string theory realization. In particular, the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence conjectures a mathematical equivalence between string theories, containing gravity, and field theories.

The AdS/CFT correspondence was first put forward in 1997 by Maldacena [4] that explicitly realizes the notion that certain field theories admit an equivalent description in terms of string theories. The most prominent and precise examples of such equivalences are: (i) the duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions and type IIB string theory on $AdS_5 \times S^5$ and (ii) $\mathcal{N} = 6$ supersymmetric Chern-Simon with gauge group $U(N)_k \times U(N)_{-k}$ coupled to matter in three dimensions and type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Generically, when the field theory is strongly coupled, the string theory description is weakly coupled and reduces to supergravity. Naturally, most of the explorations have been centered in understanding strong coupling gauge theory phenomena using a weakly coupled gravity description enhanced with classical strings and branes in the corresponding supergravity backgrounds. Going beyond the supergravity limit, that is, solving the full string theory in curved spacetimes, such as $AdS_5 \times S^5$ or $AdS_4 \times \mathbb{CP}^3$, with Ramond-Ramond fluxes presents a formidable challenge. Given these technical difficulties, it would be particularly illuminating to use AdS/CFT to understand quantum aspects of string perturbation theory in these situations. There are various platforms where this duality can be explored. More generally, our goal is to understand the AdS/CFT correspondence at the quantum level, in the context of Black Holes and Wilson Loops.

Black Holes provide an important theoretical laboratory for probing and testing the fundamental laws of the universe. In 1974, Stephen Hawking showed that black holes are thermodynamical systems with a temperature and entropy. Entropy is a measure of the number of possible microscopic states of a system in thermodynamic equilibrium. The Bekenstein-Hawking entropy of a black hole is proportional to the area of its event horizon.

$$S = k_B \frac{\text{Area}\, c^3}{4 G_N \hbar}. \tag{1.0.1}$$

This is an astonishing formula given the fact that a black hole is a solution of Einstein's equations, which are classical. Given the fundamental constants involved, its a surprising fusion of thermodynamical, relativistic, gravitational, and quantum aspects. Studying corrections to the Bekenstein-Hawking entropy is crucial for a full understanding of the microscopic degrees of freedom responsible for the macroscopic entropy.

String theory has successfully provided a framework for the microscopic counting of the Bekenstein-Hawking entropy for a class of asymptotically flat black holes in the works of Strominger and Vafa [5]. Furthermore, the quantum entropy formalism for extremal black holes has been successfully applied in the literature to study logarithmic corrections to the black hole entropy for a class of asymptotically flat black holes [6, 7, 8, 9]. However, formerly, no similar result exists for asymptotically AdS black holes. Only recently, however, has an explicit example in $\mathrm{AdS}_4/\mathrm{CFT}_3$ emerged. It has been shown that in the large-N limit the topologically twisted index of a certain Chern-Simons theory coupled to matter, known as the ABJM theory, correctly reproduces the leading term in the entropy of magnetically charged black holes in asymptotically AdS_4 spacetimes [10]. Similar matches have now been established in various other situations including: dyonic black holes [11], black holes with hyperbolic horizons [12], and black holes in massive IIA theory [13, 14, 15]. Having established the microscopic counting, it is natural to embark on an exploration of the sub-leading in N structure.

In our work, we studied logarithmic corrections to the topologically twisted index in ABJM theory using numerical and analytical techniques and the corresponding one-loop supergravity computation, focusing on both near horizon geometry and the full AdS_4 black hole background. We also conceptually clarified the holographic dictionary for this AdS_4/CFT_3 example and the role of attractor mechanism in asymptotically AdS spacetimes.

Wilson Loops are an important class of gauge invariant non-local operators, which were introduced as an order parameter for confinement. The study of quantum corrections in the case of Wilson Loops is particularly promising in the context of AdS/CFT correspondence because the expectation value of Wilson loops is determined by string worldsheets [16, 17] and consequently pushes us to confront the underpinnings of string perturbation theory more directly. The field theory side of the correspondence has recently provided a plethora of exact results by means of supersymmetric localization. For example, in $\mathcal{N}=4$ Supersymmetric Yang-Mills (SYM) and in a $\mathcal{N}=6$ Cherns-Simons theory known as ABJM [18], which are the field theory duals of string theories on $AdS_5 \times S^5$ and $\mathrm{AdS}_4 \times \mathbb{C}P^3$, respectively, exact expressions for the vacuum expectation value of some supersymmetric Wilson loops have been obtained [19, 20]. These exact results provide fertile ground to better understand the string perturbation theory in curved spaces beyond the semi-classical approximation and have predictions for the gravity results beyond the leading order [19] setting the stage for systematic explorations at quantum level.

For most observables, semiclassical physics is our only systematic approach to probe the AdS/CFT correspondence beyond the leading classical limit and thus far many questions have been dealt with on a case by case basis without a general framework. The main precept of semi-classical physics consists in integrating quadratic quantum fluctuations around a well-defined classical background. When we get down to practical evaluations, however, we must face the sometimes messy process of treating divergences, as typical of quantum field theory but now with the added intricacies of being in curved space-time. Determining the semiclassical one-loop effective action is equivalent, by definition, to the computation of infinite dimensional functional determinants. Another emergent theme of this thesis is to develop computational technology for string perturbation theory on the curved background. In particular, we study determinants of general Laplace and Dirac operators in asymptotically AdS_2 spacetimes using ζ-function regularization.

In the AdS/CFT context, it is then natural to extrapolate the exact field theory results to the regime where they could be directly compared with the supergravity and semiclassical approximations. This approach was attempted very early on in the insightful work of Drukker, Gross and Tseytlin [21]; it did not, however, led to a match with the field theory prediction. This discrepancy motivated much work [22, 23, 24, 25] that largely confirmed the original discrepancy. A recent revival of this line of effort took place in [26, 27] which considered, on the gravity side, the one-loop effective actions corresponding to the ratio of the expectation values of the $\frac{1}{4}$ to the $\frac{1}{2}$ BPS Wilson loops. Various groups have made important subsequent contributions to this question [28] and recently a precise match has been described, for the $\mathcal{N}=4$ SYM case, [29] after imposing a diffeomorphism preserving cutoff. This reveals the subtleties in the choice of regularization scheme for one-loop computations. We study the one-loop effective action for string configurations dual to latitude Wilson Loops in $\mathcal{N}=4$ SYM and ABJM theory using ζ-function regularization and perturbative heat kernel method, respectively. We get an agreement with the expected field theory result in the limit of small latitude angle.

The thesis is organized as follows:

- In chapter 2, we study the subleading logarithmic corrections in the large N limit to topologically twisted index in ABJM theory and black hole entropy in the dual one-loop quantum supergravity. We provide numerical evidence that the index contains a subleading logarithmic term of the form $-1/2\log N$. On the holographic side, this term naturally arises from a one-loop computation. However, we find that the contribution coming from the near horizon states does not reproduce the field theory answer. We give some possible reasons for this apparent discrepancy.

  In chapter 3, within eleven dimensional supergravity we compute the logarithmic correction to the entropy of magnetically charged asymptotically AdS$_4$ black holes with arbitrary horizon topology. We find perfect agreement with the expected microscopic result arising from the dual field theory computation of the topologically twisted index. Our result relies crucially on a particular limit to the extremal black hole case and clarifies some aspects of quantum corrections in asymptotically AdS spacetimes.

  These chapters are based on :

  [30] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, *Toward Microstate Counting Beyond Large N in Localization and the Dual One-loop Quantum Supergravity*, `1707.04197`

[31] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, *A One-loop Test of Quantum Black Holes in Anti de Sitter Space*, 1711.01076

- In Chapter 4, we provide a field theory interpretation of the attractor mechanism for asymptotically AdS_4 dyonic BPS black holes whose entropy is captured by the supersymmetric index of the twisted ABJM theory at Chern-Simons level one. We holographically compute the renormalized off-shell quantum effective action in the twisted ABJM theory as a function of the supersymmetric fermion masses and the arbitrary vacuum expectation values of the dimension one scalar bilinear operators and show that extremizing the effective action with respect to the vacuum expectation values of the dimension one scalar bilinears is equivalent to the attractor mechanism in the bulk. In fact, we show that the holographic quantum effective action coincides with the entropy functional and, therefore, its value at the extremum reproduces the black hole entropy.

  This chapter is based on :

  [32] A. Cabo-Bizet, U. Kol, L. A. Pando Zayas, I. Papadimitriou, and V. Rathee, *Entropy functional and the holographic attractor mechanism*, *JHEP* **05** (2018) 155, [1712.01849]

- In Chapter 5, we study the zeta-function regularization of functional determinants of Laplace and Dirac-type operators in two-dimensional Euclidean AdS_2 space. More specifically, we consider the ratio of determinants between an operator in the presence of background fields with circular symmetry and the free operator in which the background fields are absent. By Fourier-transforming the angular dependence, one obtains an infinite number of one-dimensional radial operators, the determinants of which are easy to compute. The summation over modes is then treated with care so as to guarantee that the result coincides with the two-dimensional zeta-function formalism. The method relies on some well-known techniques to compute functional determinants using contour integrals and the construction of the Jost function from scattering theory. Our work generalizes some known results in flat space. The extension to conformal AdS_2 geometries is also considered. We provide two examples, one bosonic and one fermionic, borrowed from the spectrum of fluctuations of the holographic $\frac{1}{4}$-BPS latitude Wilson loop.

This chapter is based on :

[33] J. Aguilera-Damia, A. Faraggi, L. Pando Zayas, V. Rathee, and G. A. Silva, *Functional Determinants of Radial Operators in AdS_2*, *JHEP* **06** (2018) 007, [1802.06789]

- In Chapter 6, using ζ-function regularization, we study the one-loop effective action of fundamental strings in $AdS_5 \times S^5$ dual to the latitude $\frac{1}{4}$-BPS Wilson loop in $\mathcal{N} = 4$ Super-Yang-Mills theory. To avoid certain ambiguities inherent to string theory on curved backgrounds we subtract the effective action of the holographic $\frac{1}{2}$-BPS Wilson loop. We find agreement with the expected field theory result at first order in the small latitude angle expansion but discrepancies at higher order.

  This chapter is based on :

  [34] J. Aguilera-Damia, A. Faraggi, L. A. Pando Zayas, V. Rathee, and G. A. Silva, *Zeta-function Regularization of Holographic Wilson Loops*, *Phys. Rev.* **D98** (2018), no. 4 046011, [1802.03016]

- In Chapter 7, we study the one-loop effective action of certain classical type IIA string configurations in $AdS_4 \times \mathbb{CP}^3$. These configurations are dual to Wilson loops in the $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons theory coupled to matter whose expectation values are known via supersymmetric localization. We compute the one-loop effective actions using two methods: perturbative heat kernel techniques and full ζ-function regularization. We find that the result of the perturbative heat kernel method matches the field theory prediction at the appropriate order while the ζ-function approach seems to lead to a disagreement.

  This chapter is based on :

  [35] J. Aguilera-Damia, A. Faraggi, L. A. Pando Zayas, V. Rathee, and G. A. Silva, *Toward Precision Holography in Type IIA with Wilson Loops*, *JHEP* **08** (2018) 044,

[1805.00859]

- In Chapter 8, we study the holographic configurations dual to Wilson loops in higher rank representations in the ABJM theory, which are described by branes with electric flux along their world volumes. In particular, D2 and D6 branes with electric flux play a central role as potential dual to totally symmetric and totally antisymmetric representations, respectively. We compute the spectra of excitations of these brane configurations in both, the bosonic and fermionic sectors. We highlight a number of aspects that distinguish these configurations from their D3 and D5 cousins including new peculiar mixing terms in the fluctuations and organize the spectrum of fluctuations into the corresponding supermultiplets.

  This chapter is based on :

  [36] W. Mück, L. A. Pando Zayas, and V. Rathee, *Spectra of Certain Holographic ABJM Wilson Loops in Higher Rank Representations*, *JHEP* **11** (2016) 113, [1609.06930]

CHAPTER II

Microstate Counting Beyond Large N in Localization and the Dual One-loop Quantum Supergravity

2.1 Introduction

In this chapter, we study subleading corrections to the microstate counting of the entropy of a class of magnetically charged asymptotically AdS_4 black holes.

For a class of asymptotically flat black holes, Strominger and Vafa have demonstrated that string theory provides a framework for the microstate counting of the Bekenstein-Hawking entropy [5]. Moreover, Sen and collaborators have carried a successful program of understanding logarithmic corrections to various black holes [6, 7, 8, 9].

In the context of the AdS/CFT correspondence [4], a microscopic counting of the Bekenstein-Hawking entropy of a class of black holes has recently been presented by Benini, Hristov and Zaffaroni [10, 11]. Understanding black hole entropy in this context is particularly powerful because it does provide a practical path to a fully non-perturbative definition of quantum gravity in asymptotically AdS spacetimes. The basic premise of [10] is that the topologically twisted index of ABJM, namely the supersymmetric partition function on $S^1 \times S^2$ with background magnetic flux on S^2 [37], counts the ground state degeneracy of a superconformal quantum mechanics on S^1, and that this counting enumerates the microstates of the dual magnetically charged BPS black hole in AdS.

It was in fact demonstrated in [10] that the topologically twisted index reproduces the AdS black hole entropy at leading order in the large-N expansion. Here, we wish to extend this correspondence to subleading order by examining the logarithmic corrections on both the field theory and gravity sides of the duality.

In section 2.2 we start by reviewing the field theory computation of the topologically twisted index and present numerical evidence pointing to a universal $-1/2\log N$ correction. We then turn to the gravity calculation in section 2.3, which first reviews the prescription and special status of logarithmic corrections at the one-loop level. We then discuss the dual calculation in the context of 11-dimensional supergravity, focusing on the contribution coming from the near horizon limit of the magnetically charged BPS black hole solutions. We also discuss the absence of potential contributions coming from the asymptotically AdS_4 region. In contrast with the index result, we find $-2\log N$ from the quantum gravity computation, and suggest possible reasons for this discrepancy in section 2.4.

2.2 The topologically twisted index beyond the large-N limit

The topologically twisted index for three dimensional $\mathcal{N}=2$ field theories was defined in [37] (see other related work [38, 39, 40, 41, 42]) by evaluating the supersymmetric partition function on $S^1\times S^2$ with a topological twist on S^2. When applied to the microstate counting of magnetic AdS_4 black holes, the index is computed for ABJM theory, and the topological twist arises from the magnetic fluxes on S^2 [10, 11]. Since these black holes are constructed in the STU model truncation of four-dimensional SO(8) gauged supergravity, there are a total of four U(1) gauge fields, with corresponding charges n_a satisfying the supersymmetry constraint $\sum n_a = 2$.

The topologically twisted index for ABMJ theory was worked out in [10], and reduces to the evaluation of the partition function

$$Z(y_a,n_a)=\prod_{a=1}^{4} y_a^{-\frac{1}{2}N^2 n_a}\sum_{I\in BAE}\frac{1}{\det\mathbb{B}}\frac{\prod_{i=1}^{N}x_i^N\tilde{x}_i^N\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)}{\prod_{i,j=1}^{N}\prod_{a=1,2}(\tilde{x}_j-y_a x_i)^{1-n_a}\prod_{a=3,4}(x_i-y_a\tilde{x}_j)^{1-n_a}},\tag{2.2.1}$$

where y_a are the corresponding fugacities. The summation is over all solutions I of the "Bethe Ansatz Equations" (BAE) $e^{iB_i} = e^{i\tilde{B}_i} = 1$ modulo permutations, where

$$\begin{aligned} e^{iB_i} &= x_i^k \prod_{j=1}^{N} \frac{(1-y_3\frac{\tilde{x}_j}{x_i})(1-y_4\frac{\tilde{x}_j}{x_i})}{(1-y_1^{-1}\frac{\tilde{x}_j}{x_i})(1-y_2^{-1}\frac{\tilde{x}_j}{x_i})}, \\ e^{i\tilde{B}_j} &= \tilde{x}_j^k \prod_{i=1}^{N} \frac{(1-y_3\frac{\tilde{x}_j}{x_i})(1-y_4\frac{\tilde{x}_j}{x_i})}{(1-y_1^{-1}\frac{\tilde{x}_j}{x_i})(1-y_2^{-1}\frac{\tilde{x}_j}{x_i})}. \end{aligned} \tag{2.2.2}$$

Here k is the Chern-Simons level, and the two sets of variables $\{x_i\}$ and $\{\tilde{x}_j\}$ arise from the $U(N)_k \times U(N)_{-k}$ structure of ABJM theory. Finally, the $2N \times 2N$ matrix $\mathbb{B}$ is the Jacobian relating the $\{x_i, \tilde{x}_j\}$ variables to the $\{e^{iB_i}, e^{i\tilde{B}_j}\}$ variables

$$\mathbb{B} = \begin{pmatrix} x_l \frac{\partial e^{iB_j}}{\partial x_l} & \tilde{x}_l \frac{\partial e^{iB_j}}{\partial \tilde{x}_l} \\ x_l \frac{\partial e^{i\tilde{B}_j}}{\partial x_l} & \tilde{x}_l \frac{\partial e^{i\tilde{B}_j}}{\partial \tilde{x}_l} \end{pmatrix}. \tag{2.2.3}$$

See [10] for additional details.

It is convenient to introduce the chemical potentials Δ_a according to $y_a = e^{i\Delta_a}$ and furthermore perform a change of variables $x_i = e^{iu_i}$, $\tilde{x}_j = e^{i\tilde{u}_j}$. In this case, the Bethe ansatz equations become

$$\begin{aligned} 0 &= ku_i - i\sum_{j=1}^{N}\left[\sum_{a=3,4}\log\left(1-e^{i(\tilde{u}_j-u_i+\Delta_a)}\right) - \sum_{a=1,2}\log\left(1-e^{i(\tilde{u}_j-u_i-\Delta_a)}\right)\right] - 2\pi n_i, \\ 0 &= k\tilde{u}_j - i\sum_{i=1}^{N}\left[\sum_{a=3,4}\log\left(1-e^{i(\tilde{u}_j-u_i+\Delta_a)}\right) - \sum_{a=1,2}\log\left(1-e^{i(\tilde{u}_j-u_i-\Delta_a)}\right)\right] - 2\pi \tilde{n}_j. \end{aligned} \tag{2.2.4}$$

where $n_i, \tilde{n}_j$ are integers that parametrize the angular ambiguities.

The topologically twisted index is evaluated by first solving these equations for $\{u_i, \tilde{u}_j\}$, and then inserting the resulting solution into the partition function (2.2.1). This procedure was carried out in [10] in the large-N limit with $k = 1$ by introducing the parametrization

$$u_i = iN^{1/2}\, t_i + \pi - \tfrac{1}{2}\delta v(t_i), \qquad \tilde{u}_i = iN^{1/2}\, t_i + \pi + \tfrac{1}{2}\delta v(t_i), \tag{2.2.5}$$

where we have further made use of reflection symmetry about π along the real axis. In the large-N limit, the eigenvalue distribution becomes continuous, and the set $\{t_i\}$ may be described by an eigenvalue density $\rho(t)$.

2.2.1 Evaluation of the index beyond the leading order in N

The leading order solution for $\rho(t)$ and $\delta v(t)$ was worked out in [10], and the resulting partition function exhibits the expected $N^{3/2}$ scaling of ABJM theory

$$\operatorname{Re}\log Z_0 = -\frac{N^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}\sum_a \frac{n_a}{\Delta_a}. \tag{2.2.6}$$

A similar result was extended to the context of asymptotically AdS_4 black holes with hyperbolic horizon in [12]. We are, of course, interested in taking this solution beyond the leading order. In the ABJM context, we expect the subleading behavior of the index to have the form

$$\operatorname{Re}\log Z = \operatorname{Re}\log Z_0 + N^{1/2} f_1(\Delta_a, n_a) + \log N f_2(\Delta_a, n_a) + f_3(\Delta_a, n_a) + \mathcal{O}(N^{-1/2}), \tag{2.2.7}$$

where the functions f_1, f_2 and f_3 are linear in the magnetic fluxes n_a. In principle, we would like to systematically extend the analysis beyond the leading order in order to obtain the analytic form of these functions. However, this appears to be a challenge, mainly due to the presence of the (left and right) tails of the eigenvalue distribution. (These tails correspond to the nearly vertical segments in figure 2.1.) We thus proceed with a numerical investigation.

The main setup is to arrive at a numerical solution to the BAE (2.2.4) through multidimensional root finding using the leading order distribution as the starting point. We have implemented this in Mathematica using FindRoot. The solution is first obtained either with MachinePrecision or with WorkingPrecision set to 30, and further refined using WorkingPrecision set to 200 and default settings for AccuracyGoal and PrecisionGoal. Convergence to a stable solution can be a bit delicate, since the BAE is highly sensitive to the tails; if even a single eigenvalue is sufficiently displaced, then it is easy for FindRoot to fail. In

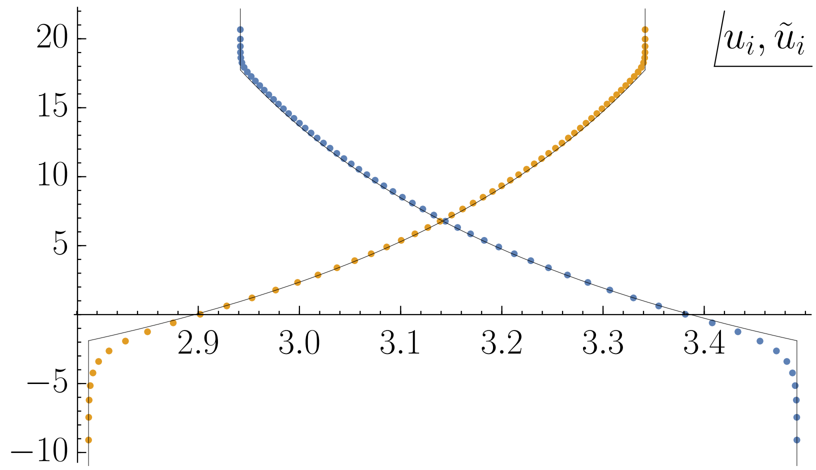

Figure 2.1: The solution to the BAE for $\Delta_a = \{0.4, 0.5, 0.7, 2\pi - 1.6\}$ and $N = 60$. The solid lines correspond to the leading order expression obtained in [10].

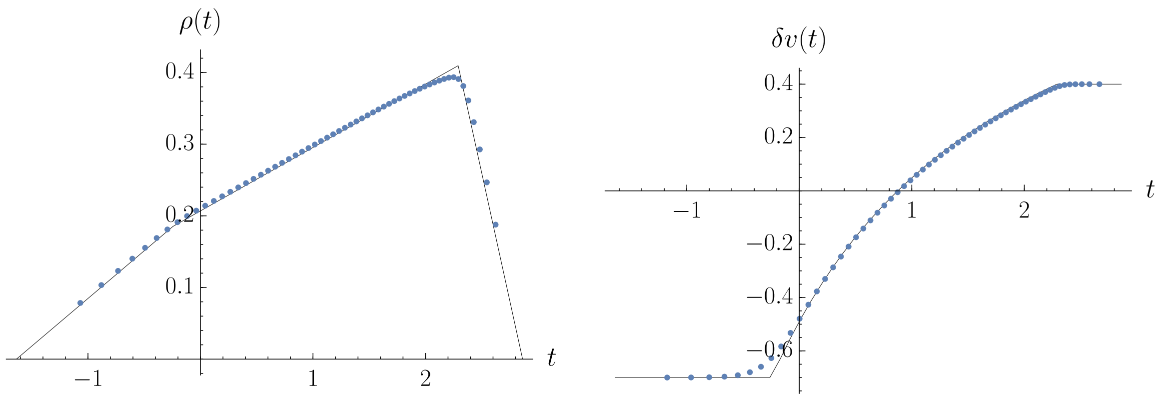

Figure 2.2: The eigenvalue density $\rho(t)$ and the function $\delta v(t)$ for $\Delta_a = \{0.4, 0.5, 0.7, 2\pi - 1.6\}$ and $N = 60$, compared with the leading order expression.

most cases, we have been able to obtain numerical solutions up to $N \approx 200$, although larger values of N are possible with some refinement of the initial distribution. As an example, the numerical solution for the u_i and $\tilde{u}_i$ eigenvalues for$\Delta_a = \{0.4, 0.5, 0.7, 2\pi - 1.6\}$ and $N = 60$ is shown in figure 2.1. The corresponding eigenvalue density $\rho(t)$ and function $\delta v(t)$ are shown in figure 2.2.

Once the eigenvalues are obtained, it is then simply a matter of numerically evaluating the index (2.2.1) on the solution to the BAE. The main challenge here is the evaluation of $\det \mathbb{B}$, as the Jacobian matrix is ill-conditioned. (This is why we work to high numerical precision when solving the BAE.) For a given set of chemical potentials Δ_a, we compute

Δ_1	Δ_2	Δ_3	f_1	f_2	f_3
$\pi/2$	$\pi/2$	$\pi/2$	3.0545	-0.4999	-3.0466
$\pi/4$	$\pi/2$	$\pi/4$	$4.2215 - 0.0491n_1$ $-0.1473n_2 - 0.0491n_3$	$-0.4996 + 0.0000n_1$ $+0.0000n_2 + 0.0000n_3$	$-4.1710 - 0.2943n_1$ $+0.0645n_2 - 0.2943n_3$
0.3	0.4	0.5	$7.9855 - 0.2597n_1$ $-0.5833n_2 - 0.6411n_3$	$-0.4994 - 0.0061n_1$ $-0.0020n_2 - 0.0007n_3$	$-9.8404 - 0.9312n_1$ $-0.0293n_2 + 0.3739n_3$
0.4	0.5	0.7	$6.6696 - 0.1904n_1$ $-0.4166n_2 - 0.4915n_3$	$-0.4986 - 0.0016n_1$ $-0.0008n_2 - 0.0001n_3$	$-7.5313 - 0.6893n_1$ $-0.1581n_2 + 0.2767n_3$

Table 2.1: Numerical fit for $\mathrm{Re}\log Z = \mathrm{Re}\log Z_0 + f_1 N^{1/2} + f_2 \log N + f_3 + \cdots$. The values of N used in the fit range from 50 to $N_{\max}$ where $N_{\max} = 290, 150, 190, 120$ for the four cases, respectively. We made use of the fact that the index is independent of the magnetic fluxes when performing the fit for the special case ($\Delta_a = \{\pi/2, \pi/2, \pi/2, \pi/2\}$).

$\log Z$ for a range of N. We then subtract out the leading behavior (2.2.6) and decompose the residuals into a sum of four independent terms

$$\mathrm{Re}\log Z = \mathrm{Re}\log Z_0 + A + B_1 n_1 + B_2 n_2 + B_3 n_3, \tag{2.2.8}$$

where we have used the condition $\sum_a n_a = 2$. At this stage, we then perform a linear least-squares fit of A and B_a to the function

$$f(N) = f_1 N^{1/2} + f_2 \log N + f_3 + f_4 N^{-1/2} + f_5 N^{-1} + f_6 N^{-3/2}. \tag{2.2.9}$$

We are, of course, mainly interested in f_2. However, since N ranges from about 50 to 200, it is important to consider the first few inverse powers of N as well. (We have confirmed numerically that the first subdominant term enters at $\mathcal{O}(N^{1/2})$, and that in particular terms of $\mathcal{O}(N)$ are absent.)

The results of the numerical fit are presented in Table 2.1. Our main result is that the numerical evidence points to the coefficient of the $\log N$ term being exactly $-1/2$. We thus have

$$\mathrm{Re}\log Z = -\frac{N^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}\sum_a \frac{n_a}{\Delta_a} + N^{1/2} f_1(\Delta_a, n_a) - \frac{1}{2}\log N + f_3(\Delta_a, n_a) + \mathcal{O}(N^{-1/2}), \tag{2.2.10}$$

where f_1 and f_3 remain to be determined. One may wonder whether their dependence on the magnetic fluxes, n_a, follows the same leading order behavior, namely $\sum_a n_a/\Delta_a$. Unfortunately, examination of the table shows that this is not the case.

Although we have been unable to discern the general behavior of f_1, for the special case we find the approximate expression

$$f_1 = 3.0545 \approx \frac{11\pi}{8\sqrt{2}} = \frac{\pi}{\sqrt{2}}\left(\frac{1}{24}+\frac{1}{3}+1\right). \tag{2.2.11}$$

We have in fact extended the special case to $k>1$. For $k \sim \mathcal{O}(1)$, the eigenvalue distribution retains the same features, but with appropriate scaling by k. Working specifically up to $k=5$ and with N up to 200, we find good evidence that in this case the partition function takes the form

$$\operatorname{Re}\log Z(\Delta_a = \pi/2) = -\frac{\pi\sqrt{2k}}{3}N^{3/2} + \frac{\pi}{\sqrt{2k}}\left(\frac{k^2}{24}+\frac{1}{3}+1\right)N^{1/2} - \frac{1}{2}\log N + \cdots\,, \tag{2.2.12}$$

which may be compared with the ABJM free energy on S^3

$$F_{\text{ABJM}} = -\frac{\pi\sqrt{2k}}{3}N^{3/2} + \frac{\pi}{\sqrt{2k}}\left(\frac{k^2}{24}+\frac{1}{3}\right)N^{1/2} - \frac{1}{4}\log N + \cdots\,. \tag{2.2.13}$$

While the leading $\mathcal{O}(N^{3/2})$ term is identical, the first subleading term in the topologically twisted index picks up an additional contribution. In addition, the coefficients of the log terms differ, and this suggests that the two expressions are capturing distinct features of the holographic dual. Some similarities between the free energy and the topologically twisted index were first pointed out in [40]. More generally, relations between partitions functions on S^3 and $S^2 \times S^1$ with a topological twist have recently been discussed in [43]. It would be interesting to place our concrete, subleading in N, results within that more formal approach.

2.2.2 Perilous $1/N$ expansion

While the numerical evidence for $-1/2\log N$ appears compelling, ideally this ought to be backed up by an analytical expansion in the large-N limit. Such an expansion would naturally shed light on the f_1 coefficient as well. However, as mentioned above, the tails make it difficult to maintain a systematic treatment of the $1/N$ expansion. In particular, the tails occur when the eigenvalues $\{u_i, \tilde{u}_j\}$ satisfy

$$\tilde{u}_i - u_i \approx \pm\Delta_a \qquad \Rightarrow \qquad \delta v(t_i) \approx \pm\Delta_a. \tag{2.2.14}$$

In this case, the logs in the BAE, (2.2.4), for j near i are evaluated near zero. The resulting large logs cause apparently subleading terms to become important, and hence mixes up orders in the superficial $1/N$ expansion, as already noted in [10].

The leading order partition function may be obtained by properly accounting for the large logs, and we suspect that a careful treatment would allow the computation to be extended to higher orders. However, this remains a technical challenge, as can be seen from the following illustration. In the large-N limit, it is natural to focus on the eigenvalue density $\rho(t)$ and the function $\delta v(t)$. In the formal large-N expansion, both functions are considered to be $\mathcal{O}(1)$, which is consistent with the plots in figure 2.2. However, their leading-order slopes are discontinuous where the left and right tails meet the inner interval. This gives rise to a δ-function divergence when working with their second derivatives. While the divergence is unimportant at leading order, it presents difficulties at higher order.

Of course, as can be seen in figure 2.2, the actual solution does not have discontinuous slope. As an estimate, we first note that the range where $\rho(t)$ changes slope is of $\mathcal{O}(1/\sqrt{N})$. As a result, $\rho''(t) \sim \mathcal{O}(\sqrt{N})$ near the transition points, and a similar estimate can be made for $\delta v(t)$. While this avoids the δ-function divergences, it nevertheless mixes up orders in the formal large-N expansion. Furthermore, it is not just the second derivative, but all higher derivatives as well that become important, even when considering just the first subleading correction to the index.

2.3 One-loop quantum supergravity

Based on our numerical evidence, we conjecture that the topologically twisted index has a universal logarithmic correction given by $-1/2 \log N$, in contrast with the ABJM free energy that has the factor $-1/4 \log N$. In the latter case, the field theory result was reproduced by a one-loop supergravity computation in [44]. In particular, the standard AdS$_4$/CFT$_3$ correspondence relates ABJM theory on S^3 to M-theory on global AdS$_4 \times S^7/\mathbb{Z}_k$ [18]. The logarithmic term then originates purely from a ghost two-form zero mode contribution on AdS$_4$.

In the present case, however, we take ABJM theory on $S^2 \times S^1$ with a topological twist generated by background magnetic flux. This topological twist relevantly deforms the ABJM theory to flow toward a superconformal quantum mechanics on S^1. Holographically, such an RG flow can be thought of as a Euclidean asymptotically AdS$_4$ BPS magnetic black hole, interpolating between the asymptotically AdS$_4$ region and an AdS$_2\times$ S^2 near horizon region. The solution can be embedded into 11-dimensional supergravity [45], and such an embedding makes it also natural to consider the quantum correction from an 11-dimensional point of view.

We are thus interested in computing the one-loop correction to the supersymmetric partition function in the BPS black hole background that interpolates between asymptotic AdS$_4 \times S^7$ and AdS$_2 \times M_9$ near the horizon, where M_9 is a S^7 bundle over S^2. As a simplification, however, we assume a decoupling limit exists, so that we can focus mainly on the AdS$_2 \times M_9$ near horizon geometry. Alternatively, corrections to the black hole entropy may be considered via the quantum entropy function in the near horizon geometry proposed in [46]. For extremal black hole with no electric charge, the quantum entropy function reduces to the partition function of 11-dimensional supergravity compactified in the near horizon geometry, and we are again led to AdS$_2 \times M_9$.

In the computation of one-loop corrections to the partition function, we focus on the logarithmic term, as such a term, in odd dimensional spaces, arises purely from zero modes (see [47] and [44] for a review). The effect of zero modes on the logarithmic term can be naturally divided into two parts: the subtraction of zero modes from the trace of the heat kernel to make the heat kernel well defined, and the integration over zero modes in the path integral. Those two parts can be summarized schematically, for a given kinetic operator D

of a physical field, as

$$\Delta F_D = (-1)^D (\beta_D - 1) n_D^0 \log L, \tag{2.3.15}$$

where β_D encodes the integration over zero modes in the path integral, and -1 is due to the subtraction in the heat kernel. We use $(-1)^D$ to distinguish bosonic/fermionic contributions. The treatment for ghosts is slightly different, and they are considered separately as in [44]. In summary, the total logarithmic correction is given by

$$\Delta F = \sum_{\{D\}} (-1)^D (\beta_D - 1) n_D^0 \log L + \Delta F_{\text{Ghost}}, \tag{2.3.16}$$

where the summation is over physical fields.

For completeness, we shall first summarize the fields that have non-trivial zero modes n_D^0 in AdS_2 and their β_D, although they are quite standard and well known in the literature (see for example, the appendix of [9]). We then compute the logarithmic correction from the physical sector and the ghost sector of 11-dimensional supergravity in the near horizon geometry $\mathrm{AdS}_2 \times M_9$.

2.3.1 The number and scalings of zero modes

The spectrum of a kinetic operator on a non-compact space, such as AdS_2, typically consists of two parts: a continuous part due to the non-compactness of the space, and possibly a discrete part that contains a countably infinite number of eigenfunctions with zero eigenvalue. The continuous part of the trace of the heat kernel in the case of AdS_N is well defined, whereas the zero modes from the discrete part, if any, should be subtracted from the heat kernel. The formal sum that counts the number of zero modes in the compact case is divergent when the space is non-compact,

$$n_0 = \sum_j \int \sqrt{g}\, d^2x |\phi_j(x)|^2, \tag{2.3.17}$$

where $\phi_j(x)$'s are normalized to 1. Thus computing n_0 requires regularization. For symmetric spaces G/H, n_0 can be evaluated by working out explicit eigenfunctions, exchanging the sum and integral, and using a regularized volume as in [9, 44].

Here, we present another way of computing n_0 using the general theorem in [48]. The number of zero modes can be associated with the formal degree of the discrete series representation of G corresponding to the given field, which occurs when G has a maximal torus that is compact. For $\mathrm{AdS}_N = SO(N,1)/SO(N)$, they occur when N is even, and they can be labeled in terms of the highest weight label (σ, n_0), where $\sigma = (n_{\frac{N-2}{2}}, n_{\frac{N-4}{2}}, \ldots, n_1)$, with $n_{\frac{N-2}{2}} > n_{\frac{N-4}{2}} > \cdots > n_1 > |n_0|$. Any vector bundle over AdS_N can be labeled by an irreducible representation of $SO(N)$ (or $\mathrm{Spin}(N)$) in terms of highest weight labels $\tau = (h_{\frac{N}{2}}, h_{\frac{N-2}{2}}, \ldots, h_1)$, and in order to determine the number of zero modes for a given field, one looks for the branching condition

$$\frac{1}{2} < |n_0| \leq |h_1| \leq n_1 \leq \cdots \leq n_{\frac{N-2}{2}} \leq h_{\frac{N}{2}}. \tag{2.3.18}$$

The number of zero modes is the sum of all degrees $P(\sigma, n_0)$ of discrete series representations (σ, n_0) that satisfies the branching condition, up to a normalization factor that only depends on the dimension:

$$n_0^\tau = \frac{\mathrm{Vol}(\mathrm{AdS}_N)}{c_N} \sum_{(\sigma,n_0)} P(\sigma, n_0). \tag{2.3.19}$$

For AdS_2, $P(n_0) = n_0 - \frac{1}{2}$ and $c_N = 2\pi$, and a field is labeled by a single highest weight label which is its spin. (General expressions for c_N and $P(\sigma, n_0)$ can be found in section 6 of [48].) The branching condition, (2.3.18), implies that fields with spin greater than $\frac{1}{2}$ have zero modes, i.e. one-form, gravitino, and graviton fields. Moreover, using (2.3.19), one has

$$\begin{aligned} n_g^0 &= 2 \times \frac{(-2\pi)}{2\pi}\left(2 - \frac{1}{2}\right) = -3, \\ n_\psi^0 &= 2 \times \frac{(-2\pi)}{2\pi}\left(\frac{3}{2} - \frac{1}{2}\right) = -2, \\ n_A^0 &= 2 \times \frac{(-2\pi)}{2\pi}\left(1 - \frac{1}{2}\right) = -1, \end{aligned} \tag{2.3.20}$$

where n_g^0, n_ψ^0, n_A^0 are respectively the number of zero modes of a graviton, a gravitino and a one form. We also used the fact that the regularized volume of AdS_2 is -2π. These values, of course, coincide with the direct evaluation performed in [9].

The logarithmic part of the integration over zero modes in the path integral can be obtained by dimensional analysis. Given a kinetic operator $\mathcal{O}$, the path integral over zero modes is given by

$$\int Df|_{\text{zero modes}} \exp\left(-\int d^d x \sqrt{g} f \mathcal{O} f\right) = \int Df|_{\text{zero modes}} \sim L^{\beta_{\mathcal{O}} n^0_{\mathcal{O}}}, \tag{2.3.21}$$

through which we define $\beta_{\mathcal{O}}$ for an operator $\mathcal{O}$. To obtain the logarithmic correction, it is enough to find the L dependence of (2.3.21), which amounts to finding the L dependence in the path integral measure. In the case of Euclidean AdS$_{2N}$, all such zero modes arise due to a non-normalizable gauge parameter λ, where $f = G\lambda$ with G representing the infinitesimal gauge transformation. For example, let $g_{\mu\nu} = L^2 g^{(0)}_{\mu\nu}$. The path integral measure of a p-form in d dimensions is normalized as

$$\int DA_{[p]} \exp\left(-L^{d-2p} \int d^d x \sqrt{g^{(0)}} g^{(0)\mu_1\nu_1} g^{(0)\mu_2\nu_2} \ldots g^{(0)\mu_p\nu_p} A_{\mu_1\ldots\mu_p} A_{\nu_1\ldots\nu_p}\right) = 1. \tag{2.3.22}$$

Therefore, the correctly normalized measure is

$$D(L^{\frac{d-2p}{2}} d\lambda_{[p-1]}), \tag{2.3.23}$$

where $\lambda_{[p-1]}$ is a non-normalizable $(p-1)$-form gauge parameter, and has no L dependence. Such a measure gives $L^{(d-2p)/2}$ per zero mode, and therefore contributes as $L^{(d-2p)n^0_p/2}$ in the path integral. Thus $\beta_{A_{[p]}} = (d-2p)/2$ in d dimensions. One can carry out similar computations for other fields, paying particular attention to the possible L dependence of the gauge parameter, as in [9] and [6]. One then finds

$$\beta_g = \frac{d}{2}, \qquad \beta_{\psi_\mu} = d-1, \qquad \beta_{A_{[p]}} = \frac{d-2p}{2}. \tag{2.3.24}$$

2.3.2 The logarithmic corrections

The 11-dimensional $\mathcal{N} = 1$ gravitational multiplet consists of $(g_{\mu\nu}, \psi_\mu, C_{\mu\nu\rho})$. The fluctuation of the metric to the lowest order can be summarized as

$$h_{\mu\nu}(x,y) = \begin{cases} h_{\alpha\beta}(x)\phi(y), \\ h_{\alpha i} = \sum_a A^a_\alpha(x) K^a_i(y), \\ \phi(x) h_{ij}(y), \end{cases} \tag{2.3.25}$$

where we use (x^α, y^i) to denote AdS_2 and M_9 coordinates, respectively, and $K^{ai}(y)\partial_i$ is a killing vector of M_9. The graviton zero modes therefore contribute in two ways: a graviton in AdS_2, and gauge fields corresponding to Killing vectors of M_9.

From the near horizon geometry in [10] one can read off the metric on M_9

$$ds_9^2 = \Delta^{\frac{2}{3}} ds^2_{S^2} + \frac{4}{\Delta^{\frac{1}{3}}} \sum_{i=1}^{4} \frac{1}{X_i} \left(d\mu_i^2 + \mu_i^2 (d\psi_i + \frac{n_i}{2}\cos\theta d\phi)^2 \right), \tag{2.3.26}$$

where we denote the coordinates on S^2 by (θ, ϕ), X_i's are constant with $\prod X_i = 1$, $\Delta = \sum_{i=1}^4 X_i \mu_i^2$, and $\sum_{i=1}^4 \mu_i^2 = 1$. The metric, (2.3.26), suggests the following seven Killing vectors:

$$\left\{ \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi + \sum_j \frac{n_j}{2}\frac{\sin\phi}{\sin\theta}\partial_{\psi_j},\ -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi + \sum_j \frac{n_j}{2}\frac{\cos\phi}{\sin\theta}\partial_{\psi_j},\ \partial_\phi \right\},$$
$$\left\{ \partial_{\psi_i} \right\}, \tag{2.3.27}$$

where $i = 1, 2, 3, 4$, and the Killing vectors span the algebra of the isometry group $SU(2) \times U(1)^4$. Thus the logarithmic correction due to the 11-dimensional graviton is given by

$$\Delta F_h = (\beta_h - 1)(n_g^0 + 7n_A^0) \log L = \left(\frac{11}{2} - 1 \right) [(-3)\times 1 + (-1)\times 7] \log L = -45 \log L. \tag{2.3.28}$$

A gravitino ψ_μ can either be an AdS_2 gravitino and a spin-1/2 fermion on M_9, or vice versa. Ideally one would find the number of killing spinors of M_9. Nevertheless, it is more convenient to reduce to four-dimensions first. In this case, the $\mathcal{N}=2$ gravitational multiplet contains two gravitinos, which further decompose to two gravitinos on AdS_2. As the number of gravitinos only concerns the number of supersymmetries that are preserved, it should be the same no matter whether one works directly in 11 dimensions, or through a reduction to four dimensions. Thus, the contribution due to the gravitino is given by

$$\Delta F_\psi = -\sum(\beta_\psi - 1)n_\psi^0 \log L = -(10-1)[(-2)\times 2]\log L = 36\log L, \tag{2.3.29}$$

where the minus sign is assigned as it is Grassmann odd.

The fluctuation of a 11 dimensional 3 form can be summarized as

$$C_3(x,y) = \begin{cases} A_0(x)\wedge B_3(y), \\ A_1(x)\wedge B_2(y), \\ A_2(x)\wedge B_1(y), \end{cases} \tag{2.3.30}$$

where the subscript represents the rank of the form, $A(x)$ represents a form on AdS_2 and $B(y)$ a form on M_9. Note for M_9 the Betti numbers $b_1 = 0$ and $b_2 = 1$. Therefore the contribution from the 3-form, from the middle line in (2.3.30), is

$$\Delta F_C = (\beta_C - 1)n_C^0 \log L = \left(\frac{5}{2} - 1\right)[(-1)\times 1]\log L = -\frac{3}{2}\log L. \tag{2.3.31}$$

We now turn to the treatment for ghosts, which requires special care. We therefore compute them separately, and we only concern ourselves with ghosts that give rise to AdS_2 zero modes. Therefore only the ghosts for the graviton, which gives a vector ghost c_μ, and the ghosts for the 3-form are considered. The BRST quantization of supergravity generally provides a kinetic term $c_\mu^*(-g^{\mu\nu}\Box - R^{\mu\nu})c_\nu$ with other off diagonal terms that are lower triangular, which do not change the eigenvalues of the kinetic operator on c_ν. In our case, $R_{\mu\nu}$ is never zero, and therefore the graviton ghosts are not relevant to the logarithmic correction.

The general action for quantizing a p-form A_p requires a set of $(p-j+1)$-form ghost fields, with $j = 2, 3, \ldots, p+1$, and the ghost is Grassmann even if j is odd and Grassmann odd if j is even [49, 50]. Although for the $(p-j+1)$-form, the Laplacian operator $(\Delta_{p-j+1})^j$ in the computation of the heat kernel requires an extra $j-1$ removal of the zero modes, the integration over the zero modes is unchanged. The result, as in Eq. (3.4) of [44], is

$$\Delta F_{\text{Ghost}} = \sum_j (-1)^j (\beta_{A_{p-j}} - j - 1) n^0_{A_{p-j}} \log L. \tag{2.3.32}$$

Note for our case that b_1 of M_9 is zero. Therefore the only non-vanishing term is $p = 3$, $j = 2$, which gives

$$\Delta F_{\text{Ghost}} = -\frac{3}{2} \log L. \tag{2.3.33}$$

Finally, adding the contributions (2.3.28), (2.3.29), (2.3.31) and (2.3.33) leads to the total logarithmic correction

$$\Delta F = \left(-45 + 36 - \frac{3}{2} - \frac{3}{2} \right) \log L = -12 \log L \sim -2 \log N, \tag{2.3.34}$$

where in the last equality we used the AdS/CFT dictionary $N \sim L^6$, and neglected L independent terms. We note that this result does not match with the logarithmic term of the topologically twisted index, (2.2.10), which instead is conjectured to have coefficient $-1/2$.

We finish this section by addressing a very natural question. In our computation we have focused exclusively on the near horizon geometry. Given that the black holes we are discussing are asymptotically AdS_4, are there contributions that come precisely from the asymptotic region? After all, the computation of [44] obtained logarithmic corrections on the gravity side by studying quantum supergravity on $\text{AdS}_4 \times S^7$ and found that the entire contribution comes from a two-form zero mode in AdS_4. The result of [44] perfectly matches field theory results corresponding to the free energy of ABJM on S^3. Our case, however, pertains to a computation of ABJM on $S^2 \times S^1$. In an elucidating discussion about boundary modes presented in [51], the authors considered global aspects of AdS_4 with S^3 and $S^2 \times S^1$ boundary conditions. In particular, they established that the Euler number depends on these boundary conditions and is, respectively, $\chi = 1$ and $\chi = 0$. This result indicates the existence of a two-form zero mode in the case of S^3 boundary conditions which is precisely the two-form responsible for the successful match with the field theory free energy. It also indicates the absence of the corresponding two-form zero mode for $S^2 \times S^1$ boundary conditions. Moreover, the crucial use of S^3 boundary conditions in the explicit construction of the non-trivial two forms [48, 52, 53], also supports our claim.

Therefore, at least to this level of scrutiny, there is no contribution coming from the asymptotically AdS_4 region. It will, of course, be interesting to develop a systematic approach to dealing with asymptotically AdS contributions in the framework of holographic renormalization.

2.4 Discussion

Given the disagreement in the computations, we shall discuss some of our underlying assumptions. On the field theory side, the topologically twisted index reproduces the Bekenstein-Hawking entropy of AdS black holes at leading order in the large-N expansion [10, 12]. It is thus tempting to expect that the index provides an complete microstate description at all orders. To explore this possibility, we have performed a numerical investigation of the topologically twisted index and obtained a logarithmic correction of $-1/2 \log N$. We have attempted to reproduce this term by computing a one-loop partition function on the supergravity side of the duality.

While AdS/CFT suggests that the corresponding one-loop partition function ought to be computed in the full magnetic AdS_4 black hole background, we made a decoupling approximation and focused instead on the $\mathrm{AdS}_2 \times S^2$ near horizon region. Given the 11-dimensional supergravity origin, only zero modes contribute to the logarithmic term, and we find instead the term $-2 \log N$ from the bulk computation. In the next chapter, we explore that the agreement would be restored working in the full black hole geometry in a certain thermal-based limit.

Let us now discuss a number of other directions that would be nice to explore. One natural question is motivated by the universality of the result of [44]. Indeed, a large class of field theory partition functions on S^3 has a $-1/4 \log N$ correction for matter Chern-Simons theories of various types [54, 55]. On the gravity side of the correspondence, the universality of this result relies on the logarithmic term being given strictly by a two-form zero mode in AdS_4; it is thus independent of the Sasaki-Einstein X_7 manifold where the supergravity is defined [44]. It would be interesting to entertain a similar universality argument for the correction we find here, namely $-1/2 \log N$.

A more challenging question is: Can one obtain the full logarithmic correction to the entropy, and not just the $\log N$ coefficient? One possibility is to tackle the theory directly in four dimensions. In this case the heat kernel, being in an even dimensional space, contributes in a more complicated way. A similar technical problem appears in the 't Hooft limit where the gravity dual theory lives on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. It is worth pointing out an added difficulty in the case of the magnetically charged black holes we are considering. For asymptotically flat black holes, a typical practice is to consider particular N-correlated scalings of the charges; this allows for the computation of corrections in various regimes. However, generic scalings of the charges are not allowed in our case because the charges are constrained, for example, by $\sum n_i = 2$. Alternatively, one could attempt a full supergravity localization following the work [56] and the more recent effort in [57].

Of course, it is worth noting that the first subleading correction to the topologically twisted index occurs at $\mathcal{O}(N^{1/2})$. In principle, it would be useful to obtain an analytic expression for this correction, which we denoted $f_1(\Delta_a, n_a)$ in (2.2.7). On the gravity side, this term presumably originates from higher-derivative corrections to the Wald entropy. While we have been as yet unable to find the analytic form of f_1, it may be possible to do so with additional numerical work.

Finally, it would be interesting to discuss other asymptotically AdS gravity configurations forming AdS/CFT dual pairs. For example, we may consider black strings in AdS_5 that are dual to topologically twisted four-dimensional field theories [58]. The topologically twisted index for the dual four-dimensional field theories on $S^2 \times T^2$ has been constructed in [59, 38] and its high temperature limit has recently been discussed in [60].

CHAPTER III

One-loop test of Quantum Black Holes in the Anti de Sitter Space

3.1 Introduction

In this chapter, we report on a computation of the one-loop effective action for a class of asymptotically AdS_4 black holes that matches precisely the coefficient of the logarithmic correction arising from a microscopic description.

In the previous chapter, we studied corrections to the topologically twisted index using a combination of numerical and analytical techniques and identified a logarithmic correction of the form $-\frac{1}{2}\log N$. A corresponding computation on the gravity side, focusing on the near horizon contribution to the one-loop effective action and following the quantum entropy formalism developed by Sen [61, 46], however, failed to match this microscopic result as showed in Chapter II. However, here we find perfect agreement when the one-loop supergravity computation is performed in the full AdS_4 black hole background, and not just in the near horizon geometry. This suggests that, in contrast with asymptotically flat black holes, the microscopic degrees of freedom of AdS black holes are sensitive to the background in which they are embedded.

3.2 Topologically Twisted Index in ABJM

On the microscopic side, the CFT dual to magnetically charged AdS_4 black holes is given by ABJM theory with background flavor fluxes turned on. ABJM theory is a three-dimensional Chern-Simons-matter theory with $U(N)_k \times U(N)_{-k}$ gauge group and opposite integer levels k and $-k$ [18]. The matter sector contains four complex scalar fields $C_I, (I = 1,2,3,4)$ in the bifundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$, together with their fermionic partners. The theory is superconformal and has $\mathcal{N} = 6$ supersymmetry generically, but for $k = 1, 2$, the symmetry is enhanced to $\mathcal{N} = 8$. Holographically, ABJM describes a stack of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, whose low energy dynamics are effectively described by 11 dimensional supergravity.

The presence of background fluxes implements a partial topological twist, and is crucial for preserving supersymmetry when the theory is defined on $\Sigma_g \times S^1$, where Σ_g is a genus-g Riemann surface corresponding to the horizon topology of the black hole. The topologically twisted index is then defined as the supersymmetric partition function of the twisted theory, $Z(n_a, \Delta_a) = \mathrm{Tr}\,(-1)^F e^{-\beta H} e^{iJ_a \Delta_a}$. It depends on the fluxes, n_a, through H and on the chemical potentials Δ_a. This index was constructed in [37] for $\mathcal{N} \geq 2$ supersymmetric theories on $S^2 \times S^1$ and computed via supersymmetric localization. It was then applied to ABJM theory in [10], and evaluated in the large-N limit.

In the large-N limit, and at genus zero, the $k = 1$ index takes the form

$$\begin{aligned} F = & -\frac{N^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}\sum_a \frac{n_a}{\Delta_a} + N^{1/2} f_1(\Delta_a, n_a) \\ & -\frac{1}{2}\log N + f_3(\Delta_a, n_a) + \mathcal{O}(N^{-1/2}), \end{aligned} \tag{3.2.1}$$

where $F = \mathrm{Re}\log Z$. The leading $\mathcal{O}(N^{3/2})$ term was obtained in [10], and exactly reproduces the Bekenstein-Hawking entropy of a family of extremal AdS_4 magnetic black holes admitting an explicit embedding into 11d supergravity [45], once extremized with respect to the flavor and R-symmetries. The $\mathcal{O}(N^{1/2})$ term can be identified with $\mathcal{O}(\alpha'^3 R^4)$ corrections in the supergravity, and does not appear to have a simple form. On the other hand, the $-\frac{1}{2}\log N$ term, obtained numerically in previous chapter, appears to be universal, and is what we wish to reproduce from the gravity side.

In fact, the topologically twisted index can be defined on Riemann surfaces with arbitrary genus [59, 42], and there is a simple relation between the index on $\Sigma_g \times S^1$ and that on $S^2 \times S^1$: $F_{\Sigma_g\times S^1}(n_a, \Delta_a) = (1-g)F_{S^2\times S^1}(\frac{n_a}{1-g}, \Delta_a)$. Since the coefficient of the logarithmic term in $F_{S^2\times S^1}$ does not depend on n_a we simply have

$$F_{\Sigma_g\times S^1}(n_a, \Delta_a) = \cdots - \frac{1-g}{2}\log N + \cdots . \tag{3.2.2}$$

We now demonstrate that this logarithmic correction naturally appears in the quantum correction to the extremal magnetically charged AdS_4 black hole.

3.3 One-loop Quantum Supergravity

Since the AdS_4 black holes may be embedded in 11d supergravity [45], we will take a 11d approach to the gravity calculation. Dimensional analysis shows that logarithmic corrections come from one-loop determinants. The standard computation of such terms for black holes in asymptotically flat spacetime reduces to the near horizon geometry [46]. However, in Chapter II, the near horizon contribution was calculated to be $-2\log N$, resulting in a mismatch with the field theory answer. Such a mismatch indicates that either somehow the near horizon geometry is not enough to compute the quantum entropy, or the index does not correctly count microstates in the sub-leading order.

In this chapter, we provide evidence for the first possibility by directly computing the logarithmic correction to the entropy from its thermodynamical definition,

$$S = \lim_{\beta\to\infty}(1-\beta\partial_\beta)\log Z[\beta,\dots], \tag{3.3.3}$$

where β is the inverse temperature. We work in the large AdS radius limit, $L \gg 1$, where $L \sim N^{\frac{1}{6}}$ by the AdS/CFT dictionary. Our focus is on the one-loop partition function, which can be written schematically as

$$Z_{\text{1-loop}}[\beta,\ldots] = \sum_D (-1)^D(\tfrac{1}{2}\log\det{}'D) + \Delta F_0, \tag{3.3.4}$$

where D stands for kinetic operators corresponding to various fluctuating fields and $(-1)^D = -1$ for bosons and 1 for fermions. The prime indicates removal of the zero modes, which are accounted for separately by

$$\Delta F_0 = \log \int [d\phi]|_{D\phi=0}, \tag{3.3.5}$$

where $\exp(-\int d^dx\sqrt{g}\phi D\phi) = 1$.

For a stationary background, the logarithmic part of the one-loop determinant comes from

$$-\tfrac{1}{2}\log\det{}'D = \left(\frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^\beta dt A_{d/2}(\beta,\dots) - n_0\right)\log L + \cdots, \tag{3.3.6}$$

where $A_{d/2}(\beta,\dots) = \int d^{d-1}x\sqrt{g}\, a_{d/2}(x,x)$. For odd dimensional spacetimes, the Seeley-De Witt coefficient $a_{\frac{d}{2}}(x,x)$ vanishes due to the lack of a diffeomorphism invariant scalar function of the metric with scaling dimension d [47]. The advantage of working in 11d is then clear, as only the zero mode contributions remain. The structure of the logarithmic term is then given by

$$\log Z[\beta,\dots] = \sum_{\{D\}}(-1)^D(\beta_D - 1)n_D^0\log L + \Delta F_{\text{Ghost}} + \cdots, \tag{3.3.7}$$

where the ghost contributions are treated separately, as in [44], and β_D is due to the integration over zero modes, Eq. (3.3.5), in the path integral, as studied in various cases of logarithmic contributions to the black hole entropy and the one-loop partition function [8, 6, 7, 44].

3.3.1 Magnetically charged AdS$_4$ Black Holes

Our task at hand is thus to enumerate the zero modes of the fluctuations in the AdS$_4$ magnetic black hole background. These black holes were originally obtained in [62], more recently discussed in [63] and reviewed in [10]. They are solutions of $\mathcal{N} = 2$ gauged supergravity with 3 vector multiplets, and with prepotential and FI gauging parameters

$$F = -2i\sqrt{X^0 X^1 X^2 X^3}, \qquad \xi_\Lambda = \frac{1}{2}, \qquad \Lambda = 1, \ldots, 4. \tag{3.3.8}$$

The family of black holes admits background fluxes F^a, $a = 1, \ldots, 4$ over a Riemann surface horizon Σ_g. The BPS condition requires

$$\frac{1}{2\pi} \sum_a \int_{\Sigma_g} F^a = \chi(\Sigma_g). \tag{3.3.9}$$

The solutions are parametrized by four fluxes n^a and the genus of the horizon, g, subject to the above BPS constraint. The metric of the solution can be put in the form

$$ds^2 = U^2(r)\, d\tau^2 + U^{-2}(r)\, dr^2 + h^2(r) ds^2_{\Sigma_g}, \tag{3.3.10}$$

where $U(r) = e^{K(r)} r^2 (1 - \frac{a}{2gr^2})^2$ and $h(r) = 2e^{K(r)} r^2$ in the extremal case. A more comprehensive review, including non-extremal solutions, is found in [64].

These black holes may be uplifted as solutions to 11d supergravity, with fields consisting of a metric $g_{\mu\nu}$, a three-form field $C_{\mu\nu\rho}$ and a gravitino Ψ_μ. From an 11d perspective, we are interested in their zero mode fluctuations on a background which is locally of the form $M_4 \times S^7$, where M_4 has metric given by Eq. (3.3.10), and the 7-sphere is squashed in the process of turning on magnetic flux. Given an 11d kinetic operator, one can decompose it to a M_4 part and a S^7 part. Since compactness of S^7 leads to non-negative eigenvalues, zero modes of the 11d supergravity fields are thus simultaneously zero modes in M_4 and S^7. As a result, we only need to consider the massless Kaluza-Klein sector, corresponding to the fields of 4d $\mathcal{N} = 8$ gauged supergravity, and to seek out their zero modes in the AdS$_4$ black hole background.

3.3.2 Metric and Fermion zero modes

From a four-dimensional perspective, the fluctuating fields we must consider include the metric, p-forms, and fermions. We first demonstrate that the metric and fermions do not have any zero modes in the black hole background. This leaves the p-forms, we we turn to below. For the metric, a zero mode requires a pure gauge mode with a non-normalizable gauge parameter. To show it cannot exist, it is enough to focus on the asymptotic metric,

$$ds^2 = \frac{dr^2}{r^2} + r^2(dt^2 + ds^2_{\Sigma_g}). \tag{3.3.11}$$

For a pure gauge deformation, $h_{\mu\nu} = \nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu$, normalizability demands

$$\begin{aligned} h_{rr} &= 2\nabla_r \eta_r \qquad\quad \sim 1/r^4, \\ h_{ri} &= \nabla_r \eta_i + \nabla_i \eta_r \sim 1/r^2, \\ h_{ij} &= \nabla_i \eta_j + \nabla_j \eta_i \sim \mathcal{O}(1). \end{aligned} \tag{3.3.12}$$

Thus asymptotically $\eta_i \sim 1/r$ and $\eta_r \sim 1/r^3$. As a result

$$\|\eta\|^2 = \int \sqrt{g} g^{\mu\nu} \eta_\mu \eta_\nu d^4x \sim \int^\infty (r^4 \eta_r^2 + \eta_i^2) dr < \infty, \tag{3.3.13}$$

and the gauge parameter is thus normalizable.

A similar argument can be made for the gravitino to show the absence of zero modes. In particular, potential gravitino zero modes correspond to would be pure gauge modes $\psi_\mu = \mathcal{D}_\mu \epsilon$ (where $\mathcal{D}_\mu$ is the supercovariant derivative), however with non-normalizable spinor ϵ. Working with the metric (3.3.11), we can see that $\epsilon \sim 1/r^2$ is required for ψ_μ to be normalizable. Since this makes ϵ normalizable as well, we conclude that there are no gravitino zero modes in this background.

3.3.3 p-form zero modes

We now turn to an examination of p-form zero modes. Recall that, for zero modes of A_p in a compact space, one requires $\langle dA_p, dA_p\rangle = 0$ with respect to the standard inner product on p-forms. This amounts to requiring A_p to be closed. But A_p and $A_p + d\alpha_{p-1}$ are gauge equivalent, and the redundant contributions in the path integral are canceled by the Faddeev-Popov procedure. Therefore the number of the zero modes is the dimension of the p-th de-Rham cohomology.

We are of course interested in a non-compact space, in which case there are several complications, especially with infinite volume. First, the physical spectrum only includes forms with finite action, as the weight in the Euclidean path integral is e^{-S}. Second, for a non-normalizable $p-1$ form, the gauge transformation $d\alpha_{p-1}$ can be normalizable and included in the physical spectrum, yet the Faddeev-Popov procedure can only cancel gauge transformations with normalizable α_{p-1}. The result is a physical spectrum with some pure gauge modes with non-normalizable gauge parameters, a situation which is ubiquitous in one-loop gravity computations in AdS [8, 7]. Third, there are usually infinitely many such modes, making the number of zero modes infinite. Mathematically, the first two complications lead one to consider L^2 cohomology, $H^p_{L^2}(M, \mathbb{R})$ by replacing the de-Rham chain complex by one consisting of L^2 p-forms whose exterior derivative is also L^2 [52]. The third complication simply states that $\dim H^p_{L^2}(M, \mathbb{R})$ can be unbounded.

A further subtlety in the non-compact case is the difference between $H^p_{L^2}(M, \mathbb{R})$ and, $\mathcal{H}^p_{L^2}(M, \mathbb{R})$, the space of L^2 harmonic p-forms. As in [48], a transverse condition on the gauge field is imposed when heat kernel method is applied. It is, therefore, more precise to identify the space of concern to be $\mathcal{H}^p_{L^2}(M, \mathbb{R})$. The number n^0_p of p-form zero modes is then given by the regularized dimension

$$n^0_p = \dim^R \mathcal{H}^p_{L^2}(M, \mathbb{R}) = \int_R \sum_n A^n_p \wedge \star A^n_p, \tag{3.3.14}$$

where $\{A^n_p\}$ is a set of orthonormal basis functions, and the integral is defined as the finite piece after regularization.

Before turning to a full accounting of zero modes, we make an observation that will prove useful below. When the manifold is compact the Euler characteristic is given by $\chi(M) = \sum_p (-1)^p \mathrm{dim}\mathcal{H}^p(M, \mathbb{R})$, and a similar relation still holds for non-compact manifolds in the class known as conformally compact manifolds (see Corollary 8.1 in [65]). A conformally compact manifold is a manifold with boundary whose metric admits expansions near the boundary

$$ds^2 = \frac{du^2}{\alpha(u)^2 u^2} + \frac{h_{ij} dx^i dx^j}{u^2}, \tag{3.3.15}$$

where the boundary is at $u = 0$, with $\alpha(0) \neq 0$ and $h_{ij}(0)$ well defined. For such a manifold of even dimension it was proved in [65] that $\mathcal{H}^i_{L^2} = H^k_{DR}(M, \partial M)$ for $i < \frac{n}{2}$ and $\mathcal{H}^i_{L^2} = H^k_{DR}(M)$ for $i > \frac{n}{2}$. The appropriate modification of the Gauss-Bonnet theorem states

$$\begin{aligned} \int^{\mathrm{Reg}} \mathrm{Pf}(R) = 2 \sum_{i<\frac{n}{2}} & (-1)^i \mathrm{dim} H^i_{DR}(M, \partial M) \\ & + (-1)^{\frac{n}{2}} \mathrm{dim}^R \mathcal{H}^{\frac{n}{2}}_{L^2}(M, \mathbb{R}), \end{aligned} \tag{3.3.16}$$

where $H^i_{DR}(M, \partial M)$ stands for the relative de-Rham cohomology, and the Gauss-Bonnet integral is regularized. It follows from the definition that an asymptotic AdS manifold is a conformally compact manifold and Eq. (3.3.16) applies to determine $\mathrm{dim}^R \mathcal{H}^{\frac{n}{2}}_{L^2}(M, \mathbb{R})$ for the AdS$_4$ black hole. Indeed, an explicit version of the above formula was applied in [51] to elucidate aspects of quantum inequivalence in AdS$_4$.

In applying the thermodynamic entropy (3.3.3), we take the extremal limit of the non-extremal AdS$_4$ black hole. In this case, the topology of the non-extremal black hole is homotopic to its horizon Σ_g due to the contractible (t, r) directions. Thus the Euler characteristic of the non-extremal black hole is simply $\chi_{\mathrm{BH}} = 2(1 - g)$. It also indicates that all but the second relative de-Rham cohomology vanish. Therefore, using Eq. (3.3.16), one obtains

$$n_2^0 = \mathrm{dim}^R \mathcal{H}^2_{L^2}(M, \mathbb{R}) = \int^{\mathrm{Reg}} \mathrm{Pf}(R) = \chi_{\mathrm{BH}} = 2(1 - g), \tag{3.3.17}$$

and moreover these are the only possible zero modes in the black hole background.

The regularized dimension, n_2^0, can be negative for higher genus. In fact, this is a general feature of regularized dimensions defined as above. For example, in the case of AdS$_2$, $\dim^R \mathcal{H}^1_{L^2}(\mathrm{AdS}_2, \mathbb{R}) = -1$ and such negative dimensions occurs in various computations of the macroscopic logarithmic contributions to BPS black holes in asymptotically flat spacetime [6, 7].

3.3.4 Two-form zero modes from 11d SUGRA

What we have seen above is that the logarithmic correction only comes from two-form zero modes in in the asymptotically AdS$_4$ black hole background. This result is essentially the same as in [44], however with the difference that here the 11d space is only locally $M_4 \times S^7$, where M_4 is the AdS black hole. (This difference manifests itself as $n_2^0 = \chi_{\mathrm{AdS}} = 1$ for global AdS$_4$ with S^3 boundary, in contrast to Eq. (3.3.17) for the black hole.) However, the Kaluza-Klein procedure, when performed properly, is equally valid in both cases.

The straightforward reduction of 11d supergravity on squashed S^7 does not yield any two-forms in four dimensions, as there are no non-trivial 1-cycles for the 11d three-form $C_{\mu\nu\rho}$ to be reduced on. However, the quantization of $C_{\mu\nu\rho}$ introduces 2 two-form ghosts that are Grassmann odd, 3 one-form ghosts that are Grassmann even and 4 scalar ghosts that are Grassmann odd [49], and the two-form ghosts will contribute to the log term.

The 11d two-form ghost A_2 has action

$$S_2 = \int A_2 \wedge \star(\delta d + d\delta)^2 A_2, \tag{3.3.18}$$

and the logarithmic term in the one-loop contribution to the entropy is thus, according to Eqs. (3.3.4)-(3.3.7),

$$\log Z_{\text{1-loop}}[\beta, \ldots] = (2 - \beta_2) n_2^0 \log L + \cdots , \tag{3.3.19}$$

where β_2 comes from integrating the zero modes in the path integral, and the minus sign takes care of the Grassmann odd nature of A_2. The zero mode path integral becomes simply $\int [dA_2]|_{\text{zero modes}}$, and to find the logarithmic contribution in this term, one looks at the L dependence by dimensional analysis, as in [44]. The properly normalized measure is $\int d[A_{\mu\nu}]\exp(-L^7\int d^{11}x\sqrt{g^{(0)}}g^{(0)\mu\nu}g^{(0)\rho\sigma}A_{\mu\rho}A_{\nu\sigma})=1$, where we single out the L dependence of the metric, $g^{(0)}_{\mu\nu}=\frac{1}{L^2}g_{\mu\nu}$. Thus the normalized measure is $\prod_x d(L^{\frac{7}{2}}A_{\mu\nu})$. For each zero mode, there is a $L^{\frac{7}{2}}$ factor. Thus in the logarithmic determinant, one has $\beta_2=\frac{7}{2}$. Combining Eqs. (3.3.17) and (3.3.19), the $\log L$ contribution to the thermal entropy in the extremal background is thus

$$\log Z_{\text{1-loop}}[\beta,\dots]=-3(1-g)\log L+\cdots\,. \tag{3.3.20}$$

3.3.5 Extremal Black Hole Entropy

The coefficient of the logarithmic term in Eq. (3.3.20) does not depend on β. In fact, due to the vanishing of the Seeley De-Witt coefficient, it can only depend on β through regularized n_p^0's, which, due to the asymptotic AdS condition, are topological. Therefore Eq. (3.3.3) gives simply $S_{\text{1-loop}}=-3(1-g)\log L+\cdots$. As this is β independent, it is also valid in the extremal limit, $\beta\to\infty$. Finally, the AdS/CFT dictionary establishes that $L\sim N^{1/6}$ leading to a logarithmic correction to the extremal black hole entropy of the form

$$S_{\text{1-loop}}=-\frac{1-g}{2}\log N+\cdots\,, \tag{3.3.21}$$

which perfectly agrees with the microscopic result, (3.2.2).

3.4 Conclusion

It is worth highlighting that the supergravity one-loop computation is universal in the sense that it applies to any asymptotically AdS$_4$ black hole that can be embedded in 11d supergravity under the mild condition that the seven-dimensional compactification manifold has vanishing first homology. There is a similar universal behavior in the one-loop effective action in AdS$_4$ [44] which matches perfectly with the logarithmic correction of the supersymmetric partition function on S^3. It would be interested to establish the universality of the logarithmic corrections to the black hole entropy from the field theory side as well.

Our precise example, when taken in conjunction with Chapter II and [66], clarifies that the quantum entropy function that has been so successful in the context of asymptotically flat black holes needs to be revisited in the context of asymptotically AdS black holes. Arguably, the connection between degrees of freedom residing at the horizon and other potential hair degrees of freedom needs to be better understood by revisiting previous approaches [67, 68].

It was crucial in our result that we took a particular thermal-based limit to the extremal black hole agreeing with some observations in the literature [46, 69]. This limiting procedure raises the specter that perhaps supersymmetric computations contain some information about slightly non-extremal systems in which case a window into capturing more dynamical information, such as Hawking radiation, could be opening.

Therefore, the non-extremal black hole background only admits 2-form zero modes, with $n_2^0 = 2(1-g)$. Our task is thus to identify the relevant 2-forms originating from 11d supergravity on $M_4 \times S^7$ and to sum up their contributions according to Eq. (3.3.7).

CHAPTER IV

Entropy functional and the holographic attractor mechanism

4.1 Introduction

Recently, there has been an explicit realization of $\mathrm{AdS}_4/\mathrm{CFT}_3$ example that yielded impressive results for the microstate counting of black hole entropy [10]. Under a series of assumptions, more crucially an identification of chemical potentials and an extremization procedure, a perfect large N match between the topologically twisted index and the black hole entropy was established [10]. Under similar assumptions matches have now been established in various other situations including: dyonic black holes [11], black holes with hyperbolic horizons [12], and black holes in massive IIA theory [13, 14, 15].

The goal of this chapter is largely motivated by a desire to conceptually clarify, within the standard AdS/CFT dictionary, the various assumptions made in [10]. Consider, for example, the role of the attractor mechanism which is a key intuition building concept in our understanding of black holes in supergravity theories [70, 71, 72]. It roughly states that the black hole entropy is determined by extremization of the central charge in the moduli space. A decade after its original formulation, the attractor mechanism intuition took an upgraded incarnation - the entropy formula [73] - which accommodates higher curvature corrections and weakens the hold of supersymmetry. There is, however, an important conceptual difference between the attractor mechanism in flat space and its counterpart in asymptotically AdS spacetimes. In asymptotically flat spacetimes the attractor mechanism is loosely associated with no-hair theorems. In asymptotically AdS spacetimes this intuition is lacking due to the natural existence of hair. Moreover, in the context of the AdS/CFT most of the key properties of the duality are precisely defined in the asymptotic region, not close to the horizon. This dichotomy between boundary and horizon data has been pointed out before and discussed in the context of Wald entropy formula in [74]. Here we address it via the AdS/CFT correspondence.

Recall that in the attractor mechanism one extremizes the central charge with respect to the moduli [72]. We demonstrate that for asymptotically AdS black holes in gauged supergravity, the attractor mechanism can be reinterpreted using exclusively boundary data. More precisely, using the AdS/CFT dictionary, we compute the renormalized off-shell quantum effective action in the twisted ABJM theory as a function of the supersymmetric fermion masses and the arbitrary vacuum expectation values of the dimension one scalar bilinear operators. This effective action coincides with the entropy functional and we show that its extremization with respect to the vacuum expectation values of the dimension one scalar bilinears is equivalent to the attractor mechanism. We thus provide a strictly field theoretic interpretation of the attractor mechanism in the context of $\mathcal{N} = 2$ gauged supergravity and a rigorous understanding of the beautiful results of [10].

This chapter is organized as follows. In section 4.2, we review the relevant structure of $\mathcal{N} = 2$ gauge supergravity and provide a universal formula for the regularized on-shell action in terms of the effective superpotential for general dyonic black holes introduced in [75]. Section 4.3 is devoted to key aspects of the holographic dictionary. We derive the supersymmetric boundary counterterms and discuss the supersymmetric boundary conditions for the scalars. Moreover, we determine the renormalized operators dual to bulk fields and we compute the renormalized quantum effective action for dyonic BPS black holes. Using this quantum effective action, in section 4.4 we obtain one of the key results of the chapter: a holographic interpretation of the attractor mechanism. We conclude in section 4.5. Some technical details are relegated to two appendices. In appendix A we explicitly discuss various parameterizations of the STU model, and in B we review the radial Hamiltonian formulation of the bulk dynamics.

4.2 Effective superpotential for dyonic black holes

We are mostly interested in black hole solutions of the Abelian $U(1)^4$ $\mathcal{N}=2$ gauged supergravity in four dimensions, often referred to as the gauged STU model, which is a consistent truncation of $\mathcal{N}=8$ gauged supergravity [76, 45]. With appropriate supersymmetric boundary conditions, this theory is holographically dual to a sector of the ABJM theory at Chern-Simons level one. Most of our analysis, however, applies broadly to $\mathcal{N}=2$ gauged supergravity and we begin by briefly reviewing some general properties.

4.2.1 $\mathcal{N}=2$ gauged supergravity

As an example of the generality of our approach we describe the $U(1)^4$ theory using the general framework of $\mathcal{N}=2$ gauged supergravity in four dimensions. In this language the $U(1)^4$ theory consists of the gravity multiplet coupled to $n_V=3$ vector multiplets and no hypermultiplets. Since the gauge group is Abelian, the scalars in the vector multiplets are neutral and so the only charged fields present are the two gravitini. This is usually referred to as Fayet-Iliopoulos (FI) gauging.[1] The gauge fields that couple to the gravitini are a linear combination of the graviphoton and the n_V vectors from the vector multiplets, $\xi_\Lambda A^\Lambda_\mu$, with $\Lambda = 0, 1, \dots, n_V$. The constants ξ_Λ are called the FI parameters. For the $U(1)^4$ theory the FI parameters are all equal, i.e.

$$\xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi > 0, \tag{4.2.1}$$

where the value of the constant ξ depends on the normalization of the vector fields in the Lagrangian. For general FI parameters we define $2\xi \equiv \sqrt{\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2}$. We keep ξ arbitrary in order to facilitate comparison with different conventions in the literature.

The complex scalars z^α in the vector multiplets, with $\alpha = 1, \dots, n_V$, parameterize a special Kähler manifold – an n_V-dimensional Hodge-Kähler manifold which is the base of

[1]Throughout this chapter we consider only purely electric gauging.

a symplectic bundle with the covariantly holomorphic sections

$$\mathfrak{V} = e^{\mathcal{K}(z,\bar{z})/2}(X^\Lambda, F_\Lambda), \tag{4.2.2}$$

where $\mathcal{K}$ is the Kähler potential. In certain symplectic frames there exist a second degree homogeneous function $F(X)$, called the prepotential, such that $F_\Lambda = \partial_\Lambda F$. For the STU model (in the duality frame of purely electric gaugings) the prepotential is

$$F = -2i\sqrt{X^0X^1X^2X^3}, \tag{4.2.3}$$

and so

$$F_\Lambda = \frac{F}{2X^\Lambda}. \tag{4.2.4}$$

The holomorphic sections define the embedding ambient space

$$\langle \mathfrak{V}, \mathfrak{V} \rangle \equiv e^{\mathcal{K}(z,\bar{z})} \left(X^\Lambda \overline{F}_\Lambda - \overline{X}^\Lambda F_\Lambda \right) = i, \tag{4.2.5}$$

which in turn defines the Kähler potential in terms of the holomorphic sections

$$\mathcal{K} = -\log\Big(i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)\Big). \tag{4.2.6}$$

The corresponding Kähler metric is given by

$$\mathcal{K}_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}. \tag{4.2.7}$$

The above data completely determines the bosonic part of the $\mathcal{N} = 2$ gauged supergravity action to be

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\Big(R - \mathcal{G}_{\alpha\bar{\beta}}\partial^\mu z^\alpha \partial_\mu \bar{z}^{\bar{\beta}} - 2\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} - \mathcal{R}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma} - \mathcal{V}\Big) + S_{\rm GH}, \tag{4.2.8}$$

where

$$S_{\rm GH} = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\;2K, \tag{4.2.9}$$

is the Gibbons-Hawking term and we have normalized the fields such that the gravitational constant $\kappa^2 = 8\pi G_4$ appears as an overall factor in front of the action, as appropriate for comparing our results with the large-N limit of the dual ABJM theory. We use the standard $-+++$ signature for the metric and we have reversed the signs of the symmetric matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ relative to the usual convention in the supergravity literature (e.g. [77, 78]) since with our definition the eigenvalues of $\mathcal{I}_{\Lambda\Sigma}$ are positive definite. Moreover, with our normalization of the vector multiplet scalars the scalar metric is related to the Kähler metric (4.2.7) as

$$\mathcal{G}_{\alpha\bar\beta} = 2\mathcal{K}_{\alpha\bar\beta} = 2\partial_\alpha\partial_{\bar\beta}\mathcal{K}. \tag{4.2.10}$$

The real symmetric matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are given by

$$\mathcal{I}_{\Lambda\Sigma} = -{\rm Im}\,\mathcal{N}_{\Lambda\Sigma}, \qquad \mathcal{R}_{\Lambda\Sigma} = -{\rm Re}\,\mathcal{N}_{\Lambda\Sigma}, \qquad \det(\mathcal{I}) > 0, \tag{4.2.11}$$

where the period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined through the relations

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma}X^\Sigma, \qquad \partial_{\bar\alpha}\overline{F}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\partial_{\bar\alpha}\overline{X}^\Sigma. \tag{4.2.12}$$

Whenever a prepotential exits the period matrix can be expressed as (see e.g. [78])

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2i\frac{{\rm Im}\,(F_{\Lambda P})X^P{\rm Im}\,(F_{\Sigma\Phi})X^\Phi}{X^\Omega{\rm Im}\,(F_{\Omega\Psi})X^\Psi}, \tag{4.2.13}$$

where

$$F_{\Lambda\Sigma} \equiv \partial_\Lambda F_\Sigma = \partial_\Lambda\partial_\Sigma F = \frac{F}{4X^\Lambda X^\Sigma}\left(1 - 2\,{\rm d}^{\Lambda\Sigma}\right). \tag{4.2.14}$$

The last equality applies only to the STU model prepotential (4.2.3).

Finally, the scalar potential is obtained from the holomorphic superpotential

$$W \equiv \sum_{\Lambda} \xi_{\Lambda} X^{\Lambda}, \tag{4.2.15}$$

through the identity

$$\xi^2 L^2 \mathcal{V} = e^{\mathcal{K}} \left(\mathcal{K}^{\alpha\bar{\beta}} \mathfrak{D}_{\alpha} W \mathfrak{D}_{\bar{\beta}} \overline{W} - 3 W \overline{W} \right), \tag{4.2.16}$$

where the Kähler covariant derivatives are defined as

$$\mathfrak{D}_{\alpha} \equiv \partial_{\alpha} + \partial_{\alpha}\mathcal{K}, \qquad \mathfrak{D}_{\bar{\alpha}} \equiv \partial_{\bar{\alpha}} + \partial_{\bar{\alpha}}\mathcal{K}, \tag{4.2.17}$$

and L is the AdS_4 radius.[2] It is also useful to introduce the *real* superpotential

$$\mathcal{W} = -\frac{\sqrt{2}}{\xi L} e^{\mathcal{K}/2} |W|, \tag{4.2.18}$$

in terms of which the scalar potential takes the form

$$\mathcal{V} = 4\mathcal{G}^{\alpha\bar{\beta}} \partial_{\alpha}\mathcal{W} \partial_{\bar{\beta}}\mathcal{W} - \frac{3}{2}\mathcal{W}^2. \tag{4.2.19}$$

Even after specifying the gauging, i.e. the FI parameters, and the prepotential F, there are still two potential ambiguities in specifying the theory completely. From a strict bulk point of view these ambiguities are loosely speaking "gauge choices", in the sense that they do not affect physical quantities, but they do change the parameterization of the solutions. Understanding these gauge freedoms, therefore, is important in order to compare different choices in the literature. More importantly, however, these gauge freedoms in the bulk are often lifted by imposing boundary conditions on the fields, and so not all choices are *a priory* directly compatible with holography.

The first gauge freedom is related to a phase factor introduced in the Killing spinor projections, and hence in the BPS equations. This is discussed in [77], as well as in [63]

[2] Notice that $1/\xi L$ corresponds to the gauge coupling, often denoted by g in the supergravity literature.

for the case of purely magnetic solutions (see around eq. (4.16) there). In [77], a constraint for the phase factor is derived and its universal solution in terms of the symplectic sections is obtained, which leads to unambiguous BPS equations, without any additional constraint on the symplectic sections. For purely magnetic solutions the constraint derived in [77] sets the phase factor to zero, in agreement with the choice made in [63]. Below we provide an alternative derivation of these BPS equations through Hamilton-Jacobi (HJ) theory, and so we implicitly treat this phase factor in the same way as [77].

The second ambiguity arises in the specification of the symplectic sections $X^\Lambda(z)$ in terms of the physical scalar fields z^α in the vector multiplets. Since there are $n_V + 1$ symplectic sections X^Λ but only n_V complex scalars z^α, there is an inherent redundancy in specifying the functions $X^\Lambda(z)$. This redundancy is eliminated by a gauge-fixing condition, that can be visualized as a choice for the embedding of the n_V-dimensional complex surface spanned by the physical scalars in the vector multiplets inside the ambient space spanned by the sections X^Λ. Different embeddings do not affect physical quantities such as the *real* superpotential, the scalar potential, the Kähler metric and the period matrix (of course up to field redefinitions of the physical scalars), but they do transform non-trivially the Kähler potential and the holomorphic superpotential. In appendix A we summarize a number of different embeddings of the STU model scalars that have been used in the literature, and we show explicitly how the $\mathcal{N} = 2$ supergravity quantities defined above transform. This is important for translating known black hole solutions to different parameterizations of the STU model, as well as for understanding the holographic dictionary.

4.2.2 Ansatz for static dyonic solutions

We are interested in static solutions of the $\mathcal{N} = 2$ supergravity action (4.2.8) that can potentially carry both magnetic and electric charge. Such solutions can be parameterized by the Ansatz

$$
\begin{aligned}
&ds_B^2 = dr^2 + e^{2A(r)}\left(-f(r)dt^2 + d\sigma_k^2\right), \qquad k = 0, \pm 1, \\
&A_B^\Lambda = a^\Lambda(r)\mathrm{d}t + p^\Lambda \Big(\int d\theta\, \omega_k(\theta)\Big)\mathrm{d}\varphi, \qquad z_B^\alpha = z_B^\alpha(r), \qquad \bar{z}_B^{\bar\beta} = \bar{z}_B^{\bar\beta}(r),
\end{aligned} \tag{4.2.20}
$$

so that the field strengths of the Abelian gauge fields take the form

$$
F_B^\Lambda = \mathrm{d}A_B^\Lambda = \dot{a}^\Lambda\, \mathrm{d}r \wedge \mathrm{d}t + p^\Lambda \omega_k(\theta)\, \mathrm{d}\theta \wedge \mathrm{d}\varphi. \tag{4.2.21}
$$

In this Ansatz $d\sigma_k^2 = d\theta^2 + \omega_k^2(\theta)d\varphi^2$ is the metric on $\Sigma_k = \{S^2, T^2, H^2\}$ respectively for $k = 1, 0, -1$, namely

$$
\omega_k(\theta) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}\,\theta) = \begin{cases} \sin\theta, & k = 1, \\ \theta, & k = 0, \\ \sinh\theta, & k = -1. \end{cases} \tag{4.2.22}
$$

In the case of H^2 the non-compact hyperbolic space must be quotiened by a discrete subgroup of the isometry group, *i.e.*, a Fuchsian group, in order to get a compact Riemann surface of genus $\mathfrak{g} > 1$.

Inserting the Ansatz (4.2.20) in the field equations following from the $\mathcal{N} = 2$ supergravity action (4.2.8) results in the following set of coupled equations

$$2\dot{A}\Big(3\dot{A}+\frac{\dot{f}}{f}\Big)-\mathcal{G}_{\alpha\bar{\beta}}\dot{z}^{\alpha}\dot{\bar{z}}^{\bar{\beta}}+\mathcal{V}-2ke^{-2A}+e^{-4A}\mathcal{I}_{\Lambda\Sigma}\left(4e^{2A}f^{-1}\dot{a}^{\Lambda}\dot{a}^{\Sigma}+p^{\Lambda}p^{\Sigma}\right)=0, \quad (4.2.23a)$$

$$\ddot{A}+\dot{A}\Big(3\dot{A}+\frac{\dot{f}}{2f}\Big)+\frac{1}{2}\Big(\mathcal{V}-2ke^{-2A}+e^{-4A}\mathcal{I}_{\Lambda\Sigma}\left(4e^{2A}f^{-1}\dot{a}^{\Lambda}\dot{a}^{\Sigma}+p^{\Lambda}p^{\Sigma}\right)\Big)=0, \quad (4.2.23b)$$

$$\ddot{f}+\dot{f}\Big(3\dot{A}-\frac{\dot{f}}{2f}\Big)+2kfe^{-2A}-2fe^{-4A}\mathcal{I}_{\Lambda\Sigma}\left(4e^{2A}f^{-1}\dot{a}^{\Lambda}\dot{a}^{\Sigma}+p^{\Lambda}p^{\Sigma}\right)=0, \quad (4.2.23c)$$

$$2\mathcal{G}_{\alpha\bar{\beta}}\ddot{\bar{z}}^{\bar{\beta}}_B+2\partial_{\gamma}\mathcal{G}_{\alpha\bar{\beta}}\dot{z}^{\gamma}_B\dot{\bar{z}}^{\bar{\beta}}_B+2\partial_{\bar{\gamma}}\mathcal{G}_{\alpha\bar{\beta}}\dot{\bar{z}}^{\bar{\gamma}}_B\dot{\bar{z}}^{\bar{\beta}}_B-2\partial_{\alpha}\mathcal{G}_{\gamma\bar{\beta}}\dot{z}^{\gamma}_B\dot{\bar{z}}^{\bar{\beta}}_B+2\mathcal{G}_{\alpha\bar{\beta}}\Big(3\dot{A}+\frac{\dot{f}}{2f}\Big)\dot{\bar{z}}^{\bar{\beta}}_B-\partial_{\alpha}\mathcal{V}$$
$$+e^{-4A}\partial_{\alpha}\mathcal{I}_{\Lambda\Sigma}\left(4e^{2A}f^{-1}\dot{a}^{\Lambda}\dot{a}^{\Sigma}-p^{\Lambda}p^{\Sigma}\right)-4f^{-1/2}e^{-3A}\partial_{\alpha}\mathcal{R}_{\Lambda\Sigma}\dot{a}^{\Lambda}p^{\Sigma}=0, \quad (4.2.23d)$$

$$2\mathcal{G}_{\alpha\bar{\beta}}\ddot{z}^{\alpha}_B+2\partial_{\gamma}\mathcal{G}_{\alpha\bar{\beta}}\dot{z}^{\gamma}_B\dot{z}^{\alpha}_B+2\partial_{\bar{\gamma}}\mathcal{G}_{\alpha\bar{\beta}}\dot{\bar{z}}^{\bar{\gamma}}_B\dot{z}^{\alpha}_B-2\partial_{\bar{\beta}}\mathcal{G}_{\alpha\bar{\gamma}}\dot{z}^{\alpha}_B\dot{\bar{z}}^{\bar{\gamma}}_B+2\mathcal{G}_{\alpha\bar{\beta}}\Big(3\dot{A}+\frac{\dot{f}}{2f}\Big)\dot{z}^{\alpha}_B-\partial_{\bar{\beta}}\mathcal{V}$$
$$+e^{-4A}\partial_{\bar{\beta}}\mathcal{I}_{\Lambda\Sigma}\left(4e^{2A}f^{-1}\dot{a}^{\Lambda}\dot{a}^{\Sigma}-p^{\Lambda}p^{\Sigma}\right)-4f^{-1/2}e^{-3A}\partial_{\bar{\beta}}\mathcal{R}_{\Lambda\Sigma}\dot{a}^{\Lambda}p^{\Sigma}=0, \quad (4.2.23e)$$

$$\partial_r\Big(2\mathcal{I}_{\Lambda\Sigma}e^{A}f^{-1/2}\dot{a}^{\Sigma}-\mathcal{R}_{\Lambda\Sigma}p^{\Sigma}\Big)=0, \quad (4.2.23f)$$

where a dot $\cdot$ denotes a derivative with respect to the radial coordinate r. The last equation, which comes from the Maxwell equation, can be integrated to obtain

$$2\mathcal{I}_{\Lambda\Sigma}e^{A}f^{-1/2}\dot{a}^{\Sigma}-\mathcal{R}_{\Lambda\Sigma}p^{\Sigma}=-q_{\Lambda}, \quad (4.2.24)$$

where the integration constants q_{Λ} are electric charges associated with the Abelian gauge fields A^{Λ}_B.

4.2.3 Effective superpotential and first order equations

First order flow equations for static solutions of $\mathcal{N}=2$ gauged supergravity are known not only for BPS black holes [77], but also for several examples of non-extremal black holes [79, 80, 81, 82, 83, 84, 78]. In all these cases, the procedure for deriving the first order equations involves writing the on-shell action as a sum of squares. Although this procedure is sufficiently general for static and spatially homogeneous solutions, in practice only a limited number of flow equations can be obtained this way. HJ theory, however, provides a systematic and general procedure for deriving first order equations, even for non-static and spatially dependent solutions.[3] For *any* solution of the action (4.2.8) these first order equations are given in (B.13), where the Hamilton principal functional $\mathcal{S}$ plays the role of a generalized effective superpotential. In particular, HJ theory provides an equation – the HJ equation – for the effective superpotential, which can therefore be determined systematically by seeking a solution to the HJ equation.

For static solutions of the form (4.2.20) the general first order equations following from HJ theory were obtained in [75, 85].[4] The result can be summarized as follows: given a solution $\mathcal{U}(z, \bar{z}, A)$ of the effective superpotential equation

$$4\mathcal{G}^{\alpha\bar{\beta}}\partial_\alpha\mathcal{U}\partial_{\bar{\beta}}\mathcal{U} - \frac{1}{2}(3 + \partial_A)\mathcal{U}^2 = \mathcal{V}_{eff}, \tag{4.2.25}$$

where

$$\mathcal{V}_{eff} = \mathcal{V} - 2ke^{-2A} + e^{-4A}\mathcal{I}_{\Lambda\Sigma}p^\Lambda p^\Sigma + e^{-4A}\mathcal{I}^{\Lambda\Sigma}\left(q_\Lambda - \mathcal{R}_{\Lambda M}p^M\right)\left(q_\Sigma - \mathcal{R}_{\Sigma N}p^N\right), \tag{4.2.26}$$

any solution of the first order equations

$$\begin{aligned} &\dot{A} = -\frac{1}{2}\mathcal{U}, \qquad \frac{\dot{f}}{f} = -\partial_A\mathcal{U}, \qquad \dot{z}^\alpha_B = 2\mathcal{G}^{\alpha\bar{\beta}}\partial_{\bar{\beta}}\mathcal{U}, \qquad \dot{z}^{\bar{\beta}}_B = 2\mathcal{G}^{\alpha\bar{\beta}}\partial_\alpha\mathcal{U}, \\ &\dot{a}^\Lambda = \frac{1}{2}e^{-A}f^{1/2}\mathcal{I}^{\Lambda\Sigma}\left(\mathcal{R}_{\Sigma M}p^M - q_\Sigma\right), \end{aligned} \tag{4.2.27}$$

automatically solves the second order equations (4.2.23).[5] As was shown in [75] (which

[3] See appendix B for the Hamiltonian formulation of the theory described by the $\mathcal{N} = 2$ Lagrangian (4.2.8).

[4] Some of the earlier works, e.g. [81, 82] also employ the HJ method, but only for special cases where the effective superpotential is a function of the scalar fields only. Another approach to first order equations for static black holes was presented in [86], but that formulation uses a scalar field as the radial coordinate and amounts to a rewriting of the second order equations of motion. In particular, the first order equations derived in [86] are strictly on-shell, in the sense that every single solution of the equations of motion is governed by a different effective superpotential.

[5] One may ask the converse question, namely whether for any solution of the second order equations (4.2.23) there is a superpotential $\mathcal{U}(z, \bar{z}, A)$ such that the first order equations (4.2.27) hold. This is an interesting and subtle question. Locally in field space this should indeed be the case. Globally, however, a different superpotential $\mathcal{U}(z, \bar{z}, A)$ may be necessary in different patches in field space in order to describe a full solution of the second order equations of motion. This happens when e.g. the variables z, $\bar{z}$ and A are not monotonic functions of the radial coordinate. However, for supersymmetric black holes the function $\mathcal{U}(z, \bar{z}, A)$ is related to the true superpotential of the theory and so it exist globally in field space.

focused on the case $k = 0$), these flow equations follow from the HJ equation associated with the Hamiltonian constraint in (B.9), using the separable ansatz

$$\mathcal{S} = -\frac{1}{\kappa^2}\int d^3x\sqrt{\sigma_k}\Big(e^{3A}f^{1/2}\mathcal{U}(z,\bar{z},A) + 2q_\Lambda a^\Lambda\Big). \tag{4.2.28}$$

The HJ equation then reduces to the superpotential equation (4.2.25) for the function $\mathcal{U}(z,\bar{z},A)$, and the flow equations (B.13) reduce to the first order equations (4.2.27).

4.2.4 Regularized on-shell action

Given Hamilton's principal function (4.2.28) we can easily evaluate the on-shell action with a radial UV cutoff for any solution of the form (4.2.20). We first observe that the only term containing second order derivatives in the Lagrangian (4.2.8) is the bulk Ricci scalar. Using the decomposition (B.3) of the bulk Ricci scalar allows one to isolate the terms that contain two derivatives in the radial coordinate. Assuming there is a horizon at $r = r_h$ the on-shell action (4.2.8) evaluated with a radial cutoff r_o takes the form

$$S_{\text{reg}} = \frac{1}{\kappa^2}\int_{r_h} d^3x\sqrt{-\gamma}\;K + \int_{r_h}^{r_o} dr\;L, \tag{4.2.29}$$

where L is the radial Lagrangian (B.6) and the total derivative term from the Ricci scalar evaluated on the cutoff has canceled against the Gibbons-Hawking term. Since the Hamiltonian (B.8) vanishes on-shell, the regularized on-shell action becomes

$$\begin{aligned}
S_{\text{reg}} &= \frac{1}{\kappa^2}\int_{r_h} d^3x\sqrt{-\gamma}\;K + \int_{r_h}^{r_o} dr\int d^3x\Big(\pi^{ij}\dot{\gamma}_{ij} + \pi_\alpha\dot{z}^\alpha + \pi_{\bar{\beta}}\dot{\bar{z}}^{\bar{\beta}} + \pi^i_\Lambda\dot{A}^\Lambda_i\Big)\\
&= \frac{1}{\kappa^2}\int_{r_h} d^3x\sqrt{-\gamma}\;K + \int_{r_h}^{r_o} dr\int d^3x\Big(\frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\gamma_{ij}}\dot{\gamma}_{ij} + \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,z^\alpha}\dot{z}^\alpha + \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\bar{z}^{\bar{\beta}}}\dot{\bar{z}}^{\bar{\beta}} + \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,A^\Lambda_i}\dot{A}^\Lambda_i\Big)\\
&= \frac{1}{\kappa^2}\int_{r_h} d^3x\sqrt{-\gamma}\;K + \mathcal{S}|_{r_o} - \mathcal{S}|_{r_h}\,,
\end{aligned} \tag{4.2.30}$$

where we have used the expressions (B.11) for the canonical momenta. We should point out that this expression for the regularized on-shell action holds for any diffeomorphism invariant two-derivative theory of gravity coupled to matter fields and for any solution of the equations of motion. It follows solely from HJ theory.

For static solutions of the form (4.2.20) we have seen that the HJ functional $\mathcal{S}$ is given by (4.2.28), and so the only term remaining to evaluate is the trace of the extrinsic curvature

$$K = 3\dot{A} + \frac{\dot{f}}{2f}, \tag{4.2.31}$$

on the horizon. $\dot{A}$ vanishes on the horizon, while the blackening factor f behaves as

$$f = 4\pi T(u_h - u) + \mathcal{O}(u_h - u)^2, \tag{4.2.32}$$

where the domain wall coordinate u is related to the radial coordinate r through the definition [75]

$$\partial_r = -\sqrt{f}e^{-A}\partial_u. \tag{4.2.33}$$

It follows that

$$e^{3A}f^{1/2}\frac{\dot{f}}{2f}\bigg|_{r_h} = -\frac{1}{2}e^{2A}\partial_u f\bigg|_{u_h} = 2\pi T\, e^{2A}\big|_{u_h}. \tag{4.2.34}$$

Hence, the *Lorentzian* regularized on-shell action for any solution of the form (4.2.20) is given by

$$S_{\rm reg} = -\frac{1}{\kappa^2}\int_{r_o} d^3x\sqrt{\sigma_k}\, e^{3A}f^{1/2}\mathcal{U} + \left(a^\Lambda(r_o) - a^\Lambda(r_h)\right)Q_\Lambda\int dt + \frac{2\pi T}{\kappa^2}\int dt A_h, \tag{4.2.35}$$

where

$$A_h = \mathrm{Vol}(\Sigma_k)\, e^{2A}\big|_h, \tag{4.2.36}$$

is the area of the horizon,

$$Q_\Lambda \equiv -\frac{2q_\Lambda}{\kappa^2}\mathrm{Vol}(\Sigma_k), \tag{4.2.37}$$

are the electric charges, and $\mathrm{Vol}(\Sigma_k)$ is the area of the compact surface Σ_k.

4.2.5 Supersymmetric superpotential and BPS equations for dyonic black holes

The superpotential equation (4.2.25) admits the exact solution

$$\mathcal{U}_{\mathrm{BPS}} = -\frac{\sqrt{2}}{\xi L} e^{\mathcal{K}/2} |W + ie^{-2A} Z|, \tag{4.2.38}$$

where W is the holomorphic superpotential given in (4.2.15) and Z is the central charge

$$Z = -\sqrt{2}\, \xi L \left(p^{\Lambda} F_{\Lambda} + q_{\Lambda} X^{\Lambda} \right), \tag{4.2.39}$$

provided the magnetic charges satisfy the Dirac quantization condition

$$p^{\Lambda} = -\frac{L}{\sqrt{2}} \mathfrak{n}^{\Lambda}, \qquad \sum_{\Lambda} \mathfrak{n}^{\Lambda} = 2k. \tag{4.2.40}$$

This is precisely the superpotential obtained in [77] for dyonic BPS black holes of the $U(1)^4$ gauged supergravity by using the Bogomol'nyi argument of writing the on-shell action as a sum of squares. Our derivation, however, is entirely different, and relies solely on HJ theory. A similar derivation of this superpotential using HJ theory was given in [78, 85]. The identification of the exact superpotential (4.2.38) with the true superpotential coming from the supersymmetry variation of the fermionic fields, together with the flow equations (4.2.27), imply that supersymmetric solutions of the action (4.2.8) satisfy the BPS equations

$$\begin{aligned}
\dot{A} &= \frac{1}{\sqrt{2}\xi L} e^{\mathcal{K}/2} |W + ie^{-2A} Z| \,, \\
\frac{\dot{f}}{f} &= \frac{\sqrt{2}}{\xi L} e^{\mathcal{K}/2} \partial_A |W + ie^{-2A} Z| \,, \\
\dot{z}^{\alpha}_B &= -\frac{\sqrt{2}}{\xi L} \mathcal{K}^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \left(e^{\mathcal{K}/2} |W + ie^{-2A} Z| \right), \\
\dot{z}^{\bar{\beta}}_B &= -\frac{\sqrt{2}}{\xi L} \mathcal{K}^{\alpha\bar{\beta}} \partial_{\alpha} \left(e^{\mathcal{K}/2} |W + ie^{-2A} Z| \right), \\
\dot{a}^{\Lambda} &= \frac{1}{2} e^{-A} f^{1/2} \mathcal{I}^{\Lambda\Sigma} \left(\mathcal{R}_{\Sigma M} p^{M} - q_{\Sigma} \right).
\end{aligned} \tag{4.2.41}$$

Recall that the dots $\cdot$ in these equations denote a derivative with respect to the radial coordinate r defined through the ansatz (4.2.20).

Near extremal superpotential The ambiguities in taking extremal limits of black holes are well known [87, 69]. It is also understood that to capture certain aspects of extremal black holes, such as the thermodynamics, it is necessary to start from the corresponding non-extremal solutions and approach the extremal ones in a limiting process, as has been done, for example, in computations of the entropy function [61]. In particular, in order to evaluate the on-shell action for BPS solutions using the regularized expression (4.2.35) it is necessary to evaluate it first on near extremal solutions and then take the extremal limit. The reason for this is that the temperature $T \to 0$ in the extremal limit, while the integral over the Euclidean time gives a factor of $\beta = 1/T \to \infty$. Starting with near extremal solutions renders β finite and T non-zero, leading to an expression that admits a well defined limit as $T \to 0$.

One of the advantages of the HJ method is that it provides an equation for the effective superpotential $\mathcal{U}$, for both supersymmetric and non-supersymmetric black holes. In order to determine the superpotential for near-extremal black holes, therefore, one can solve (4.2.25) in perturbation theory around the BPS superpotential. Inserting the near extremal superpotential

$$\mathcal{U} = \mathcal{U}_{\text{BPS}} + \varepsilon \Delta \mathcal{U}, \tag{4.2.42}$$

where ε is the near extremality parameter, in the superpotential equation (4.2.25) one finds that the first order correction away from extremality satisfies the linear equation

$$4\mathcal{G}^{\alpha\bar{\beta}}\partial_\alpha \mathcal{U}_{\text{BPS}}\partial_{\bar{\beta}}\Delta\mathcal{U} + 4\mathcal{G}^{\alpha\bar{\beta}}\partial_\alpha \Delta\mathcal{U}\partial_{\bar{\beta}}\mathcal{U}_{\text{BPS}} - 3\mathcal{U}_{\text{BPS}}\Delta\mathcal{U} - \partial_A(\mathcal{U}_{\text{BPS}}\Delta\mathcal{U}) = 0. \tag{4.2.43}$$

However, for the purpose of regularizing the extremal limit of the on-shell action we need not solve this equation to determine the functional form of the first order correction

$\Delta\mathcal{U}$. Instead, it suffices to prove the following two properties:

$$(i) \quad \varepsilon \propto T^\nu, \quad \nu > 1, \qquad (ii) \quad \Delta\mathcal{U} = \mathcal{O}(e^{-3r/L}) \quad \text{as} \quad r \to \infty. \tag{4.2.44}$$

The second property is straightforward to prove. Using the first order equations (4.2.27) to replace the superpotential $\mathcal{U}_{\text{BPS}}$ and its derivatives in (4.2.43) with the radial derivatives of the fields, as well as the asymptotic identities $A \sim r/L$ and $\phi \sim 1$ as $r \to \infty$ for asymptotically AdS_4 solutions, the linear equation (4.2.27) becomes

$$(\partial_r + 3/L)\,\Delta\mathcal{U} = 0, \tag{4.2.45}$$

which implies condition (ii) in (4.2.44). Condition (i) can be translated to a statement about the near extremal mass. Namely, our analysis in section 4.3 implies that

$$M - M_{\text{BPS}} = \mathcal{O}(T^\nu), \tag{4.2.46}$$

where ν is the same exponent as in condition (i). However, it is known that $\nu = 2$ for near extremal black holes with an AdS_2 near horizon geometry [88].

4.3 Holographic renormalization and the quantum effective action

We now have the necessary ingredients in order to construct the holographic dictionary for the theory defined by the bulk action (4.2.8) and appropriate supersymmetric boundary conditions. We will later identify this theory with a sector of the topologically twisted ABJM theory at Chern-Simons level one. We begin this section by determining the boundary counterterms that render the Dirichlet variational problem well posed. We then derive the additional *finite* terms required to impose the desired supersymmetric boundary conditions on the scalars. Having determined all necessary boundary terms, we identify the renormalized operators dual to the bulk fields and obtain general expressions for the renormalized partition function and effective action for any solutions of the form (4.2.20).

4.3.1 Supersymmetric boundary counterterms and boundary conditions

The solution (4.2.38) to the superpotential equation implies that the local boundary counterterms compatible with supersymmetry are given by [89, 90]

$$S_{\text{ct}} = \frac{1}{\kappa^2} \int d^3x \sqrt{-\gamma}\, \mathcal{W} \left(1 - \frac{k}{2} \text{Im}\, \left(W^{-1} Z_m\right) R[\gamma]\right), \tag{4.3.47}$$

where the real superpotential $\mathcal{W}$ is defined in (4.2.18) and Z_{m} denotes the magnetic part of the central charge (4.2.39), i.e.

$$Z_m \equiv -\sqrt{2}\, \xi L p^\Lambda F_\Lambda. \tag{4.3.48}$$

Several comments are in order here. Firstly, to obtain this expression from (4.2.38) we have Taylor expanded for small e^{-2A} and truncated the resulting expansion to $\mathcal{O}(e^{-2A})$, since higher order terms vanish as the UV cutoff is removed. Moreover, we have covariantized the warp factor by replacing $ke^{-2A} \to R[\gamma]/2$ and set the electric charges to zero since they contribute terms finite and non-local in the gauge potentials A_i^Λ. In contrast, the magnetic charges contribute to the divergent terms, but they are local in the gauge potentials and therefore are acceptable as local covariant counterterms. Despite setting the electric charges to zero in (4.3.47), we should stress that these local counterterms are valid for any solution of the theory, charged or uncharged, supersymmetric or not, since these counterterms coincide with the asymptotic solution of the HJ equation for any value of the electric and magnetic charges subject to the quantization condition (4.2.40).

A second remark concerns the fact that in the counterterms (4.3.47) we have included, besides the divergent terms, all finite *local* terms dictated by the supersymmetric superpotential (4.2.38). This choice of finite local counterterms renders the boundary term (4.3.47) invariant under reparameterizations of the symplectic sections X^Λ and hence applicable to any parameterization of the STU model. More importantly, as we argue below, this choice is also dictated by supersymmetry.

In order to write down the possible local finite counterterms it is necessary to pick a specific parameterization of the symplectic sections $X^{\Lambda}(z)$. From now on we will mostly work in the Pufu-Freedman (PF) parameterization summarized in appendix A, since this parameterization is compatible with supersymmetric boundary conditions and the holographic dictionary, but it is also particularly convenient for discussing the dual theory. Using the Fefferman-Graham expansions of the vector multiplet scalars in the PF parameterization given in (A.13) and decomposing the scalars in real and imaginary part as

$$z^{\alpha} = \mathcal{X}^{\alpha} + i\mathcal{Y}^{\alpha}, \qquad \bar{z}^{\bar{\alpha}} = \mathcal{X}^{\alpha} - i\mathcal{Y}^{\alpha}, \qquad \mathcal{X}^{\alpha},\ \mathcal{Y}^{\alpha} \in \mathbb{R}, \tag{4.3.49}$$

one can easily conclude that the finite terms in (4.3.47) are schematically of the form

$$(a)\ \mathcal{X}^{\alpha}\mathcal{X}^{\beta}\mathcal{X}^{\gamma}, \quad \mathcal{X}^{\alpha}\mathcal{X}^{\beta}\mathcal{Y}^{\gamma}, \quad \mathcal{X}^{\alpha}\mathcal{Y}^{\beta}\mathcal{Y}^{\gamma}, \quad \mathcal{Y}^{\alpha}\mathcal{Y}^{b}\mathcal{Y}^{\gamma}, \qquad (b)\ \mathcal{X}^{\alpha}R[\gamma], \quad \mathcal{Y}^{\alpha}R[\gamma]. \tag{4.3.50}$$

Were we to impose Dirichlet boundary conditions on all scalars $\mathcal{X}^{\alpha}$ and $\mathcal{Y}^{\alpha}$, such terms would correspond to a choice of renormalization scheme, since the induced fields $\mathcal{X}^{\alpha}$ and $\mathcal{Y}^{\alpha}$ would be identified with the covariant sources of the dual operators. However, supersymmetry requires that $\mathcal{X}^{\alpha}$ and $\mathcal{Y}^{\alpha}$ be quantized in opposite quantizations [91] and comparing with the ABJM theory further specifies that the real part $\mathcal{X}^{\alpha}$ of the vector multiplet scalar is dual to dimension one scalar operators, while the imaginary part $\mathcal{Y}^{\alpha}$ is dual to dimension two operators [92]. It follows that $\mathcal{X}^{\alpha}$ should satisfy Neumann or mixed boundary conditions, while $\mathcal{Y}^{\alpha}$ must satisfy Dirichlet boundary conditions. Hence, the finite terms involving $\mathcal{X}^{\alpha}$ do not correspond to scheme dependence, since $\mathcal{X}^{\alpha}$ is identified with the dual operator instead of its source. As we will see shortly, the freedom of choosing the coefficient of finite local terms of the form (4.3.50) involving $\mathcal{X}^{\alpha}$ corresponds instead to the freedom of interpreting the boundary conditions $\mathcal{X}^{\alpha}$ as Neumann or mixed.

Since the real and imaginary parts of the vector multiplet scalars should satisfy different boundary conditions, it is necessary to formulate the variational problem in terms of $\mathcal{X}^{\alpha}$

and $\mathcal{Y}^\alpha$. To this end we decompose the scalar canonical momenta defined in appendix B as

$$\pi_\alpha \,\mathrm{d}\, z^\alpha + \pi_{\bar\beta} \,\mathrm{d}\, \bar z^{\bar\beta} = \pi_\alpha (\mathrm{d}\,\mathcal{X}^\alpha + i\,\mathrm{d}\,\mathcal{Y}^\alpha) + \pi_{\bar\beta}(\mathrm{d}\,\mathcal{X}^{\bar\beta} - i\,\mathrm{d}\,\mathcal{Y}^{\bar\beta}) = \pi^{\mathcal{X}}_\alpha \,\mathrm{d}\,\mathcal{X}^\alpha + \pi^{\mathcal{Y}}_\alpha \,\mathrm{d}\,\mathcal{Y}^\alpha, \quad (4.3.51)$$

and so the canonical momenta conjugate to $\mathcal{X}^\alpha$ and $\mathcal{Y}^\alpha$ are respectively

$$\pi^{\mathcal{X}}_\alpha = (\pi_\alpha + \pi_{\bar\alpha}), \qquad \pi^{\mathcal{Y}}_\alpha = i(\pi_\alpha - \pi_{\bar\alpha}). \quad (4.3.52)$$

Using the counterterms (4.3.47) we then define the renormalized canonical momenta

$$\Pi^{ij} = \pi^{ij} + \frac{\mathrm{d}\, S_{\mathrm{ct}}}{\mathrm{d}\,\gamma_{ij}}, \qquad \Pi^i_\Lambda = \pi^i_\Lambda + \frac{\mathrm{d}\, S_{\mathrm{ct}}}{\mathrm{d}\, A^\Lambda_i}, \qquad \Pi^{\mathcal{X}}_\alpha = \pi^{\mathcal{X}}_\alpha + \frac{\mathrm{d}\, S_{\mathrm{ct}}}{\mathrm{d}\,\mathcal{X}^\alpha}, \qquad \Pi^{\mathcal{Y}}_\alpha = \pi^{\mathcal{Y}}_\alpha + \frac{\mathrm{d}\, S_{\mathrm{ct}}}{\mathrm{d}\,\bar{\mathcal{Y}}^\alpha}, \quad (4.3.53)$$

which are associated with the variational principle

$$\mathrm{d}(S_{\mathrm{reg}} + S_{\mathrm{ct}}) = \int d^3x \Big(\Pi^{ij}\,\mathrm{d}\,\gamma_{ij} + \Pi^i \,\mathrm{d}\, A^\Lambda_i + \Pi^{\mathcal{X}}_\alpha \,\mathrm{d}\,\mathcal{X}^\alpha + \Pi^{\mathcal{Y}}_\alpha \,\mathrm{d}\,\mathcal{Y}^\alpha\Big). \quad (4.3.54)$$

This variational principle corresponds to Dirichlet boundary conditions on the scalars $\mathcal{Y}^\alpha$ and so we must not add any other boundary term that changes the variational problem for these fields. However, we need to add a very specific *finite* boundary term in order to impose Neumann or mixed boundary conditions on the scalars $\mathcal{X}^\alpha$, while at the same time preserving supersymmetry.

An important point that is often confusing is that Neumann and mixed boundary conditions can in fact refer to the *same* boundary conditions – one must first specify the Dirichlet theory with respect to which the Neumann boundary conditions are obtained via a Legendre transformation. Different renormalization schemes in the Dirichlet problem correspond to different definitions of what we refer to as the Neumann theory. This should become clear from the general procedure for imposing Neumann or mixed boundary conditions in the renormalized theory [93, 94], which we now review in the context of supersymmetry.

In order to impose generic mixed boundary conditions on $\mathcal{X}^\alpha$ we must start with the *renormalized* action corresponding to the Dirichlet problem (4.3.54). As we mentioned earlier, this variational problem picks a specific set of finite local counterterms that would correspond to a choice of renormalization scheme, had we imposed Dirichlet boundary conditions. Any choice of such finite terms is in principle acceptable for the Dirichlet problem, unless there are additional constraints, e.g. supersymmetry. At this point it is useful to summarize the results of [90] in relation to supersymmetric boundary conditions for scalar fields.

- Finite terms of type (a) in (4.3.50) are Weyl invariant but generically cannot be supersymmetrized individually. As a result, the coefficient of such terms is fixed to the value dictated by the supersymmetric superpotential and does not correspond to a choice of supersymmetric scheme. This result was shown for a general field theory background in [92] and [90], but in flat space it is well known that in order to make the vacuum energy zero (as required by supersymmetry) the supersymmetric superpotential must be used as a counterterm.[6]

- Finite terms of the form (b) in (4.3.50) can be made Weyl invariant by replacing the Ricci scalar with the conformal Laplacian, and they can also be supersymmetrized. Therefore, supersymmetry alone does not fix the coefficient of such terms and they correspond to a choice of supersymmetric scheme in the Dirichlet problem.

- It was shown in [90] that starting with a supersymmetric Dirichlet problem, the corresponding Neumann problem is supersymmetric as well. This amounts to the statement that the relevant Legendre transformation can be supersymmetrized.

Combining these results for supersymmetric Dirichlet and Neumann boundary conditions with the procedure for imposing mixed boundary conditions in the renormalized theory [93, 94], it is straightforward to see how mixed boundary conditions interplay with supersymmetry. Recall that starting with the Dirichlet problem (4.3.54), imposing mixed

[6]There is a caveat to this rule, however, related to the quadratic term in the Taylor expansion of the superpotential around the fixed point. The supersymmetric superpotential can be used as a counterterm iff the coefficient of the quadratic term is proportional to Δ_-, and not Δ_+ [95].

boundary conditions on the scalars $\mathcal{X}^\alpha$ requires adding a finite boundary term of the form[7] (see Table 2 in [93])

$$S_v = \int d^3x\, \sqrt{-\gamma}\; J^v_\alpha \mathcal{X}^\alpha + \int d^3x \sqrt{-\gamma}\; v(\mathcal{X}), \tag{4.3.55}$$

where

$$J^v_\alpha \equiv -\frac{1}{\sqrt{-\gamma}} \Pi^{\mathcal{X}}_\alpha - \partial_\alpha v(\mathcal{X}), \tag{4.3.56}$$

is identified with the source of the dual scalar operator and $v(\mathcal{X})$ is an arbitrary (polynomial) function. Adding this term to (4.3.54) leads to the variational principle

$$\mathrm{d}(S_{\text{reg}}+S_{\text{ct}}+S_v) = \int d^3x \left(\left(\Pi^{ij} + \frac{1}{2} \left(J^v_\alpha \mathcal{X}^\alpha + v(\mathcal{X}) \right) \gamma^{ij} \right) \mathrm{d}\,\gamma_{ij} + \Pi^i_\Lambda\, \mathrm{d}\, A^\Lambda_i + \Pi^{\mathcal{Y}}_\alpha\, \mathrm{d}\, \mathcal{Y}^\alpha + \mathcal{X}^\alpha\, \mathrm{d}\, J^v_\alpha \right). \tag{4.3.57}$$

From these expressions we can draw the following general conclusions.

Finite terms of type (a): Supersymmetry aside, a choice of scheme in the Dirichlet problem specified by terms of type (a) in (4.3.50) is mapped to a shift in the function $v(\mathcal{X})$ for mixed boundary conditions. Adding, for example, the finite term

$$\int d^3x \sqrt{-\gamma}\; \lambda_{\alpha\beta\gamma} \mathcal{X}^\alpha \mathcal{X}^\beta \mathcal{X}^\gamma, \tag{4.3.58}$$

where $\lambda_{\alpha\beta\gamma}$ are arbitrary constants specifying a choice of scheme in the Dirichlet problem, leads to a shift in the renormalized canonical momenta according to

$$\Pi^{ij} \to \Pi^{ij} + \frac{1}{2} \sqrt{-\gamma}\; \lambda_{\alpha\beta\gamma} \mathcal{X}^\alpha \mathcal{X}^\beta \mathcal{X}^\gamma \gamma^{ij}, \qquad \Pi^{\mathcal{X}}_\alpha \to \Pi^{\mathcal{X}}_\alpha + 3\sqrt{-\gamma}\; \lambda_{\alpha\beta\gamma} \mathcal{X}^\beta \mathcal{X}^\gamma. \tag{4.3.59}$$

Keeping the scalar source J_v fixed, we see that this shift in the renormalized canonical momenta is equivalent to shifting the function $v(\mathcal{X})$ defining the mixed boundary conditions

[7] More general mixed boundary conditions are possible, where the deformation function v is allowed to depend on other fields present, provided Dirichlet boundary conditions are imposed on these fields. In the present theory we could take $v(\mathcal{X}, \mathcal{Y}, \gamma_{ij})$. We will comment on such more general mixed boundary conditions below.

as

$$v(\mathcal{X}) \to v(\mathcal{X}) + \lambda_{\alpha\beta\gamma}\mathcal{X}^\alpha\mathcal{X}^\beta\mathcal{X}^\gamma. \tag{4.3.60}$$

Therefore, one can move this type of terms between the counterterms and the function $v(\mathcal{X})$ freely, but the total value of this cubic coupling in the renormalized action is fixed and uniquely determined by the scalar boundary conditions, which require J^v to be kept fixed. The same holds also for the other terms of type (a) in (4.3.50), but those terms correspond to a shift $v(\mathcal{X})$ by a more general function $\Delta v(\mathcal{X}, \mathcal{Y})$ (see comment in footnote 7), and they also modify the momenta $\Pi^{\mathcal{Y}}_\alpha$.

There are two simple corollaries of this observation. Firstly, *marginal* mixed boundary conditions on the scalars can also be viewed as Neumann boundary conditions. In particular, if the function $v(\mathcal{X})$ corresponds to a marginal deformation, then it can be entirely absorbed in a choice of scheme for the original Dirichlet problem. Secondly, in combination with the results of [90] summarized above, the *total* value of the marginal scalar terms is fixed by supersymmetry, but these terms can still be moved between the counterterms and the function $v(\mathcal{X})$. Namely, starting with a supersymmetric Dirichlet problem, i.e. the value of the couplings $\lambda_{\alpha\beta\gamma}$ is fixed by the superpotential of the theory, then the corresponding Neumann boundary condition is supersymmetric, but any mixed boundary condition breaks supersymmetry. However, starting with a generic value of the cubic couplings in the Dirichlet problem such that supersymmetry is broken, then the corresponding Neumann problem is not supersymmetric, but a very specific mixed boundary condition is. Therefore, with our choice to include the full supersymmetric superpotential in the counterterms as in (4.3.47), we have to impose Neumann boundary conditions on the scalars $\mathcal{X}^\alpha$ since any mixed boundary condition will break supersymmetry. Hence, supersymmetry dictates that starting with the counterterms (4.3.47), we must set the function $v(\mathcal{X})$ to zero.

Finally, an interesting situation arises specifically for the finite local terms of the form $\mathcal{X}^\alpha\mathcal{Y}^\beta\mathcal{Y}^\gamma$. Since such terms are linear in the scalars $\mathcal{X}^\alpha$, they drop out of the Legendre transform with respect to $\mathcal{X}^\alpha$. This can be seen immediately from (4.3.55) and (4.3.56) by taking $v(\mathcal{X}, \mathcal{Y}) = \lambda_{\alpha\beta\gamma}\mathcal{X}^\alpha\mathcal{Y}^\beta\mathcal{Y}^\gamma$, or equivalently, by changing the scheme by such a term in the original Dirichlet problem. It follows that starting with the value of this coupling in the Dirichlet problem dictated by the superpotential of the theory, such that the Dirichlet problem is supersymmetric, adding a term of the form $\lambda_{\alpha\beta\gamma}\mathcal{X}^\alpha\mathcal{Y}^\beta\mathcal{Y}^\gamma$ *and* performing the Legendre transform with respect to $\mathcal{X}^\alpha$ trivially preserves supersymmetry since the final result of the Legendre transformation is identical with that obtained from the original supersymmetric Dirichlet problem. This is precisely the observation made in section 3.8 of [96]. However, adding such a term is not trivial because it changes the definition of the scalar source J_α, which affects the calculation of physical observables, e.g. correlation functions. Indeed, it was by looking at the three-point function $\langle\mathcal{O}_{\Delta=1}\mathcal{O}_{\Delta=2}\mathcal{O}_{\Delta=2}\rangle$ that the authors of [96] were able to determine the correct value of this cubic coupling.

Finite terms of type (b): The above observations apply to terms of type (b) in (4.3.50) as well, except that such terms correspond to a choice of supersymmetric scheme in the Dirichlet problem, and so supersymmetry does not fix their overall coefficient. Moreover, in contrast to the analogous terms in four dimensions considered in [90], in three dimensions the type (b) terms involving the scalars $\mathcal{X}^\alpha$ are linear in $\mathcal{X}^\alpha$ and so they drop out of the Legendre transform, exactly as the terms $\mathcal{X}^\alpha\mathcal{Y}^\beta\mathcal{Y}^\gamma$ we just discussed. It follows that such terms trivially preserve supersymmetry in the Legendre transformed theory, but they do affect the definition of the sources J_α of the dimension one operators and, hence, some argument is required in order to fix the coefficients of such terms.

Such an argument is provided by the requirement that the sources of the dimension one operators, namely

$$J_\alpha \equiv -\frac{1}{\sqrt{-\gamma}}\Pi^{\mathcal{X}}_\alpha, \tag{4.3.61}$$

vanish on supersymmetric solutions. Notice that we choose to impose Neumann boundary conditions since, as in the case of type (a) terms, mixed boundary conditions can be traded for a choice of scheme in the original Dirichlet problem, without affecting the actual value of the source J_α. Terms of type (b) in the counterterms contribute a constant multiple of the Ricci curvature $R[\gamma]$ in the renormalized canonical momenta $\Pi^{\mathcal{X}}_\alpha$. Hence, for solutions with $k=\pm1$ the sources J_α are shifted by a constant non-zero term. Requiring that the

BPS equations coincide with the condition of vanishing scalar sources, i.e. $J_\alpha|_{\text{BPS}} = 0$, unambiguously determines the coefficients of the type (b) terms proportional to $\mathcal{X}^\alpha$ in the counterterms to be the ones given in (4.3.47). This amounts to including the full supersymmetric effective superpotential for magnetic BPS solutions in the counterterms.[8] Notice that we are able to use this argument to determine the coefficient of the finite terms proportional to the Ricci curvature in the counterterms because we are considering BPS solutions with a non-zero boundary curvature, i.e. $k = \pm 1$. As we pointed out earlier, these terms are analogous to the terms $\mathcal{X}^\alpha\mathcal{Y}^\beta\mathcal{Y}^\gamma$ discussed in [96], except that those terms contribute to the sources J_α a term proportional to $\mathcal{Y}^\beta\mathcal{Y}^\gamma$, i.e. to the square of the sources of the dimension two operators. Since the sources $\mathcal{Y}^\alpha$ vanish in the background solutions considered in [96], this shift in the source J_α is not visible in the BPS equations, which is why the authors of [96] have to use the scalar three-point functions to determine the coefficient of this term.

4.3.2 Renormalized holographic observables

The outcome of the analysis in the previous subsection is that the renormalized generating function in the dual supersymmetric theory is given by

$$\mathbb{W}\big[g_{(0)ij}, A^\Lambda_{(0)i}, \mathcal{Y}^\alpha_-, J^+_\alpha\big] \equiv S_{\text{ren}} = \lim_{r_o\to\infty} \left(S_{\text{reg}} + S_{\text{ct}} + S_{v=0}\right), \tag{4.3.62}$$

where S_{reg} is the regularized on-shell action, including the Gibbons-Hawking term, S_{ct} are the boundary counterterms defined in (4.3.47), and $S_{v=0}$ is the Legendre transform (4.3.55),

[8]It may be useful to point out that [92] sets the coefficients of the finite terms of type (b) in (4.3.50) to zero in the choice of supersymmetric counterterms, while [97] does not specify them arguing that they drop out of the Legendre transform. Although the coefficient of the finite terms $\mathcal{Y}^\alpha R[\gamma]$ is indeed a choice of supersymmetric scheme, since Dirichlet boundary conditions are imposed on $\mathcal{Y}^\alpha$, we have argued that the coefficient of the terms $\mathcal{X}^\alpha R[\gamma]$ is in fact determined by the value of the source of the dimension one operators in BPS solutions. Despite the fact that, as correctly pointed out in [97], supersymmetry is (trivially) preserved for any value of the coefficient of $\mathcal{X}^\alpha R[\gamma]$ because such terms cancel out in the Legendre transform, demanding that the single trace source of the dimension one operators in BPS solutions vanishes uniquely determines the coefficients of $\mathcal{X}^\alpha R[\gamma]$ in the supersymmetric counterterms to be those dictated by the supersymmetric superpotential, as in (4.3.47).

with the function $v(\mathcal{X})$ set to zero. This generating function depends on the sources

$$
\begin{aligned}
g_{(0)ij} &= \lim_{r\to\infty}(e^{-2r/L}\gamma_{ij}), & A^{\Lambda}_{(0)i} &= \lim_{r\to\infty} A^{\Lambda}_{i}, \\
\mathcal{Y}^{\alpha}_{-} &= \lim_{r\to\infty}(e^{r/L}\mathcal{Y}^{\alpha}), & J^{+}_{\alpha} &= \lim_{r\to\infty}(e^{2r/L}J^{v=0}_{\alpha}).
\end{aligned} \tag{4.3.63}
$$

Differentiating $\mathbb{W}\big[g_{(0)ij}, A^{\Lambda}_{(0)i}, \mathcal{Y}^{\alpha}_{-}, J^{+}_{\alpha}\big]$ with respect to these sources gives the corresponding one-point functions in the presence of sources. Namely,

$$
\begin{aligned}
\langle \mathcal{T}^{ij}\rangle &= \lim_{r\to\infty}\left(e^{2r/L}\Big(\frac{2}{\sqrt{-\gamma}}\Pi^{ij} + J^{v=0}_{\alpha}\mathcal{X}^{\alpha}\gamma^{ij}\Big)\right) = \frac{2}{\sqrt{-g_{(0)}}}\frac{\mathrm{d}\,\mathbb{W}}{\mathrm{d}\,g_{(0)ij}}, \\
\langle \mathcal{J}^{i}_{\Lambda}\rangle &= \lim_{r\to\infty}\left(\frac{1}{\sqrt{-\gamma}}\Pi^{i}_{\Lambda}\right) = \frac{1}{\sqrt{-g_{(0)}}}\frac{\mathrm{d}\,\mathbb{W}}{\mathrm{d}\,A^{\Lambda}_{(0)i}}, \\
\langle \mathcal{O}^{\Delta=2}_{\alpha}\rangle &= \lim_{r\to\infty}\left(e^{-r/L}\frac{1}{\sqrt{-\gamma}}\pi^{\mathcal{Y}}_{\alpha}\right) = \frac{1}{\sqrt{-g_{(0)}}}\frac{\mathrm{d}\,\mathbb{W}}{\mathrm{d}\,\mathcal{Y}^{\alpha}_{-}}, \\
\langle \mathcal{O}^{\alpha}_{\Delta=1}\rangle &= \lim_{r\to\infty}\left(e^{r/L}\mathcal{X}^{\alpha}\right) = \mathcal{X}^{\alpha}_{-} = \frac{1}{\sqrt{-g_{(0)}}}\frac{\mathrm{d}\,\mathbb{W}}{\mathrm{d}\,J^{+}_{\alpha}}.
\end{aligned} \tag{4.3.64}
$$

These one-point functions satisfy the following Ward identities, which can be deduced from the first class constraints (B.9) in the radial Hamiltonian formulation of the bulk dynamics [94]

$$
- D^{j}_{(0)}\langle \mathcal{T}_{ij}\rangle + \langle \mathcal{O}^{\alpha}_{\Delta=1}\rangle\partial_i J^{+}_{\alpha} + \langle \mathcal{O}^{\alpha}_{\Delta=2}\rangle\partial_i\mathcal{Y}^{\alpha}_{-} + \left(\langle \mathcal{J}^{j}_{\Lambda}\rangle + \frac{2}{\kappa^2}\epsilon^{jkl}_{(0)}\mathcal{R}_{\Lambda\Sigma}(0)F^{\Sigma}_{(0)kl}\right)F^{\Lambda}_{(0)ij} = 0, \tag{4.3.65a}
$$

$$
D_{(0)i}\langle \mathcal{J}^{i}_{\alpha}\rangle = 0, \tag{4.3.65b}
$$

$$
- \langle \mathcal{T}^{i}_{i}\rangle + 2J^{+}_{\alpha}\langle \mathcal{O}^{\alpha}_{\Delta=1}\rangle + \mathcal{Y}^{\alpha}_{-}\langle \mathcal{O}^{\Delta=2}_{\alpha}\rangle = 0. \tag{4.3.65c}
$$

The Legendre transform of the generating functional (4.3.62) with respect to any of the sources is the quantum effective action for the vacuum expectation value (VEV) of the corresponding operator. As we will show in the next section, the entropy functional for BPS black holes of the supergravity action (4.2.8) is related to the effective action obtained by Legendre transforming the generating functional (4.3.62) with respect to the source J^{+}_{α} of the dimension one scalar operators. The resulting quantum effective action is a functional of the VEVs $\mathcal{X}_-^\alpha = \langle \mathcal{O}^\alpha_{\Delta=1}\rangle$ and takes the form

$$\Gamma[(g_{(0)ij}, A^\Lambda_{(0)i}, \mathcal{Y}^\alpha_-, \mathcal{X}^\alpha_-)] = \mathbb{W}\big[g_{(0)ij}, A^\Lambda_{(0)i}, \mathcal{Y}^\alpha_-, J^+_\alpha\big] - \int d^3x\ \mathcal{X}^\alpha_- J^+_\alpha = \lim_{r_o\to\infty}(S_{\rm reg} + S_{\rm ct})\,. \tag{4.3.66}$$

Notice that the Legendre transform simply removes the term $S_{v=0}$ in the generating function and so the effective action of the Neumann theory coincides with the generating function of the Dirichlet one [93]. Earlier computations of this effective action, up to two derivatives in the derivative expansion, appeared for a number of different examples in [98, 93, 99, 94].

4.3.3 The BPS limit and black hole thermodynamics

For solutions of the form (4.2.20) all the renormalized observables can be related to the corresponding effective superpotential $\mathcal{U}$. Using the counterterms (4.3.47) and the expression (4.2.35) for the regularized on-shell action one finds that the renormalized partition function is given by

$$\begin{aligned}\mathbb{W} =\ & -\frac{1}{\kappa^2}\int_{r_o\to\infty} d^3x\sqrt{\sigma_k}\ e^{3A} f^{1/2}(\mathcal{U} - \mathcal{U}^{q=0}_{\rm BPS}) + \mu^\Lambda Q_\Lambda \int dt + \frac{2\pi T}{\kappa^2}\int dt A_h \\ & + \frac{1}{\kappa^2}\int_{r_o\to\infty} d^3x\sqrt{\sigma_k}\ e^{3A} f^{1/2}\mathcal{X}^\alpha\frac{\partial}{\partial\mathcal{X}^\alpha}(\mathcal{U} - \mathcal{U}^{q=0}_{\rm BPS}),\end{aligned} \tag{4.3.67}$$

where $\mathcal{U}^{q=0}_{\rm BPS}$ stands for the supersymmetric superpotential (4.2.38), with the electric charges set to zero. Moreover, the area of the horizon A_h and the electric charges Q_Λ were defined respectively in (4.2.36) and (4.2.37), and we have introduced the electric chemical potentials

$$\mu^\Lambda \equiv a^\Lambda(\infty) - a^\Lambda(r_h). \tag{4.3.68}$$

Notice that the last term in (4.3.67) corresponds to the Legendre transform $S_{v=0}$ and, therefore, the effective action (4.3.66) becomes

$$\Gamma = -\frac{1}{\kappa^2}\int_{r_o\to\infty} d^3x\sqrt{\sigma_k}\ e^{3A} f^{1/2}(\mathcal{U} - \mathcal{U}^{q=0}_{\rm BPS}) + \mu^\Lambda Q_\Lambda\int dt + \frac{2\pi T}{\kappa^2}\int dt\ A_h. \tag{4.3.69}$$

The one-point functions (4.3.64) can also be evaluated in terms of the effective superpotential. The general expressions for the one-point functions for any background of the form (4.2.20) are given in eq. (3.34) of [75]. Using the supersymmetric counterterms (4.3.47) these expressions become

$$\begin{aligned}
\langle \mathcal{T}_{tt} \rangle &= \frac{1}{\kappa^2} \lim_{r\to\infty} e^{3A} (\mathcal{U} - \mathcal{U}_{\text{BPS}}^{q=0}), \\
\langle \mathcal{T}_{aa} \rangle &= -\frac{1}{\kappa^2} \lim_{r\to\infty} e^{3A} \Big(1 + \frac{1}{2}\partial_A\Big) (\mathcal{U} - \mathcal{U}_{\text{BPS}}^{q=0}), \\
\langle \mathcal{J}^i_\Lambda \rangle &= -\frac{2}{\kappa^2} q_\Lambda \, \mathrm{d}^{it}, \\
\langle \mathcal{O}^{\Delta=2}_\alpha \rangle &= -\frac{1}{\kappa^2} \lim_{r\to\infty} e^{2r/L} \frac{\partial}{\partial \mathcal{Y}^\alpha} \left(1 - \mathcal{X}^\alpha \frac{\partial}{\partial \mathcal{X}^\alpha}\right) (\mathcal{U} - \mathcal{U}_{\text{BPS}}^{q=0}), \\
\langle \mathcal{O}^\alpha_{\Delta=1} \rangle &= \lim_{r\to\infty} \big(e^{r/L} \mathcal{X}^\alpha\big).
\end{aligned} \tag{4.3.70}$$

The extremal limit These quantities can be evaluated explicitly in the extremal limit, corresponding to the exact superpotential (4.2.38). As we pointed out earlier, in order to evaluate some of these observables in the extremal limit, it is necessary to start from near extremal solutions and take the zero temperature limit in the end. Since all observables are expressed in terms of a generic effective superpotential $\mathcal{U}$, evaluating them on near extremal solutions amounts to using the near extremal superpotential (4.2.42). For large radial cutoff r_o this can be expanded to obtain

$$\begin{aligned}
\mathcal{U} - \mathcal{U}_{\text{BPS}}^{q=0} &= \mathcal{U}_{\text{BPS}} - \mathcal{U}_{\text{BPS}}^{q=0} + \varepsilon \Delta \mathcal{U} \\
&= -\frac{\sqrt{2}}{\xi L} e^{\mathcal{K}/2} |W| \big(-e^{-2A} \mathrm{Im}\,(W^{-1} Z_e) + \mathcal{O}(e^{-4A})\big) + \varepsilon \Delta \mathcal{U},
\end{aligned} \tag{4.3.71}$$

where Z_e denotes the electric part of the central charge (4.2.39), i.e.

$$Z_e \equiv -\sqrt{2}\, \xi L q_\Lambda X^\Lambda. \tag{4.3.72}$$

Using this expansion, and the Pufu-Freedman parameterization of the STU model discussed in appendix A, we find that the effective action (4.3.69) for near extremal black holes takes the form

$$\Gamma=\frac{2}{\kappa^2}\int d^3x\sqrt{\sigma_k}\,(\mathfrak{m}^\Lambda-\mu^\Lambda)q_\Lambda+\frac{2\pi T}{\kappa^2}\int dt\;A_h-\frac{1}{\kappa^2}\int_{r_o\to\infty}d^3x\sqrt{\sigma_k}\;e^{3A}f^{1/2}\varepsilon\Delta\mathcal{U},\quad(4.3.73)$$

where

$$\mathfrak{m}^0\equiv\frac{1}{8}(\mathcal{Y}^1_-+\mathcal{Y}^2_-+\mathcal{Y}^3_-),\quad \mathfrak{m}^\alpha\equiv-\frac{1}{8}\Big((-1)^{\mathrm{d}_{\alpha 1}}\mathcal{Y}^1_-+(-1)^{\mathrm{d}_{\alpha 2}}\mathcal{Y}^2_-+(-1)^{\mathrm{d}_{\alpha 3}}\mathcal{Y}^3_-\Big),\quad \alpha=1,2,3.\quad(4.3.74)$$

Notice that

$$\sum_\Lambda\mathfrak{m}^\Lambda=0.\quad(4.3.75)$$

Moreover, the second property in (4.2.44) of the near extremal superpotential ensures that the last term in (4.3.73), which is proportional to $\Delta\mathcal{U}$, is finite as the cutoff is removed at fixed ε. Moreover, the first result in (4.2.44) implies that as $\varepsilon\to 0$, this term gives a zero contribution to the Euclidean effective action, and so only the first two terms in (4.3.73) can potentially contribute in the extremal limit. The term involving the area of the horizon has a finite extremal limit, but the integrand of the first term in (4.3.73) is not proportional to the temperature and so it seems to lead to a divergent contribution to the extremal Euclidean effective action due to the infinite periodicity of the Euclidean time, i.e. $\beta\to\infty$. We therefore conclude that supersymmetric solutions must satisfy the boundary condition

$$\mathfrak{m}^\Lambda=\mu^\Lambda.\quad(4.3.76)$$

This condition relates the sources $\mathcal{Y}^\alpha_-$ of the dimension two scalar operators to the electric chemical potentials and, therefore, is an additional requirement for the Dirichlet boundary conditions on the scalars $\mathcal{Y}^\alpha$ and the gauge fields A^Λ_i to be supersymmetric. Provided the condition (4.3.76) holds, therefore, the effective action for BPS solutions is given by the area of the horizon, namely

$$\Gamma_{\mathrm{BPS}}[\mathcal{X}^\alpha_-;\mathfrak{m}^\Lambda,\mathfrak{n}^\Lambda]=\frac{2\pi T}{\kappa^2}\int dt\;A_h[\mathcal{X}^\alpha_-;\mathfrak{m}^\Lambda,\mathfrak{n}^\Lambda],\quad(4.3.77)$$

where we have kept the temperature as a regulator in the off-shell effective action. It is only after Wick rotation to Euclidean signature that the temperature will cancel against the perimeter of the Euclidean time circle. It should be stressed that at this point the horizon area is not equal to the extremal entropy since it is evaluated at *arbitrary* VEVs $\mathcal{X}_-^\alpha = \langle \mathcal{O}^\alpha_{\Delta=1}\rangle$. $\Gamma_{\rm BPS}[\mathcal{X}_-^\alpha;\mathfrak{m}^\Lambda,\mathfrak{n}^\Lambda]$ is the field theory quantum effective action for these VEVs. As we will show in the next section, the extremization of this effective action, at fixed magnetic fluxes $\mathfrak{n}^\Lambda$, is the field theory realization of the attractor mechanism in the bulk.

The area of the horizon can be evaluated explicitly for BPS black holes using the exact superpotential (4.2.38). This is because the effective superpotential vanishes on the horizon [77], i.e.[9]

$$\mathcal{U}_{\rm BPS}|_h = 0, \tag{4.3.78}$$

and therefore

$$e^{2A}\big|_h = -\,iW^{-1}Z\big|_h = i\sqrt{2}\,\xi L\,W^{-1}(p^\Lambda F_\Lambda + q_\Lambda X^\Lambda)\big|_h\,. \tag{4.3.79}$$

Inserting this expression for the warp factor in (4.3.77) we arrive at the following general expression for the effective action of dyonic BPS black holes of the $U(1)^4$ gauge supergravity:

$$\Gamma_{\rm BPS}[\mathcal{X}_-^\alpha;\mathfrak{m}^\Lambda,\mathfrak{n}^\Lambda] = \frac{2\pi T}{\kappa^2}\int dt\int d^2x\sqrt{\sigma_k}\;\; i\sqrt{2}\,\xi L W^{-1}(p^\Lambda F_\Lambda + q_\Lambda X^\Lambda)\Big|_h\,. \tag{4.3.80}$$

In the next section we will evaluate this effective action explicitly, first on magnetic and then dyonic BPS black holes. Notice that this effective action depends on the UV parameters $\mathcal{X}_-^\alpha$, $\mathfrak{m}^\Lambda$, $\mathfrak{n}^\Lambda$ and, therefore, it is necessary to know the full black hole solutions to correctly evaluate it. In particular, it is *not* sufficient to evaluate the effective action using the near horizon solutions, since this does not determine the relation between the parameters of the near horizon solutions to the physical UV modes $\mathcal{X}_-^\alpha$, $\mathfrak{m}^\Lambda$ and $\mathfrak{n}^\Lambda$.

Using the exact superpotential (4.2.38), we can also evaluate the one-point function

[9] The fact that the effective superpotential $\mathcal{U}$ vanishes on the horizon, even for non-BPS black holes, follows from the first order equations (4.2.27) and the near horizon behavior of the blackening factor in (4.2.32).

(4.3.70) for BPS black holes. A straightforward calculation determines

$$\begin{aligned}&\langle \mathcal{T}_{tt}\rangle = -\frac{2}{\kappa^2}\mathfrak{m}^\Lambda q_\Lambda, \qquad \langle \mathcal{T}_{ta}\rangle = \langle \mathcal{T}_{ab}\rangle = 0, \qquad \langle \mathcal{J}^i_\Lambda\rangle = -\frac{2}{\kappa^2} q_\Lambda \, \mathrm{d}^{it}, \\ &\langle \mathcal{O}^{\Delta=2}_\alpha\rangle = \frac{1}{4\kappa^2}\Big(q_0 - (-1)^{\mathrm{d}_{\alpha 1}} q_1 - (-1)^{\mathrm{d}_{\alpha 2}} q_2 - (-1)^{\mathrm{d}_{\alpha 3}} q_3\Big), \qquad \langle \mathcal{O}^\alpha_{\Delta=1}\rangle = \mathcal{X}^\alpha_{(0)}.\end{aligned} \tag{4.3.81}$$

These expressions for the one-point functions of BPS solutions have a number of important consequences. Firstly, using the definition of the fermion masses $\mathfrak{m}^\Lambda$ in (4.3.74) and the VEVs of the dimension two operators in (4.3.81) we deduce that

$$\langle \mathcal{O}^{\Delta=2}_\alpha\rangle \, \mathrm{d}\, \mathcal{Y}^\alpha_- = \frac{2}{\kappa^2} q_\Lambda \, \mathrm{d}\, \mathfrak{m}^\Lambda. \tag{4.3.82}$$

In combination with the effective action (4.3.73) this result implies that

$$\frac{\partial}{\partial \mathfrak{m}^\Lambda}\int dt \; A_h[\mathcal{X}^\alpha_-; \mathfrak{m}^\Lambda, \mathfrak{n}^\Lambda] = 0, \tag{4.3.83}$$

or equivalently

$$\frac{\partial}{\partial \mathcal{Y}^\alpha_-}\int dt \; A_h[\mathcal{X}^\alpha_-; \mathfrak{m}^\Lambda, \mathfrak{n}^\Lambda] = 0, \tag{4.3.84}$$

and so the BPS effective action is extremized with respect to the sources of the dimension two operators, with the extremal values given by the chemical potentials as in (4.3.76). This observation will play a central role in our field theory interpretation of the attractor mechanism.

Another implication of the supersymmetric one-point functions (4.3.81) is that the supersymmetric mass of dyonic BPS black holes is [100, 94]

$$M = -\frac{2}{\kappa^2}\mathfrak{m}^\Lambda q_\Lambda \mathrm{Vol}(\Sigma_k), \tag{4.3.85}$$

and, as a direct consequence of the relation (4.3.76), satisfies the BPS relation

$$M - \mu^\Lambda Q_\Lambda = 0. \tag{4.3.86}$$

Finally, collecting the above results, we can evaluate the Euclidean on-shell action, which is holographically identified with the grand canonical potential, i.e. the Gibbs free energy, for dyonic BPS black holes:

$$\mathcal{I} = -\mathbb{W}^{\mathrm{E}}_{\mathrm{BPS}} = -\Gamma^{\mathrm{E}}_{\mathrm{BPS}} = -S, \tag{4.3.87}$$

where S is the extremal entropy, evaluated at the extremum of the effective action. This result agrees with that obtained in [97], as well as [13] in the case of AdS_4 black holes without scalars. Our derivation, however, provides an explicit proof that, as anticipated in [97], the free energy of extremal asymptotically AdS black holes is a direct consequence of imposing the BPS relation (4.3.86) in the quantum statistical relation for general asymptotically AdS black holes [100]

$$\mathcal{I} = \beta(M - ST - \mu_\Lambda Q^\Lambda), \tag{4.3.88}$$

and taking the extremal limit.

4.4 Holographic attractor mechanism and the entropy functional

In this section we will demonstrate that extremizing the holographic quantum effective action (4.3.80) for BPS black holes with respect to the VEVs of the dimension one scalar operators determines the correct supersymmetric values for these VEVs. Moreover, we will show that the value of the effective action at the extremum coincides with the black hole entropy. This provides a purely field theoretic extremization principle, which we dub *the holographic attractor mechanism.*

Evaluating the effective action (4.3.80) explicitly as a function of the UV parameters $\mathcal{X}^\alpha_-$, $\mathfrak{m}^\Lambda$ and $\mathfrak{n}^\Lambda$ is a formidable task: it requires knowledge of the general – not necessarily regular in the interior – solution of the BPS equations, with arbitrary $\mathcal{X}^\alpha_-$, in closed form. This is necessary in order to express explicitly the area of the horizon in terms of the arbitrary scalar VEVs $\mathcal{X}^\alpha_-$ at the UV. However, we will see that knowledge of this solution is not necessary in order to obtain the extremal entropy. In particular, we will demonstrate by means of a concrete example that extremizing the effective action with respect to the scalar VEVs is equivalent to extremizing the expression (4.3.80) for the effective action with respect to the values of the physical scalars on the horizon. If one knowns the exact BPS black hole solution corresponding to the extremum of the effective potential, then the values of the UV VEVs at the extremum can be determined as well.

The BPS swampland The metric ansatz (4.2.20) is designed so that the first order equations (4.2.27) take the simplest form. However, to obtain explicit black hole solutions it is convenient to reparameterize the metric by defining

$$e^{2A} = \bar{r}^2 h(\bar{r}), \qquad f = \frac{b(\bar{r})}{\bar{r}^2 h^2(\bar{r})}, \qquad dr = \frac{h^{\frac{1}{2}}(\bar{r})}{b^{\frac{1}{2}}(\bar{r})} d\bar{r}. \tag{4.4.89}$$

The ansatz (4.2.20) then becomes

$$\begin{aligned} ds^2 &= h(\bar{r}) b^{-1}(\bar{r}) d\bar{r}^2 - h^{-1}(\bar{r}) b(\bar{r}) dt^2 + h(\bar{r}) \bar{r}^2 d\sigma_k^2, \\ A^\Lambda &= a^\Lambda(\bar{r}) \mathrm{d}t + p^\Lambda \Big(\int d\theta\, \omega_k(\theta) \Big) \mathrm{d}\varphi, \qquad z^\alpha = z^\alpha(\bar{r}), \end{aligned} \tag{4.4.90}$$

while the BPS equations (4.2.41) take the form

$$\begin{aligned} b^{\frac{1}{2}}(\bar{r}) h^{-\frac{1}{2}}(\bar{r}) A' &= \frac{1}{\sqrt{2}\xi L} e^{\mathcal{K}/2} |W + i e^{-2A} Z| \,, \\ b^{\frac{1}{2}}(\bar{r}) h^{-\frac{1}{2}}(\bar{r}) \frac{f'}{f} &= \frac{\sqrt{2}}{\xi L} e^{\mathcal{K}/2} \partial_A |W + i e^{-2A} Z| \,, \\ b^{\frac{1}{2}}(\bar{r}) h^{-\frac{1}{2}}(\bar{r}) z'^\alpha &= -\frac{\sqrt{2}}{\xi L} \mathcal{K}^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \left(e^{\mathcal{K}/2} |W + i e^{-2A} Z| \right), \\ b^{\frac{1}{2}}(\bar{r}) h^{-\frac{1}{2}}(\bar{r}) \bar{z}'^{\bar{\beta}} &= -\frac{\sqrt{2}}{\xi L} \mathcal{K}^{\alpha\bar{\beta}} \partial_\alpha \left(e^{\mathcal{K}/2} |W + i e^{-2A} Z| \right), \\ 2\bar{r}^2 h(\bar{r}) a'^\Lambda &= \mathcal{I}^{\Lambda\Sigma} \left(\mathcal{R}_{\Sigma M} p^M - q_\Sigma \right), \end{aligned} \tag{4.4.91}$$

where $'$ denotes a derivative with respect to the radial coordinate $\bar{r}$.

Since the BPS equations (4.4.91) are first order, their general solution contains $n_V + 3$ real and n_V complex integration constants, i.e. one for each equation. One of the real integration constants is related to rescaling of the radial coordinate $\bar{r}$ and is fixed by requiring that the solution is asymptotically AdS_4 with radius L. A second real integration constant is fixed by a suitable regularity condition in the interior, such as the existence of a smooth horizon. Moreover, as we have seen in the previous section in eq. (4.3.76), supersymmetry relates the n_V+1 electric chemical potentials to the n_V sources of the dimension two operators. The general supersymmetric asymptotically AdS_4 solution is therefore parameterized by $2n_V$ real integration constants: n_V independent electric chemical potentials and n_V VEVs for the dimension one operators. The chemical potentials, however, are a boundary condition and can therefore be set to any desired value. The VEVs of the dimension one operators, on the other hand, are dynamically determined by the theory. Namely, they are fixed by extremizing the quantum effective action, evaluated on the *BPS swampland*, i.e. the general supersymmetric solution with n_V arbitrary scalar VEVs, for given chemical potentials.

4.4.1 Magnetic BPS black holes

For real scalars and vanishing electric charges the BPS equations (4.4.91) are sufficiently simple to be written explicitly. In the Pufu-Freedman parameterization, discussed in appendix A, they take the form

$$
\begin{aligned}
b^{\frac{1}{2}}(\bar{r})h^{-\frac{1}{2}}(\bar{r})A' &= \frac{1}{4L\sqrt{(1-(z^1)^2)(1-(z^2)^2)(1-(z^3)^2)}}\Big[4\left(1+z^1z^2z^3\right) \\
&\quad -\sqrt{2}\,Le^{-2A}\Big(p^0(1-z^1)(1-z^2)(1-z^3)+p^1(1-z^1)(1+z^2)(1+z^3) \\
&\quad +p^2(1+z^1)(1-z^2)(1+z^3)+p^3(1+z^1)(1+z^2)(1-z^3)\Big)\Big], \\
b^{\frac{1}{2}}(\bar{r})h^{-\frac{1}{2}}(\bar{r})\frac{f'}{f} &= \frac{\sqrt{2}\,e^{-2A}}{\sqrt{(1-(z^1)^2)(1-(z^2)^2)(1-(z^3)^2)}}\Big(p^0(1-z^1)(1-z^2)(1-z^3) \\
&\quad +p^1(1-z^1)(1+z^2)(1+z^3)+p^2(1+z^1)(1-z^2)(1+z^3)+p^3(1+z^1)(1+z^2)(1-z^3)\Big), \\
b^{\frac{1}{2}}(\bar{r})h^{-\frac{1}{2}}(\bar{r})z'^{\alpha} &= -\frac{(1-(z^\alpha)^2)}{4L\sqrt{(1-(z^1)^2)(1-(z^2)^2)(1-(z^3)^2)}}\Big[4\left(z^\alpha+z^1z^2z^3/z^\alpha\right) \\
&\quad +\sqrt{2}\,Le^{-2A}\Big(p^0(1-z^1)(1-z^2)(1-z^3)-(-1)^{\mathrm{d}_{\alpha1}}p^1(1-z^1)(1+z^2)(1+z^3) \\
&\quad -(-1)^{\mathrm{d}_{\alpha2}}p^2(1+z^1)(1-z^2)(1+z^3)-(-1)^{\mathrm{d}_{\alpha3}}p^3(1+z^1)(1+z^2)(1-z^3)\Big)\Big], \\
a'^{\Lambda} &= 0.
\end{aligned}
\tag{4.4.92}
$$

General magnetic BPS solution The general solution of the BPS equations (4.4.92) can be sought in the form of an asymptotic expansion in the UV. In particular, ensuring that the solution is asymptotically AdS_4 with radius L, we write

$$\begin{aligned} h(\bar{r}) &= \sqrt{1 + h^{(1)} L/\bar{r} + h^{(2)} (L/\bar{r})^2 + \cdots}\,, \\ b(\bar{r}) &= \frac{\bar{r}^2}{L^2} + b^{(-1)} \bar{r}/L + b^{(0)} + b^{(1)} L/\bar{r} + b^{(2)} (L/\bar{r})^2 + \cdots\,, \\ v_\alpha(\bar{r}) &= 1 + v_\alpha^{(1)} L/\bar{r} + v_\alpha^{(2)} (L/\bar{r})^2 + v_\alpha^{(3)} (L/\bar{r})^3 + \cdots\,, \end{aligned} \tag{4.4.93}$$

where the functions v_α determine the scalars z^α through the relations

$$z^1 = \frac{1 - v_1}{1 + v_1}, \qquad z^2 = \frac{1 - v_2}{1 + v_2}, \qquad z^3 = \frac{1 - v_3}{1 + v_3}. \tag{4.4.94}$$

Inserting these expansions in the BPS equations (4.4.92) we find up to the first two subleading orders

$$\begin{aligned} & b^{(-1)} = h^{(1)}, \quad b^{(0)} = h^{(2)} + k, \quad h^{(2)} = \frac{1}{8}\Big(3(h^{(1)})^2 - 4\sum_\alpha (v_\alpha^{(1)})^2\Big), \\ & v_1^{(2)} = \frac{1}{4}\Big(4k - 4\mathfrak{n}_2 - 4\mathfrak{n}_3 - h^{(1)} v_1^{(1)} + 2(v_1^{(1)})^2 - 2 v_2^{(1)} v_3^{(1)}\Big), \\ & v_2^{(2)} = \frac{1}{4}\Big(4k - 4\mathfrak{n}_1 - 4\mathfrak{n}_3 - h^{(1)} v_2^{(1)} + 2(v_2^{(1)})^2 - 2 v_1^{(1)} v_3^{(1)}\Big), \\ & v_3^{(2)} = \frac{1}{4}\Big(4k - 4\mathfrak{n}_2 - 4\mathfrak{n}_1 - h^{(1)} v_3^{(1)} + 2(v_3^{(1)})^2 - 2 v_2^{(1)} v_1^{(1)}\Big). \end{aligned} \tag{4.4.95}$$

We have determined these expansions up to the terms $h^{(5)}$, $b^{(3)}$, $v_\alpha^{(5)}$, but there is no good reason to reproduce the lengthy expressions for the coefficients here. The crucial property of this solution of the BPS equations is that the coefficients $h^{(1)}$ and $v_\alpha^{(1)}$ are arbitrary integration constants, while all higher order terms are uniquely determined in terms of these. Notice that the undetermined coefficients $v_\alpha^{(1)}$ correspond to the VEVs of the dimension one operators, namely

$$\mathcal{X}_-^\alpha = -\frac{L}{2} v_\alpha^{(1)}, \tag{4.4.96}$$

and hence, provided we find a way to fix the integration constant $h^{(1)}$, this solution is the desired BPS swampland that we should use to evaluate the effective action (4.3.80).

Series resummation and the Cacciatori-Klemm solution Remarkably, setting the arbitrary integration constants in the solution (4.4.93) to $h^{(1)} = 0$ and

$$
\begin{aligned}
v_1^{(1)} &= \pm \frac{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}, \\
v_2^{(1)} &= \pm \frac{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}, \\
v_3^{(1)} &= \pm \frac{(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}},
\end{aligned}
\tag{4.4.97}
$$

where the signs are correlated, the expansions for $h(\bar{r})$ and $b(\bar{r})$ truncate, while those for $v_\alpha(\bar{r})$ can be resummed. The result is the Cacciatori-Klemm solution [62, 63]

$$
h(\bar{r}) = \sqrt{\prod_\Lambda \left(\alpha_\Lambda + \frac{\beta_\Lambda}{\bar{r}}\right)}, \qquad g(\bar{r}) = \left(\frac{\bar{r}}{L} + \frac{c\, L}{\bar{r}}\right)^2, \tag{4.4.98}
$$

$$
v_1 = \sqrt{\frac{(\alpha_2 + \beta_2/\bar{r})(\alpha_3 + \beta_3/\bar{r})}{(\alpha_0 + \beta_0/\bar{r})(\alpha_1 + \beta_1/\bar{r})}}, \quad v_2 = \sqrt{\frac{(\alpha_1 + \beta_1/\bar{r})(\alpha_3 + \beta_3/\bar{r})}{(\alpha_0 + \beta_0/\bar{r})(\alpha_2 + \beta_2/\bar{r})}}, \tag{4.4.99}
$$

$$
v_3 = \sqrt{\frac{(\alpha_2 + \beta_2/\bar{r})(\alpha_1 + \beta_1/\bar{r})}{(\alpha_0 + \beta_0/\bar{r})(\alpha_3 + \beta_3/\bar{r})}},
$$

where the real constants α_Λ, β_Λ and c satisfy the constraints [62, 63, 10]

$$
\alpha_\Lambda = 1, \qquad \sum_\Lambda \beta_\Lambda = 0, \qquad \mathfrak{n}^\Lambda = c + \frac{\beta_\Lambda^2}{L^2}. \tag{4.4.100}
$$

The effective action and its extremization We now show that the Cacciatori-Klemm solution is obtained by evaluating the effective action (4.3.80) on the solution (4.4.93) with $h^{(1)} = 0$, and extremizing with respect to the VEVs $v_\alpha^{(1)}$. Since we do not know the swampland solution in cosed form, we cannot explicitly obtain the effective action as a function of the VEVs $v_\alpha^{(1)}$. However, from the expression (4.3.80) follows that the effective action depends on the VEVs $v_\alpha^{(1)}$ only through the value of the vector multiplet scalars on the horizon, namely

$$
z^\alpha|_h = z_h^\alpha(v_\beta^{(1)}). \tag{4.4.101}
$$

From the counterterms (4.3.47) and the definition (4.3.61) of the sources of the dimension one scalar operators follows that on supersymmetric vacua ($\mathfrak{m}^\Lambda = 0$ because we consider purely magnetic solutions here)

$$\frac{\mathrm{d}\,\Gamma_{\mathrm{BPS}}[\mathcal{X}_-^\alpha;\mathfrak{m}^\Lambda=0,\mathfrak{n}^\Lambda]}{\mathrm{d}\,\mathcal{X}_-^\alpha} = -J_\alpha^+ = 0, \tag{4.4.102}$$

and so such vacua correspond to the extrema of the effective action. Moreover, provided

$$\det\left(\frac{\partial z_h^\beta}{\partial v_\alpha^{(1)}}\right) \neq 0, \tag{4.4.103}$$

which we will assume, the chain rule

$$\frac{\partial \Gamma_{\mathrm{BPS}}}{\partial v_\alpha^{(1)}} = \frac{\partial z_h^\beta}{\partial v_\alpha^{(1)}}\frac{\partial \Gamma_{\mathrm{BPS}}}{\partial z_h^\beta}, \tag{4.4.104}$$

implies that the extrema of the effective action as function the VEVs $v_\alpha^{(1)}$ correspond to its extrema as function of the values z_h^α of the scalars on the horizon. This result provides a holographic interpretation of the attractor mechanism, as the extremization of the quantum effective action in the dual theory with respect to the VEVs of the dimension one operators.

For the purely magnetic black holes we can verify explicitly that extremizing the effective action with respect to the scalars on the horizon, or equivalently the scalar VEVs, reproduces the values z_h^α on the horizon obtained from the Cacciatori-Klemm solution. From the expression (4.3.80) for the effective action follows that as a function of the values z_h^α of the scalars on the horizon it takes the form

$$\begin{aligned}\Gamma_{\mathrm{BPS}} \propto (1+z_h^1z_h^2z_h^3)^{-1}\Big(&p^0(1-z_h^1)(1-z_h^2)(1-z_h^3)+p^1(1-z_h^1)(1+z_h^2)(1+z_h^3)\\&+p^2(1+z_h^1)(1-z_h^2)(1+z_h^3)+p^3(1+z_h^1)(1+z_h^2)(1-z_h^3)\Big).\end{aligned} \tag{4.4.105}$$

The extrema of this function are

$$z_*^1 = \frac{1-\sqrt{\frac{x^2x^3}{x^0x^1}}}{1+\sqrt{\frac{x^2x^3}{x^0x^1}}}, \qquad z_*^2 = \frac{1-\sqrt{\frac{x^1x^3}{x^0x^2}}}{1+\sqrt{\frac{x^1x^3}{x^0x^2}}}, \qquad z_*^3 = \frac{1-\sqrt{\frac{x^2x^1}{x^0x^3}}}{1+\sqrt{\frac{x^2x^1}{x^0x^3}}}, \tag{4.4.106}$$

where

$$
\begin{aligned}
x^0 &= 1 + \frac{4(\mathfrak{n}_0 - k/2)^2 + 1 - \mathfrak{n}_0^2 - \mathfrak{n}_1^2 - \mathfrak{n}_2^2 - \mathfrak{n}_3^2}{2\sqrt{(1 - \mathfrak{n}_0\mathfrak{n}_1 - \mathfrak{n}_0\mathfrak{n}_2 - \mathfrak{n}_0\mathfrak{n}_3 - \mathfrak{n}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{n}_3 - \mathfrak{n}_2\mathfrak{n}_3)^2 - 4\mathfrak{n}_0\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}}, \\
x^1 &= 1 + \frac{4(\mathfrak{n}_1 - k/2)^2 + 1 - \mathfrak{n}_0^2 - \mathfrak{n}_1^2 - \mathfrak{n}_2^2 - \mathfrak{n}_3^2}{2\sqrt{(1 - \mathfrak{n}_0\mathfrak{n}_1 - \mathfrak{n}_0\mathfrak{n}_2 - \mathfrak{n}_0\mathfrak{n}_3 - \mathfrak{n}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{n}_3 - \mathfrak{n}_2\mathfrak{n}_3)^2 - 4\mathfrak{n}_0\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}}, \\
x^2 &= 1 + \frac{4(\mathfrak{n}_2 - k/2)^2 + 1 - \mathfrak{n}_0^2 - \mathfrak{n}_1^2 - \mathfrak{n}_2^2 - \mathfrak{n}_3^2}{2\sqrt{(1 - \mathfrak{n}_0\mathfrak{n}_1 - \mathfrak{n}_0\mathfrak{n}_2 - \mathfrak{n}_0\mathfrak{n}_3 - \mathfrak{n}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{n}_3 - \mathfrak{n}_2\mathfrak{n}_3)^2 - 4\mathfrak{n}_0\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}}, \\
x^3 &= 1 + \frac{4(\mathfrak{n}_3 - k/2)^2 + 1 - \mathfrak{n}_0^2 - \mathfrak{n}_1^2 - \mathfrak{n}_2^2 - \mathfrak{n}_3^2}{2\sqrt{(1 - \mathfrak{n}_0\mathfrak{n}_1 - \mathfrak{n}_0\mathfrak{n}_2 - \mathfrak{n}_0\mathfrak{n}_3 - \mathfrak{n}_1\mathfrak{n}_2 - \mathfrak{n}_1\mathfrak{n}_3 - \mathfrak{n}_2\mathfrak{n}_3)^2 - 4\mathfrak{n}_0\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}}.
\end{aligned}
\tag{4.4.107}
$$

Using the solution of the conditions (4.4.100), namely

$$
\begin{aligned}
\beta_\Lambda =& \pm \frac{L}{4}\left(\frac{4(\mathfrak{n}_\Lambda - \frac{k}{2})^2 + 1 - (\mathfrak{n}_0^2 + \mathfrak{n}_1^2 + \mathfrak{n}_2^2 + \mathfrak{n}_3^2)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}\right), \quad k = \pm 1, \\
c =& \frac{k}{2} - \frac{\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_1\beta_2 + \beta_2 + \beta_3 + \beta_1\beta_3}{2L^2},
\end{aligned}
\tag{4.4.108}
$$

it is straightforward to verify that the values (4.4.106) of the scalars on the horizon are exactly those obtained from the solution (4.4.98), and hence, the corresponding value of the effective action coincides with the black hole entropy. Moreover, since we know this solution in closed form, we determine that the scalar VEVs that extremize the effective potential are given by (4.4.97), or equivalently

$$
\begin{aligned}
\langle \mathcal{O}^1_{\Delta=1} \rangle =& \pm \frac{L}{2}\left(\frac{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}\right), \\
\langle \mathcal{O}^2_{\Delta=1} \rangle =& \pm \frac{L}{2}\left(\frac{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}\right), \\
\langle \mathcal{O}^3_{\Delta=1} \rangle =& \pm \frac{L}{2}\left(\frac{(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}{\sqrt{(k - \mathfrak{n}_1 - \mathfrak{n}_2)(k - \mathfrak{n}_1 - \mathfrak{n}_3)(k - \mathfrak{n}_2 - \mathfrak{n}_3)}}\right),
\end{aligned}
\tag{4.4.109}
$$

with the overall signs correlated.

The above analysis demonstrates that the entropy functional for purely magnetic BPS solutions of the $U(1)^4$ theory should be identified with the quantum effective action for the dimension one scalar operators in the twisted ABJM model at Chern-Simons level one. In particular, the purely magnetic black holes correspond to zero chemical potentials for the currents $\mathcal{J}^i_\Lambda$. The value of the quantum effective action on its extremum coincides with the extremal black hole entropy, as well as the Witten index in the twisted ABJM model. In order to turn on the chemical potentials in the supersymmetric index discussed in [37, 10, 11] it is mandatory to consider dyonic black holes in the bulk – it is not possible for the chemical potentials to be non-zero for electrically neutral black holes. With non-zero electric chemical potentials μ^Λ, the supersymmetric index coincides with the quantum effective action for the dimension one operators in the twisted ABJM model at Chern-Simons level one, *deformed* by the supersymmetric fermion masses $\mathfrak{m}^\Lambda = \mu^\Lambda$. This effective action, given by (4.3.80), coincides with the black hole entropy at the extremal value of the scalar VEVs for the dimension one operators.

4.4.2 Dyonic BPS black holes

As we have just argued, the supersymmetric index discussed in [37, 10, 11], with arbitrary fugacities, should be matched to the quantum effective action (4.3.80), evaluated on the BPS swampland with non-zero chemical potentials. The corresponding dyonic BPS solution can be sought in the form of a UV expansion, analogous to (4.4.93) for the purely magnetic solutions. As we discussed at the beginning of this section, at fixed chemical potentials this solution depends on n_V arbitrary VEVs for the dimension one scalar operators.

The identity (4.3.84) implies that the BPS effective action is extremized with respect to the sources $\mathcal{Y}^\alpha_-$ of the dimension two operators, with the extremal values of these sources related to the electric chemical potentials as in (4.3.76). It follows that further extremizing the effective action with respect to the VEVs $\mathcal{X}^\alpha_-$ of the dimension one operators amounts to extremizing the effective action with respect to the complex modes $z^\alpha_- = \mathcal{X}^\alpha_- + i\mathcal{Y}^\alpha_-$, which appear at leading order in the Fefferman-Graham expansion of the scalars z^α, given in (A.13). The same argument as for the magnetic black holes and real scalars above then implies that extremizing the effective action with respect to $z^\alpha_- = \mathcal{X}^\alpha_- + i\mathcal{Y}^\alpha_-$ is equivalent to extremizing it with respect to the *complex* values z^α_h of the scalars on the horizon. The latter corresponds to the attractor mechanism [11] and this result, therefore, extends our holographic interpretation of the attractor mechanism as the extremization of the quantum effective action to general dyonic BPS solutions. The BPS black solutions corresponding to the extremum of the effective action in the dyonic case are those found in [101] and the corresponding entropy functional was discussed in [102]. These solutions can be used to obtain the values of the VEVs of the dimension one operators at the extremum of the effective action in terms of the magnetic and electric charges, but we will not compute these VEVs explicitly here.

4.5 Conclusion

One of our main results is a clarification of the holographic renormalization paradigm for asymptotically AdS_4 black holes in $\mathcal{N} = 2$ gauged supergravity. Along these lines and with the hope of providing a purely field theoretic interpretation for some of the assumptions made in the comparison with the microscopic entropy via topologically twisted index computations, we have found a boundary interpretation for the attractor mechanism.

Our conceptual home for the attractor mechanism in asymptotically AdS spacetimes that are solutions of $\mathcal{N} = 2$ gauged supergravity shows that it is equivalent, on the field theory side, to extremizing the quantum effective action with respect to certain VEVs. Our formulation of the mechanism retains some features of the original formulation in asymptotically flat spacetimes but exploits the inner workings of the AdS/CFT dictionary. For example, instead of extremizing with respect to moduli, as in the asymptotically flat case, we extremize with respect to VEVs in the asymptotically AdS case. Rather than extremizing the central charge that appears in the original formulation of attractor mechanism, we extremize the quantum effective action as follows from the entropy formula formalism.

We have resolved some conceptually challenging issues in the path toward the identification of the topologically twisted index and black hole entropy. In particular, we have clarified the nature of: (i) the field theoretic need for extremization and its connection with the attractor mechanism, (ii) the proper identification of scalar VEVs and the precise relations to the chemical potentials.

There are a number of open problems that would be interesting to tackle using the results we have obtained in this work. An obvious but technically challenging problem is to repeat our analysis for rotating asymptotically AdS_4 black holes and, more importantly, asymptotically AdS_5 ones. For rotating black holes it is much more difficult to obtain first order equations and to derive general expressions for the on-shell action, but AdS_5 black holes introduce additional complications of a completely different nature. In particular, supersymmetry on four dimensional curved backgrounds is generically anomalous [103, 90][10], which leads to anomalous contributions in the BPS relations among the conserved charges of supersymmetric AdS_5 black holes.

Moreover, having clarified the connection with the entropy formula, it would be quite interesting to extend our findings to include an interpretation of the quantum entropy formula [46]. Indeed, after a preliminary discussion in Chapter II and [66] focusing on the near horizon degrees of freedom, some quantum corrections to the black hole entropy have been matched using an approach that focuses on the asymptotic degrees of freedom in Chapter III.

[10]This anomaly was implicitly present in the analysis of [104] as well, but was not recognized as such.

CHAPTER V

Functional Determinants of Radial Operators in AdS_2

5.1 Introduction

There are many situations in the AdS/CFT correspondence where one ends up comptuting determinants in AdS_2 and its generalizations. The original discussion of the holographic dual to the $\frac{1}{2}$-BPS Wilson loop made used of AdS_2 determinants for the first time [21]. The list of one-loop effective action problems that can be tackled exploiting the fact that AdS_2 is a homogeneous space is rather large. For example, it naturally includes the one-loop effective actions of supersymmetric D3 and D5 branes dual to Wilson loops in $\mathcal{N} = 4$ SYM in the symmetric and anti-symmetric representations, respectively [105, 106]. Given that the worldvolume of these configurations are $AdS_2 \times S^2$ and $AdS_2 \times S^4$, the one-loop effective actions reduce also to determinants on AdS_2 [107, 108, 25]. A similar class of one-loop effective action appears also in the context of ABJM as shown in Chapter VIII. In the context of localization of supersymmetric field theories there have been some natural appearances of AdS_2 [109, 12, 110, 111]. Determinants of AdS_2 operators have also figured prominently in logarithmic corrections to the entropy of extremal black holes [8]. When the worldvolume geometry is not AdS_2 new methods need to be developed; we have discussed in fair detail the case of the $\frac{1}{4}$-BPS holographic Wilson loop in Chapter VI, using the results of the this chapter.

Motivated by the above richness of applications, in this chapter we discuss determinants of general Laplace and Dirac operators in asymptotically AdS_2 spacetimes. We use the regularization method chosen *par excellence* in curved spaces: ζ-function regularization. These methods have a long an fruitful history, dating back over four decades, starting with the pioneering works of [112, 113]; for a more complete list of references see [114]. Much of our exposition and results follows quite closely the vast literature in the subject of functional determinants which has a very solid branch anchored in the more mathematical tradition starting in [115]; for a more complete list of references see [116]. In the bulk of the chapter we make an effort to help the interested reader find the original versions of our arguments in the literature. We owe a particularly great debt to the work of Dunne and Kirsten [117]. This work could be simply described as an extension of theirs to the case of asymptotically AdS_2 spacetime rather than flat space.

The chapter is organized as follows. In section 5.2 we summarize the main results of our work, namely, we present ζ-function regularize of radial Laplace-like operators. In section 5.3 we present a number of explicit examples. The systematic derivation of our results is developed in section 5.4. We conclude in section 5.5 where we also point out some interesting directions that can be pursued in relations to the current work.

5.2 Main results and discussion

5.2.1 Preamble

Throughout this chapter we will work on the disk model of Euclidean AdS_2 (or $\mathbb{H}_2$) with metric

$$ds^2 = L^2 \left(d\rho^2 + \sinh^2 \rho \, d\tau^2\right) \,, \qquad \rho \geq 0 \,, \quad \tau \sim \tau + 2\pi \,. \tag{5.2.1}$$

For simplicity we set $L = 1$ but we will reinstate the radius in the final expressions. We are interested in Laplace and Dirac-type operators defined in the geometry (5.2.1) in the presence of additional background fields. Specifically, we consider operators of the form

$$\mathcal{O} = -g^{\mu\nu} D_\mu D_\nu + m^2 + V \,, \qquad \text{(bosons)} \tag{5.2.2}$$

$$\mathcal{O} = -i \left(\not{D} + \not{\partial}\Omega \right) - i\Gamma_{\underline{01}} (m + V) + W \,, \qquad \text{(fermions)} \tag{5.2.3}$$

where the covariant derivative $D_\mu = \nabla_\mu - iq\mathcal{A}_\mu$ includes a $U(1)$ gauge field. Here m and q are arbitrary mass and charge parameters, respectively. It should be clear from the outset that, even though we use the same notation, m, q, V and $\mathcal{A}_\mu$ need not be the same for bosons and fermions. In the latter case we have included an extra *connection*, $d\Omega$ (notice the absence of i, thus implying it cannot be gauged away), whose origin is motivated by thinking of these operators as coming from some other geometry that is conformal to AdS_2. We also clarify that W and V are not matrix-valued. Rather, they are scalar functions.

Our goal is to compute the ratio of determinants of the operators (5.2.2) and (5.2.3) with the corresponding free operators obtained by setting $\mathcal{A}_\mu = \Omega = V = W = 0$. For generic choices of the background fields, this is an extremely difficult task and can only be handled on a case by case basis. Considerable progress can be made, however, if one assumes circular symmetry. Consequently, we restrict ourselves to configurations where $\mathcal{A}_\rho = 0$ and $\mathcal{A}_\tau = \mathcal{A}(\rho)$, as well as $V = V(\rho)$, $W = W(\rho)$ and $\Omega = \Omega(\rho)$. The condition $\mathcal{A}_\rho = 0$ is actually a gauge choice, while the remaining assumptions imply circular symmetry.

A recurring notion in the following sections is the regularity of the eigenfunctions of the operators in question. Accordingly, the background fields must also be regular. Given the topology of AdS_2, this translates to

$$\mathcal{A}(\rho) \underset{\rho\to 0}{\longrightarrow} \rho^{1+\epsilon} \,, \qquad \partial_\rho \Omega(\rho) \underset{\rho\to 0}{\longrightarrow} \rho^{\epsilon} \,, \qquad \epsilon \geq 0 \,, \tag{5.2.4}$$

so that the 1-forms $\mathcal{A}(\rho) d\tau$ and $\partial_\rho \Omega(\rho) d\rho$ are well-defined at the origin. At infinity the gauge field and connection behave like

$$\mathcal{A}(\rho) \underset{\rho\to\infty}{\longrightarrow} \mathcal{A}_\infty \,, \qquad \partial_\rho \Omega(\rho) \underset{\rho\to\infty}{\longrightarrow} 0 \,. \tag{5.2.5}$$

On the other hand, the potentials are assumed to decay at least as

$$V(\rho) \underset{\rho\to\infty}{\longrightarrow} \frac{e^{-\rho}}{\rho^{2+\epsilon}} \qquad W(\rho) \underset{\rho\to\infty}{\longrightarrow} \frac{e^{-\frac{\rho}{2}}}{\rho^{1+\epsilon}}\,. \tag{5.2.6}$$

Simply put, the background fields must behave in such a way that all the integrals appearing below are finite. These fall-off conditions imply that the operators become effectively free for large ρ,

$$\mathcal{O} \underset{\rho\to\infty}{\longrightarrow} \mathcal{O}^{\text{free}}\,, \tag{5.2.7}$$

except for the presence of a constant gauge field, which does not affect in any substantial way the validity of the results.

The spectral problem at hand is intrinsically two-dimensional but the assumption of circular symmetry reduces it to a one-dimensional calculation. Upon Fourier-transforming the τ dependence the relevant radial operators become

$$\mathcal{O}_l = -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{(l-q\mathcal{A})^2}{\sinh^2\rho} + m^2 + V\,, l\in\mathbb{Z}\,, \text{(bosons)} \tag{5.2.8}$$

$$\mathcal{O}_l = -i\Gamma_{\underline{1}}\left(\partial_\rho + \frac{1}{2}\coth\rho + \partial_\rho\Omega\right) + \Gamma_{\underline{0}}\frac{(l-q\mathcal{A})}{\sinh\rho} - i\Gamma_{\underline{01}}\left(m+V\right) + W\,, l\in\mathbb{Z}+\frac{1}{2}\,.\text{(fermions)} \tag{5.2.9}$$

As a first attempt to reconstruct the full determinant one could write

$$\ln\frac{\det\mathcal{O}}{\det\mathcal{O}^{\text{free}}} \overset{?}{=} \sum_{l=-\infty}^{\infty}\ln\frac{\det\mathcal{O}_l}{\det\mathcal{O}_l^{\text{free}}}\,. \tag{5.2.10}$$

The trouble with this expression, however, is that, even though the ratio $\frac{\det\mathcal{O}_l}{\det\mathcal{O}_l^{\text{free}}}$ is well defined, the sum over Fourier modes typically diverges. To give it meaning one could, for example, regulate the sum by imposing a sharp cutoff at $|l|=\Lambda$ and subtract the divergent pieces. In some contexts, an underlying symmetry might even cancel the divergences altogether. A cutoff regularization, however, might conflict with symmetries of curved spaces, in particular diffeomorphism invariance, rendering this approach not entirely satisfying. A more geometric approach is desirable.

One would like to insist on the idea of reconstructing the two-dimensional determinants as a product over one-dimensional ones, since the latter are relatively easy to compute. The purpose of this work is to provide a regularization scheme that coincides with the two-dimensional ζ-function formalism, that is,

$$\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} \equiv -\hat{\zeta}'_{\mathcal{O}}(0) - \ln(\mu^2)\hat{\zeta}_{\mathcal{O}}(0)\,, \qquad \hat{\zeta}_{\mathcal{O}}(s) \equiv \zeta_{\mathcal{O}}(s) - \zeta_{\text{free}}(s)\,, \tag{5.2.11}$$

where μ is a mass scale that parametrizes the ambiguity in the renormalization of the determinant. The same definitions apply to the radial operators $\mathcal{O}_l$, although the renormalization scale is absent in one dimension. For fermions, we define the determinant and ζ-function of the first order operator in terms of the squared one as

$$\det \mathcal{O} \equiv \left(\det \mathcal{O}^2\right)^{\frac{1}{2}}\,, \qquad \zeta_{\mathcal{O}}(s) \equiv \frac{1}{2}\zeta_{\mathcal{O}^2}(s)\,. \tag{5.2.12}$$

In this context, the correct version of (5.2.10) is

$$\zeta_{\mathcal{O}}(s) = \sum_{l=-\infty}^{\infty} \zeta_{\mathcal{O}_l}(s)\,. \tag{5.2.13}$$

This relation is as usual generically not well-defined in the entire complex s-plane, only for large enough $\operatorname{Re} s$. The problem in the present work then boils down to finding the analytic continuation to $s = 0$ of the whole sum and not each individual term separately.

5.2.2 Results

Concerning the bosonic case, our main result is

$$\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= \ln \frac{\det \mathcal{O}_0}{\det \mathcal{O}_0^{\text{free}}} + \sum_{l=1}^{\infty} \left(\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} + \ln \frac{\det \mathcal{O}_{-l}}{\det \mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l}\hat{\zeta}_{\mathcal{O}}(0) \right) - 2\left(\ln(\mu L) + \gamma\right)\hat{\zeta}_{\mathcal{O}}(0) \\
&\quad + \int_0^\infty d\rho\, \sinh\rho \ln\left(\frac{\sinh\rho}{2}\right) V - q^2 \int_0^\infty d\rho\, \frac{\mathcal{A}^2}{\sinh\rho} \\
\hat{\zeta}_{\mathcal{O}}(0) &= -\frac{1}{2}\int_0^\infty d\rho\, \sinh\rho\, V\,,
\end{aligned}$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In turn, the ratio of radial determinants for each Fourier mode can be computed as

$$\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} = \lim_{\rho\to\infty} \ln \frac{\psi_l(\rho)}{\psi_l^{\text{free}}(\rho)}\,, \tag{5.2.14}$$

where $\psi_l(\rho)$ is the solution to the homogeneous equation for $\mathcal{O}_l$ that is regular at $\rho = 0$,

$$\mathcal{O}_l\psi_l = 0\,, \qquad \psi_l(\rho) \underset{\rho\to 0}{\longrightarrow} \rho^{|l|}\,. \tag{5.2.15}$$

The normalization is chosen so that the leading coefficient in the small ρ expansion matches that of the free solution appearing in the denominator[1] of (5.2.14).

Similarly, for fermionic operators we get

$$\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= \sum_{l=\frac{1}{2}}^{\infty}\left(\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} + \ln \frac{\det \mathcal{O}_{-l}}{\det \mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l+\frac{1}{2}}\hat{\zeta}_{\mathcal{O}}(0)\right) - 2\left(\ln\left(\mu L\right)+\gamma\right)\hat{\zeta}_{\mathcal{O}}(0)\\
&+\int_0^\infty d\rho\, \sinh\rho \ln\left(\frac{\sinh\rho}{2}\right)\left((m+V)^2 - W^2 - m^2\right) - q^2\int_0^\infty d\rho\, \frac{\mathcal{A}^2}{\sinh\rho}\\
&-\int_0^\infty d\rho\, \sinh\rho\, W^2\,,\\
\hat{\zeta}_{\mathcal{O}}(0) &= -\frac{1}{2}\int_0^\infty d\rho\, \sinh\rho\left((m+V)^2 - W^2 - m^2\right)\,,
\end{aligned}$$

where

$$\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} = \lim_{\rho\to\infty}\left(\ln \frac{\psi_l^{(i)}(\rho)}{\psi_l^{(i)\,\text{free}}(\rho)} + \Omega(\rho) - \Omega(0)\right)\,. \tag{5.2.16}$$

Here $\psi_l^{(i)}(\rho)$ is any of the two components of the regular spinor solution to the *first order* homogeneous equation,

$$\mathcal{O}_l\psi_l = 0\,, \qquad \psi_l(\rho) \underset{\rho\to 0}{\longrightarrow} \rho^{|l|-\frac{1}{2}}\,. \tag{5.2.17}$$

The small ρ behavior is displayed only for the leading component[2]. As for bosons, this

[1] This is completely analogous to the usual initial conditions $\psi(0) = 0$, $\psi'(0) = 1$ imposed on the homogeneous functions appearing in the Gelfand-Yaglom method. In two and higher dimensions, however, the centrifugal barrier implies that the regular solution actually vanishes as a power law depending on the Fourier mode, so $\psi'(0) = 1$ must be generalized.

[2] The other component goes as $\rho^{|l|+\frac{1}{2}}$ with a coefficient that depends on the behavior of the

component should be normalized so that its behavior at the origin coincides with that of the free solution to be inserted in (5.2.16). We stress that any of the two components can be used in (5.2.16).

A few comments are in order. Our results are simple generalizations of those in flat space [117]; mainly replace $\rho \to \sinh\rho$ for the radial dependence and $\rho\, d\rho \to \sinh\rho\, d\rho$ in the integration measure. This is related to the fact that, by construction, zeta-function regularization is diffeomorphism invariant, even though expressions (5.2.14) and (5.2.16) are written in a particular coordinate system. Also, it is reassuring to check that $\hat{\zeta}_{\mathcal{O}}(0)$ coincides with the general formula in terms of the Seeley coefficient [21, 47] (see also appendix C)

$$\hat{\zeta}_{\mathcal{O}}(0) = a_2(1|\mathcal{O}) - a_2(1|\mathcal{O}^{\text{free}})\,. \tag{5.2.18}$$

Another important point is that in an infinite space such as AdS_2 there is actually no freedom in choosing the boundary conditions once one imposes that the eigenfunctions are regular everywhere. An intermediate step in the derivation (5.2.14) and (5.2.16) involves putting the system in a finite box of radius R where boundary conditions are indeed relevant. However, the $R \to \infty$ limit eliminates all traces of these.

As one would expect from circular symmetry, the two-dimensional determinants can be written as a sum of one-dimensional radial determinants. It is important to emphasize, however, that all results are finite and do not require further regularization. It is still useful to compare with the momentum cut-off prescription widely used in context of holographic Wilson loops [23][26][27][118]. To that end, we notice that the sums over Fourier modes in (5.2.14) and (5.2.16) are rendered finite by the presence of the term $\frac{1}{l}\hat{\zeta}_{\mathcal{O}}(0)$,

$$\hat{\zeta}_{\mathcal{O}}(0)\sum_{l=1}^{\Lambda}\frac{1}{l} = \hat{\zeta}(0)\,(\ln\Lambda + \gamma) + O(\Lambda^{-1})\,, \tag{5.2.19}$$

which cancels a $\ln\Lambda$ divergence in (5.2.10). It was not obvious a priori that the correct coefficient was $\hat{\zeta}_{\mathcal{O}}(0)$. In the fermionic case, it is also crucial to include the Ω term in (5.2.16)

potentials at the origin (see (5.4.128)).

so that the sum is free of linear divergences. In retrospect, this justifies the rescaling of the boundary conditions done in [27]. Finally, zeta-function regularization systematically fixes all the finite terms in (5.2.14) and (5.2.16) that depend explicitly on the background fields, which a cut-off method could not possibly foresee.

5.2.3 Conformal AdS_2 spaces

A simple generalization of the methods presented here include functional determinants defined on spaces that are conformally equivalent to AdS_2, namely,

$$ds^2_M = M ds^2 \,, \tag{5.2.20}$$

where the conformal factor M is smooth everywhere so as to not change the topology[3]. The Laplace and Dirac operators in the two geometries, are related by using

$$e^{\underline{a}}_{_M} = \sqrt{M} e^{\underline{a}}, \qquad w^{\underline{ab}}_{_M} = w^{\underline{ab}} - \frac{1}{2M}\left(\partial^{\underline{a}} M e^{\underline{b}} - \partial^{\underline{b}} M e^{\underline{a}}\right) \,, \tag{5.2.21}$$

where $\partial^{\underline{a}} M = e^{\underline{a}\mu}\partial_\mu M$ and $e^{\underline{a}}_{\mu} e^{\mu}_{\underline{b}} = \delta^{\underline{a}}_{\underline{b}}$. Some Dirac matrix algebra then shows

$$\nabla^2_{_M} = \frac{1}{M}\nabla^2, \qquad \nabla\!\!\!\!/_{_M} = \frac{1}{\sqrt{M}}\left(\nabla\!\!\!\!/ + \frac{\partial\!\!\!/ M}{4M}\right) \,. \tag{5.2.22}$$

This leads us to consider more general operators of the form

$$\mathcal{O}_M = M^{-1}\mathcal{O} \,, \qquad \mathcal{O} = -g^{\mu\nu} D_\mu D_\nu + m^2 + V \,, \qquad \text{(bosons)} \tag{5.2.23}$$

$$\mathcal{O}_M = M^{-\frac{1}{2}}\mathcal{O} \,, \qquad \mathcal{O} = -i\left(D\!\!\!\!/ + \partial\!\!\!/\Omega\right) - i\Gamma_{\underline{01}}\left(m + V\right) + W \,, \qquad \text{(fermions)} \tag{5.2.24}$$

where $\mathcal{O}$ is defined in the AdS_2 geometry as before. Notice that any potential terms originally appearing in $\mathcal{O}_M = -D^2_M + \cdots$ or $\mathcal{O}_M = -iD\!\!\!\!/_M + \cdots$ will need to be rescaled by M or $M^{\frac{1}{2}}$ in order to write them in this fashion. In the fermionic case there is an additional con-

[3]Of course, any two-dimensional geometry is conformally equivalent to any other two-dimensional geometry. This is, however, a local statement. The emphasis here is that the conformal factor does not blow up anywhere so the topology is still that of a disk.

tribution $\frac{1}{4}\not{\partial}\ln M$ coming from the spin connection in (5.2.22), which we have absorbed in $\not{\partial}\Omega$. As before we assume that the conformal factor depends only on the radial coordinate; circular symmetry would otherwise be lost. The gauge field is unaffected by the rescaling.

The determinants of $\mathcal{O}_M$ and $\mathcal{O}$ are connected by the standard Weyl anomaly calculation (see appendix C). Taking the ratio with the free operator on AdS_2 we find

$$\ln\left(\frac{\det\mathcal{O}_M}{\det\mathcal{O}^{\text{free}}}\right) = \ln\left(\frac{\det\mathcal{O}}{\det\mathcal{O}^{\text{free}}}\right) + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[m^2 + V - \frac{1}{6}R + \frac{1}{12}\nabla^2\ln M\right] \tag{5.2.25}$$

for bosons, while for fermions the anomaly reads

$$\ln\left(\frac{\det\mathcal{O}_M}{\det\mathcal{O}^{\text{free}}}\right) = \ln\left(\frac{\det\mathcal{O}}{\det\mathcal{O}^{\text{free}}}\right) + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right]. \tag{5.2.26}$$

In each expression the first term on the right hand side can be computed using the results of the previous section. The second term accounts for the rescaling. We have assumed that $M \to 1$ as $\rho \to \infty$ so the space is asymptotically AdS_2, which explains the absence of boundary terms.

5.3 Examples

In this section we apply the methods developed here to two examples borrowed from the literature on holographic Wilson loops [119, 26, 27]. See Chapter VI.

5.3.1 Bosons

For the bosonic case we take

$$\mathcal{O}_M = M^{-1}\mathcal{O}\,, \qquad \mathcal{O} = -g^{\mu\nu}D_\mu D_\nu + V\,, \qquad D_\mu = \nabla_\mu + i\mathcal{A}_\mu\,, \tag{5.3.27}$$

with

$$M(\rho) = 1 + \frac{\sin^2\theta(\rho)}{\sinh^2\rho}\,, \qquad \mathcal{A}(\rho) = 1 - \frac{1+\cosh\rho\cos\theta(\rho)}{\cosh\rho + \cos\theta(\rho)}\,, \qquad V(\rho) = -\frac{\partial_\rho \mathcal{A}(\rho)}{\sinh\rho}\,. \quad (5.3.28)$$

The function $\theta(\rho)$ is given by

$$\sin\theta(\rho) = \frac{\sinh\rho\sin\theta_0}{\cosh\rho + \cos\theta_0}\,, \qquad (5.3.29)$$

where $0 \leq \theta_0 \leq \frac{\pi}{2}$ is a parameter. The free operator corresponds to

$$\mathcal{O}^{\text{free}} \equiv \mathcal{O}\big|_{\theta_0=0} = \mathcal{O}_M\big|_{\theta_0=0} = -\nabla^2\,. \qquad (5.3.30)$$

Let us use our result (5.2.14) to compute the ratio of determinants between $\mathcal{O}$ and $\mathcal{O}^{\text{free}}$. We will include the effect of the Weyl anomaly in (5.2.25) at the end. First,

$$\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho\, \sinh\rho\, V = \sin^2\frac{\theta_0}{2}\,. \qquad (5.3.31)$$

Similarly,

$$\int_0^\infty d\rho\, \sinh\rho\, \ln\left(\frac{\sinh\rho}{2}\right) V = -\frac{1}{2}\theta_0\sin\theta_0 + \cos\theta_0 \ln\cos\frac{\theta_0}{2}\,, \qquad (5.3.32)$$

and

$$\int_0^\infty d\rho\, \frac{\mathcal{A}^2}{\sinh\rho} = -\sin^2\frac{\theta_0}{2} - 2\ln\cos\frac{\theta_0}{2}\,. \qquad (5.3.33)$$

Next, notice that the general solution to the differential equation

$$\mathcal{O}_l\psi_l = 0\,, \qquad \mathcal{O}_l = -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{(l+\mathcal{A})^2}{\sinh^2\rho} - \frac{\partial_\rho\mathcal{A}}{\sinh\rho}\,, \qquad l \in \mathbb{Z}\,, \qquad (5.3.34)$$

is

$$\psi_l(\rho) = \left(\tanh\frac{\rho}{2}\right)^{-l} e^{-\mathcal{W}(\rho)} \left(C_1 + C_2 \int d\rho \left(\tanh\frac{\rho}{2}\right)^{2l} \frac{e^{2\mathcal{W}(\rho)}}{\sinh\rho}\right), \qquad \partial_\rho \mathcal{W}(\rho) = \frac{\mathcal{A}(\rho)}{\sinh\rho}\,. \tag{5.3.35}$$

Since $\mathcal{W}(\rho)$ is finite at $\rho = 0$, we see that for $l < 0$ the regular solution corresponds to $C_2 = 0$, whereas for $l > 0$ we must set $C_1 = 0$. Making sure that the normalization is the same as for the free solution we find

$$\psi_l(\rho) = \begin{cases} \dfrac{\cos\frac{\theta_0}{2}\left(2\tanh\frac{\rho}{2}\right)^{-l}(\cosh\rho+1)}{\sqrt{\cosh^2\rho + 2\cosh\rho\cos\theta_0 + 1}} & l \le 0 \\[2ex] \dfrac{\left(2\tanh\frac{\rho}{2}\right)^{l}\sqrt{\cosh^2\rho + 2\cosh\rho\cos\theta_0 + 1}}{(l+2)\cos\frac{\theta_0}{2}(\cosh\rho+1)}\left(l + \dfrac{2(\cosh\rho+1)^2\cos^2\frac{\theta_0}{2}}{\cosh^2\rho + 2\cosh\rho\cos\theta_0 + 1}\right) & l \ge 0 \end{cases}\,. \tag{5.3.36}$$

Thus,

$$\ln\frac{\det\mathcal{O}_l}{\det\mathcal{O}_l^{\text{free}}} = \begin{cases} \ln\cos\dfrac{\theta_0}{2} & l \le 0 \\[2ex] -\ln\cos\dfrac{\theta_0}{2} + \ln\left(\dfrac{l + 2\cos^2\frac{\theta_0}{2}}{l+2}\right) & l \ge 0 \end{cases}\,. \tag{5.3.37}$$

Happily, the sum over Fourier modes can be computed in closed form. Indeed,

$$\begin{aligned} \sum_{l=1}^{\infty}\left(\ln\frac{\det\mathcal{O}_l}{\det\mathcal{O}_l^{\text{free}}} + \ln\frac{\det\mathcal{O}_{-l}}{\det\mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l}\hat{\zeta}_{\mathcal{O}}(0)\right) &= \sum_{l=1}^{\infty}\left(\ln\left(\frac{l+2\cos^2\frac{\theta_0}{2}}{l+2}\right) + \frac{2}{l}\sin^2\frac{\theta_0}{2}\right) \\ &= -\ln\Gamma\left(2\cos^2\frac{\theta_0}{2}\right) - 2\ln\cos\frac{\theta_0}{2} + 2\gamma\sin^2\frac{\theta_0}{2}\,. \end{aligned}$$

Notice that were it not for the $\hat{\zeta}_{\mathcal{O}}(0)$-term, the sum would have been divergent, which is precisely the situation faced in [23, 26, 27]. Putting everything together we arrive at

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= -\ln \Gamma\left(2\cos^2\frac{\theta_0}{2}\right) + 2\cos^2\frac{\theta_0}{2}\ln\cos\frac{\theta_0}{2} + \sin^2\frac{\theta_0}{2} - \frac{1}{2}\theta_0\sin\theta_0 \\
&= -\frac{\gamma}{2}\theta_0^2 + \left(\frac{19}{96} + \frac{\gamma}{24} - \frac{\pi^2}{48}\right)\theta_0^4 + O\left(\theta_0^6\right) ,
\end{aligned} \tag{5.3.38}
$$

where we have set $\mu = 1$ for simplicity. Finally, we compute the Weyl anomaly relating the determinants of $\mathcal{O}_M$ and $\mathcal{O}$. It reads

$$
\begin{aligned}
&\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[V - \frac{1}{6}R + \frac{1}{12}\nabla^2\ln M\right] \\
&= \left(\frac{1}{3} + 2\cos^2\frac{\theta_0}{2}\right)\ln\cos\frac{\theta_0}{2} - \frac{1}{2}\sin^2\frac{\theta_0}{2} + \frac{1}{2}\theta_0\sin\theta_0 \,.
\end{aligned} \tag{5.3.39}
$$

Combining this with the previous expression we find

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}_M}{\det \mathcal{O}^{\text{free}}} &= -\ln \Gamma\left(2\cos^2\frac{\theta_0}{2}\right) + \left(\frac{1}{3} + 4\cos^2\frac{\theta_0}{2}\right)\ln\cos\frac{\theta_0}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2} \\
&= \left(\frac{1}{12} - \frac{\gamma}{2}\right)\theta_0^2 + \left(\frac{101}{576} + \frac{\gamma}{24} - \frac{\pi^2}{48}\right)\theta_0^4 + O\left(\theta_0^6\right) .
\end{aligned} \tag{5.3.40}
$$

The reason we have expanded our results for small θ_0 is to compare them against the perturbative technique developed in [28]. While we spare the details of the calculation, we confirm that the leading terms in (5.3.38) and (5.3.40) are in fact reproduced, independently, by this method. It would be interesting to extend the perturbative method to next order in the expansion parameter and check that it also reproduces the $O\left(\theta_0^4\right)$ terms.

5.3.2 Fermions

As a fermionic example we consider the operator

$$
\mathcal{O}_M = M^{-\frac{1}{2}}\mathcal{O}\,, \qquad \mathcal{O} = -i\left(\not{D} + \frac{1}{4}\not{\partial}\ln M\right) - i\Gamma_{\underline{01}}\left(1+V\right) + W\,, \qquad D_\mu = \nabla_\mu + \frac{i}{2}\mathcal{A}_\mu\,,
\tag{5.3.41}
$$

where $M(\rho)$ and $\mathcal{A}(\rho)$ are the same as before and

$$V(\rho) = \frac{1}{\sqrt{M(\rho)}} - 1\,, \qquad W(\rho) = \frac{\sin^2\theta(\rho)}{\sqrt{M(\rho)}\sinh^2\rho}\,. \tag{5.3.42}$$

The free operator reads

$$\mathcal{O}^{\text{free}} = \mathcal{O}\big|_{\theta_0=0} = \mathcal{O}_M\big|_{\theta_0=0} = -\nabla\!\!\!/ - i\Gamma_{\underline{01}}\,. \tag{5.3.43}$$

This time the relevant formulas are (5.2.16) and (5.2.26). We find

$$\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho\,\sinh\rho\left((m+V)^2 - m^2 - W^2\right) \tag{5.3.44}$$

$$= \sin^2\frac{\theta_0}{2}\,, \tag{5.3.45}$$

as well as

$$\int_0^\infty d\rho\,\sinh\rho\ln\left(\frac{\sinh\rho}{2}\right)\left((m+V)^2 - W^2 - m^2\right) = 2\cos\theta_0\ln\cos\frac{\theta_0}{2}\,, \tag{5.3.46}$$

together with

$$\int_0^\infty d\rho\,\sinh\rho\,W^2 = 2\sin^2\frac{\theta_0}{2} - \frac{1}{2}\theta_0\sin\theta_0\,, \tag{5.3.47}$$

and

$$\lim_{\rho\to\infty}\left(\Omega(\rho) - \Omega(0)\right) = \lim_{\rho\to\infty}\frac{1}{4}\ln\left(\frac{M(\rho)}{M(0)}\right) = \frac{1}{2}\ln\cos\frac{\theta_0}{2} \tag{5.3.48}$$

The integral involving the gauge field is the same as in the bosonic example. Solving the differential equation, however, is more involved in this case given the spinor structure of the fields. The radial problem is

$$\mathcal{O}_l\psi_l = 0\,, \qquad \mathcal{O}_l = -i\sigma_1\left(\partial_\rho + \frac{1}{2}\coth\rho + \frac{1}{4}\partial_\rho\ln M\right) - \frac{1}{\sinh\rho}\sigma_2\left(l + \frac{1}{2}\mathcal{A}\right) + \sigma_3\left(1+V\right) + W\,, \tag{5.3.49}$$

with $l \in \mathbb{Z} + \frac{1}{2}$. Letting

$$\psi_l(\rho) = \begin{pmatrix} u_l(\rho) \\ v_l(\rho) \end{pmatrix} , \tag{5.3.50}$$

we can solve algebraically for $u_l(\rho)$ to find[4]

$$-\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho v_l(\rho)\right) + \frac{(l+\mathcal{B})^2}{\sinh^2\rho}v_l(\rho) - \frac{\partial_\rho\mathcal{B}}{\sinh\rho}v_l(\rho) = 0\,, \tag{5.3.51}$$

where

$$\mathcal{B} = \frac{1}{2}\mathcal{A} - \sinh\rho\left(\frac{1}{2}\coth\rho + \frac{1}{4}\partial_\rho\ln M\right)\,. \tag{5.3.52}$$

Equation (5.3.51) has the same form as its bosonic counterpart (5.3.34), but we write its general solution slightly differently,

$$v_l(\rho) = \left(\tanh\frac{\rho}{2}\right)^{-l+\frac{1}{2}} e^{-\mathcal{W}(\rho)}\left(C_1 + C_2\int d\rho\,\left(\tanh\frac{\rho}{2}\right)^{2l-1}\frac{e^{2\mathcal{W}(\rho)}}{\sinh\rho}\right)\,, \qquad \partial_\rho\mathcal{W}(\rho) = \frac{\mathcal{B}(\rho)+\frac{1}{2}}{\sinh\rho}\,. \tag{5.3.53}$$

When defined in this way, the prepotential $\mathcal{W}$ is finite at $\rho = 0$, making the analysis simpler. We then get

$$\begin{aligned} u_l^{(-)}(\rho) &= \frac{\left(2\tanh\frac{\rho}{2}\right)^{-l-\frac{1}{2}}}{\left(l-\frac{1}{2}\right)}\sqrt{\frac{2\left(\cosh\rho+\cos\theta_0\right)}{\cosh^2\rho+2\cosh\rho\cos\theta_0+1}}\left(l+\frac{1}{2}-\frac{\cosh^2\rho+2\cosh\rho\cos\theta_0+1}{2\left(\cosh\rho+\cos\theta_0\right)}\right)\,, \\ v_l^{(-)}(\rho) &= \frac{i\left(2\tanh\frac{\rho}{2}\right)^{-l-\frac{1}{2}}\sinh\rho}{2\left(l-\frac{1}{2}\right)}\sqrt{\frac{2}{\cosh\rho+\cos\theta_0}}\,, \end{aligned} \tag{5.3.54}$$

[4]Notice that $M = (1+V+W)^2$ which considerably simplifies the calculations.

for $l \leq -\frac{1}{2}$, and

$$u_l^{(+)}(\rho) = \frac{i\left(2\tanh\frac{\rho}{2}\right)^{l+\frac{1}{2}}}{2\cos\frac{\theta_0}{2}\sqrt{\left(\cos\theta_0+\cosh\rho\right)\left(1+2\cos\theta_0\cosh\rho+\cosh^2\rho\right)}(2l+1)(2l+3)}$$
$$\times\left(2\cos\theta_0+(2l+1)\cos\theta_0+\cosh\rho\left(2l+1+2\cos\theta_0\right)\left(2\cos\theta_0+\cosh\rho\right)\right)\,,$$

(5.3.55)

$$v_l^{(+)}(\rho) = \frac{\left(2\tanh\frac{\rho}{2}\right)^{l+\frac{1}{2}}(2l-1)}{\cos\frac{\theta_0}{2}\sinh\rho\sqrt{2\left(\cos\theta_0+\cosh\rho\right)(2l+1)}}$$
$$\times\left(\cos\theta_0+\frac{\left(2l+1+\cos\theta_0\right)\left(1+(2l+1)\cosh\rho+\cosh^2\rho\right)}{(2l-1)(2l+3)}\right)\,,$$

(5.3.56)

for $l \geq \frac{1}{2}$. The overall normalization constants have been chosen so that the behavior at the origin coincides with (5.4.128) for $l \geq \frac{1}{2}$ and (5.4.129) for $l \leq -\frac{1}{2}$.

Expanding for $\rho \to \infty$ and making the quotient with the free solutions we can compute the sum over Fourier modes, which yields

$$\sum_{l=\frac{1}{2}}^{\infty}\left(\ln\frac{\det\mathcal{O}_l}{\det\mathcal{O}_l^{\text{free}}}+\ln\frac{\det\mathcal{O}_{-l}}{\det\mathcal{O}_{-l}^{\text{free}}}+\frac{2}{l+\frac{1}{2}}\hat{\zeta}_{\mathcal{O}}(0)\right) = \sum_{l=\frac{1}{2}}^{\infty}\left(\ln\left(\frac{l+\frac{1}{2}+\cos\theta_0}{l+\frac{3}{2}}\right)+\frac{2}{l+\frac{1}{2}}\sin^2\frac{\theta_0}{2}\right)$$
$$= -\ln\Gamma\left(2\cos^2\frac{\theta_0}{2}\right)+2\gamma\sin^2\frac{\theta_0}{2}\,.$$

(5.3.57)

Note that the sum is rendered finite due to the presence of both the $\hat{\zeta}_{\mathcal{O}}(0)$ and the $(\Omega(\infty)-\Omega(0))$ terms.

Collecting all the pieces, we finally obtain

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= -\ln \Gamma\left(2\cos^2 \frac{\theta_0}{2}\right) + \left(\frac{1}{2} + 2\cos\theta_0\right) \ln\cos\frac{\theta_0}{2} - \frac{7}{4}\sin^2\frac{\theta_0}{2} + \frac{\theta_0}{2}\sin\theta_0 \\
&= \frac{1}{2}\left(\frac{1}{2} - \gamma\right)\theta_0^2 + \frac{1}{384}\left(57 + 16\gamma - 8\pi^2\right)\theta_0^4 + O\left(\theta_0^6\right) , \qquad (5.3.58)
\end{aligned}
$$

where we have set $\mu = 1$ for simplicity. In order to obtain the determinant of $\mathcal{O}_M(\theta_0)$, we still have to compute the Weyl anomaly contribution, which in this case reads

$$
\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2 \ln M\right] = \frac{7}{4}\sin^2\frac{\theta_0}{2} + \frac{11}{6}\ln\cos\frac{\theta_0}{2} \qquad (5.3.59)
$$

thus arriving to the following expression

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}_M}{\det \mathcal{O}^{\text{free}}} &= -\ln \Gamma\left(2\cos^2 \frac{\theta_0}{2}\right) + 2\cos\theta_0 \ln\cos\frac{\theta_0}{2} + \frac{7}{3}\ln\cos\frac{\theta_0}{2} + \frac{\theta_0}{2}\sin\theta_0 \\
&= \frac{1}{2}\left(\frac{11}{12} - \gamma\right)\theta_0^2 + \frac{1}{576}\left(59 + 24\gamma - 12\pi^2\right)\theta_0^4 + O\left(\theta_0^6\right) . \qquad (5.3.60)
\end{aligned}
$$

Note the first term is in perfect agreement with the perturbative result reported in [28]. As in the bosonic case, it would be interesting to check the next order in (5.3.60) by extending the perturbative analysis proposed in [28] up to $O\left(\theta_0^4\right)$.

5.4 Derivation

Having discussed the results of the chapter and some simple examples, in this section we provide a detailed derivation of equations (5.2.14) and (5.2.16). The procedure essentially mimics the approach taken for flat space in [117]. For the treatment of fermionic determinants we follow [120, 121]. We point the reader to these references for any omitted details, although we do try to make the discussion self-contained. See also [122, 117, 121, 114, 123].

The main goal is to find the analytic continuation of expression (5.2.13) to $s = 0$. This is achieved in three steps: i) finding a useful integral representation of the radial zeta functions using scattering data; ii) give meaning to the sum over Fourier modes when evaluated at $s = 0$ by an appropriate subtraction; iii) analytically continue the subtracted terms via

Riemann zeta-function.

Before we proceed, a brief comment on notation. It is customary to parametrize the eigenvalues of the AdS_2 operators (5.2.2) and (5.2.3) by

$$\lambda(\nu) = \nu^2 + \nu_0^2\,, \qquad \nu_0 = \sqrt{\frac{1}{4} + m^2}\,, \qquad \text{(bosons)} \tag{5.4.61}$$

$$\lambda(\nu) = \pm\sqrt{\nu^2 + \nu_0^2}\,, \qquad \nu_0 = m\,, \qquad \text{(fermions)} \tag{5.4.62}$$

and we adhere to this notation through the rest of the chapter. As will become clear below, the variable ν has the interpretation of a radial momentum.

5.4.1 ζ-function as a contour integral

Consider the bosonic operator (5.2.2). We assume it to be Hermitian and positive definite. Suppose for the moment that the eigenvalues are discrete. This can be achieved by putting the system in a finite spherical box of radius R and eventually taking $R \to \infty$. For simplicity, we exclude the possibility of zero modes. The spectrum then consists of a finite number of (bound) states with $0 < \lambda < \nu_0^2$ and an infinite number of (scattering) states with $\lambda > \nu_0^2$. The zeta-function is symbolically defined as

$$\zeta_{\mathcal{O}}(s) \equiv \sum_n \lambda_n^{-s}\,, \tag{5.4.63}$$

where n runs over the full spectrum. Although obviously not valid at $s = 0$, this representation of $\zeta_{\mathcal{O}}(s)$ does have meaning in regions of the complex s-plane where the sum converges[5], and motivates the definition (5.2.11) of the regularized determinant[6]. However, in order to compute the quantities $\zeta_{\mathcal{O}}(0)$ and $\zeta'_{\mathcal{O}}(0)$ one must first analitically continue the sum to an expression that is well-defined at the origin. Precisely, the main objective in this section is to provide the details of the continuation procedure for operators in AdS_2 displaying circular symmetry. Under these conditions the spectral problem is separable and

[5] If $\lambda_n \sim n^k$, $k > 0$ for $n \to \infty$, then $\mathrm{Re}\, s > \frac{1}{k}$.

[6] The mass scale μ appears because of the rescaling $\lambda \to \mu^2\lambda$ needed to make the eigenvalues dimensionless in (5.4.63).

the zeta-function can always be written as

$$\zeta_{\mathcal{O}}(s) = \sum_{l \in \mathbb{Z}} \zeta_{\mathcal{O}_l}(s)\,, \qquad \zeta_{\mathcal{O}_l}(s) \equiv \sum_i \lambda_{(l,i)}^{-s}\,, \tag{5.4.64}$$

where i labels the eigenvalues of the radial operators $\mathcal{O}_l$ given in (5.2.8). In general, it is not enough to simply continue $\zeta_{\mathcal{O}_l}(s)$ to $s = 0$ and then sum over Fourier modes since the resulting series will be divergent.

The first step is to find a more suitable representation of the zeta-function. This can be done by trading the sum over i in (5.4.64) for a contour integral via the residue theorem. In terms of the momentum ν introduced in (5.4.61), the zeta-function for the radial operators can be written as [124]

$$\zeta_{\mathcal{O}_l}(s) = \oint_\gamma \frac{d\nu}{2\pi i} \left(\nu^2 + \nu_0^2\right)^{-s} \partial_\nu \ln f_l(\nu)\,, \tag{5.4.65}$$

where $f_l(\nu)$ is a holomorphic function that has simple zeros at the location of the eigenvalues $\lambda_{(l,i)} = \nu_{(l,i)}^2 + \nu_0^2$ and γ is a path enclosing them all (see figure 5.1). The logarithm is there

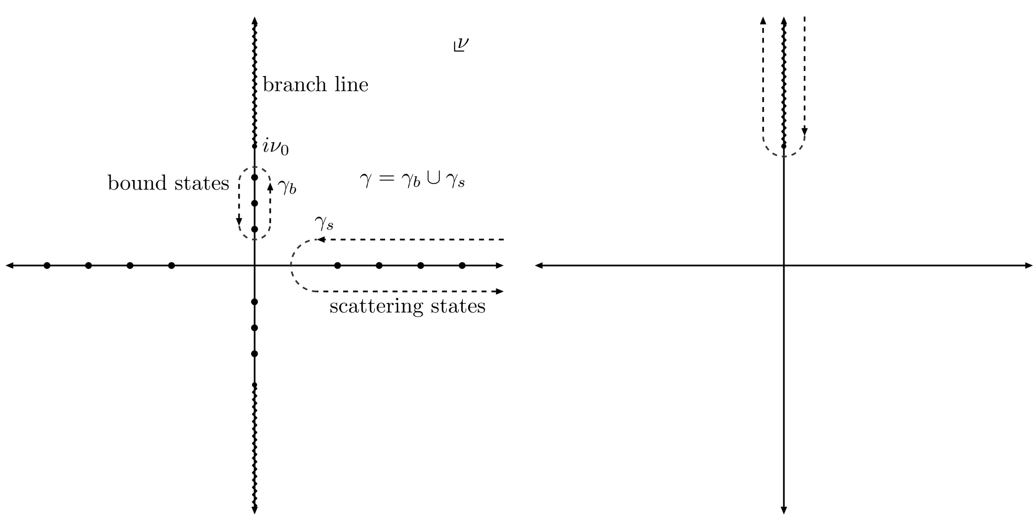

Figure 5.1: Left: contour in the complex ν-plane for the integral (5.4.65). Right: after deforming the contour, the integral is performed over the branch cut at the positive imaginary axis.

to ensure that the residue at each pole is equal to 1. How do we find such a function $f_l(\nu)$?

Imagine solving the differential equation $\mathcal{O}_l\psi = \lambda(\nu)\psi$. Being second order, it will have two independent solutions. These will depend on ν, which at this point is an unspecified parameter. The first consideration we need to make is that we restrict the spectral problem to functions that are smooth everywhere. In particular, for AdS_2, this means regularity[7] at $\rho = 0$. Up to an overall normalization, there is a unique solution satisfying this requirement. Call it $\phi_{(l,\nu)}(\rho)$. The second observation is that the actual eigenvalues are determined by the boundary conditions. For the Dirichlet case, for example, we impose $\phi_{(l,\nu)}(R) = 0$. This relation should be understood as an equation for ν, having in general infinitely many solutions $\nu = \nu_{(l,i)}$. Extending the domain to the entire complex ν-plane, we identify $f_l(\nu) \equiv \phi_{(l,\nu)}(R)$. Indeed, this function has a simple zero whenever ν corresponds to one of the eigenvalues of the operator $\mathcal{O}_l$.

The countour integral can be manipulated using standard techniques of complex analysis. To that end, notice that the function $\left(\nu^2+\nu_0^2\right)^{-s}$ has branch points at $\nu = \pm i\nu_0$. We choose to place the branch cuts along the imaginary axis, as shown in figure 5.1. Taking into account the symmetry $\nu \to -\nu$ we can deform the path so that it surrounds one of the cuts. The integrand then picks up a phase $e^{\pm i\pi s}$ on each side of the cut and we find

$$\zeta_{\mathcal{O}_l}(s) = \frac{\sin\pi s}{\pi}\int_{\nu_0}^{\infty} d\nu\,\left(\nu^2-\nu_0^2\right)^{-s}\partial_\nu \ln\phi_{(l,i\nu)}(R)\,. \tag{5.4.66}$$

The above representation of the zeta-function is typically not defined at $s = 0$ due to the large ν behavior of $\phi_{(l,i\nu)}$, and its analytic continuation will depend on the details of the operator at hand.

The behavior improves if we subtract the contribution of some reference (free/solvable) operator[8] so that the difference becomes

$$\hat{\zeta}_{\mathcal{O}_l}(s) \equiv \zeta_{\mathcal{O}_l}(s) - \zeta_{\text{free}}(s) = \frac{\sin\pi s}{\pi}\int_{\nu_0}^{\infty} d\nu\,\left(\nu^2-\nu_0^2\right)^{-s}\partial_\nu \ln\frac{\phi_{(l,i\nu)}(R)}{\phi^{\text{free}}_{(l,i\nu)}(R)}\,. \tag{5.4.67}$$

[7]Moreover, near the origin the operator reduces to that in flat space and the AdS features become irrelevant.

[8]At large energies the interactions become irrelevant and one expects $\phi_{(l,i\nu)}(R)$ to be proportional to $\phi^{\text{free}}_{(l,i\nu)}(R)$.

This subtraction is further justified by remembering that we are mainly interested in the $R \to \infty$ limit, where additional divergences related to the IR cutoff R appear. The integral at $s = 0$ is now finite and we can write

$$\hat{\zeta}'_{\mathcal{O}_l}(0) = -\ln \frac{\phi_{(l,i\nu_0)}(R)}{\phi^{\text{free}}_{(l,i\nu_0)}(R)} + \lim_{\nu\to\infty} \ln \frac{\phi_{(l,i\nu)}(R)}{\phi^{\text{free}}_{(l,i\nu)}(R)} , \qquad \hat{\zeta}_{\mathcal{O}_l}(0) = 0 \,. \tag{5.4.68}$$

Such a simple expression for the derivative of the zeta-function is valid only because the radial operators $\mathcal{O}_l$ are one-dimensional. Notice from (5.4.61) that $\lambda(i\nu_0) = 0$, so the function $\phi_{(l,i\nu_0)}(\rho)$ is the regular solution to the homogeneous equation $\mathcal{O}_l\psi = 0$. This equation is typically much easier to solve than the full eigenvalue problem, if not analytically, numerically. The large ν limit, on the other hand, will be shown to vanish in the bosonic case after a proper normalization. Of course, this is nothing but the Gelfand-Yaglom representation of one-dimensional determinants [125, 124]. For $d = 2$ we still need to sum over Fourier modes. As mentioned above, the sum is divergent at $s = 0$, so we are not ready yet. Nonetheless, $\hat{\zeta}'_{\mathcal{O}_l}(0)$ will appear in the final answer.

A similar line of reasoning can be followed for other boundary conditions, even in presence of zero modes, leading to analogous formulas for $\hat{\zeta}_{\mathcal{O}}(s)$ [124, 117, 125]. Indeed, with a few modifications, it can also be applied for the fermionic operators (5.2.3) [121, 126]. In this case, since the differential equation is first order, only half of the components of the spinor eigenfunctions can be constrained by the (local) boundary conditions. A standard choice are bag boundary conditions [47]. Another subtlety is that fermionic operators usually posses negative eigenvalues, leading to an ambiguity in the definition of the zeta-function. This ambiguity can be avoided by considering instead the squared operator, which is second order and is assumed to have a strictly positive spectrum. It is important to emphasize, however, that the eigenvalues of $\mathcal{O}^2$ should already be determined by those of $\mathcal{O}$. In other words, no additional or incompatible boundary conditions should be imposed on the second half of the eigenspinors when dealing with the second order operator. This last statement means that in the countour representation of $\zeta_{\mathcal{O}_l^2}(s)$, it is enough to consider the regular solution to the eigenvalue problem $\mathcal{O}_l\psi = \lambda(\nu)\psi$ and not $\mathcal{O}_l^2\psi = \lambda(\nu)^2\psi$. For convenience we explicitly separate the positive and negative eigenvalue sectors and write

$$\hat{\zeta}_{\mathcal{O}_l^2}(s) = \frac{\sin \pi s}{\pi} \int_{\nu_0}^{\infty} d\nu \, \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \left(\ln \frac{\phi^{+}_{(l,i\nu)}(R)}{\phi^{+\,\mathrm{free}}_{(l,i\nu)}(R)} + \ln \frac{\phi^{-}_{(l,i\nu)}(R)}{\phi^{-\,\mathrm{free}}_{(l,i\nu)}(R)} \right) . \tag{5.4.69}$$

Here $\phi^{\pm}_{(l,\nu)}(R)$ is some combination, determined by the choice of boundary conditions, of the components of the regular solution to the first order equation $\mathcal{O}_l \psi^{\pm} = \pm\sqrt{\nu^2 + \nu_0^2}\, \psi^{\pm}$. The spectrum of the free massive Dirac operator is symmetric, so $\phi^{+\,\mathrm{free}}_{(l,\nu)}(R) = \phi^{-\,\mathrm{free}}_{(l,\nu)}(R)$, but this is not necessarily the case for interacting operators. Notice the appearance of $\lambda(\nu)^{-2s}$ as opposed to $\lambda(\nu)^{-s}$, meaning that we are squaring the eigenvalues and therefore computing $\hat{\zeta}_{\mathcal{O}_l^2}(s)$. Evaluating at $s = 0$ we get

$$\begin{aligned}
\hat{\zeta}'_{\mathcal{O}_l^2}(0) &= -\ln \frac{\phi^{+}_{(l,i\nu_0)}(R)}{\phi^{+\,\mathrm{free}}_{(l,i\nu_0)}(R)} - \ln \frac{\phi^{-}_{(l,i\nu_0)}(R)}{\phi^{-\,\mathrm{free}}_{(l,i\nu_0)}(R)} + \lim_{\nu\to\infty} \left(\ln \frac{\phi^{+}_{(l,i\nu)}(R)}{\phi^{+\,\mathrm{free}}_{(l,i\nu)}(R)} + \ln \frac{\phi^{-}_{(l,i\nu)}(R)}{\phi^{-\,\mathrm{free}}_{(l,i\nu)}(R)} \right) , \\
\hat{\zeta}_{\mathcal{O}_l^2}(0) &= 0 \, .
\end{aligned} \tag{5.4.70}$$

Again, the computation of the zeta-function for the full fermionic operator requires a summation over the (half-integer) Fourier modes, so we are not allowed to take $s = 0$ at this moment.

5.4.2 Free eigenfunctions, Jost function and boundary conditions

We are interested in operators of the form (5.2.2) and (5.2.3) for which the background fields decay sufficiently fast at infinity, so that they become effectively free. Therefore, it is not surprising that the free eigenfunctions play a preponderant role in the analysis. Their exact form will be displayed below. For the moment we focus on some of their properties.

Let $h^{(l,\nu)}_{\pm}(\rho)$ be the two linearly independent eigenfunctions of the operator $\mathcal{O}_l^{\mathrm{free}}$. They satisfy

$$\mathcal{O}_l^{\mathrm{free}} h^{(l,\nu)}_{\pm}(\rho) = \lambda(\nu) h^{(l,\nu)}_{\pm}(\rho) \, , \tag{5.4.71}$$

where the eigenvalues are parametrized as in (5.4.61). In the fermionic case these are actually two-component eigenspinors and should carry an additional label specifying the sign of the eigenvalues. The notation $\pm$ refers to the fact that, asymptotically, these solutions become in- and out-going waves,

$$h_{\pm}^{(l,\nu)}(\rho) \sim e^{\left(-\frac{1}{2}\pm i\nu\right)\rho}\,, \qquad \rho \to \infty\,, \tag{5.4.72}$$

as follows directly from the differential equation. Square-integrability requires that $\nu \in \mathbb{R}$; the modulating factor $e^{-\frac{\rho}{2}}$ is compensated by the integration measure $\sqrt{g} = \sinh\rho \sim e^{\rho}$, yielding a plane wave orthogonality relation. It is important to mention, however, that neither $h_{+}^{(l,\nu)}(\rho)$ nor $h_{-}^{(l,\nu)}(\rho)$ are regular at $\rho = 0$, and therefore not actually square-integrable. Rather, after an appropriate choice of relative normalizations, the free regular solution is given by the combination

$$\phi_{(l,\nu)}^{\text{free}}(\rho) = \frac{i}{2}\left(h_{-}^{(l,\nu)}(\rho) - h_{+}^{(l,\nu)}(\rho)\right)\,. \tag{5.4.73}$$

Its small ρ expansion is again dictated by the differential equation and reads

$$\phi_{(l,\nu)}^{\text{free}}(\rho) \sim \rho^{|l|}\,, \qquad \rho \to 0\,, \qquad \text{(bosons)} \tag{5.4.74}$$

$$\phi_{(l,\nu)}^{\text{free}}(\rho) \sim \rho^{|l|-\frac{1}{2}}\,, \qquad \rho \to 0\,, \qquad \text{(fermions)} \tag{5.4.75}$$

For fermions only for the leading component is shown; the other component goes like $\rho^{|l|+\frac{1}{2}}$. The overall constant will depend on the exact normalization of $h_{\pm}^{(l,\nu)}$, the choice of which is arbitrary.

Consider now the interacting case. In general, finding the regular solution is prohibitively complicated. Nevertheless, there are two statements that are generally true. The first is that, precisely because it is regular, the behavior of $\phi_{(l,\nu)}(\rho)$ at $\rho = 0$ is the same as for the free solution. The second property stems from the previous observation that the operators become free for large ρ, meaning that the regular solution can be expanded as[9]

$$\phi_{(l,\nu)}(\rho) \underset{\rho\to\infty}{\longrightarrow} \frac{i}{2}\left(g_l(\nu)h_-^{(l,\nu)}(\rho) - \bar{g}_l(\nu)h_+^{(l,\nu)}(\rho)\right) . \tag{5.4.76}$$

This is only true asymptotically, of course. The coefficient $g_l(\nu)$ is called Jost function and plays a central part in the calculation of functional determinants. In fact, the the ratio $\ln(g_l(\nu)/\bar{g}_l(\nu))$ is precisely the phase shift from scattering theory that determines the density of eigenvalues. In the free case the above relation becomes exact with $g_l^{\text{free}}(\nu) = 1$.

Let us use the properties we have just discussed to see what happens to the zeta-function when we take the infinite space limit $R \to \infty$. To this purpose, note that for imaginary values of the radial momentum, the function $h_+^{(l,i\nu)}(R)$ is exponentially decaying, whereas $h_-^{(l,i\nu)}(R)$ blows up. Therefore, the ratio between the regular interacting solution and the free one becomes

$$\lim_{R\to\infty} \frac{\phi_{(l,i\nu)}(R)}{\phi^{\text{free}}_{(l,i\nu)}(R)} = g_l(i\nu) . \tag{5.4.77}$$

This gives the following expression for the zeta-function of the bosonic operator (5.2.2)

$$\hat{\zeta}_{\mathcal{O}}(s) = \frac{\sin\pi s}{\pi} \sum_{l\,\in\,\mathbb{Z}} \int_{\nu_0}^{\infty} d\nu\, \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \ln g_l(i\nu) . \tag{5.4.78}$$

A similar simplification occurs in the fermionic case (5.2.3), yielding

$$\hat{\zeta}_{\mathcal{O}^2}(s) = \frac{\sin\pi s}{\pi} \sum_{l\,\in\,\mathbb{Z}+\frac{1}{2}} \int_{\nu_0}^{\infty} d\nu\, \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \ln \mathfrak{g}_l(i\nu) , \tag{5.4.79}$$

where $\ln \mathfrak{g}_l(i\nu) \equiv \ln g_l^+(i\nu) + \ln g_l^-(i\nu)$ includes the contribution from the positive and negative eigenvalue sectors. Technically, the above expressions define the zeta function in terms of scattering data.

Besides the introduction of the Jost function in the two formulas above, the $R \to \infty$ limit has another, crucial, consequence on the zeta function: it makes the dependence on

[9] Given that the gauge field goes to a constant $\mathcal{A}(\rho) \to \mathcal{A}_\infty$ for $\rho \to \infty$, the asymptotics of the regular solution is more naturally expanded in terms of the shifted eigenfunctions $h_\pm^{(l-\mathcal{A}_\infty,\nu)}(\rho)$. At large ρ, however, these differ from their un-shifted version only by a normalization, making the definition (5.4.76) of the Jost function still viable.

the specific choice of boundary conditions disappear. Take for example the case of Neumann boundary conditions. The only modification one needs to make in $\hat{\zeta}_{\mathcal{O}}(s)$ is the replacement $\phi_{(l,i\nu)}(R) \to \partial_\rho \phi_{(l,i\nu)}(R)$. It is easy to see that upon taking the ratio with the corresponding free solution, the large R limit will again be given by the Jost function. The same is true for more general boundary conditions and for spinor fields. We then conclude that the determinants in AdS_2 are insensitive to the choice of boundary conditions one makes in the intermediate step of putting the system in a finite box.

As pointed out several times already, the sum over Fourier modes is ill-defined for $s = 0$. In what follows, we will perform the analytic continuation of (5.4.78) and (5.4.79). The general strategy is to subtract as many terms as necessary inside the integral such that the series becomes convergent at $s = 0$. The dangerous region is obviously $l \to \infty$, but also $\nu \sim l \to \infty$, so the calculation involves extracting the asymptotic behavior of $g_l(i\nu)$ in this regime. This can be done by constructing a representation of the Jost function in terms of the free eigenfunctions $h_{\pm}^{(l,\nu)}(\rho)$, the Green's function for the free operator and the background fields. The subtracted terms need to be added back and the analytic continuation is done using the well-known properties of the Riemann zeta-function.

5.4.3 Bosons

In this section we exhibit the derivation of (5.2.14). We split the radial operator (5.2.8) into a free part and an interaction,

$$\mathcal{O}_l = \mathcal{O}_l^{\text{free}} + U(\rho)\,, \qquad U(\rho) = V(\rho) + \frac{\mathcal{A}(\rho)^2}{\sinh^2\rho} - \frac{2l\mathcal{A}(\rho)}{\sinh^2\rho}\,. \tag{5.4.80}$$

where the free operator is given by

$$\mathcal{O}_l^{\text{free}} = -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{l^2}{\sinh^2\rho} + m^2\,, \qquad l \in \mathbb{Z}\,. \tag{5.4.81}$$

It will be important in what follows to keep in mind that there is a l-dependent term in the potential $U(\rho)$.

5.4.3.1 Free eigenfunctions

The bosonic free eigenfunctions satisfying (5.4.71) read

$$h_{\pm}^{(l,\nu)}(\rho) = \sqrt{\frac{2}{\pi\nu}} \left| \frac{\Gamma(1+i\nu)}{\Gamma\left(\frac{1}{2}+i\nu+|l|\right)} \right| e^{-i\pi|l|} Q^{|l|}_{-\frac{1}{2}\mp i\nu}(\cosh\rho) \,, \qquad \left(h_{\pm}^{(l,\nu)}\right)^* = h_{\mp}^{(l,\nu)} \,, \quad (5.4.82)$$

where $Q^{|l|}_{-\frac{1}{2}\mp i\nu}(\cosh\rho)$ are associated Legendre functions of the second kind. The condition that $\nu \in \mathbb{R}$ is necessary for square-integrability, as can be seen from the asymptotic expansions

$$h_{\pm}^{(l,\nu)}(\rho) \approx \sqrt{\frac{2}{\nu}} \left| \frac{\Gamma(1+i\nu)}{\Gamma\left(\frac{1}{2}+i\nu+|l|\right)} \right| \frac{\Gamma\left(\frac{1}{2}\mp i\nu+|l|\right)}{\Gamma(1\mp i\nu)} e^{\left(-\frac{1}{2}\pm i\nu\right)\rho} \,, \qquad \rho\to\infty \,. \qquad (5.4.83)$$

The combination

$$\phi^{\text{free}}_{(l,\nu)}(\rho) \equiv \frac{i}{2}\left(h_{-}^{(l,\nu)}(\rho) - h_{+}^{(l,\nu)}(\rho)\right) = \sqrt{\frac{\pi\nu}{2}} \left| \frac{\Gamma\left(\frac{1}{2}+i\nu+|l|\right)}{\Gamma(1+i\nu)} \right| P^{-|l|}_{-\frac{1}{2}\pm i\nu}(\cosh\rho) \,, \quad (5.4.84)$$

namely, the imaginary part of the eigenfunctions, is proportional to the associated Legendre function of the first kind and is regular at $\rho = 0$ with

$$\phi^{\text{free}}_{(l,\nu)}(\rho) \approx \sqrt{\frac{\pi\nu}{2}} \left| \frac{\Gamma\left(\frac{1}{2}+i\nu+|l|\right)}{\Gamma(1+i\nu)} \right| \frac{1}{\Gamma(1+|l|)} \left(\frac{\rho}{2}\right)^{|l|} \,, \qquad \rho\to 0 \,. \qquad (5.4.85)$$

As a matter of convenience, the normalization of the eigenfunctions has been chosen so that their Wronskian is independent of ν:

$$h_{-}^{(l,\nu)}(\rho)\partial_\rho h_{+}^{(l,\nu)}(\rho) - h_{+}^{(l,\nu)}(\rho)\partial_\rho h_{-}^{(l,\nu)}(\rho) = \frac{2i}{\sinh\rho} \,. \qquad (5.4.86)$$

Regardless of the normalization, this property allows us to construct the Green's function

$$\begin{aligned} G^{(l,\nu)}(\rho,\rho') &= \frac{i}{2}\sinh\rho' \left(h_{-}^{(l,\nu)}(\rho)h_{+}^{(l,\nu)}(\rho') - h_{+}^{(l,\nu)}(\rho)h_{-}^{(l,\nu)}(\rho')\right)\theta(\rho-\rho') \qquad (5.4.87) \\ &= \sinh\rho' \left(\phi^{\text{free}}_{(l,\nu)}(\rho)h_{+}^{(l,\nu)}(\rho') - \phi^{\text{free}}_{(l,\nu)}(\rho)h_{-}^{(l,\nu)}(\rho')\right)\theta(\rho-\rho') \,, \end{aligned}$$

which satisfies

$$\left(\mathcal{O}_l^{\text{free}} - \lambda(\nu)\right) G^{(l,\nu)}(\rho,\rho') = -\delta(\rho,\rho')\,. \tag{5.4.88}$$

Finally, we need to continue the eigenfunctions to imaginary momentum, $\nu \to i\nu$, and extract their asymptotic behavior for $l \to \infty$ and fixed $\alpha \equiv \frac{\nu}{|l|}$ with $0 < \alpha < 1$. We find

$$h_+^{(l,i\nu)}(\rho) \approx \sqrt{\frac{\alpha}{\pi|\sin(\pi\nu)|}}\left(1-\alpha^2\right)^{\frac{\nu+|l|}{2}}\left(\alpha^2\sinh^2\rho+1\right)^{-\frac{1}{4}} e^{-|l|\eta(\rho)}\,, \tag{5.4.89}$$

$$\phi_{(l,i\nu)}^{\text{free}}(\rho) \approx i\sqrt{\frac{\alpha|\sin(\pi\nu)|}{\pi}}\left(1-\alpha^2\right)^{-\frac{\nu+|l|}{2}}\left(\alpha^2\sinh^2\rho+1\right)^{-\frac{1}{4}} e^{|l|\eta(\rho)}\,, \tag{5.4.90}$$

where

$$\eta(\rho) = \alpha\ln\left(\alpha\cosh\rho + \sqrt{1+\alpha^2\sinh^2\rho}\right) - \ln\left(\cosh\rho + \sqrt{1+\alpha^2\sinh^2\rho}\right) + \ln\sinh\rho\,. \tag{5.4.91}$$

5.4.3.2 Regular solution and Jost function

In order to compute the zeta-function using (5.4.78), we first need to construct a solution to the eigenvalue problem that is regular at the origin. With the help of the free Green's function (5.4.87), we can invert the differential equation and write it in Lippmann-Schwinger form,

$$\phi_{(l,\nu)}(\rho) = \phi_{(l,\nu)}^{\text{free}}(\rho) + \int_0^\rho d\rho' G^{(l,\nu)}(\rho,\rho')U(\rho')\phi_{(l,\nu)}(\rho')\,. \tag{5.4.92}$$

In principle the integral above extends to $\rho' \to \infty$, but our choice of Green's function truncates it to $\rho' \leq \rho$. This choice is dictated by the fact that we want to control the behavior of the solution at $\rho = 0$ to ensure that it is regular. Notice that $G^{(l,\nu)}(\rho,\rho) = 0$, so the normalization $\phi_{(l,\nu)}(\rho) \approx \phi_{(l,\nu)}^{\text{free}}(\rho)$, with the same leading coefficient in the series expansion, is fixed by the integral equation.

Replacing the Green's function (5.4.87) in (5.4.92), taking $\rho \to \infty$ and by means of (5.4.76), we arrive to the following expression for the Jost function

$$g_l(\nu) = 1 + \int_0^\infty d\rho\,\sinh\rho\, h_+^{(l,\nu)}(\rho)U(\rho)\phi_{(l,\nu)}(\rho)\,. \tag{5.4.93}$$

Of course, this expression still involves the unknown function $\phi^{(l,\nu)}(\rho)$ and can be solved iteratively as an expansion in powers of the potential U. However, as we will confirm below, it is sufficient to solve for the regular solution only up to second order. After some algebra one gets[10]

$$\ln g_l(\nu) = \int_0^\infty d\rho \, \sinh\rho \, h_+^{(l,\nu)}(\rho) U(\rho) \phi^{\text{free}}_{(l,\nu)}(\rho)$$
$$- \int_0^\infty d\rho \, \sinh\rho \, \left(h_+^{(l,\nu)}(\rho)\right)^2 U(\rho) \int_0^\rho d\rho' \, \sinh\rho' \, \left(\phi^{\text{free}}_{(l,\nu)}(\rho')\right)^2 U(\rho') + O(U^3) \,, \tag{5.4.94}$$

where we have taken the logarithm since that is what actually enters in the ζ-function.

The next step involves continuing the Jost function to imaginary values of the radial momentum and extracting its limiting behavior for large ν and large l. Remember that the goal is to subtract from $\ln g_l(i\nu)$ as many terms as necessary so that the sum over Fourier modes in (5.4.78) becomes convergent at $s = 0$. Clearly we can discard all terms that decay faster than l^{-1}. Introducing the asymptotic expansions of the eigenfunctions given in (5.4.89) and (5.4.90) into (5.4.94) we obtain

$$\ln g_l(i\nu) = \frac{1}{2|l|} \int_0^\infty d\rho \frac{\sinh\rho \, U(\rho)}{\sqrt{\alpha^2 \sinh^2\rho + 1}} \tag{5.4.95}$$
$$- \frac{1}{4l^2} \int_0^\infty d\rho \, \frac{\sinh\rho \, U(\rho) e^{-2|l|\eta(\rho)}}{\sqrt{\alpha^2 \sinh^2\rho + 1}} \int_0^\rho d\rho' \, \frac{\sinh\rho' U(\rho') e^{2|l|\eta(\rho')}}{\sqrt{\alpha^2 \sinh^2\rho' + 1}} + O(l^{-2}) \,.$$

Notice that the first line involves a term of order $O(l^0)$ coming from (5.4.80). However, this will cancel when summing over positive and negative Fourier modes. By the same token, subleading contributions to eigenfunctions where not considered in (5.4.89) and (5.4.90), as they are insensitive to the sign of l. A priori, the second line also involves a $O(l^0)$ term, but this is really not so. It can be seen that in the saddle point approximation, which is

[10]Use $\ln\left(1 + ax + bx^2\right) = ax + \left(b - \frac{1}{2}a^2\right) x^2 + O(x^3)$ and $\int_a^b dx f(x) \int_a^x dy f(y) = \frac{1}{2}\left(\int_a^b dx f(x)\right)^2$.

justified in the limit we are studying, the integral over ρ' yields

$$\int_0^\rho d\rho' \, \frac{\sinh\rho' U(\rho') e^{2|l|\eta(\rho')}}{\sqrt{\alpha^2 \sinh^2\rho' + 1}} \approx \frac{1}{2|l|} \frac{\sinh^2\rho \, U(\rho) e^{2|l|\eta(\rho)}}{\alpha^2 \sinh^2\rho + 1} + O(l^{-2}) \, . \tag{5.4.96}$$

Since each nested integral results in a factor of $1/l$, higher orders in U in the Lippmann-Schwinger expansion (5.4.92) are not necessary for the subtraction. This way we arrive at the following expression for the asymptotic behavior of the Jost function

$$\ln g_l^{\text{asym}}(i\nu) + \ln g_{-l}^{\text{asym}}(i\nu) \equiv \frac{1}{|l|} \int_0^\infty d\rho \, \frac{\sinh\rho \, V(\rho)}{\left(1 + \alpha^2 \sinh^2\rho\right)^{\frac{1}{2}}} + \frac{\alpha^2}{|l|} \int_0^\infty d\rho \, \frac{\sinh\rho \, \mathcal{A}(\rho)^2}{\left(1 + \alpha^2 \sinh^2\rho\right)^{\frac{3}{2}}} \, . \tag{5.4.97}$$

Recall that the dependence on the radial momentum enters through $\alpha = \nu/|l|$. One can easily see that

$$\lim_{\nu\to\infty} \left(\ln g_l^{\text{asym}}(i\nu) + \ln g_{-l}^{\text{asym}}(i\nu) \right) = 0 \, . \tag{5.4.98}$$

Similarly, expanding 5.4.94 for large ν and fixed l one finds[11]

$$\lim_{\nu\to\infty} \ln g_l(i\nu) = 0 \, . \tag{5.4.99}$$

The fact that this limit vanishes is a consequence of the choice of normalization of the regular solution.

5.4.3.3 Analytic continuation

The analytic continuation of the zeta-function (5.4.78) to $s = 0$ is achieved by splitting the sum as

$$\hat{\zeta}_{\mathcal{O}}(s) = \hat{\zeta}_f(s) + \hat{\zeta}_d(s) \, , \tag{5.4.100}$$

[11] We omit the explicit expansions of the eigenfunctions in this limit since they are even simpler than the ones presented above.

where

$$
\begin{aligned}
\hat{\zeta}_f(s) =& \frac{\sin \pi s}{\pi} \int_{\nu_0}^{\infty} d\nu \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \ln g_0(i\nu) \\
&+ \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \int_{\nu_0}^{\infty} d\nu \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \left(\ln g_l(i\nu) + \ln g_{-l}(i\nu) - \ln g_l^{\text{asym}}(i\nu) - \ln g_{-l}^{\text{asym}}(i\nu)\right) , \\
\hat{\zeta}_d(s) =& \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \int_{\nu_0}^{\infty} d\nu \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \left(\ln g_l^{\text{asym}}(i\nu) + \ln g_{-l}^{\text{asym}}(i\nu)\right) .
\end{aligned}
$$

Here we have separated the mode $l = 0$ and combined the $l > 0$ and $l < 0$ terms into a single sum. The main point is that $\zeta_f(s)$ is now convergent at $s = 0$, since by construction of $g_l^{\text{asym}}(i\nu)$ it goes as $O(l^{-2})$ for $l \to \infty$. Thus, we can simply take its derivative and evaluate

$$
\begin{aligned}
&\hat{\zeta}_f(0) = 0 \,, \\
&\hat{\zeta}_f'(0) = -\ln g_0\left(i\nu_0\right) - \sum_{l=1}^{\infty} \left(\ln g_l\left(i\nu_0\right) + \ln g_{-l}\left(i\nu_0\right) - \ln g_l^{\text{asym}}\left(i\nu_0\right) - \ln g_{-l}^{\text{asym}}\left(i\nu_0\right)\right) .
\end{aligned}
\tag{5.4.101}
$$

Again, $\hat{\zeta}_f'(0)$ is guaranteed to be finite. On the other hand, $\zeta_d(s)$ is still divergent at $s = 0$ and needs continuation. The improvement is that this sum is easier to handle. Indeed, the general formulas

$$
\int_a^{\infty} dx \left(x^2 - a^2\right)^{-s} \frac{d}{dx} \left(\left(1 + b^2 x^2\right)^{-n/2}\right) = -\frac{\Gamma\left(s + \frac{n}{2}\right) \Gamma\left(1 - s\right) b^{2s}}{\Gamma\left(\frac{n}{2}\right) \left(1 + a^2 b^2\right)^{s + \frac{n}{2}}} \,, \tag{5.4.102}
$$

$$
\int_a^{\infty} dx \left(x^2 - a^2\right)^{-s} \frac{d}{dx} \left(x^2 \left(1 + b^2 x^2\right)^{-n/2}\right) = -\frac{\Gamma\left(s + \frac{n}{2} - 1\right) \Gamma\left(1 - s\right) b^{2(s-1)} \left(\left(n - 2\right) a^2 b^2 - 2s\right)}{2\Gamma\left(\frac{n}{2}\right) \left(1 + a^2 b^2\right)^{s + \frac{n}{2}}} \,, \tag{5.4.103}
$$

allow us to explicitly perform the integration over the radial momentum and find

$$
\zeta_d(s) = -\frac{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(1 - s\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^{\infty} d\rho \, \left(\sinh \rho\right)^{2s+1} \left(V(\rho) R_1(s, \rho) + \frac{\mathcal{A}(\rho)^2}{\sinh^2 \rho} R_2(s, \rho)\right) \,, \tag{5.4.104}
$$

where

$$R_1(s,\rho) = \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{1}{2}\right)} , \tag{5.4.105}$$

$$R_2(s,\rho) = \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left(\frac{\nu_0^2 \sinh^2 \rho}{l^2} - 2s\right) \left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{3}{2}\right)} . \tag{5.4.106}$$

In order to continue these sums, we again subtract and add back the asymptotic behavior of the summand that makes the series divergent when $s = 0$, namely,

$$\begin{aligned} R_1(s,\rho) &= \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left[\left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{1}{2}\right)} - 1\right] + \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} , \\ R_2(s,\rho) &= \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left[\left(\frac{\nu_0^2 \sinh^2 \rho}{l^2} - 2s\right) \left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{3}{2}\right)} + 2s\right] \\ &\quad - \frac{2s \sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} . \end{aligned}$$

Recognizing the last term in each expression as the Riemann zeta function, we arrive at

$$\begin{aligned} R_1(s,\rho) =& \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left[\left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{1}{2}\right)} - 1\right] + \frac{\sin \pi s}{\pi} \zeta_R(2s+1) , \\ R_2(s,\rho) =& \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} \frac{1}{l^{1+2s}} \left[\left(\frac{\nu_0^2 \sinh^2 \rho}{l^2} - 2s\right) \left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\left(s+\frac{3}{2}\right)} + 2s\right] \\ & - \frac{2s \sin \pi s}{\pi} \zeta_R(2s+1) . \end{aligned}$$

Since each sum in square brackets is now convergent for $s = 0$, we readily find[12]

$$R_1(0,\rho) = \frac{1}{2} , \qquad R_1'(0,\rho) = \sum_{l=1}^{\infty} \frac{1}{l} \left[\left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\frac{1}{2}} - 1\right] + \gamma , \tag{5.4.107}$$

$$R_2(0,\rho) = 0 , \qquad R_2'(0,\rho) = \nu_0^2 \sinh^2 \rho \sum_{l=1}^{\infty} \frac{1}{l^3} \left(1 + \frac{\nu_0^2 \sinh^2 \rho}{l^2}\right)^{-\frac{3}{2}} - 1 . \tag{5.4.108}$$

[12]Actually, $R_2(s,\rho)$ was already convergent at $s = 0$. However, its term by term derivative was not, so the procedure was still necessary.

This is the desired continuation. Then,

$$\hat{\zeta}_d(0) = -\frac{1}{2}\int_0^\infty d\rho\, \sinh\rho\, V(\rho)\,, \tag{5.4.109}$$

and

$$\begin{aligned}
\hat{\zeta}_d'(0) = &-\int_0^\infty d\rho\, \sinh\rho \left(\ln\left(\frac{\sinh\rho}{2}\right) + \gamma\right) V(\rho) + \int_0^\infty d\rho\, \frac{\mathcal{A}(\rho)^2}{\sinh\rho} \\
&- \sum_{l=1}^\infty \frac{1}{l}\int_0^\infty d\rho\, \sinh\rho \left[\left(1+\frac{\nu_0^2\sinh^2\rho}{l^2}\right)^{-\frac{1}{2}} V(\rho) - V(\rho) - \frac{\nu_0^2}{l^2}\left(1+\frac{\nu_0^2\sinh^2\rho}{l^2}\right)^{-1}\mathcal{A}(\rho)^2\right] \\
= &-\int_0^\infty d\rho\, \sinh\rho \left(\ln\left(\frac{\sinh\rho}{2}\right) + \gamma\right) V(\rho) + \int_0^\infty d\rho\, \frac{\mathcal{A}(\rho)^2}{\sinh\rho} \\
&- \sum_{l=1}^\infty \left(\ln g_l^{\mathrm{asym}}(i\nu_0) + \ln g_{-l}^{\mathrm{asym}}(i\nu_0) - \frac{1}{l}\int_0^\infty d\rho\, \sinh\rho\, V(\rho)\right) .
\end{aligned}$$

In the last step we have recognized the asymptotic form (5.4.97) of the Jost function evaluated at $\nu = \nu_0$. Combining the expressions for $\hat{\zeta}_f(0)$, $\hat{\zeta}_d(0)$, $\hat{\zeta}_f'(0)$ and $\hat{\zeta}_d'(0)$ we arrive at

$$\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho\, \sinh\rho\, V(\rho)\,, \tag{5.4.110}$$

$$\begin{aligned}
\hat{\zeta}_{\mathcal{O}}'(0) = &-\ln g_0\,(i\nu_0) - \sum_{l=1}^\infty \left(\ln g_l\,(i\nu_0) + \ln g_{-l}\,(i\nu_0) + \frac{2}{l}\hat{\zeta}(0)\right) + 2\gamma\hat{\zeta}(0) \\
&- \int_0^\infty d\rho\, \sinh\rho \ln\left(\frac{\sinh\rho}{2}\right) V(\rho) + \int_0^\infty d\rho\, \frac{\mathcal{A}(\rho)^2}{\sinh\rho}\,.
\end{aligned} \tag{5.4.111}$$

Notice that $\ln g_l^{\mathrm{asym}}(i\nu_0)$ cancels out at the end so it is no longer needed. Finally, by means of (5.4.77), (5.4.68) and (5.4.99), $g_l(i\nu_0)$ is identified with the determinant of the radial operator $\mathcal{O}_l$ and the full renormalized determinant (5.2.11) becomes our main result (5.2.14). Once the radius of AdS_2 is reinstated, the dimensionless quantity $L\mu$ appears.

5.4.4 Fermions

We now move on to the derivation of the fermionic expression (5.2.16). As in the bosonic case, the full operator splits into

$$\mathcal{O}_l = \mathcal{O}_l^{\text{free}} - i\Gamma_{\underline{01}} U(\rho) \quad , \quad U(\rho) = -\Gamma_{\underline{0}}\, \partial_\rho \Omega(\rho) - i\, q\, \Gamma_{\underline{1}} \frac{\mathcal{A}(\rho)}{\sinh\rho} + V(\rho) - i\Gamma_{\underline{01}} W(\rho)\,. \tag{5.4.112}$$

The matrix $-i\Gamma_{\underline{01}}$ in front of U is a matter of convenience. The free fermionic radial operator is

$$\mathcal{O}_l^{\text{free}} = -i\Gamma_{\underline{1}} \left(\partial_\rho + \frac{1}{2}\coth\rho\right) + \Gamma_{\underline{0}} \frac{l}{\sinh\rho} - i\Gamma_{\underline{01}} m\,, \qquad l \in \mathbb{Z} + \frac{1}{2}\,. \tag{5.4.113}$$

From now on we will work with the following representation of the Dirac matrices,

$$\Gamma_{\underline{0}} = -\sigma_2\,, \quad \Gamma_{\underline{0}} = \sigma_1 \quad \Rightarrow \quad -i\Gamma_{\underline{01}} = \sigma_3\,. \tag{5.4.114}$$

5.4.4.1 Free eigenfunctions

Unlike the bosonic case, the free operator (5.4.113) has positive and negative eigenvalues. It is sufficient, however, to restrict ourselves to $\lambda > 0$, since the $\lambda < 0$ sector can be obtained from the former by a simple operation. The eigenfunctions for $l \geq \frac{1}{2}$ and $l \leq -\frac{1}{2}$ are also related to each other, so we will work with strictly positive Fourier modes. This is not to say that we are neglecting three out of the four possible sectors.

The spinor eigenfunctions satisfying (5.4.71) with $\lambda > 0$ and $l \geq \frac{1}{2}$ read

$$h_\pm^{(l,\nu)}(\rho) = \sqrt{\frac{\Gamma\left(l+\frac{1}{2} \mp i\nu\right)\Gamma\left(\frac{1}{2} \pm i\nu\right)}{\Gamma\left(l+\frac{1}{2} \pm i\nu\right)\Gamma\left(\frac{1}{2} \mp i\nu\right)}} \sqrt{2} \left(\tanh\frac{\rho}{2}\right)^{l-\frac{1}{2}} \left(2\cosh\frac{\rho}{2}\right)^{-1\pm 2i\nu} \psi_\pm^{(l,\nu)}(\rho)\,, \tag{5.4.115}$$

where

$$\psi_{\pm}^{(l,\nu)}(\rho) = \begin{pmatrix} \left(\dfrac{\lambda(\nu)+m}{\lambda(\nu)-m}\right)^{\frac{1}{4}} \tanh\dfrac{\rho}{2} F\left(l+\dfrac{1}{2}\mp i\nu, 1\mp i\nu; 1\mp 2i\nu; \dfrac{1}{\cosh^2\frac{\rho}{2}}\right) \\ \pm\left(\dfrac{\lambda(\nu)-m}{\lambda(\nu)+m}\right)^{\frac{1}{4}} F\left(l+\dfrac{1}{2}\mp i\nu, \mp i\nu; 1\mp 2i\nu; \dfrac{1}{\cosh^2\frac{\rho}{2}}\right) \end{pmatrix}. \tag{5.4.116}$$

The combination

$$\begin{aligned} \phi^{\text{free}}_{(l,\nu)}(\rho) &\equiv \frac{i}{2}\left(h_-^{(l,\nu)}(\rho) - h_+^{(l,\nu)}(\rho)\right) \\ &= \frac{1}{\Gamma\left(l+\frac{1}{2}\right)}\sqrt{\frac{\pi}{2}}\left|\frac{\Gamma\left(l+\frac{1}{2}\mp i\nu\right)}{\Gamma\left(\frac{1}{2}\mp i\nu\right)}\right|\left(\tanh\frac{\rho}{2}\right)^{l-\frac{1}{2}}\left(\cosh\frac{\rho}{2}\right)^{-1+2i\nu}\psi^{(l,\nu)}(\rho), \end{aligned} \tag{5.4.117}$$

with

$$\psi^{(l,\nu)}(\rho) = \begin{pmatrix} -\dfrac{\nu}{l+\frac{1}{2}}\left(\dfrac{\lambda(\nu)+m}{\lambda(\nu)-m}\right)^{\frac{1}{4}} \tanh\dfrac{\rho}{2} F\left(l+\dfrac{1}{2}- i\nu, 1- i\nu; l+\dfrac{3}{2}; \tanh^2\dfrac{\rho}{2}\right) \\ i\left(\dfrac{\lambda(\nu)-m}{\lambda(\nu)+m}\right)^{\frac{1}{4}} F\left(l+\dfrac{1}{2}- i\nu, - i\nu; l+\dfrac{1}{2}; \tanh^2\dfrac{\rho}{2}\right) \end{pmatrix}, \tag{5.4.118}$$

is regular at the origin. As before, the condition $\nu \in \mathbb{R}$ is imposed by square-integrability. The solutions for the remaining three sectors can be obtained by simple operations, namely,

$$\begin{aligned} &l \leq -\frac{1}{2}, \quad \lambda(\nu) > 0 \quad \longrightarrow \quad (i\sigma_1)\, h_{\pm}^{(-l,\nu)}(\rho)\Big|_{m\to -m}, \\ &l \geq \frac{1}{2}, \quad \lambda(\nu) < 0 \quad \longrightarrow \quad (i\sigma_2)\, h_{\pm}^{(l,\nu)}(\rho), \\ &l \leq -\frac{1}{2}, \quad \lambda(\nu) < 0 \quad \longrightarrow \quad (i\sigma_3)\, h_{\pm}^{(-l,\nu)}(\rho)\Big|_{m\to -m}. \end{aligned} \tag{5.4.119}$$

The normalization of the eigenspinors has been chosen so that they satisfy

$$h_-^{(l,\nu)}(\rho) h_+^{(l,\nu)}(\rho)^T - h_+^{(l,\nu)}(\rho) h_-^{(l,\nu)}(\rho)^T = \frac{2i\sigma_2}{\sinh\rho}, \tag{5.4.120}$$

in all four sectors. This identity allow us to construct the Green's function

$$G^{(l,\nu)}(\rho,\rho') = \frac{i}{2}\sinh\rho' \left[h_-^{(l,\nu)}(\rho) h_+^{(l,\nu)}(\rho')^T - h_+^{(l,\nu)}(\rho) h_-^{(l,\nu)}(\rho')^T \right] \sigma_3\, \theta(\rho-\rho')\,, \tag{5.4.121}$$
$$= \sinh\rho' \left[\phi^{\rm free}_{(l,\nu)}(\rho) h_+^{(l,\nu)}(\rho')^T - h_+^{(l,\nu)}(\rho) \phi^{\rm free}_{(l,\nu)}(\rho')^T \right] \sigma_3\, \theta(\rho-\rho')\,,$$

which satisfies

$$\left(\mathcal{O}_l^{\rm free} - \lambda(\nu)\right) G^{(l,\nu)}(\rho,\rho') = -\delta(\rho,\rho')\,. \tag{5.4.122}$$

Notice that

$$G^{(l,\nu)}(\rho,\rho) = -\frac{i}{2}\sigma_1\,, \tag{5.4.123}$$

as follows from the coincidence limit of the step function. Since we will need them shortly, we present the asymptotic behavior of the solutions $h_+^{(l,i\nu)}(\rho)$ and $\phi^{\rm free}_{(l,i\nu)}(\rho)$ in the region where $(l+\frac{1}{2})\to\infty$ and $\nu=\alpha(l+\frac{1}{2})$ with $0<\alpha<1$,

$$h_+^{(l,i\nu)}(\rho) \approx \mathcal{F}(\rho)\, e^{-(l+\frac{1}{2})\eta(\rho)} \begin{pmatrix} 1+\frac{1}{l+\frac{1}{2}}\left(A(\rho)-\frac{im}{2\alpha}\right) \\ \frac{-1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}\left(1+\frac{1}{l+\frac{1}{2}}\left(B(\rho)+\frac{im}{2\alpha}\right)\right) \end{pmatrix}\,, \tag{5.4.124}$$

$$\phi^{\rm free}_{(l,i\nu)}(\rho) \approx \mathcal{G}(\rho)\, e^{(l+\frac{1}{2})\eta(\rho)} \begin{pmatrix} 1+\frac{1}{l+\frac{1}{2}}\left(C(\rho)-\frac{im}{2\alpha}\right) \\ -\frac{1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}\left(1+\frac{1}{l+\frac{1}{2}}\left(D(\rho)+\frac{im}{2\alpha}\right)\right) \end{pmatrix}\,, \tag{5.4.125}$$

where $\eta(\rho)$ was defined in (5.4.91) and the rest of the functions involved satisfy the relations

$$\mathcal{F}(\rho)\mathcal{G}(\rho) = \frac{i\alpha}{2\sqrt{1+\alpha^2\sinh^2\rho}}\,, \qquad B(\rho) = A(\rho) + \frac{1+\sqrt{1+\alpha^2\sinh^2\rho}}{2(1+\alpha^2\sinh^2\rho)} \tag{5.4.126}$$
$$C(\rho) = -A(\rho)\,, \qquad D(\rho) = -A(\rho) - \frac{-1+\sqrt{1+\alpha^2\sinh^2\rho}}{2(1+\alpha^2\sinh^2\rho)}$$

As we will show below, the explicit forms of the functions $\mathcal{F}(\rho)$, $\mathcal{G}(\rho)$ and $A(\rho)$ do not play any role in the computation, so we do not present them here. Notice that we have included the first sub-dominant term.

5.4.4.2 Regular Solution and Jost function

We now consider the eigenvalue problem for the full operator (5.4.112). The regular solution is constructed using the Lippmann-Schwinger equation, with the help of the free Green's function (5.4.121),

$$\phi_{(l,\nu)}(\rho) = \phi^{\text{free}}_{(l,\nu)}(\rho) + \int_0^\rho d\rho' G^{(l,\nu)}(\rho,\rho')\,\sigma_3\,U(\rho')\phi_{(l,\nu)}(\rho')\,. \tag{5.4.127}$$

Naively one would think that $\phi_{(l,\nu)}(\rho) \longrightarrow \phi^{\text{free}}_{(l,\nu)}(\rho)$ as $\rho \to 0$. However, a more careful analysis reveals that[13]

$$\begin{aligned}\phi_{(l,\nu)}(\rho) &\approx \phi^{\text{free}}_{(l,\nu)}(\rho) + G^{(l,\nu)}(\rho,\rho)\,\sigma_3\,U(\rho)\int_0^\rho d\rho' \phi^{\text{free}}_{(l,\nu)}(\rho') \\ &\approx \frac{i}{\Gamma\left(l+\frac{1}{2}\right)}\sqrt{\frac{\pi}{2}}\left|\frac{\Gamma\left(l+\frac{1}{2}-i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)}\right|\left(\frac{\rho}{2}\right)^{l-\frac{1}{2}}\left(\frac{\lambda-m}{\lambda+m}\right)^{\frac{1}{4}}\begin{pmatrix} i\dfrac{\lambda+m+V(0)-W(0)}{2l+1}\rho \\ 1\end{pmatrix}.\end{aligned} \tag{5.4.128}$$

This is consistent with the behavior obtained by studying the differential equation near the origin. Accor dingly, for $l \leq -\frac{1}{2}$ and $\lambda > 0$, we have

$$\phi_{(l,\nu)}(\rho) \approx \frac{1}{\Gamma\left(|l|+\frac{1}{2}\right)}\sqrt{\frac{\pi}{2}}\left|\frac{\Gamma\left(|l|+\frac{1}{2}-i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)}\right|\left(\frac{\rho}{2}\right)^{|l|-\frac{1}{2}}\left(\frac{\lambda+m}{\lambda-m}\right)^{\frac{1}{4}}\begin{pmatrix} 1 \\ i\dfrac{\lambda-m-V(0)-W(0)}{2|l|+1}\rho\end{pmatrix}, \tag{5.4.129}$$

and similarly for the remaining two sectors. At any rate, the normalization of the regular solution is fixed by the normalization of the free eigenfunctions (5.4.115).

The Jost function can be extracted from the large ρ behavior of the solution by means

[13] Both $G^{(l,\nu)}(\rho,\rho)$ and $U(\rho)$ are finite at $\rho = 0$, so the leading behavior is dictated by $\phi^{\text{free}}_{(l,\nu)}(\rho)$.

of its definition (5.4.76). A direct evaluation yields[14]

$$g_l(\nu) = 1 + \int_0^\infty d\rho' \, \sinh\rho' h_+^{(l,\nu)}(\rho')^T U(\rho')\phi_{(l,\nu)}(\rho') \,. \tag{5.4.130}$$

As in the bosonic case, it will be sufficient to retain terms up to second order in the potential $U(\rho)$ so that

$$\begin{aligned} \ln g_l(\nu) = &\int_0^\infty d\rho \, \sinh\rho \, h_+^{(l,\nu)}(\rho)^T U(\rho)\phi_{(l,\nu)}^{\text{free}}(\rho) \\ &- \int_0^\infty d\rho \, \sinh\rho \, h_+^{(l,\nu)}(\rho)^T U(\rho) h_+^{(l,\nu)}(\rho) \int_0^\rho d\rho' \, \sinh\rho' \, \phi_{(l,\nu)}^{\text{free}}(\rho')^T U(\rho')\phi_{(l,\nu)}^{\text{free}}(\rho') + O(U^3) \,. \end{aligned} \tag{5.4.131}$$

We now need to continue the Jost function to imaginary radial momentum and extract its asymptotic behavior in the region $\left|l+\frac{1}{2}\right| \to \infty$ and $\nu = \alpha\left|l+\frac{1}{2}\right|$ $(0 < \alpha < 1)$. In the sector of positive l and positive λ we can make use of the asymptotic expansions presented above. The calculation proceeds much like the bosonic case with the proviso that the eigenfunctions have spinorial structure. However, the fermionic potential is l-independent and now subleading orders in (5.4.124)-(5.4.125) do contribute. Again resorting to a saddle point approximation we find

$$\ln g_l^+(i\nu) = \frac{i\alpha}{2}\int d\rho \frac{\sinh\rho\left(U_{h\phi}^{(0)} + \frac{1}{l+\frac{1}{2}}U_{h\phi}^{(1)}\right)}{\sqrt{1+\alpha^2\sinh^2\rho}} + \frac{\alpha^2}{4(2l+1)}\int_0^\infty d\rho \, \frac{\sinh^3\rho \, U_{hh} \, U_{\phi\phi}}{(1+\alpha^2\sinh^2\rho)^{\frac{3}{2}}} + O\left(l^{-2}\right) \,, \tag{5.4.132}$$

[14] As in the bosonic case, the effect of the shift in the Fourier mode due to the constant asymptotic value of the gauge field can be absorbed in the definition of the Jost function.

where

$$\begin{aligned}
U_{h\phi}^{(0)} &= (U_{11}-U_{22}) - \frac{1}{\alpha\sinh\rho}(U_{12}+U_{21}) - \frac{\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}(U_{12}-U_{21})\,, \\
U_{h\phi}^{(1)} &= -\frac{im}{\alpha}(U_{11}+U_{22}) - \frac{U_{22}}{1+\alpha^2\sinh^2\rho} + \frac{\alpha\sinh\rho}{2(1+\alpha^2\sinh^2\rho)}(U_{12}+U_{21})\,, \\
U_{hh} &= U_{11} + U_{22}\left(\frac{-1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}\right)^2 + \frac{-1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}(U_{12}+U_{21})\,, \\
U_{\phi\phi} &= U_{11} + U_{22}\left(\frac{1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}\right)^2 - \frac{1+\sqrt{1+\alpha^2\sinh^2\rho}}{\alpha\sinh\rho}(U_{12}+U_{21})\,.
\end{aligned} \tag{5.4.133}$$

As was previously mentioned, these expressions are independent of the function $A(\rho)$ appearing in the asymptotic expansions of $h_+^{(l,\nu)}(\rho)$ and $\phi_{(l,\nu)}^{\mathrm{free}}(\rho)$.

The remaining three sectors of solutions are obtained by performing the operations (5.4.119), which amount to the substitutions $U \to (i\sigma_i)^T U (i\sigma_i)$ and $m \to \pm m$ in the above formulæ. After summing over all four sectors and discarding a ν-independent term we identify the potentially divergent part as

$$\begin{aligned}
\ln \mathfrak{g}_l^{\mathrm{asym}}(i\nu) + \ln \mathfrak{g}_{-l}^{\mathrm{asym}}(i\nu) \equiv{}& \frac{2}{l+\frac{1}{2}}\int_0^\infty d\rho \sinh\rho \frac{(U_{11}+m)(U_{22}+m)-m^2}{\sqrt{1+\alpha^2\sinh^2\rho}} \\
&+ \frac{\alpha^2}{2(l+\frac{1}{2})}\int_0^\infty d\rho \sinh^3\rho \frac{(U_{11}-U_{22})^2-(U_{12}+U_{21})^2}{\left(1+\alpha^2\sinh^2\rho\right)^{\frac{3}{2}}}\,,
\end{aligned} \tag{5.4.134}$$

where we made use of the definition below (5.4.79). Note that

$$\lim_{\nu\to\infty}\left(\ln \mathfrak{g}_l^{\mathrm{asym}}(i\nu) + \ln \mathfrak{g}_{-l}^{\mathrm{asym}}(i\nu)\right) = 0\,. \tag{5.4.135}$$

On the other hand, a similar calculation but in the limit of large ν and fixed l yields

$$\lim_{\nu\to\infty}\left(\ln \mathfrak{g}_l\,(i\nu) + \ln \mathfrak{g}_{-l}\,(i\nu)\right) = 2i\int d\rho\,(U_{21}-U_{12})\;, \tag{5.4.136}$$

which is non-vanishing. This is an effect of the normalization (5.4.128).

5.4.4.3 Analytic continuation

The analytic continuation of (5.4.79) proceeds much in the same way as for bosons. We split the sum over Fourier modes as

$$\hat{\zeta}_{\mathcal{O}^2}(s) = \hat{\zeta}_f(s) + \hat{\zeta}_d(s)\,, \tag{5.4.137}$$

where

$$\begin{aligned}
\hat{\zeta}_f(s) &= \frac{\sin \pi s}{\pi} \sum_{l=\frac{1}{2}}^{\infty} \int_{\nu_0}^{\infty} d\nu \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \left(\ln \mathfrak{g}_l(i\nu) + \ln \mathfrak{g}_{-l}(i\nu) - \ln \mathfrak{g}_l^{\text{asym}}(i\nu) - \ln \mathfrak{g}_{-l}^{\text{asym}}(i\nu)\right)\,, \\
\hat{\zeta}_d(s) &= \frac{\sin \pi s}{\pi} \sum_{l=\frac{1}{2}}^{\infty} \int_{\nu_0}^{\infty} d\nu \left(\nu^2 - \nu_0^2\right)^{-s} \partial_\nu \left(\ln \mathfrak{g}_l^{\text{asym}}(i\nu) + \ln \mathfrak{g}_{-l}^{\text{asym}}(i\nu)\right)\,.
\end{aligned} \tag{5.4.138}$$

The series in $\zeta_f(s)$ is now convergent at $s=0$ and we find

$$\begin{aligned}
\hat{\zeta}_f(0) &= 0\,, \\
\hat{\zeta}_f'(0) &= -\sum_{l=\frac{1}{2}}^{\infty} \left(\ln \mathfrak{g}_l(i\nu_0) + \ln \mathfrak{g}_{-l}(i\nu_0) - \ln \mathfrak{g}_l^{\text{asym}}(i\nu_0) - \ln \mathfrak{g}_{-l}^{\text{asym}}(i\nu_0) - 2i\left(U_{12} - U_{21}\right)\right)\,,
\end{aligned} \tag{5.4.139}$$

Were it not for the last term, coming from (5.4.136), the sum over Fourier modes would suffer from a linear divergence. In turn, to compute $\zeta_d(s)$ we make use of the asymptotic form of the Jost function given in (5.4.134) and the results (5.4.102)-(5.4.103) to perform the momentum integrals, thus obtaining

$$\begin{aligned}
\hat{\zeta}_d(s) = &-\frac{2\Gamma\left(s+\frac{1}{2}\right)\Gamma(1-s)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty d\rho\, (\sinh\rho)^{2s+1} \left(\left(U_{11}+m\right)\left(U_{22}+m\right) - m^2\right) R_1(s,\rho) \\
&-\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma(1-s)}{2\Gamma\left(\frac{1}{2}\right)} \int_0^\infty d\rho\, (\sinh\rho)^{2s+1} \left(\left(U_{11}-U_{22}\right)^2 - \left(U_{12}+U_{21}\right)^2\right) R_2(s,\rho)\,.
\end{aligned} \tag{5.4.140}$$

The sums $R_1(s,\rho)$ and $R_2(s,\rho)$ become equal to (5.4.105) and (5.4.106), respectively, after shifting $l \to l - \frac{1}{2} \in \mathbb{N}^+$ and using $\nu_0 = m$. The shift is a legal operation since we have not set $s=0$ yet and the sums are convergent. Surely, the continuation of $R_1(s,\rho)$ and $R_2(s,\rho)$

is the same as before. Hence we arrive at

$$
\begin{aligned}
\zeta_d(0) =& -\int_0^\infty d\rho \, \sinh\rho \left((U_{11}+m)(U_{22}+m) - m^2 \right) \\
\hat{\zeta}_d'(0) =& -2\int_0^\infty d\rho \, \sinh\rho \left(\ln\left(\frac{\sinh\rho}{2}\right) + \gamma \right) \left((U_{11}+m)(U_{22}+m) - m^2 \right) \\
&+ \frac{1}{2}\int_0^\infty d\rho \, \sinh\rho \, \left((U_{11}-U_{22})^2 - (U_{12}+U_{21})^2 \right) \\
&- \sum_{l=\frac{1}{2}}^{\infty} \left(\ln \mathfrak{g}_l^{\text{asym}}(i\nu_0) + \ln \mathfrak{g}_{-l}^{\text{asym}}(i\nu_0) - \frac{2}{l+\frac{1}{2}} \int_0^\infty d\rho \, \rho \, \left((U_{11}+m)(U_{22}+m) - m^2 \right) \right) ,
\end{aligned}
$$

where we have used the expression (5.4.134) to recognize $\ln \mathfrak{g}_l^{\text{asym}}(i\nu_0) + \ln \mathfrak{g}_{-l}^{\text{asym}}(i\nu_0)$. Collecting all the pieces we obtain

$$
\begin{aligned}
\hat{\zeta}_{\mathcal{O}^2}(0) =& -\int_0^\infty d\rho \, \sinh\rho \left((U_{11}+m)(U_{22}+m) - m^2 \right) \\
\hat{\zeta}_{\mathcal{O}^2}'(0) =& -2\sum_{l=\frac{1}{2}}^{\infty} \left(\ln g_l^+(i\nu_0) + \ln g_{-l}^+(i\nu_0) - i\int d\rho \, (U_{21}-U_{12}) + \frac{1}{l+\frac{1}{2}}\zeta_{\mathcal{O}^2}(0) \right) + 2\gamma\hat{\zeta}_{\mathcal{O}^2}(0) \\
&- 2\int_0^\infty d\rho \, \sinh\rho \ln\left(\frac{\sinh\rho}{2}\right) \left((U_{11}+m)(U_{22}+m) - m^2 \right) \\
&+ \frac{1}{2}\int_0^\infty d\rho \, \sinh\rho \, \left((U_{11}-U_{22})^2 - (U_{12}+U_{21})^2 \right) ,
\end{aligned}
\tag{5.4.141}
$$

where we have made explicit that since $\lambda(i\nu_0) = 0$, the Jost functions $g_l^+(i\nu_0)$ and $g_l^-(i\nu_0)$ coincide. Finally, through (5.4.77), (5.4.70) and (5.4.136) we identify

$$
\hat{\zeta}_{\mathcal{O}_l^2}'(0) = -2\ln g_l^+(i\nu_0) - i\int_0^\infty d\rho \, (U_{12}-U_{21}) \, . \tag{5.4.142}
$$

Writing the potential components in terms of the background fields and recalling that $\hat{\zeta}_{\mathcal{O}}(s) = \frac{1}{2}\hat{\zeta}_{\mathcal{O}^2}(s)$ we arrive at our main result (5.2.16) for the determinant of a fermionic operator.

5.5 Conclusion

In this chapter we have explicitly computed the determinants for a general class of circularly-symmetric bosonic and fermionic operators in AdS_2 and spaces that are conformally AdS_2. In this context there are a number of options depending on the regularization technique used. Some widely used regularization techniques are not explicitly diffeormophism invariant. Our main result is to have obtained answers that are completely aligned with the zeta-function regularization method. Consequently, and importantly, we now have diffeormphic-invariant expressions for such determinants.

Our driving motivation has been to enlarge the arsenal of tools required to push the AdS/CFT correspondence into its precision regime. An important limitation of our computation is that it exploits, in a crucial manner, the angular symmetry of the problem. Namely, we are able to turn the problem into effectively a one-dimensional one due to the symmetry. There are many problems in this class, some we have mentioned but others are less obvious such as the one-loop correction to the anti-parallel lines. It would be interesting, however, to have a better understanding of the form of the determinant independently of the symmetries and ultimately a computational approach that is intrinsically two-dimensional. The drive to less symmetric situations is not merely an academic goal. There are examples which are under control from the localization point of view but where the symmetry is not preserved [127]. More general methods are still needed and it would be valuable to develop them.

Precision holography has largely focused on the results provided by supersymmetric localization. It would be great to connect with the efforts developed in the context of integrability [128],[129]. Integrability provides a wide field to explore from the point of semi-classical gravity computations. Ultimately, one would hope to tackle questions with less or no supersymmetry and where integrability does not play a role. We also expect that our methods will find use in other problems possibly related to one-loop gravity computations in the context of corrections to black hole entropy, as determinants in AdS_2 have already been found in many works starting with [8] and its sequels.

CHAPTER VI

Zeta-function Regularization of Holographic Wilson Loops

6.1 Introduction

The most studied examples at the quantum level are the holographic duals of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS Wilson loops in $\mathcal{N} = 4$ SYM. In the semi-classical approximation the one-loop corrections are equivalent to computations of determinants of certain Laplace-like operators in curved spaces. Determinant of operators in curved space have a long history in physics and also in mathematics as sources of spectral information. There are, indeed, various computational methods that have already been applied in the context of holographic Wilson loops. For example, the expectation value of the holographic $\frac{1}{2}$-BPS Wilson loop was originally computed using ζ-function techniques in [21] and subsequently revisited using the Gelfand-Yaglom approach in [23]. More recently the better-defined problem of computing the difference of the effective actions of the holographic $\frac{1}{4}$- and $\frac{1}{2}$-BPS strings has received particular attention since supersymmetric localization provides a precise answer. The first attempts were reported in [26, 27]. These two groups used a Gelfand-Yaglom based method to tackle the problem but did not find a match with the field theory prediction. Ultimately, after a careful analysis, the mismatch was traced back to a change in topology from the disk to the cylinder and the use of a diffeomorphism-invariant IR cutoff [29].

A more immediate motivation for developing ζ-function regularization techniques stems from the fact that using perturbative heat kernel techniques to the first nontrivial order in the latitude angle, the authors of [28] found a match between the gauge and gravity calculations for the expectation value of the $\frac{1}{4}$-BPS latitude Wilson loop. This suggests that ζ-function might be the correct framework to compute the one-loop determinants for the spectrum of fluctuations of the string; it also attacks the problem directly on the disk rather than mapping it to the cylinder as done in [23, 26, 27, 29]. The holographic dual to the $\frac{1}{2}$-BPS Wilson loop is a fundamental string with AdS_2 worldsheet. For this homogeneous space one can address its one-loop effective action with results dating back to [130, 131] as was done in [21, 25]. For the $\frac{1}{4}$-BPS, however, the space is no longer homogeneous and new technology is required to evaluate the determinants. In this chapter we approach the computation of one-loop determinants using recent results of ζ-function regularization of Laplace-like operators in conformally AdS_2 spaces that are reported in Chapter V. There is a strong general motivation to develop ζ-function regularization. Starting with the insightful works of [112],[113], ζ-function regularization methods have shown to be highly reliable in various areas of applications [114]; we hope that generalizing such methods will find natural applications in several contexts.

We show that the ζ-function regularized answer matches at leading order in the small latitude angle but receives correction at higher order, leading to a mismatch with the expected field theory answer.

The rest of the chapter is organized as follows. In section 6.2 we briefly review some of the most salient features of the semiclassical approach to holographic Wilson loops. Section 6.3 presents a summary of the result of our companion paper where we obtained explicit expressions for determinants of general Laplace-like operators in conformally AdS_2 spacetimes. Section 6.4 determines the ratio of the latitude to the $\frac{1}{2}$-BPS holographic Wilson loops. We conclude in section 6.5.

6.2 Latitude Wilson loops

For this chapter to be self-contained we briefly review some of the most salient features of the holographic Wilson loops we discuss. This subject has been the center of a lot of investigation recently and we refer the reader to the works [26, 27] for omitted details.

The $\frac{1}{4}$-BPS latitude Wilson loop (in the fundamental representation of $SU(N)$) is defined as [132, 119]

$$W(C) = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \oint_C ds \left(i A_\mu \dot{x}^\mu + |\dot{x}| \Phi_I \, n^I(s) \right) ,$$

where $\mathcal{P}$ denotes path ordering along the loop and C labels a curve parametrized as

$$x^\mu(s) = (\cos s, \sin s, 0, 0)\,, \qquad n^I(s) = (\sin\theta_0 \cos s, \sin\theta_0 \sin s, \cos\theta_0, 0, 0, 0), \qquad s \in (0, 2\pi)$$

For $\theta_0 = 0$, this operator was shown to preserve half of the supersymmetries and its expectation value was evaluated exactly, under certain conjectures [Gaussian], by [133] and [134]. The definitive proof was provided by Pestun via the by now thoroughly exploited supersymmetric localization technique [19]. The answer, exact in the gauge group rank N and the t' Hooft coupling λ, is

$$\langle W \rangle_{\text{circle}} \;\; = \;\; \frac{1}{N} L^1_{N-1} \left(-\frac{\lambda}{4N} \right) e^{\lambda/8N}. \tag{6.2.1}$$

More generally, for arbitrary values of θ_0, the vacuum expectation value of this operator is conjectured to be given by a simple re-scaling of the 't Hooft coupling $\lambda \to \lambda' = \lambda \cos^2\theta_0$ in the above exact expression [132, 119, 135].

The dual $\frac{1}{2}$-BPS string has an $AdS_2 \subset AdS_5$ worlsheet with disk topology,

$$ds^2 = d\rho^2 + \sinh^2\!\rho \; d\tau^2, \quad \rho \geq 0\,, \quad \tau \sim \tau + 2\pi\,. \tag{6.2.2}$$

On the other hand, the $\frac{1}{4}$-BPS string worldsheet forms a cap through the north pole of

$S^2 \subset S^5$ and the induced geometry is asymptotic to AdS_2,

$$ds^2_M = M(\rho)\, ds^2, \qquad M(\rho) = 1 + \frac{\sin^2\theta(\rho)}{\sinh^2\rho}\,, \qquad \sin\theta(\rho) = \frac{\sinh\rho\sin\theta_0}{\cosh\rho + \cos\theta_0}\,, \tag{6.2.3}$$

where $0 \leq \theta_0 \leq \frac{\pi}{2}$ is the latitude angle. The $\frac{1}{2}$-BPS solution corresponds to $\theta_0 = 0$. The string action can be evaluated on-shell on this classical solution. The result, after an appropriate renormalization, is [132]

$$S^{(0)} = -\sqrt{\lambda}\cos\theta_0\,. \tag{6.2.4}$$

Since $\langle W\rangle \simeq \exp\left(-S^{(0)}\right) = \exp\left(\sqrt{\lambda}\cos\theta_0\right)$, we recover, at the leading classical level, the expectation (6.2.5) from field theory.

Comparing the one-loop effective actions of the $\frac{1}{4}$ and $\frac{1}{2}$-BPS strings, as discussed in [26, 27], and anticipated in [23] leads to a better defined string theory problem since both dual strings have world-sheets with disk topology. The general expectation is that the issues related to ghost zero modes and other aspects of string perturbation theory on curved spacetimes might cancel upon considering the difference of effective actions. The exact field theory answer at large λ is

$$\frac{\langle W\rangle_{\text{latitude}}}{\langle W\rangle_{\text{circle}}} \simeq \exp\left(\sqrt{\lambda}(\cos\theta_0 - 1) - \frac{3}{2}\ln\cos\theta_0 + \ldots\right)\,. \tag{6.2.5}$$

The leading order term in the large λ expansion was matched against a particular string worldsheet identified in [132]. In recent years, there has been a strong effort in computing the $-(3/2)\ln\cos\theta_0$ term from the string theory one-loop effective action [26, 27, 28, 29]. In this work we approach this question using ζ-function regularization.

At the semiclassical level, the fluctuations of the fundamental string dual to the $\frac{1}{4}$-BPS

Wilson loop were thoroughly studied in [26, 27]. The spectrum involves the operators

$$\begin{aligned}
&\mathcal{O}_1(\theta_0) = M^{-1}\left(-g^{\mu\nu}\nabla_\mu\nabla_\nu + 2\right)\,, \qquad \mathcal{O}_2(\theta_0) = M^{-1}\left(-g^{\mu\nu}\nabla_\mu\nabla_\nu + V_2\right)\,,\\
&\mathcal{O}_{3\pm}(\theta_0) = M^{-1}\left(-g^{\mu\nu}D_\mu D_\nu + V_3\right)\,, \qquad D_\mu = \nabla_\mu \pm i\mathcal{A}_\mu\,,\\
&\mathcal{O}_\pm(\theta_0) = M^{-\frac{1}{2}}\left(-i\left(\not{D} + \frac{1}{4}\not{\partial}\ln M\right) - i\Gamma_{\underline{01}}\left(1+V\right) \pm W\right)\,, \qquad D_\mu = \nabla_\mu \pm \frac{i}{2}\mathcal{A}_\mu\,,
\end{aligned} \tag{6.2.6}$$

with $g_{\mu\nu}$ and ∇_μ evaluated for the AdS_2 metric (6.2.2), $\mathcal{A}_\rho = 0$, $\mathcal{A}_\tau = \mathcal{A}$ and

$$\begin{aligned}
&V_2(\rho) = -\frac{2\sin^2\theta(\rho)}{\sinh^2\rho}\,, \qquad V_3(\rho) = -\frac{\partial_\rho\mathcal{A}(\rho)}{\sinh\rho}\,, \qquad V(\rho) = \frac{1}{\sqrt{M(\rho)}} - 1\,,\\
&W(\rho) = \frac{\sin^2\theta(\rho)}{\sqrt{M(\rho)}\sinh^2\rho}\,, \qquad \mathcal{A}(\rho) = 1 - \frac{1+\cosh\rho\cos\theta(\rho)}{\cosh\rho + \cos\theta(\rho)}\,.
\end{aligned} \tag{6.2.7}$$

The difference in 1-loop effective actions with the $\frac{1}{2}$-BPS string is then

$$e^{-\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0)} = \left[\frac{\left(\frac{\det\mathcal{O}_+(\theta_0)}{\det\mathcal{O}_+(0)}\right)^4\left(\frac{\det\mathcal{O}_-(\theta_0)}{\det\mathcal{O}_-(0)}\right)^4}{\left(\frac{\det\mathcal{O}_1(\theta_0)}{\det\mathcal{O}_1(0)}\right)^3\left(\frac{\det\mathcal{O}_2(\theta_0)}{\det\mathcal{O}_2(0)}\right)^3\left(\frac{\det\mathcal{O}_{3+}(\theta_0)}{\det\mathcal{O}_{3+}(0)}\right)^1\left(\frac{\det\mathcal{O}_{3-}(\theta_0)}{\det\mathcal{O}_{3-}(0)}\right)^1}\right]^{\frac{1}{2}}\,. \tag{6.2.8}$$

The powers in the fermionic determinants reflect the Majorana nature of the spinors in Lorentzian signature.

The main difficulty in evaluating the above determinants is that the space is not homogeneous as is the case for $\theta_0 = 0$ where the results of [130, 131] are readily applied. A perturbative approach, valid for small values of θ_0, was taken in [28] leading to the following evaluation of the one-loop effective action

$$\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = -\frac{3}{4}\theta_0^2 + O\left(\theta_0^4\right)\,, \tag{6.2.9}$$

which coincides, to this order, with the expected field theory answer $\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \frac{3}{2}\ln\cos\theta_0$ as follows from Eq. 6.2.5. We will reproduce the perturbative result in this chapter and consider the more general problem at arbitrary θ_0.

6.3 Zeta-function regularization on AdS_2

In this section we recall a number of results for determinants of Laplace- and Dirac-like operators in AdS_2 from Chapter V. The method applies to operators defined on the AdS_2 geometry (6.2.2) and in the presence of external fields. Concretely, we consider general operators of the form:

$$\bar{\mathcal{O}} = -g^{\mu\nu} D_\mu D_\nu + m^2 + V\,, \qquad \text{(bosons)} \tag{6.3.10}$$

$$\bar{\mathcal{O}} = -i\left(\not{D} + \not{\partial}\Omega\right) - i\Gamma_{\underline{01}}\,(m+V) + W\,, \qquad \text{(fermions)} \tag{6.3.11}$$

with $D_\mu = \nabla_\mu - iq\mathcal{A}_\mu$. Under the assumption of circular symmetry, these operators can be expanded into their Fourier components

$$\begin{aligned}
\bar{\mathcal{O}}_l &= -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{(l-q\mathcal{A})^2}{\sinh^2\rho} + m^2 + V\,, \quad l\in\mathbb{Z}\,, \quad \text{(bosons)}\\
\bar{\mathcal{O}}_l &= -i\Gamma_{\underline{1}}\left(\partial_\rho + \frac{1}{2}\coth\rho + \partial_\rho\Omega\right) + \Gamma_{\underline{0}}\frac{(l-q\mathcal{A})}{\sinh\rho} - i\Gamma_{\underline{01}}\,(m+V) + W\,, \quad l\in\mathbb{Z}+\frac{1}{2}\,, \quad \text{(fermions)}
\end{aligned} \tag{6.3.12}$$

where we have set $\mathcal{A}_\rho = 0$ and $\mathcal{A}_\tau = \mathcal{A}(\rho)$, as well as $V = V(\rho)$, $W = W(\rho)$ and $\Omega = \Omega(\rho)$. Appropriate regularity conditions at the origin and fall-off conditions at infinity are required for the background fields (see chapter V for further details).

The ratio of determinants between the operators (6.3.10)-(6.3.11) and their free counterparts, obtained by setting $\mathcal{A} = \Omega = V = W = 0$, is defined using ζ-function regularization

$$\ln\frac{\det\bar{\mathcal{O}}}{\det\bar{\mathcal{O}}^{\text{free}}} \equiv -\hat{\zeta}'_{\bar{\mathcal{O}}}(0) - \ln(\mu^2)\hat{\zeta}_{\bar{\mathcal{O}}}(0)\,, \qquad \hat{\zeta}_{\bar{\mathcal{O}}}(s) \equiv \zeta_{\bar{\mathcal{O}}}(s) - \zeta_{\text{free}}(s)\,, \tag{6.3.13}$$

where μ is the renormalization parameter. Extending previous results [117] it is shown in Chapter V that these ratios are given by simple expressions. The result for bosons reads

$$\begin{aligned}
\ln\frac{\det\bar{\mathcal{O}}}{\det\bar{\mathcal{O}}^{\text{free}}} &= \ln\frac{\det\bar{\mathcal{O}}_0}{\det\bar{\mathcal{O}}_0^{\text{free}}} + \sum_{l=1}^{\infty}\left(\ln\frac{\det\bar{\mathcal{O}}_l}{\det\bar{\mathcal{O}}_l^{\text{free}}} + \ln\frac{\det\bar{\mathcal{O}}_{-l}}{\det\bar{\mathcal{O}}_{-l}^{\text{free}}} + \frac{2}{l}\hat{\zeta}_{\bar{\mathcal{O}}}(0)\right) - 2\left(\gamma + \ln\frac{\mu}{2}\right)\hat{\zeta}_{\bar{\mathcal{O}}}(0)\\
&\quad + \int_0^\infty d\rho\,\sinh\rho\ln\left(\sinh\rho\right)V - q^2\int_0^\infty d\rho\,\frac{\mathcal{A}^2}{\sinh\rho}\,,\\
\hat{\zeta}_{\bar{\mathcal{O}}}(0) &= -\frac{1}{2}\int_0^\infty d\rho\,\sinh\rho\,V\,,
\end{aligned}$$

whereas for fermions we have

$$
\begin{aligned}
\ln \frac{\det \bar{\mathcal{O}}}{\det \bar{\mathcal{O}}^{\text{free}}} &= \sum_{l=\frac{1}{2}}^{\infty} \left(\ln \frac{\det \bar{\mathcal{O}}_l}{\det \bar{\mathcal{O}}_l^{\text{free}}} + \ln \frac{\det \bar{\mathcal{O}}_{-l}}{\det \bar{\mathcal{O}}_{-l}^{\text{free}}} + \frac{2}{l+\frac{1}{2}} \hat{\zeta}_{\bar{\mathcal{O}}}(0) \right) - 2\left(\gamma + \ln \frac{\mu}{2}\right) \hat{\zeta}_{\bar{\mathcal{O}}}(0) \\
&\quad + \int_0^{\infty} d\rho \, \sinh\rho \ln(\sinh\rho) \Big((m+V)^2 - W^2 - m^2 \Big) - q^2 \int_0^{\infty} d\rho \, \frac{\mathcal{A}^2}{\sinh\rho} - \int_0^{\infty} d\rho \, \sinh\rho \, W^2, \\
\hat{\zeta}_{\bar{\mathcal{O}}}(0) &= -\frac{1}{2} \int_0^{\infty} d\rho \, \sinh\rho \Big((m+V)^2 - W^2 - m^2 \Big) \, ,
\end{aligned}
$$

with $\gamma \approx 0.57721$, the Euler-Mascheroni constant. In turn, the ratio for Fourier modes is computed as

$$
\frac{\det \bar{\mathcal{O}}_l}{\det \bar{\mathcal{O}}_l^{\text{free}}} = \begin{cases} \dfrac{\psi_l(\infty)}{\psi_l^{\text{free}}(\infty)} & , \quad \text{(bosons)} \\[4ex] \dfrac{\psi_l^{(i)}(\infty)}{\psi_l^{(i)\,\text{free}}(\infty)} e^{\Omega(\infty)-\Omega(0)} & , \quad \text{(fermions)} \end{cases} \tag{6.3.14}
$$

where $\psi_l(\rho)$ is the solution to the homogeneous equation that is regular at $\rho = 0$,

$$
\bar{\mathcal{O}}_l \psi_l = 0 \, , \qquad \psi_l(\rho) \underset{\rho \to 0}{\longrightarrow} \begin{cases} \rho^{|l|} & , \quad \text{(bosons)} \\[3ex] \rho^{|l|-\frac{1}{2}} & , \quad \text{(fermions).} \end{cases} \tag{6.3.15}
$$

For fermions $\psi_l^{(i)}(\rho)$ is one (either) of the two components of the regular spinor solution to the *first* order homogeneous equation. The overall normalization of ψ_l in (6.3.14) is not important as long as the leading coefficient of the small ρ expansion matches that of the free solution[1] ψ_l^{free}.

For the $\frac{1}{4}$-BPS strings we are interested in, the operators do not precisely take the form

[1] This is analogous to the usual conditions $\psi(0) = 0$, $\psi'(0) = 1$ imposed on the homogeneous solutions in the application of the Gelfand-Yaglom method to 1d determinants with Dirichlet boundary condition at the origin. In two and higher dimensions, the centrifugal barrier imposes the regular solution to vanish as a power law depending on the angular momentum, therefore generically $\psi'(0) = 0$.

(6.3.10) or (6.3.11), but rather they are conformally related to them

$$\mathcal{O} = M^{-1}\bar{\mathcal{O}}\,, \qquad \text{(bosons)} \tag{6.3.16}$$

$$\mathcal{O} = M^{-\frac{1}{2}}\bar{\mathcal{O}}\,. \qquad \text{(fermions)} \tag{6.3.17}$$

This can be understood as the effect of a Weyl rescaling of the metric by a function $M(\rho)$, which we assume to be smooth everywhere with $M(\rho) \to 1$ for $\rho \to \infty$. Happily, the determinants of $\mathcal{O}$ and $\bar{\mathcal{O}}$ are related by an anomaly calculation (cf. appendix A of [21]). Indeed[2],

$$\ln\left(\frac{\det\mathcal{O}}{\det\bar{\mathcal{O}}^{\text{free}}}\right) = \ln\left(\frac{\det\bar{\mathcal{O}}}{\det\bar{\mathcal{O}}^{\text{free}}}\right) + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[m^2 + V - \frac{1}{6}R + \frac{1}{12}\nabla^2\ln M\right] \tag{6.3.18}$$

for bosons, while for fermions the result is

$$\ln\left(\frac{\det\mathcal{O}}{\det\bar{\mathcal{O}}^{\text{free}}}\right) = \ln\left(\frac{\det\bar{\mathcal{O}}}{\det\bar{\mathcal{O}}^{\text{free}}}\right) + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right]. \tag{6.3.19}$$

6.4 One-loop effective action

In this section we apply the general results quoted in the previous section to the holographic description of the $\frac{1}{4}$-BPS latitude Wilson loops in $\mathcal{N} = 4$ SYM [119]. We refer the reader to the extensive literature for details; in particular to [26, 27, 28, 29].

Before plunging into the calculation of each individual ratio in (6.2.8), it is useful to combine the full spectrum of operators and gain some insight into the cancellations that occur in the one-loop effective action. Recall that, according to the discussion in section 6.3 (see eqns. (6.3.18) and (6.3.19)), the computation of each determinant is divided into two parts: an anomaly due to the Weyl transformation that maps the induced geometry (6.2.3) to AdS_2, and the ratio for the corresponding rescaled operators. Notice that for the

[2]Boundary terms involving the extrinsic curvature and the normal derivative of the conformal factor do not contribute in the present case (see Chapter V for details).

operators (6.2.6),

$$\mathcal{O}(\theta_0 = 0) = \bar{\mathcal{O}}(\theta_0 = 0) = \bar{\mathcal{O}}^{\text{free}}. \tag{6.4.20}$$

Let us focus on the Weyl anomaly first. One can check that the potential and mass terms for the $\frac{1}{4}$-BPS operators (6.2.6) satisfy

$$8 \times \left((1+V)^2 - W^2\right) - 3 \times 2 - 3 \times V_2 - 2 \times V_3 = -R + \nabla^2 \ln M\,, \tag{6.4.21}$$

a relation which is in fact a general feature of the gauge-fixed Nambu-Goto string, where the right hand side is recognized as the curvature of the induced metric, $R[Mg] = M^{-1}\left(R[g] - \nabla^2 \ln M\right)$. The contribution from the curvature and conformal factor terms in (6.3.18)-(6.3.19) is

$$\left(8 \times \left(\frac{1}{12}\right) - 8 \times \left(-\frac{1}{6}\right)\right) R + \nabla^2 \ln M \left(8 \times \left(-\frac{1}{24}\right) - 8 \times \left(\frac{1}{12}\right)\right) = 2R - \nabla^2 \ln M\,. \tag{6.4.22}$$

We then find that the modification to the ratio of determinants due to the rescaling of the metric is[3]

$$\text{anomaly}: \qquad \frac{1}{4\pi}\int d^2\sigma \sqrt{g}\, R \ln M = -\left(\theta_0 \sin\theta_0 + 4\cos^2\frac{\theta_0}{2} \ln\cos\frac{\theta_0}{2}\right)\,. \tag{6.4.23}$$

Unlike the case where one maps the induced worldsheet metric to flat space [29], the anomaly is non-vanishing[4]. This is an effect of the curvature of AdS_2 and is perfectly compatible with the conformal invariance of the string action [21] (see also appendix B of [136]).

We now move on to the computation of the determinants on AdS_2 using (6.3.14) and (6.3.14), starting with the total zeta-function at the origin

$$\hat{\zeta}_{\text{tot}}(0) = 3\hat{\zeta}_{\bar{\mathcal{O}}_1}(0) + 3\hat{\zeta}_{\bar{\mathcal{O}}_2}(0) + \hat{\zeta}_{\bar{\mathcal{O}}_{3+}}(0) + \hat{\zeta}_{\bar{\mathcal{O}}_{3-}}(0) - 4\hat{\zeta}_{\bar{\mathcal{O}}_+}(0) - 4\hat{\zeta}_{\bar{\mathcal{O}}_-}(0)\,. \tag{6.4.24}$$

[3]The attentive reader may notice an unexpected non-trigonometric (linear) dependence θ_0 in (6.4.23). This comes about because the primitive involves inverse trigonometric functions which when evaluated at the endpoints and for $\theta_0 \in (0, \frac{\pi}{2})$ simplify to the above expression.

[4]In that case the conformal factor is $M(\rho)\sinh^2\rho$. This becomes singular as $\rho \to \infty$, which forces the introduction of a large cut-off to regulate the divergences. Consequently, boundary terms must be added. These cancel, as does the bulk contribution since $R = 0$.

This quantity determines the dependence of the one-loop effective action on the renormalization scale. Equations (6.3.14) and (6.3.14) show a slightly different combination of potentials than in (6.4.21), namely,

$$8 \times \Big((1+V)^2 - W^2 - 1\Big) - 3 \times V_2 - 2 \times V_3 = \nabla^2 \ln M \,, \tag{6.4.25}$$

which by itself does not vanish. When integrated, however, it does,

$$\int_0^\infty d\rho \, \sinh\rho \, \nabla^2 \ln M = \sinh\rho \, \partial_\rho \ln M \Big|_0^\infty = 0 \qquad \Rightarrow \qquad \hat{\zeta}_{\rm tot}(0) = 0 \,. \tag{6.4.26}$$

As a consequence, no ambiguity related to the choice of renormalization scale, μ, affects the effective action. The above cancellation also means that the Fourier sum of the combined bosons and fermions one-dimensional radial determinants does not need regularization[5], in accordance with the calculations of [26, 27].

A related quantity involving the same combination of potentials as $\hat{\zeta}_{\rm tot}(0)$ is the sum of $\ln(\sinh\rho)$ integrals in (6.3.14) and (6.3.14), which when added to the Weyl anomaly gives

$$\text{anomaly} + \ln\sinh\rho : \qquad \int_0^\infty d\rho \, \sinh\rho \left(\frac{1}{2} R \ln M + \ln(\sinh\rho) \, \nabla^2 \ln M \right) = -2 \ln\cos\frac{\theta_0}{2} \,. \tag{6.4.27}$$

As we will see, this terms cancels the reminder that was found in [26, 27]. We can also keep track of the contribution coming from the gauge field, easily seen to vanish:

$$\mathcal{A}^2 : \qquad 1 \times (1)^2 + 1 \times (-1)^2 - 4 \times \left(\frac{1}{2}\right)^2 - 4 \times \left(-\frac{1}{2}\right)^2 = 0 \,. \tag{6.4.28}$$

In contrast, the last term in (6.3.14) involving the fermionic potential gives

$$W^2 : \qquad -8 \times \int_0^\infty d\rho \, \sinh\rho \, W^2 = 4\theta_0 \sin\theta_0 - 16 \sin^2\frac{\theta_0}{2} \,. \tag{6.4.29}$$

[5] Each term in (6.4.24) is responsible for subtracting the divergence in the sum over Fourier modes in each individual determinant (see (6.3.14) and (6.3.14)).

Ultimately, this expression accounts for the mismatch with the gauge theory prediction (6.2.5).

Finally, one can check that the radial determinants at fixed Fourier mode coincide with those presented in [26, 27]. Therefore,

$$\sum_l \ln \frac{\det \bar{\mathcal{O}}_l}{\det \bar{\mathcal{O}}_l^{\text{free}}} : \qquad -3 \ln \cos \theta_0 + 2 \ln \cos \frac{\theta_0}{2} \, . \tag{6.4.30}$$

In hindsight this was to be expected since the calculation involves solving a set of homogeneous equations in AdS which translate into those of [26, 27] after an appropriate Weyl transformation of the metric and properly adjusting the potentials and connection terms. The difference in the present case is that instead of imposing a sharp Dirichlet boundary condition at small but finite value of ρ as in [26, 27], here we only require regularity of the solutions at the center of the disk. Nevertheless, the answer is the same.

Putting all the above results together, the final expression for the difference in the one-loop effective actions of the $\frac{1}{4}$ and $\frac{1}{2}$-BPS strings is

$$\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \frac{3}{2} \ln \cos \theta_0 + 2 \left(4 \sin^2 \frac{\theta_0}{2} - \theta_0 \sin \theta_0 \right) = -\frac{3}{4}\theta_0^2 + O\left(\theta_0^4\right) \, . \tag{6.4.31}$$

As indicated above, when taking the small θ_0 limit, our holographic answer coincides with the field theory prediction (6.2.5), just as in the perturbative ζ-function computation of [28].

Let us briefly comment on this result. Recall that the works of [26, 27] computed the effective action by looking only at the sum of the radial determinants, finding the reminder $\ln \cos \frac{\theta_0}{2}$ in (6.4.30). Recently, it was argued in [29] that this term is corrected for if a diffeomorphism-invariant regulator is used in the calculation, producing a match between the string theory calculation and the gauge theory prediction. In contrast, the ζ-function formalism is automatically diffeomorphism-invariant, and we see that this reminder disappears due to the combination (6.4.27). Alas, there is an extra contribution coming from the fermionic potential W^2 that yields a mismatch with the gauge theory calculation. At the moment we dare not speculate about the origin of this term.

For completeness, we present the results for each individual determinant in the spectrum. Taking into account (6.4.20)

$$
\begin{aligned}
\ln\left(\frac{\det\,\mathcal{O}_1(\theta_0)}{\det\,\mathcal{O}_1(0)}\right) &= \theta_0\sin\theta_0 + \frac{1}{2}\sin^2\frac{\theta_0}{2} + \left(\frac{7}{3} + 2\cos\theta_0\right)\ln\left(\cos\frac{\theta_0}{2}\right) \\
&= \frac{7}{12}\theta_0^2 + O\left(\theta_0^4\right)\,, \\
\ln\left(\frac{\det\,\mathcal{O}_2(\theta_0)}{\det\,\mathcal{O}_2(0)}\right) &= -\theta_0\sin\theta_0 + \frac{9}{2}\sin^2\frac{\theta_0}{2} + \left(\frac{7}{3} + 2\cos\theta_0\right)\ln\left(\cos\frac{\theta_0}{2}\right) - 2\ln\left(\Gamma\left(\cos\theta_0\right)\right) - \ln\left(\cos\theta_0\right) \\
&= \left(\frac{1}{12} - \gamma\right)\theta_0^2 + O\left(\theta_0^4\right)\,, \\
\ln\left(\frac{\det\,\mathcal{O}_{3\pm}(\theta_0)}{\det\,\mathcal{O}_{3\pm}(0)}\right) &= \frac{1}{2}\sin^2\frac{\theta_0}{2} + \left(\frac{7}{3} + 2\cos\theta_0\right)\ln\left(\cos\frac{\theta_0}{2}\right) - \ln\left(\Gamma\left(\cos\theta_0\right)\right) - \ln\left(\cos\theta_0\right) \\
&= \frac{1}{2}\left(\frac{1}{6} - \gamma\right)\theta_0^2 + O\left(\theta_0^4\right)\,, \\
\ln\left(\frac{\det\,\mathcal{O}_{\pm}(\theta_0)}{\det\,\mathcal{O}_{\pm}(0)}\right) &= \frac{1}{2}\theta_0\sin\theta_0 + \left(\frac{7}{3} + 2\cos\theta_0\right)\ln\left(\cos\frac{\theta_0}{2}\right) - \ln\left(\Gamma\left(\cos\theta_0\right)\right) - \ln\left(\cos\theta_0\right) \\
&= \frac{1}{2}\left(\frac{11}{12} - \gamma\right)\theta_0^2 + O\left(\theta_0^4\right)\,.
\end{aligned}
$$

Our results match the perturbative heat kernel calculation of [28]. Notice that the first ratio is entirely an effect of the Weyl anomaly, since the rescaled operators for the $\frac{1}{4}$-BPS and the $\frac{1}{2}$-BPS solutions coincide. Actually, we have checked that all the ratios for the rescaled operators, without including the anomaly, also match with the perturbative method for a fixed AdS_2 metric. It would be interesting to extend the perturbative heat kernel results of [28] to the next order in θ_0.

6.5 Conclusion

In this chapter we have computed the difference of one-loop effective actions of the $\frac{1}{4}$- and $\frac{1}{2}$-BPS strings using ζ-function regularization. We were encouraged and motivated by a previous perturbative heat kernel computation reporting agreement with the field theory prediction at the first nontrivial order in the latitude angle θ_0 [28]. It is worth highlighting that we tackled the computation directly on the hyperbolic disk rather than mapping the problem to a cylinder, as has been traditionally done [23, 26, 27, 29]. Along these lines, it would be an elucidating step to adapt our results to compute the ζ-function for circularly symmetric operators defined on the flat cylinder geometry. This would shed some light on the role of the diffeomorpism-invariant regulator advocated in [29]. We hope to pursue these directions in the near future.

Alas, our complete computation shows that at higher order in θ_0 the agreement is lost. We are thus, left facing a puzzle. Armed with the supersymmetric localization answer we can indulge in a form of answer analysis. As stated before, the remainder of previous calculations does not appear in our approach since ζ-function regularization is explicitly diffeormorphism invariant. One identifiable culprit for the discrepancy we now faced is the term proportional to W^2 in the expression for the fermions. We suspect that ultimately some aspects of chiral symmetry might be at play, as suggested in [137] in a different context. Another potential problem underlying our discrepancy could be supersymmetry. We do not see how to move forward in this direction at the moment but find it quit plaussible to be the cause of the discrepancy. This work is a push in understanding the role of technical methods needed to tackle precision computations in holography and we are certain that its application will go beyond the one presented here.

CHAPTER VII

Precision Holography in Type IIA with Wilson Loops

7.1 Introduction

The advent of localization techniques has provided a plethora of exact results relevant for the field theory sides of AdS/CFT correspondence, that is, for $\mathcal{N} = 4$ SYM [19] and for ABJM [138]. In this context, it is then natural to extrapolate the exact field theory results to the regime where they could be directly compared with the supergravity and semiclassical approximations. This approach was attempted very early on in the insightful work of Drukker, Gross and Tseytlin [21]; it did not, however, led to a match with the field theory prediction. This discrepancy motivated much work [22, 23, 24, 25] that largely confirmed the original discrepancy. A recent revival of this line of effort took place in [26, 27] which considered, on the gravity side, the one-loop effective actions corresponding to the ratio of the expectation values of the $\frac{1}{4}$ to the $\frac{1}{2}$ BPS Wilson loops. Various groups have made important subsequent contributions to this question [28] and recently a precise match has been described, for the $\mathcal{N} = 4$ SYM case, [29] after imposing a diffeomorphism preserving cutoff.

In this chapter we take one step in the direction of extending some of the techniques developed thus far to the context of the AdS/CFT pair $AdS_4 \times \mathbb{CP}^3$/ABJM. We hope that by turning our attention to the AdS_4/ABJM pair we can gather complementary information to the one already available and ultimately learn about string perturbation theory in curved backgrounds with Ramond-Ramond fluxes. There are, indeed, a number of exact results obtained via localization of the ABJM theory starting with the free energy of the theory on S^3 [138] but most importantly to us there are various exact results for supersymmetric Wilson loops for the $\frac{1}{2}$ BPS [20] and, more recently, for the $\frac{1}{6}$ BPS configuration [139]. We consider one-loop effective actions of string configurations dual to those supersymmetric Wilson loops in ABJM. Our focus is in understanding some aspects of the picture of precision holography, that is, the matching of sub-leading corrections on the string theory side with the prediction of field theory. As the first step in attacking the $\mathcal{N}=6$ case, in this first work we provide all the details to set the wheels of precision holography in $AdS_4 \times \mathbb{CP}^3$/ABJM with Wilson loops in motion.

The rest of the chapter is organized as follows. We briefly review the field theory status of the expectation values of the relevant Wilson loops in section 7.2. In section 7.3 we discuss the classical string configurations and in section 7.4 we present the quadratic fluctuations. The string theory semiclassical one-loop effective action is equivalent to the computations of quotients of determinants. In section 7.5 we consider the perturbative computation of determinants to first non-trivial order in the latitude angle θ_0. Section 6 tackles the computation of the one-loop effective actions using ζ-function regularization. We conclude with some comments and open problems in section 7.6. We relegate a number of more technical aspects to a series of appendices, including: conventions D, geometric data E, an explicit discussion of regularity conditions for the gauge fields F, and details of the fermionic reduction G.

7.2 The $\frac{1}{6}$-BPS Latitude Wilson Loop

The ABJM theory is a three-dimensional Chern-Simons-matter theory with $U(N)_k \times U(N)_k$ gauge group where the subindices indicate the Chern-Simons level [18]. The matter sector contains four complex scalar fields $C_I, (I = 1, 2, 3, 4)$ in the bifundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ and the corresponding complex conjugate in the $(\bar{\mathbf{N}}, \mathbf{N})$ representation; the theory also contains fermionic superpartners (see [18] for details).

To build $\frac{1}{6}$ supersymmetric Wilson loops, one starts considering only one of the gauge fields of the whole $U(N) \times U(N)$ gauge group, denoted by A_μ. To preserve supersymmetry we need to include a contribution from the matter sector. The main intuition comes from the construction of supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM. However, in the absence of adjoint fields, an appropriate combination of bi-fundamentals, C_I, namely [140, 141, 142] is required:

$$W_{\mathcal{R}} = \frac{1}{\dim[\mathcal{R}]} \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \int \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}|\, M^I_J C_I \bar{C}^J \right) ds, \tag{7.2.1}$$

where $\mathcal{R}$ denotes the representation. It was shown in [140, 141, 142] that the above operator preserves $\frac{1}{6}$ of the 24 supercharges when the loop is a straight line or a circle and the matrix takes the form $M^I_J = \mathrm{diag}\,(1, 1, -1, -1)$.

A remarkable result of [138] was to show that the computation of the vacuum expectation values of these Wilson loops reduces to a matrix model. Namely, the Wilson loop vev is obtained by inserting $\mathrm{Tr}_{\mathcal{R}}\, e^{\mu_i}$ inside the following partition function:

$$Z(N,k) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \frac{d\nu_i}{2\pi} \frac{\prod_{i<j} \left(2\sinh \frac{\mu_i - \mu_j}{2}\right)^2 \left(2\sinh \frac{\nu_i - \nu_j}{2}\right)^2}{\prod_{i,j} \left(2\cosh \frac{\mu_i - \nu_j}{2}\right)^2} \exp\left[\frac{ik}{4\pi} \sum_i (\mu_i^2 - \nu_i^2)\right]. \tag{7.2.2}$$

A particularly impressive exact result was the computation of the supersymmetric free energy of ABJM on S^3 in terms of Airy functions [143, 144] which elucidated various aspects of the interpolation between week and strong coupling in the context of ABJM. The results that are more relevant for our current work pertain exact evaluations of Wilson loops. The construction of the Wilson loop presented above in equation (7.2.1) does not capture the $\frac{1}{2}$ BPS string configurations. These involved the introduction of a superconnection [145]. The exact expectation values of the $\frac{1}{2}$- and certain $\frac{1}{6}$-BPS Wilson loops were presented in [20] and take the general form

$$\langle W^{\frac{1}{2}}_{\square} \rangle = \frac{1}{4} \csc\left(\frac{2\pi}{k}\right) \frac{\mathrm{Ai}\left[\left(\frac{2}{\pi^2 k}\right)^{-1/3} \left(N - \frac{k}{24} - \frac{7}{3k}\right)\right]}{\mathrm{Ai}\left[\left(\frac{2}{\pi^2 k}\right)^{-1/3} \left(N - \frac{k}{24} - \frac{1}{3k}\right)\right]}, \tag{7.2.3}$$

where the denominator is recognized as the partition function of the ABJM theory obtained in [54, 55]. The above result and many others in this class are exact to all orders in $1/N$, up to exponentially small corrections in N. Recently, in [139], a matrix model for the exact evaluation of the latitude BPS Wilson loops has been proposed. The expectation value for any genus of the fermionic (in the sense of the superconnection [145]) latitude Wilson loop is given in terms of Airy functions by (see equations (1.3) and (5.44) in [139]),

$$\langle W_F^{\frac{1}{6}}(\nu)\rangle_\nu = -\frac{\nu\,\Gamma(-\frac{\nu}{2})\,\mathrm{Ai}\big[\big(\frac{2}{\pi^2 k}\big)^{-1/3}\big(N-\frac{k}{24}-\frac{6\nu+1}{3k}\big)\big]}{2^{\nu+2}\,\sqrt{\pi}\,\Gamma\big(\frac{3-\nu}{2}\big)\,\sin\big(\frac{2\pi\nu}{k}\big)\,\mathrm{Ai}\big[\big(\frac{2}{\pi^2 k}\big)^{-1/3}\big(N-\frac{k}{24}-\frac{1}{3k}\big)\big]}, \tag{7.2.4}$$

where $\nu=\sin(2\alpha)\,\cos\theta_0$, the angle α can be freely chosen and determines the coupling to matter, the geometric parameter we are interested in is θ_0, and $0\le\nu\le 1$. The beautiful result above is the culmination of an impressive series of papers [146, 147, 148, 149] (see also[150, 151]).

The fermionic latitude Wilson loop maps to a type IIA string configuration in the $AdS_4\times\mathbb{CP}^3$ background with endpoints moving in a circle inside $\mathbb{CP}^3$. When expanded to the regime of validity of the holographic computation, namely, taking the leading genus-zero expansion in the above, it has been shown to coincide with the semi-classical string computation of the $\frac{1}{6}$-BPS Wilson loop expectation value [152].

$$\langle W_F^{\frac{1}{6}}(\nu)\rangle_\nu|_{g=0} = -\iota\frac{2^{-\nu-2}\,\kappa^\nu\,\Gamma\big(-\frac{\nu}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{3}{2}-\frac{\nu}{2}\big)} \tag{7.2.5}$$

We will consider the ratio of $\frac{1}{6}$-BPS Wilson loop expectation value with the $\frac{1}{2}$-BPS one, dual to a circular Wilson loop. Therefore, the field theory prediction to be matched to our computation of the one-loop effective action of the string configuration takes the form ($\nu=\cos\theta_0$)

$$\begin{aligned}\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \ln\left[\frac{\langle W_F^{\frac{1}{6}}(\nu)\rangle_\nu}{\langle W_F^{\frac{1}{2}}(1)\rangle_1}\right] &= \ln\left(\frac{1}{\pi}\,\cot\left(\pi\frac{\cos\theta_0}{2}\right)\right) - \ln\left(\sin^2\frac{\theta_0}{2}\right)\\ &+ 2\,\ln\Gamma\left(\cos^2\frac{\theta_0}{2}\right) - \ln\big(\Gamma(\cos\theta_0)\big) - \ln\big(\cos\theta_0\big)\\ &= \frac{1}{2}\,\theta_0^2 + O(\theta_0^4).\end{aligned} \tag{7.2.6}$$

Anticipating the use of a perturbative result using heat kernel techniques, in the last line above we have expanded the field theory answer for small latitude angle θ_0.

7.3 String Configurations Dual to Supersymmetric Wilson Loops

In this section we review the classical string configurations dual to the fermionic latitude family of BPS Wilson loops. We present these results for the convenience of the reader and to set up our notation but refer the interested reader to the original literature [140, 141, 142] for the $\frac{1}{2}$ BPS cofiguration and [152] for the latitude $\frac{1}{6}$ BPS configuration.

7.3.1 The $AdS_4 \times \mathbb{CP}^3$ background

The Euclidean AdS_4 ($EAdS_4$) metric is written as an $\mathbb{H}_2 \times S^1$ foliation,

$$ds^2_{EAdS_4} = \cosh^2 u \left(\sinh^2 \rho \, d\psi^2 + d\rho^2\right) + \sinh^2 u \, d\phi^2 + du^2 \,. \tag{7.3.7}$$

Similarly, the metric on $\mathbb{CP}^3$ is taken to be

$$\begin{aligned} ds^2_{\mathbb{CP}^3} = \frac{1}{4} \Big[& d\alpha^2 + \cos^2 \frac{\alpha}{2} \left(d\vartheta_1^2 + \sin^2 \vartheta_1 \, d\varphi_1^2\right) + \sin^2 \frac{\alpha}{2} \left(d\vartheta_2^2 + \sin^2 \vartheta_2 \, d\varphi_2^2\right) \\ & + \cos^2 \frac{\alpha}{2} \sin^2 \frac{\alpha}{2} \left(d\chi + \cos\vartheta_1 \, d\varphi_1 - \cos\vartheta_2 \, d\varphi_2\right)^2 \Big] \,. \end{aligned} \tag{7.3.8}$$

The full metric is

$$ds^2 = L^2 \left(ds^2_{EAdS_4} + 4 \, ds^2_{\mathbb{CP}^3}\right) \,, \qquad L^2 = \frac{R^3}{4k} \,. \tag{7.3.9}$$

Finally, the remaining background fields are

$$e^{\Phi} = \frac{2L}{k} \,, \qquad F_{(4)} = -\frac{3ikL^2}{2} \mathrm{vol}\left(AdS_4\right) \,, \qquad F_{(2)} = \frac{k}{4} dA \,, \tag{7.3.10}$$

where

$$\mathrm{vol}\,(AdS_4) = \cosh^2 u \sinh u \sinh\rho\, d\psi \wedge d\rho \wedge du \wedge d\phi\,, \tag{7.3.11}$$

$$A = \cos\alpha\, d\chi + 2\cos^2\frac{\alpha}{2}\cos\vartheta_1\, d\varphi_1 + 2\sin^2\frac{\alpha}{2}\cos\vartheta_2\, d\varphi_2\,. \tag{7.3.12}$$

The factor of i in $F_{(4)}$ is due to the Euclidean continuation. The 2-form is proportional to the Kahler form in $\mathbb{CP}^3$.

7.3.2 Classical String Solution

The classical 1/6-BPS string solution we are interested in has

$$\begin{aligned} u &= 0\,, \qquad \rho' = -\sinh\rho\,, \qquad \psi = \tau\,, \\ \alpha &= 0\,, \qquad \vartheta_1' = -\sin\vartheta_1\,, \qquad \varphi_1 = \tau\,. \end{aligned} \tag{7.3.13}$$

The induced metric is then

$$ds^2 = L^2 A\left(d\tau^2 + d\sigma^2\right)\,, \qquad A = \sinh^2\rho + \sin^2\vartheta_1^2 = \rho'^2 + \vartheta_1'^2\,. \tag{7.3.14}$$

The solution to (7.3.13) involves the latitude parameter θ_0. We write,

$$\sinh\rho = \frac{1}{\sinh\sigma}\,, \qquad \sin\vartheta_1 = \frac{1}{\cosh(\sigma+\sigma_0)}\,, \qquad \cos\theta_0 = \tanh\sigma_0. \tag{7.3.15}$$

The induced geometry is disk shaped and asymptotes AdS_2 at the boundary. The 1/2-BPS limit corresponds to $\sigma_0 \to \infty$ for which the induced geometry becomes exactly AdS_2.

7.3.3 Symmetries of the classical solution

We start by recalling that the background geometry is constructed out from coset spaces $AdS_4 = SO(2,3)/SO(1,3)$ and $\mathbb{CP}^3 = SU(4)/SU(3)\times U(1)$.

Before gauge-fixing, the string embedding is characterized by 10 worldsheet scalars $x^m(\tau,\sigma)$ and a 10-dimensional Majorana spinor θ whose dynamics is determined by the type IIA Green-Schwarz action (more details below). The symmetries of the theory are:

- Local:
  - Diffeomorphisms:

$$\delta_\xi x^m = \xi^a \partial_a x^m \,, \qquad \delta_\xi \theta = \xi^a \partial_a \theta \,, \tag{7.3.16}$$

    where ξ^a is an arbitrary worldsheet vector field.

  - κ-symmetry:

$$\delta_\kappa x^m = \frac{i}{2}\bar{\theta}\Gamma^m \delta_\kappa \theta \,, \qquad \delta_\kappa \theta = (1 + \Gamma_F)\,\kappa \,, \qquad \Gamma_F = \frac{\epsilon^{ab}}{2\sqrt{-g}}\Gamma_{ab}\Gamma_{11} \,, \tag{7.3.17}$$

    where κ is an arbitrary 10-dimensional Majorana spinor and worldsheet scalar.

- Global:
  - Target space isometries:

$$\delta_\lambda x^m = K^m \,, \qquad \delta_\lambda \theta = K^a \partial_a \theta - \frac{1}{4}\left(\nabla_m K_n - \nabla_n K_m\right)\Gamma^{mn}\theta \,, \tag{7.3.18}$$

    where K^m is any target space Killing vector and $K_a = \partial_a x^m K_m$.

  - Target space supersymmetries:

$$\delta_\epsilon x^m = -\frac{i}{2}\bar{\theta}\Gamma^m \delta_\epsilon \theta \,, \qquad \delta_\epsilon \theta = \epsilon \,, \qquad D_m \epsilon = 0 \,, \tag{7.3.19}$$

    where ϵ is any target space Killing spinor.

Given a classical solution (with fermions set to zero, $\theta = 0$), the preserved bosonic symmetries correspond to the set of transformations satisfying

$$\delta x^m = 0 \qquad \Rightarrow \qquad K^m + \epsilon^a \partial_a x^m = 0 \,. \tag{7.3.20}$$

In other words, the target space isometries inherited by the solution are those that leave the embedding invariant up to worldsheet diffeomorphisms. Contracting this condition with

$g_{mn}\partial_a x^n$ we can solve

$$\epsilon^a = -K^a\,, \tag{7.3.21}$$

where $K_a = \partial_a x^m K_m$. This in turn implies that, in order to generate a symmetry, the Killing vector must satisfy

$$K^m = g^{ab}\partial_a x^m \partial_b x^n K_n\,. \tag{7.3.22}$$

The logic for the fermionic symmetries is the same. The ones preserved by the background are those satisfying

$$\delta\theta = 0 \qquad \Rightarrow \qquad \epsilon + (1+\Gamma_F)\,\kappa = 0\,. \tag{7.3.23}$$

These are target space supersymmetries which can be compensated by a local κ-symmetry transformation. Multiplying by $(1-\Gamma_F)$, we find that

$$(1-\Gamma_F)\,\epsilon = 0\,. \tag{7.3.24}$$

This is the usual condition for preserved supersymmetries. This condition is in fact sufficient since then we can solve

$$\kappa = -\frac{1}{2}\epsilon\,. \tag{7.3.25}$$

For the case at hand, we find that the $AdS_4 \times \mathbb{CP}^3$ Killing vectors

$$\begin{aligned}
K_1 &= \partial_\psi + \partial_{\varphi_1}\,, \qquad (7.3.26)\\
K_2 &= \partial_\phi\,,\\
K_3 &= -\cos\varphi_2\,\partial_{\vartheta_2} + \cot\vartheta_2\sin\varphi_2\,\partial_{\varphi_2} + \frac{\sin\varphi_2}{\sin\vartheta_2}\partial_\chi\,,\\
K_4 &= \sin\varphi_2\,\partial_{\vartheta_2} + \cot\vartheta_2\cos\varphi_2\,\partial_{\varphi_2} + \frac{\cos\varphi_2}{\sin\vartheta_2}\partial_\chi\,,\\
K_5 &= \partial_{\varphi_2}\\
K_6 &= \partial_\chi\,,
\end{aligned}$$

generate a symmetry of the solution. The first Killing vector must be accompanied by a translation in the worldsheet coordinate τ such that $\epsilon^\tau_{cl} = -\lambda_{cl}$ and $\epsilon^\sigma_{cl} = 0$; it corresponds to an isometry of the induced geometry. The rest have zero norm on the worldsheet so $\epsilon^a_{cl} = 0$. Altogether we have a $\underbrace{U(1)}_{K_1} \times \underbrace{U(1)}_{K_2} \times \underbrace{SU(2)}_{K_3,K_4,K_5} \times \underbrace{U(1)}_{K_6}$ symmetry.

The geometric interpretation of the symmetries is most easily seen in the embedding coordinates of $EAdS_4 \subset \mathbb{R}^5$ and the Hopf fibration $S^1 \hookrightarrow S^7 \to \mathbb{CP}^3$:

$$
\begin{aligned}
1 &= X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2\,, \\
ds^2 &= -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2\,,
\end{aligned}
\qquad
\begin{aligned}
X_0 &= \cosh u \cosh \rho\,, \\
X_1 &= \cosh u \sinh \rho \cos \psi\,, \\
X_2 &= \cosh u \sinh \rho \sin \psi\,, \\
X_3 &= \sinh u \cos \phi\,, \\
X_4 &= \sinh u \sin \phi\,,
\end{aligned}
\tag{7.3.27}
$$

$$
z_1 = \cos\frac{\alpha}{2}\cos\frac{\vartheta_1}{2} e^{\frac{i}{2}(\varphi_1 + \frac{\chi}{2})}\,, \qquad z_3 = \sin\frac{\alpha}{2}\cos\frac{\vartheta_2}{2} e^{\frac{i}{2}(\varphi_2 - \frac{\chi}{2})}\,, \tag{7.3.28}
$$

$$
z_2 = \cos\frac{\alpha}{2}\sin\frac{\vartheta_1}{2} e^{\frac{i}{2}(-\varphi_1 + \frac{\chi}{2})}\,, \qquad z_4 = \sin\frac{\alpha}{2}\sin\frac{\vartheta_2}{2} e^{\frac{i}{2}(-\varphi_2 - \frac{\chi}{2})}\,. \tag{7.3.29}
$$

The worldsheet has $z_3 = z_4 = 0$.

In the next section we will consider perturbations of the string embedding around the classical solution and look at the transformation properties of the fluctuations under the preserved symmetries. It will prove convenient to take linear combinations of K_3, K_4 and K_5 that have a simple action on the fluctuations. We find that such combinations are

$$
K_3' = \cos(\vartheta_2^{cl})\left(\sin(\varphi_2^{cl})K_3 + \cos(\varphi_2^{cl})K_4\right) + \sin(\vartheta_2^{cl})K_5\,, \tag{7.3.30}
$$

$$
K_4' = \cos(\varphi_2^{cl})K_3 - \sin(\varphi_2^{cl})K_4\,, \tag{7.3.31}
$$

$$
K_5' = \sin(\vartheta_2^{cl})\left(\sin(\varphi_2^{cl})K_3 + \cos(\varphi_2^{cl})K_4\right) - \cos(\vartheta_2^{cl})K_5\,, \tag{7.3.32}
$$

$$
\begin{aligned}
K_3' = {} & \cos(\vartheta_2^{cl})\sin(\varphi_2 - \varphi_2^{cl})\partial_{\vartheta_2} + \left(\cot\vartheta_2\cos(\vartheta_2^{cl})\cos(\varphi_2 - \varphi_2^{cl}) + \sin\vartheta_2^{cl}\right)\partial_{\varphi_2} \\
& + \frac{\cos(\vartheta_2^{cl})\cos(\varphi_2 - \varphi_2^{cl})}{\sin\vartheta_2}\partial_\chi\,,
\end{aligned}
\tag{7.3.33}
$$

where ϑ_2^{cl} and φ_2^{cl} are the (constant) values that the coordinates ϑ_2 and φ_2 take on the classical solution. We shall drop the primes henceforth.

7.4 Quadratic Fluctuations

Having reviewed the classical solution dual to the $\frac{1}{6}$-BPS latitude Wilson loop and its symmetries, in this section we derive the corresponding spectrum of quadratic fluctuations. There has already been some previous work for the case of the $\frac{1}{2}$-BPS configuration in [153] and [118] whose spectrum is a limit of our result. We will start by giving a general expression for the quadratic fluctuations of the type IIA string in $AdS_4 \times \mathbb{CP}^3$ and then specialize to the case of the $\frac{1}{6}$ BPS string dual to the latitude Wilson loop. In what follows, target-space indices are denoted by $m, n, \ldots$, world-sheet indices are $a, b, \ldots$, while the directions orthogonal to the string are represented by $i, j, \ldots$. All corresponding tangent space indices are underlined.

7.4.1 Type IIA strings on $AdS_4 \times \mathbb{CP}^3$

In the bosonic sector, the string dynamics is dictated by the Nambu-Goto (NG) action

$$S_{\rm NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-g}\,, \tag{7.4.34}$$

where g_{ab} is the induced metric on the world sheet and $g = \det g_{ab}$. Our first goal in this section is to consider perturbations $x^m \to x^m + \varepsilon y^m$, $\varepsilon \ll 1$, around any given classical embedding and find the quadratic action that governs them. To this purpose, let us choose convenient vielbeins for the $AdS_4 \times \mathbb{CP}^3$ metric that are properly adapted to the study of fluctuations. Using the local $SO(9,1)$ symmetry, we can always pick a frame $E^{\underline{m}} = (E^{\underline{a}}, E^{\underline{i}})$ such that the pullback of $E^{\underline{a}}$ onto the world-sheet forms a vielbein for the induced metric, while the pullback of $E^{\underline{i}}$ vanishes. Of course, these are nothing but the 1-forms dual to the tanget and normal vectors fields, respectively. The Lorentz symmetry is consequently broken to $SO(1,1) \times SO(8)$. Having made this choice we may define the fields

$$\chi^{\underline{m}} \equiv E^{\underline{m}}{}_m y^m \,, \tag{7.4.35}$$

and gauge fix the diffeomorphism invariance by freezing the tangent fluctuations, namely, by requiring

$$\chi^{\underline{a}} = 0 \,. \tag{7.4.36}$$

The physical degrees of freedom are then parameterized by the normal directions $\chi^{\underline{i}}$. In this gauge the variation of the induced metric is

$$\varepsilon^{-1}\delta g_{ab} = -2H_{\underline{i}ab}\chi^{\underline{i}} + \nabla_a\chi^{\underline{i}}\nabla_b\chi^{\underline{j}}\delta_{\underline{ij}} + \left(H_{\underline{i}a}{}^{c}H_{\underline{j}bc} - R_{m\underline{i}n\underline{j}}\partial_a x^m \partial_b x^n\right)\chi^{\underline{i}}\chi^{\underline{j}} \,, \tag{7.4.37}$$

where $H^{\underline{i}}{}_{ab}$ is the extrinsic curvature of the embedding and

$$\nabla_a\chi^{\underline{i}} = \partial_a\chi^{\underline{i}} + \mathcal{A}^{\underline{ij}}{}_a\chi_{\underline{j}} \tag{7.4.38}$$

is the world-sheet covariant derivative, which includes the $SO(8)$ normal bundle connection $\mathcal{A}^{\underline{ij}}{}_a$. These objects, as well as the world-sheet spin connection $w^{\underline{ab}}$, are related to the pullback of the target-space spin connection $\Omega^{\underline{mn}}$ by

$$w^{\underline{ab}} = P[\Omega^{\underline{ab}}] \,, \qquad H^{\underline{i}}{}_{ab} = P[\Omega^{\underline{i}}{}_{\underline{a}}]_a e^{\underline{a}}{}_b \,, \qquad \mathcal{A}^{\underline{ij}} = P[\Omega^{\underline{ij}}] \,, \tag{7.4.39}$$

where $e^{\underline{a}}{}_a = P\left[E^{\underline{a}}\right]_a$ is the induced geometry vielbein. Using the well-known expansion of the square root of a determinant, a short calculation shows that, to quadratic order, the NG action becomes

$$S^{(2)}_{\rm NG} = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-g}\left(g^{ab}\nabla_a\chi^{\underline{i}}\nabla_b\chi^{\underline{j}}\delta_{\underline{ij}} - \left(g^{ab}H_{\underline{i}a}{}^{c}H_{\underline{j}bc} + \delta^{\underline{ab}}R_{\underline{aibj}}\right)\chi^{\underline{i}}\chi^{\underline{j}}\right) \,, \tag{7.4.40}$$

where we have used the equations of motion $g^{ab}H^{\underline{i}}{}_{ab} = 0$ and written $g^{ab}R_{m\underline{i}n\underline{j}}\partial_a x^m \partial_b x^n = \delta^{\underline{ab}}R_{\underline{aibj}}$. The continuation of this expression to Euclidean signature is straightforward.

Let us now discuss the fermionic degrees of freedom. In Lorentzian signature, the type IIA string involves a single 10-dimensional Majorana spinor θ. At quadratic order, the Green-Schwarz (GS) action that controls its dynamics on $AdS_4 \times \mathbb{CP}^3$ is given by

$$S_{\text{GS}} = \frac{i}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\,\overline{\theta}\left(g^{ab} - \frac{\epsilon^{ab}}{\sqrt{-g}}\Gamma_{11}\right)\Gamma_a D_b\theta\,, \tag{7.4.41}$$

where the symbol ϵ^{ab} is a density with $\epsilon^{\tau\sigma} = 1$, $\Gamma_a = \Gamma_m\partial_a x^m$ is the pullback of the 10-dimensional Dirac matrices and $\Gamma_{11} \equiv \Gamma_{\underline{0123456789}}$. Also, $D_a = \partial_a x^m D_m$ is the pullback of the spacetime covariant derivative appearing in the supersymmetry variation of the gravitino, which includes the contribution from the RR fluxes. Explicitly,

$$D_a = \partial_a x^m \nabla_m + \frac{1}{8}e^{\Phi}\left[\not{F}_{(2)}\Gamma_{11} + \not{F}_{(4)}\right]\Gamma_a\,. \tag{7.4.42}$$

The above action can be simplified considerably. Indeed, given our choice of vielbein we have

$$D_a = \nabla_a - \frac{1}{2}H^{\underline{i}}{}_a{}^{\underline{a}}\Gamma_{\underline{ai}} + \frac{1}{8}e^{\Phi}\left[\not{F}_{(2)}\Gamma_{11} + \not{F}_{(4)}\right]\Gamma_a\,, \tag{7.4.43}$$

where the world-sheet covariant derivative ∇_a includes the normal bundle connection $\mathcal{A}^{\underline{ij}}{}_a$, that is,

$$\nabla_a = \partial_a + \frac{1}{4}w^{\underline{ab}}{}_a\Gamma_{\underline{ab}} + \frac{1}{4}\mathcal{A}^{\underline{ij}}{}_a\Gamma_{\underline{ij}}\,. \tag{7.4.44}$$

Using the relation $\epsilon^{ab}\Gamma_a = \sqrt{-g}\,\Gamma_{\underline{01}}\Gamma^b$, it is easy to see that the terms proportional to the extrinsic curvature drop out from the action because of the equations of motion $H^{\underline{i}}{}_{ab}\Gamma^a\Gamma^b = H^{\underline{i}}{}_{ab}g^{ab} = 0$. Then,

$$S_{\text{GS}} = \frac{i}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-g}\,\overline{\theta}\left(1 - \Gamma_{\underline{01}}\Gamma_{11}\right)\Gamma^a\left(\nabla_a + \frac{1}{8}e^{\Phi}\left[\not{F}_{(2)}\Gamma_{11} + \not{F}_{(4)}\right]\Gamma_a\right)\theta\,. \tag{7.4.45}$$

Now, in addition to diffeomorphism invariance and local Lorentz rotations, the full string action enjoys the local κ-symmetry

$$\delta_\kappa \theta = \frac{1}{2}\left(1+\Gamma_{\underline{01}}\Gamma_{11}\right)\kappa\,, \qquad \delta_\kappa x^m = \frac{i}{2}\overline{\theta}\Gamma^m \delta_\kappa \theta\,. \tag{7.4.46}$$

It is then possible to gauge fix to

$$\frac{1}{2}\left(1-\Gamma_{\underline{01}}\Gamma_{11}\right)\theta = \theta \qquad \Leftrightarrow \qquad \frac{1}{2}\overline{\theta}\left(1-\Gamma_{\underline{01}}\Gamma_{11}\right) = \overline{\theta}\,, \tag{7.4.47}$$

resulting in

$$S_{\mathrm{GS}} = \frac{i}{2\pi\alpha'}\int d\tau d\sigma\,\sqrt{-g}\,\overline{\theta}\,\Gamma^a\left(\nabla_a + \frac{1}{8}e^{\Phi}\left[-\not{F}_{(2)}\Gamma_{\underline{01}} + \not{F}_{(4)}\right]\Gamma_a\right)\theta\,. \tag{7.4.48}$$

Finally, we will need the Euclidean continuation of the action:

$$S_{\mathrm{GS}} = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\,\sqrt{g}\,\overline{\theta}\,\Gamma^a\left(\nabla_a + \frac{1}{8}e^{\Phi}\left(i\not{F}_{(2)}\Gamma_{\underline{01}} + \not{F}_{(4)}\right)\Gamma_a\right)\theta\,. \tag{7.4.49}$$

The κ-symmetry fixing becomes $i\Gamma_{\underline{01}}\Gamma_{11}\theta = \theta$ where now $\Gamma_{11} \equiv -i\Gamma_{\underline{0123456789}}$. We will take this expression as our starting point; all quantities involved are intrinsically Euclidean, including the fluxes and Dirac matrices.

7.4.2 Bosonic Fluctuations

Putting everything together we find that the action that governs the bosonic fluctuations is

$$\begin{aligned}
S_{(2,3)} &= \frac{L^2}{\pi\alpha'}\int d\tau d\sigma\sqrt{g}\left(g^{ab}\left(\partial_a\chi^{\underline{23}}\right)^*\partial_b\chi^{\underline{23}} + \frac{2\sinh^2\rho}{\sqrt{g}}\left|\chi^{\underline{23}}\right|^2\right)\,,\ \chi^{\underline{23}} = \frac{1}{\sqrt{2}}\left(\chi^{\underline{2}} + i\chi^{\underline{3}}\right)\,,\\
S_{(4,5)} &= \frac{L^2}{\pi\alpha'}\int d\tau d\sigma\sqrt{g}\left(g^{ab}\left(D^{\mathcal{A}}_a\chi^{\underline{45}}\right)^* D^{\mathcal{A}}_b\chi^{\underline{45}} - \frac{2m^2}{\sqrt{g}}\left|\chi^{\underline{45}}\right|^2\right)\,,\ \chi^{\underline{45}} = \frac{1}{\sqrt{2}}\left(\chi^{\underline{4}} + i\chi^{\underline{5}}\right)\,,\\
S_{(6,7)} &= \frac{L^2}{\pi\alpha'}\int d\tau d\sigma\sqrt{g}\left(g^{ab}\left(D^{\mathcal{B}}_a\chi^{\underline{67}}\right)^* D^{\mathcal{B}}_b\chi^{\underline{67}} - \frac{\sin^2\vartheta_1}{2\sqrt{g}}\left|\chi^{\underline{67}}\right|^2\right)\,,\ \chi^{\underline{67}} = \frac{1}{\sqrt{2}}\left(\chi^{\underline{6}} + i\chi^{\underline{7}}\right)\,,\\
S_{(8,9)} &= \frac{L^2}{\pi\alpha'}\int d\tau d\sigma\sqrt{g}\left(g^{ab}\left(D^{\mathcal{B}}_a\chi^{\underline{89}}\right)^* D^{\mathcal{B}}_b\chi^{\underline{89}} - \frac{\sin^2\vartheta_1}{2\sqrt{g}}\left|\chi^{\underline{89}}\right|^2\right)\,,\ \chi^{\underline{89}} = \frac{1}{\sqrt{2}}\left(\chi^{\underline{8}} + i\chi^{\underline{9}}\right)\,,
\end{aligned} \tag{7.4.50}$$

where

$$m = \frac{\sinh\rho \sin\vartheta_1(\rho)}{\cosh\rho - \cos\vartheta_1(\rho)}, \tag{7.4.51}$$

and the $U(1)$ covariant derivatives read

$$D^{\mathcal{A}} = d + i\mathcal{A}\,, \qquad D^{\mathcal{B}} = d + i\mathcal{B}\,, \tag{7.4.52}$$

with

$$\begin{aligned} \mathcal{A} \equiv \mathcal{A}^{\underline{45}} &= \left(1 - \frac{\cosh\rho\cos\vartheta_1(\rho) + 1}{\cosh\rho + \cos\vartheta_1(\rho)}\right) d\tau, \\ \mathcal{B} \equiv \mathcal{A}^{\underline{67}} = \mathcal{A}^{\underline{89}} &= \frac{1}{2}\left(\cos\vartheta_1(\rho) - 1\right) d\tau. \end{aligned} \tag{7.4.53}$$

We have factored out the AdS radius L from the metric and the fluctuations. Notice that the $U(1) \times U(1) \times SU(2) \times U(1)$ symmetry structure is evident, with $\chi^{\underline{67}}$ and $\chi^{\underline{89}}$ forming a doublet.

7.4.3 Fermionic Fluctuations

For the case at hand, the fermionic action reads

$$S_{\text{GS}} = \frac{L^2}{\pi\alpha'} \int d\tau d\sigma \,\sqrt{g}\,\overline{\theta}\left(\Gamma^a \nabla_a + M\right)\theta\,, \tag{7.4.54}$$

where

$$\nabla_\tau = \partial_\tau + \frac{1}{2}\Gamma^{\underline{01}}\,w + \frac{1}{2}\Gamma^{\underline{45}}\,\mathcal{A} + \frac{1}{2}\left(\Gamma^{\underline{67}} + \Gamma^{\underline{89}}\right)\mathcal{B}\,, \tag{7.4.55}$$

$$\nabla_\sigma = \partial_\sigma\,, \tag{7.4.56}$$

$$M = \frac{i\Gamma^{\underline{01}}}{4A}\left(\left(3\Gamma^{\underline{23}} - \Gamma^{\underline{45}}\right)\left(\sinh^2\rho - \sin^2\vartheta_1(\rho)\,\Gamma^{\underline{0145}}\right) + \left(\Gamma^{\underline{67}} + \Gamma^{\underline{89}}\right)A\right)\,. \tag{7.4.57}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are the connections defined above in equation (7.4.53), A is the conformal factor of the induced worldsheet metric defined in (7.3.14) and w the worldsheet spin connection given by (E.18).

As for the bosons we have extracted the radius L from the metric and rescaled the fermionic fields by $L^{1/2}$. The symmetry of the action under the $U(1)\times U(1)\times SU(2)\times U(1)$ bosonic subgroup follows from the fact that all the objects involved commute with the preserved generators (7.3.30).

7.4.4 One-loop Effective Action

The induced world-sheet geometry is that of the 2d Euclidean manifold $\mathcal{M}$ with the metric[1]

$$\begin{aligned} ds^2_{\mathcal{M}} &= M(\rho)\,\big(d\rho^2+\sinh^2\rho\,d\tau^2\big),\\ M(\rho) &= 1+\frac{\sin^2\theta(\rho)}{\sinh^2\rho},\quad \sin\theta(\rho)=\frac{\sinh\rho\,\sin\theta_0}{\cosh\rho+\cos\theta_0}\end{aligned}\tag{7.4.58}$$

where $0\le\theta_0\le\frac{\pi}{2}$ is the latitude angle. $\theta_0=0$ corresponds to the $\frac{1}{2}$- BPS solution.

The difference in 1-loop effective actions of $\frac{1}{6}$-BPS string withrespect to the $\frac{1}{2}$-BPS is

$$e^{-\Delta\,\Gamma^{\text{1-loop}}_{\text{eff}}(\theta_0)}=\left[\frac{\left(\frac{\det\mathcal{O}_{4+}(\theta_0)}{\det\mathcal{O}_{4+}(0)}\right)^2\left(\frac{\det\mathcal{O}_{4-}(\theta_0)}{\det\mathcal{O}_{4-}(0)}\right)^2\left(\frac{\det\mathcal{O}_{5+}(\theta_0)}{\det\mathcal{O}_{5+}(0)}\right)\left(\frac{\det\mathcal{O}_{5-}(\theta_0)}{\det\mathcal{O}_{5-}(0)}\right)\left(\frac{\det\mathcal{O}_{6+}(\theta_0)}{\det\mathcal{O}_{6+}(0)}\right)\left(\frac{\det\mathcal{O}_{6-}(\theta_0)}{\det\mathcal{O}_{6-}(0)}\right)}{\left(\frac{\det\mathcal{O}_{1}(\theta_0)}{\det\mathcal{O}_{1}(0)}\right)^2\left(\frac{\det\mathcal{O}_{2+}(\theta_0)}{\det\mathcal{O}_{2+}(0)}\right)\left(\frac{\det\mathcal{O}_{2-}(\theta_0)}{\det\mathcal{O}_{2-}(0)}\right)\left(\frac{\det\mathcal{O}_{3+}(\theta_0)}{\det\mathcal{O}_{3+}(0)}\right)^2\left(\frac{\det\mathcal{O}_{3-}(\theta_0)}{\det\mathcal{O}_{3-}(0)}\right)^2}\right]^{\frac{1}{2}}$$

where the bosonic spectrum of operators is

$$\begin{aligned}\mathcal{O}_1(\theta_0)&=M^{-1}\big(-g^{\mu\nu}\,\nabla_\mu\nabla_\nu+2\big),\\ \mathcal{O}_{2\pm}(\theta_0)&=M^{-1}\big(-g^{\mu\nu}\,D^a_\mu\,D^a_\nu+V_2\big),\qquad D^a_\mu=\nabla_\mu\pm\iota\,\mathcal{A}_\mu,\\ \mathcal{O}_{3\pm}(\theta_0)&=M^{-1}\big(-g^{\mu\nu}\,D^b_\mu\,D^b_\nu+V_3\big),\qquad D^b_\mu=\nabla_\mu\pm\iota\,\mathcal{B}_\mu.\end{aligned}\tag{7.4.59}$$

Effective 2d fermionic operators $\mathcal{O}_{i\pm}$ $(i=4,5,6)$ are obtained by a judicious choice of the 10d Gamma matrices (see (G.2)). Calling α,β,γ the eigenvalues of $\Gamma^{\underline{45}},\Gamma^{\underline{67}},\Gamma^{\underline{89}}$ respectively, the 10d operator appearing in (7.4.54) take a block diagonal form with entries

$$\mathcal{O}_{\alpha,\beta,\gamma}(\theta_0)=M^{-\frac{1}{2}}\left(-\iota\left(\not{D}+\frac{1}{4}\not{\partial}\,\ln M\right)-\iota\,\Gamma_{\underline{01}}\,\big(m+V\big)+\alpha W\right),\tag{7.4.60}$$

[1] To simplify the notation, in the following sections we renamed $\theta(\rho)\equiv\vartheta_1(\rho)$.

The operators in (7.4.59) are defined as:

$$\mathcal{O}_{4,\alpha} \equiv \mathcal{O}_{\alpha,\beta,-\beta}, \qquad \mathcal{O}_{5,\alpha} \equiv \mathcal{O}_{\alpha,\alpha,\alpha}, \qquad \mathcal{O}_{6,\alpha} \equiv \mathcal{O}_{\alpha,-\alpha,-\alpha}\,. \tag{7.4.61}$$

Explicitly we have $\mathcal{A}_\rho = \mathcal{B}_\rho = 0$, $\mathcal{A}_\tau = \mathcal{A}(\rho)$, $\mathcal{B}_\tau = \mathcal{B}(\rho)$ with $g_{\mu\nu}$ and ∇_μ evaluated for the AdS_2 metric,

$$D_\mu = \nabla_\mu + \iota\frac{\alpha}{2}\,\mathcal{A}_\mu + \iota\frac{\beta+\gamma}{2}\,\mathcal{B}_\mu, \tag{7.4.62}$$

and

$$\begin{aligned}
\mathcal{A}(\rho) &= 1 - \frac{1+\cosh\rho\,\cos\theta(\rho)}{\cosh\rho + \cos\theta(\rho)}, \quad \mathcal{B}(\rho) = \frac{1}{2}\big(\cos\theta(\rho) - 1\big), && (7.4.63)\\
V_2(\rho) &= -\frac{\partial_\rho\mathcal{A}(\rho)}{\sinh\rho}, \quad V_3(\rho) = -\frac{\partial_\rho\mathcal{B}(\rho)}{\sinh\rho}, && (7.4.64)\\
V(\rho) &= \frac{(1-3\,\beta\,\gamma)}{4}\,\frac{1}{\sqrt{M(\rho)}} - \frac{\alpha(\beta+\gamma)}{4}\,\sqrt{M(\rho)} - m, && (7.4.65)\\
W(\rho) &= \frac{1-3\,\beta\,\gamma}{4}\,\frac{\sin^2\theta(\rho)}{\sqrt{M(\rho)}\,\sinh^2\rho}. && (7.4.66)
\end{aligned}$$

Here m corresponds to the value of potential, V, at $\rho = \infty$.

$$m = \frac{(1-3\,\beta\,\gamma)}{4} - \frac{\alpha(\beta+\gamma)}{4} \tag{7.4.67}$$

For completeness we quote that

$$\cos\theta(\rho) = \frac{1+\cosh\rho\,\cos\theta_0}{\cosh\rho+\cos\theta_0}. \tag{7.4.68}$$

It is important to mention that the $\mathcal{O}_{6,\alpha}$ operators give rise to asymptotically massless fermions.

7.5 One-loop Effective Action: Perturbative Heat Kernel

We now proceed to evaluate fluctuations determinant using the heat kernel techniques. To evaluate the determinants we will exploit the fact that heat kernel techniques for AdS_2 are well-developed [130, 131, 25]. More precisely, we will use perturbation theory valid in the limit when the induced world-sheet geometry can be considered as a small deformation of AdS_2 govern by the deformation parameter θ_0. This approach has been successfully applied the holographic perturbative computation of a ratio of Wilson loops expectation values [28]. Namely, we will expand around the parameter $\alpha = \theta_0^2$, where the near AdS_2 geometry corresponds to the latitude in $S^2 \subset S^5$ parametrized by angle θ_0. For $\theta_0 = 0$, the worldsheet metric reduces to AdS_2. Under the conditions clarified below we will be able to determine the first leading order correction to the string partition function by the perturbative expansion of the heat kernels.

Let $\mathcal{M}$ be a d dimensional smooth compact Riemannian manifold with metric g_{ij} and $\mathcal{O}$ be a second order elliptic operator of the Laplace type. Then, we can define the logarithm of the determinant using ζ-function regularization as,

$$\log \mathrm{Det}_{\mathcal{M}}\, \mathcal{O} = -\zeta'_{\mathcal{O}}(0), \tag{7.5.69}$$

The ζ function is related to the integrated heat kernel by the Mellin transform,

$$\zeta_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} K_{\mathcal{O}}(t), \qquad K_{\mathcal{O}}(t) = \int d^d x\, \sqrt{g}\, \mathrm{tr} K_{\mathcal{O}}(x,x;t),$$

where by construction, $K_{\mathcal{O}}(x,x';t)$ satisfies the heat conduction equation

$$\left(\partial_t + \mathcal{O}_x\right) K_{\mathcal{O}}(x,x';t) = 0, \tag{7.5.70}$$

with the initial condition

$$K_{\mathcal{O}}(x,x';0)=\frac{1}{\sqrt{g}}\,\delta^{(d)}\left(x-x'\right)\mathbb{I}. \tag{7.5.71}$$

Let us now assume that the manifold $\mathcal{M}$ can be viewed as a deformation of another manifold $\bar{\mathcal{M}}$. Namely, for $\alpha=0$ we have $\bar{\mathcal{M}}$ with metric $\bar{g}_{ij}$; we further assume that in this limit the spectral problem can be solved exactly and seek to construct the solution for $\mathcal{M}$. We can expand $K_{\mathcal{O}}$ and subsequently $\mathrm{Det}_{\mathcal{M}}\mathcal{O}$ in perturbation theory in α:

$$\begin{aligned} g_{ij} &= \bar{g}_{ij}+\alpha\,\tilde{g}_{ij}+O(\alpha^2),\\ \mathcal{O} &= \bar{\mathcal{O}}+\alpha\,\tilde{\mathcal{O}}+O(\alpha^2),\\ K_{\mathcal{O}}(x,x';t) &= \bar{K}_{\mathcal{O}}(x,x';t)+\alpha\,\tilde{K}_{\mathcal{O}}(x,x';t)+O(\alpha^2), \end{aligned} \tag{7.5.72}$$

such that $\bar{K}_{\mathcal{O}}(x,x';t)$ satisfies (7.5.70) and (7.5.71) .

It can be shown [28], that $\tilde{K}_{\mathcal{O}}(x,x';t)$ can be solved from

$$\left(\partial_t+\bar{\mathcal{O}}_x\right)\tilde{K}_{\mathcal{O}}(x,x';t)+\tilde{\mathcal{O}}_x\,\bar{K}_{\mathcal{O}}(x,x';t)=0, \tag{7.5.73}$$

with the initial condition

$$\tilde{K}_{\mathcal{O}}(x,x';t)=-\frac{\tilde{g}}{2\,\bar{g}^{3/2}}\,\delta^{(d)}\left(x-x'\right)\mathbb{I}. \tag{7.5.74}$$

The trace of heat kernel can be written as;

$$\tilde{K}_{\mathcal{O}}(t) \;=\; -t\int d^dx\,\sqrt{\bar{g}}\,\mathrm{tr}\big[\tilde{\mathcal{O}}_x\,\bar{K}_{\mathcal{O}}(x,x';t)\big]_{x=x'}. \tag{7.5.75}$$

In perturbation theory, the ζ-function and the determinant takes the form

$$\log\mathrm{Det}_{\mathcal{M}}\mathcal{O} \;=\; -\bar{\zeta}'_{\mathcal{O}}(0)+-\alpha\,\tilde{\zeta}'_{\mathcal{O}}(0)+O(\alpha^2), \tag{7.5.76}$$

$$\tilde{\zeta}'_{\mathcal{O}}(s) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\tilde{K}_{\mathcal{O}}(t). \tag{7.5.77}$$

In our context, the string partition function corresponding to the Wilson loop in the gauge theory is given by

$$Z = \langle W(\lambda, \alpha) \rangle \equiv e^{-\Gamma}, \qquad \Gamma = \sqrt{\lambda}\, \Gamma^{(0)}(\alpha) + \Gamma^{(1)}(\alpha) + O(\lambda^{-1/2}) \tag{7.5.78}$$

where $\Gamma^{(0)}(\alpha)$ is the classical piece and object of current interest is $\Gamma^{(1)}(\alpha)$, which corresponds to the one-loop corrections to the string action. In particular, we are interested in evaluating $\tilde{\Gamma}^{(1)}(0)$.

7.5.1 Circular Wilson Loop

In the limit $\theta_0 = 0$, or $\sigma_0 = \infty$, the operators take the following form;

$$\begin{aligned} &\text{Bosons:} \quad \bar{\mathcal{O}}_1 = -\Delta_{\rho,\tau} + 2, \quad \bar{\mathcal{O}}_{2\pm} = \bar{\mathcal{O}}_{3\pm} = -\Delta_{\rho,\tau} \\ &\text{Fermions:} \quad \bar{\mathcal{O}}_{\alpha,\beta,\gamma} = -\iota\, \nabla\!\!\!\!/\,_{\rho,\tau} + \iota\, m\, \sigma_3 \end{aligned} \tag{7.5.79}$$

where $4\,m = \alpha + \beta + \gamma - 3\,\alpha\,\beta\,\gamma$ with $\alpha, \beta, \gamma = \pm 1$ as follows from the spinor reduction described in appendix G.

The integrated AdS_2 heat kernel and ζ-function for the massive Laplace operator $-\Delta + m^2$ is known to be,

$$\begin{aligned} \bar{K}_{-\Delta+m^2}(t) &= \frac{V_{AdS_2}}{2\,\pi} \int_0^\infty dv\, v\, \tanh(\pi v)\, e^{-t\,(v^2+m^2+\frac{1}{4})} \\ \bar{\zeta}_{-\Delta+m^2}(s) &= \frac{V_{AdS_2}}{\pi} \left[\frac{(m^2+\frac{1}{4})^{1-s}}{2\,(s-1)} - 2 \int_0^\infty dv\, \frac{v}{(e^{2\pi v}+1)\,(v^2+m^2+\frac{1}{4})^s} \right]. \end{aligned} \tag{7.5.80}$$

The regularized determinants for $\theta_0 = 0$ bosonic operators becomes

$$\bar{\zeta}'_{\mathcal{O}_1}(0) \;=\; -\frac{25}{12} + \frac{3}{2}\,\log 2\pi - 2\,\log \mathsf{A}, \tag{7.5.81}$$

$$\bar{\zeta}'_{\mathcal{O}_{2\pm}}(0) = \bar{\zeta}'_{\mathcal{O}_{3\pm}}(0) \;=\; -\frac{1}{12} + \frac{1}{2}\,\log 2\pi - 2\,\log \mathsf{A}, \tag{7.5.82}$$

where A is the Glaisher constant. The spectrum of the bosonic fluctuations correspond to

2 massive scalars ($m^2 = 2$) and 6 massless scalars, thus,

$$\begin{aligned}\bar{\Gamma}_B^{(1)}(0) &= -\frac{2}{2}\,\bar{\zeta}'_{\mathcal{O}_1}(0) - \frac{6}{2}\,\bar{\zeta}'_{\mathcal{O}_{2\pm,3\pm}}(0)\\ &= \frac{7}{3} - 3\,\log\,2\pi + 8\,\log\mathsf{A}\end{aligned} \tag{7.5.83}$$

The standard expression for the AdS_2 heat kernel corresponding to the square of the massive Dirac operator $-\nabla\!\!\!/ + m\,\Gamma^3$ is,

$$\bar{K}_{-\nabla\!\!\!/^2+m^2}(t) = \frac{V_{AdS_2}}{\pi}\int_0^\infty dv\,v\,\coth(\pi v)\,e^{-t\,(v^2+m^2)} \tag{7.5.84}$$

and the ζ-function is given by

$$\bar{\zeta}_{-\nabla\!\!\!/^2+m^2}(s) = \frac{V_{AdS_2}}{\pi}\left[\frac{(m^2)^{1-s}}{2\,(s-1)} + 2\int_0^\infty dv\,\frac{v}{(e^{2\pi v}-1)\,(v^2+m^2)^s}\right]. \tag{7.5.85}$$

In the present case, the fermionic excitations involve 2 modes with $m^2 = 0$ and 6 modes with $m^2 = 1$. Then,

$$\bar{\zeta}'_{m^2=0}(0) = \frac{1}{3} - 4\,\log\mathsf{A} \tag{7.5.86}$$

$$\bar{\zeta}'_{m^2=1}(0) = -\frac{5}{3} - 4\,\log\mathsf{A} + 2\,\log\,2\pi \tag{7.5.87}$$

The final contribution from fermions results,

$$\begin{aligned}\bar{\Gamma}_F^{(1)}(0) &= -\frac{2}{2}\,\bar{\zeta}'_{m^2=0}(0) - \frac{6}{2}\,\bar{\zeta}'_{m^2=1}(0)\\ &= 2\left(\frac{7}{3} + 8\,\log\mathsf{A} - 3\,\log\,2\pi\right).\end{aligned} \tag{7.5.88}$$

Thus, the one-loop correction in the circular Wilson loop case becomes

$$\bar{\Gamma}^{(1)}(0) = \bar{\Gamma}_B^{(1)}(0) - \frac{1}{2}\,\bar{\Gamma}_F^{(1)}(0) = 0 \tag{7.5.89}$$

This result certainly requires further scrutiny[2]. Here we simply note that, as it stands, it

[2]We acknowledge various discussions in the summer of 2015 with Jewel Ghosh regarding the heat kernel approach to the one-loop effective action of the half BPS configuration.

does not agree with the field theory prediction of (7.2.3) in the string theory limit given by

$$\langle W_{\square}^{\frac{1}{2}} \rangle = \frac{e^{\pi\sqrt{2\lambda}}}{8\pi\lambda} + O(\lambda^{-1/2}) \tag{7.5.90}$$

It also does not agree with a Gelfand-Yaglom based computation which further involved numerical evaluation [118]. We leave a proper treatment of the expectation value of the half BPS Wilson loop to a separate work. Here we are mostly concerned with the ratio of expectation values.

7.5.2 Difference of one-loop effective actions

The perturbative expansion of the relevant operators here,

$$\begin{aligned}
\mathcal{O}_i(\theta_0) &= \bar{\mathcal{O}}_i + \tilde{\mathcal{O}}_i\,\theta_0^2 + O(\theta_0^4), \qquad i = 1, 2\pm, 3\pm && (7.5.91)\\
\mathcal{O}_{\alpha,\beta,\gamma}(\theta_0) &= \bar{\mathcal{O}}_{\alpha,\beta,\gamma} + \tilde{\mathcal{O}}_{\alpha,\beta,\gamma}\,\theta_0^2 + O(\theta_0^4), && (7.5.92)\\
\mathcal{O}^2_{\alpha,\beta,\gamma}(\theta_0) &= \bar{\mathcal{O}}^2_{\alpha,\beta,\gamma} + \theta_0^2\,\{\bar{\mathcal{O}}_{\alpha,\beta,\gamma}, \tilde{\mathcal{O}}_{\alpha,\beta,\gamma}\} + O(\theta_0^4). && (7.5.93)
\end{aligned}$$

where $\{..\}$ denotes the anticommutator of two differential operators.

In the expansion scheme of (7.5.72), the corresponding perturbative operator is

$$\begin{aligned}
\tilde{\mathcal{O}}_1 &= \frac{1}{(1+\cosh\rho)^2}\,\big(\Delta_{\rho,\tau} - 2\big),\\
\tilde{\mathcal{O}}_{2\pm} &= \frac{1}{(1+\cosh\rho)^2}\left[\Delta_{\rho,\tau} - \frac{1}{2}\,\big(1 \pm \iota\,\partial_\tau\big)\right]\\
\tilde{\mathcal{O}}_{3\pm} &= \frac{1}{(1+\cosh\rho)^2}\left[\Delta_{\rho,\tau} - \frac{\sinh^2\rho}{(1+\cosh\rho)^2}\,(2 \pm \iota\,\partial_\tau)\right],
\end{aligned} \tag{7.5.94}$$

for the bosonic second order operators. While, for the first order fermionic operator, we have,

$$\begin{aligned}
\tilde{\mathcal{O}}_{\alpha,\beta,\gamma}(\theta_0) = \frac{1}{2\,(1+\cosh\rho)^2}\Bigg[\iota\nabla\!\!\!\!/ + \frac{\sinh\rho}{1+\cosh\rho}\big(\iota\Gamma^{\underline{0}}\big) + \Gamma^{\underline{1}}\left(\frac{\alpha\,(1-\cosh\rho)^2}{2} - \frac{\beta+\gamma}{4}\,\sinh^2\rho\right)\\
- \frac{(-1+3\,\beta\,\gamma)}{2}\,\big(\iota\Gamma_{\underline{01}}\big) + \frac{\alpha(1-3\beta\gamma)}{2}\Bigg].
\end{aligned} \tag{7.5.95}$$

Bosons: Substituting the $\tilde{O}_1$ in (7.5.75), we get,

$$\tilde{K}_{\mathcal{O}_1}(t) = -t \int_0^{2\pi} d\tau \int_0^{\Lambda} d\rho \, \frac{\sinh\rho}{(1+\cosh\rho)^2} \left[\left(\Delta_{\rho,\tau} - 2\right) \bar{K}_{-\Delta+2}\left(\rho,\tau,\rho',\tau';t\right) \right]_{\rho=\rho',\tau=\tau'} \tag{7.5.96}$$

We know that $\bar{K}$ satisfies, the following equation,

$$\left(\partial_t - \Delta_{\rho,\tau} + 2\right) \bar{K}_{\mathcal{O}_1}\left(\rho,\tau,\rho',\tau';t\right) = 0 \tag{7.5.97}$$

Thus, plugging it back in (7.5.96), we obtain

$$\tilde{K}_{\mathcal{O}_1}(t) = -t \int_0^{2\pi} d\tau \int_0^{\Lambda} d\rho \, \frac{\sinh\rho}{(1+\cosh\rho)^2} \, \partial_t \, \bar{K}_{\mathcal{O}_1}\left(\rho,\tau,\rho,\tau;t\right) \tag{7.5.98}$$

Now we can take the limit $\Lambda \to \infty$ and using the integral representation of heat kernel $\bar{K}$

$$\tilde{K}_{\mathcal{O}_1}(t) = \frac{t}{2} \int_0^{\infty} dv \, v \, \tanh(\pi \, v) \left(v^2 + \frac{9}{4}\right) e^{-t\left(v^2+9/4\right)} \tag{7.5.99}$$

Using $\tanh(\pi \, v) = 1 - 2/(e^{2\pi v}+1)$ and we can write the corresponding ζ-function as,

$$\tilde{\zeta}_{\mathcal{O}_1}(s) = \int_0^{\infty} dv \, \frac{s\, v}{2\left(v^2+9/4\right)^s} - \int_0^{\infty} dv \, \frac{s\, v}{\left(e^{2\pi v}+1\right)\left(v^2+9/4\right)^s} \tag{7.5.100}$$

The first integral converges only for Re s > 1, we can first integrate over v and then analytically continue to all values of s

$$\tilde{\zeta}_{\mathcal{O}_1}(s) = \frac{s}{4\,(s-1)} \left(\frac{9}{4}\right)^{1-s} - s \int_0^{\infty} dv \, \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+9/4\right)^s}. \tag{7.5.101}$$

The final result is

$$\tilde{\zeta}'_{\mathcal{O}_1}(0) = -\frac{7}{12}. \tag{7.5.102}$$

In the case of $\mathcal{O}_{2\pm}$, we add the contribution from $\mathcal{O}_{2+}$ and $\mathcal{O}_{2-}$ to get rid of the ∂_τ term, this substantially simplifies the calculation. Then,

$$\tilde{K}_{\mathcal{O}_{2+}}(t) + \tilde{K}_{\mathcal{O}_{2-}}(t) = t \int_0^\infty dv \left[\left(v^2 + \frac{5}{4} \right) v \, \tanh(\pi v) \, e^{-t \left(v^2 + \frac{1}{4} \right)} \right], \tag{7.5.103}$$

$$\tilde{\zeta}_{\mathcal{O}_{2+}}(s) + \tilde{\zeta}_{\mathcal{O}_{2-}}(s) = s \int_0^\infty dv \, v \, \frac{\left(v^2 + \frac{5}{4}\right)}{\left(v^2 + \frac{1}{4}\right)^{1+s}} - 2\, s \int_0^\infty dv \, \frac{v}{e^{2\pi v} + 1} \, \frac{\left(v^2 + \frac{5}{4}\right)}{\left(v^2 + \frac{1}{4}\right)^{1+s}}.$$

So,

$$\tilde{\zeta}'_{\mathcal{O}_{2+}}(0) + \tilde{\zeta}'_{\mathcal{O}_{2-}}(0) = -\frac{1}{6} + \gamma. \tag{7.5.104}$$

Similarly, for the operator $\mathcal{O}_{3\pm}$, we get

$$\tilde{K}_{\mathcal{O}_{3+}}(t) + \tilde{K}_{\mathcal{O}_{3-}}(t) = t \int_0^\infty dv \left[\left(v^2 + \frac{3}{4} \right) v \, \tanh(\pi v) \, e^{-t \left(v^2 + \frac{1}{4} \right)} \right]. \tag{7.5.105}$$

Then,

$$\tilde{\zeta}_{\mathcal{O}_{3+}}(s) + \tilde{\zeta}_{\mathcal{O}_{3-}}(s) = \int_0^\infty dv \, sv \, \frac{\left(v^2 + \frac{3}{4}\right)}{\left(v^2 + \frac{1}{4}\right)^{1+s}} - 2\, s \int_0^\infty dv \, \frac{v}{e^{2\pi v} + 1} \, \frac{\left(v^2 + \frac{3}{4}\right)}{\left(v^2 + \frac{1}{4}\right)^{1+s}},$$

which gives

$$\tilde{\zeta}'_{\mathcal{O}_{3+}}(0) + \tilde{\zeta}'_{\mathcal{O}_{3-}}(0) = -\frac{1}{6} + \frac{\gamma}{2}, \tag{7.5.106}$$

where γ is the Euler-Mascheroni constant.

The total contribution for bosonic operators is simply given by

$$\begin{aligned} \tilde{\Gamma}_B^{(1)} &= -\frac{2}{2}\, \tilde{\zeta}'_{\mathcal{O}_1}(0) - \frac{1}{2}\, \tilde{\zeta}'_{\mathcal{O}_{2+}}(0) - \frac{1}{2}\, \tilde{\zeta}'_{\mathcal{O}_{2-}}(0) - \frac{2}{2}\, \tilde{\zeta}'_{\mathcal{O}_{3+}}(0) - \frac{2}{2}\, \tilde{\zeta}'_{\mathcal{O}_{3-}}(0) \\ &= \frac{5}{6} - \gamma. \end{aligned} \tag{7.5.107}$$

Fermions: an important computational ingredient in case of fermions is

$$\{\bar{\mathcal{O}}_{\alpha,\beta,\gamma}, \tilde{\mathcal{O}}_{\alpha,\beta,\gamma}\} = \{\bar{\mathcal{O}}, \tilde{\mathcal{O}}\} = \frac{1}{(1+\cosh\rho)^2}\nabla^2_{\rho,\tau} - \frac{m\,(1-3\beta\gamma)}{2(1+\cosh\rho)^2} + \frac{\iota X(\rho)}{\sinh^2\rho(1+\cosh\rho)^2}\,\partial_\tau, \tag{7.5.108}$$

where

$$X(\rho) = \frac{\alpha\,(1-\cosh\rho)^2}{2} - \frac{\beta+\gamma}{4}\,\sinh^2\rho. \tag{7.5.109}$$

One can derive formal expressions which can be evaluated for the cases of interest, we skip some intermediate steps that involve Mellin transform from the heat kernel to the zeta function. In particular, we obtain

$$\begin{aligned}\delta\zeta_F(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\delta K(t) = \int_0^\infty dv \frac{sv\left(v^2+2m^2+m\frac{\alpha(\beta+\gamma)}{2}\right)}{(v^2+m^2)^{s+1}}\coth\pi v \\ &= \int_0^\infty dv\frac{sv(v^2+2m^2)}{(v^2+m^2)^{s+1}} + 2\int_0^\infty dv\frac{sv(v^2+2m^2+m\frac{\alpha(\beta+\gamma)}{2})}{(v^2+m^2)^{s+1}(e^{2\pi v}-1)} \\ &= \frac{m^{1-2s}\left(m(-1+2s)+\frac{\alpha(\beta+\gamma)}{2}(s-1)\right)}{2(s-1)} + 2\int_0^\infty dv\frac{sv(v^2+2m^2)}{(v^2+m^2)^{s+1}(e^{2\pi v}-1)},\end{aligned} \tag{7.5.110}$$

thus giving

$$\begin{aligned}\delta\zeta_F'(0) &= -\frac{1}{2}m(m+\left(m+\frac{\alpha(\beta+\gamma)}{2})\ln m^2\right) + 2\int_0^\infty dv\frac{v\left(v^2+2m^2+m\frac{\alpha(\beta+\gamma)}{2}\right)}{(v^2+m^2)(e^{2\pi v}-1)} \\ &= -\frac{1}{2}m(m+\left(m+\frac{\alpha(\beta+\gamma)}{2})\ln m^2\right) + 2\int_0^\infty dv\frac{v}{(e^{2\pi v}-1)} \\ &\quad + 2m\left(m+\frac{\alpha(\beta+\gamma)}{2}\right)\int_0^\infty dv\frac{v}{(v^2+m^2)(e^{2\pi v}-1)} \\ &= -\frac{1}{2}m(m+\left(m+\frac{\alpha(\beta+\gamma)}{2})\ln m^2\right) + \frac{1}{12} + m\left(m+\frac{\alpha(\beta+\gamma)}{2}\right) \\ &\left(\frac{1}{2}\ln m^2 - \frac{1}{2|m|} - \psi(|m|)\right),\end{aligned} \tag{7.5.111}$$

where $\psi(x) = \frac{d}{dx}\Gamma(x)$ is the digamma function.

In particular, for $\mathcal{O}_{6,\alpha}$ operators, which have $m = 0$, we obtain

$$\delta\zeta'_F(0) = -\frac{5}{12}. \tag{7.5.112}$$

While operators $\mathcal{O}_{4,\alpha}$ having $m = 1$ lead to

$$\delta\zeta'_F(0) = -\frac{11}{12} + \gamma\,. \tag{7.5.113}$$

Finally $\mathcal{O}_{5,\alpha}$ operators have $m = -1$ and give

$$\delta\zeta'_F(0) = -\frac{5}{12}\,. \tag{7.5.114}$$

Adding the fermionic contributions leads to

$$\delta\zeta_F^{\text{tot}}(s) = \frac{1}{2}\left[2 \times \left(-\frac{5}{12}\right) + 4 \times \left(-\frac{11}{12} + \gamma\right) + 2 \times \left(-\frac{5}{12}\right)\right] = -\frac{8}{3} + 2\gamma. \tag{7.5.115}$$

Since the total bosonic contribution (7.5.107) follows from

$$\delta\zeta_B^{\text{tot}}(s) = -\frac{5}{3} + 2\gamma, \tag{7.5.116}$$

the total one-loop perturbative contribution results

$$\delta\zeta_B^{\text{tot}}(0) - \delta\zeta_F^{\text{tot}}(0) = \Big(-\frac{5}{3} + 2\gamma\Big) - \Big(-\frac{8}{3} + 2\gamma\Big) = 1. \tag{7.5.117}$$

Finally,

$$\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \frac{1}{2}\theta_0^2, \tag{7.5.118}$$

which agrees, at the given order, with the field theory prediction (cf. (7.2.6)).

7.6 One-loop Effective Action: Zeta Function Regularization

In this section we follow work in Chapter V and VI where we developed a regularization in the case of radial determinants that coincides with ζ-function regularization in various cases. There are various reasons to tackle the problem using these methods. First, one would obviously like to go beyond the small θ_0 limit and obtain and expression that is valid in the whole range of θ_0. Second, by construction, our regularization is diffeomorphic invariant and works directly on the disk; other approaches [26, 27, 29] rely on mapping the problem from the disk to the cylinder. Although these latter methods have proven to be quite effective it is conceptually satisfying to deal with the problem directly on the disk.

The main outcome of Chapter V is a prescription for computing ζ-function regularized determinants of radial operators in asymptotically AdS_2 spacetimes. The result for bosons is

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= \ln \frac{\det \mathcal{O}_0}{\det \mathcal{O}_0^{\text{free}}} + \sum_{l=1}^{\infty} \left(\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} + \ln \frac{\det \mathcal{O}_{-l}}{\det \mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l}\, \hat{\zeta}_{\mathcal{O}}(0) \right) \\
&\quad - 2\left(\gamma + \ln \frac{\mu}{2}\right) \hat{\zeta}_{\mathcal{O}}(0) + \int_0^\infty d\rho\, \sinh\rho\, \ln(\sinh\rho)\, V - q^2 \int_0^\infty d\rho\, \frac{\mathcal{A}^2}{\sinh\rho},
\end{aligned}
\tag{7.6.119}
$$

$$
\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho \sinh\rho\, V, \tag{7.6.120}
$$

whereas for fermions, it reads

$$
\begin{aligned}
\ln \frac{\det \mathcal{O}}{\det \mathcal{O}^{\text{free}}} &= \sum_{l=\frac{1}{2}}^{\infty} \left(\ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} + \ln \frac{\det \mathcal{O}_{-l}}{\det \mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l+\frac{1}{2}}\, \hat{\zeta}_{\mathcal{O}}(0) \right) - 2\left(\gamma + \ln \frac{\mu}{2}\right) \hat{\zeta}_{\mathcal{O}}(0) \\
&\quad + \int_0^\infty d\rho\, \sinh\rho\, \ln(\sinh\rho) \left((m+V)^2 - W^2 - m^2\right) \\
&\quad - q^2 \int_0^\infty d\rho\, \frac{\mathcal{A}^2}{\sinh\rho} - \int_0^\infty d\rho\, \sinh\rho\, W^2,
\end{aligned}
\tag{7.6.121}
$$

$$
\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho \sinh\rho \left((m+V)^2 - W^2 - m^2\right). \tag{7.6.122}
$$

7.6.1 Bosons

We now proceed to apply the prescription above to the different bosonic operators.

7.6.1.1 $\mathcal{O}_1(\theta_0)$

The action for these fluctuations is

$$\mathcal{O}_1(\theta_0) \quad = M^{-1}\big(- g^{\mu\nu}\, \nabla_\mu \nabla_\nu + 2\big) \tag{7.6.123}$$

We see that the rescaled operator does not depend on θ_0, meaning that these fluctuations contribute only with an anomaly as showed in Chapter VI,

$$\begin{aligned} \ln\left(\frac{\det \mathcal{O}_1(\theta_0)}{\det \mathcal{O}_1(0)}\right) &= \theta_0 \sin\theta_0 + \frac{1}{2}\,\sin^2\frac{\theta_0}{2} + \left(\frac{7}{3} + 2\,\cos\theta_0\right)\ln\cos\frac{\theta_0}{2} \\ &= \frac{7}{12}\,\theta_0^2 + O\big(\theta_0^4\big) \end{aligned} \tag{7.6.124}$$

7.6.1.2 $\mathcal{O}_{2\pm}(\theta_0)$

For these charged fluctuations, we have,

$$\ln\left(\frac{\det \mathcal{O}_{AdS_2}(\theta_0)}{\det \mathcal{O}_{AdS_2}(0)}\right) = \ln\frac{\psi_0(\theta_0)}{\psi_0(0)} + \sum_{l=1}^{\infty}\left(\ln\frac{\psi_l(\theta_0)}{\psi_l(0)} + \ln\frac{\psi_{-l}(\theta_0)}{\psi_{-l}(0)} - \frac{D}{l}\right) + F + D\gamma\,, \tag{7.6.125}$$

where

$$D \equiv \int_0^\infty d\rho\, \sinh\rho\, V_{AdS_2}(\rho) \tag{7.6.126}$$

$$F \equiv \int_0^\infty d\rho\left(\sinh\rho\, V_{AdS_2}(\rho)\ln\left(\frac{\sinh\rho}{2}\right) - \frac{\mathcal{A}(\rho)^2}{\sinh\rho}\right)\,, \tag{7.6.127}$$

Explicitly, the relevant operator in AdS_2 is

$$\mathcal{O}_{AdS_2} = -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{(l-\mathcal{A}(\rho))^2}{\sinh^2\rho} + V_{AdS_2}\,, \tag{7.6.128}$$

where the gauge field and the potential read

$$\mathcal{A}(\rho) = -\frac{(\cosh\rho - 1)^2\,(1-\cos\theta_0)}{\cosh^2\rho + 2\cosh\rho\cos\theta_0 + 1}\,, \qquad V_{AdS_2}(\rho) = \frac{\partial_\rho \mathcal{A}(\rho)}{\sinh\rho}\,. \tag{7.6.129}$$

Notice that we can write this as

$$\mathcal{A}(\rho) = \sinh\rho\,\partial_\rho W(\rho)\,, \qquad W(\rho) = \frac{1}{2}\ln\left(\frac{(\cosh\rho+1)^2}{\cosh^2\rho + 2\cosh\rho\cos\theta_0 + 1}\right). \tag{7.6.130}$$

This fact allows us to write the solution to the equation of motion as

$$f_l(\rho) = \tanh^{-l}\left(\frac{\rho}{2}\right) e^{W(\rho)}\left(A + B\int d\rho \frac{\tanh^{2l}\left(\frac{\rho}{2}\right) e^{-2W(\rho)}}{\sinh\rho}\right). \tag{7.6.131}$$

For the case at hand, the regular solution at $\rho = 0$ is

$$f_l(\rho) = \begin{cases} 2^{-\left(l+\frac{1}{2}\right)}\sqrt{1+\cos\theta_0}\,\tanh^{-l}\left(\frac{\rho}{2}\right)\dfrac{\cosh\rho+1}{\sqrt{\cosh^2\rho + 2\cosh\rho\cos\theta_0+1}} & l<0 \\[3ex] \dfrac{2^{l+\frac{1}{2}}\tanh^{l}\left(\frac{\rho}{2}\right)}{(l+2)\sqrt{1+\cos\theta_0}}\,\dfrac{\sqrt{\cosh^2\rho + 2\cosh\rho\cos\theta_0+1}}{\cosh\rho+1}\left(l + \dfrac{(\cosh\rho+1)^2\,(1+\cos\theta_0)}{\cosh^2\rho+2\cosh\rho\cos\theta_0+1}\right) & l>0 \end{cases}.$$

We then find

$$\psi_l(\theta_0) = \begin{cases} \left(\dfrac{1+\cos\theta_0}{2}\right)^{\frac{1}{2}} & l \le 0 \\[3ex] \left(\dfrac{1+\cos\theta_0}{2}\right)^{-\frac{1}{2}}\left(\dfrac{l+1+\cos\theta_0}{l+2}\right) & l \ge 0 \end{cases}. \tag{7.6.132}$$

Next, we compute the integrals

$$
\begin{aligned}
D &\equiv \int_0^\infty d\rho \, \sinh \rho \, V_{AdS_2}(\rho) \\
&= -2 \sin^2 \frac{\theta_0}{2} \\
F &\equiv \int_0^\infty d\rho \left(\sinh \rho \, V_{AdS_2}(\rho) \ln \left(\frac{\sinh \rho}{2} \right) - \frac{A(\rho)^2}{\sinh \rho} \right) , \\
&= -\frac{\theta_0}{2} \sin \theta_0 + (2 + \cos \theta_0) \ln \cos \frac{\theta_0}{2} + \sin^2 \frac{\theta_0}{2} .
\end{aligned}
\tag{7.6.133}
$$

The anomaly contribution is given by

$$
\begin{aligned}
I &\equiv \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \, \ln M \left[m^2 + V_{AdS_2} - \frac{1}{6} R + \frac{1}{12} \nabla^2 \ln M \right] \\
&= \frac{1}{2} \sin^2 \frac{\theta_0}{2} + \frac{1}{3} \ln \cos \frac{\theta_0}{2} + \frac{1}{2} \int d\rho \, \sinh \rho \ln M \, V_{AdS_2} \\
&= -\frac{1}{2} \sin^2 \frac{\theta_0}{2} + \frac{1}{3} \ln \cos \frac{\theta_0}{2} + \frac{1}{2} \theta_0 \sin \theta_0 + 2 \cos^2 \frac{\theta_0}{2} \ln \cos \frac{\theta_0}{2} .
\end{aligned}
\tag{7.6.134}
$$

Putting everything together we get

$$
\begin{aligned}
\ln \left(\frac{\det \mathcal{O}_{2\pm}(\theta_0)}{\det \mathcal{O}_{2\pm}(0)} \right) &= \ln \frac{\psi_0(\theta_0)}{\psi_0(0)} + \sum_{l=1}^{\infty} \left(\ln \frac{\psi_l(\theta_0)}{\psi_l(0)} + \ln \frac{\psi_{-l}(\theta_0)}{\psi_{-l}(0)} - \frac{D}{l} \right) + F + D\gamma + I \\
&= \ln \cos \frac{\theta_0}{2} - \ln \Gamma \left(2 \cos^2 \frac{\theta_0}{2} \right) - 2 \ln \cos \frac{\theta_0}{2} + 2\gamma \sin^2 \frac{\theta_0}{2} \\
&\quad - \frac{\theta_0}{2} \sin \theta_0 + (2 + \cos \theta_0) \ln \cos \frac{\theta_0}{2} + \sin^2 \frac{\theta_0}{2} - 2\gamma \sin^2 \frac{\theta_0}{2} \\
&\quad - \frac{1}{2} \sin^2 \frac{\theta_0}{2} + \frac{1}{3} \ln \cos \frac{\theta_0}{2} + \frac{1}{2} \theta_0 \sin \theta_0 + 2 \cos^2 \frac{\theta_0}{2} \ln \cos \frac{\theta_0}{2} \\
&= - \ln \Gamma \left(\cos \theta_0 \right) - \ln \cos \theta_0 + \left(\frac{7}{3} + 2 \cos \theta_0 \right) \ln \cos \frac{\theta_0}{2} + \frac{1}{2} \sin^2 \frac{\theta_0}{2} .
\end{aligned}
\tag{7.6.135}
$$

As before, the small θ_0 expansion coincides with the results of [28] and (7.5.104)

$$
\ln \left(\frac{\det \mathcal{O}_{2\pm}(\theta_0)}{\det \mathcal{O}_{2\pm}(0)} \right) = \frac{1}{2} \left(\frac{1}{6} - \gamma \right) \theta_0^2 + O\left(\theta_0^4 \right) . \tag{7.6.136}
$$

7.6.1.3 $\mathcal{O}_{3\pm}(\theta_0)$

The relevant operator in AdS_2 is now

$$\mathcal{O}_{AdS_2} = -\frac{1}{\sinh\rho}\partial_\rho\left(\sinh\rho\,\partial_\rho\right) + \frac{\left(l-\mathcal{B}(\rho)\right)^2}{\sinh^2\rho} + V_{AdS_2}\,, \tag{7.6.137}$$

where the gauge field and the potential read

$$\mathcal{B}(\rho) = \frac{1}{2}\frac{\left(\cosh\rho-1\right)\left(1-\cos\theta_0\right)}{\cosh\rho+\cos\theta_0}\,, \qquad V_{AdS_2}(\rho) = -\frac{\partial_\rho\mathcal{B}(\rho)}{\sinh\rho}\,. \tag{7.6.138}$$

Notice that we can write this as

$$\mathcal{B}(\rho) = \sinh\rho\,\partial_\rho W(\rho)\,, \qquad W(\rho) = \frac{1}{2}\ln\left(\frac{\left(\cosh\rho-1\right)\left(\cosh\rho+\cos\theta_0\right)}{\sinh^2\rho}\right)\,. \tag{7.6.139}$$

This fact allows us to write the solution to the equation of motion as

$$f_l(\rho) = \tanh^l\left(\frac{\rho}{2}\right)e^{-W(\rho)}\left(A + B\int d\rho\frac{\tanh^{-2l}\left(\frac{\rho}{2}\right)e^{2W(\rho)}}{\sinh\rho}\right)\,. \tag{7.6.140}$$

For the case at hand, the regular solution at $\rho=0$ is

$$f_l(\rho) = \begin{cases} 2^l\cos\dfrac{\theta_0}{2}\tanh^{l-\frac{1}{2}}\left(\dfrac{\rho}{2}\right)\sqrt{\dfrac{\sinh\rho}{\cosh\rho+\cos\theta_0}} & l>0 \\[2ex] \dfrac{\tanh^{-l+\frac{1}{2}}\left(\frac{\rho}{2}\right)}{2^{l+1}\left(l-1\right)\cos\frac{\theta_0}{2}}\sqrt{\dfrac{\cosh\rho+\cos\theta_0}{\sinh\rho}}\left(2l-\dfrac{\left(\cosh\rho-1\right)\left(1+\cos\theta_0\right)}{\cosh\rho+\cos\theta_0}\right) & l<0 \end{cases}\,. \tag{7.6.141}$$

We then find

$$\psi_l(\theta_0) = \begin{cases} \left(\dfrac{1+\cos\theta_0}{2}\right)^{\frac{1}{2}} & l \le 0 \\ \\ \left(\dfrac{1+\cos\theta_0}{2}\right)^{-\frac{1}{2}} \left(\dfrac{l - \frac{1+\cos\theta_0}{2}}{l-1}\right) & l \le 0 \end{cases} . \tag{7.6.142}$$

Next, we compute the integrals

$$\begin{aligned} D &\equiv \int_0^\infty d\rho \, \sinh\rho \, V_{AdS_2}(\rho) = -\sin^2\frac{\theta_0}{2} \, , \\ F &\equiv \int_0^\infty d\rho \left(\sinh\rho \, V_{AdS_2}(\rho) \ln\left(\frac{\sinh\rho}{2}\right) - \frac{A(\rho)^2}{\sinh\rho}\right) = 2\cos^2\frac{\theta_0}{2}\ln\cos\frac{\theta_0}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2} \, , \\ I &\equiv \frac{1}{2}\sin^2\frac{\theta_0}{2} + \frac{1}{3}\ln\cos\frac{\theta_0}{2} + \frac{1}{2}\int d\rho \, \sinh\rho \ln M \, V_{AdS_2} = \frac{3}{2}\sin^2\frac{\theta_0}{2} + \frac{1}{3}\ln\cos\frac{\theta_0}{2} \\ &\quad - \frac{1}{4}\theta_0 \sin\theta_0 + \sin^2\frac{\theta_0}{2}\ln\cos\frac{\theta_0}{2} \, . \end{aligned} \tag{7.6.143}$$

Putting everything together we get

$$\begin{aligned} \ln\left(\frac{\det \mathcal{O}_{3\pm}(\theta_0)}{\det \mathcal{O}_{3\pm}(0)}\right) &= \ln\frac{\psi_0(\theta_0)}{\psi_0(0)} + \sum_{l=1}^{\infty}\left(\ln\frac{\psi_l(\theta_0)}{\psi_l(0)} + \ln\frac{\psi_{-l}(\theta_0)}{\psi_{-l}(0)} - \frac{D}{l}\right) + F + D\gamma + I \\ &= \ln\cos\frac{\theta_0}{2} - \ln\Gamma\left(\cos^2\frac{\theta_0}{2}\right) - 2\ln\cos\frac{\theta_0}{2} + \gamma\sin^2\frac{\theta_0}{2} \\ &+ 2\cos^2\frac{\theta_0}{2}\ln\cos\frac{\theta_0}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2} - \gamma\sin^2\frac{\theta_0}{2} \\ &+ \frac{3}{2}\sin^2\frac{\theta_0}{2} + \frac{1}{3}\ln\cos\frac{\theta_0}{2} - \frac{1}{4}\theta_0\sin\theta_0 + \sin^2\frac{\theta_0}{2}\ln\cos\frac{\theta_0}{2} \\ &= -\ln\Gamma\left(\cos^2\frac{\theta_0}{2}\right) + \frac{1}{2}\left(\frac{5}{3} + \cos\theta_0\right)\ln\cos\frac{\theta_0}{2} - \frac{1}{4}\theta_0\sin\theta_0 + 2\sin^2\frac{\theta_0}{2} \, . \end{aligned} \tag{7.6.144}$$

The small θ_0 expansion is

$$\ln\left(\frac{\det \mathcal{O}_{3\pm}(\theta_0)}{\det \mathcal{O}_{3\pm}(0)}\right) = \frac{1}{2}\left(\frac{1}{6} - \frac{\gamma}{2}\right)\theta_0^2 + O\left(\theta_0^4\right) \, , \tag{7.6.145}$$

which coincides with the perturbative heat kernel approach (7.5.106).

Summary: The total bosonic contribution is,

$$\frac{1}{2}\left[2\ \ln\left(\frac{\det\mathcal{O}_1(\theta_0)}{\det\mathcal{O}_1(0)}\right)+2\ \ln\left(\frac{\det\mathcal{O}_{2\pm}(\theta_0)}{\det\mathcal{O}_{2\pm}(0)}\right)+4\ \ln\left(\frac{\det\mathcal{O}_{3\pm}(\theta_0)}{\det\mathcal{O}_{3\pm}(0)}\right)\right]=$$

$$\begin{aligned}
&=\frac{\theta_0}{2}\sin\theta_0+5\ \sin^2\frac{\theta_0}{2}+\left(\frac{19}{3}+5\ \cos\theta_0\right)\ \ln\cos\frac{\theta_0}{2}-2\ \ln\ \Gamma\left(\cos^2\frac{\theta_0}{2}\right)\\
&-\ln\left(\Gamma(\cos\theta_0)\right)-\ln\left(\cos\theta_0\right)\\
&=\left(\frac{5}{6}-\gamma\right)\theta_0^2+O(\theta_0^4)
\end{aligned}\tag{7.6.146}$$

which matches the perturbative heat kernel calculation (7.5.107).

7.6.2 Fermions

The effective 2d fermions operators (7.4.60) involve gauge couplings to the normal bundle (see (7.4.62)). Three different operators operators (7.4.61) appear in the computation of the 1-loop effective action.

7.6.2.1 $\mathcal{O}_{4\pm}(\theta_0)$

This case corresponds to vanishing coupling to the $\mathcal{B}_\mu$ gauge field. Then, the following quantities simplifies to:

$$D_\mu=\nabla_\mu+\iota\frac{\alpha}{2}\,\mathcal{A}_\mu,\quad V(\rho)=\frac{1}{\sqrt{M(\rho)}}-1,\quad W(\rho)=\frac{\sin^2\theta(\rho)}{\sqrt{M(\rho)}\ \sinh^2\rho}.\tag{7.6.147}$$

Take $\Gamma^{\underline{0}}=\sigma_1$, $\Gamma^{\underline{1}}=\sigma_2$, and consider operators of the form,

$$\mathcal{O}_\alpha(\theta_0)=-\iota\not{D}+V_1\tag{7.6.148}$$

where

$$V_1=-\frac{\partial_\rho M}{4\,M}\ \iota\sigma_1+\frac{1}{\sqrt{M}}\left(\sigma_3+\alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\right).\tag{7.6.149}$$

Using circular symmetry, we expand into Fourier components. Explicitly,

$$\iota\,\mathcal{O}_l = \begin{pmatrix} \frac{\iota}{\sqrt{M}}\Big(1+\alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\Big) & \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} - \frac{l}{\sinh\rho} - \frac{\alpha\,\mathcal{A}}{2\sinh\rho} \\ \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} + \frac{l}{\sinh\rho} + \frac{\alpha\,\mathcal{A}}{2\sinh\rho} & \frac{\iota}{\sqrt{M}}\Big(-1+\alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\Big) \end{pmatrix}$$

The relevant integrals in (7.6.121)-(7.6.122) give

$$\hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho\ \sinh\rho\,\left((m+V)^2 - m^2 - W^2\right) = \sin^2\frac{\theta_0}{2} \tag{7.6.150}$$

$$\int_0^\infty d\rho\ \sinh\rho\ \ln\left(\frac{\sinh\rho}{2}\right)\left((m+V)^2 - m^2 - W^2\right) = 2\cos\theta_0\ \ln\cos\frac{\theta_0}{2} \tag{7.6.151}$$

$$\int_0^\infty d\rho\ \sinh\rho\, W^2 = -\frac{1}{2}\theta_0\ \sin\theta_0 + 2\ \sin^2\frac{\theta_0}{2} \tag{7.6.152}$$

$$\int_0^\infty d\rho\frac{\mathcal{A}^2}{\sinh\rho} = -\sin^2\frac{\theta_0}{2} - 2\ \log\cos\frac{\theta_0}{2} \tag{7.6.153}$$

$$\int_0^\infty d\rho\frac{\mathcal{B}^2}{\sinh\rho} = -\frac{1}{2}\sin^2\frac{\theta_0}{2} - \log\cos\frac{\theta_0}{2} \tag{7.6.154}$$

The Weyl anomaly contribution results,

$$\begin{aligned} &\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right] \\ &= \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[2 - M + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right] \\ &= \frac{7}{4}\sin^2\frac{\theta_0}{2} + \frac{11}{6}\ln\cos\frac{\theta_0}{2} \end{aligned} \tag{7.6.155}$$

For $\alpha = 1$ we obtain

$$\ln\left(\frac{\det\mathcal{O}_{4+}(\theta_0)}{\det\mathcal{O}_{4+}(0)}\right) = \frac{\theta_0}{2}\ \sin\theta_0 + \left(\frac{7}{3} + 2\ \cos\theta_0\right)\ \ln\cos\frac{\theta_0}{2} - \ln\Gamma(\cos\theta_0) - \ln\cos\theta_0 \tag{7.6.156}$$

7.6.2.2 $\mathcal{O}_{5\pm}(\theta_0)$

In this case,

$$D_\mu = \nabla_\mu + \iota\alpha\left(\frac{\mathcal{A}_\mu}{2} + \mathcal{B}_\mu\right), \quad V(\rho) = -\frac{1}{2\sqrt{M}} - \frac{1}{2}\sqrt{M} + 1, \quad W(\rho) = -\frac{1}{2}\frac{\sin^2\theta(\rho)}{\sqrt{M}\,\sinh^2\rho}.$$

Using circular symmetry, we can expand the operator into Fourier components. Let

$$\psi_l(\rho) = \begin{bmatrix} u_l(\rho) \\ v_l(\rho) \end{bmatrix} \tag{7.6.157}$$

with $l \in \mathbb{Z} + \frac{1}{2}$. The radial problem becomes, $\mathcal{O}_l\psi_l = 0$, where

$$\mathcal{O}_l = -\iota\sigma_1\left(\partial_\rho + \frac{1}{2}\coth\rho + \frac{1}{4}\partial_\rho \ln M\right) - \frac{1}{\sinh\rho}\sigma_2\left(l + \frac{\alpha}{2}\mathcal{A} + \alpha\mathcal{B}\right) + \sigma_3(-1+V) + \alpha W.$$

Explicitly,

$$\iota\,\mathcal{O}_l^\alpha = \begin{pmatrix} \frac{\iota}{2\sqrt{M}}\left(-1 - M - \alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\right) & \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} - \frac{l}{\sinh\rho} - \frac{\alpha\,(\mathcal{A}+2\,\mathcal{B})}{2\,\sinh\rho} \\ \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} + \frac{l}{\sinh\rho} + \frac{\alpha\,(\mathcal{A}+2\,\mathcal{B})}{2\sinh\rho} & \frac{\iota}{2\sqrt{M}}\left(1 + M - \alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\right). \end{pmatrix}$$

For $\alpha = 1$, the operator reduces to

$$\iota\,\mathcal{O}_l = \begin{pmatrix} -\iota\sqrt{M} & \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} - \frac{l}{\sinh\rho} - \frac{(\mathcal{A}+2\,\mathcal{B})}{2\,\sinh\rho} \\ \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} + \frac{l}{\sinh\rho} + \frac{(\mathcal{A}+2\,\mathcal{B})}{2\sinh\rho} & \frac{\iota}{\sqrt{M}} \end{pmatrix}$$

and the system of equations become,

$$\left(\partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} - \frac{l}{\sinh\rho} - \frac{(\mathcal{A}+2\,\mathcal{B})}{2\,\sinh\rho}\right) v_l(\rho) - \iota\sqrt{M}\,u_l(\rho) = 0, \tag{7.6.158}$$

$$\left(\partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} + \frac{l}{\sinh\rho} + \frac{(\mathcal{A}+2\,\mathcal{B})}{2\,\sinh\rho}\right) u_l(\rho) + \iota\,\frac{1}{\sqrt{M}}\,v_l(\rho) = 0. \tag{7.6.159}$$

Introducing

$$D^\pm \equiv \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4M} \pm \left(\frac{l}{\sinh\rho} + \frac{(\mathcal{A}+2\,\mathcal{B})}{2\,\sinh\rho}\right), \tag{7.6.160}$$

we start solving the second order equation for $v_l(\rho)$. It takes the form,

$$\sqrt{M}\, D^+\bigg(\frac{1}{\sqrt{M}}\, D^- v_l\bigg) - v_l = 0, \tag{7.6.161}$$

which we rewrite as,

$$-\frac{1}{\sinh\rho}\partial_\rho\big(\sinh\rho\,\partial_\rho\, v_l(\rho)\big) + \frac{(l+\mathcal{X})^2}{\sinh^2\rho}v_l(\rho) - \frac{\partial_\rho\mathcal{X}}{\sinh\rho}v_l(\rho) = 0, \tag{7.6.162}$$

with

$$\mathcal{X} = \sinh\rho\,\bigg(-\frac{\coth\rho}{2} - \frac{\partial_\rho M}{4\,M}\bigg) + \frac{\mathcal{A}+2\mathcal{B}}{2}. \tag{7.6.163}$$

The solution is

$$v_l(\rho) = \bigg(\tanh\frac{\rho}{2}\bigg)^{-l+\frac{1}{2}} e^{-\mathcal{W}(\rho)}\,\bigg(C_1 + C_2\int d\rho\bigg(\tanh\frac{\rho}{2}\bigg)^{2l-1}\frac{e^{2\mathcal{W}(\rho)}}{\sinh\rho}\bigg),\quad \partial_\rho\mathcal{W}(\rho) = \frac{\mathcal{X}(\rho)+\frac{1}{2}}{\sinh\rho}.$$

As before, we fix constants C_1 and C_2 by demanding regularity at the origin ($\rho = 0$). For $l \geq 1/2$, we find

$$v_l^+(\rho) = C_2\,\frac{(2l+\cosh\rho)}{(4l^2-1)\,\sinh\frac{\rho}{2}}\bigg(\tanh\frac{\rho}{2}\bigg)^{l+\frac{1}{2}} \tag{7.6.164}$$

which gives

$$u_l^+(\rho) = -C_2\,\frac{2\iota\,\sinh\frac{\rho}{2}\,(\cos\theta_0+\cosh\rho)}{(4l^2-1)\,\sqrt{1+\cosh^2\rho+2\cos\theta_0\,\cosh\rho}}\,\bigg(\tanh\frac{\rho}{2}\bigg)^{l-\frac{1}{2}}. \tag{7.6.165}$$

For $l \leq -1/2$,

$$v_l^-(\rho) = C_1\,\cosh\frac{\rho}{2}\,\bigg(\tanh\frac{\rho}{2}\bigg)^{-l+\frac{1}{2}} \tag{7.6.166}$$

and

$$u_l^-(\rho) = C_1 \, \frac{\iota\,(2l - \cosh\rho)\,(\cos\theta_0 + \cosh\rho)}{2\,\cosh\frac{\rho}{2}\,\sqrt{1 + \cosh^2\rho + 2\cos\theta_0\,\cosh\rho}} \left(\tanh\frac{\rho}{2}\right)^{-l-\frac{1}{2}} \tag{7.6.167}$$

The relevant formulas in this case, corresponding to $m = -1$, are

$$\left((m+V)^2 - m^2 - W^2\right) = 0 \tag{7.6.168}$$

$$\hat{\zeta}_{\mathcal{O}}(0) \;\; = \;\; -\frac{1}{2}\int_0^\infty d\rho\,\sinh\rho\,\left((m+V)^2 - m^2 - W^2\right) = 0 \tag{7.6.169}$$

and

$$\int_0^\infty d\rho\,\sinh\rho\,\ln\left(\frac{\sinh\rho}{2}\right)\left((m+V)^2 - m^2 - W^2\right) = 0 \tag{7.6.170}$$

together with

$$\int_0^\infty d\rho\,\sinh\rho\,W^2 = -\frac{1}{8}\theta_0\,\sin\theta_0 + \frac{1}{2}\,\sin^2\frac{\theta_0}{2} \tag{7.6.171}$$

The Weyl anomaly contribution in this case results,

$$\begin{aligned}
&\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right]\\
&= \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[1 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right]\\
&= \frac{\theta_0}{2}\sin\theta_0 + 2\cos^2\frac{\theta}{2}\,\log\cos\frac{\theta_0}{2} - \frac{1}{4}\sin^2\frac{\theta_0}{2} - \frac{1}{6}\log\cos\frac{\theta_0}{2}
\end{aligned} \tag{7.6.172}$$

7.6.2.3 $\mathcal{O}_{6\pm}(\theta_0)$

In this case,

$$D_\mu = \nabla_\mu + \iota\alpha\left(\frac{\mathcal{A}_\mu}{2} - \mathcal{B}_\mu\right), \quad V(\rho) = -\frac{1}{2\sqrt{M}} + \frac{1}{2}\sqrt{M}, \quad W(\rho) = -\frac{1}{2}\,\frac{\sin^2\theta(\rho)}{\sqrt{M}\,\sinh^2\rho}.$$

Consider an operator of form,

$$\mathcal{O}_\alpha(\theta_0) = -\iota \not{D} + V, \tag{7.6.173}$$

where

$$V = -\frac{\partial_\rho M}{4\,M}\,\iota\sigma_1 + \frac{1}{2\,\sqrt{M}}\Big(\big(-1+M\big)\sigma_3 - \alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\Big). \tag{7.6.174}$$

Using circular symmetry, we can expand this into Fourier components. Explicitly,

$$\iota\,\mathcal{O}_l^\alpha = \begin{pmatrix} \frac{\iota}{2\sqrt{M}}\Big(-1+M-\alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\Big) & \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} - \frac{l}{\sinh\rho} - \frac{\alpha\,(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho} \\ \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} + \frac{l}{\sinh\rho} + \frac{\alpha\,(\mathcal{A}-2\,\mathcal{B})}{2\sinh\rho} & \frac{\iota}{2\sqrt{M}}\Big(1-M-\alpha\frac{\sin^2\theta(\rho)}{\sinh^2\rho}\Big) \end{pmatrix}$$

For $\alpha = 1$, the system of equations decouples,

$$\iota\,\mathcal{O}_l = \begin{pmatrix} 0 & \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} - \frac{l}{\sinh\rho} - \frac{(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho} \\ \partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} + \frac{l}{\sinh\rho} + \frac{(\mathcal{A}-2\,\mathcal{B})}{2\sinh\rho} & \frac{\iota\,(1-M)}{\sqrt{M}} \end{pmatrix}.$$

Taking

$$\psi_l(\rho) = \begin{bmatrix} u_l(\rho) \\ v_l(\rho) \end{bmatrix}, \tag{7.6.175}$$

the equation then becomes:

$$\Big(\partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} - \frac{l}{\sinh\rho} - \frac{(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho}\Big)\,v_l(\rho) = 0, \tag{7.6.176}$$

$$\Big(\partial_\rho + \frac{\coth\rho}{2} + \frac{\partial_\rho M}{4\,M} + \frac{l}{\sinh\rho} + \frac{(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho}\Big)\,u_l(\rho) + \iota\,\frac{1-M}{\sqrt{M}}\,v_l(\rho) = 0. \tag{7.6.177}$$

Solving for $v_l(\rho)$ gives,

$$v_l(\rho) = C_1\,\Big(\sinh\frac{\rho}{2}\Big)^{l-\frac{1}{2}}\,\Big(\cosh\frac{\rho}{2}\Big)^{-l-\frac{5}{2}}\,\big(\cos\theta_0 + \cosh\rho\big), \tag{7.6.178}$$

where C_1 is a constant. Using this solution, we can now solve equation for $u_l(\rho)$,

$$u_l^{'}(\rho)+\left(\frac{\coth\rho}{2}+\frac{\partial_\rho M}{4\,M}+\frac{l}{\sinh\rho}+\frac{(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho}\right)u_l(\rho)+\iota\,\frac{1-M}{\sqrt{M}}\,v_l(\rho)=0. \quad (7.6.179)$$

The integrating factor for this equation is,

$$\begin{aligned} I(\rho)&=\mathrm{Exp}\bigg[\int d\rho\left(\frac{\coth\rho}{2}+\frac{\partial_\rho M}{4\,M}+\frac{l}{\sinh\rho}+\frac{(\mathcal{A}-2\,\mathcal{B})}{2\,\sinh\rho}\right)\bigg]\\ &=\left(-\,\iota\sinh\frac{\rho}{2}\right)^{l+\frac{1}{2}}\left(\cosh\frac{\rho}{2}\right)^{-l-\frac{3}{2}}\sqrt{3+4\,\cos\theta_0\cosh\rho+\cosh(2\rho)}. \end{aligned} \quad (7.6.180)$$

Then, full solution takes the form,

$$\begin{aligned} u_l(\rho)&=\frac{1}{I(\rho)}\bigg[\int d\rho\,I(\rho)\left(-\,\iota\,\frac{1-M}{\sqrt{M}}\,v_l(\rho)\right)+C_2\bigg]\\ &=\bigg[C_1\,\frac{2^{\frac{3}{2}+l}\,\iota\,(2+2l+\cosh\rho)\left(\sinh\frac{\rho}{2}\right)^{\frac{1}{2}+2l}\,\sin^2\theta_0}{(3+8l+4l^2)\left(\cosh\frac{\rho}{2}\right)^{\frac{3}{2}}\left(\sinh\rho\right)^{l}\sqrt{3+4\cos\theta_0\cosh\rho+\cosh(2\rho)}}\\ &\quad+C_2\,\frac{\left(\cosh\frac{\rho}{2}\right)^{\frac{3}{2}+l}\left(-\,\iota\sinh\frac{\rho}{2}\right)^{-\frac{1}{2}-l}}{\sqrt{3+4\cos\theta_0\cosh\rho+\cosh 2\rho}}\bigg]. \end{aligned} \quad (7.6.181)$$

Demanding the solution to be regular at the origin fixes $C_2=0$ for $l\geq 1/2$ and $C_1=0$ for $l\leq -1/2$. For $l\leq -1/2$,

$$u_l^{-}(\rho)=C_2\,\frac{\left(\cosh\frac{\rho}{2}\right)^{\frac{3}{2}+l}\left(-\,\iota\sinh\frac{\rho}{2}\right)^{-\frac{1}{2}-l}}{\sqrt{3+4\cos\theta_0\cosh\rho+\cosh 2\rho}}, \quad (7.6.182)$$

and for $l\geq 1/2$,

$$u_l^{+}(\rho)=C_1\,\frac{2^{\frac{3}{2}+l}\,\iota\,(2+2l+\cosh\rho)\left(\sinh\frac{\rho}{2}\right)^{\frac{1}{2}+2l}\,\sin^2\theta_0}{(3+8l+4l^2)\left(\cosh\frac{\rho}{2}\right)^{\frac{3}{2}}\left(\sinh\rho\right)^{l}\sqrt{3+4\cos\theta_0\cosh\rho+\cosh(2\rho)}}, \quad (7.6.183)$$

$$v_l^{+}(\rho)=C_1\left(\sinh\frac{\rho}{2}\right)^{l-\frac{1}{2}}\left(\cosh\frac{\rho}{2}\right)^{-l-\frac{5}{2}}\left(\cos\theta_0+\cosh\rho\right). \quad (7.6.184)$$

The relevant formulas in the present case $m = 0$ case are,

$$\left((m+V)^2 - m^2 - W^2\right) = 0 \quad \Rightarrow \quad \hat{\zeta}_{\mathcal{O}}(0) = -\frac{1}{2}\int_0^\infty d\rho \, \sinh\rho \left((m+V)^2 - m^2 - W^2\right) \tag{7.6.185}$$

$$\Rightarrow \quad \int_0^\infty d\rho \, \sinh\rho \, \ln\left(\frac{\sinh\rho}{2}\right)\left((m+V)^2 - m^2 - W^2\right) = 0 \tag{7.6.186}$$

together with

$$\int_0^\infty d\rho \, \sinh\rho \, W^2 = -\frac{1}{8}\theta_0 \, \sin\theta_0 + \frac{1}{2}\,\sin^2\frac{\theta_0}{2} \tag{7.6.187}$$

The Weyl anomaly contribution in this case follows from,

$$\begin{aligned}
&\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[(m+V)^2 - W^2 + \frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right] \\
&= \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\left[\frac{1}{12}R - \frac{1}{24}\nabla^2\ln M\right] \\
&= -\frac{1}{4}\sin^2\frac{\theta_0}{2} - \frac{1}{6}\log\cos\frac{\theta_0}{2}
\end{aligned} \tag{7.6.188}$$

The total contribution from the $\mathcal{O}_{5\pm}$ and $\mathcal{O}_{6\pm}$ cases is

$$\begin{aligned}
\ln\left(\frac{\det O}{\det O^{\mathrm{free}}}\right) &= -2\left(\frac{1}{2}\right)^2\int_0^\infty d\rho\,\frac{(\mathcal{A}+2\mathcal{B})^2}{\sinh\rho} - 2\left(-\frac{1}{2}\right)^2\int_0^\infty d\rho\,\frac{(\mathcal{A}+2\mathcal{B})^2}{\sinh\rho} \\
&- 2\left(\frac{1}{2}\right)^2\int_0^\infty d\rho\,\frac{(\mathcal{A}-2\mathcal{B})^2}{\sinh\rho} - 2\left(-\frac{1}{2}\right)^2\int_0^\infty d\rho\,\frac{(\mathcal{A}-2\mathcal{B})^2}{\sinh\rho} - 4\int_0^\infty d\rho\sinh\rho\, W^2 \\
&= -\int_0^\infty d\rho\,\frac{\mathcal{A}^2}{\sinh\rho} - 4\int_0^\infty d\rho\,\frac{\mathcal{B}^2}{\sinh\rho} - 4\int_0^\infty d\rho\sinh\rho\, W^2
\end{aligned}$$

Thus,

$$\ln\left(\frac{\det O}{\det O^{\mathrm{free}}}\right) = \frac{1}{2}\theta_0\,\sin\theta_0 + \sin^2\frac{\theta_0}{2} + 6\,\log\left(\cos\frac{\theta_0}{2}\right) \tag{7.6.189}$$

7.6.3 One-loop effective action

The total zeta-function at the origin is

$$\hat{\zeta}_{\rm tot}(0) = 2\,\hat{\zeta}_{\mathcal{O}_1}(0) + \hat{\zeta}_{\mathcal{O}_{2+}}(0) + \hat{\zeta}_{\mathcal{O}_{2-}}(0) + 2\,\hat{\zeta}_{\mathcal{O}_{3+}}(0) + 2\,\hat{\zeta}_{\mathcal{O}_{3-}}(0) - 2\hat{\zeta}_{\mathcal{O}_{+}}(0) - 2\hat{\zeta}_{\mathcal{O}_{-}}(0)$$

where $\mathcal{O}_{\pm}$ are fermionic contributions arising from the $\mathcal{O}_{4\pm}(\theta_0)$ operators. Adding up the pieces we find

$$4\left((1+V)^2 - W^2 - 1\right) - 2\,V_2 - 4\,V_3 = \nabla^2 \ln M \qquad (7.6.190)$$

which vanishes when integrated,

$$\int_0^\infty d\rho \sinh\rho\, \nabla^2\, \ln M = \sinh\rho\,\partial_\rho \ln M|_0^\infty = 0, \qquad \hat{\zeta}_{\rm tot}(0) = 0. \qquad (7.6.191)$$

The contributions from gauge field are seen to vanish,

$$\begin{aligned}
&1\times(1)^2\mathcal{A}^2 + 1\times(-1)^2\mathcal{A}^2 + 2\times(1)^2\mathcal{B}^2 + 2\times(-1)^2\mathcal{B}^2 - 2\times(\frac{1}{2})^2\mathcal{A}^2 - 2\times(-\frac{1}{2})^2\mathcal{A}^2 \\
&- 1\times(\frac{1}{2})^2\times(\mathcal{A}+2\mathcal{B})^2 - 1\times(-\frac{1}{2})^2(\mathcal{A}+2\mathcal{B})^2 - 1\times(\frac{1}{2})^2(\mathcal{A}-2\mathcal{B})^2 - 1\times(-\frac{1}{2})^2(\mathcal{A}-2\mathcal{B})^2 \\
&= 0
\end{aligned}$$

The contribution from W^2 term in the fermionic potentital is non-trivial

$$\begin{aligned}
W^2:\quad & -4\times\int_0^\infty d\rho\sinh\rho\left(\frac{\sin^2\theta(\rho)}{\sqrt{M}\,\sinh^2\rho}\right)^2 - 4\times\int_0^\infty d\rho\sinh\rho\left(-\frac{\sin^2\theta(\rho)}{2\sqrt{M}\,\sinh^2\rho}\right)^2 \\
&= \frac{5}{2}\theta_0\,\sin\theta_0 - 10\sin^2\frac{\theta_0}{2}
\end{aligned}$$

The Weyl anomaly has different contributions, they are:

- Potential and mass terms

$$4\left((1+V)^2 - W^2\right) + 2\times 1 - 2\times 2 - 2\times V_2 - 4\times V_3 = -R + \nabla^2 \ln M$$

- Curvature and conformal terms

$$\left(8\times\left(\frac{1}{12}\right)-8\times\left(-\frac{1}{6}\right)R+\nabla^2\ln M\left(8\times\left(-\frac{1}{24}\right)-8\times\left(\frac{1}{12}\right)\right)=2R-\nabla^2\ln M.\right.$$

The contribution from the conformal factor cancels and the total contribution from Weyl anomaly results siimply from the curvature term,

$$\text{anomaly}: \qquad \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R\ln M=-\left(\theta_0\sin\theta_0+4\,\cos^2\frac{\theta_0}{2}\,\ln\cos\frac{\theta_0}{2}\right)$$

The contribution from $\ln\left(\sinh\rho\right)$ integrals involve the same combination of potentials as $\hat{\zeta}_{\text{tot}}(0)$, which when added to the Weyl anomaly gives,

$$\text{anomaly}+\ln\sinh\rho: \quad \int_0^\infty d\rho\,\sinh\rho\left(\frac{1}{2}R\,\ln M+\ln\left(\frac{\sinh\rho}{2}\right)\nabla^2\ln M\right)=-2\ln\cos\frac{\theta_0}{2}.$$

The final result for the 1-loop effective action results

$$\begin{aligned}\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0)&=\frac{5}{4}\theta_0\sin\theta_0-5\,\sin^2\frac{\theta_0}{2}+2\,\ln\cos\frac{\theta_0}{2}+2\,\ln\,\Gamma\Big(\cos^2\frac{\theta_0}{2}\Big)\\&-\ln\big(\Gamma(\cos\theta_0)\big)-\ln\big(\cos\theta_0\big)\\&=\frac{1}{4}\theta_0^2+O(\theta_0^4)\end{aligned}\tag{7.6.192}$$

This result does not agree with the field theory expectation. Although our regularization is diffeomorphic invariant there might be ambiguities that need to be understood better. At the moment we can track the discrepancy between the two methods to an ambiguity in the treatment of the $m=0$ fermionic modes, we will return to this question elsewhere. It seems that a more expeditious way to get at the exact answer might follow the approach of [29, 154] who mapped the spectral problems from the disk to the cylinder with the incorporation of an explicit diffeomorphic invariant cutoff; we hope to report on such explorations in an upcoming publication.

7.7 Conclusion

In this chapter we have discussed in detail the construction of the quadratic fluctuations for the string configuration dual to the general latitude Wilson loop in ABJM theory. We have paid particular attention to the various symmetries of the configurations and shown how they serve as a guiding avatar in the structure of fluctuations. At the semiclassical level the computation of the one-loop effective action is equivalent to the computation of determinants. We employed two methods for computing such determinants. The perturbative heat kernel method has lead to agreement with the expected field theory answer in the limit of small latitude angle. The ζ-function regularization method is non-perturbative but does not seem to lead to the expected field theory answer as it stands. We have previously developed the ζ-function approach in Chapter V and applied it to the $\mathcal{N} = 4$ context in Chapter VI, motivated by the goal of constructing a regularization that is explicitly diffeomorphic invariant. The key new ingredient in this work that introduces extra ambiguities with respect to our earlier efforts is the fact that some of the modes correspond to massless fermions. The situation is not completely satisfactory but sheds light on deficiencies and advantages of the various methods used to tackle questions of precision holography with Wilson loops. For example, some of the puzzles we face were confronted in the realm of $\mathcal{N} = 4$ SYM and paved the way leading to perfect matching with the field theory answer in [29]. There the computations of the determinants was mapped from the disk to the cylinder. We hope to revisit our computations using a similar approach.

One interesting property of the duality pair we discuss is that it admits two very natural limits. Here we focused on the 't Hooft limit where $\lambda = N/k$ is kept fixed as N is taken very large. It would be interesting to explore the M-theory limit, where k is kept fixed, beyond the leading order as well; some preliminary results were reported in [153]. Exploring quantum corrections in this context might ultimately shed light on various intricate quantum properties of M2 branes.

It would also be interesting to explore similar issues for Wilson loops in higher dimensional representations. Classical results were presented in [140, 141]; at the quantum level some preliminary results have been presented in Chapter VIII for the gravity configurations and a sub-leading analysis of the matrix model was presented in [155]. The prospects for precision holography in this case are improved due to the fact that the corresponding quadratic fluctuations live in the odd-dimensional world-volumes of the corresponding D2 and D6 branes, studied in Chapter VIII. Heat kernel techniques are considerably simplified in odd-dimensional spaces since the contributions arise exclusively from zero or boundary modes.

Recently, in the case of $\mathcal{N} = 4$ SYM, the expectation value of the $\frac{1}{2}$-BPS Wilson loop has been computed on the gravity side by taking the ratio of two of the limits of the latitude string [154]. We hope that a similar analysis in the case of ABJM Wilson loops will shed light on various aspects of precision holography in IIA, our work provides most of the required ingredients.

CHAPTER VIII

Holographic ABJM Wilson Loops in Higher Rank Representations

8.1 Introduction

In the context of AdS/CFT correspondence, there has been a concerted effort toward matching the holographic one-loop corrections with subleading terms in the field theory side [156, 21, 22, 23, 24]. One of the main motivation is to continue to construct ever more stringgent tests that will clarify the nature and aspects of the corrections. There is also an ongoing program of extending one-loop corrections to holographic configurations dual to Wilson loops in higher rank representations of the $SU(N)$ gauge group in $\mathcal{N} = 4$ SYM [107, 108, 25, 157].

In this chapter we take a step towards the understanding, beyond the leading order, of holographic configurations that are expected to correspond to supersymmetric Wilson loops in higher rank representations [140] in the ABJM theory. Namely, we construct the spectra of quantum fluctuations of a D6 and a D2 brane discussed in [140]. We present a complete analysis including the bosonic and fermionic excitations, thus completing some preliminary attempts undertaken in the literature. We find that the systems present some peculiar couplings not seen before in similar situations.

The rest of the chapter is organized as follows. In section 8.2 we review the supergravity background and reproduce the leading, classical value of the corresponding D-brane actions.

In section 8.3 we study the fluctuations of the D6 brane in both its bosonic and fermionic sectors and summarize the spectrum of dual operators. The analysis of the D2 fluctuations and the calculation of the corresponding spectrum of dual operators is carried out in section 8.4. In section 8.5, we relate our findings to the structure of supersymmetric multiplets known from the literature, and we conclude in section 8.6. We treat a number of more technical and additional aspects in a series of appendices. In particular, in appendix H we review the metric representations of $\mathbb{C}P^n$ needed in the main text. In appendix I we recall some details of the representation of $OSp(4|2)$. The harmonic analysis on the coset space $\tilde{T}^{1,1}$, which we need for the D6 fluctuations, is presented in appendix J.

8.2 Background configurations

8.2.1 SUGRA background

We start by reviewing the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ solution of type-IIA SUGRA, which is the dual of the ABJM theory [18]. This solution was described more than three decades ago by Nilsson and Pope [158] and we rely heavily on their presentation.

Our conventions will be as follows. We work with a Minkowski metric with $(-+\ldots+)$ signature. The AdS_4 and $\mathbb{C}P^3$ coordinates are denoted by the sets of indices $0,1,2,3$ and $4,\ldots,9$, respectively. The corresponding flat indices are underlined. Moreover, we set $\alpha' = 1$.

For our analysis, we shall use the string frame expressions for the background geometry given in [140], but we find it more convenient to work with dimensionless fields. Given the scope of the work we start by considering the bosonic Dp-brane action, which, in Minkowski signature, is given by

$$S^B_{\mathrm{D}p} = -T_p \int d^{p+1}\xi \, \epsilon - \Phi \sqrt{-\det(g_{ab} + \mathcal{F}_{ab})} + T_p \int \epsilon \mathcal{F} \wedge \sum_q C_q \ . \tag{8.2.1}$$

with $\mathcal{F}_{ab} = B_{ab} + 2\pi F_{ab}$, and $T_p = (2\pi)^{-p}$ is the Dp-brane tension. The metric g_{ab}, the 2-form B_{ab} and the RR fields C_q are intended as the pull-backs of the respective $10d$ background fields.

The $\mathrm{AdS}_4 \times \mathbb{C}P^3$ solution is given by [140]

$$ds_{10}^2 = \frac{1}{\beta}\left(ds_{\mathrm{AdS}_4}^2 + d\Sigma_3^2\right) \ , \quad \epsilon 2\Phi_0 = \frac{4}{\beta k^2} \ , \quad F_4 = \frac{3k}{2\beta}\epsilon_{\mathrm{AdS}_4} \ , \quad F_2 = \frac{k}{2}J_3 \ . \tag{8.2.2}$$

Here, $d\Sigma_3$ and J_3 are the line element and the Kähler form of unit-2 $\mathbb{C}P^3$, respectively, see appendix H for the definitions. AdS_4 is of unit radius. The relations to the dual field theory parameters and to the parameters used in [140] are

$$\beta = \frac{4k}{R^3} = \left(\pi\sqrt{2\lambda}\right)^{-1} \ , \qquad \lambda = \frac{N}{k}. \tag{8.2.3}$$

This suggests the following rescalings,

$$d\hat{s}_{10}^2 = \beta ds_{10}^2 \ , \quad \hat{\Phi} = \Phi - \Phi_0 \ , \quad \hat{\mathcal{F}} = \beta\mathcal{F} \ , \quad \hat{C}_p = \epsilon\Phi_0\beta^{\frac{p}{2}}C_p \ . \tag{8.2.4}$$

Thus, the action (8.2.1) becomes

$$S_{\mathrm{D}p}^B = -\hat{T}_p \int d^{p+1}\xi \, \epsilon - \hat{\Phi}\sqrt{-\det(\hat{g}_{ab} + \hat{\mathcal{F}}_{ab})} + \hat{T}_p \int \epsilon\hat{\mathcal{F}} \wedge \sum_q \hat{C}_q \ , \tag{8.2.5}$$

where the Dp-brane tension $\hat{T}_p$ is now

$$\hat{T}_p = T_p\epsilon - \Phi_0\beta^{-\frac{p+1}{2}} \ . \tag{8.2.6}$$

In particular,

$$\hat{T}_2 = \frac{N}{4\pi\sqrt{2\lambda}} = \frac{1}{4}\beta N \ , \qquad \hat{T}_6 = \frac{N\sqrt{2\lambda}}{(4\pi)^3} = \frac{N}{(4\pi)^3\pi\beta} \ . \tag{8.2.7}$$

The same rescaling procedure can be applied on the fermion action. Henceforth, we shall omit the hats for simplicity.

Applying (8.2.4) to (8.2.2), we find the dimensionless expressions

$$ds_{10}^2 = ds_{\mathrm{AdS}_4}^2 + d\Sigma_3^2 \ , \quad \Phi = 0 \ , \quad F_4 = 3\epsilon_{\mathrm{AdS}_4} \ , \quad F_2 = J_3 \ . \tag{8.2.8}$$

The dual field strengths are given by[1]

$$F_6 = *F_4 = -3\epsilon_{\mathbb{C}P^3}\ , \qquad F_8 = *F_2 = -\frac{1}{2}\epsilon_{\mathrm{AdS}_4}\wedge F_2\wedge F_2\ , \tag{8.2.10}$$

where $\epsilon_{\mathbb{C}P^3}$ denotes the volume form of the unit-2 $\mathbb{C}P^3$.

To conclude this review, we recall from [140] the explicit expressions for the metric which we will use in this chapter[2]

$$\begin{aligned} ds^2_{\mathrm{AdS}_4} &= \cosh^2 u\, ds^2_{\mathrm{AdS}_2} + du^2 + \sinh^2 u\, d\phi^2\ , \\ d\Sigma_3 &= d\alpha^2 + \cos^2\frac{\alpha}{2}(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2) + \sin^2\frac{\alpha}{2}(d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2) \\ &\quad + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1\, d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2\ , \end{aligned} \tag{8.2.11}$$

and the C-forms

$$\begin{aligned} C_1 &= \frac{1}{2}(\cos\alpha - 1)d\chi + \cos^2\frac{\alpha}{2}\cos\vartheta_1 d\varphi_1 + \sin^2\frac{\alpha}{2}\cos\vartheta_2 d\varphi_2\ , \\ C_3 &= \left(\cosh^3 u - 1\right)\epsilon_{\mathrm{AdS}_2}\wedge d\phi\ , \\ C_5 &= \frac{1}{8}(\sin^2\alpha\cos\alpha + 2\cos\alpha - 2)\sin\vartheta_1\sin\vartheta_2\, d\vartheta_1\wedge d\varphi_1\wedge d\vartheta_2\wedge d\varphi_2\wedge d\chi\ , \\ C_7 &= -\frac{1}{6}\left(\cosh^3 u - 1\right)\epsilon_{\mathrm{AdS}_2}\wedge d\phi\wedge F_2\wedge F_2\ . \end{aligned} \tag{8.2.12}$$

Our conventions for the volume forms are

$$\begin{aligned} \epsilon_{(10)} &= \epsilon_{\mathrm{AdS}_4}\wedge\epsilon_{4\mathbb{C}P^3}\ , \\ \epsilon_{\mathrm{AdS}_4} &= \cosh^2 u\,\sinh u\,\epsilon_{\mathrm{AdS}_2}\wedge du\wedge d\phi\ , \\ \epsilon_{\mathbb{C}P^3} &= \frac{1}{8}\sin^3\alpha\sin\vartheta_1\sin\vartheta_2\, d\alpha\wedge d\vartheta_1\wedge d\varphi_1\wedge d\vartheta_2\wedge d\varphi_2\wedge d\chi\ . \end{aligned} \tag{8.2.13}$$

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[1] In our conventions, the Kähler form is

$$J_3 = -\left(e^{\underline{4}}\wedge e^{\underline{9}} + e^{\underline{5}}\wedge e^{\underline{6}} + e^{\underline{7}}\wedge e^{\underline{8}}\right)\ . \tag{8.2.9}$$

[2] For $\mathbb{C}P^3$, this is the $m = n = 1$ foliation (H.10). The $\mathbb{C}P^3$ coordinates take values $\alpha, \vartheta_{1,2} \in (0,\pi)$, $\varphi_{1,2} \in (0, 2\pi)$, $\chi \in (0, 4\pi)$.

It is known that there are two inequivalent $\mathrm{AdS}_4 \times \mathbb{C}P^3$ solutions of IIA SUGRA, one $\mathcal{N} = 6$ supersymmetric, the other one without supersymmetries [158]. We are, of course, interested in the $\mathcal{N} = 6$ solution. The difference between the two solutions lies in a relative sign of F_2 and F_4, and one is well advised, in view of diverse conventions, to check the supersymmetry of the above configuration. For doing so, we use the supersymmetry transformations given in [159], because we will rely on that paper for the construction of the fermion action. The supersymmetry transformation of the gravitino and dilatino are

$$\delta_\epsilon \psi_m = D_m \epsilon \ , \qquad \delta_\epsilon \lambda = \Delta \epsilon \ , \tag{8.2.14}$$

where (dropping terms that vanish in our case)

$$D_m = \nabla_m - \frac{1}{8}\left(\frac{1}{2} F_{np}\Gamma^{np}\Gamma_{(10)} + \frac{1}{4!} F_{npqr}\Gamma^{npqr}\right)\Gamma_m \ , \tag{8.2.15}$$

$$\Delta = \frac{1}{8}\left(\frac{3}{2} F_{np}\Gamma^{np}\Gamma_{(10)} - \frac{1}{4!} F_{npqr}\Gamma^{npqr}\right) \ . \tag{8.2.16}$$

The $10d$ chirality matrix is defined by $\Gamma_{(10)} = \Gamma^{\underline{0 \cdots 9}}$. To check whether (8.2.8) is supersymmetric, one first considers the dilatino variation in (8.2.14). Defining

$$Q = \frac{1}{2} F_{mn}\Gamma^{mn}\Gamma^{\underline{456789}} = \Gamma^{\underline{5678}} + \Gamma^{\underline{4569}} + \Gamma^{\underline{4789}} \ , \tag{8.2.17}$$

and using (8.2.8), (8.2.16) can be written as

$$\Delta = \frac{3}{8}\Gamma^{\underline{0123}}(Q-1) \ . \tag{8.2.18}$$

Moreover, it follows from (8.2.17) that Q satisfies

$$(Q+3)(Q-1) = 0 \ , \tag{8.2.19}$$

and has the eigenvalues $(-3, -3, 1, 1, 1, 1, 1, 1)$. The degeneracies follow from $\mathrm{tr} Q = 0$.

There are, therefore, six $\mathbb{C}P^3$ spinors that solve

$$Q\epsilon = \epsilon \ . \tag{8.2.20}$$

Comparing with [158] we find that this is indeed the $\mathcal{N} = 6$ solution. We also recall from [158] that the AdS$_4$ components of (8.2.15) yield four AdS$_4$ Killing spinors, and that by virtue of (8.2.20) the integrability condition for the $\mathbb{C}P^3$ components of (8.2.15) is satisfied.

8.2.2 D6 and D2-branes

The D6-brane purportedly dual to the 1/6 BPS totally antisymmetric Wilson loop wraps AdS$_2$ $\subset$ AdS$_4$ at the point $u = 0$ and $\tilde{T}^{1,1} \subset \mathbb{C}P^3$ at constant α. The latter is a squashed $T^{1,1}$ space [160]. The internal gauge field $\mathcal{F}$ has electric flux only in the AdS$_2$ factor, $\mathcal{F} = E\epsilon_{\text{AdS}_2}$, where E is conjugate to the fundamental string charge p. Because the latter is fixed, the potential that yields the Wilson loop expectation value is the Legendre transform of the D6-brane action [140]. It is straightforward to obtain[3]

$$S_{\text{WL}} = S^B_{\text{D6}} - \frac{1}{\beta}pE = \frac{N}{4\beta}\left[\sin^3\alpha\sqrt{1-E^2} - E\left(\sin^2\alpha\cos\alpha + 2\cos\alpha - 2\right)\right] - \frac{1}{\beta}pE \ . \tag{8.2.21}$$

The equation of motion for α fixes

$$E = -\cos\alpha \ , \tag{8.2.22}$$

and that for E yields

$$p = \beta\frac{\delta S^B_{\text{D6}}}{\delta E} = \frac{N}{2}(1-\cos\alpha) \ . \tag{8.2.23}$$

The fact that p ranges from 0 to N is a signature of the antisymmetric representation. This evidence for the anti-symmetric representation is a typical phenomenon in many brane configurations originally understood in the case of the giant gravitons [161, 162]. Finally,

[3]The renormalized volume of the unit AdS$_2$ is $V_{\text{AdS}_2} = -2\pi$ [108].

the expectation value of the Wilson loop is found as

$$S_{\rm WL} = \frac{p(N-p)}{\beta N}. \tag{8.2.24}$$

Note the symmetry under $p \leftrightarrow N-p$. It was shown in [140] that this D6-brane is 1/6-BPS.

The D2-brane dual to the 1/6 BPS symmetric Wilson loop wraps $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$ at the point $u=0$ and the circle $S^1 \subset \mathbb{C}P^3$ along χ. Again, $\mathcal{F} = E\epsilon_{\mathrm{AdS}_2}$. With this configuration, the Wilson loop potential is

$$S_{WL} = S^B_{\rm D2} - \frac{1}{\beta}pE = \beta N\pi^2 \left[\sin\alpha\sqrt{1-E^2} - E\left(\cos\alpha - 1\right)\right] - \frac{1}{\beta}pE \ . \tag{8.2.25}$$

The field equation for α yields again (8.2.22), while the equation for E yields

$$p = \beta\frac{\delta S^B_{\rm D2}}{\delta E} = \beta^2 N\pi^2 = \frac{1}{2}k \ , \tag{8.2.26}$$

corresponding to $k/2$ fundamental strings. Finally, the Wilson loop expectation is

$$S_{WL} = \frac{k}{2}\sqrt{2\lambda}\pi \ . \tag{8.2.27}$$

It was shown in [140] that a single D2-brane is 1/3-BPS. Smearing on $\mathbb{C}P^1$ reduces supersymmetry to 1/6-BPS. There are outstanding questions as to in which precise higher rank representation each of the classical solutions discussed here and their generalizations reside. Let us simply note that other possible classical configurations do not seem to fit nicely with their $AdS_5 \times S^5$ counter-part. For example, the symmetric representation in that case corresponds to a D3 brane discussed in [163] whose spectrum of quantum excitations was presented in [107]. This D3 branes wraps $AdS_2 \times S^2 \subset AdS_5$ and the value of its electric flux can be arbitrarily large. We have verified that the analogous D2 configuration wrapping the $AdS_2 \times S^1 \subset AdS_4$ does not seem to have the expected properties.

The beautiful construction of the 1/2 BPS Wilson loop on the field theory side [145] and some of its generalizations discussed in [146] are still largely unexplored on the holographic side; the gap is particularly glaring in the case of higher rank representations. Let us advance a few observations we have briefly explored in this regard. On grounds of the supergroup symmetries, one expects that the 1/2 BPS D6 configuration should wrap $\mathbb{CP}^2 \subset \mathbb{CP}^3$ as to have $U(3)$ symmetry realized in its worldvolume. Correspondingly, there are potential D2 configurations that wrap a circle transverse to $\mathbb{CP}^2 \subset \mathbb{CP}^3$ and therefore, contain the action of $U(3)$ in the flucutations transverse to the worldvolume. A very preliminary exploration of these possibilities also yields puzzling results and we will report on these configurations separately.

8.3 D6-brane fluctuations

In this section, we consider the bosonic and fermionic fluctuations of the 1/6-BPS D6-branes. The notation in this section will be as follows. The $10d$ curved coordinates are denoted by Latin indices from the middle of the alphabet, $m, n = 0, \ldots, 9$. Latin indices from the beginning of the alphabet denote generic D6-brane coordinates, $a, b = 0, 1, 5, 6, 7, 8, 9$. When the worldvolume is split into $\mathrm{AdS}_2 \times \mathcal{M}_5$, $\alpha, \beta = 0, 1$ are used for the AdS_2 part, while Greek indices from the middle of the alphabet, $\mu, \nu = 5, \ldots, 9$, are reserved for the factor $\mathcal{M}_5 \subset \mathbb{C}P^3$. Latin indices $i, j = 2, 3, 4$ denote the normal directions. Flat indices are underlined.

8.3.1 Bosonic fluctuations

For the bosonic fluctuations, we start with the action (8.2.5). We follow the procedure described in detail in [108], which relies on the geometry of embedded manifolds and renders all expressions manifestly covariant. We refer the reader to Sec. 3 and Appendix B of that paper for the relevant formulae. Following this strategy, the fluctuations of the D6-brane worldvolume are parameterized by three scalars $\chi^{\underline{i}}$ corresponding to the three normal directions. They consist of a doublet ($i = 2, 3$) characterizing the normals of $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$ and a singlet ($i = 4$) for the normal within $\mathbb{C}P^3$. The worldvolume displacement is described by a geodesic map,

$$x^m \to (\exp_x y)^m \ , \qquad y^m = N_{\underline{i}}^m \chi^{\underline{i}} \ . \tag{8.3.28}$$

In addition, there are the fluctuations of the 2-form gauge field,

$$\mathcal{F}_{ab} \to \mathcal{F}_{ab} + f_{ab} \ , \quad f = da \ . \tag{8.3.29}$$

Defining $M_{ab} = g_{ab} + \mathcal{F}_{ab}$, we have to second order [cf. (3.10) of [108]]

$$\delta M_{ab} = -2H_{\underline{i}ab}\chi^{\underline{i}} + f_{ab} + \nabla_a \chi^{\underline{i}} \nabla_b \chi^{\underline{j}} \delta_{\underline{ij}} + \left(H_{\underline{i}a}{}^c H_{\underline{j}bc} - R_{mpnq} x_a^m x_b^n N_{\underline{i}}^p N_{\underline{j}}^q \right) \chi^{\underline{i}} \chi^{\underline{j}} \ . \tag{8.3.30}$$

Here, $H_{ab}^{\underline{i}}$ is the extrinsic curvature (second fundamental form) of the embedding. The expansion up to second order of the Born-Infeld (BI) term may be obtained from the general formula

$$\sqrt{-\det M} \to \sqrt{-\det M} \left[1 + \frac{1}{2}\mathrm{tr}X + \frac{1}{8}(\mathrm{tr}X)^2 - \frac{1}{4}\mathrm{tr}X^2 \right] \ , \tag{8.3.31}$$

where $X = M^{-1}\delta M$. This yields

$$\begin{aligned} \sqrt{-\det M_{ab}} \to \sqrt{-\det g_{ab}} \sin\alpha \Bigg\{ & 1 + 3\cot\alpha\, \chi^{\underline{4}} - \frac{\cos\alpha}{\sin^2\alpha} \left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta} \right) \\ & + \frac{1}{2\sin^2\alpha} \nabla^\alpha \chi^{\underline{i}} \nabla_\alpha \chi_{\underline{i}} + \frac{1}{2} \nabla^\mu \chi^{\underline{i}} \nabla_\mu \chi_{\underline{i}} \\ & + \frac{1}{\sin^2\alpha} \left[(\chi^{\underline{2}})^2 + (\chi^{\underline{3}})^2 \right] + \left(\frac{3}{\sin^2\alpha} - \frac{9}{2} \right) (\chi^{\underline{4}})^2 \\ & + \frac{1}{4\sin^4\alpha} f_{\alpha\beta} f^{\alpha\beta} + \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2\sin^2\alpha} f^{\alpha\mu} f_{\alpha\mu} - \frac{3\cos^2\alpha}{\sin^3\alpha} \chi^{\underline{4}} \left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta} \right) \Bigg\} \ . \end{aligned} \tag{8.3.32}$$

Here we have used

$$H_{\underline{i}\alpha\beta} = 0 \ , \qquad H_{\underline{4}}{}^\mu{}_\mu = -3\cot\alpha \ , \qquad H_{\underline{4}}{}^{\mu\nu} H_{\underline{4}\mu\nu} = 3\cot^2\alpha + 1 \ , \tag{8.3.33}$$

and the fact that $\mathbb{C}P^3$ is Einstein, $R_{mn}^{4\mathbb{C}P^3} = \frac{2\cdot 3+2}{4} g_{mn}^{4\mathbb{C}P^3} = 2 g_{mn}^{4\mathbb{C}P^3}$.

The Wess-Zumino (WZ) terms are obtained taking into account the expansion of the form fields and the tangent vectors for the pull-back, cf. (3.3) and (3.4) of [108]. The C_7 WZ term gives

$$P[C_7] \to d^7\xi\sqrt{-\det g_{ab}}\frac{1}{2}e^{\mu}_{\underline{9}}\left(\chi^{\underline{2}}\nabla_\mu\chi^{\underline{3}} - \chi^{\underline{3}}\nabla_\mu\chi^{\underline{2}}\right) \ , \tag{8.3.34}$$

where the indices $\underline{2}$ and $\underline{3}$ denote the normals in the u– and ϕ–directions, respectively. This contribution is somewhat unexpected, because both C_7 and its first u–derivative vanish for $u = 0$. However, one must carefully consider the small–u behaviour, because the normal component $N^{\phi}_{\underline{3}}$ goes like $1/u$. This leads to the finite result (8.3.34), which is absent in previous discussions of similar classical configurations.

The C_5 WZ term leads to

$$\begin{aligned}\mathcal{F}\wedge P[C_5] \to d^7\xi\sqrt{-\det g_{ab}}\Bigg\{ & -\cos\alpha\, C(\alpha) + 3\cos\alpha\,\chi^{\underline{4}} - C(\alpha)\left(\frac{1}{2}\epsilon^{\alpha\beta}f_{\alpha\beta}\right) \\ & + 3\chi^{\underline{4}}\left(\frac{1}{2}\epsilon^{\alpha\beta}f_{\alpha\beta}\right) + \frac{9\cos^2\alpha}{2\sin\alpha}(\chi^{\underline{4}})^2\Bigg\} \ ,\end{aligned} \tag{8.3.35}$$

where

$$C(\alpha) = \sin^{-3}\alpha\left(\sin^2\alpha\cos\alpha + 2\cos\alpha - 2\right) \ . \tag{8.3.36}$$

The C_3 WZ term vanishes, and the C_1 WZ term gives a contribution, which is found easily after an integration by parts

$$\frac{1}{6}\mathcal{F}^3\wedge P[C_1] = \frac{1}{2}\mathcal{F}\wedge f\wedge f\wedge P[C_1] \to -\frac{1}{2}\cos\alpha\,\epsilon_{\mathrm{AdS}_2}\wedge a\wedge f\wedge P[F_2] \ , \tag{8.3.37}$$

where $f = da$. This form has the advantage of being independent of any exact terms in C_1. Using (8.2.9), one finds

$$\frac{1}{6}\mathcal{F}^3\wedge P[C_1] \to d^7\xi\sqrt{-\det g_{ab}}\,\frac{1}{2}\cos\alpha\,\mathcal{E}^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \ , \tag{8.3.38}$$

where $\mathcal{E}^{\mu\nu\rho}$ is the totally antisymmetric tensor known as the Betti 3-form [164, 165],

$$\frac{1}{6}\mathcal{E}_{\mu\nu\rho}\,d\xi^\mu\wedge d\xi^\nu\wedge d\xi^\rho = e^{\underline{4}}\wedge\left(e^{\underline{5}}\wedge e^{\underline{6}} + e^{\underline{7}}\wedge e^{\underline{8}}\right) \ . \tag{8.3.39}$$

Finally, we sum the contributions (8.3.32), (8.3.34), (8.3.35) and (8.3.38), drop total derivatives and express the resulting quadratic action in terms of the open string metric, which rescales the AdS_2 part to the radius $\sin\alpha$,

$$d\tilde{s}^2 = \sin^2\alpha\, g_{\alpha\beta} d\xi^\alpha d\xi^\beta + g_{\mu\nu} d\xi^\mu d\xi^\nu \ . \tag{8.3.40}$$

This yields

$$\begin{aligned} S^{B,2}_{\mathrm{D6}} = -\frac{T_6}{\sin\alpha}\int d^7\xi\,\sqrt{-\det\tilde{g}_{ab}}\Bigg\{ &\frac{1}{2}\tilde{\nabla}^a\chi^{\underline{i}}\tilde{\nabla}_a\chi^{\underline{i}} + \frac{1}{\sin^2\alpha}\left[(\chi^{\underline{2}})^2 + (\chi^{\underline{3}})^2\right] - \frac{3}{2\sin^2\alpha}(\chi^{\underline{4}})^2 \\ &+ \frac{1}{\sin\alpha} e^\nu_{\underline{9}}\chi^{\underline{3}}\nabla_\nu\chi^{\underline{2}} + \frac{1}{4}\tilde{f}^{ab}\tilde{f}_{ab} - \frac{3}{\sin\alpha}\chi^{\underline{4}}\left(\frac{1}{2}\tilde{\epsilon}^{\alpha\beta} f_{\alpha\beta}\right) - \frac{1}{2}\cot\alpha\,\mathcal{E}^{\mu\nu\rho} a_\mu\partial_\nu a_\rho \Bigg\} \ , \end{aligned} \tag{8.3.41}$$

which is our final result for the bosonic action of the 1/6 BPS D6-brane. Note that our result completes a preliminary discussion of the quadratic excitations presented in [160].

8.3.2 Fermionic fluctuations

For the fermionic fluctuations, our starting point is Eq. (17) of [159],

$$S^F_{\mathrm{D6}} = \frac{T_6}{2}\int d^7\xi \epsilon{-}\Phi\sqrt{-\det M_{ab}}\,\bar{\theta}\,(1-\Gamma_{\mathrm{D6}})\left[(\tilde{M}^{-1})^{ab}\Gamma_b D_a - \Delta\right]\theta \ , \tag{8.3.42}$$

where θ is a 32-component, $10d$ Majorana spinor, $\bar{\theta} = i\theta^\dagger\Gamma^{\underline{0}}$, $\tilde{M}_{ab} = g_{ab} + \Gamma_{(10)}\mathcal{F}_{ab}$, $D_a = \partial_a X^m D_m$, D_m and Δ were defined in (8.2.15) and (8.2.16), respectively, and Γ_{D6} is

$$\begin{aligned} \Gamma_{\mathrm{D6}} &= \frac{\sqrt{-\det g_{ab}}}{\sqrt{-\det(g_{ab}+\mathcal{F}_{ab})}}\left(-\Gamma^{\underline{0156789}}\right)\sum_q \frac{(-\Gamma_{(10)})^q}{q!2^q}\Gamma^{b_1\ldots b_{2q}}\mathcal{F}_{b_1b_2}\ldots\mathcal{F}_{b_{2q-1}b_{2q}} \\ &= \frac{1}{\sin\alpha}\left(-\Gamma^{\underline{0156789}}\right)\left(1+\cos\alpha\,\Gamma^{\underline{01}}\Gamma_{(10)}\right) \ . \end{aligned} \tag{8.3.43}$$

The pullback of the covariant derivative on spinors is given by [108]

$$\partial_a X^m\nabla_m = \nabla_a - \frac{1}{2}H_{\underline{i}ab}\Gamma^b\Gamma^{\underline{i}} + \frac{1}{4}A_{\underline{ij}a}\Gamma^{\underline{ij}} \ . \tag{8.3.44}$$

The combinations we need are

$$\partial_\alpha X^m \nabla_m = \nabla_\alpha \ , \quad \Gamma^\mu \partial_\mu X^m \nabla_m = \Gamma^\mu \nabla_\mu + \frac{3}{2} \cot\alpha \, \Gamma^{\underline{4}} \ . \tag{8.3.45}$$

Direct evaluation of the operator in squared brackets in (8.3.42) yields

$$\begin{aligned}(\tilde{M}^{-1})^{ab}\Gamma_b D_a - \Delta = & \frac{1}{\sin^2\alpha}\left(1 - \cos\alpha\,\Gamma^{\underline{01}}\Gamma_{(10)}\right)\Gamma^\alpha\nabla_\alpha + \Gamma^\mu\nabla_\mu + \frac{3}{2}\cot\alpha\,\Gamma^{\underline{4}} \\ & + \frac{1}{4\sin^2\alpha}\left(1 - \cos\alpha\,\Gamma^{\underline{01}}\Gamma_{(10)}\right)\left[-\left(\Gamma^{\underline{49}} + \Gamma^{\underline{56}} + \Gamma^{\underline{78}}\right)\Gamma_{(10)} + 3\Gamma^{\underline{0123}}\right] \\ & + \frac{1}{4}\left(\Gamma^{\underline{56}} + \Gamma^{\underline{78}}\right)\Gamma_{(10)} - \frac{3}{2}\Gamma^{\underline{0123}} \ .\end{aligned} \tag{8.3.46}$$

To proceed, we fix the κ-symmetry by imposing θ to be chiral. What matters here is that only terms in (8.3.42) with an odd number of Γ-matrices survive the chiral projection. In fact, the chirality is irrelevant. Hence, we find

$$\begin{aligned}&\bar{\theta}\,(1 - \Gamma_{\mathrm{D6}})\left[(\tilde{M}^{-1})^{ab}\Gamma_b D_a - \Delta\right]\theta = \\ &\quad \bar{\theta}\epsilon R\Gamma^{\underline{01}}\Gamma_{(10)}\left[\tilde{\Gamma}^a\tilde{\nabla}_a - \frac{1}{4}\cot\alpha\left(\Gamma^{\underline{569}} + \Gamma^{\underline{789}}\right) + \frac{1}{4\sin\alpha}\Gamma^{\underline{239}}\left(1 - 3\Gamma^{\underline{5678}}\right)\right]\epsilon R\Gamma^{\underline{01}}\Gamma_{(10)}\theta \ ,\end{aligned} \tag{8.3.47}$$

where the spinor rotation parameter R is determined by $\sinh 2R = -\cot\alpha$. In what follows, we simply work with the rotated spinor, $\epsilon R\Gamma^{\underline{01}}\Gamma_{(10)}\theta \to \theta$. The Dirac operator in (8.3.47) is the one corresponding to the open string metric (8.3.40).

To proceed, it is necessary to decompose the 32×32 Γ-matrices into a $7d$ representation. We shall use

$$\begin{aligned}\Gamma^a &= \gamma^a \otimes \mathbb{I}_2 \otimes \sigma_1 \ , && (a = 0,1,5,6,7,8,9) \\ \Gamma^{\underline{i}} &= \mathbb{I}_8 \otimes \tau_{i-1} \otimes \sigma_2 \ , && (i = 2,3,4) \ ,\end{aligned} \tag{8.3.48}$$

where γ^a, τ_i and σ_i denote $7d$ Minkowski Gamma matrices and two copies of Pauli matrices, respectively. The representation (8.3.48) is chiral with,

$$\Gamma_{(10)} = -\gamma^{\underline{0156789}} \otimes \mathbb{I}_2 \otimes \sigma_3 = \pm \mathbb{I}_8 \otimes \mathbb{I}_2 \otimes \sigma_3 \ , \tag{8.3.49}$$

where the sign depends on the representation of the $7d$ gamma matrices. Hence, a $10d$ chiral spinor (16 components) decomposes into a doublet of $7d$ spinors, and the matrices τ_i act on the doublet.

The Majorana condition on θ translates into a symplectic Majorana condition on the $7d$ spinor doublet. To see this, decompose the the Majorana intertwiner [166] into

$$B_{+(9,1)} = B_{+(6,1)} \otimes B_{-(3,0)} \otimes \mathbb{I}_2 \ . \tag{8.3.50}$$

Finally, after applying the decomposition (8.3.48) to (8.3.47) and substituting the result into (8.3.42), we obtain the fermionic action

$$S^F_{\mathrm{D6}} = \frac{T_6}{2\sin\alpha} \int d^7\xi \sqrt{-\det \tilde{g}_{ab}}\, \bar{\theta}_\pm \left[\tilde{\gamma}^a \tilde{\nabla}_a - \frac{1}{4} \cot\alpha \left(\gamma^{\underline{569}} + \gamma^{\underline{789}} \right) \pm \frac{i}{4\sin\alpha} \gamma^{\underline{9}} \left(1 - 3\gamma^{\underline{5678}} \right) \right] \theta_\pm \ . \tag{8.3.51}$$

There is an implicit sum over the spinor doublet index ($\pm$), and the sign of the last term in the brackets agrees with the doublet index.

We conclude this section by writing Eq. (8.3.51) in a $2+5$ form, which is useful for the calculation of the spectrum. We shall use the decomposition

$$\gamma^\alpha = \gamma^\alpha \otimes \mathbb{I}_4 \ , \qquad \gamma^\mu = \gamma^{\underline{01}} \otimes \gamma^\mu \ , \tag{8.3.52}$$

where the matrices γ^α and γ^μ on the right hand sides are intended as $2d$ and $5d$ gamma matrices, respectively. Hence, we can rewrite (8.3.51) as

$$S^F_{\mathrm{D6}} = \frac{T_6}{2\sin\alpha} \int d^7\xi \sqrt{-\det \tilde{g}_{ab}}\, \bar{\theta}_\pm \left(\tilde{\gamma}^\alpha \tilde{\nabla}_\alpha \otimes \mathbb{I}_4 + \gamma^{\underline{01}} \otimes \mathcal{D}_\pm \right) \theta_\pm \ , \tag{8.3.53}$$

where the differential operators $\mathcal{D}_\pm$ acting on the $\tilde{T}^{1,1}$ part are

$$\mathcal{D}_\pm = \tilde{\gamma}^\mu \tilde{\nabla}_\mu - \frac{1}{4}\cot\alpha\left(\gamma^{\underline{569}} + \gamma^{\underline{789}}\right) \pm \frac{i}{4\sin\alpha}\gamma^{\underline{9}}\left(1 - 3\gamma^{\underline{5678}}\right) \ . \tag{8.3.54}$$

8.3.3 Field equations

For completeness, we list here the field equations deriving from the actions (8.3.41) and (8.3.53). The doublet of scalars $\chi^{\underline{i}}$, $i = 2, 3$, satisfy

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha}\right)\chi^{\underline{2}} - \frac{1}{\sin\alpha}e^\mu_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{3}} = 0 \ , \tag{8.3.55}$$

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha}\right)\chi^{\underline{3}} + \frac{1}{\sin\alpha}e^\mu_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{2}} = 0 \ . \tag{8.3.56}$$

Introducing $\chi^\pm = \chi^{\underline{2}} \pm i\chi^{\underline{3}}$, (8.3.55) and (8.3.56) become

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha} \pm \frac{i}{\sin\alpha}e^\mu_{\underline{9}}\tilde{\nabla}_\mu\right)\chi^\pm = 0 \ . \tag{8.3.57}$$

It is worth noting that this is a generalization of what would traditionally be a couple of massive fields describing the embedding of $AdS_2 \subset AdS_4$. Namely, in the absence of the last term above, one has two scalar fields with $m^2 = 2$ just as in the case [153]. Similarly for the embedding of supersymmetric branes in $AdS_5 \times S^5$, one gets three $m^2 = 2$ modes from $AdS_2 \subset AdS_5$ for the D3 and D5 respectively [107, 108]. It is easy to track this term to the C_7 contribution from the WZ part of the action (see Eq. 8.3.34); we will see that there is a corresponding C_3 contribution to the D2 fluctuations, thus leading to a sort of universality.

The scalar $\chi^{\underline{4}}$ couples to the AdS$_2$-components a_α of the vector field. Their field equations are given by

$$\left(\tilde{\nabla}_a\tilde{\nabla}^a + \frac{3}{\sin^2\alpha}\right)\chi^{\underline{4}} + \frac{3}{\sin\alpha}f = 0 \ , \tag{8.3.58}$$

$$\tilde{\nabla}_a(\tilde{\nabla}^a a^\alpha - \tilde{\nabla}^\alpha a^a) + \frac{3}{\sin\alpha}\tilde{\epsilon}^{\alpha\beta}\partial_\beta\chi^{\underline{4}} = 0 \ , \tag{8.3.59}$$

where f stands for $f = \frac{1}{2}\tilde{\epsilon}^{\alpha\beta} f_{\alpha\beta}$. We adopt the Lorentz gauge, $\tilde{\nabla}_a a^a = 0$. The remaining gauge freedom can be used to further impose $\tilde{\nabla}_\alpha a^\alpha = \tilde{\nabla}_\mu a^\mu = 0$ on-shell. Acting with $\tilde{\nabla}^\gamma \tilde{\epsilon}_{\gamma\alpha}$ on (8.3.59), one obtains

$$\tilde{\nabla}_a \tilde{\nabla}^a f + \frac{3}{\sin\alpha} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \chi^{\underline{4}} = 0 \ . \tag{8.3.60}$$

Hence, we can write (8.3.58) and (8.3.60) in the matrix form

$$\begin{pmatrix} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + \tilde{\nabla}_\mu \tilde{\nabla}^\mu + \frac{3}{\sin^2\alpha} & \frac{3}{\sin\alpha} \\ \frac{3}{\sin\alpha} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha & \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + \tilde{\nabla}_\mu \tilde{\nabla}^\mu \end{pmatrix} \begin{pmatrix} \chi^{\underline{4}} \\ f \end{pmatrix} = 0 \ . \tag{8.3.61}$$

The vector components a^μ satisfy, in Lorentz gauge,

$$-\left(\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + \tilde{\nabla}_\nu \tilde{\nabla}^\nu \right) a^\mu + R^\mu{}_\nu a^\nu - \cot\alpha\, \mathcal{E}^{\mu\nu\rho} \partial_\nu a_\rho = 0 \ . \tag{8.3.62}$$

The field equations for the spinors are simply

$$\left(\tilde{\gamma}^\alpha \tilde{\nabla}_\alpha \otimes \mathbb{I}_4 + \gamma^{\underline{01}} \otimes \mathcal{D}_\pm \right) \theta_\pm \ = 0, \tag{8.3.63}$$

where $\mathcal{D}_\pm$ is defined by (8.3.54).

8.3.4 Spectrum of D6-brane fluctuations

In this section, we calculate the spectrum of fluctuations of the D6-brane and obtain the conformal dimensions of the dual operators. The bosonic fluctuations were considered in [160], but the result is partially incorrect because of missing terms in the quadratic action. To obtain the spectrum, the equations of motion listed in subsection 8.3.3 must be solved. This requires to construct the (generalized) harmonics on the $\tilde{T}^{1,1}$ factor of the D6 world volume, which we defer to appendix J due to its rather technical nature.

We start with the doublet of scalars, $\chi^{\underline{i}}$, $(i = 2, 3)$. The field equation for the combinations $\chi^\pm = \chi^{\underline{2}} \pm i\chi^{\underline{3}}$ is given by (8.3.57). Substituting (J.31) and (J.29), it becomes a field equation on AdS_2,

$$\left(\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha - m_\pm^2\right) \chi^{\underline{\pm}} = 0 \ , \tag{8.3.64}$$

where

$$m_\pm^2 = \frac{C_{j,l} + 1 \pm r}{\sin^2 \alpha} \ . \tag{8.3.65}$$

Because the radius of AdS_2 in the open string metric is $\sin\alpha$, the standard relation between m^2 and the conformal dimension of the dual operator yields

$$\Delta^{(\pm)} = \frac{1}{2} + \sqrt{\frac{5}{4} + C_{j,l} \pm r} \ . \tag{8.3.66}$$

We recall the definition (J.29) of $C_{j,l}$,

$$C_{j,l} = \sin^2 \frac{\alpha}{2}(2j+1)^2 + \cos^2 \frac{\alpha}{2}(2l+1)^2 \ . \tag{8.3.67}$$

As explained in appendix J.0.3, j, l are either both integer or half-integer, and $|r| \leq \bar{l}$, where

$$\bar{l} = 2\min(j,l) \ . \tag{8.3.68}$$

The field equations of the scalar $\chi^{\underline{4}}$ and the AdS_2-components of the vector field are given by (8.3.61). Substituting the eigenvalues of the scalar Laplacian on $\tilde{T}^{1,1}$ (J.27), one obtains

$$\begin{pmatrix} -\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + \frac{C_{j,l}-4}{\sin^2\alpha} & -\frac{3}{\sin\alpha} \\ -\frac{3}{\sin\alpha}\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha & -\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + \frac{C_{j,l}-1}{\sin^2\alpha} \end{pmatrix} \begin{pmatrix} \chi^{\underline{4}} \\ f \end{pmatrix} = 0 \ . \tag{8.3.69}$$

The characteristic polynomial of this matrix is equivalent to the product of two massive Klein-Gordon equations on AdS_2, with two mass values. To these correspond the following conformal dimensions of the two dual operators,

$$\Delta^{(\underline{4})} \in \left\{ \sqrt{C_{j,l}} + 2; \sqrt{C_{j,l}} - 1 \right\} \ . \tag{8.3.70}$$

The eigenvalues are $(\bar{l}+1)$-fold degenerate, because they are independent of r. From the second value one must exclude the case $j = l = 0$ ($C_{j,l} = 1$), because the corresponding bulk mode is not dynamical [160, 108]. (It is the gauge mode that allows to impose $\tilde{\nabla}_\alpha a^\alpha = \tilde{\nabla}_\mu a^\mu = 0$, which is more restrictive than the Lorentz gauge $\tilde{\nabla}_a a^a = 0$.)

Consider the $\tilde{T}^{1,1}$ components of the vector field. Their field equations are given by (8.3.62), which becomes a massive Klein-Gordon equation on AdS_2 of the form (8.3.64) (the $\tilde{T}^{1,1}$ vector is an AdS_2 scalar) once the results of the harmonic analysis on $\tilde{T}^{1,1}$ have been used. The mass-square is simply given by the eigenvalues of the modified vector Laplacian, which are listed in appendix J.0.4. The conformal dimension of the dual operator then follows from the standard formula. We list the results in Tables 8.1 and 8.2 for the generic case $j \neq l$ and the special case $j = l$, respectively.

The conformal dimensions of the operators dual to the spinor fields are found from the spinor field equation (8.3.63). After using the results of the harmonic analysis, one may consider

$$\left(\tilde{\gamma}^\alpha \tilde{\nabla}_\alpha + \lambda \gamma^{\underline{01}}\right) \vartheta \otimes \theta_\lambda \ , \tag{8.3.71}$$

where $\lambda = ich$ represents the eigenvalue of $\mathcal{D}_\pm$ corresponding to the eigenvector θ_λ, which is a $\tilde{T}^{1,1}$ spinor, while ϑ is a spinor on AdS_2. Denoting by ϑ_μ ($\mu \geq 0$) a solution of the AdS_2 Dirac equation

$$\left(\tilde{\gamma}^\alpha \tilde{\nabla}_\alpha - \mu\right) \vartheta_\mu = 0 \ , \tag{8.3.72}$$

and using $\gamma^{\underline{01}} \vartheta_\mu = \vartheta_{-\mu}$, one finds that (8.3.71) is solved by $\vartheta = \vartheta_\mu + i\vartheta_{-\mu}$, with $\mu = ch$. It follows from the standard formula that the conformal dimension of the dual fermionic operators are simply $\Delta_f = \frac{1}{2} + h$. The values of h that can be found in the tables in Appendix J.0.4. Again, we list the results in Tables 8.1 and 8.2 for the generic case $j \neq l$ and the special case $j = l$, respectively.

8.4 D2-brane fluctuations

In this section we consider the bosonic and fermionic fluctuations of the classical 1/3-BPS D2-brane discussed in Sec. 8.2. The procedure that leads to the quadratic action is the same as the one used in Sec. 8.3 for the D6-brane. The notation remains essentially the

Table 8.1: Conformal dimensions and supermultiplet structure in the generic case $j \neq l$.

$2(\bar{l}+1)$ fermion supermultiplets ($n = -\bar{l}, -\bar{l}+2, \ldots, \bar{l}$)			
boson/fermion#	f	$b * 2$	f
Δ_n	$\Delta_{n0} = \sqrt{C_{j,l} + \frac{5}{4} + n}$	$\Delta_{n0} + \frac{1}{2}$	$\Delta_{n0} + 1$
$2(\bar{l}+1)$ boson supermultiplets			
boson/fermion#	b	$f * 2$	b
Δ	$\Delta_1 = \sqrt{C_{j,l}} + 1$	$\Delta_1 + \frac{1}{2}$	$\Delta_1 + 1$
Δ	$\Delta_2 = \sqrt{C_{j,l}} - 1$	$\Delta_2 + \frac{1}{2}$	$\Delta_2 + 1$

Table 8.2: Conformal dimensions and supermultiplet structure in the special case $j = l$.

$4j$ fermion supermultiplets ($n = -2j, -2j+2, \ldots, 2j-2$)			
boson/fermion#	f	$b * 2$	f
Δ_n	$\Delta_{n0} = \sqrt{(2j+1)^2 + \frac{5}{4} + n}$	$\Delta_{n0} + \frac{1}{2}$	$\Delta_{n0} + 1$
2 fermion supermultiplets			
boson/fermion#	f	b	—
Δ	$2j + \frac{3}{2}$	$2j + 2$	
boson supermultiplets			
boson/fermion#	$b * (2j+1)$	$f * (4j+2)$	$b * (2j+1)$
Δ	$2j + 2$	$2j + \frac{5}{2}$	$2j + 3$
boson/fermion#	$b * (2j+1)$	$f * (4j)$	$b * (2j-1)$
Δ	$2j$	$2j + \frac{1}{2}$	$2j + 1$

same, with the following logical differences due to dimensionality. Generic D2-brane indices are denoted by $a, b = 0, 1, 9$. When the worldvolume is split into $\mathrm{AdS}_2 \times S^1$, $\alpha, \beta = 0, 1$ are used for the AdS_2 part, while $\mu = 9$ refers to the S^1 part. Latin indices $i, j = 2, 3, 4, 5, 6, 7, 8$ denote the normal directions.

8.4.1 Bosonic fluctuations

The starting point is, again, the action (8.2.5). For the D2-brane, there are three terms, the BI term and two CS terms (C_3 and $\mathcal{F} \wedge C_1$). Expanding the BI term to quadratic order, one obtains

$$\sqrt{-\det M_{ab}} \to \sqrt{-\det g_{ab}} \sin\alpha \Bigg\{ 1 + \cot\alpha\, \chi^{\underline{4}} - \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta}\right) \tag{8.4.73}$$
$$+ \frac{1}{2\sin^2\alpha}\nabla^\alpha\chi^{\underline{i}}\nabla_\alpha\chi_{\underline{i}} + \frac{1}{2}\nabla^\mu\chi^{\underline{i}}\nabla_\mu\chi_{\underline{i}}$$
$$+ \frac{1}{\sin^2\alpha}\left[(\chi^{\underline{2}})^2 + (\chi^{\underline{3}})^2\right] - \frac{1}{2}(\chi^{\underline{4}})^2 - \frac{1}{8}\left[(\chi^{\underline{5}})^2 + (\chi^{\underline{6}})^2 + (\chi^{\underline{7}})^2 + (\chi^{\underline{8}})^2\right]$$
$$+ \frac{1}{4\sin^4\alpha} f_{\alpha\beta} f^{\alpha\beta} + \frac{1}{2\sin^2\alpha} f^{\alpha\mu} f_{\alpha\mu} - \frac{\cos^2\alpha}{\sin^3\alpha}\chi^{\underline{4}}\left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta}\right)\Bigg\} \, .$$

Note that the covariant derivative contains the normal bundle connection,

$$\nabla_a \chi^{\underline{i}} = \partial_a \chi^{\underline{i}} + A_a{}^{\underline{i}}{}_{\underline{j}} \chi^{\underline{j}} \, , \tag{8.4.74}$$

which, in contrast to the D6-brane case, has non-zero components

$$A_{\mu\underline{56}} = \frac{1}{2}\sin^2\frac{\alpha}{2} \, , \qquad A_{\mu\underline{78}} = -\frac{1}{2}\cos^2\frac{\alpha}{2} \, . \tag{8.4.75}$$

The only non-zero component of the second fundamental form is

$$H_{\underline{4}}{}^\mu{}_\mu = -\cot\alpha \, . \tag{8.4.76}$$

The WZ term with C_3 is similar to the C_7 term in the D6-brane case, and leads to the following contribution

$$P[C_3] \to d^3\xi \sqrt{-\det g_{ab}} \frac{3}{2} e^\mu_{\underline{9}} \left(\chi^{\underline{2}}\nabla_\mu\chi^{\underline{3}} - \chi^{\underline{3}}\nabla_\mu\chi^{\underline{2}}\right) \, . \tag{8.4.77}$$

The C_1 WZ term is similar to the C_5 term in the D6-brane case, but contains some additional terms,

$$\mathcal{F} \wedge P[C_1] \to d^3\xi \sqrt{-\det g_{ab}} \Bigg\{ \cot\alpha(1-\cos\alpha) + \cos\alpha\,\chi^{\underline{4}} + \frac{1-\cos\alpha}{\sin\alpha}\left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta}\right) \tag{8.4.78}$$
$$+ \chi^{\underline{4}}\left(\frac{1}{2}\epsilon^{\alpha\beta} f_{\alpha\beta}\right) + \frac{\cos^2\alpha}{2\sin\alpha}(\chi^{\underline{4}})^2$$
$$+ \frac{1}{2}\cos\alpha\, e_{\underline{9}}^{\mu}\left(\chi^{\underline{5}}\nabla_\mu\chi^{\underline{6}} - \chi^{\underline{6}}\nabla_\mu\chi^{\underline{5}} + \chi^{\underline{7}}\nabla_\mu\chi^{\underline{8}} - \chi^{\underline{8}}\nabla_\mu\chi^{\underline{7}}\right) \Bigg\}\ .$$

Finally, we sum the three contributions (8.4.73), (8.4.77) and (8.4.78), drop total derivatives and express the resulting quadratic action in terms of the open string metric, which again rescales the AdS_2 part to have radius $\sin\alpha$,

$$d\tilde{s}^2 = \sin^2\alpha\, g_{\alpha\beta} d\xi^\alpha d\xi^\beta + g_{\mu\nu} d\xi^\mu d\xi^\nu\ . \tag{8.4.79}$$

The final action is:

$$S_{\mathrm{D2}}^{B,2} = -\frac{T_2}{\sin\alpha}\int d^3\xi\,\sqrt{-\det\tilde{g}_{ab}}\Bigg\{\frac{1}{2}\tilde{\nabla}^a\chi^{\underline{i}}\tilde{\nabla}_a\chi^{\underline{i}} + \frac{1}{\sin^2\alpha}\left[(\chi^{\underline{2}})^2 + (\chi^{\underline{3}})^2\right] + \frac{3}{\sin\alpha} e_{\underline{9}}^{\mu}\chi^{\underline{3}}\nabla_\mu\chi^{\underline{2}} \tag{8.4.80}$$
$$-\frac{1}{8}\left[(\chi^{\underline{5}})^2 + (\chi^{\underline{6}})^2 + (\chi^{\underline{7}})^2 + (\chi^{\underline{8}})^2\right] + \cot\alpha\, e_{\underline{9}}^{\mu}\left(\chi^{\underline{6}}\nabla_\mu\chi^{\underline{5}} + \chi^{\underline{8}}\nabla_\mu\chi^{\underline{7}}\right)$$
$$-\frac{1}{2\sin^2\alpha}(\chi^{\underline{4}})^2 + \frac{1}{4}\tilde{f}^{ab}\tilde{f}_{ab} - \frac{1}{\sin\alpha}\chi^{\underline{4}}\left(\frac{1}{2}\tilde{\epsilon}^{\alpha\beta} f_{\alpha\beta}\right)\Bigg\}\ .$$

Note that, as in the D6 case, there are a number of terms describing a modification of the naive embedding of $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$. The fluctuations $\chi^{\underline{2}}$ and $\chi^{\underline{3}}$ contain an extra mixing term that arises from the C_3 contribution to the WZ action, see Eq. 8.4.77. In addition, there are mixing terms for the pairs of scalars $(\chi^{\underline{5}}, \chi^{\underline{6}})$ and $(\chi^{\underline{7}}, \chi^{\underline{8}})$, and these pairs of scalars are affected by the non-zero connections in the normal bundle.

8.4.2 Fermionic fluctuations

The construction of the fermionic action for the D2-brane is similar to the D6-brane case. We start with Eq. (17) of [159],

$$S_{\rm D2}^{(F)} = \frac{T_2}{2} \int d^3\xi \epsilon{-}\Phi \sqrt{-\det M_{ab}}\, \bar{\theta} \left(1 - \Gamma_{\rm D2}\right) \left[(\tilde{M}^{-1})^{ab} \Gamma_b D_a - \Delta \right] \theta, \tag{8.4.81}$$

where Γ_a is the pullback of the gamma matrices Γ_m, the fermionic field θ is a $10d$ Majorana spinor, and $\Gamma_{\rm D2}$ is given by

$$\Gamma_{\rm D2} = \frac{1}{\sin\alpha} \left(-\Gamma^{\underline{019}}\right) \left(1 + \cos\alpha \Gamma_{(10)} \Gamma^{\underline{01}}\right) \ . \tag{8.4.82}$$

The pullback of the covariant derivative is again given by (8.3.44). Explicitly, using (8.4.76) and (8.4.75), we have

$$\partial_\alpha X^m \nabla_m = \nabla_\alpha \ , \qquad \Gamma^\mu \partial_\mu X^m \nabla_m = \Gamma^\mu \nabla_\mu + \frac{1}{2} \cot\alpha \Gamma^{\underline{4}} + \frac{1}{4} \Gamma^\mu A_{\underline{ij}\mu} \Gamma^{\underline{ij}} \ , \tag{8.4.83}$$

where

$$A_{\underline{ij}\mu} \Gamma^{\underline{ij}} = \sin^2 \frac{\alpha}{2} \Gamma^{\underline{56}} - \cos^2 \frac{\alpha}{2} \Gamma^{\underline{78}} \ . \tag{8.4.84}$$

The κ-symmetry is fixed by taking θ to be chiral, which implies that only terms with an odd number of Γ-matrices survive in the action. The result for the fermionic action after a straightforward calculation, expressed in terms of the open string metric (8.4.79), is

$$\begin{aligned} S_{\rm D2}^{(F)} = & \frac{T_2}{2\sin\alpha} \int d^3\xi \sqrt{-\det \tilde{g}_{ab}}\, \bar{\theta} \epsilon R \Gamma^{\underline{01}} \Gamma_{(10)} \Big\{ \tilde{\Gamma}^a \tilde{\nabla}_a \\ & + \frac{1}{4\sin\alpha} \left[\Gamma^{\underline{569}} - \Gamma^{\underline{789}} + \Gamma^{\underline{239}} \left(3 - \Gamma^{\underline{5678}}\right) \right] \Big\} \epsilon R \Gamma^{\underline{01}} \Gamma_{(10)} \theta \ , \end{aligned} \tag{8.4.85}$$

where the spinor rotation parameter R is given by $\sinh 2R = -\cot\alpha$. In what follows, we shall simply work with the rotated spinor, $\epsilon R \Gamma^{\underline{01}} \Gamma_{(10)} \theta \to \theta$.

Given the symmetries of our problem, it is convenient to decompose the $10d$ Lorentz group as

$$SO(9,1) \subset SO(2,1) \times SO(2) \times SO(5), \tag{8.4.86}$$

corresponding to the (0,1,9), (2,3) and (4,5,6,7,8) directions, respectively. A representation of the $10d$ gamma matrices compatible with the above decomposition is

$$\begin{aligned}
\Gamma^{a} &= \gamma^{a} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \sigma_1 \ , && (a = 0,1,9) \ , \\
\Gamma^{\underline{i}} &= \mathbb{I} \otimes \tau^{i-1} \otimes \mathbb{I} \otimes \sigma_2 \ , && (i = 2,3) \ , \\
\Gamma^{\underline{j}} &= \mathbb{I} \otimes \tau^{3} \otimes \lambda^{\underline{j}} \otimes \sigma_2 \ , && (j = 4,5,6,7,8) \ ,
\end{aligned} \tag{8.4.87}$$

where σ^i and τ^i are two sets of Pauli matrices, and λ^i are $5d$ Euclidean γ-matrices. The representation (8.4.87) is chiral,

$$\Gamma_{(10)} = \pm \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \sigma_3 \ , \tag{8.4.88}$$

where the sign depends on the representations of the $SO(2,1)$ and $SO(5)$ Clifford algebras. To be specific, let us choose the γ^a such that $\gamma^{\underline{9}} = \gamma^{\underline{01}}$, i.e., $\gamma^{\underline{019}} = 1$.

Hence, under the decomposition (8.4.87), the 16-component chiral θ becomes an octet of 2-component $3d$ spinors. It is useful to decompose this octet into eigenspinors of the three mutually commuting matrices τ^3, $\lambda^{\underline{56}}$ and $\lambda^{\underline{78}}$,

$$\lambda^{\underline{56}}\theta_{abc} = ip\theta_{pqr} \ , \quad \lambda^{\underline{78}}\theta_{abc} = iq\theta_{pqr} \ , \quad \tau^3\theta_{pqr} = r\theta_{pqr} \ , \quad (p,q,r = \pm 1) \ . \tag{8.4.89}$$

The action (8.4.85) now becomes

$$S^{(F)}_{\mathrm{D2}} = \frac{T_2}{2\sin\alpha} \int d^3\xi \sqrt{-\det \tilde{g}_{ab}}\, \bar{\theta}_{pqr} \left\{ \tilde{\gamma}^a \tilde{\nabla}_a + \frac{i}{4\sin\alpha} \gamma^{\underline{01}} [p - q + r(3 - pq)] \right\} \theta_{pqr} \ , \tag{8.4.90}$$

where the sum over the octet is implicit.

8.4.3 Spectrum of D2-brane fluctuations

The doublet of scalars $\chi^{\underline{i}}$, $i = 2, 3$, satisfies

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha}\right)\chi^{\underline{2}} - \frac{3}{\sin\alpha}e^{\mu}_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{3}} = 0 \ , \tag{8.4.91}$$

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha}\right)\chi^{\underline{3}} + \frac{3}{\sin\alpha}e^{\mu}_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{2}} = 0 \ . \tag{8.4.92}$$

The system is diagonalized by introducing $\chi^{\pm} = \chi^{\underline{2}} \pm i\chi^{\underline{3}}$, for which (8.4.91) and (8.4.92) become

$$\left(-\tilde{\nabla}_a\tilde{\nabla}^a + \frac{2}{\sin^2\alpha} \pm \frac{3i}{\sin\alpha}e^{\mu}_{\underline{9}}\tilde{\nabla}_\mu\right)\chi^{\pm} = 0 \ . \tag{8.4.93}$$

Decomposing into the modes on the S^1 factor of the D2-brane worldvolume, which are characterized by an integer n, (8.4.93) gives rise to

$$\left(\Box - \frac{n^2 \mp 3n + 2}{\sin^2\alpha}\right)\chi^{\pm}_n = 0 \ , \tag{8.4.94}$$

where $\Box = \tilde{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta$. The conformal dimensions of the dual operators are obtained from the standard formula,

$$\Delta^{\pm}_n = \frac{1}{2} + \left|n \mp \frac{3}{2}\right| \ . \tag{8.4.95}$$

These are positive integers.

As for the D6-brane the scalar $\chi^{\underline{4}}$ couples to the AdS$_2$-components a_α of the vector field. Their field equations are

$$\left(\tilde{\nabla}_a\tilde{\nabla}^a + \frac{1}{\sin^2\alpha}\right)\chi^{\underline{4}} + \frac{1}{\sin\alpha}f = 0 \ , \tag{8.4.96}$$

$$\tilde{\nabla}_a(\tilde{\nabla}^a a^\alpha - \tilde{\nabla}^\alpha a^a) + \frac{1}{\sin\alpha}\tilde{\epsilon}^{\alpha\beta}\partial_\beta\chi^{\underline{4}} = 0 \ , \tag{8.4.97}$$

where f stands again for $f = \frac{1}{2}\tilde{\epsilon}^{\alpha\beta}f_{\alpha\beta}$. Proceeding as in the D6-brane case gives rise to

$$\begin{pmatrix} \Box + \tilde{\nabla}_\mu\tilde{\nabla}^\mu + \frac{1}{\sin^2\alpha} & \frac{1}{\sin\alpha} \\ \frac{1}{\sin\alpha}\Box & \Box + \tilde{\nabla}_\mu\tilde{\nabla}^\mu \end{pmatrix}\begin{pmatrix}\chi^{\underline{4}} \\ f\end{pmatrix} = 0 \ . \tag{8.4.98}$$

Expanding into modes on S^1, (8.4.98) yields

$$\begin{pmatrix} \Box + \frac{1-n^2}{\sin^2\alpha} & \frac{1}{\sin\alpha} \\ \frac{1}{\sin\alpha}\Box & \Box - \frac{n^2}{\sin^2\alpha} \end{pmatrix} \begin{pmatrix} \chi^{\underline{4}}_n \\ f_n \end{pmatrix} = 0\ . \tag{8.4.99}$$

To obtain the conformal dimensions of the dual operators, one formally solves the characteristic equation of (8.4.99) for $\Box$ and translates the two AdS_2 mass eigenvalues into the dual conformal dimensions. The result is

$$\Delta_n^{\pm} = \frac{1}{2} + \left| |n| \pm \frac{1}{2} \right|\ . \tag{8.4.100}$$

Consider the doublet of scalars $(\chi^{\underline{5}}, \chi^{\underline{6}})$. Their field equations are given by

$$\left(\tilde{\nabla}_a\tilde{\nabla}^a + \frac{1}{4}\right)\chi^{\underline{5}} + \cot\alpha\, e^{\mu}_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{6}} = 0\ , \tag{8.4.101}$$

$$\left(\tilde{\nabla}_a\tilde{\nabla}^a + \frac{1}{4}\right)\chi^{\underline{6}} - \cot\alpha\, e^{\mu}_{\underline{9}}\tilde{\nabla}_\mu\chi^{\underline{5}} = 0\ . \tag{8.4.102}$$

Remember that the covariant derivative ∇_μ contains the normal connection (8.4.75). Introducing $\chi^{\pm} = \chi^{\underline{5}} \pm i\chi^{\underline{6}}$, we diagonalize the covariant derivative

$$\nabla_\mu\chi^{\pm} = \left[\partial_\mu \pm \frac{i}{4}(\cos\alpha - 1)\right]\chi^{\pm} \tag{8.4.103}$$

and the field equations, which become

$$\left[\Box + \tilde{g}^{\mu\mu}\partial_\mu^2 \mp \frac{i}{\sin\alpha} e^{\mu}_{\underline{9}}\partial_\mu\right]\chi^{\pm} = 0\ . \tag{8.4.104}$$

After the decomposition into S^1 modes and using the standard dimension formula, one obtains the dual operator conformal dimensions

$$\Delta_n^{\pm} = \frac{1}{2} + \left| n \mp \frac{1}{2} \right|\ . \tag{8.4.105}$$

The analysis for the doublet $(\chi^{\underline{7}}, \chi^{\underline{8}})$ proceeds in an identical fashion and yields the same result.

Table 8.3: Bosonic Spectrum

Doublet	$\Delta_n^{\pm}$
(χ_n^2, χ_n^3)	$\frac{1}{2} + \lvert n \mp \frac{3}{2}\rvert$
(χ_n^4, f_n)	$\frac{1}{2} + \lvert\lvert n\rvert \pm \frac{1}{2}\rvert$
(χ_n^5, χ_n^6)	$\frac{1}{2} + \lvert n \mp \frac{1}{2}\rvert$
(χ_n^7, χ_n^8)	$\frac{1}{2} + \lvert n \mp \frac{1}{2}\rvert$

To obtain the fermionic spectrum, consider the field equations for the octet of $3d$ spinors arising from the action (8.4.90), in which we split the Dirac operator into the $\mathrm{AdS}_2 \times S^1$ parts,

$$\left[\tilde{\gamma}^{\alpha}\tilde{\nabla}_{\alpha} + \frac{1}{\sin\alpha}\gamma^{\underline{01}}\left(2\partial_{\chi} + \frac{i}{2}D_{pqr}\right)\right]\theta_{pqr} \ , \tag{8.4.106}$$

where

$$D_{pqr} = \frac{1}{2}\left[p - q + r(3 - pq)\right] \tag{8.4.107}$$

takes the odd integer values $D_{pqr} \in (-3, -1, -1, -1, 1, 1, 1, 3)$. The S^1 dependence is solved by the a simple exponential,

$$\theta \sim \epsilon i\left(n + \frac{1}{2}\right)\frac{\chi}{2} \ , \tag{8.4.108}$$

where n is an integer. (Remember $\chi \in (0, 4\pi)$.) Hence, (8.4.106) reduces to the form

$$\left(\tilde{\gamma}^{\alpha}\tilde{\nabla}_{\alpha} + \frac{i\lambda_{npqr}}{\sin\alpha}\gamma^{\underline{01}}\right)\theta_{npqr} \ , \tag{8.4.109}$$

which is familiar from the D6-brane case. The resulting dual conformal dimensions

$$\Delta_{npqr} = \frac{1}{2} + \lvert\lambda_{npqr}\rvert \tag{8.4.110}$$

are positive half-integers $(1/2, 3/2, \cdots)$, which nicely complement the bosonic spectrum to fill supersymmetric multiplets. (It may be useful to shift the value of n depending on the value of D_{pqr}.)

Table 8.4: Fermionic Spectrum

θ_{npqr}	λ_{npqr}	Δ_n
$\theta_{n+++}, \theta_{n-++}, \theta_{n--+}$	$n+1$	$\frac{1}{2}+\lvert n+1\rvert$
$\theta_{n---}, \theta_{n+--}, \theta_{n++-}$	n	$\frac{1}{2}+\lvert n\rvert$
θ_{n-+-}	$n-1$	$\frac{1}{2}+\lvert n-1\rvert$
θ_{n+-+}	$n+2$	$\frac{1}{2}+\lvert n+2\rvert$

8.5 Comments on supersymmetry and the spectrum

The ABJM theory is a three-dimensional Chern-Simons theory with $U(N)\times U(N)$ gauge group. It contains four complex scalar fields $C_I, (I = 1, 2, 3, 4)$ in the bifundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$, the corresponding complex conjugates in the $(\bar{\mathbf{N}}, \mathbf{N})$ representation, as well as the fermionic superpartners. The gauge fields are governed by a Chern-Simons action with opposite integer levels for the two gauge groups, k and $-k$ (see [18] for details). The bosonic symmetry subgroups of this theory are the conformal group in three dimensions $SO(3, 2)$ and the R-symmetry group $SU(4)_R \sim SO(6)_R$; these combine into the supergroup $OSp(6|4)$. In the 't Hooft limit (large N with fixed N/k ratio) the ABJM theory is conjectured to be dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. The bosonic subgroups act as isometries of AdS_4 and of $\mathbb{CP}^3$.

Let us now discuss the supersymmetric operator whose dual gravity configurations we have studied in this chapter. To build these type of Wilson loops one considers only one of the gauge fields of the whole $U(N)\times U(N)$ gauge group, we call it A_μ. We are mostly guided by the construction of similar operators in $\mathcal{N} = 4$ SYM but in the absence of adjoint fields one considers the appropriate combination of bi-fundamentals, C_I. Namely [140, 141, 142],

$$W = \frac{1}{N}\mathrm{Tr}_R\,\mathcal{P}\int\left(iA_\mu\dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|\,M^I_J C_I\bar{C}^J\right)ds. \tag{8.5.111}$$

It was shown in [140, 141, 142] that the above operator preserves a 1/6 of the 24 supercharges when the loop is a straight line or a circle, and the matrix takes the form $M^I_J = \mathrm{diag}\,(1, 1, -1, -1)$. It is worth mentioning that 1/2 BPS Wilson loops have also been constructed and have a very different pattern of symmetry breaking [145]. The Wilson loops

(8.5.111) are invariant under an $SL(2,\mathbb{R})\times U(1)\subset SO(3,2)$. The $SL(2,\mathbb{R})$ part of this subgroup is generated by translation along the line P_0, dilatation D and a special conformal transformation K_0; the $U(1)$ symmetry is generated by rotations around the line, J_{12}. Of the R-symmetry, the Wilson loop preserves an $SU(2)\times SU(2)\subset SU(4)$, as follows from the explicit form of the matrix M^I_J, which admits $C_1\leftrightarrow C_2$ and $C_3\leftrightarrow C_4$. The classification of AdS superalgebras that are of interest to us was presented in [167]. One supergroup in that list that contains the bosonic symmetries discussed here is OSp(4|2). In the original classification list of [167], this is series (i) using the algebra isomorphism $so(4)\sim su(2)\times su(2)$. In appendix I we recall details of the representations of $OSp(4|2)$; in the main text we use a slightly modified notation more akin to our considerations.

Let us first consider the spectrum of the D2 brane which is given in tables 8.3 and 8.4. We see that the degeneracies agree precisely with those of the multiplet of $OSp(4|2)$ presented in table 8.5. Here supersymmetry plays a crucial role. Notice that the D2 brane preserves 1/3 of the 24 bulk supersymmetries. At the level of the multiplet representation we denote the supercharges by $Q,Q^\dagger$; four can be interpreted as creation operators.

Table 8.5: Supermultiplet for the D2 brane fluctuations

Representation	Δ	$(2p_1+1,2p_2+1)$	Degeneracies
$\lvert\Phi\rangle$	h	$(1,1)$	1
$Q^\dagger\lvert\Phi\rangle$	$h+\frac{1}{2}$	$(2,2)$	1 3
$Q^\dagger Q^\dagger\lvert\Phi\rangle$	$h+1$	$(1,3)+(3,1)$	3 3
$Q^\dagger Q^\dagger Q^\dagger\lvert\Phi\rangle$	$h+\frac{3}{2}$	$(2,2)$	3 1
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Phi\rangle$	$h+2$	$(1,1)$	1

There are a total of 16 states in the multiplet: 8 bosons + 8 fermions. The degeneracies follow directly from states being singlets or triplets of the respective $su(2)$ as indicated in the last column of the table. We found it necessary to shift some of the AdS_2 quantum numbers to fit in one multiplet, but the spacing was respected. Thus, the spectrum of excitations of the D2 brane falls neatly into long representations of $OSp(4|2)$.

Let us now consider the spectrum of the D6 brane. This configuration is 1/6 BPS, meaning that there are only four supercharges, two of which can be considered as creation operators in the representation, more precisely, they raise the AdS_2 quantum number. Given that these supercharges are a doublet of $Sp(2)$ we obtain generic multiplets of operators with dimensions $(h, h+\frac{1}{2}, h+1)$. This is nicely respected by the values of h that are listed in tables 8.1 and 8.2, with the exception of two short fermion multiplets. We emphasize that, generically, the dimensions of bosonic operators are not integers. This is a non-trivial result of our calculation. Because all the states in a given row in tables 8.1 and 8.2 have the same values of the $SO(4)$ quantum numbers (j, l), we see that the supercharges are singlets under $SO(4)$ in contrast to the situation for the D2, where the supercharges were vectors under $SO(4)$. In any case, the fact that the spectra for the D6 fluctuations can be organized into supermultiplets is a nice check of our calculation.

8.6 Conclusion

We have computed the spectra of quantum fluctuations of particular embeddings of D6 and D2 branes with electric flux in their worldvolumes in the background of $AdS_4 \times \mathbb{C}P^3$, which is dual to ABJM theory. These brane configurations are expected to be dual to supersymmetric Wilson loops in higher dimensional representations of the gauge group of ABJM theory.

The results represent by themselves interesting progress within a well-defined class of holographic problems. In particular, regardless of the field theory motivation, the general question of semiclassical quantization of certain brane configurations in string theory backgrounds is of great interest. In this respect we have found a peculiar mixing term that are induced by the top, with respect to the worldvolume dimension, RR potential C_p form in the WZ part of the D-brane action.

The construction of supersymmetric field theories in curved spacetimes plays a central role in localization. In this respect, our results provide explicit constructions of supersymmetric field theories living in curved spaces containing an AdS_2 factor. Arguably, the simplest example in this class is provided by the spectrum of excitations of a supersymmetric D3 brane in $AdS_5 \times S^5$ which was obtained in [107] and later identified as an $\mathcal{N}=4$ Abelian vector multiplet living in $AdS_2 \times S^2$ in [25]. The study of supersymmetric field theories on non-compact spaces is an important problem from the field theoretic point of view and presents a, hopefully surmountable, challenge to the program of supersymmetric localization. To first approximation, the supersymmetric field theory describing the quadratic fluctuations constructed here is more similar to the one for D5 brane fluctuations obtained in [108], which lead to a field theory on $AdS_2 \times S^4$ with non-canonical couplings between the scalars and the Abelian gauge field. In this work, in comparison with [108], we have found an interesting new mixing term of the embedding that has not been seen before in any of the embeddings in $AdS_5 \times S^5$ analyzed in [107, 108]. It is worth highlighting that the mixing is intrinsic to brane embeddings; clearly the string, as discussed in [153] cannot contain this type of mixing term.

One set of questions that clearly deserves further investigation is the precise classification of all supersymmetric brane configurations with flux on their worldvolume embedded in $AdS_4 \times \mathbb{CP}^3$. In particular, there should be other classical solutions corresponding precisely to the 1/2 BPS configurations where the nature of $\mathbb{CP}^2$ is manifest as a realization of the unbroken $SU(3)$ R-symmetry group. One particular candidate which we studied preliminarily (but chose not to report on it here) is a D2 brane that wraps $AdS_2 \times S^1 \subset AdS_4$. Another configuration is a D6 whose worldvolume contains $\mathbb{CP}^2 \subset \mathbb{CP}^3$. We expect to report on such matters systematically in a future publication.

A logical continuation of our work would be the computation of the one-loop effective actions of the D2 and D6 configurations we considered in this chapter. In the context of the AdS/CFT correspondence such calculation yields the one-loop correction to the vacuum expectation value of Wilson loops in the strong 't Hooft coupling limit of ABJM. Indeed, such an effective action computation was undertaken for the fundamental string in [118] based on the spectrum obtained in [153]. Since the results for the fundamental representation, as they currently stand, do not seem to agree with the field theory side, we defer a systematic analysis of the one-loop effective action to a separate publication. It is worth noting that there has been some success in matching the holographic one-loop corrections to field theory results for certain Wilson loops in ABJM [136]. On the field theory side, to the best of our knowledge, some of the vacuum expectation values of Wilson loops in higher rank representations have not been systematically studied, although some results for representations with a small number of boxes were reported in [168]. The configurations we consider here are dual to Wilson loops in representations whose Young tableaux have a number of boxes of the same order as the rank of the gauge group N. To the best of our knowledge the expectation values of such Wilson loops have not been systematically computed on the field theory side. Having the corresponding exact field theory results will ultimately provide grounds for a precision holographic comparison between ABJM theory and strings and branes in $AdS_4 \times \mathbb{C}P^3$.

APPENDICES

APPENDIX A

Parameterizations of the 4D $\mathcal{N} = 2$ $U(1)^4$ gauged supergravity

In this appendix we summarize some of the common choices for the symplectic sections of the $U(1)^4$ $\mathcal{N} = 2$ gauged supergravity, and we provide the explicit relations for the physical scalars between different parameterizations. As we pointed out in section 4.2.1, the Kähler potential and the holomorphic superpotential transform non-trivially under reparameterizations of the symplectic sections and, therefore, their expressions in two different parameterizations of the symplectic sections should not be identified. Moreover, as is evident from section 4.3, not all choices of holomorphic sections are compatible with a particular choice of boundary conditions on the scalars. As a result, only certain choices for the symplectic sections are compatible with supersymmetric boundary conditions and/or holography.

A.0.1 Cvetič et al. gauge

The choice of symplectic sections that leads to the original parameterization of the STU model in [45] is summarized in section 8 of [63], and for the special case of real X^Λ also in appendix A.1 of [10]. The relevant parameterization is

$$\frac{X^1}{X^0} \equiv \tau_2\tau_3, \qquad \frac{X^2}{X^0} \equiv \tau_1\tau_3, \qquad \frac{X^3}{X^0} \equiv \tau_1\tau_2, \tag{A.1}$$

together with the gauge fixing condition

$$X^0 X^1 X^2 X^3 = 1, \tag{A.2}$$

or equivalently

$$X^0 = \frac{1}{\sqrt{\tau_1 \tau_2 \tau_3}}. \tag{A.3}$$

With this choice of symplectic sections, the prepotential, the Kähler potential, the Kähler metric, the holomorphic superpotential and the scalar potential defined in section 4.2.1 take respectively the form

$$\begin{aligned}
F =& -2i, \\
\mathcal{K} =& -\log\left(\frac{(\tau_1+\bar{\tau}_1)(\tau_2+\bar{\tau}_2)(\tau_3+\bar{\tau}_3)}{|\tau_1\tau_2\tau_3|}\right), \\
\mathcal{K}_{\alpha\bar{\beta}} =& \frac{\mathrm{d}_{\alpha\bar{\beta}}}{(\tau_\alpha+\bar{\tau}_{\bar{\beta}})^2}, \\
W =& \, \xi\left(\frac{1+\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3}{\sqrt{\tau_1\tau_2\tau_3}}\right), \\
\mathcal{V} =& -\frac{2}{L^2}\left(\frac{1+|\tau_1|^2}{\tau_1+\bar{\tau}_1}+\frac{1+|\tau_2|^2}{\tau_2+\bar{\tau}_2}+\frac{1+|\tau_3|^2}{\tau_3+\bar{\tau}_3}\right).
\end{aligned} \tag{A.4}$$

Setting further

$$\tau_\alpha = e^{-\varphi_\alpha} + i\gamma_\alpha, \tag{A.5}$$

it is straightforward to show that the $\mathcal{N} = 2$ action (4.2.8) reduces to the STU model action given in [45]. Notice that this parameterization is particularly convenient in the case of real τ^α because the Kähler potential becomes a constant. This in turn implies that the holomorphic superpotential coincides with the real superpotential (4.2.18), given in eq. (3.1) or [169] (see also eq. (2.15) in [76]).

A very important property of the parameterization of the STU model in terms of the scalars φ_α and γ_α is that it is compatible with the holographic dictionary, since these scalars have the correct Fefferman-Graham expansions for fields dual to dimension one or dimension two operators. Namely,

$$\begin{aligned}\varphi_\alpha(r,x) &= \varphi_\alpha^-(x)e^{-r/L} + \varphi_\alpha^+(x)e^{-2r/L} + \cdots\,,\\ \gamma_\alpha(r,x) &= \gamma_\alpha^-(x)e^{-r/L} + \gamma_\alpha^+(x)e^{-2r/L} + \cdots\,,\end{aligned} \tag{A.6}$$

in the Fefferman-Graham coordinates defined by the metric (B.1) and the gauge-fixing conditions (B.12). This parameterization is also compatible with supersymmetry, which requires that Neumann boundary conditions be imposed on the dilatons φ_α and Dirichlet on the axions γ_α (or vice versa) [91].

A.0.2 Cacciatori-Klemm gauge

A related parameterization of the symplectic sections is used in section 3.2 of [62], where the ratios

$$\frac{X^1}{X^0} \equiv \tau_2\tau_3, \qquad \frac{X^2}{X^0} \equiv \tau_1\tau_3, \qquad \frac{X^3}{X^0} \equiv \tau_1\tau_2, \tag{A.7}$$

are parameterized exactly as in (A.1), but the gauge condition (A.2) is replaced with

$$X^0 = 1. \tag{A.8}$$

This choice leads to

$$\begin{aligned}F =&\ -2i\tau_1\tau_2\tau_3,\\ \mathcal{K} =&\ -\log\Big((\tau_1+\bar\tau_1)(\tau_2+\bar\tau_2)(\tau_3+\bar\tau_3)\Big),\\ \mathcal{K}_{\alpha\bar\beta} =&\ \frac{\mathrm{d}_{\alpha\bar\beta}}{(\tau_\alpha+\bar\tau_{\bar\beta})^2},\\ W =&\ \xi(1+\tau_2\tau_3+\tau_1\tau_3+\tau_1\tau_2),\\ \mathcal{V} =&\ -\frac{2}{L^2}\left(\frac{1+|\tau_1|^2}{\tau_1+\bar\tau_1}+\frac{1+|\tau_2|^2}{\tau_2+\bar\tau_2}+\frac{1+|\tau_3|^2}{\tau_3+\bar\tau_3}\right).\end{aligned} \tag{A.9}$$

Notice that the Kähler metric and the scalar potential are identical to those in (A.4), and so using the identification (A.5) one again obtains the STU model Lagrangian in the form given in [45]. However, the Kähler potential and the holomorphic superpotential are not the same, which is expected since neither of these quantities is invariant under reparameterizations of the symplectic sections. Despite the different gauge fixing condition for the symplectic sections, the Cacciatori-Klemm parameterization leads to the same physical scalars as those in Cvetič et al., and so it is also compatible with both the holographic dictionary and supersymmetry.

A.0.3 Pufu-Freedman gauge

Another choice for the symplectic sections that is compatible with both holography and supersymmetry is the one implicitly used in [91, 92]. In that parameterization the sections are given by

$$\begin{aligned}
X^0 &= (1+z^1)(1+z^2)(1+z^3),\\
X^1 &= (1+z^1)(1-z^2)(1-z^3),\\
X^2 &= (1-z^1)(1+z^2)(1-z^3),\\
X^3 &= (1-z^1)(1-z^2)(1+z^3),
\end{aligned} \tag{A.10}$$

from which we obtain

$$\begin{aligned}
F =& -2i(1-(z^1)^2)(1-(z^2)^2)(1-(z^3)^2),\\
\mathcal{K} =& -\log\Big(8(1-|z^1|^2)(1-|z^2|^2)(1-|z^3|^2)\Big),\\
\mathcal{K}_{\alpha\bar{\beta}} =& \frac{\mathrm{d}_{\alpha\bar{\beta}}}{(1-|z^\alpha|^2)^2},\\
W =& 4\xi(1+z^1z^2z^3),\\
\mathcal{V} =& \frac{2}{L^2}\Big(3-2\sum_{\alpha=1}^{3}\frac{1}{1-|z^\alpha|^2}\Big).
\end{aligned} \tag{A.11}$$

The physical scalars z^α in this parameterization are related to the variables τ^α in the Cvetič et al. and Cacciatori-Klemm gauges as

$$z^1 = \frac{1-\tau_1}{1+\tau_1}, \qquad z^2 = \frac{1-\tau_2}{1+\tau_2}, \qquad z^3 = \frac{1-\tau_3}{1+\tau_3}. \tag{A.12}$$

These scalars also admit the correct Fefferman-Graham expansions for fields dual to dimension one or two operators, namely

$$z^\alpha(r,x) = z^\alpha_-(x)e^{-r/L} + z^\alpha_+(x)e^{-2r/L} + \cdots . \tag{A.13}$$

The relations (A.12) then imply that the modes $z^\alpha_-(x)$ and $z^\alpha_+(x)$ can be expressed in terms of the modes of the Fefferman-Graham expansions (A.6) as

$$\begin{aligned} z^\alpha_-(x) &= \frac{1}{2}(\varphi^-_\alpha(x) - i\gamma^-_\alpha(x)), \\ z^\alpha_+(x) &= \frac{1}{2}\Big[\Big(\varphi^+_\alpha(x) - \frac{1}{2}(\gamma^-_\alpha(x))^2\Big) - i\Big(\gamma^+_\alpha(x) + \varphi^-_\alpha(x)\gamma^-_\alpha(x)\Big)\Big], \end{aligned} \tag{A.14}$$

where no summation over the index α is implied. Hence, the boundary conditions for φ_α and γ_α map respectively to the real and imaginary parts of z^α.

A.0.4 Hristov-Vandoren gauge

As a final example of a choice of symplectic sections for the STU model we should discuss the parameterization used in [63, 10], namely

$$\begin{aligned} X^0 &= \frac{1}{1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3}, \\ X^1 &= \frac{\tilde{z}^1}{1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3}, \\ X^2 &= \frac{\tilde{z}^2}{1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3}, \\ X^3 &= \frac{\tilde{z}^3}{1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3}, \end{aligned} \tag{A.15}$$

together with the gauge fixing condition (see discussion around eq. (4.16) in [63] and eq. (C.4) in [10])

$$X^0 + X^1 + X^2 + X^3 = 1, \tag{A.16}$$

as well as the reality condition

$$\operatorname{Im} X^\Lambda = 0. \tag{A.17}$$

In order to compute the various $\mathcal{N} = 2$ supergravity quantities one needs to start with general complex $\tilde{z}^\alpha$ and impose the reality condition only at the end. This procedure gives

$$\begin{aligned}
F &= \frac{-2i\sqrt{\tilde{z}^1\tilde{z}^2\tilde{z}^3}}{(1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3)^2},\\
\mathcal{K} &= -\log\left(\frac{8\sqrt{\tilde{z}^1\tilde{z}^2\tilde{z}^3}}{(1+\tilde{z}^1+\tilde{z}^2+\tilde{z}^3)^2}\right), \quad \tilde{z}^1,\tilde{z}^2,\tilde{z}^3\in\mathbb{R},\\
\mathcal{K}_{\alpha\bar{\beta}} &= \frac{1}{16}\begin{pmatrix} 3/(\tilde{z}^1)^2 & -1/\tilde{z}^1\tilde{z}^2 & -1/\tilde{z}^1\tilde{z}^3\\ -1/\tilde{z}^1\tilde{z}^2 & 3/(\tilde{z}^2)^2 & -1/\tilde{z}^2\tilde{z}^3\\ -1/\tilde{z}^1\tilde{z}^3 & -1/\tilde{z}^3\tilde{z}^2 & 3/(\tilde{z}^3)^2\end{pmatrix}, \quad \tilde{z}^1,\tilde{z}^2,\tilde{z}^3\in\mathbb{R},\\
W &= \xi,\\
\mathcal{V} &= -\frac{1}{L^2}\left(\frac{\tilde{z}^1+\tilde{z}^2+\tilde{z}^3+\tilde{z}^1\tilde{z}^2+\tilde{z}^1\tilde{z}^3+\tilde{z}^2\tilde{z}^2}{\sqrt{\tilde{z}^1\tilde{z}^2\tilde{z}^3}}\right), \quad \tilde{z}^1,\tilde{z}^2,\tilde{z}^3\in\mathbb{R},
\end{aligned} \tag{A.18}$$

where we have given the Kähler potential, the Kähler metric and the scalar potential only for real $\tilde{z}^\alpha$ since the expressions with complex scalars are far too lengthy. The expressions for the prepotential and the holomorphic superpotential hold for complex scalars.

The scalars $\tilde{z}^\alpha$ are related to the variables τ^α in the Cvetič et al. and Cacciatori-Klemm parameterizations as

$$\tilde{z}^1 = \tau_2\tau_3, \qquad \tilde{z}^2 = \tau_1\tau_3, \qquad \tilde{z}^3 = \tau_1\tau_2. \tag{A.19}$$

Taking τ^α to be real and inserting these expressions in the scalar potential in (A.18) one easily sees that it coincides with the scalar potential in (A.4) or (A.9). It follows that the parameterization used in [63, 10] agrees with all other parameterizations of the STU model discussed above, but only provided the scalars are *real*. It is in fact a very convenient parameterization for obtaining the purely magnetic solutions discussed in [63, 10], but it is not suitable for supersymmetric dyonic solutions that are necessarily supported by complex scalars.

APPENDIX B

Radial Hamiltonian Formalism

In order to formulate the supergravity theory described by the action (4.2.8) in radial Hamiltonian language we parameterize the bulk metric in terms of the lapse function N, the shift function N_i and the induced metric γ_{ij} on the radial slices, namely

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j. \tag{B.1}$$

Similarly, the Abelian gauge fields are decomposed in radial and transverse components as

$$A^\Lambda = \alpha^\Lambda dr + A_i^\Lambda dx^i. \tag{B.2}$$

Using the decomposition (B.1) of the metric the bulk Ricci scalar becomes

$$R[g] = R[\gamma] + K^2 - K_{ij}K^{ij} + \nabla_\mu \left(-2Kn^\mu + 2n^\nu \nabla_\nu n^\mu\right), \tag{B.3}$$

where

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right), \tag{B.4}$$

is the extrinsic curvature and $n^\mu = (1/N, -N^i/N)$ is the unit outward normal vector of the radial slices. As in the main text, a dot $\dot{}$ denotes a derivative with respect to the radial coordinate r, and D_i is the covariant derivative with respect to the induced metric γ_{ij}.

Using these expressions, together with the identities

$$\sqrt{-g} = N\sqrt{-\gamma}, \qquad g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^i}{N^2} & \gamma^{ij} + \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{B.5}$$

the action (4.2.8) can be written in the form $S = \int dr L$, where the radial Lagrangian L is

$$\begin{aligned} L = \frac{1}{2\kappa^2} \int d^3x N\sqrt{-\gamma}\Big\{ & R[\gamma] + K^2 - K_{ij}K^{ij} - \frac{1}{N^2}\mathcal{G}_{\alpha\bar{\beta}}\big(\dot{z}^\alpha - N^i\partial_i z^\alpha\big)\big(\dot{\bar{z}}^{\bar{\beta}} - N^i\partial_i \bar{z}^{\bar{\beta}}\big) \\ & - \frac{4}{N^2}\mathcal{I}_{\Lambda\Sigma}\gamma^{ij}(\dot{A}_i^\Lambda - \partial_i\alpha^\Lambda - N^k F_{ki}^\Lambda)(\dot{A}_j^\Sigma - \partial_j\alpha^\Sigma - N^l F_{lj}^\Sigma) - \frac{4}{N}\mathcal{R}_{\Lambda\Sigma}\epsilon^{ijk}(\dot{A}_i^\Lambda - \partial_i\alpha^\Lambda)F_{jk}^\Sigma \\ & - \mathcal{G}_{\alpha\bar{\beta}}\gamma^{ij}\partial_i z^\alpha \partial_j \bar{z}^{\bar{\beta}} - 2\mathcal{I}_{\Lambda\Sigma}F_{ij}^\Lambda F^{\Sigma ij} - \mathcal{V}\Big\}. \end{aligned} \tag{B.6}$$

The canonical momenta following from this Lagrangian take the form

$$\pi^{ij} = \frac{\mathrm{d}\,L}{\mathrm{d}\,\dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}\big(K\gamma^{ij} - K^{ij}\big), \tag{B.7a}$$

$$\pi_\alpha = \frac{\mathrm{d}\,L}{\mathrm{d}\,\dot{z}^\alpha} = -\frac{1}{2\kappa^2}\frac{\sqrt{-\gamma}}{N}\mathcal{G}_{\alpha\bar{\beta}}\big(\dot{\bar{z}}^{\bar{\beta}} - N^i\partial_i\bar{z}^{\bar{\beta}}\big), \tag{B.7b}$$

$$\pi_{\bar{\beta}} = \frac{\mathrm{d}\,L}{\mathrm{d}\,\dot{\bar{z}}^{\bar{\beta}}} = -\frac{1}{2\kappa^2}\frac{\sqrt{-\gamma}}{N}\mathcal{G}_{\alpha\bar{\beta}}\big(\dot{z}^\alpha - N^i\partial_i z^\alpha\big), \tag{B.7c}$$

$$\pi^i_\Lambda = \frac{\mathrm{d}\,L}{\mathrm{d}\,\dot{A}_i^\Lambda} = -\frac{4}{\kappa^2}\frac{\sqrt{-\gamma}}{N}\mathcal{I}_{\Lambda\Sigma}\big(\gamma^{ij}(\dot{A}_j^\Sigma - \partial_j\alpha^\Sigma) - N_j F^{\Sigma ji}\big) - \frac{2}{\kappa^2}\sqrt{-\gamma}\mathcal{R}_{\Lambda\Sigma}\epsilon^{ijk}F_{jk}^\Sigma. \tag{B.7d}$$

Notice that the canonical momenta conjugate to the variables N, N_i and α^Λ vanish identically and, hence, these fields are non-dynamical. Given the canonical momenta (B.7), a short calculation determines the Hamiltonian, namely

$$H = \int d^3x\Big(\pi^{ij}\dot{\gamma}_{ij} + \pi_\alpha\dot{z}^\alpha + \pi_{\bar{\beta}}\dot{\bar{z}}^{\bar{\beta}} + \pi^i_\Lambda\dot{A}_i^\Lambda\Big) - L = \int d^3x\left(N\mathcal{H} + N_i\mathcal{H}^i + \alpha^\Lambda\mathcal{F}_\Lambda\right), \tag{B.8}$$

where

$$
\begin{aligned}
\mathcal{H} = & - \frac{\kappa^2}{\sqrt{-\gamma}} \left(2 \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) \pi^{ij}\pi^{kl} \right. \\
& + \frac{1}{8}\mathcal{I}^{\Lambda\Sigma} \left(\pi_{\Lambda i} + \frac{2}{\kappa^2}\sqrt{-\gamma}\, \mathcal{R}_{\Lambda K}\epsilon_i{}^{kl}F^K_{kl} \right) \left(\pi^i_\Sigma + \frac{2}{\kappa^2}\sqrt{-\gamma}\, \mathcal{R}_{\Sigma M}\epsilon^{ipq}F^M_{pq} \right) \left. + 2\mathcal{G}^{\alpha\bar{\beta}}\pi_\alpha\pi_{\bar{\beta}} \right) \\
& + \frac{\sqrt{-\gamma}}{2\kappa^2} \left(-R[\gamma] + 2\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{ij}F^{\Sigma ij} + \mathcal{G}_{\alpha\bar{\beta}}\partial_i z^\alpha \partial^i \bar{z}^{\bar{\beta}} + \mathcal{V} \right), \qquad \text{(B.9a)} \\
\mathcal{H}^i = & - 2D_j\pi^{ij} + F^{\Lambda ij} \left(\pi_{\Lambda j} + \frac{2}{\kappa^2}\sqrt{-\gamma}\, \mathcal{R}_{\Lambda\Sigma}\epsilon_j{}^{kl}F^\Sigma_{kl} \right) + \pi_\alpha\partial^i z^\alpha + \pi_{\bar{\beta}}\partial^i\bar{z}^{\bar{\beta}}, \qquad \text{(B.9b)} \\
\mathcal{F}_\Lambda = & - D_i\pi^i_\Lambda. \qquad \text{(B.9c)}
\end{aligned}
$$

Since the canonical momenta conjugate to the fields N, N_i and α^Λ vanish identically, Hamilton's equations for these fields impose the first class constraints

$$
\mathcal{H} = \mathcal{H}^i = \mathcal{F}_\Lambda = 0, \tag{B.10}
$$

which reflect the diffeomorphism and gauge invariance of the bulk theory. It follows that the Hamiltonian (B.8) vanishes identically on-shell.

Finally, HJ theory allows us to express the canonical momenta as gradients of the so called Hamilton's principal function $\mathcal{S}[\gamma, A^\Lambda, z^\alpha, \bar{z}^{\bar{\beta}}]$, i.e.

$$
\pi^{ij} = \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\gamma_{ij}}, \quad \pi^i_\Lambda = \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,A^\Lambda_i}, \quad \pi_\alpha = \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,z^\alpha}, \quad \pi_{\bar{\beta}} = \frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\bar{z}^{\bar{\beta}}}. \tag{B.11}
$$

Inserting these expressions for the momenta in the constraints (B.9) leads to a set of functional partial differential equations, the HJ equations, for the functional $\mathcal{S}[\gamma, A^\Lambda, z^\alpha, \bar{z}^{\bar{\beta}}]$. Given a solution of the HJ equations, equating the expressions (B.11) and (B.7) for the canonical momenta leads to a set of first order flow equations for the fields $\gamma_{ij}(r,x)$, $A^\Lambda_i(r,x)$, $z^\alpha(r,x)$, $\bar{z}^{\bar{\beta}}(r,x)$. In the radial (or Fefferman-Graham) gauge

$$
N = 1, \qquad N_i = 0, \qquad \alpha^\Lambda = 0, \tag{B.12}
$$

these first order equations take the form

$$
\begin{aligned}
\dot{\gamma}_{ij} &= -\frac{4\kappa^2}{\sqrt{-\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\gamma_{kl}}, && \text{(B.13a)}\\
\dot{z}^{\alpha} &= -\frac{2\kappa^2}{\sqrt{-\gamma}}\mathcal{G}^{\alpha\bar{\beta}}\frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,\bar{z}^{\bar{\beta}}}, && \text{(B.13b)}\\
\dot{\bar{z}}^{\bar{\beta}} &= -\frac{2\kappa^2}{\sqrt{-\gamma}}\mathcal{G}^{\alpha\bar{\beta}}\frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,z^{\alpha}}, && \text{(B.13c)}\\
\dot{A}_i^{\Lambda} &= -\frac{\kappa^2}{4\sqrt{-\gamma}}\mathcal{I}^{\Lambda\Sigma}\gamma_{ij}\frac{\mathrm{d}\,\mathcal{S}}{\mathrm{d}\,A_j^{\Sigma}} - \frac{1}{2}\mathcal{I}^{\Lambda\Sigma}\mathcal{R}_{\Sigma M}\epsilon_i{}^{jk}F_{jk}^{M}. && \text{(B.13d)}
\end{aligned}
$$

As we discuss in section 4.2.3, these general flow equations lead to first order BPS-like equations for any solution of the form (4.2.20), including non supersymmetric solutions.

APPENDIX C

Weyl Anomaly

In two dimensions, for an operator of the form

$$\mathcal{O}_M = M^{-1}\mathcal{O}\,, \qquad \mathcal{O} = -g^{\mu\nu}D_\mu D_\nu + X\,, \tag{C.1}$$

the dependence of $\det \mathcal{O}_M$ on M is determined by [21, 47, 170]

$$\delta_M \left(\ln\det\mathcal{O}_M\right) = -a_2\left(\delta\ln M|\mathcal{O}_M\right)\,, \tag{C.2}$$

where a_2 is the Seeley coefficient

$$a_2(F|\mathcal{O}_M) = \frac{1}{4\pi}\mathrm{Tr}\left[\int_{\mathcal{M}} d^2\sigma\sqrt{g}\,F\,b_2(\mathcal{O}_M) + \int_{\partial\mathcal{M}} ds\sqrt{\gamma}\left(F\,c_2(\mathcal{O}_M) \mp \frac{1}{2}\partial_n F\right)\right]\,, \tag{C.3}$$

$$b_2(\mathcal{O}_M) = -X + \frac{1}{6}R - \frac{1}{6}\nabla^2\ln M\,, \qquad c_2(\mathcal{O}_M) = \frac{1}{3}\left(K - \frac{1}{2}\partial_n\ln M\right)\,, \tag{C.4}$$

and the trace is taken over all degrees of freedom. For AdS_2 the unit normal vector and the extrinsic curvature are given by $n = \partial_\rho$ and $K = g^{\mu\nu}\nabla_\mu n_\nu = \coth\rho$. Integrating this relation yields

$$\ln\left(\frac{\det\mathcal{O}_M}{\det\mathcal{O}}\right) = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\ln M\,\mathrm{Tr}\left(X - \frac{1}{6}R + \frac{1}{12}\nabla^2\ln M\right)\,. \tag{C.5}$$

Here we have discarded boundary terms, which is justified as long as the conformal factor is everywhere smooth with $M \to 1$ sufficiently fast as $\rho \to \infty$. This is all that is needed for the scalar case. The treatment of fermionic fluctuations is similar, except that the anomaly argument only works for second order operators. So, given instead

$$\mathcal{O}_M = M^{-\frac{1}{2}}\mathcal{O}\,, \qquad \mathcal{O} = -i\not{D} + Y\,. \tag{C.6}$$

we must relate the determinants of $\mathcal{O}_M^2$ and $\mathcal{O}^2$. Directly squaring leads to

$$\mathcal{O}_M^2 = M^{-1}\mathcal{O}'\,, \qquad \mathcal{O}' = -g^{\mu\nu}D'_\mu D'_\nu + X'\,, \tag{C.7}$$

where

$$D'_\mu = D_\mu + \frac{i}{2}\theta_\mu\,, \qquad \theta_\mu = \Gamma_\mu Y + Y\Gamma_\mu + \frac{i\not{\partial}M}{2M}\Gamma_\mu\,. \tag{C.8}$$

and

$$\begin{aligned} X' = &-\frac{1}{4}\left(\Gamma^\mu Y\Gamma_\mu Y + Y\Gamma^\mu Y\Gamma_\mu + \Gamma^\mu Y^2\Gamma_\mu - 2Y^2 + \Gamma_\mu Y\frac{i\not{\partial}M}{2M}\Gamma^\mu - \frac{i\not{\partial}M}{M}Y + \frac{i\not{\partial}M}{2M}\Gamma^\mu Y\Gamma_\mu\right) \\ &+ \frac{i}{2}\left(-\Gamma^\mu D_\mu Y + D_\mu Y\Gamma^\mu + \frac{i}{2}\nabla^2 \ln M\right) + \frac{1}{4}R - iq\not{F}\,. \end{aligned} \tag{C.9}$$

The corresponding Seeley coefficient reads

$$\operatorname{Tr} b_2(\mathcal{O}_M^2) = \operatorname{Tr}\left(-X' + \frac{1}{6}R - \frac{1}{6}\nabla^2 \ln M\right) = \operatorname{Tr}\left(\frac{1}{2}\Gamma^\mu Y\Gamma_\mu Y - \frac{1}{12}R + \frac{1}{12}\nabla^2 \ln M\right)\,. \tag{C.10}$$

Integrating the anomaly equation yields

$$\ln\left(\frac{\det \mathcal{O}_M^2}{\det \mathcal{O}^2}\right) = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\ln M\,\operatorname{Tr}\left(-\frac{1}{2}\Gamma^\mu Y\Gamma_\mu Y + \frac{1}{12}R - \frac{1}{24}\nabla^2 \ln M\right)\,. \tag{C.11}$$

APPENDIX D

Conventions and Notation

Ten-dimensional target-space indices are denoted by $m, n, \ldots$, two-dimensional world-sheet indices are $a, b, \ldots$, while the directions orthogonal to the string are represented by $i, j, \ldots$. All corresponding tangent space indices are underlined.

In Euclidean signature the Dirac matrices satisfy

$$\Gamma_{\underline{m}}^{\dagger} = \Gamma_{\underline{m}}\,, \qquad \Gamma_{\underline{m}}^{2} = 1\,, \tag{D.1}$$

and the chirality matrix is

$$\Gamma_{11} \equiv -i\Gamma_{\underline{0123456789}}\,, \qquad \Gamma_{11}^{\dagger} = \Gamma_{11}\,, \qquad \Gamma_{11}^{2} = 1\,. \tag{D.2}$$

The charge conjugation intertwiners $C_{\pm}$ are such that

$$C_{\pm}\Gamma_{\underline{m}}C_{\pm}^{-1} = \pm\Gamma_{\underline{m}}^{T}\,, \qquad C_{\pm}\Gamma_{11}C_{\pm}^{-1} = -\Gamma_{11}^{T}\,, \qquad C_{\pm}^{T} = \pm C_{\pm}\,. \tag{D.3}$$

Majorana spinors are defined as

$$\psi^{T}C_{\pm} = \psi^{\dagger} \qquad \Leftrightarrow \qquad \psi^{*} = \pm C_{\pm}\psi\,. \tag{D.4}$$

In Lorentzian signature we have

$$\Gamma_{\underline{m}}^{\dagger} = \Gamma_{\underline{0}}\Gamma_{\underline{m}}\Gamma_{\underline{0}}\,, \qquad \Gamma_{\underline{0}}^{2} = -1\,, \qquad \Gamma_{\underline{m}\neq 0}^{2} = 1\,. \tag{D.5}$$

and the chirality matrix reads

$$\Gamma_{11} \equiv \Gamma_{\underline{0123456789}}\,, \qquad \Gamma_{11}^{\dagger} = \Gamma_{11}\,, \qquad \Gamma_{11}^{2} = 1\,. \tag{D.6}$$

APPENDIX E

Geometric Data on $AdS_4 \times \mathbb{CP}^3$

In this appendix we collect all the geometric formulae necessary to compute the spectrum of excitations of the 1/6-BPS string.

We start by writing the target space fields. The Euclidean AdS_4 ($EAdS_4$) metric is written as an $\mathbb{H}_2 \times S^1$ foliation,

$$ds^2_{EAdS_4} = \cosh^2 u \left(\sinh^2 \rho \, d\psi^2 + d\rho^2\right) + \sinh^2 u \, d\phi^2 + du^2 \,, \tag{E.1}$$

with $u \geq 0$, $\rho \geq 0$, $\psi \sim \psi + 2\pi$ and $\phi \sim \phi + 2\pi$. The metric on $\mathbb{CP}^3$ is taken to be

$$\begin{aligned} ds^2_{\mathbb{CP}^3} = \frac{1}{4} \Big[& d\alpha^2 + \cos^2 \frac{\alpha}{2} \left(d\vartheta_1^2 + \sin^2 \vartheta_1 \, d\varphi_1^2\right) + \sin^2 \frac{\alpha}{2} \left(d\vartheta_2^2 + \sin^2 \vartheta_2 \, d\varphi_2^2\right) \\ & + \cos^2 \frac{\alpha}{2} \sin^2 \frac{\alpha}{2} \left(d\chi - (1 - \cos \vartheta_1) \, d\varphi_1 + (1 - \cos \vartheta_2) \, d\varphi_2\right)^2 \Big] \,, \end{aligned} \tag{E.2}$$

where $0 \leq \alpha \leq \pi$, $0 \leq \vartheta_1 \leq \pi$, $0 \leq \vartheta_1 \leq \pi$, $\varphi_1 \sim \varphi_1 + 2\pi$, $\varphi_2 \sim \varphi_2 + 2\pi$ and $\chi \sim \chi + 4\pi$. The full $EAdS_4 \times \mathbb{CP}^3$ metric with radius L is then

$$ds^2 = L^2 \left(ds^2_{EAdS_4} + 4 \, ds^2_{\mathbb{CP}^3}\right) \,. \tag{E.3}$$

The other background fields read

$$e^{\Phi} = \frac{2L}{k}\,, \qquad F_{(4)} = -\frac{3ikL^2}{2}\mathrm{vol}\,(AdS_4)\;, \qquad F_{(2)} = \frac{k}{4}J\,, \tag{E.4}$$

where

$$\mathrm{vol}\,(AdS_4) = \cosh^2 u \sinh u \sinh\rho\, d\psi\wedge d\rho\wedge du\wedge d\phi\,, \tag{E.5}$$

$$\begin{aligned} J = & -2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\,d\alpha\wedge\left(d\chi - \left(1-\cos\vartheta_1\right)\,d\varphi_1 + \left(1-\cos\vartheta_2\right)\,d\varphi_2\right) \\ & -2\cos^2\frac{\alpha}{2}\sin\vartheta_1\,d\vartheta_1\wedge d\varphi_1 - 2\sin^2\frac{\alpha}{2}\sin\vartheta_2\,d\vartheta_1\wedge d\varphi_2\,. \end{aligned} \tag{E.6}$$

The factor of i in $F_{(4)}$ is due to the Euclidean continuation. The 2-form is proportional to the Kahler form in $\mathbb{CP}^3$.

Target space indices are labeled by $m, n, \ldots\ldots$, worldvolume indices are $a, b, \ldots.$, directions orthogonal to the string are denoted by $i, j, \ldots.$ The corresponding target space indices are underlined.

The choice of adapted $EAdS_4\times\mathbb{CP}^3$ vielbein $E^{\underline{m}} = \left(E^{\underline{a}}, E^{\underline{i}}\right)$ is

$$\begin{aligned}
E^{\underline{0}} &= L\,A^{-\frac{1}{2}}\left(\cosh^2 u\sinh^2\rho\,\dot{\psi}\,d\psi + \cos^2\frac{\alpha}{2}\sin^2\vartheta_1\,\dot{\varphi}_1\,d\varphi_1\right)\,,\\
E^{\underline{1}} &= L\,B^{-\frac{1}{2}}\left(\cosh^2 u\,\rho'\,d\rho + \cos^2\frac{\alpha}{2}\,\vartheta_1'\,d\vartheta_1\right)\,,\\
E^{\underline{2}} &= L\,du\,,\\
E^{\underline{3}} &= L\,\sinh u\,d\phi\,,
\end{aligned}$$

$$\begin{pmatrix} E^{\underline{4}} \\ E^{\underline{5}} \end{pmatrix} = \begin{pmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{pmatrix}\begin{pmatrix} L\,B^{-\frac{1}{2}}\cosh u\cos\frac{\alpha}{2}\left(\rho'\,d\vartheta_1 - \vartheta_1'\,d\rho\right) \\ L\,A^{-\frac{1}{2}}\cosh u\sinh\rho\cos\frac{\alpha}{2}\sin\vartheta_1\left(\dot{\psi}\,d\varphi_1 - \dot{\varphi}_1\,d\psi\right)\end{pmatrix},$$

$$\begin{aligned}
E^{\underline{6}} &= L\,\sin\frac{\alpha}{2}\,d\vartheta_2\,,\\
E^{\underline{7}} &= L\,\sin\frac{\alpha}{2}\sin\vartheta_2\,d\varphi_2\,,\\
E^{\underline{8}} &= L\,d\alpha\,,\\
E^{\underline{9}} &= L\,\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\left(d\chi - \left(1-\cos\vartheta_1\right)\,d\varphi_1 + \left(1-\cos\vartheta_2\right)\,d\varphi_2\right)
\end{aligned} \tag{E.7}$$

where

$$A(u,\rho,\alpha,\vartheta_1) = \cosh^2 u \sinh^2 \rho\, \dot{\psi}^2 + \cos^2 \frac{\alpha}{2} \sin^2 \vartheta_1\, \dot{\varphi}_1{}^2\,,$$
$$B(u,\rho,\alpha,\vartheta_1) = \cosh^2 u\, \rho'^2 + \cos^2 \frac{\alpha}{2}\, \vartheta_1'^2\,. \tag{E.8}$$

Here $\dot{\psi} = \frac{d\psi}{d\tau}$ and $\dot{\varphi}_1 = \frac{d\varphi_1}{d\tau}$ are constant numbers while $\rho' = \frac{d\rho}{d\sigma}$ and $\vartheta_1' = \frac{d\vartheta_1}{d\sigma}$ are understood as functions of ρ and ϑ_1, respectively. Also, Δ is an arbitrary function of ψ and φ_1 describing and $SO(2)$ rotation of the canonical frames and it is to be chosen at our convenience. The standard $EAdS_4 \times \mathbb{CP}^3$ vielbein is recovered for $\rho' = 1$, $\vartheta_1' = 0$, $\dot{\psi} = 1$ and $\dot{\varphi}_1 = 0$, and $\Delta = 0$. For the 1/6-BPS solution, $\rho' = -\sinh\rho$, $\vartheta_1' = -\sin\vartheta_1$ and $\dot{\psi} = \dot{\varphi}_1 = 1$. The standard and the adapted vielbein are related by the local Lorentz transformation

$$S = e^{\Delta J_{\underline{45}}} e^{a J_{\underline{05}}} e^{b J_{\underline{14}}}\,,$$

where

$$\cos a = \frac{\cosh u \sinh\rho\, \dot{\psi}}{\sqrt{A}}\,, \qquad \sin a = \frac{\cos\frac{\alpha}{2} \sin\vartheta_1\, \dot{\varphi}_1}{\sqrt{A}}\,, \tag{E.9}$$
$$\cos b = \frac{\cosh u\, \rho'}{\sqrt{B}}\,, \qquad \sin b = \frac{\cos\frac{\alpha}{2}\, \vartheta_1'}{\sqrt{B}}\,. \tag{E.10}$$

Notice that for $\rho' = -\dot{\psi}\sinh\rho$ and $\vartheta_1' = -\dot{\varphi}_1 \sin\vartheta_1$ we have

$$b = a + \pi\,. \tag{E.11}$$

For reasons to be explained below, we shall set Δ such that $\Delta = \tau$ on the worldsheet.

The adapted vielbein has the desired property that upon taking the pullback onto the worldsheet

$$P[E^{\underline{a}}] = e^{\underline{a}}\,, \qquad \underline{a} = 0, 1\,, \tag{E.12}$$
$$P[E^{\underline{i}}] = 0\,, \qquad \underline{i} = 2, \ldots, 9\,, \tag{E.13}$$

where

$$e^{\underline{0}} = \sqrt{A}\, d\tau\,, \quad e^{\underline{1}} = \sqrt{A}\, d\sigma\,, \tag{E.14}$$

is a vielbein for the induced geometry

$$ds^2_{\text{ind}} = A\left(d\tau^2 + d\sigma^2\right) . \tag{E.15}$$

The conformal factor reads

$$A(\sigma) = \sinh^2\rho + \sin^2\vartheta_1 = \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma+\sigma_0)} . \tag{E.16}$$

The worldsheet spin connection, the extrinsic curvature and the normal bundle gauge fields are given by, respectively,

$$w^{\underline{ab}} = P[\Omega^{\underline{ab}}] , \qquad H^{\underline{i}}_{\;ab} = P[\Omega^{\underline{i}}{}_{\underline{a}}]_a e^{\underline{a}}{}_b , \qquad \mathcal{A}^{\underline{ij}} = P[\Omega^{\underline{ij}}] , \tag{E.17}$$

where $\Omega^{\underline{mn}}$ is the target space spin connection. For the $\frac{1}{6}$-BPS string we find

$$w^{\underline{01}} = \frac{A'}{2A}\, d\tau \equiv w\, d\tau , \tag{E.18}$$

$$\mathcal{A}^{\underline{45}} = \frac{\cosh\rho\cos\vartheta_1 + 1}{\cosh\rho + \cos\vartheta_1}\, d\tau - P[d\Delta] = \left(\tanh(2\sigma+\sigma_0) - \dot{\Delta}\right) d\tau , \tag{E.19}$$

$$\mathcal{A}^{\underline{67}} = \frac{1}{2}\left(1-\cos\vartheta_1\right) d\tau = \frac{1}{2}\left(1-\tanh(\sigma+\sigma_0)\right) d\tau , \tag{E.20}$$

$$\mathcal{A}^{\underline{89}} = \frac{1}{2}\left(1-\cos\vartheta_1\right) d\tau = \frac{1}{2}\left(1-\tanh(\sigma+\sigma_0)\right) d\tau , \tag{E.21}$$

and

$$H^{\underline{4}}{}_a{}^b = \frac{m}{\sqrt{A}}\begin{pmatrix} -\cos\Delta & \sin\Delta \\ \sin\Delta & \cos\Delta \end{pmatrix} , \qquad H^{\underline{5}}{}_a{}^b = \frac{m}{\sqrt{A}}\begin{pmatrix} \sin\Delta & \cos\Delta \\ \cos\Delta & -\sin\Delta \end{pmatrix} , \tag{E.22}$$

where

$$m = \frac{\sinh\rho\sin\vartheta_1}{\cosh\rho - \cos\vartheta_1} = \frac{1}{\cosh(2\sigma+\sigma_0)} . \tag{E.23}$$

For the purpose of computing the spectrum of fluctuations we will chose Δ such that

$$P[d\Delta] = d\tau \qquad (\text{e.g. } \Delta = \psi) . \tag{E.24}$$

The reason for this choice is that the gauge fields

$$\mathcal{A} \equiv \mathcal{A}^{\underline{45}} = (\tanh(2\sigma+\sigma_0) - 1)\ d\tau\,, \tag{E.25}$$

$$\mathcal{B} \equiv \mathcal{A}^{\underline{67}} = \mathcal{A}^{\underline{89}} = \frac{1}{2}\,(1 - \tanh(\sigma+\sigma_0))\ d\tau\,, \tag{E.26}$$

are then regular at the center of the disk $\sigma \to \infty$, where the 1-form $d\tau$ is not well defined. Indeed[1] $\mathcal{A} \sim e^{-4\sigma}$ and $\mathcal{B} \sim e^{-2\sigma}$ as $\sigma \to \infty$. They also vanish in the 1/2-BPS limit $\sigma_0 \to \infty$. Notice that

$$w - \mathcal{A} = 1 - \cosh\rho - \cos\vartheta_1\,, \qquad \partial_\sigma \mathcal{A} = 2m^2\,, \qquad \partial_\sigma \mathcal{B} = -\frac{1}{2}\sin^2\vartheta_1\,. \tag{E.27}$$

These relations prove to be useful when casting the equations of motion in a simple form.

Finally, the contractions involving the Riemann tensor that we need are

$$\delta^{\underline{ab}} R_{\underline{aibj}} = \begin{cases} -\dfrac{2\sinh^2\rho}{A} & \underline{i}=\underline{j}=\underline{2},\underline{3} \\[2ex] \dfrac{\sin^2\vartheta_1}{2A} & \underline{i}=\underline{j}=\underline{6},\underline{7},\underline{8},\underline{9} \\[2ex] 0 & \text{otherwise} \end{cases} . \tag{E.28}$$

It is useful to invert the vielbein in order to write the RR fields that enter in the spinor action and Killing equation. We will set $\Delta = 0$ in this computation and then argue that some of the results do not depend on Δ. For generality we leave ρ', ϑ_1', $\dot\psi$ and $\dot\varphi_1$ arbitrary.

[1] Near the center of the disk the metric becomes $ds^2 = dr^2 + r^2 d\tau^2$, with $r = 2e^{-\sigma}\sqrt{1+e^{-2\sigma_0}}$. Regularity of the gauge fields requires that $d\tau$ be multiplied by r^n, $n \geq 2$, as $r \to 0$.

We have,

$$\cosh u \sinh\rho\, d\psi = \frac{1}{L\sqrt{A}}\left(\cosh u\sinh\rho\,\dot{\psi}\,E^{\underline{0}} - \cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,E^{\underline{5}}\right)\,, \tag{E.29}$$

$$\cos\frac{\alpha}{2}\sin\vartheta_1\, d\varphi_1 = \frac{1}{L\sqrt{A}}\left(\cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,E^{\underline{0}} + \cosh u\sinh\rho\,\dot{\psi}\,E^{\underline{5}}\right)\,, \tag{E.30}$$

$$\cosh u\, d\rho = \frac{1}{L\sqrt{B}}\left(\cosh u\,\rho'\,E^{\underline{1}} - \cos\frac{\alpha}{2}\,\vartheta_1'\,E^{\underline{4}}\right)\,, \tag{E.31}$$

$$\cos\frac{\alpha}{2}\, d\vartheta_1 = \frac{1}{L\sqrt{B}}\left(\cos\frac{\alpha}{2}\,\vartheta_1'\,E^{\underline{1}} + \cosh u\,\rho'\,E^{\underline{4}}\right)\,. \tag{E.32}$$

These relations imply that

$$\begin{aligned}
F_{(4)} &= -\frac{3ik}{2L^2\sqrt{AB}}\left(\cosh u\sinh\rho\,\dot{\psi}\,E^{\underline{0}} - \cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,E^{\underline{5}}\right)\wedge\left(\cosh u\,\rho'\,E^{\underline{1}} - \cos\frac{\alpha}{2}\,\vartheta_1'\,E^{\underline{4}}\right)\\
&\quad\wedge E^{\underline{2}}\wedge E^{\underline{3}}\,,\\
F_{(2)} &= -\frac{k}{2L^2\sqrt{AB}}\left(-\left(\cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,E^{\underline{0}} + \cosh u\sinh\rho\,\dot{\psi}\,E^{\underline{5}}\right)\wedge\left(\cos\frac{\alpha}{2}\,\vartheta_1'\,E^{\underline{1}} + \cosh u\,\rho'\,E^{\underline{4}}\right)\right.\\
&\quad\left.+\sqrt{AB}\left(E^{\underline{6}}\wedge E^{\underline{7}} + E^{\underline{8}}\wedge E^{\underline{9}}\right)\right)\,,
\end{aligned} \tag{E.33}$$

which allows us to compute the following quantities needed for the fermionic fluctuations:

$$\begin{aligned}
\not{F}_{(4)} &= -\frac{3ik}{2L^2\sqrt{AB}}\left(\cosh u\sinh\rho\,\dot{\psi}\,\Gamma^{\underline{0}} - \cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,\Gamma^{\underline{5}}\right)\left(\cosh u\,\rho'\,\Gamma^{\underline{1}} - \cos\frac{\alpha}{2}\,\vartheta_1'\,\Gamma^{\underline{4}}\right)\Gamma^{\underline{23}}\\
\not{F}_{(2)} &= -\frac{k}{2L^2\sqrt{AB}}\left(-\left(\cos\frac{\alpha}{2}\sin\vartheta_1\,\dot{\varphi}_1\,\Gamma^{\underline{0}} + \cosh u\sinh\rho\,\dot{\psi}\,\Gamma^{\underline{5}}\right)\left(\cos\frac{\alpha}{2}\,\vartheta_1'\,\Gamma^{\underline{1}} + \cosh u\,\rho'\,\Gamma^{\underline{4}}\right)\right.\\
&\quad\left.+\sqrt{AB}\left(\Gamma^{\underline{67}} + \Gamma^{\underline{89}}\right)\right)\,,
\end{aligned} \tag{E.34}$$

and

$$\begin{aligned}
\frac{1}{8}e^{\Phi}\Gamma^{\underline{a}}\not{F}_{(4)}\Gamma_{\underline{a}} &= \frac{3i}{4L\sqrt{AB}}\left(\cosh^2 u\sinh\rho\,\rho'\,\dot{\psi}\,\Gamma^{\underline{01}} + \cos^2\frac{\alpha}{2}\sin\vartheta_1\,\vartheta_1'\,\dot{\varphi}_1\Gamma^{\underline{45}}\right)\Gamma^{\underline{23}}\,,\\
\frac{1}{8}e^{\Phi}\Gamma^{\underline{a}}\not{F}_{(2)}\Gamma_{11}\Gamma_{\underline{a}} &= \frac{1}{4L\sqrt{AB}}\left(\cos^2\frac{\alpha}{2}\sin\vartheta_1\vartheta_1'\,\dot{\varphi}_1\,\Gamma^{\underline{01}} + \cosh^2 u\sinh\rho\,\rho'\,\dot{\psi}\,\Gamma^{\underline{45}}\right.\\
&\quad\left.+\sqrt{AB}\left(\Gamma^{\underline{67}} + \Gamma^{\underline{89}}\right)\right)\Gamma_{11}\,.
\end{aligned} \tag{E.35}$$

On the $\frac{1}{6}$-BPS solution the fermionic mass term becomes

$$\frac{1}{8}e^{\Phi}\Gamma^{\underline{a}}\left(\not{F}_{(2)}\Gamma_{11}+\not{F}_{(4)}\right)\Gamma_{\underline{a}}=\frac{1}{4LA}\left(\sinh^2\rho\left(-3i\Gamma^{\underline{0123}}+\left(-\Gamma^{\underline{45}}+\Gamma^{\underline{67}}+\Gamma^{\underline{89}}\right)\Gamma_{11}\right)\right.$$
$$\left.+\sin^2\vartheta_1\left(-3i\Gamma^{\underline{2345}}+\left(-\Gamma^{\underline{01}}+\Gamma^{\underline{67}}+\Gamma^{\underline{89}}\right)\Gamma_{11}\right)\right)$$

Notice that only quantities that are invariant under rotations in the $4-5$, $6-7$ and $8-9$ planes appear in the last two expressions. Therefore, these are also valid for arbitrary choices of Δ. In particular, they hold in the rotated frame where the connections are regular.

APPENDIX F

Regular gauge fields and spinors

The discussion about the regularity of the gauge fields is important because it is coupled to the periodicity of the fields. On general grounds, we expect regular bosonic/fermionic fields to be periodic/anti-periodic. Since a gauge transformation can change the periodicity of the fields, we must make sure that we are working in a regular gauge when we Fourier expand.

Let us see how the analysis of regularity works out in the present case. The wordsheet metric is

$$ds^2 = A(\sigma)\left(d\tau^2 + d\sigma^2\right) , \qquad A(\sigma) = \sinh^2\rho(\sigma) + \sin^2\vartheta_1(\sigma) , \tag{F.1}$$

where the functions $\rho(\sigma)$ and $\vartheta_1(\sigma)$ are defined by

$$\sinh\rho = \frac{1}{\sinh\sigma} , \qquad \sin\vartheta_1 = \frac{1}{\cosh(\sigma+\sigma_0)} . \tag{F.2}$$

The topology is that of a disk with $0 < \sigma$ and $\tau \sim \tau + 2\pi$. The center of the disk is $\sigma \to \infty$ where the geometry is flat. To see this, expand near $\sigma = \infty$ to get

$$ds^2 \approx 4e^{-2\sigma}\left(1 + e^{-2\sigma_0}\right)\left(d\tau^2 + d\sigma^2\right) . \tag{F.3}$$

Now let

$$r = 2e^{-\sigma}\sqrt{1 + e^{-2\sigma_0}}\,. \tag{F.4}$$

Then,

$$ds^2 \approx dr^2 + r^2 d\tau^2\,. \tag{F.5}$$

This is flat space indeed.

Switching to Cartesian coordinates we have

$$x = r\cos\tau\,, \qquad y = r\sin\tau\,. \tag{F.6}$$

The 1-forms transform accordingly:

$$dr = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}\,, \qquad d\tau = \frac{-ydx + xdy}{x^2 + y^2}\,, \tag{F.7}$$

The important fact to remember is that the coordinates (x, y), as well as the 1-forms dx and dy are everywhere well defined. Notice then that neither dr nor $d\tau$ are well defined as $r \to 0$, but the combination $dr^2 + r^2 d\tau^2$ is. Also, the 1-form rdr is well defined as $r \to 0$ with $rdr \to 0$. In contrast,

$$rd\tau = \frac{-ydx + xdy}{\sqrt{x^2 + y^2}}\,, \tag{F.8}$$

is ill-defined as $r \to 0$ since the value of the limit depends on the direction in which we approach the origin. This means that only 1-forms involving the combinations

$$r^n d\tau\,, \qquad n \geq 2\,, \tag{F.9}$$

are well defind at $r = 0$, where they vanish.

Going back to the worldsheet, the above discussion means that the 1-form $d\tau$ must appear as

$$e^{-n\sigma} d\tau\,, \qquad n \geq 2\,, \tag{F.10}$$

in the gauge fields. In our case we find that

$$\mathcal{A} = \tanh(2\sigma + \sigma_0)d\tau \approx \left(1 - 2e^{-4\sigma - 2\sigma_0}\right) d\tau \,, \tag{F.11}$$

$$\mathcal{B} = -\frac{1}{2}\tanh(\sigma + \sigma_0)d\tau \approx \left(-\frac{1}{2} + e^{-2\sigma - 2\sigma_0}\right) d\tau \,, \tag{F.12}$$

where we have expanded at large σ. We see that these gauge fields are not regular at the center of the disk. However, after a gauge transformation we have

$$\mathcal{A} = \left(\tanh(2\sigma + \sigma_0) - 1\right) d\tau \approx -2e^{-4\sigma - 2\sigma_0} d\tau \,, \tag{F.13}$$

$$\mathcal{B} = -\frac{1}{2}\left(\tanh(\sigma + \sigma_0) - 1\right) d\tau \approx e^{-2\sigma - 2\sigma_0} d\tau \,. \tag{F.14}$$

These gauge fields are then regular.

APPENDIX G

Dimensional reduction of spinors

Given the symmetries of our problem, the natural way to decompose the ten-dimensional rotations group is

$$SO(10) \supset \underbrace{SO(2)}_{\gamma} \times \underbrace{SO(2)}_{\rho} \times \underbrace{SO(2)}_{\tau} \times \underbrace{SO(2)}_{\lambda} \times \underbrace{SO(2)}_{\kappa}, \tag{G.1}$$

corresponding to the $(0,1)$, $(2,3)$, $(4,5)$, $(6,7)$ and $(8,9)$ tangent directions, respectively. Under this decomposition, a possible representation of the 10-dimensional gamma matrices is

$$\begin{aligned}
\Gamma_{\underline{a}} &= \gamma_{\underline{a}} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,, && \underline{a} = 0,\,1\,,\\
\Gamma_{\underline{i}} &= \left(-i\gamma_{\underline{01}}\right) \otimes \rho_{\underline{i}} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,, && \underline{i} = 2,\,3\,,\\
\Gamma_{\underline{i}} &= \left(-i\gamma_{\underline{01}}\right) \otimes \left(-i\rho_{\underline{23}}\right) \otimes \tau_{\underline{i}} \otimes \mathbb{1} \otimes \mathbb{1}\,, && \underline{i} = 4,\,5\,,\\
\Gamma_{\underline{i}} &= \left(-i\gamma_{\underline{01}}\right) \otimes \left(-i\rho_{\underline{23}}\right) \otimes \left(-i\tau_{\underline{45}}\right) \otimes \lambda_{\underline{i}} \otimes \mathbb{1}\,, && \underline{i} = 6,\,7\,,\\
\Gamma_{\underline{i}} &= \left(-i\gamma_{\underline{01}}\right) \otimes \left(-i\rho_{\underline{23}}\right) \otimes \left(-i\tau_{\underline{45}}\right) \otimes \left(-i\lambda_{\underline{67}}\right) \otimes \kappa_{\underline{i}}\,, && \underline{i} = 8,\,9\,,
\end{aligned} \tag{G.2}$$

where we named the Dirac matrices associated to each factor as displayed above. This basis is tailored for the choice (σ_1, σ_2, σ_3 are Pauli matrices)

$$\gamma_{\underline{0}} = \rho_{\underline{2}} = \tau_{\underline{4}} = \lambda_{\underline{6}} = \kappa_{\underline{8}} = \sigma_1\,, \qquad \gamma_{\underline{1}} = \rho_{\underline{3}} = \tau_{\underline{5}} = \lambda_{\underline{7}} = \kappa_{\underline{9}} = \sigma_2\,. \tag{G.3}$$

The chirality operator is then

$$\begin{aligned}\Gamma_{11} &\equiv -i\Gamma_{\underline{0123456789}} \\ &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \,,\end{aligned} \tag{G.4}$$

and the charge conjugation intertwiners $C_\pm$ become

$$\begin{aligned}C_+ &= \Gamma_{\underline{02468}} \\ &= \sigma_1 \otimes (-i\sigma_2) \otimes \sigma_1 \otimes (-i\sigma_2) \otimes \sigma_1\end{aligned} \,, \qquad \begin{aligned}C_- &= \Gamma_{\underline{13579}} \\ &= \sigma_2 \otimes (i\sigma_1) \otimes \sigma_2 \otimes (i\sigma_1) \otimes \sigma_2\end{aligned} \,. \tag{G.5}$$

A 10-dimensional spinor can be decomposed in terms of 2-dimensional ones as

$$\psi = \sum_{s_i=\pm} \psi_{s_2 s_4 s_6 s_8} \otimes \eta_{s_2} \otimes \eta_{s_4} \otimes \eta_{s_6} \otimes \eta_{s_8} \,, \tag{G.6}$$

where

$$\eta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \qquad \eta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,. \tag{G.7}$$

This provides and explicit projection onto $\Gamma^{\underline{23}}$, $\Gamma^{\underline{45}}$, $\Gamma^{\underline{67}}$ and $\Gamma^{\underline{89}}$ eigenspaces, with corresponding eigenvalues $-i\alpha\beta\gamma$, $i\alpha$, $i\beta$ and $i\gamma$ which we use in the main body of the text[1].

The Majorana conjugate is

$$\overline{\psi}^M = \psi^T C_+ \tag{G.8}$$

$$= \sum_{s_i=\pm} s_2 s_6 \, \overline{\psi}^M_{s_2 s_4 s_6 s_8} \otimes \eta^T_{-s_2} \otimes \eta^T_{-s_4} \otimes \eta^T_{-s_6} \otimes \eta^T_{-s_8} \,, \tag{G.9}$$

with

$$\overline{\psi}^M_{s_2 s_4 s_6 s_8} \equiv \psi^T_{s_2 s_4 s_6 s_8} \sigma_1 \,. \tag{G.10}$$

[1]The κ-symmetry fixing in Euclidean language is $i\Gamma_{\underline{01}}\Gamma_{11}\theta = \theta$, where $\Gamma_{11} = -i\Gamma_{\underline{0123456789}}$. This translates to $\Gamma^{\underline{23}}\theta = -i\alpha\beta\gamma\theta$.

Thus, Majorana spinors satisfy

$$\psi^\dagger = \overline{\psi}^M \qquad \Longleftrightarrow \qquad s_2 s_6\, \overline{\psi}^M_{s_2 s_4 s_6 s_8} = {\psi^\dagger}_{-s_2 -s_4 -s_6 -s_8}\,. \tag{G.11}$$

APPENDIX H

Representations of $\mathbb{C}P^n$

Our starting point is a recursion formula for unit $\mathbb{C}P^n$ spaces [171]. In that paper, unit $\mathbb{C}P^n$ is defined as the $\mathbb{C}P^n$ space that arises from the Hopf fibration of a unit S^{2n+1}. Hence, unit $\mathbb{C}P^1$ is a 2-sphere of radius $\frac{1}{2}$. Let $d\hat{\Sigma}_n$ and $\hat{J}_m = \frac{1}{2}d\hat{A}_m$ be the line element and the Kähler form of unit $\mathbb{C}P^n$, respectively. Then, for any m and n, the following formulas hold [171],

$$d\hat{\Sigma}^2_{m+n+1} = d\xi^2 + c^2 d\hat{\Sigma}^2_m + s^2 d\hat{\Sigma}^2_n + c^2 s^2 (d\psi + \hat{A}_m - \hat{A}_n)^2 \ , \tag{H.1}$$

$$\hat{A}_{m+n+1} = c^2 \hat{A}_m + s^2 \hat{A}_n + \frac{1}{2}(c^2 - s^2) d\psi \ , \tag{H.2}$$

where $c = \cos\xi$, $s = \sin\xi$, $\xi \in (0, \pi/2)$, $\psi \in (0, 2\pi)$.

In the present paper, we deal with $\mathbb{C}P^n$ spaces with line elements $d\Sigma_n = 2d\hat{\Sigma}_n$. Let us call these unit-2 $\mathbb{C}P^n$ spaces, because they arise from the Hopf fibration of an S^{2n+1} of radius 2. Therefore, unit-2 $\mathbb{C}P^1$ is just a unit S^2. Let $d\Sigma_n = 2d\hat{\Sigma}_n$, $A_n = 2\hat{A}_n$ and introduce two new angles by $\alpha = 2\xi \in (0, \pi)$, $\chi = 2\psi \in (0, 4\pi)$. In terms of these, (H.1) and (H.2) become

$$d\Sigma^2_{m+n+1} = d\alpha^2 + c^2 d\Sigma^2_m + s^2 d\Sigma^2_n + c^2 s^2 (d\chi + A_m - A_n)^2 \ , \tag{H.3}$$

$$A_{m+n+1} = c^2 A_m + s^2 A_n + \frac{1}{2}(c^2 - s^2) d\chi \ , \tag{H.4}$$

where

$$c = \cos\frac{\alpha}{2}\ , \qquad s = \sin\frac{\alpha}{2}\ . \tag{H.5}$$

The Kähler form of unit-2 $\mathbb{C}P^n$ is $J_n = 4\hat{J}_n = 2d\hat{A}_n = dA_n$, i.e., there is no factor of 2 now. Explicitly, from (H.4),

$$J_{m+n+1} = c^2 J_m + s^2 J_n - cs d\alpha \wedge (d\chi + A_m - A_n)\ . \tag{H.6}$$

With the help of the above formulas we can recursively construct various coordinate systems of unit-2 $\mathbb{C}P^n$. One starts with the unit-2 $\mathbb{C}P^1$, which is a unit 2-sphere,

$$d\Sigma_1^2 = d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2\ , \quad A_1 = \cos\vartheta d\varphi\ , \quad J_1 = -\sin\vartheta d\vartheta \wedge d\varphi\ . \tag{H.7}$$

$\mathbb{C}P^2$ is obtained for $m = 1$, $n = 0$,[1]

$$d\Sigma_2^2 = d\alpha^2 + \cos^2\frac{\alpha}{2}d\Omega^2 + \cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{2}(d\chi + \cos\vartheta d\varphi)^2\ , \tag{H.8}$$

$$A_2 = \cos^2\frac{\alpha}{2}\cos\vartheta d\varphi + \frac{1}{2}\cos\alpha d\chi\ . \tag{H.9}$$

For $\mathbb{C}P^3$, one has two choices. One is $m = n = 1$, which yields the representation used in [140].

$$d\Sigma_3^2 = d\alpha^2 + \cos^2\frac{\alpha}{2}d\Omega_1^2 + \sin^2\frac{\alpha}{2}d\Omega_2^2 + \cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{2}(d\chi + \cos\vartheta_1 d\varphi_1 - \cos\vartheta_2 d\varphi_2)^2\ , \tag{H.10}$$

$$A_3 = \cos^2\frac{\alpha}{2}\cos\vartheta_1 d\varphi_1 + \sin^2\frac{\alpha}{2}\cos\vartheta_2 d\varphi_2 + \frac{1}{2}\cos\alpha d\chi\ . \tag{H.11}$$

The other choice is $m = 2$, $n = 0$, which gives

$$d\Sigma_3^2 = d\alpha^2 + \cos^2\frac{\alpha}{2}d\Sigma_2^2 + \cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{2}(d\chi + A_2)^2\ , \tag{H.12}$$

$$A_3 = \cos^2\frac{\alpha}{2}A_2 + \frac{1}{2}\cos\alpha d\chi\ . \tag{H.13}$$

[1] The alternative $m = 0$, $n = 1$ is equivalent by a change of coordinate $\alpha \to \pi - \alpha$.

As a corollary of the recursion formula with $n = 0$ one easily derives the volume of the unit-2 $\mathbb{C}P^n$,

$$V_n = \frac{(4\pi)^n}{n!} \ . \tag{H.14}$$

APPENDIX I

Representations of $OSp(4|2)$

The supergroup $OSp(4|2)$ with bosonic subgroup $Sp(2)$ and $SO(4)$ is the relevant supergroup for the classification of 1/3 BPS states in ABJM theory, i.e., of states that preserve 8 supercharges. The representation theory of this supergroup has been discussed in various articles. Some key general remarks on the construction of unitary super $OSp(2N|2)$ representations were given, for example, in [167]. A dedicated publication to the representations of $OSp(4|2)$ appeared, for example, in [172]. The key quantum nubers arise from the following embedding and isomorphism:

$$OSp(4|2,\mathbb{R}) \supset \; Sp(2,\mathbb{R}) \times SO(4) \cong \; Sp(2,\mathbb{R}) \times SO(3) \times SO(3) \; . \tag{I.1}$$

We can relate the $SO(4)$ labels (p_1, p_2) to $SO(3) \times SO(3)$ labels (j, l),

$$j = \frac{1}{2}(p_1 + p_2), \qquad l = \frac{1}{2}(p_1 - p_2) \; . \tag{I.2}$$

The irreducible representations of $OSp(4|2)$ are as follows, with the conditions for the existence of each multiplet given below the corresponding labels (we quote from [172]):

$$
\begin{aligned}
&(h,j,l) \\
&\oplus \underset{2h-j-l\neq 0}{(h+\tfrac{1}{2},j+\tfrac{1}{2},l+\tfrac{1}{2})} \oplus \underset{l\neq 0}{(h+\tfrac{1}{2},j+\tfrac{1}{2},l-\tfrac{1}{2})} \oplus \underset{j\neq 0}{(h+\tfrac{1}{2},j-\tfrac{1}{2},l+\tfrac{1}{2})} \oplus \underset{j\neq 0,l\neq 0}{(h+\tfrac{1}{2},j-\tfrac{1}{2},l-\tfrac{1}{2})} \\
&\oplus \underset{2h-j-l\neq 0}{(h+1,j+1,l)} \oplus \underset{j\neq 0,2h-j-l\neq 0}{(h+1,j,l)} \oplus \underset{j\neq 0,\frac{1}{2}}{(h+1,j-1,l)} \\
&\oplus \underset{2h-j-l\neq 0}{(h+1,j,l+1)} \oplus \underset{l\neq 0,2h+j-l\neq 0}{(h+1,j,l)} \oplus \underset{l\neq 0,\frac{1}{2}}{(h+1,j,l-1)} \\
&\oplus \underset{2h-j-l\neq 0}{(h+\tfrac{3}{2},j+\tfrac{1}{2},l+\tfrac{1}{2})} \oplus \underset{l\neq 0,2h+j-l\neq 0}{(h+\tfrac{3}{2},j+\tfrac{1}{2},l-\tfrac{1}{2})} \oplus \underset{j\neq 0,2h-j-l\neq 0}{(h+\tfrac{3}{2},j-\tfrac{1}{2},l+\tfrac{1}{2})} \oplus \underset{j\neq 0,l\neq 0}{(h+\tfrac{3}{2},j-\tfrac{1}{2},l-\tfrac{1}{2})} \\
&\oplus \underset{2h-j-l\neq 0}{(h+2,j,l)}
\end{aligned}
$$

This is the long multiplet in which we accommodated the spectrum of excitations of the D2 brane.

APPENDIX J

Harmonic Analysis on $\tilde{T}^{1,1}$

The field equations listed at the end of the previos section involve certain differential operators on the $\tilde{T}^{1,1}$ part of the D6-brane world volume. To deal with these operators, it is appropriate to view $\tilde{T}^{1,1}$ as a coset manifold [173, 165, 160], $\tilde{T}^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$, and to apply the powerful technique of *harmonic expansion* [174]. In this way, their spectrum is obtained in a purely algebraic fashion. The spectrum of Laplace-Beltrami operators on $\tilde{T}^{1,1}$ was found in [173, 165, 160], but the operators arising in our field equations are slightly different. To be self contained, we include a brief review of the geometry of coset manifolds. For a pedagogical introduction to the subject we refer to van Nieuwenhuizen's lectures [175]. Our signature and curvature conventions agree with those of [175]. In this section, our notation regarding indices is independent of the other sections.

J.0.1 Geometry of coset manifolds

Consider a Lie group G with a subgroup H and their respective Lie algebras $\mathbb{G}$ and $\mathbb{H}$. Decompose $\mathbb{G}$ into $\mathbb{G} = \mathbb{H} + \mathbb{K}$, such that, for the generators $T_a \in \mathbb{K}$ and $T_i \in \mathbb{H}$ and assuming $\mathbb{H}$ to be compact or semi-simple, the structure equations of $\mathbb{G}$ take the form

$$\begin{aligned}
[T_i, T_j] &= C_{ij}{}^k T_k \ , \\
[T_i, T_a] &= C_{ia}{}^b T_b \ , \\
[T_a, T_b] &= C_{ab}{}^c T_c + C_{ab}{}^i T_i \ .
\end{aligned} \tag{J.1}$$

Starting from any coset representative $L(x)$, define the Lie-algebra valued one-form

$$V(x) = L^{-1}(x)dL(x) = r(a)V^a(x)T_a + \Omega^i(x)T_i \ . \tag{J.2}$$

Here, V^a are the (rescaled) vielbeins, $r(a)$ denote scale factors, which are independent for each irreducible block of $C_{ia}{}^b$, and Ω^i are the H-connections. The Maurer-Cartan equation for V yields

$$dV^a + \frac{1}{2}\frac{r(b)r(c)}{r(a)}C_{bc}{}^a V^b \wedge V^c + C_{ib}{}^a \Omega^i \wedge V^b = 0 \ , \tag{J.3}$$

$$d\Omega^i + \frac{1}{2}r(a)r(b)C_{ab}{}^i V^a \wedge V^b + \frac{1}{2}C_{jk}{}^i \Omega^j \wedge \Omega^k = 0 \ . \tag{J.4}$$

Indices will be lowered and raised using a flat coset metric η_{ab} and its inverse η^{ab}, respectively. Later, we shall choose η_{ab} to be positive definite Euclidean, but for the time being it is sufficient to state that η^{ab} is pseudo-Euclidean with arbitrary signature.

The geometry of the coset manifold is characterized, as usual, by a torsionless connection defined by

$$dV^a + \mathcal{B}^a{}_b \wedge V^b = 0 \ , \qquad \mathcal{B}^{ab} = -\mathcal{B}^{ba} \ . \tag{J.5}$$

The Riemann curvature 2-form is

$$\mathcal{R}^a{}_b = d\mathcal{B}^a{}_b + \mathcal{B}^a{}_c \wedge \mathcal{B}^c{}_b \ . \tag{J.6}$$

Comparison of (J.3) and (J.5) yields

$$\mathcal{B}^a{}_b = \frac{1}{2}\mathbb{C}_{cb}{}^a V^c + C_{ib}{}^a \Omega^i \ , \tag{J.7}$$

where

$$\mathbb{C}_{cb}{}^a = \frac{r(b)r(c)}{r(a)}C_{cb}{}^a + \frac{r(a)r(c)}{r(b)}C^a{}_{cb} + \frac{r(a)r(b)}{r(c)}C^a{}_{bc} \ . \tag{J.8}$$

The $SO(d)$ covariant derivative is defined by

$$D = d + \frac{1}{2}\mathcal{B}^{ab}\mathbb{D}(T_{ab}) \ , \tag{J.9}$$

where $\mathbb{D}$ is a representation of $SO(d)$ satisfying

$$[\mathbb{D}(T_{ab}), \mathbb{D}(T_{cd})] = \eta_{bc}\mathbb{D}(T_{ad}) + \eta_{ad}\mathbb{D}(T_{bc}) - \eta_{ac}\mathbb{D}(T_{bd}) - \eta_{bd}\mathbb{D}(T_{ac}) \ . \tag{J.10}$$

A coset harmonic is given, in an arbitrary representation of G, by the inverse of a coset representative,

$$Y(x) = L^{-1}(x) \ . \tag{J.11}$$

By definition, it satisfies

$$dY = -VY = -\left[r(a)V^a T_a + \Omega^i(x) T_i\right] Y \ , \tag{J.12}$$

where the algebra elements act on Y by *right* action. Y also forms a representation of $SO(d)$, if the action of T_i is given by

$$\left[T_i + \frac{1}{2}C_i{}^{ab}\mathbb{D}(T_{ab})\right] Y = 0 \ . \tag{J.13}$$

As a consequence, the covariant derivative (J.9) of an harmonic reduces to

$$DY = V^a D_a Y = -V^a \left[r(a)T_a + \frac{1}{4}\mathbb{C}_a{}^{bc}\mathbb{D}(T_{bc})\right] Y \ . \tag{J.14}$$

J.0.2 Geometry of $\tilde{T}^{1,1}$

Let us now apply these general results to $\tilde{T}^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$. Take T_1, T_2, T_3 and $T_{\hat{1}}, T_{\hat{2}}, T_{\hat{3}}$ to be the generators of the first and second $SU(2)$, respectively, let $i = 1, 2$, $\hat{i} = \hat{1}, \hat{2}$, and define

$$T_5 = T_3 - T_{\hat{3}} \ , \qquad T_H = T_3 + T_{\hat{3}} \ , \tag{J.15}$$

where T_H generates the $U(1)$. In this basis, the structure equations of $G = SU(2) \times SU(2)$ read

$$
\begin{aligned}
&[T_i, T_j] = \frac{1}{2}\epsilon_{ij}(T_H + T_5) \ , && \left[T_{\hat{i}}, T_{\hat{j}}\right] = \frac{1}{2}\epsilon_{\hat{i}\hat{j}}(T_H - T_5) \ , \\
&[T_H, T_i] = [T_5, T_i] = \epsilon_i{}^j T_j \ , && \left[T_H, T_{\hat{i}}\right] = -\left[T_5, T_{\hat{i}}\right] = \epsilon_{\hat{i}}{}^{\hat{j}} T_{\hat{j}} \ .
\end{aligned}
\tag{J.16}
$$

Defining the scale parameters of the irreducible blocks by

$$
r(i) = a \ , \qquad r(\hat{i}) = b \ , \qquad r(5) = c \ , \tag{J.17}
$$

the spin connections (J.7) are found as

$$
\begin{aligned}
B^{5i} &= \frac{a^2}{4c} V^j \epsilon_j{}^i \ , & B^{ij} &= -\epsilon^{ij}\left[\omega + \left(c - \frac{a^2}{4c}\right) V^5\right] \ , \\
B^{5\hat{i}} &= -\frac{b^2}{4c} V^{\hat{j}} \epsilon_{\hat{j}}{}^{\hat{i}} \ , & B^{\hat{i}\hat{j}} &= -\epsilon^{\hat{i}\hat{j}}\left[\omega - \left(c - \frac{b^2}{4c}\right) V^5\right] \ .
\end{aligned}
\tag{J.18}
$$

The Ricci tensor $R_{ab} = \mathcal{R}^c{}_{acb}$ turns out to be block-diagonal,

$$
R^i{}_j = \delta^i_j\left(a^2 - \frac{a^4}{8c^2}\right) \ , \quad R^{\hat{i}}{}_{\hat{j}} = \delta^{\hat{i}}_{\hat{j}}\left(b^2 - \frac{b^4}{8c^2}\right) \ , \quad R^5{}_5 = \frac{a^4 + b^4}{8c^2} \ . \tag{J.19}
$$

In is convenient to work in a complex basis, with

$$
x^{\pm} = \frac{1}{2}(x^1 \pm i x^2) \ , \qquad x^{\hat{\pm}} = \frac{1}{2}(x^{\hat{1}} \pm i x^{\hat{2}}) \ , \tag{J.20}
$$

such that the positive definite Euclidean metric η_{ab} is given by

$$
\eta_{+-} = \eta_{\hat{+}\hat{-}} = 2 \ , \qquad \eta_{55} = 1 \ , \tag{J.21}
$$

and the components of the ϵ tensors are

$$
\epsilon_{\pm}{}^{\pm} = \epsilon_{\hat{\pm}}{}^{\hat{\pm}} = \pm i \ . \tag{J.22}
$$

In this basis, the covariant derivatives (J.14) are given by

$$D_\pm = -aT_\pm \pm \frac{ia^2}{4c}\mathbb{D}(T_{5\pm}) \ , \tag{J.23}$$
$$D_{\hat\pm} = -bT_{\hat\pm} \mp \frac{ib^2}{4c}\mathbb{D}(T_{5\hat\pm}) \ ,$$
$$D_5 = -cT_5 + \frac{i}{2}\left(c - \frac{a^2}{4c}\right)\mathbb{D}(T_{+-}) - \frac{i}{2}\left(c - \frac{b^2}{4c}\right)\mathbb{D}(T_{\hat+\hat-}) \ .$$

A suitable representation (by right action) of the $SU(2)\times SU(2)$ generators is[1]

$$T_\pm Y_q^{j,l,r} = -i\left(j \pm \frac{q+r}{2}\right)Y_{q\mp1}^{j,l,r\mp1} \ , \tag{J.24}$$
$$T_{\hat\pm} Y_q^{j,l,r} = -i\left(l \pm \frac{q-r}{2}\right)Y_{q\mp1}^{j,l,r\pm1} \ ,$$
$$T_5 Y_q^{j,l,r} = irY_q^{j,l,r} \ ,$$
$$T_H Y_q^{j,l,r} = iqY_q^{j,l,r} \ .$$

J.0.3 Spectrum of operators on $\tilde T^{1,1}$

We are interested in the spectrum of the differential operators on $\tilde T^{1,1}$, which appear in the field equations listed in subsection 8.3.3. The scale parameters a, b and c are related to the angle α by

$$a^2 = \frac{1}{\cos^2\frac{\alpha}{2}} \ , \qquad b^2 = \frac{1}{\sin^2\frac{\alpha}{2}} \ , \qquad c^2 = \frac{1}{\sin^2\alpha} \ . \tag{J.25}$$

This leaves a sign ambiguity, which will be resolved shortly. Notice that (J.25) implies

$$a^2 + b^2 = 4c^2 \ , \tag{J.26}$$

which will simplify many expressions in the sequel.

Scalar fields Scalar fields transform trivially under $SO(d)$, which implies $q = 0$ by (J.13). Vectors (with covariant indices) transform under $\mathbb{D}(T_{ab})_c{}^d = \eta_{ac}\delta_b^d - \eta_{bc}\delta_a^d$. Notice that D_aY is a vector. We can now calculate the Laplacian $\Box_0 = D_aD^a$ of a scalar harmonic,

[1]Notice that the role of $T_\pm$ and $T_{\hat\pm}$ as $SU(2)$ raising and lowering operators is the opposite compared to what is indicated by their indices. This is a consequence of right action.

which results in

$$-\,\Box_0 Y_0^{j,l,r} = H_0 Y_0^{j,l,r}\ , \tag{J.27}$$

where

$$H_0 = a^2 j(j+1) + b^2 l(l+1) - \frac{r^2}{4}\left(a^2+b^2-4c^2\right)\ . \tag{J.28}$$

This is independent of r by virtue of (J.26). Using (J.25), let us rewrite it as

$$H_0 = c^2(C_{j,l}-1)\ , \qquad C_{j,l} = \sin^2\frac{\alpha}{2}(2j+1)^2 + \cos^2\frac{\alpha}{2}(2l+1)^2\ , \tag{J.29}$$

Because of the relations $q = m_3 + m_{\hat{3}} = 0$ and $r = m_3 - m_{\hat{3}}$, where m_3 and $m_{\hat{3}}$ are $SU(2)$ quantum numbers, it must hold that j and l are either both integer or half-integer. Accordingly, r is an even or odd integer with $|r| \le \bar{l} = 2\min(j,l)$.

The field equation (8.3.57) contains, however, the operator

$$-\,\Box_0' Y = \left(-\Box_0 \pm icD_5\right) Y\ , \tag{J.30}$$

where the sign depends on whether χ^+ or χ^- is considered (and on the still ambiguous sign of c). It is straightforward to obtain

$$-\,\Box_0' Y_0^{j,l,r} = \left(H_0 \pm c^2 r\right) Y_0^{j,l,r}\ . \tag{J.31}$$

Vector fields Consider vector fields with covariant indices. The Laplace-Beltrami operator is given by

$$-\,\Box_1 Y_a = \left(-\delta_a^b D_c D^c + R_a^b\right) Y_b\ , \tag{J.32}$$

From (J.13) and (J.24) we deduce that the components of Y_a must carry the follwing quantum numbers,

$$Y = \begin{pmatrix} Y_{\mp1}^{j,l,r\mp1} \\ Y_{\mp1}^{j,l,r\pm1} \\ Y_0^{j,l,r} \end{pmatrix}\ . \tag{J.33}$$

After evaluating the covariant derivatives and using (J.19), one obtains the matrix form

$$-\Box_1 Y_a = \begin{pmatrix} H_0 \pm \frac{a^2}{2} r & 0 & \pm\frac{a^3}{4c}(2j \pm r) \\ 0 & H_0 \mp \frac{b^2}{2} r & \mp\frac{b^3}{4c}(2l \mp r) \\ \pm\frac{a^3}{8c}(2j+2\mp r) & \mp\frac{b^3}{8c}(2l+2\pm r) & H_0 + \frac{a^4+b^4}{4c^2} \end{pmatrix} Y \ . \tag{J.34}$$

We remark that this result corrects some opf the results of Benincasa and Ramallo [160]. In fact, in contrast to what was found in [160], H_0 always is an eigenvalue of this matrix, belonging to the longitudinal vector $D_a Y_0^{j,l,r}$.

For the field equation (8.3.62) we need the operator

$$-\Box_1' Y_a = -\Box_1 Y_a - \cot\alpha\, \mathcal{E}_a{}^{cb} D_c Y_b \ . \tag{J.35}$$

Direct evaluation yields

$$\mathcal{E}_a{}^{cb} D_c Y_b = -\begin{pmatrix} \pm rc & 0 & \pm\frac{a}{2}(2j \pm r) \\ 0 & \pm rc & \pm\frac{b}{2}(2l \mp r) \\ \pm\frac{a}{4}(2j+2\mp r) & \pm\frac{b}{4}(2l+2\pm r) & \frac{a^2-b^2}{2c} \end{pmatrix} Y \tag{J.36}$$

The factor $\cot\alpha$ is determined (J.25) up to a sign, which is related to the (unfixed) frame orientation. One realizes that the terms in (J.34) and (J.36) combine very nicely (cancelling the asymmetries in a and b), if the sign is fixed such that[2]

$$c = \frac{1}{\sin\alpha} \qquad \Rightarrow \qquad \cot\alpha = \frac{b^2-a^2}{4c} \ . \tag{J.37}$$

Therefore, simplifying also by (J.26), we obtain

$$-\Box_1' Y_a = \begin{pmatrix} H_0 \pm rc^2 & 0 & \pm\frac{ac}{2}(2j \pm r) \\ 0 & H_0 \mp rc^2 & \mp\frac{bc}{2}(2l \mp r) \\ \pm\frac{ac}{4}(2j+2\mp r) & \mp\frac{bc}{4}(2l+2\pm r) & H_0 + 2c^2 \end{pmatrix} Y \ . \tag{J.38}$$

[2]In [165], the sign was fixed imposing supersymmetry on $\tilde{T}^{1,1}$. In our case $\tilde{T}^{1,1}$ is not Einstein, so there are no Killing spinors.

It is straightforward to calculate the eigenvalues of this matrix, but we have to be slightly more detailed in the analysis of the spectrum. The fact that each non-zero component of the vector (J.33) must be a valid representation of $SU(2) \times SU(2)$ poses a number of restrictions. As for scalar fields, j and l must both be integers or half-integers, with r even or odd, respectively. The restrictions on the range of r that arise from the non-zero vector components are summarized in Tab. J.1. The overall range of r for a given eigenvector is obtained as the intersection of all the restrictions, taking care of vanishing vector components. Our results for the eigenvectors, eigenvalues, and ranges of r are listed in Appendix J.0.4.

Table J.1: Restrictions on r for non-zero components of the vector (J.33).

component	$SU(2) \times SU(2)$ rep.	restrictions on r	
$+$	$Y_{-1}^{j,l,r-1}$	$-2j+2 \le r \le 2j+2$	$-2l \le r \le 2l$
$-$	$Y_{1}^{j,l,r+1}$	$-2j-2 \le r \le 2j-2$	$-2l \le r \le 2l$
$\hat{+}$	$Y_{-1}^{j,l,r+1}$	$-2j \le r \le 2j$	$-2l-2 \le r \le 2l-2$
$\hat{-}$	$Y_{1}^{j,l,r-1}$	$-2j \le r \le 2j$	$-2l+2 \le r \le 2l+2$
5	$Y_{0}^{j,l,r}$	$-2j \le r \le 2j$	$-2l \le r \le 2l$

Spinor fields In our conventions, the $SO(d)$ generators acting on spinors are $\mathbb{D}(T_{ab}) = \Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, where the Dirac matrices satisfy $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$. We choose them as

$$\gamma_i = \sigma_i \times \mathbb{I} \ , \qquad \gamma_{\hat{i}} = \sigma_3 \times \sigma_i \ , \qquad \gamma_5 = \sigma_3 \times \sigma_3 \ . \tag{J.39}$$

Notice that they satisfy $\gamma_{12\hat{1}\hat{2}5} = -1$. Furthermore, in the complex basis (J.20), we have

$$\sigma_+ = \sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \ , \qquad \sigma_- = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \ . \tag{J.40}$$

This implies that the $SO(d)$ generators needed in the covariant derivatives (J.23) are

$$\begin{aligned} \Sigma_{5\pm} &= \mp\frac{1}{2}\sigma_\pm \times \sigma_3 \ , & \Sigma_{5\hat{\pm}} &= \mp\frac{1}{2}\mathbb{I} \times \sigma_\pm \ , \\ \Sigma_{+-} &= -\sigma_3 \times \mathbb{I} \ , & \Sigma_{\hat{+}\hat{-}} &= -\mathbb{I} \times \sigma_3 \ . \end{aligned} \tag{J.41}$$

The branching of this representation into representations of $U(1)$ is given by

$$-\frac{1}{2}C_H{}^{ab}\Sigma_{ab} = i\begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} . \tag{J.42}$$

We can now construct the Dirac operator $\not{D} = \gamma^a D_a$. Direct evaluation yields

$$\begin{aligned}
\not{D} &= -\frac{a}{2}(\sigma_- T_+ + \sigma_+ T_-) \times \mathbb{I} - \frac{b}{2}\sigma_3 \times (\sigma_{\hat{-}} T_{\hat{+}} + \sigma_{\hat{+}} T_{\hat{-}}) - cT_5 \sigma_3 \times \sigma_3 \\
&\quad - \frac{i}{2}\left(c + \frac{a^2}{4c}\right)\mathbb{I} \times \sigma_3 + \frac{i}{2}\left(c + \frac{b^2}{4c}\right)\sigma_3 \times \mathbb{I} \\
&= \begin{pmatrix} -cT_5 & -bT_{\hat{+}} & -aT_+ & 0 \\ -bT_{\hat{-}} & cT_5 & 0 & -aT_+ \\ -aT_- & 0 & cT_5 & bT_{\hat{+}} \\ 0 & -aT_- & bT_{\hat{-}} & -cT_5 \end{pmatrix} \\
&\quad + \frac{i}{8c}\begin{pmatrix} -(a^2-b^2) & & & \\ & 8c^2+a^2+b^2 & & \\ & & -(8c^2+a^2+b^2) & \\ & & & a^2-b^2 \end{pmatrix} .
\end{aligned} \tag{J.43}$$

The field equation (8.3.63) contains the operators (8.3.54). They become, in the notation of this section,

$$\mathcal{D}_\pm = \not{D} + \frac{i}{4}\cot\alpha\,(\Sigma_{+-} + \Sigma_{\hat{+}\hat{-}}) \pm \frac{i}{4\sin\alpha}(\gamma_5 + 3) . \tag{J.44}$$

Using (J.37), the additional terms have the following matrix form,

$$\frac{i}{8c}(a^2-b^2)\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \pm \frac{ic}{2}\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} . \tag{J.45}$$

As for the vector case, we realize that the sign of c implied by (J.37) is such that (J.45) cancels the asymmetries between a and b in the Dirac operator (J.43).

By inspection of (J.43), (J.42) and (J.24), we can establish that the spinor components must carry the following quantum numbers,

$$Y = \begin{pmatrix} Y_{-1}^{j,l,r} \\ Y_0^{j,l,r-1} \\ Y_0^{j,l,r+1} \\ Y_1^{j,l,r} \end{pmatrix} . \tag{J.46}$$

This makes it possible to replace the coset generators in (J.43) by numerical values. Using also (J.26) we obtain

$$\mathcal{D}_\pm Y = \frac{i}{2} \begin{pmatrix} c(-2r \pm 2) & b(2l+1-r) & a(2j+1+r) & 0 \\ b(2l+1+r) & c(2r+1\pm 1) & 0 & a(2j+1+r) \\ a(2j+1-r) & 0 & c(2r-1\pm 1) & -b(2l+1-r) \\ 0 & a(2j+1-r) & -b(2l+1+r) & c(-2r\pm 2) \end{pmatrix} Y . \tag{J.47}$$

We proceed as for the vectors, evaluating first the restrictions the $SU(2) \times SU(2)$ representations of the single spinor components impose. Here, j and l are both integer or half-integer, with r odd or even respectively (vice versa with respect to the scalar and vector cases). The restrictions arising from the non-zero components are listed in Table J.2.

Table J.2: Restrictions on r for non-zero components of the spinor (J.46).

$SU(2) \times SU(2)$ rep.	restrictions on r	
$Y_{-1}^{j,l,r}$	$-2j+1 \le r \le 2j+1$	$-2l-1 \le r \le 2l-1$
$Y_0^{j,l,r-1}$	$-2j+1 \le r \le 2j+1$	$-2l+1 \le r \le 2l+1$
$Y_0^{j,l,r+1}$	$-2j-1 \le r \le 2j-1$	$-2l-1 \le r \le 2l-1$
$Y_1^{j,l,r}$	$-2j-1 \le r \le 2j-1$	$-2l+1 \le r \le 2l+1$

J.0.4 Tables of harmonics and eigenvalues

The following tables list the solutions of the harmonic analysis on $\tilde{T}^{1,1}$ for the vector and spinor fields. One must distinguish the generic case $j \neq l$, from the special case $j = l$, for which $C_{j,l}$ simplifies to $C_{j,l} = (2j+1)^2$. Some of the generic solutions simplify in the special case $j = l$, because common factors can be pulled out of the vectors and spinors. As a consequence, the associated range of r may be smaller than in the generic case.

As discussed in the main text, j and l are both non-negative integer or half-integer, with r even or odd (odd or even), respectively, for vectors (spinors). We define $\bar{l} = 2\min(j,l)$.

Table J.3: Eigenvectors and eigenvalues of the modified vector Laplacian, $-\Box'_1$, defined in (J.35) and given in (J.38) in matrix form. Generic case $j \neq l$.

$-\Box'_1$	eigenvector	eigenvalue	range of r
$j \neq l$	$\begin{pmatrix} a(2j+r) \\ a(2j-r) \\ b(2l-r) \\ b(2l+r) \\ -2cr \end{pmatrix}$	H_0	$\|r\| \leq \bar{l}$
	$\begin{pmatrix} a(2j+r)(r+h) \\ a(2j-r)(r-h) \\ b(2l-r)(r-h) \\ b(2l+r)(r+h) \\ 2c(h^2-r^2) \end{pmatrix}$	$H_0 + hc^2$ $h = 1 \pm \sqrt{C_{j,l}}$	$\|r\| \leq \bar{l}$
	$\begin{pmatrix} b(2l+2-r) \\ 0 \\ 0 \\ -a(2j+2-r) \\ 0 \end{pmatrix}$	$H_0 + rc^2$	$\|r-2\| \leq \bar{l}$
	$\begin{pmatrix} 0 \\ b(2l+2+r) \\ -a(2j+2+r) \\ 0 \\ 0 \end{pmatrix}$	$H_0 - rc^2$	$\|r+2\| \leq \bar{l}$

Table J.4: Eigenvectors and eigenvalues of the modified vector Laplacian, $-\Box_1'$. Special case $j = l$. Only the $h = 2j+2$ solution exists for $j = 0$, while the $h = -2j$ solution does not exist for $j = \frac{1}{2}$.

$-\Box_1'$	eigenvector	eigenvalue	range of r
$j = l$	$\begin{pmatrix} a(2j+r) \\ a(2j-r) \\ b(2j-r) \\ b(2j+r) \\ -2cr \end{pmatrix}$	$4j(j+1)c^2$	$\lvert r\rvert \leq 2j > 0$
	$\begin{pmatrix} a(2j+r)(r+h) \\ a(2j-r)(r-h) \\ b(2j-r)(r-h) \\ b(2j+r)(r+h) \\ 2c(h^2-r^2) \end{pmatrix}$	$[4j(j+1)+h]c^2$ $h = 2j+2$	$\lvert r\rvert \leq 2j$
	$\begin{pmatrix} a \\ -a \\ -b \\ b \\ -2c \end{pmatrix}$	$[4j(j+1)+h]c^2$ $h = -2j$	$\lvert r\rvert \leq 2j-2$
	$\begin{pmatrix} b \\ 0 \\ 0 \\ -a \\ 0 \end{pmatrix}$	$[(2j+1)^2+(r-1)]c^2$	$\lvert r-1\rvert \leq 2j-1$
	$\begin{pmatrix} 0 \\ b \\ -a \\ 0 \\ 0 \end{pmatrix}$	$[(2j+1)^2-(r+1)]c^2$	$\lvert r+1\rvert \leq 2j-1$

Table J.5: Eigenvectors and eigenvalues of the spinor operators $\mathcal{D}_\pm$, defined in (J.44) and given in (J.47) in matrix form. The eigenvalue is related to h by $\lambda = ich$. Generic case $j \neq l$.

	eigenvector	h	range of r
$\mathcal{D}_+$	$\begin{pmatrix} b(2l+1-r) \\ 2c(h+r-1) \\ 0 \\ a(2j+1-r) \end{pmatrix}$	$1 \pm \sqrt{C_{j,l}}$	$\lvert r-1\rvert \leq \bar{l}$
	$\begin{pmatrix} a(2j+r+1) \\ 0 \\ 2c(h+r-1) \\ -b(2l+r+1) \end{pmatrix}$	$\frac{1}{2} \pm \sqrt{\frac{1}{4} + C_{j,l} - r}$	$\lvert r+1\rvert \leq \bar{l}$
$\mathcal{D}_-$	$\begin{pmatrix} a(2j+r+1) \\ 0 \\ 2c(h+r+1) \\ -b(2l+r+1) \end{pmatrix}$	$-1 \pm \sqrt{C_{j,l}}$	$\lvert r+1\rvert \leq \bar{l}$
	$\begin{pmatrix} b(2l+1-r) \\ 2c(h+r+1) \\ 0 \\ a(2j+1-r) \end{pmatrix}$	$-\frac{1}{2} \pm \sqrt{\frac{1}{4} + C_{j,l} + r}$	$\lvert r-1\rvert \leq \bar{l}$

Table J.6: Eigenvectors and eigenvalues of the spinor operator $\mathcal{D}_+$. The eigenvalue is related to h by $\lambda = ich$. Special case $j = l$. Notice that the range of r depends on the sign in the eigenvalue.

	eigenvector	h	range of r
$\mathcal{D}_+$	$\begin{pmatrix} b(2j+1-r) \\ 2c(h+r-1) \\ 0 \\ a(2j+1-r) \end{pmatrix}$	$1 \pm (2j+1)$	$\lvert r-1\rvert \leq 2j$ $(+)$ $\lvert r\rvert \leq 2j-1$ $(-)$
	$\begin{pmatrix} a(2j+r+1) \\ 0 \\ 2c(h+r-1) \\ -b(2j+r+1) \end{pmatrix}$	$\frac{1}{2} \pm \sqrt{\frac{1}{4} + (2j+1)^2 - r}$	$\lvert r\rvert \leq 2j-1$ $(+)$ $\lvert r+1\rvert \leq 2j$ $(-)$
$\mathcal{D}_-$	$\begin{pmatrix} a(2j+r+1) \\ 0 \\ 2c(h+r+1) \\ -b(2j+r+1) \end{pmatrix}$	$-1 \pm (2j+1)$	$\lvert r\rvert \leq 2j-1$ $(+)$ $\lvert r+1\rvert \leq 2j$ $(-)$
	$\begin{pmatrix} b(2j+1-r) \\ 2c(h+r+1) \\ 0 \\ a(2j+1-r) \end{pmatrix}$	$-\frac{1}{2} \pm \sqrt{\frac{1}{4} + (2j+1)^2 + r}$	$\lvert r-1\rvert \leq 2j$ $(+)$ $\lvert r\rvert \leq 2j-1$ $(-)$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Y. Nambu, *QUARK MODEL AND THE FACTORIZATION OF THE VENEZIANO AMPLITUDE*, in *Symmetries and Quark Models*, pp. 269–278, 1970.

[2] L. Susskind, *Structure of hadrons implied by duality*, *Phys. Rev.* **D1** (1970) 1182–1186.

[3] J. Scherk and J. H. Schwarz, *Dual models for non-hadrons*, *Nuclear Physics B* **81** (Oct., 1974) 118–144.

[4] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[5] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029].

[6] S. Banerjee, R. K. Gupta, and A. Sen, *Logarithmic Corrections to Extremal Black Hole Entropy from Quantum Entropy Function*, *JHEP* **1103** (2011) 147, [1005.3044].

[7] S. Banerjee, R. K. Gupta, I. Mandal, and A. Sen, *Logarithmic Corrections to N=4 and N=8 Black Hole Entropy: A One Loop Test of Quantum Gravity*, *JHEP* **1111** (2011) 143, [1106.0080].

[8] A. Sen, *Logarithmic Corrections to N=2 Black Hole Entropy: An Infrared Window into the Microstates*, *Gen. Rel. Grav.* **44** (2012), no. 5 1207–1266, [1108.3842].

[9] A. Sen, *Logarithmic Corrections to Rotating Extremal Black Hole Entropy in Four and Five Dimensions*, *Gen. Rel. Grav.* **44** (2012) 1947–1991, [1109.3706].

[10] F. Benini, K. Hristov, and A. Zaffaroni, *Black hole microstates in AdS_4 from supersymmetric localization*, *JHEP* **05** (2016) 054, [1511.04085].

[11] F. Benini, K. Hristov, and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS4*, *Phys. Lett.* **B771** (2017) 462–466, [1608.07294].

[12] A. Cabo-Bizet, V. I. Giraldo-Rivera, and L. A. Pando Zayas, *Microstate counting of AdS_4 hyperbolic black hole entropy via the topologically twisted index*, *JHEP* **08** (2017) 023, [1701.07893].

[13] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, *A Universal Counting of Black Hole Microstates in AdS_4*, 1707.04257.

[14] F. Benini, H. Khachatryan, and P. Milan, *Black hole entropy in massive Type IIA*, `1707.06886`.

[15] S. M. Hosseini, K. Hristov, and A. Passias, *Holographic microstate counting for AdS_4 black holes in massive IIA supergravity*, *JHEP* **10** (2017) 190, [`1707.06884`].

[16] J. M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859–4862, [`hep-th/9803002`].

[17] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C22** (2001) 379–394, [`hep-th/9803001`].

[18] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091, [`0806.1218`].

[19] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71–129, [`0712.2824`].

[20] A. Klemm, M. Marino, M. Schiereck, and M. Soroush, *Aharony-Bergman-Jafferis-Maldacena Wilson loops in the Fermi gas approach*, *Z. Naturforsch.* **A68** (2013) 178–209, [`1207.0611`].

[21] N. Drukker, D. J. Gross, and A. A. Tseytlin, *Green-Schwarz string in AdS(5) x S**5: Semiclassical partition function*, *JHEP* **04** (2000) 021, [`hep-th/0001204`].

[22] M. Sakaguchi and K. Yoshida, *A Semiclassical string description of Wilson loop with local operators*, *Nucl.Phys.* **B798** (2008) 72–88, [`0709.4187`].

[23] M. Kruczenski and A. Tirziu, *Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling*, *JHEP* **05** (2008) 064, [`0803.0315`].

[24] C. Kristjansen and Y. Makeenko, *More about One-Loop Effective Action of Open Superstring in $AdS_5 \times S^5$*, *JHEP* **09** (2012) 053, [`1206.5660`].

[25] E. I. Buchbinder and A. A. Tseytlin, *1/N correction in the D3-brane description of a circular Wilson loop at strong coupling*, *Phys. Rev.* **D89** (2014), no. 12 126008, [`1404.4952`].

[26] V. Forini, V. Giangreco M. Puletti, L. Griguolo, D. Seminara, and E. Vescovi, *Precision calculation of 1/4-BPS Wilson loops in $AdS_5 \times S^5$*, *JHEP* **02** (2016) 105, [`1512.00841`].

[27] A. Faraggi, L. A. Pando Zayas, G. A. Silva, and D. Trancanelli, *Toward precision holography with supersymmetric Wilson loops*, *JHEP* **04** (2016) 053, [`1601.04708`].

[28] V. Forini, A. A. Tseytlin, and E. Vescovi, *Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $AdS_5\times S^5$*, *JHEP* **03** (2017) 003, [`1702.02164`].

[29] A. Cagnazzo, D. Medina-Rincon, and K. Zarembo, *String corrections to circular Wilson loop and anomalies*, 1712.07730.

[30] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, *Toward Microstate Counting Beyond Large N in Localization and the Dual One-loop Quantum Supergravity*, 1707.04197.

[31] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, *A One-loop Test of Quantum Black Holes in Anti de Sitter Space*, 1711.01076.

[32] A. Cabo-Bizet, U. Kol, L. A. Pando Zayas, I. Papadimitriou, and V. Rathee, *Entropy functional and the holographic attractor mechanism*, *JHEP* **05** (2018) 155, [1712.01849].

[33] J. Aguilera-Damia, A. Faraggi, L. Pando Zayas, V. Rathee, and G. A. Silva, *Functional Determinants of Radial Operators in AdS_2*, *JHEP* **06** (2018) 007, [1802.06789].

[34] J. Aguilera-Damia, A. Faraggi, L. A. Pando Zayas, V. Rathee, and G. A. Silva, *Zeta-function Regularization of Holographic Wilson Loops*, *Phys. Rev.* **D98** (2018), no. 4 046011, [1802.03016].

[35] J. Aguilera-Damia, A. Faraggi, L. A. Pando Zayas, V. Rathee, and G. A. Silva, *Toward Precision Holography in Type IIA with Wilson Loops*, *JHEP* **08** (2018) 044, [1805.00859].

[36] W. Mück, L. A. Pando Zayas, and V. Rathee, *Spectra of Certain Holographic ABJM Wilson Loops in Higher Rank Representations*, *JHEP* **11** (2016) 113, [1609.06930].

[37] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, *JHEP* **07** (2015) 127, [1504.03698].

[38] M. Honda and Y. Yoshida, *Supersymmetric index on $T^2 \times S^2$ and elliptic genus*, 1504.04355.

[39] C. Closset, S. Cremonesi, and D. S. Park, *The equivariant A-twist and gauged linear sigma models on the two-sphere*, *JHEP* **06** (2015) 076, [1504.06308].

[40] S. M. Hosseini and A. Zaffaroni, *Large N matrix models for 3d $\mathcal{N} = 2$ theories: twisted index, free energy and black holes*, *JHEP* **08** (2016) 064, [1604.03122].

[41] S. M. Hosseini and N. Mekareeya, *Large N topologically twisted index: necklace quivers, dualities, and Sasaki-Einstein spaces*, *JHEP* **08** (2016) 089, [1604.03397].

[42] C. Closset and H. Kim, *Comments on twisted indices in 3d supersymmetric gauge theories*, *JHEP* **08** (2016) 059, [1605.06531].

[43] C. Closset, H. Kim, and B. Willett, *Supersymmetric partition functions and the three-dimensional A-twist*, *JHEP* **03** (2017) 74, [1701.03171].

[44] S. Bhattacharyya, A. Grassi, M. Marino, and A. Sen, *A One-Loop Test of Quantum Supergravity*, *Class. Quant. Grav.* **31** (2014) 015012, [1210.6057].

[45] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati, and T. A. Tran, *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, *Nucl. Phys.* **B558** (1999) 96–126, [hep-th/9903214].

[46] A. Sen, *Quantum Entropy Function from AdS(2)/CFT(1) Correspondence*, *Int. J. Mod. Phys.* **A24** (2009) 4225–4244, [0809.3304].

[47] D. V. Vassilevich, *Heat kernel expansion: User's manual*, *Phys. Rept.* **388** (2003) 279–360, [hep-th/0306138].

[48] R. Camporesi and A. Higuchi, *The Plancherel measure for p-forms in real hyperbolic spaces*, *Journal of Geometry and Physics* **15** (1994), no. 1 57 – 94.

[49] W. Siegel, *Hidden Ghosts*, *Phys. Lett.* **B93** (1980) 170–172.

[50] E. J. Copeland and D. J. Toms, *Quantized Antisymmetric Tensor Fields and Selfconsistent Dimensional Reduction in Higher Dimensional Space-times*, *Nucl. Phys.* **B255** (1985) 201–230.

[51] F. Larsen and P. Lisbao, *Divergences and boundary modes in $\mathcal{N} = 8$ supergravity*, *JHEP* **01** (2016) 024, [1508.03413].

[52] J. Dodziuk, *L^2 Harmonic Forms on Rotationally Symmetric Riemannian Manifolds*, *Proceedings of the American Mathematical Society* **77** (1979), no. 3 395–400.

[53] H. Donnelly, *The differential form spectrum of hyperbolic space*, *Manuscripta Math.* **33** (1981), no. 3 365–385.

[54] H. Fuji, S. Hirano, and S. Moriyama, *Summing Up All Genus Free Energy of ABJM Matrix Model*, *JHEP* **08** (2011) 001, [1106.4631].

[55] M. Marino and P. Putrov, *ABJM theory as a Fermi gas*, *J. Stat. Mech.* **1203** (2012) P03001, [1110.4066].

[56] A. Dabholkar, N. Drukker, and J. Gomes, *Localization in supergravity and quantum AdS_4/CFT_3 holography*, *JHEP* **10** (2014) 90, [1406.0505].

[57] J. Nian and X. Zhang, *Entanglement Entropy of ABJM Theory and Entropy of Topological Black Hole*, *JHEP* **07** (2017) 096, [1705.01896].

[58] F. Benini and N. Bobev, *Two-dimensional SCFTs from wrapped branes and c-extremization*, *JHEP* **06** (2013) 005, [1302.4451].

[59] F. Benini and A. Zaffaroni, *Supersymmetric partition functions on Riemann surfaces*, *Proc. Symp. Pure Math.* **96** (2017) 13–46, [1605.06120].

[60] S. M. Hosseini, A. Nedelin, and A. Zaffaroni, *The Cardy limit of the topologically twisted index and black strings in AdS_5*, *JHEP* **04** (2017) 014, [1611.09374].

[61] A. Sen, *Entropy Function and AdS_2/CFT_1 Correspondence*, *JHEP* **11** (2008) 075, [0805.0095].

[62] S. L. Cacciatori and D. Klemm, *Supersymmetric AdS(4) black holes and attractors*, *JHEP* **01** (2010) 085, [0911.4926].

[63] K. Hristov and S. Vandoren, *Static supersymmetric black holes in AdS_4 with spherical symmetry*, *JHEP* **04** (2011) 047, [1012.4314].

[64] D. D. K. Chow and G. Compre, *Dyonic AdS black holes in maximal gauged supergravity*, *Phys. Rev.* **D89** (2014), no. 6 065003, [1311.1204].

[65] P. Albin, *A renormalized index theorem for some complete asymptotically regular metrics: The Gauss–Bonnet theorem*, *Advances in Mathematics* **213** (2007), no. 1 1 – 52.

[66] I. Jeon and S. Lal, *Logarithmic Corrections to Entropy of Magnetically Charged AdS4 Black Holes*, *Phys. Lett.* **B774** (2017) 41–45, [1707.04208].

[67] N. Banerjee, I. Mandal, and A. Sen, *Black Hole Hair Removal*, *JHEP* **07** (2009) 091, [0901.0359].

[68] D. P. Jatkar, A. Sen, and Y. K. Srivastava, *Black Hole Hair Removal: Non-linear Analysis*, *JHEP* **02** (2010) 038, [0907.0593].

[69] S. M. Carroll, M. C. Johnson, and L. Randall, *Extremal limits and black hole entropy*, *JHEP* **11** (2009) 109, [0901.0931].

[70] S. Ferrara, R. Kallosh, and A. Strominger, *N=2 extremal black holes*, *Phys. Rev.* **D52** (1995) R5412–R5416, [hep-th/9508072].

[71] A. Strominger, *Macroscopic entropy of N=2 extremal black holes*, *Phys. Lett.* **B383** (1996) 39–43, [hep-th/9602111].

[72] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D54** (1996) 1514–1524, [hep-th/9602136].

[73] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038, [hep-th/0506177].

[74] S. Dutta and R. Gopakumar, *On Euclidean and Noetherian entropies in AdS space*, *Phys. Rev.* **D74** (2006) 044007, [hep-th/0604070].

[75] J. Lindgren, I. Papadimitriou, A. Taliotis, and J. Vanhoof, *Holographic Hall conductivities from dyonic backgrounds*, *JHEP* **07** (2015) 094, [1505.04131].

[76] M. J. Duff and J. T. Liu, *Anti-de Sitter black holes in gauged $N = 8$ supergravity*, *Nucl. Phys.* **B554** (1999) 237–253, [hep-th/9901149].

[77] G. Dall'Agata and A. Gnecchi, *Flow equations and attractors for black holes in $N = 2$ U(1) gauged supergravity*, *JHEP* **03** (2011) 037, [1012.3756].

[78] D. Klemm, N. Petri, and M. Rabbiosi, *Symplectically invariant flow equations for $N = 2$, $D = 4$ gauged supergravity with hypermultiplets*, *JHEP* **04** (2016) 008, [1602.01334].

[79] A. Ceresole and G. Dall'Agata, *Flow Equations for Non-BPS Extremal Black Holes*, *JHEP* **03** (2007) 110, [hep-th/0702088].

[80] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, *First order description of black holes in moduli space*, *JHEP* **11** (2007) 032, [0706.0712].

[81] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, *First Order Description of D=4 static Black Holes and the Hamilton-Jacobi equation*, *Nucl. Phys.* **B833** (2010) 1–16, [0905.3938].

[82] M. Trigiante, T. Van Riet, and B. Vercnocke, *Fake supersymmetry versus Hamilton-Jacobi*, *JHEP* **05** (2012) 078, [1203.3194].

[83] A. Gnecchi and C. Toldo, *On the non-BPS first order flow in N=2 U(1)-gauged Supergravity*, *JHEP* **03** (2013) 088, [1211.1966].

[84] A. Gnecchi and C. Toldo, *First order flow for non-extremal AdS black holes and mass from holographic renormalization*, *JHEP* **10** (2014) 075, [1406.0666].

[85] D. Klemm and M. Rabbiosi, *First order flow equations for nonextremal black holes in AdS (super)gravity*, *JHEP* **10** (2017) 149, [1706.05862].

[86] E. Kiritsis and V. Niarchos, *The holographic quantum effective potential at finite temperature and density*, *JHEP* **08** (2012) 164, [1205.6205].

[87] G. W. Gibbons and R. E. Kallosh, *Topology, entropy and Witten index of dilaton black holes*, *Phys. Rev.* **D51** (1995) 2839–2862, [hep-th/9407118].

[88] A. Almheiri and B. Kang, *Conformal Symmetry Breaking and Thermodynamics of Near-Extremal Black Holes*, *JHEP* **10** (2016) 052, [1606.04108].

[89] I. Papadimitriou, *Holographic Renormalization of general dilaton-axion gravity*, *JHEP* **08** (2011) 119, [1106.4826].

[90] O. S. An, *Anomaly-corrected supersymmetry algebra and supersymmetric holographic renormalization*, 1703.09607.

[91] P. Breitenlohner and D. Z. Freedman, *Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity*, *Phys. Lett.* **115B** (1982) 197–201.

[92] D. Z. Freedman and S. S. Pufu, *The holography of F-maximization*, *JHEP* **03** (2014) 135, [1302.7310].

[93] I. Papadimitriou, *Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT*, *JHEP* **05** (2007) 075, [hep-th/0703152].

[94] M. M. Caldarelli, A. Christodoulou, I. Papadimitriou, and K. Skenderis, *Phases of planar AdS black holes with axionic charge*, *JHEP* **04** (2017) 001, [1612.07214].

[95] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, *JHEP* **10** (2004) 075, [hep-th/0407071].

[96] D. Z. Freedman, K. Pilch, S. S. Pufu, and N. P. Warner, *Boundary Terms and Three-Point Functions: An AdS/CFT Puzzle Resolved*, *JHEP* **06** (2017) 053, [1611.01888].

[97] N. Halmagyi and S. Lal, *On the On-Shell: The Action of AdS_4 Black Holes*, 1710.09580.

[98] S. de Haro, I. Papadimitriou, and A. C. Petkou, *Conformally Coupled Scalars, Instantons and Vacuum Instability in AdS(4)*, *Phys. Rev. Lett.* **98** (2007) 231601, [hep-th/0611315].

[99] E. Kiritsis, W. Li, and F. Nitti, *Holographic RG flow and the Quantum Effective Action*, *Fortsch. Phys.* **62** (2014) 389–454, [1401.0888].

[100] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004, [hep-th/0505190].

[101] N. Halmagyi, *Static BPS black holes in AdS_4 with general dyonic charges*, *JHEP* **03** (2015) 032, [1408.2831].

[102] P. Goulart, *Dyonic AdS_4 black hole entropy and attractors via entropy function*, *JHEP* **09** (2016) 003, [1512.05399].

[103] I. Papadimitriou, *Supercurrent anomalies in 4d SCFTs*, *JHEP* **07** (2017) 038, [1703.04299].

[104] P. Benetti Genolini, D. Cassani, D. Martelli, and J. Sparks, *Holographic renormalization and supersymmetry*, *JHEP* **02** (2017) 132, [1612.06761].

[105] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074, [hep-th/0604007].

[106] J. Gomis and F. Passerini, *Wilson loops as D3-branes*, *JHEP* **01** (2007) 097, [hep-th/0612022].

[107] A. Faraggi and L. A. Pando Zayas, *The Spectrum of Excitations of Holographic Wilson Loops*, *JHEP* **05** (2011) 018, [1101.5145].

[108] A. Faraggi, W. Mueck, and L. A. Pando Zayas, *One-loop Effective Action of the Holographic Antisymmetric Wilson Loop*, *Phys.Rev.* **D85** (2012) 106015, [1112.5028].

[109] J. R. David, E. Gava, R. K. Gupta, and K. Narain, *Localization on $AdS_2\times S^1$*, *JHEP* **03** (2017) 050, [1609.07443].

[110] D. Rodriguez-Gomez and J. G. Russo, *Free energy and boundary anomalies on $\mathbb{S}^a \times \mathbb{H}^b$ spaces*, *JHEP* **10** (2017) 084, [1708.00305].

[111] J. R. David, E. Gava, R. K. Gupta, and K. Narain, *Boundary conditions and localization on AdS. Part I*, *JHEP* **09** (2018) 063, [1802.00427].

[112] J. S. Dowker and R. Critchley, *Effective Lagrangian and Energy Momentum Tensor in de Sitter Space*, *Phys. Rev.* **D13** (1976) 3224.

[113] S. W. Hawking, *Zeta Function Regularization of Path Integrals in Curved Space-Time*, *Commun. Math. Phys.* **55** (1977) 133.

[114] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta regularization techniques with applications.* 1994.

[115] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds*, *Annals Math.* **98** (1973) 154–177.

[116] P. B. Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, .

[117] G. V. Dunne and K. Kirsten, *Functional determinants for radial operators*, *J. Phys.* **A39** (2006) 11915–11928, [`hep-th/0607066`].

[118] H. Kim, N. Kim, and J. H. Lee, *One-loop corrections to holographic Wilson loop in AdS4xCP3*, *J.Korean Phys.Soc.* **61** (2012) 713–719, [`1203.6343`].

[119] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, *Supersymmetric Wilson loops on S**3*, *JHEP* **05** (2008) 017, [`0711.3226`].

[120] M. Bordag and K. Kirsten, *Vacuum energy in a spherically symmetric background field*, *Phys. Rev.* **D53** (1996) 5753–5760, [`hep-th/9608070`].

[121] M. Bordag and K. Kirsten, *The Ground state energy of a spinor field in the background of a finite radius flux tube*, *Phys. Rev.* **D60** (1999) 105019, [`hep-th/9812060`].

[122] K. Kirsten, *Spectral functions in mathematics and physics*, *AIP Conf. Proc.* **484** (1999), no. 1 106–146, [`hep-th/0005133`].

[123] K. Kirsten, *Spectral functions in mathematics and physics.* PhD thesis, Leipzig U., 1999. `hep-th/0007251`.

[124] K. Kirsten and P. Loya, *Computation of determinants using contour integrals*, *Am. J. Phys.* **76** (2008) 60–64, [`0707.3755`].

[125] K. Kirsten and A. J. McKane, *Functional determinants by contour integration methods*, *Annals Phys.* **308** (2003) 502–527, [`math-ph/0305010`].

[126] M. Bordag and I. Drozdov, *Fermionic vacuum energy from a Nielsen-Olesen vortex*, *Phys. Rev.* **D68** (2003) 065026, [`hep-th/0305002`].

[127] S. Giombi and V. Pestun, *Correlators of Wilson Loops and Local Operators from Multi-Matrix Models and Strings in AdS*, *JHEP* **01** (2013) 101, [`1207.7083`].

[128] N. Gromov and P. Vieira, *The AdS(5) x S**5 superstring quantum spectrum from the algebraic curve*, *Nucl. Phys.* **B789** (2008) 175–208, [`hep-th/0703191`].

[129] N. Beisert *et. al.*, *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3–32, [`1012.3982`].

[130] R. Camporesi and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, *J. Math. Phys.* **35** (1994) 4217–4246.

[131] R. Camporesi and A. Higuchi, *On the Eigen functions of the Dirac operator on spheres and real hyperbolic spaces*, *J. Geom. Phys.* **20** (1996) 1–18, [gr-qc/9505009].

[132] N. Drukker, *1/4 BPS circular loops, unstable world-sheet instantons and the matrix model*, *JHEP* **09** (2006) 004, [hep-th/0605151].

[133] J. K. Erickson, G. W. Semenoff, and K. Zarembo, *Wilson loops in N=4 supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B582** (2000) 155–175, [hep-th/0003055].

[134] N. Drukker and D. J. Gross, *An Exact prediction of N=4 SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896–2914, [hep-th/0010274].

[135] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, *Wilson loops: From four-dimensional SYM to two-dimensional YM*, *Phys. Rev.* **D77** (2008) 047901, [0707.2699].

[136] J. Aguilera-Damia, D. H. Correa, and G. A. Silva, *Semiclassical partition function for strings dual to Wilson loops with small cusps in ABJM*, *JHEP* **03** (2015) 002, [1412.4084].

[137] J. Avan and H. J. de Vega, *Instantons of 2d fermionic effective actions by inverse scattering transformation*, *Commun. Math. Phys.* **102** (1985) 463.

[138] A. Kapustin, B. Willett, and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **1003** (2010) 089, [0909.4559].

[139] M. S. Bianchi, L. Griguolo, A. Mauri, S. Penati, and D. Seminara, *A matrix model for the latitude Wilson loop in ABJM theory*, *JHEP* **08** (2018) 060, [1802.07742].

[140] N. Drukker, J. Plefka, and D. Young, *Wilson loops in 3-dimensional N=6 supersymmetric Chern-Simons Theory and their string theory duals*, *JHEP* **0811** (2008) 019, [0809.2787].

[141] S.-J. Rey, T. Suyama, and S. Yamaguchi, *Wilson Loops in Superconformal Chern-Simons Theory and Fundamental Strings in Anti-de Sitter Supergravity Dual*, *JHEP* **03** (2009) 127, [0809.3786].

[142] B. Chen and J.-B. Wu, *Supersymmetric Wilson Loops in N=6 Super Chern-Simons-matter theory*, *Nucl. Phys.* **B825** (2010) 38–51, [0809.2863].

[143] M. Marino and P. Putrov, *Exact Results in ABJM Theory from Topological Strings*, *JHEP* **06** (2010) 011, [0912.3074].

[144] N. Drukker, M. Marino, and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511–563, [1007.3837].

[145] N. Drukker and D. Trancanelli, *A Supermatrix model for N=6 super Chern-Simons-matter theory*, *JHEP* **02** (2010) 058, [0912.3006].

[146] V. Cardinali, L. Griguolo, G. Martelloni, and D. Seminara, *New supersymmetric Wilson loops in ABJ(M) theories*, *Phys.Lett.* **B718** (2012) 615–619, [1209.4032].

[147] M. S. Bianchi, L. Griguolo, M. Leoni, S. Penati, and D. Seminara, *BPS Wilson loops and Bremsstrahlung function in ABJ(M): a two loop analysis*, *JHEP* **06** (2014) 123, [1402.4128].

[148] M. Bonini, L. Griguolo, M. Preti, and D. Seminara, *Surprises from the resummation of ladders in the ABJ(M) cusp anomalous dimension*, *JHEP* **05** (2016) 180, [1603.00541].

[149] M. S. Bianchi, L. Griguolo, A. Mauri, S. Penati, M. Preti, and D. Seminara, *Towards the exact Bremsstrahlung function of ABJM theory*, *JHEP* **08** (2017) 022, [1705.10780].

[150] L. Bianchi, L. Griguolo, M. Preti, and D. Seminara, *Wilson lines as superconformal defects in ABJM theory: a formula for the emitted radiation*, *JHEP* **10** (2017) 050, [1706.06590].

[151] L. Bianchi, M. Preti, and E. Vescovi, *Exact Bremsstrahlung functions in ABJM theory*, *JHEP* **07** (2018) 060, [1802.07726].

[152] D. H. Correa, J. Aguilera-Damia, and G. A. Silva, *Strings in $AdS_4 \times \mathbb{CP}^3$ Wilson loops in $\mathcal{N}$ =6 super Chern-Simons-matter and bremsstrahlung functions*, *JHEP* **06** (2014) 139, [1405.1396].

[153] M. Sakaguchi, H. Shin, and K. Yoshida, *Semiclassical Analysis of M2-brane in AdS_4xS^7/Z_k*, *JHEP* **1012** (2010) 012, [1007.3354].

[154] D. Medina-Rincon, A. A. Tseytlin, and K. Zarembo, *Precision matching of circular Wilson loops and strings in AdS_5 S^5*, *JHEP* **05** (2018) 199, [1804.08925].

[155] J. Cookmeyer, J. T. Liu, and L. A. Pando Zayas, *Higher Rank ABJM Wilson Loops from Matrix Models*, *JHEP* **11** (2016) 121, [1609.08165].

[156] S. Forste, D. Ghoshal, and S. Theisen, *Stringy corrections to the Wilson loop in N=4 superYang-Mills theory*, *JHEP* **9908** (1999) 013, [hep-th/9903042].

[157] A. Faraggi, J. T. Liu, L. A. Pando Zayas, and G. Zhang, *One-loop structure of higher rank Wilson loops in AdS/CFT*, *Phys. Lett.* **B740** (2015) 218–221, [1409.3187].

[158] B. E. W. Nilsson and C. N. Pope, *Hopf Fibration of Eleven-dimensional Supergravity*, *Class. Quant. Grav.* **1** (1984) 499.

[159] L. Martucci, J. Rosseel, D. Van den Bleeken, and A. Van Proeyen, *Dirac actions for D-branes on backgrounds with fluxes*, *Class. Quant. Grav.* **22** (2005) 2745–2764, [hep-th/0504041].

[160] P. Benincasa and A. V. Ramallo, *Fermionic impurities in Chern-Simons-matter theories*, *JHEP* **1202** (2012) 076, [1112.4669].

[161] J. McGreevy, L. Susskind, and N. Toumbas, *Invasion of the giant gravitons from Anti-de Sitter space*, *JHEP* **06** (2000) 008, [hep-th/0003075].

[162] A. Hashimoto, S. Hirano, and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051, [hep-th/0008016].

[163] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010, [hep-th/0501109].

[164] E. Witten, *Fermion Quantum Numbers in Kaluza-Klein Theory*, in *Shelter Island II Shelter Island, New York, June 1-2, 1983*, 1983. [Conf. Proc.C8306011,227(1983)].

[165] A. Ceresole, G. Dall'Agata, and R. D'Auria, *K K spectroscopy of type IIB supergravity on AdS(5) x T**11*, *JHEP* **11** (1999) 009, [hep-th/9907216].

[166] A. Van Proeyen, *Tools for supersymmetry*, in *Proceedings, Spring School on Quantum Field Theory. Supersymmetries and Superstrings*, vol. 9, pp. I.1–I.48, 1999. hep-th/9910030.

[167] M. Gunaydin, G. Sierra, and P. K. Townsend, *The unitary supermultiplets of $d = 3$ Anti-de-Sitter and $d = 2$ conformal superalgebras*, *Nucl. Phys.* **B274** (1986) 429.

[168] Y. Hatsuda, M. Honda, S. Moriyama, and K. Okuyama, *ABJM Wilson Loops in Arbitrary Representations*, *JHEP* **10** (2013) 168, [1306.4297].

[169] I. Papadimitriou, *Non-Supersymmetric Membrane Flows from Fake Supergravity and Multi-Trace Deformations*, *JHEP* **02** (2007) 008, [hep-th/0606038].

[170] A. S. Schwarz, *The Partition Function of a Degenerate Functional*, *Commun. Math. Phys.* **67** (1979) 1–16.

[171] M. Cvetic, H. Lu, and C. N. Pope, *Consistent warped space Kaluza-Klein reductions, half maximal gauged supergravities and CP**n constructions*, *Nucl. Phys.* **B597** (2001) 172–196, [hep-th/0007109].

[172] H. A. Schmitt, P. Halse, B. R. Barrett, and A. B. Balantekin, *Positive Discrete Series Representations of the Noncompact Superalgebra Osp(4/2,R)*, *J. Math. Phys.* **30** (1989) 2714–2720.

[173] A. Ceresole, G. Dall'Agata, R. D'Auria, and S. Ferrara, *Spectrum of type IIB supergravity on AdS(5) x T**11: Predictions on N=1 SCFT's*, *Phys. Rev.* **D61** (2000) 066001, [hep-th/9905226].

[174] A. Salam and J. A. Strathdee, *On Kaluza-Klein Theory*, *Annals Phys.* **141** (1982) 316–352.

[175] P. van Nieuwenhuizen, *General Theory of Coset Manifolds and Antisymetric Tensors Applied to Kaluza-Klein Supergravity* , in *In *Trieste 1984, Proceedings, Supersymmetry and Supergravity '84*, 239-323*, 1984.