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Vehicle Cruise: Improved Fuel Economy by Periodic Control*†

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A simple dynamic model, formulated for vehicles in cruise, shows that time-dependent periodic control gives better fuel economy than optimum steady-state control under appropriate conditions on the drag and fuel-consumption functions.

Summary—It is shown that time-dependent periodic control can improve the fuel economy of vehicles in cruise. The time-dependent controls considered are relaxed steady-state (RSS) control, quasi-steady-state (QSS) control and quasi-relaxed steady-state (QRSS) control. Examples are given which show that QRSS control may give better performance than either RSS or QSS control. Properties of optimal cost functions, dependent on the minimum required average speed, are derived. The possibility or impossibility of improved performance through the use of QRSS, QSS and RSS control is investigated in terms of assumptions on the vehicle drag and fuel-consumption functions.

1. INTRODUCTION

It is traditional to operate many dynamic processes in an optimal steady-state mode where the controls and system state are constant and chosen to extremize a performance function subject to process constraints and equilibrium equations. While this approach has considerable intuitive appeal it is not always best. There are many examples, taken mostly from the field of chemical engineering [1], where time-dependent periodic control of the process yields improved performance. The problem of vehicle cruise presented here shows that periodic control can, under appropriate conditions, yield better fuel economy than conventional optimal steady-state control. In particular, both relaxed steady-state control and quasi-steady-state control can do better. The problem also illustrates, for the first time, that quasi-relaxed steady-state control [2] may be better than either relaxed steady-state control or quasi-steady-state control.

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The vehicle cruise problem is formulated as follows. The performance function

$$J(T(\cdot), V(\cdot), \tau) = V_{avg}(F_{avg})^{-1} = \text{specific range} \qquad (1)$$

depends on the thrust $T(\cdot)$ (measurable on $[0,\tau)$), the speed $V(\cdot)$, and the period $\tau > 0$ which satisfy the following constraints:

$$\dot{V} = -D(V) + T(t), \quad V(0) = V(\tau) \geqslant 0, \qquad (2)$$

$$0 \leqslant T(t) \leqslant 1, \quad a.a.\ t \in [0, \tau], \qquad (3)$$

$$F_{avg} = \frac{1}{\tau}\int_0^\tau F(T(t))\,dt = \text{average fuel rate}, \qquad (4)$$

$$V_{avg} = \frac{1}{\tau}\int_0^\tau V(t)\,dt = \text{average speed}, \qquad (5)$$

$$V_{avg} \geqslant V_{min} \geqslant 0. \qquad (6)$$

The condition $V(0) = V(\tau)$ assures that both $V(\cdot)$ and $T(\cdot)$ are periodic when the domain of these functions is extended to $(-\infty, +\infty)$ by

$$V(t+\tau) = V(t) \quad \text{and} \quad T(t+\tau) = T(t).$$

It is assumed that: the drag function $D(\cdot)$: $[0, +\infty) \to [0, +\infty)$ is Lipschitz continuous and the equation $D(V) = 1$ has a unique solution $V = V_{max}$. Thus V_{max} is the maximum steady-state speed. Moreover, it is not difficult to show that $V(\cdot)$ satisfies (2) and (3) only if $0 \leqslant V(t) \leqslant V_{max}$, $0 \leqslant t \leqslant \tau$. To assure that the speed constraint (6) can be satisfied it is required that $0 \leqslant V_{min} \leqslant V_{max}$. The fuel-consumption function $F(\cdot)$: $[0, 1] \to (0, +\infty)$ is assumed to be lower semi-continuous and non-decreasing ($F(T_2) \geqslant F(T_1)$, $0 \leqslant T_1 \leqslant T_2 \leqslant 1$). Positive jumps in $F(T)$ permit the modelling of multi-engine propulsion systems where the fuel flow rate may jump discontinuously with the turning-on of additional engines. The requirement of lower semi-continuity is for mathematical convenience; it implies the existence of appropriate minima. In any case the assumptions on $D(\cdot)$ and $F(\cdot)$ are not physically unreasonable and they allow for a rich supply of interesting examples. Inspection shows that the specific range is bounded:

$$0 \leqslant J \leqslant V_{\max}(F(0))^{-1}.$$

The maximization of $J(T(\cdot), V(\cdot), \tau)$ subject to the constraints is an optimal periodic control (OPC) problem [1–3]. Solutions of OPC may be sought through the application of necessary conditions [3, 4], but in what follows the main interest is in specialized optimal steady-state controls which simplify the analysis and yield, at least in an approximate sense, $T(\cdot)$, $V(\cdot)$, and τ which still satisfy (2–6). Hopefully, the study of these special controls helps in understanding the mechanisms by which time-dependent control improves performance. The optimal values of J for these steady-state controls also shed light on the solution of OPC since they never exceed J^*, the optimal J for OPC.

2. THE FOUR STEADY-STATE PROBLEMS

The optimal steady-state (SS) problem is obtained from OPC by restricting $T(\cdot)$ and $V(\cdot)$ to be constant ($T(t) \equiv T$, $V(t) \equiv V$). In this case the value of τ makes no difference and (1–6) reduce to

$$J = V(F(T))^{-1}, \quad D(V) = T, \quad 0 \leqslant T \leqslant 1, \quad V_{\min} \leqslant V. \tag{7}$$

Using $T = D(V)$ to eliminate T, gives

$$J = V(f_{SS}(V))^{-1}, \quad V_{\min} \leqslant V \leqslant V_{\max}, \tag{8}$$

where

$$f_{SS}(V) = F(D(V)). \tag{9}$$

The maximum of J exists (because $f_{SS}(\cdot)$ is lower semi-continuous and J is bounded from above) and depends on the value of $V_{\min}$. Thus for $V_{\min} \in [0, V_{\max}]$

$$J_{SS}^*(V_{\min}) = \max V(f_{SS}(V))^{-1} \quad \text{subject to } V \in [V_{\min}, V_{\max}] \tag{10}$$

is defined and characterizes the solution of the optimal SS problem for all $V_{\min}$. If $J^*(V_{\min})$ is the optimum cost for OPC (let $J^*(V_{\min})$ be the supremum if the maximum does not exist) then $J^*(\cdot) \geqslant J_{SS}^*(\cdot)$. Here and in what follows the notation $f_1(\cdot) \geqslant f_2(\cdot)$ is used if and only if $f_1(V) \geqslant f_2(V)$ for all $V \in [0, V_{\max}]$. The three remaining steady-state problems correspond to limiting forms of time-dependent periodic control.

Relaxed steady-state (RSS) controls are obtained by defining

$$T(t) = \begin{cases} T_1, & 0 \leqslant t < \lambda\tau \\ T_2, & \lambda\tau \leqslant t < \tau, \end{cases} \tag{11}$$

where $0 \leqslant \lambda \leqslant 1$, and letting $\tau \to 0$. The periodicity condition and differential equation (2) then require that $-\int_0^\tau D(V(t))\,dt + \tau(\lambda T_1 + (1-\lambda)T_2) = 0$. As $\tau \to 0$, $(V(t) - V_{avg})$ tends to zero on $[0, \tau]$ and thus $D(V_{avg}) \to \lambda T_1 + (1-\lambda) T_2$. Since

$$F_{avg} = \lambda F(T_1) + (1-\lambda) F(T_2)$$

this leads to the following RSS optimization problem:

$$\left.\begin{aligned} J &= V(\lambda F(T_1) + (1-\lambda) F(T_2))^{-1}, \\ D(V) &= T_1 \lambda + (1-\lambda) T_2, \\ &0 \leqslant T_1 \leqslant 1, \quad 0 \leqslant T_2 \leqslant 1, \quad 0 \leqslant \lambda \leqslant 1, \\ &V_{\min} \leqslant V, \end{aligned}\right\} \tag{12}$$

where for simplicity V_{avg} has been replaced by V. Define the convex hull of $F(\cdot)$, $(\operatorname{conv} F)(\cdot)$, by

$$(\operatorname{conv} F)(T) = \min \lambda F(T_1) + (1-\lambda) F(T_2) \quad \text{subject to } \lambda T_1 + (1-\lambda) T_2 = T \quad \text{and } \lambda, T_1, T_2 \in [0, 1]. \tag{13}$$

This function is the 'largest' convex function which is a lower bound for $F(\cdot)$ [5]. It follows that maximizing J in (12) corresponds to maximizing J in

$$J = V(f_{RSS}(V))^{-1}, \quad V_{\min} \leqslant V \leqslant V_{\max}, \tag{14}$$

where

$$f_{RSS}(V) = (\operatorname{conv} F)(D(V)). \tag{15}$$

Since $(\operatorname{conv} F)(\cdot)$ is convex and therefore continuous, $f_{RSS}(\cdot)$ is continuous, though not necessarily convex. This and $f_{RSS}(V) \geqslant F(0) > 0$ implies J has a maximum. Thus as in (10) define

$$J_{RSS}^*(V_{\min}) = \max V(f_{RSS}(V))^{-1} \quad \text{subject to } V \in [V_{\min}, V_{\max}]. \tag{16}$$

Since $f_{RSS}(\cdot) \leqslant f_{SS}(\cdot)$ it follows that

$$J^*(\cdot) \geqslant J_{RSS}^*(\cdot) \geqslant J_{SS}^*(\cdot).$$

Quasi-steady-state (QSS) controls are obtained by switching from one constant speed V_1 to another constant speed V_2 and letting the time between switches become large. Assume $0 < V_1 \leqslant V_2 < V_{\max}$ and define

$$T(t) = \begin{cases} 0, & 0 \leqslant t < \Delta t_1, \\ D(V_1), & \Delta t_1 \leqslant t < \Delta t_1 + \lambda\bar{\tau}, \\ +1, & \Delta t_1 + \lambda\bar{\tau} \leqslant t < \Delta t_1 + \lambda\bar{\tau} + \Delta t_2, \\ D(V_2), & \Delta t_1 + \lambda\bar{\tau} + \Delta t_2 \leqslant t < \tau, \end{cases} \tag{17}$$

where $\tau = \Delta t_1 + \Delta t_2 + \bar{\tau}$, $0 \leqslant \lambda \leqslant 1$, Δt_1 is the time (finite because $V_1 > 0$) required in (2) to go from $V = V_2$ to $V = V_1$ with $T(t) \equiv 0$, Δt_2 is the time (finite because $V_2 < 1$) required in (2) to go from

$V=V_1$ to $V=V_2$ with $T(t)\equiv 1$. As $\bar{\tau}\to+\infty$ it is clear that

$$F_{avg}\to\lambda F(D(V_1))+(1-\lambda)F(D(V_2))=\lambda f_{SS}(V_1)+(1-\lambda)f_{SS}(V_2)$$

and $V_{avg}\to\lambda V_1+(1-\lambda)V_2$. If either $V_1=0$ or $V_2=V_{max}$ the result is the same but the argument is a little more complicated because V_1 or V_2 can only be approached. Thus QSS controls lead to the following optimization problem:

$$\left.\begin{aligned}J&=(\lambda V_1+(1-\lambda)V_2)(\lambda f_{SS}(V_1)\\&\quad+(1-\lambda)f_{SS}(V_2))^{-1},\\0&\leqslant V_1\leqslant V_2\leqslant 1,\quad 0\leqslant\lambda\leqslant 1.\end{aligned}\right\}\tag{18}$$

Define

$$f_{QSS}(V)=(\text{conv}f_{SS})(V)=\min\lambda f_{SS}(V_1)+(1-\lambda)f_{SS}(V_2)\quad\text{subject to }\lambda V_1+(1-\lambda)V_2=V,\quad\lambda\in[0,1]\text{ and }V_1,V_2\in[0,V_{max}]\tag{19}$$

and the maximizing of J in (18) is equivalent to maximizing J in

$$J=V(f_{QSS}(V))^{-1},\quad V_{min}\leqslant V\leqslant V_{max}.\tag{20}$$

Since $f_{QSS}(\cdot)$ is continuous and bounded away from zero the maximum exists and can be written

$$J_{QSS}{}^*(V_{min})=\max V(f_{QSS}(V))^{-1}\quad\text{subject to }V\in[V_{min},V_{max}].\tag{21}$$

Because of (19), $f_{QSS}(\cdot)\leqslant f_{SS}(\cdot)$ and therefore $J^*(\cdot)\geqslant J_{QSS}{}^*(\cdot)\geqslant J_{SS}{}^*(\cdot)$.

Finally, it is possible to switch, with the time between switchings becoming large, between two speeds V_1 and V_2 where either or both V_1 and V_2 are maintained by RSS controls. This is the quasi-relaxed steady-state (QRSS) case. Without writing down the details it should be obvious how to proceed. The effective fuel consumption at V_i is $f_{RSS}(V_i)$ instead of $f_{SS}(V_i)$. Making this substitution in the analysis of the previous paragraph means that (20) is replaced by

$$J=V(f_{QRSS}(V))^{-1},\quad V_{min}\leqslant V\leqslant V_{max},\tag{22}$$

where

$$f_{QRSS}(V)=(\text{conv}f_{RSS})(V).\tag{23}$$

As before the maximum of J in (22) exists and it is possible to define

$$J_{QRSS}{}^*(V_{min})=\max V(f_{QRSS}(V))^{-1}\quad\text{subject to }V\in[V_{min},V_{max}].\tag{24}$$

From (23), $f_{QRSS}(\cdot)\leqslant f_{RSS}(\cdot)$ and therefore $J^*(\cdot)\geqslant J_{QRSS}{}^*(\cdot)\geqslant J_{RSS}{}^*(\cdot)\geqslant J_{SS}{}^*(\cdot)$. Since $f_1(\cdot)\leqslant f_2(\cdot)$ implies $(\text{conv}f_1)(\cdot)\leqslant(\text{conv}f_2)(\cdot)$ it follows from $f_{RSS}(\cdot)\leqslant f_{SS}(\cdot)$ that $f_{QRSS}(\cdot)\leqslant f_{QSS}(\cdot)$. Thus $J^*(\cdot)\geqslant J_{QRSS}{}^*(\cdot)\geqslant J_{QSS}{}^*(\cdot)\geqslant J_{SS}{}^*(\cdot)$.

Figure 1 illustrates the preceding definitions and results for representative drag and fuel consumption functions. To interpret Fig. 1(c) it helps to

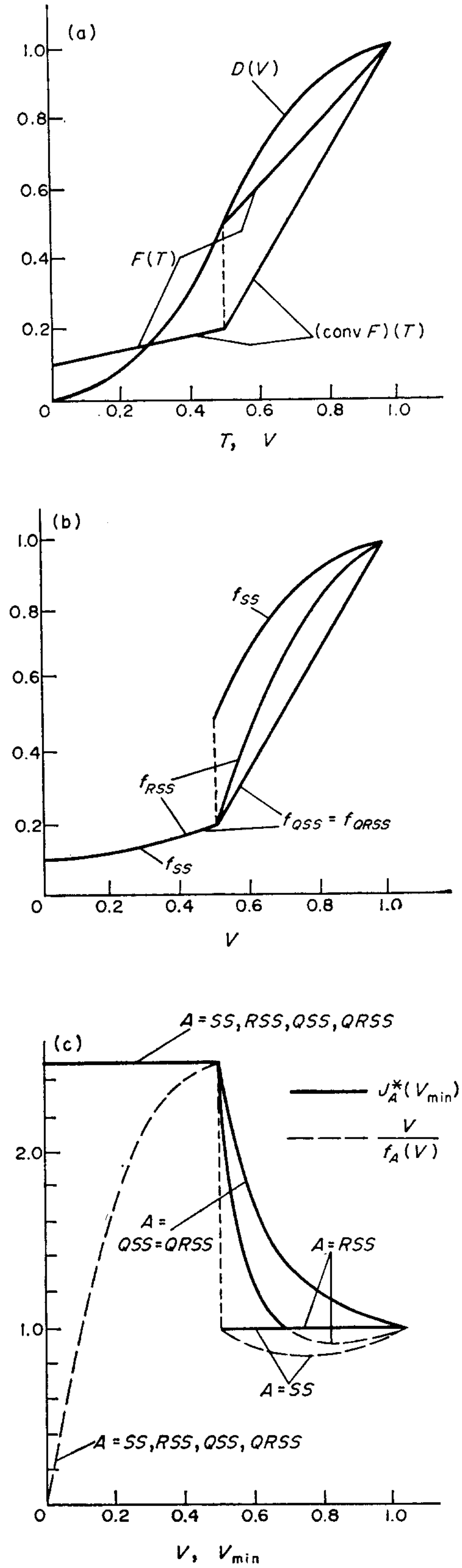

FIG. 1. Example illustrating definitions and results of Section 2.

notice in Fig. 1(b) that lines of fixed specific range are of the form $f = J^{-1} V$. Thus lines of least slope correspond to maximum specific range.

3. EXAMPLES WHERE QRSS CONTROL IS BETTER

In this section two examples are considered where $J_{QRSS}^*(V_{min})$ is greater than both $J_{RSS}^*(V_{min})$ and $J_{QSS}^*(V_{min})$. For each example the functions $f_{SS}, f_{RSS}, f_{QSS}, f_{QRSS}, J_{SS}^*, J_{RSS}^*, J_{QSS}^*$ and J_{QRSS}^* are characterized and there is some general discussion. The details of the derivations entail simple analytic geometry and are therefore omitted. It is easy to be convinced that the form of the results is correct by sketching the functions involved.

Example 1

The drag and fuel-consumption functions are given by

$$D(V) = \begin{cases} 0{\cdot}1V + 2{\cdot}7V^2 - 1{\cdot}8V^3, & 0 \leqslant V \leqslant 1, \\ 0{\cdot}9 + 0{\cdot}1V, & 1 \leqslant V \end{cases} \tag{25}$$

and

$$F(T) = \begin{cases} 0{\cdot}05 + 1{\cdot}2T, & 0 \leqslant T \leqslant 0{\cdot}5, \\ 0{\cdot}25 + 0{\cdot}8T, & 0{\cdot}5 \leqslant T \leqslant 1. \end{cases} \tag{26}$$

Since $D(1) = 1$ and $D(0{\cdot}5) = 0{\cdot}5$, $V_{max} = 1$ and

$$f_{SS}(V) = \begin{cases} 0{\cdot}05 + 1{\cdot}2D(V), & 0 \leqslant V \leqslant 0{\cdot}5, \\ 0{\cdot}25 + 0{\cdot}8D(V), & 0{\cdot}5 \leqslant V \leqslant 1. \end{cases} \tag{27}$$

Note that $f_{SS}(\cdot)$ is continuous because $F(\cdot)$ is continuous. Since $F(\cdot)$ is concave $(\text{conv}\, F)(T) = 0{\cdot}05 + T < F(T)$ for $0 < T < 1$. Thus

$$f_{RSS}(V) = 0{\cdot}05 + D(V) \tag{28}$$

and $f_{RSS}(V) < f_{SS}(V)$ for $0 < V < 1$. For $0 < V < 1$ the RSS control switches at 'high' frequency between $T_1 = 1$ and $T_2 = 0$ and λ in (11) is determined by $D(V) = \lambda$. The function $f_{SS}(\cdot)$ is convex on $[0, 0{\cdot}5]$ but concave on $[0{\cdot}5, 1]$. Thus $f_{QSS}(V) < f_{SS}(V)$ for some V. In fact

$$f_{QSS}(V) = \begin{cases} 0{\cdot}05 + 1{\cdot}2D(V) = f_{SS}(V), \\ \quad 0 \leqslant V \leqslant V_1, \\ 1{\cdot}05 + (1{\cdot}0805\ldots)(V-1) \\ \quad < f_{SS}(V), \quad V_1 < V < 1, \end{cases} \tag{29}$$

where $V_1 = (0{\cdot}1809\ldots)$. For $V_1 < V < 1$ 'low' frequency switching between V_1 and $V_2 = 1$ occurs and λ is determined by $V = \lambda V_1 + 1 - \lambda$. Inspection also shows that $f_{RSS}(\cdot)$ is not convex and it turns out that

$$f_{QRSS}(V) = \begin{cases} 0{\cdot}05 + D(V) = f_{RSS}(V), \\ \quad 0 \leqslant V \leqslant 0{\cdot}25, \\ 1{\cdot}05 + 1{\cdot}1125(V-1) < f_{RSS}(V), \\ \quad 0{\cdot}25 < V < 1. \end{cases} \tag{30}$$

For $0{\cdot}25 < V < 1$ there is 'low' frequency switching between $V_1 = 0{\cdot}25$ and $V_2 = 1$ with λ determined by $V = 0{\cdot}25\lambda + 1 - \lambda = 1 - 0{\cdot}75\lambda$. The speed V_1 is maintained by 'high' frequency switching between $T_1 = 1$ and $T_2 = 0$ with λ_R, the fraction of time spent at $T(t) = T_1$, being determined by $\lambda_R = D(0{\cdot}25) = 0{\cdot}165625$. The speed $V_2 = 1$ is maintained by a constant $T(t) \equiv +1$. Clearly, $f_{QRSS}(V) < f_{QSS}(V)$ for $0 < V < 1$. In the neighborhood of $V = 0$, $f_{RSS}(V) < f_{QSS}(V)$; in the neighborhood of $V = 1$, $f_{QSS}(V) < f_{RSS}(V)$. Therefore neither $f_{RSS}(\cdot) \leqslant f_{QSS}(\cdot)$ nor $f_{QSS}(\cdot) \leqslant f_{RSS}(\cdot)$ is possible.

The implications of the above results with respect to $J_{SS}^*(\cdot)$, $J_{RSS}^*(\cdot)$, $J_{QSS}^*(\cdot)$ and $J_{QRSS}^*(\cdot)$ is shown in Fig. 2. The values of a, b, c, d and e are: (0·1374...), (0·1524...), (0·2897...), (0·3833...) and (0·4363...). Each of the four types of steady-state control apply for V_{min} in certain intervals: QRSS for (0·25, 1), QSS for (0·1809..., 1), RSS for $[0, e)$ and SS for $[0, 1]$. For $V_{min} \in [0, a]$, SS control requires $T = D(a)$; for $V_{min} \in [0, b]$, RSS control requires $\lambda = D(b)$; for $V_{min} \in [c, 1]$, SS control requires $T = 1$. Neither $J_{RSS}^*(\cdot) \leqslant J_{QSS}^*(\cdot)$ nor $J_{QSS}^*(\cdot) \leqslant J_{RSS}^*(\cdot)$ holds true.

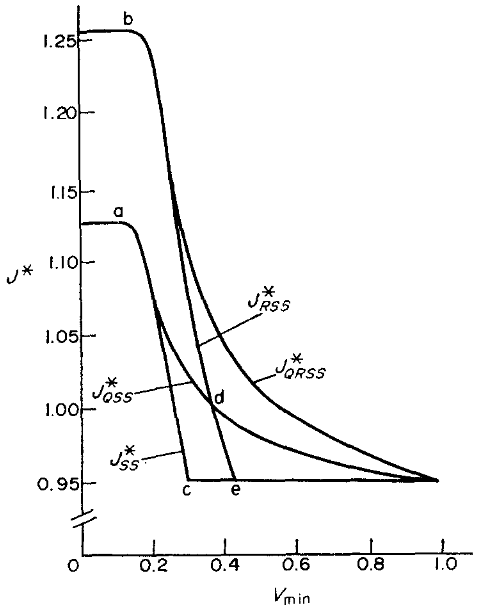

FIG. 2. Optimal cost functions for Example 1.

Example 2

The drag and fuel-consumption functions are given by

$$D(V) = \begin{cases} 0{\cdot}25V + 1{\cdot}25V^2 - 0{\cdot}5V^3, & 0 \leqslant V \leqslant 1, \\ -0{\cdot}25V + 1{\cdot}25V, & 1 \leqslant V, \end{cases} \tag{31}$$

and

$$F(T) = 0{\cdot}1 + D^{-1}(T) + 0{\cdot}3(D^{-1}(T))^2, \tag{32}$$

where $D^{-1}(T)$ is the inverse function corresponding to $D(\cdot)$. Since $D(1) = 1$, $V_{max} = 1$. Clearly,

$$f_{SS}(V) = 0{\cdot}1 + V + 0{\cdot}3V^2. \tag{33}$$

It can be shown that $D^{-1}(T) \geqslant W(T)$ where

$$W(T) = \begin{cases} \frac{4}{3}T, & 0 \leqslant T \leqslant \frac{3}{8}, \\ \frac{1}{5} + \frac{4}{5}T, & \frac{3}{8} \leqslant T \leqslant 1. \end{cases} \quad (34)$$

Using this inequality it follows that $F(T) > 0{\cdot}1 + 1{\cdot}3T$ for $0 < T < 1$. Thus $(\text{conv}\, F)(T) = 0{\cdot}1 + 1{\cdot}3T$ and

$$f_{RSS}(V) = 0{\cdot}1 + 1{\cdot}3D(V). \quad (35)$$

Hence $f_{RSS}(V) < f_{SS}(V)$ for $0 < V < 1$. Since $f_{SS}(\cdot)$ is convex

$$f_{QSS}(V) = f_{SS}(V). \quad (36)$$

Because the second derivative of $D(\cdot)$ changes sign in $[0, 1]$, $f_{RSS}(\cdot)$ is not convex and this leads to

$$f_{QRSS}(V) = \begin{cases} f_{RSS}(V), & 0 \leqslant V \leqslant 0{\cdot}75, \\ 1{\cdot}4 + 1{\cdot}665625(V-1) < f_{RSS}(V), & \\ \quad 0{\cdot}75 < V < 1. \end{cases} \quad (37)$$

These results yield the following expressions:

$$J_{SS}^*(V) = J_{QSS}^*(V) = \begin{cases} (0{\cdot}7427\ldots), & \\ \quad 0 \leqslant V \leqslant 3^{-0{\cdot}5}, & \\ V(f_{SS}(V))^{-1}, & \\ \quad 3^{-0{\cdot}5} \leqslant V \leqslant 1, \end{cases} \quad (38)$$

$$J_{RSS}^*(V) = \begin{cases} (0{\cdot}9206\ldots), & 0 \leqslant V \leqslant (0{\cdot}2818\ldots), \\ V(f_{RSS}(V))^{-1}, & (0{\cdot}2818\ldots) \leqslant V \leqslant 1, \end{cases} \quad (39)$$

$$J_{QRSS}^*(V) = \begin{cases} J_{RSS}^*(V), & 0 \leqslant V \leqslant 0{\cdot}75, \\ V(f_{QRSS}(V))^{-1}, & \\ \quad 0{\cdot}75 \leqslant V \leqslant 1. \end{cases} \quad (40)$$

Thus

$$J_{SS}^*(V) = J_{QSS}^*(V) < J_{RSS}^*(V) = J_{QRSS}^*(V) \quad \text{for} \quad V \in [0, 0{\cdot}75]$$

and

$$J_{SS}^*(V) = J_{QSS}^*(V) < J_{RSS}^*(V) < J_{QRSS}^*(V) \quad \text{for } V \in (0{\cdot}75, 1).$$

Therefore RSS control, i.e. 'high' frequency switching between $T = 1$ and $T = 0$, gives better results than SS control for all $0 < V_{min} < 1$. Even though QSS control offers no improvement over SS control, QRSS control, i.e. 'low' frequency switching between $T = 1$ and a relaxed T(t) ('high' frequency switching between $T_1 = 1$ and $T_2 = 0$ with $\lambda_R = D(0{\cdot}75)$), provides an improvement over RSS control.

4. SOME GENERAL PROPERTIES OF THE OPTIMAL COST FUNCTIONS

In this section general properties of the functions $J_{SS}^*(\cdot)$, $J_{RSS}^*(\cdot)$, $J_{QSS}^*(\cdot)$ and $J_{QRSS}^*(\cdot)$ are derived. Some of these properties depend on additional assumptions concerning $D(\cdot)$ and $F(\cdot)$ and give information about the possibility (or impossibility) of performance improvement by means of RSS, QSS and QRSS controls.

All four of the SS problems are of the form

$$J_A = V(f_A(V))^{-1}, \quad V_{min} \leqslant V \leqslant V_{max} \quad (41)$$

with $J_A^*(\cdot)$ defined by

$$J_A^*(V_{min}) = \max V(f_A(V))^{-1} \quad \text{subject to} \quad V \in [V_{min}, V_{max}]. \quad (42)$$

Obviously, $J_A^*(\cdot)$ is a non-increasing function on $[0, V_{max}]$. Lower semi-continuity of $f_A(\cdot)$ implies upper semi-continuity of $J_A^*(\cdot)$. Continuity of $f_A(\cdot)$ implies continuity of $J_A^*(\cdot)$, but the converse is not true. Clearly $J_{RSS}^*(\cdot)$, $J_{QSS}^*(\cdot)$ and $J_{QRSS}^*(\cdot)$ are all continuous.

To obtain additional properties of $J_A^*(\cdot)$ define

$$\hat{V}_A = \max V \quad \text{subject to } V(f_A(V))^{-1} = J_A^*(0). \quad (43)$$

The maximum exists because the upper semi-continuity of $V(f_A(V))^{-1}$ implies that the set corresponding to $V(f_A(V))^{-1} = J_A^*(0)$ is closed. Clearly

$$J_A^*(V) = J_A^*(0), \quad 0 \leqslant V \leqslant \hat{V}_A$$
$$< J_A^*(0), \quad \hat{V}_A < V < V_{max}. \quad (44)$$

With an additional assumption on $f_A(\cdot)$ more can be said.

Theorem 1

Let $f_A(\cdot)$ be convex on $[\hat{V}_A, V_{max}]$. Then $J_A^*(\cdot)$ is continuous on $[0, V_{max}]$, $J_A^*(\cdot)$ is strictly decreasing on $[\hat{V}_A, V_{max}]$ and

$$J_A^*(V) = V(f_A(V))^{-1}, \quad \hat{V}_A \leqslant V \leqslant V_{max}. \quad (45)$$

The proof depends on the following lemma whose proof is straightforward and is therefore omitted.

Lemma

Let

$$g(V) = aV + b \quad (46)$$

have the following properties: $f_A(V_0) = g(V_0)$ where $V_0 \in (\hat{V}_A, V_{max}]$, $f_A(\cdot) \geqslant g(\cdot)$. That is, $g(\cdot)$ is a line of support to $f_A(\cdot)$ at $V_0 \in (V_A, V_{max}]$. Then $a > (J_A^*(0))^{-1}$ and $b < 0$.

Proof. Applying the lemma gives

$$\begin{aligned} V(f_A(V))^{-1} - V_0(f_A(V_0))^{-1} &= (Vf_A(V_0) - V_0 f_A(V)) \\ &\quad \times (f_A(V_0) f_A(V))^{-1} \\ &\leqslant (V(aV_0 + b) \\ &\quad - V_0(aV + b)) \\ &\quad \times (f_A(V_0) f_A(V))^{-1} \\ &\leqslant b(V - V_0)(f_A(V_0) \\ &\quad \times f_A(V))^{-1}, \\ &\quad 0 \leqslant V \leqslant V_{max} \\ &< 0, \quad V > V_0. \end{aligned} \quad (47)$$

Thus $V(f_A(V))^{-1}$ is strictly decreasing on $(\hat{V}_A, V_{\max}]$. From (42) it then follows that (45) is true and that $J_A{}^*(\cdot)$ is continuous because $f_A(\cdot)$ is continuous.

Corollary 2

For $A = \text{QSS}$ and $A = \text{QRSS}$ the results of Theorem 1 are true. Moreover $\hat{V}_{\text{QSS}} = \hat{V}_{\text{SS}}$ and $\hat{V}_{\text{QRSS}} = \hat{V}_{\text{RSS}}$.

Proof. The first results are consequences of the convexity of $f_{\text{QSS}}(\cdot)$ and $f_{\text{QRSS}}(\cdot)$. The geometric interpretation of $J_A{}^*(0)$ is that $V(J_A{}^*(0))^{-1}$ is a line of support of $f_A(\cdot)$ as indicated at the end of Section 2. But $(\text{conv} f_{\text{SS}})(\cdot) = f_{\text{QSS}}(\cdot)$ and $f_{\text{SS}}(\cdot)$ have identical lines of support [5]. Thus $J_{\text{SS}}{}^*(0) = J_{\text{QSS}}{}^*(0)$ and clearly $\hat{V}_{\text{QSS}} \geqslant V_{\text{SS}}$. But $\hat{V}_{\text{QSS}} > \hat{V}_{\text{SS}}$ implies

$$V(f_{\text{QSS}}(V))^{-1} = V(f_{\text{SS}}(V))^{-1} \quad \text{for} \quad V = \hat{V}_{\text{QSS}}$$

which contradicts the definition of $\hat{V}_{\text{SS}}$. Therefore $\hat{V}_{\text{QSS}} = \hat{V}_{\text{SS}}$. The identical argument applies to $\hat{V}_{\text{QRSS}} = \hat{V}_{\text{RSS}}$.

Remark 1

Note that

$$J_{\text{QRSS}}{}^*(0) = J_{\text{RSS}}{}^*(0) \geqslant J_{\text{QSS}}{}^*(0) = J_{\text{SS}}{}^*(0).$$

Thus if V_{avg} is not constrained ($V_{\min} = 0$) low frequency switching produces no improvement. This is in agreement with [6].

Under what circumstances does RSS control produce an improvement? The following theorem seems to be about the most that can be said.

Theorem 3

Let

$$Q = \{V\colon J_{\text{SS}}{}^*(V) = V(f_{\text{SS}}(V))^{-1},\ V \in [0, V_{\max}]\}. \quad (48)$$

(i) If $(\text{conv} F)(T) < F(T)$ for some $T \in D(Q)$ then $J_{\text{RSS}}{}^*(V) > J_{\text{SS}}{}^*(V)$ for some $V \in [0, V_{\max}]$. (ii) If $J_{\text{RSS}}{}^*(V) > J_{\text{SS}}{}^*(V)$ for some $V \in [0, V_{\max}]$ then $(\text{conv} F)(T) < F(T)$ for some $T \in [0, 1]$.

Proof. Result (ii) is obvious. Result (i) follows because for V such that $D(V) = T$, $V(f_{\text{RSS}}(V))^{-1} > J_{\text{SS}}{}^*(V)$.

Remark 2

It is easy to construct examples which demonstrate the gap between (i) and (ii). Consider, for example, $V(f_{\text{SS}}(V))^{-1} < V_2(f_{\text{SS}}(V_2))^{-1}$ for all $V \in (V_1, V_2)$, $0 \leqslant V_1 < V_2 \leqslant V_{\max}$, and $f_{\text{RSS}}(\cdot)$ such that $f_{\text{RSS}}(V) < f_{\text{SS}}(V)$ for some $V \in (V_1, V_2)$ and $V(f_{\text{RSS}}(V))^{-1} \leqslant V_2(f_{\text{SS}}(V_2))^{-1}$ for all $V \in [V_1, V_2]$.

Remark 3

If $F(\cdot)$ is convex on $[0, 1]$ then $J_{\text{RSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot)$ and $J_{\text{QRSS}}{}^*(\cdot) = J_{\text{QSS}}{}^*(\cdot)$.

In the case of QSS control and QRSS control both necessary and sufficient conditions are available for improvement (or non-improvement).

Theorem 4

$J_{\text{QSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot)$ $(J_{\text{QRSS}}{}^*(\cdot) = J_{\text{RSS}}{}^*(\cdot))$ if and only if $f_{\text{SS}}(\cdot)$ is convex on $[\hat{V}_{\text{SS}}, V_{\max}]$ ($f_{\text{RSS}}(\cdot)$ is convex on $[\hat{V}_{\text{RSS}}, V_{\max}]$).

Proof. With an appropriate change in notation the two cases have the same proof so consider only the QSS case. If $f_{\text{SS}}(\cdot)$ is convex on $[\hat{V}_{\text{SS}}, V_{\max}]$ then $f_{\text{SS}}(\cdot) \geqslant h(\cdot)$ where $h(V) = (J_{\text{SS}}{}^*(0))^{-1} V$ for $0 \leqslant V \leqslant \hat{V}_{\text{SS}}$, $= f_{\text{SS}}(V)$ for $\hat{V}_{\text{SS}} \leqslant V \leqslant V_{\max}$. Since $h(\cdot)$ is convex $f_{\text{SS}}(\cdot) \geqslant (\text{conv} f_{\text{SS}})(\cdot) \geqslant h(\cdot)$. Thus $f_{\text{SS}}(\cdot) = (\text{conv} f_{\text{SS}})(\cdot)$ on $[\hat{V}_{\text{SS}}, V_{\max}]$ and because of Corollary 2, $J_{\text{QSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot)$. Now suppose $J_{\text{QSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot)$ and $f_{\text{SS}}(\cdot)$ is not convex on $[\hat{V}_{\text{SS}}, V_{\max}]$. Then there is a line of support to $f_{\text{SS}}(\cdot)$ which contacts $f_{\text{SS}}(\cdot)$ at V_1 and V_2, $\hat{V}_{\text{SS}} \leqslant V_1 < V_2 \leqslant V_{\max}$ and does not contact $f_{\text{SS}}(\cdot)$ on (V_1, V_2), i.e. $g(V_i) = f_{\text{SS}}(V_i)$, $i = 1, 2$ and $g(V) = f_{\text{SS}}(V)$ for $V \in (V_1, V_2)$. Moreover, $g(V) = f_{\text{QSS}}(V)$ for $V \in [V_1, V_2]$ and $g(V) \leqslant f_{\text{SS}}(V)$ for $V \in [0, V_{\max}]$. Take $V_{\min} \in (V_1, V_2)$. Because $J_{\text{QSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot)$

$$V(f_{\text{SS}}(V))^{-1} = J_{\text{QSS}}{}^*(V_{\min}) \quad (49)$$

has a solution for some $V \in [V_{\min}, V_{\max}]$. Using $g(\cdot) \leqslant f_{\text{SS}}(\cdot)$ this means that the system

$$V(aV + b)^{-1} \leqslant V_{\min}(aV_{\min} + b)^{-1}, \quad V \in [V_{\min}, V_{\max}] \quad (50)$$

has a solution. But this can be written

$$0 \leqslant b(V - V_{\min}),\ V \in [V_{\min}, V_{\max}]. \quad (51)$$

Since by the lemma $b < 0$, $V = V_{\min}$. Substituting into (49), $f_{\text{SS}}(V_{\min}) = f_{\text{QSS}}(V_{\min})$ which contradicts $f_{\text{QSS}}(V) = g(V) < f_{\text{SS}}(V)$ for $V \in (V_1, V_2)$.

Remark 4

If $f_{\text{SS}}(\cdot)$ has a discontinuity on $[\hat{V}_{\text{SS}}, V_{\max}]$, $f_{\text{SS}}(\cdot)$ is not convex on $[\hat{V}_{\text{SS}}, V_{\max}]$. Thus $J_{\text{QSS}}{}^*(V) > J_{\text{SS}}{}^*(V)$ for some $V \in [0, V_{\max}]$.

Corollary 5

If $F(\cdot)$ is convex on $[0, 1]$ and $D(\cdot)$ is convex on $[\hat{V}_{\text{SS}}, V_{\max}]$ then

$$J_{\text{QRSS}}{}^*(\cdot) = J_{\text{QSS}}{}^*(\cdot) = J_{\text{RSS}}{}^*(\cdot) = J_{\text{SS}}{}^*(\cdot).$$

Proof. The first and last equalities follow from Remark 3. Since both $F(\cdot)$ and $D(\cdot)$ are convex and $F(\cdot)$ is non-decreasing it turns out [5] that $f_{\text{SS}}(V) = F(D(V))$ is convex on $[\hat{V}_{\text{SS}}, V_{\max}]$. Thus the theorem gives middle equality.

Corollary 6

If $D(\cdot)$ is convex on $[\hat{V}_{RSS}, V_{max}]$ then

$$J_{QRSS}^*(\cdot) = J_{RSS}^*(\cdot).$$

Proof. $f_{RSS}(\cdot)$ is a non-decreasing convex function of a convex function and is therefore convex on $[\hat{V}_{RSS}, V_{max}]$. The theorem applies immediately.

Remark 5

There are examples which show that $J_{QSS}^*(V) > J_{SS}^*(V)$ and $J_{RSS}^*(V) > J_{SS}^*(V)$ are both possible when $D(\cdot)$ is convex. Thus it is not possible to draw a stronger conclusion that Corollary 6.

There is another conclusion which can be made about the lack of superiority of $J_{QRSS}^*(\cdot)$.

Theorem 7

If $J_{QSS}^*(\cdot) \geq J_{RSS}^*(\cdot)$ then $J_{QRSS}^*(\cdot) = J_{QSS}^*(\cdot)$.

Proof. By Corollary 2 and the definition of $J_{RSS}^*(\cdot)$, $V(f_{RSS}(V))^{-1} \leq V_{min}(f_{QSS}(V_{min}))^{-1}$ for all $V \geq V_{min} \geq \hat{V}_{SS}$. Thus $f_{RSS}(V) \geq f_{QSS}(V)$ for $V \in [\hat{V}_{SS}, V_{max}]$. Also $f_{RSS}(V) \geq (J_{QSS}^*(0))^{-1} V$ for $V \in [0, \hat{V}_{SS}]$. Let $f_A(V) = f_{QSS}(V)$ for $V \in [\hat{V}_{SS}, V_{max}]$, $= (J_{QSS}^*(0))^{-1} V$ for $V \in [0, \hat{V}_{SS}]$. Since $f_A(\cdot)$ is convex and $f_{RSS}(\cdot) \geq f_A(\cdot)$, $f_{QRSS}(\cdot) = (\text{conv} f_{RSS})(\cdot) \geq f_A(\cdot)$. But $J_A^*(\cdot) = J_{QSS}^*(\cdot)$ and hence $J_{QRSS}^*(\cdot) \leq J_A^*(\cdot) = J_{QSS}^*(\cdot)$. Since $J_{QRSS}^*(\cdot) \geq J_{QSS}^*(\cdot)$ the proof is complete.

Remark 6

Theorem 7 shows that QRSS control can be better than both RSS and QSS control only if $J_{QSS}^*(\cdot) \geq J_{RSS}^*(\cdot)$ is not true, i.e. $J_{QSS}^*(V) < J_{RSS}^*(V)$ for some $V \in [0, V_{max}]$. Examples 1 and 2 of the previous section demonstrate this requirement. In Example 1 neither $J_{QSS}^*(\cdot) \geq J_{RSS}^*(\cdot)$ nor $J_{RSS}^*(\cdot) \geq J_{QSS}^*(\cdot)$ holds. In Example 2 $J_{RSS}^*(\cdot) \geq J_{QSS}^*(\cdot)$ and in addition $J_{QSS}^*(\cdot) = J_{SS}^*(\cdot)$. It is not difficult to construct examples where $J_{RSS}^*(\cdot) \geq J_{QSS}^*(\cdot)$ and $J_{QSS}^*(\cdot) \neq J_{SS}^*(\cdot)$. Thus all the possibilities of $J_{QRSS}^*(V) > J_{RSS}^*(V)$ and $J_{QRSS}^*(V) > J_{QSS}^*(V)$ allowed by Theorem 7 do in fact exist.

5. OTHER PERIODIC CONTROLS, SOLUTION OF OPC

In Section 2 $J^*(V_{min})$ was defined as the supremum of (1) subject to constraints (2–6) and it was noted that $J^*(\cdot) \geq J_{QRSS}^*(\cdot) \geq J_{RSS}^*(\cdot)$, $J_{QSS}^*(\cdot) \geq J_{SS}^*(\cdot)$. This raises two questions. (i) Do there exist periodic controls which perform better than QRSS controls ($J^*(V) > J_{QRSS}^*(V)$ for some $V \in [0, V_{max}]$)? (ii) Does OPC have a solution ($J^*(V_{min}) = \max J(T(\cdot), V(\cdot), \tau)$)? In general, question (ii) must be answered negatively. It is easy to give examples where $J^*(V) = J_{RSS}^*(V)$ or $J^*(V) = J_{QSS}^*(V)$ and there is no triple $(T(\cdot), V(\cdot), \tau)$ which satisfies (2–6) and achieves the supremum.

To answer question (i), at least in part, an upper bound for $J(T(\cdot), V(\cdot), \tau)$ will be derived. Define

$$D_{avg} = \frac{1}{\tau}\int_0^\tau D(V(t))\,dt \tag{52}$$

and note from (2) that $D_{avg} = T_{avg}$. Write $V(t) = V_{avg} + \Delta V(t)$ and observe that

$$D(V(t)) \geq (\text{conv}\, D)(V_{avg} + \Delta V(t))$$
$$\geq (\text{conv}\, D)(V_{avg}) + q\Delta V(t)$$

where q is the slope of a line of support to $(\text{conv}\, D)(\cdot)$ at V_{avg}. By definition the average of $\Delta V(t)$ over the interval $[0, \tau]$ is zero and thus

$$D_{avg} \geq \frac{1}{\tau}\int_0^\tau [(\text{conv}\, D)(V_{avg}) + q\Delta V(t)]\,dt$$
$$\geq (\text{conv}\, D)(V_{avg}). \tag{53}$$

In the same manner it follows that

$$F_{avg} \geq (\text{conv}\, F)(T_{avg}). \tag{54}$$

Using (53), (54), $T_{avg} = D_{avg}$, and the fact that $(\text{conv} F)(\cdot)$ is a non-decreasing function gives $F_{avg} \geq (\text{conv}\, F)((\text{conv}\, D)(V_{avg}))$. This yields the desired upper bound:

$$J(V(\cdot), T(\cdot), \tau) \leq [(\text{conv}\, F)((\text{conv}\, D) \times (V_{avg}))]^{-1} V_{avg}. \tag{55}$$

If $D(\cdot)$ is convex (55) gives

$$J^*(V_{min}) = \sup J(V(\cdot), T(\cdot), \tau)$$
$$\leq \max V_{avg}(f_{RSS}(V_{avg}))^{-1}$$
$$\text{subject to } V_{avg} \in [V_{min}, V_{max}]$$
$$\leq J_{RSS}^*(V_{min}). \tag{56}$$

Since $J_{RSS}^*(V_{min}) \leq J^*(V_{min})$ this proves the following.

Theorem 8.

If $D(\cdot)$ is convex $J^*(\cdot) = J_{RSS}^*(\cdot)$.

Remark 7.

When $D(\cdot)$ is convex $J^*(\cdot) = J_{QRSS}^*(\cdot) = J_{RSS}^*(\cdot) \geq J_{QSS}^*(\cdot) \geq J_{SS}^*(\cdot)$ (see Corollary 6). Thus there is no triple $(V(\cdot), T(\cdot), \tau)$ which does better than RSS control.

Remark 8.

When $D(\cdot)$ and $F(\cdot)$ are convex

$$J^*(\cdot) = J_{QRSS}{}^*(\cdot) = J_{RSS}{}^*(\cdot) = J_{QSS}{}^*(\cdot) = J_{SS}{}^*(\cdot)$$

per Corollary 5. Thus the SS control is a solution of OPC.

In searching through many examples it has not been possible to show that the following is false.

Conjecture

$$J^*(\cdot) = J_{QRSS}{}^*(\cdot).$$

This conjecture answers questions (i) negatively with no additional assumptions on D and F. If the conjecture is true it is a strong result which depends on the special structure of the OPC problem stated in Section 1. There is no reason to expect that it would hold in more complex vehicle cruise problems.

6. CONCLUSIONS, POSSIBLE APPLICATIONS

A simple model (1–6) for optimizing vehicle cruise has been presented and it has been shown that QRSS, RSS and QSS control may all produce better results than conventional SS control. The potential advantage of QRSS control has been demonstrated, although such advantage requires that both $D(\cdot)$ and $F(\cdot)$ be non-convex as stated in Remark 3 and Corollary 6. Furthermore, if both $D(\cdot)$ and $F(\cdot)$ are convex all modes of steady-state control are equivalent to conventional SS control, per Corollary 5, which is a solution of OPC as indicated in Remark 8.

The simple model is probably not an accurate description of most practical vehicle cruise problems, but it is sufficiently good to suggest practical applications. Consider, for instance, Fig. 3, which shows fuel-flow rate vs speed for a typical model of a gas turbine ship [7]. Although the dynamic model of the ship is much more complex than (2), $f(V)$ plays the same role as $f_{SS}(V)$ in this paper. Thus for $V_{min} \leqslant V_1$ maximum specific range (steady-state conditions) is obtained at $V = V_1$ with one engine operating. However, for $V_{min} \in (V_2, V_3)$ quasi-steady-state operation produces better fuel economy than two-engine steady-state operation as implied by Remark 4. For the model in [7] the difference may be as much as 16%. The quasi-steady-state mode requires cycling between the speed V_2 (one engine) and the speed V_3 (two engines), but the period is very long, perhaps hours, and thus the cycling should not be difficult to implement. Another possible application is to aircraft cruise. If an optimal periodic control problem is formulated for the aircraft ($F-4$) discussed in [8] it is possible to show that relaxed-steady-state control gives improved cruise performance. However, there are questions concerning the accuracy of the airplane model which must be investigated further.

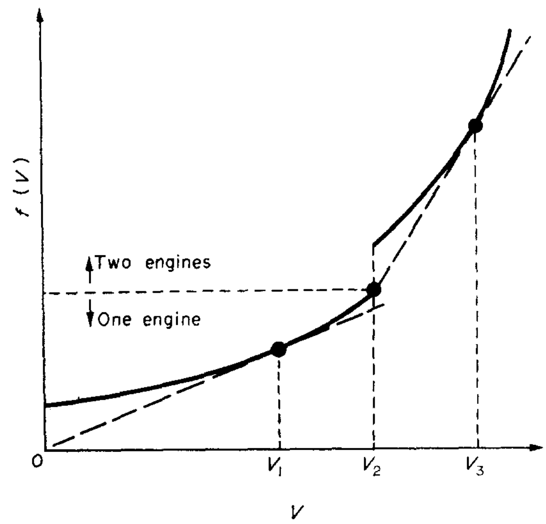

FIG. 3. Steady-state fuel rate for ship.

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