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On the Vacuum State for the Isentropic Gas Dynamics Equations

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1. INTRODUCTION

The equations of gas dynamics have been studied by several authors, (e.g., [1, 3, 6–8, 13, 14, 16, 17]) but always in regions bounded away from a vacuum. In this paper, we investigate some properties of solutions containing the vacuum state. For simplicity, we assume that the gas is isentropic, so that the dynamics are modeled by the following equations in Eulerian coordinates:

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = 0, \qquad -\infty < x < \infty, t > 0. \end{aligned} \tag{1}$$

Here $\rho = \rho(x,t)$ and $u = u(x,t)$ denote respectively, the density and speed of the gas, and p, the pressure, is a given function of ρ. We assume that p satisfies

$$p(0) = p'(0) = 0, \qquad \text{and} \qquad p' > 0, p'' > 0 \qquad \text{in} \qquad \rho > 0. \tag{2}$$

For example, one can take $p(\rho) = \rho^\gamma, \gamma > 1$.

By definition, a vacuum state is any portion of the $x - t$ plane in which $\rho = 0$. In Section 2 we show that a vacuum state must be bounded by rarefaction waves. We then study the ways in which it is possible to go from a nonvacuum to a vacuum state (see [15]). There are, in fact two distinct types of vacuums which we call compression and rarefaction vacuums; the compression vacuums differ from the rarefaction vacuums in that they give rise to compression waves which eventually form shock

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waves. Having considered these notions, we solve the Riemann problem when the initial data contains a vacuum, and we show that our solution is stable with respect to perturbations of the initial data. In Section 4, we observe some distinct new qualitative features of the solutions in the presence of a vacuum. Thus, for initial data consisting of a vacuum in a neighborhood of $x = -\infty$, we show by examples that eventually, only a single rarefaction wave bordering the vacuum survives, and all other shock waves of any strength, disappear; indeed, some even in finite time. Finally, we observe that due to the existence of bounded invariant regions in $\rho - u$ space, the Glimm difference scheme [2] for approximate solutions is globally defined. However, we are unable to prove the convergence of the approximate solutions, due to the fact that near a vacuum state, wave interactions produce such strong nonlinear effects that Glimm's existence theorem cannot be applied—the main estimate simply fails. Indeed, we believe that a resolution of this difficulty will be a major step in solving (1) with "big" initial data.

2. Preliminaries

We shall begin our study of the vacuum state by first showing that a shock wave cannot enter into a vacuum region; i.e., that a vacuum cannot be adjacent to a shock wave. In order to see this, we consider the Rankine–Hugoniot jump conditions for (1):

$$\sigma(\rho_0 - \rho) = \rho_0 u_0 - \rho u, \sigma(\rho_0 u_0 - \rho u) = \left(\rho_0 u_0^2 - \rho u^2 + p(\rho_0) - p(\rho)\right), \tag{3}$$

where σ is the shock speed. If we assume that $\rho_0 \neq 0$ and $\rho = 0$, then, if $|u| < \infty$, the equations become $\sigma = u_0$ and $\sigma \rho_0 u_0 = \rho_0 u_0^2 + p(\rho_0)$, so $p(\rho_0) = 0$, and thus from (2), $\rho_0 = 0$. It follows that $|u| = \infty$; that is, the shock curve can never meet the line $\rho = 0$.

We next consider the rarefaction-wave curves. To compute them, we set $v = \rho u$ in (1) to get

$$\rho t + v_x = 0, \qquad v_t + \left(v^2 \rho^{-1} + p(\rho)\right)_x = 0.$$

If we let $F = (v, v^2\rho^{-1} + p(\rho))$, then

$$dF = \begin{bmatrix} 0 & 1 \\ -v^2\rho^{-2} + p' & 2v\rho^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + p' & 2u \end{bmatrix}$$

and so dF has eigenvalues

$$\lambda_{\pm} = u \pm \sqrt{p'(\rho)}\,,$$

with corresponding (right) eigenvectors

$$r_\pm = \left(1, u \pm \sqrt{p'(\rho)}\right)^t.$$

The rarefaction-wave curves, are, by definition, integral curves of the vector fields $r_\pm$. Thus, they satisfy the ordinary differential equations

$$\frac{d(u\rho)}{d\rho} = u \pm \sqrt{p'(\rho)},$$

whose solutions are

$$u - u_0 = \int_{\rho_0}^{\rho} \frac{1}{x}\sqrt{p'(x)}\, dx.$$

Now in order that these curves meet the line $\rho = 0$, we make the assumption (see [15])

$$\int_0^{\rho_0} \frac{1}{x}\sqrt{p'(x)}\, dx < \infty \tag{4}$$

for any $\rho_0 > 0$.[1] Then the $R_\pm$ rarefaction-wave curves are those points in $\rho \geq 0$ which satisfy

$$R_+ : u - u_0 = \int_{\rho_0}^{\rho} \frac{1}{x}\sqrt{p'(x)}\, dx, \qquad \rho \geq \rho_0 \geq 0$$

$$R_- : u - u_0 = -\int_{\rho_0}^{\rho} \frac{1}{x}\sqrt{p'(x)}\, dx, \qquad \rho_0 \geq \rho \geq 0. \tag{5}$$

Thus, we see at once that the vacuum can be connected to a rarefaction wave of either family, provided that (4) holds. In this case, the vacuum must be on the "right" of an R_- rarefaction wave and on the "left" of an R_+ rarefaction wave.

Now for later use, we shall compute the shock-wave curves. They satisfy the Rankine–Hugoniot conditions (3), and are thus given by

$$u - u_0 = \pm\sqrt{(\rho\rho_0)^{-1}(\rho - \rho_0)(p(\rho) - p(\rho_0))}.$$

In order to satisfy the "entropy inequalities" the $S_\pm$ shockwave curves are

[1]Note that if $p(\rho) = \rho^\gamma$, then (3) holds provided that $\gamma > 1$.

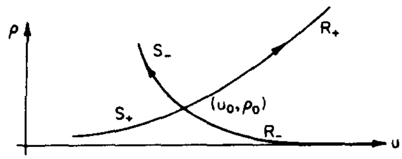

FIGURE 1

those points in $\rho \geq 0$ which satisfy the following:

$$S_+ : u - u_0 = -\sqrt{\frac{(p(\rho) - p(\rho_0))(\rho - \rho_0)}{\rho\rho_0}} \quad \rho_0 \geq \rho > 0,$$

$$S_- : u - u_0 = -\sqrt{\frac{(p(\rho) - p(\rho_0))(\rho - \rho_0)}{\rho\rho_0}} \quad \rho \geq \rho_0 > 0. \tag{6}$$

Using (5) and (6), we can depict the shock and rarefaction wave curves as in Fig. 1.

3. Compression and Rarefaction Vacuums

Our objectives in this section are to describe the possible vacuum states which are bounded by nonvacuum states, and to solve the Riemann problem when one state contains a vacuum.

That there are two different types of vacuums is due to the fact that in Eulerian coordinates, in the presence of a vacuum, it is possible that the "head" of an R_+ rarefaction wave can travel *slower* than the "tail" of an R_- rarefaction wave. Thus, suppose that a vacuum $\rho = 0$ appears in a region $a \leq x \leq b$, and that there is no vacuum in both $a - \epsilon < x < a$ and $b < x < b + \epsilon$, for some $\epsilon > 0$. Then the relative velocities at $x = a - 0$ and $x = b + 0$, determine the two types of vacuums, in the sense of the following definition.

DEFINITION. A vacuum state in a region $a \leq x \leq b$ (with $\rho \neq 0$ in both $a - \epsilon < x < a$ and $b < x < b + \epsilon$ for some $\epsilon > 0$) is called a compression vacuum if $u(a - 0) > u(b + 0)$; otherwise it is called a *rarefaction vacuum*.

The two distinctly different types of vacuums are illustrated in Figs. 2 and 3 below.

The above notions of compression and rarefaction vacuums make sense only when the vacuum state is bounded by nonvacuum states. In any case, the values of u need not be specified for a vacuum state since obviously

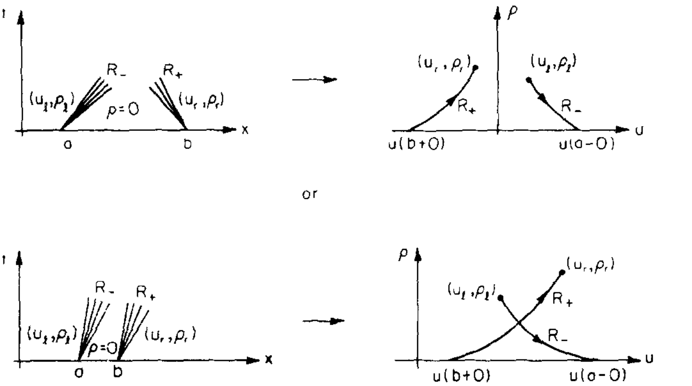

FIG. 2. Compression vacuum.

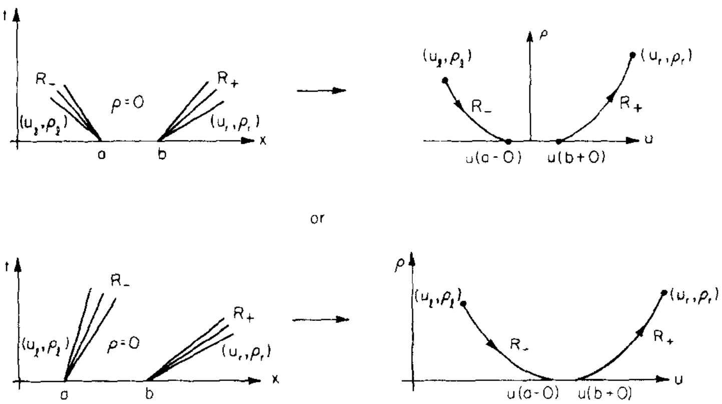

FIG. 3. Rarefaction vacuum.

$\rho = 0$ is always a solution of (1), no matter how the velocity u is defined. Furthermore, there is clearly no physical significance in speaking of the "speed" u of the gas in the vacuum region.

Keeping these notions in mind, we shall show how to solve the Riemann problem when one state is a vacuum. Thus we, consider the initial value problem for the system (1) with initial data consisting of two constant states

$$\begin{aligned} (u_0(x), \rho_0(x)) &= (u_l, \rho_l), \qquad x < 0, \\ &= (u_r, 0), \qquad x > 0, \end{aligned} \tag{7}$$

with $\rho_l > 0$. To solve this problem, we first construct the R_- curve through (u_l, ρ_l), and denote by $(\bar{u}, 0)$, the point where this curve meets $\rho = 0$. Now there are two cases to consider; namely if $\bar{u} > u_r$ or if $\bar{u} \le u_r$. If $\bar{u} > u_r$,

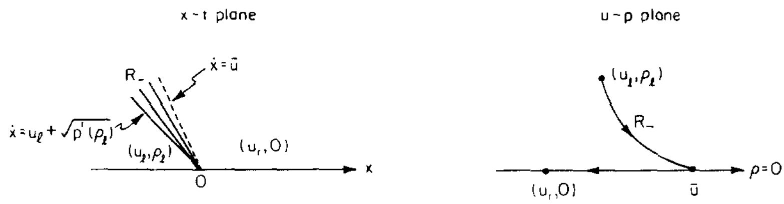

FIGURE 4

then the solution is denoted in Fig. 4. The interesting feature in this solution is that the R_- rarefaction wave enters into the vacuum region with speed $\bar{u}$, and the line having this speed separates the two vacuum states $(\bar{u}, 0)$ and $(u_r, 0)$ from each other. This choice of solution is forced upon us, if we want the solution to depend continuously on the initial data. Thus, if we perturb the state $(u_r, 0)$, to a state (u, ρ), where $\rho > 0$, near 0, and u is near u_r, we see that the Riemann problem for (1) with data (u_l, ρ_l) in $x < 0$, and (u, ρ) in $x > 0$ is solved by an R_- rarefaction wave, followed by an S_+ shock; see Fig. 5. Now it is easy to see that as $(u, \rho) \to (u_r, 0)$, $(u', \rho') \to (\bar{u}, 0)$. Moreover, the speed σ of the S_+ shock connecting (u', ρ') to (u, ρ) satisfies the equation $\sigma(\rho - \rho') = \rho u - \rho' u'$. Hence we have

$$\sigma - u = \frac{\rho'(u' - u)}{\rho' - \rho} > u' - u$$

since $\rho' > \rho$ and $u' > u$. Thus $\sigma > u'$. But the Lax stability condition [5] implies that $u' + \sqrt{p'(\rho')} > \sigma > u + \sqrt{p'(\rho)}$. Thus, in the limit, as ρ' and ρ tend to zero, we have $u' \to \bar{u}$ and $u \to u_r$, and σ tends to a limit σ_* which satisfies both $\bar{u} \geq \sigma_* \geq u_r$ and $\sigma_* \geq \bar{u}$; hence $\sigma_* = \bar{u}$.

Next, consider the case where $\bar{u} \leq u_r$. Here we obtain the solution depicted in Fig. 6. In the region $\bar{u} \leq x/t \leq u_r$, we define $u(x, t) = x/t$, and of course, $\rho(x, t) = 0$. This choice of solution is again forced upon us by stability considerations, as is illustrated in Fig. 7.

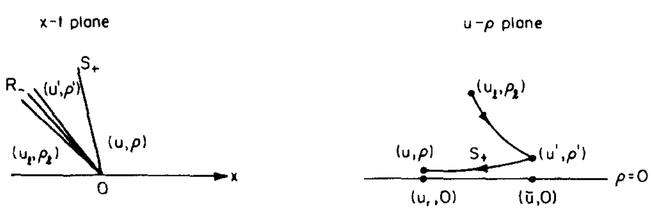

FIGURE 5

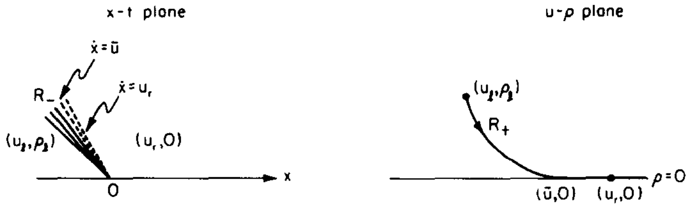

FIGURE 6

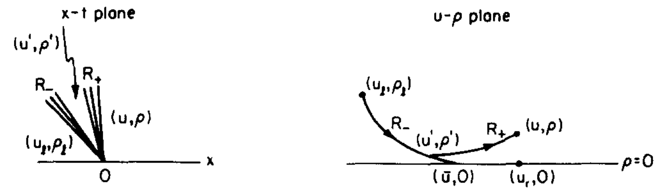

FIGURE 7

Note that we have tacitly assumed here that $u_r > \bar{u}$; in the case where $u_r = \bar{u}$, the (illusory) rarefaction wave disappears, and the solution consists of an R_- rarefaction wave connecting the state (u_l, ρ_l) to the state $(\bar{u}, 0)$.

Next, we note that if the initial data is of the form

$$\begin{aligned}(u_0(x), \rho_0(x)) = (u_l, \rho_l), \qquad & x < 0,\\ \rho = 0, \qquad & x > 0,\end{aligned}$$

then the solution is given in Fig. 4; namely, we connect (u_l, ρ_l) to the vacuum state by the complete rarefaction wave starting at (u_l, ρ_l) and ending at $(\bar{u}, 0)$. Lastly, if the data on the left of $x = 0$ is $(u_l, 0)$, then the solution of the Riemann problem is similar to the one we have discussed; merely replace the R_- curves by R_+ curves and S_+ curves by S_- curves.

We shall now briefly describe the interaction of rarefaction waves in the presence of a compression vacuum (see Fig. 1.) Here again there are two cases to consider; namely the *strong* compression vacuum $(u_l > u_r)$, and the *weak* compression vacuum, $(u_l \leq u_r)$. It is clear that the R_- and R_+ rarefaction waves interact with each other at a finite time $T > 0$. The problem consists of obtaining the solution for times $t \geq T$. This problem is quite difficult, and does not seem to be solvable by any of the known methods, (see Section 4). Thus, we shall merely give a plausability argument describing the dominant waves in the solution. In this argument, we shall use Glimm's technique (see [2]).

Thus, first consider the case of a strong compression vacuum, where $u_l \gg u_r$. Then referring to Fig. 2, if we choose "Riemann-type" data, $(\bar{u}_l, \bar{\rho}_l)$, $(\bar{u}_r, \bar{\rho}_r)$, where $(\bar{u}_l, \bar{\rho}_l)$ lies on the R_- curve between $\rho = \rho_l$ and $\rho = 0$, and $(\bar{u}_r, \bar{\rho}_r)$ lies on the R_+ curve between $\rho = \rho_r$ and $\rho = 0$, then we

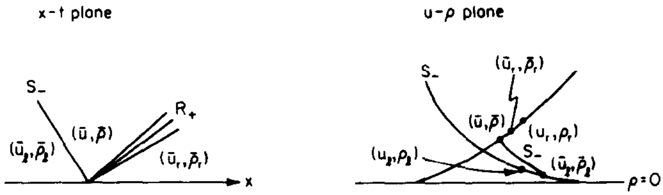

FIGURE 8

see that this Riemann problem is solved by an S_- shock followed by an S_+ shock. That is, every approximating solution contains two strong shock waves. It is thus reasonable to expect that the asymptotic state to which the solution approaches as $t \to +\infty$ will contain two shock waves moving away from each other.

In the case of a strong compression vacuum with $u_l > u_r$, but u_l close to u_r, we can only say that it is plausable to expect that the asymptotic state of the solution will contain an S_- shock wave; this follows since some of the approximating solutions will consist of an S_- shock wave and an R_+ rarefaction wave; see Fig. 8.

Finally, if we consider the case of weak compression vacuums, it is easy to see that the initial interaction is approximated by either an (S_-, R_+) solution, or an (R_-, R_+) solution. Thus it seems reasonable to expect that the solution will contain an R_+ rarefaction wave.

4. SOME EXAMPLES

In this section we shall discuss two examples; our point is to illustrate rigorously the sharp differences in the behavior of solutions which contain vacuum states, as opposed to the usually considered solutions away from a vacuum. Thus, as we saw in Section 2, if the gas contains a vacuum, say to the left of $x = M$, then the solution of the Riemann problem consists of a single rarefaction wave. We now give an example of initial data giving rise to strong shock waves, which also contain a vacuum, say to the left of some point, whose solution of the initial-value problem tends in finite time, to a single rarefaction wave. That is, the vacuum cancels shocks of any strength in finite time. On the other hand, the asymptotic behavior of the solution is still governed completely by the extreme states $(u_0(\pm\infty), \rho_0(\pm\infty))$; see [9, 10].

Thus, consider piecewise constant initial data $(u_0(x), \rho_0(x))$, which can be resolved into a finite number of R_+ rarefaction waves and S_- shock waves in $x > M$, while to the left of $x = M$, the data is a vacuum $\rho = 0$. From our results in the last section, we know that the first wave bordering

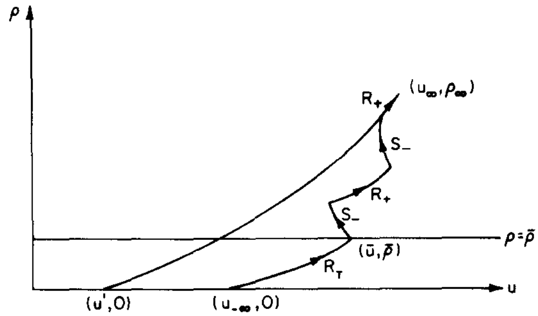

FIGURE 9

the vacuum is an R_+ rarefaction wave. We shall show now, using the Glimm difference scheme, that a solution to the initial-value problem exists. By investigating the elementary wave interactions (see [4, 17]), the Glimm approximate solutions consist of only R_+ rarefaction waves, and S_- shock waves; see Fig. 9. It follows at once that for the approximate solutions, the total variation of the density ρ is a constant; namely $\rho_\infty = \lim_{x\to\infty}\rho_0(x)$. It remains to show that the total variation of the velocity u, of the approximate solutions are also uniformly bounded.[2] To this end, we first note that the limiting values $(u_{-\infty}, 0) \equiv \lim_{x\to-\infty}(u_0(x), \rho_0(x))$, and $(u_\infty, \rho_\infty) \equiv \lim_{x\to\infty}(u_0(x), \rho_0(x))$ of the approximate solutions at $x = \pm\infty$ remains unchanged for all time. Since the velocity increases across R_+ rarefaction waves, and decreases across S_- shock waves, we need only show that the strength of S_- shock waves in the approximate solutions remains bounded for all time. This is true since the total variation is majorized by twice the decreasing variation plus the difference in the limits $u_\infty - u_{-\infty}$.

Now given any S_- shock wave, $(u_1, \rho_1; u_2, \rho_2)$, we find the unique point (u_0, ρ_1) on the R_+ rarefaction wave curve through (u_2, ρ_2) and define the strength of this shock wave to be $u_1 - u_0 > 0$; see Fig. 10. With this definition of the strength of S_- shock waves, it is easy to see (by investigating the elementary interaction of S_- shock waves, and R_+ rarefaction waves) that the total strength of S_- shock waves in the approximate solutions in fact stay constant, and thus bounded. This establishes the existence of the solution (see [2, 8]).

We now investigate the asymptotic behavior of the solution. We shall first consider the case when there is only a single S_- shock wave to the right of the R_+ rarefaction wave bordering the vacuum. These two waves must interact with each other in finite time, say $t = T_0$. Furthermore, the

[2] Note that near $\rho = 0$, the jump in u need not be majorized by the jump in ρ.

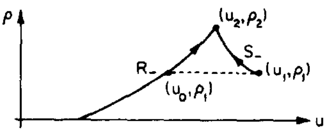

FIGURE 10

shock penetrates the rarefaction wave, producing no other waves, and the shock is a smooth curve, across which the solution has limits satisfying the usual jump conditions (see [4]). Indeed, we shall show that the shock wave passes through the rarefaction wave and disappears in finite time; see Figs. 11 and 12. Note too that the speed of the characteristic bordering the vacuum state changes abruptly at the instant the shock wave disappears.

Referring to Figs. 11 and 12, for any $t > T_0$ for which the shock exists, we denote by $(u_-(t), \rho_-(t))$ and $(u_+(t), \rho_+(t))$ the limits of the solution on both sides of the shock. We will derive an ordinary differential equation for $\rho_-(t)$, and show that $\rho_-(t)$ becomes zero in finite time. It will then follow that the shock must disappear in finite time; see Section 2. The reason behind this is due to the fact that the shock always makes a positive angle with the characteristics in the rarefaction wave. However, near the vacuum state, the characteristic speed and the shock speed both approach the speed of the gas. Since the shock speed tends to u' (see Section 3) and the characteristic speed bordering the vacuum is $u_{-\infty}$, the shock meets this

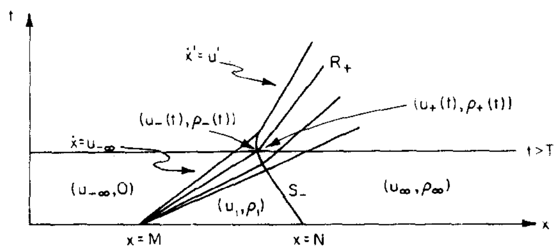

FIGURE 11

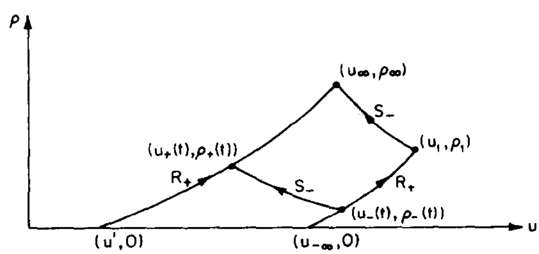

FIGURE 12

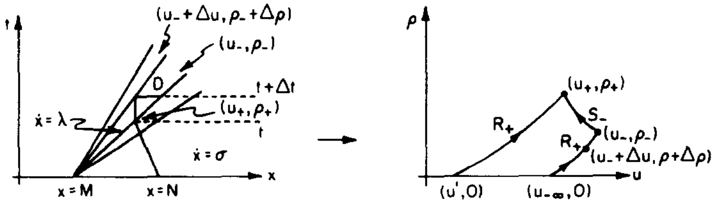

FIGURE 13

characteristic of speed $u_{-\infty}$ in finite time, since $u_{-\infty} > u'$. Moreover, the position of the shock can be explicitly calculated with the aid of the differential equation for $\rho_-(t)$. We proceed with the details.

Let $t > T$, and for $\Delta t > 0$, we consider the horizontal distance D in the R_+ rarefaction wave determined by the S_- shock at times t and $t + \Delta t$; see Fig. 13. (For brevity, we use the notation $(u_\pm(t), \rho_\pm(t)) = (u_\pm, \rho_\pm)$.) We have $\Delta\rho < 0$, $\Delta u < 0$, and $\sigma = u_- + \rho_+(u_+ - u_-)(\rho_+ - \rho_-)^{-1}$, $\lambda = u_- + \sqrt{p'(\rho_-)}$, so that from the x–t plane, we have

$$D = (\lambda - \sigma)\Delta t = \left(\sqrt{p'(\rho_-)} - \frac{\rho_+(u_+ - u_-)}{\rho_+ - \rho_-}\right)\Delta t. \tag{8}$$

On the other hand, D is an x distance, at fixed time $t + \Delta t$, so using the properties of rarefaction waves, we have $D = (t + \Delta t)(u_- + \sqrt{p'(\rho_-)}) - (t + \Delta t)(u_- + \Delta u + \sqrt{p'(\rho_- + \Delta\rho_-)})$. Hence, up to first order in Δt, we have

$$\begin{aligned} D &= t\left\{-\Delta u + \sqrt{p'(\rho_-)} - \sqrt{p'(\rho_- + \Delta\rho_-)}\right\} \\ &= t\left\{-\Delta u - \tfrac{1}{2}\Delta\rho_-\,(p'(\rho_-))^{-1/2}p''(\rho_-)\right\}. \end{aligned} \tag{9}$$

But, from (5),

$$\Delta u = \rho_-^{-1}(p'(\rho_-))^{1/2}\Delta\rho_- . \tag{10}$$

Thus (9) and (10) imply that, up to first order

$$D = -t\Delta\rho_-\left\{\rho_-^{-1}\sqrt{p'(\rho_-)} + \tfrac{1}{2}(p'(\rho_-))^{-1/2}p''(\rho_-)\right\}. \tag{11}$$

Combining this with (8), we get

$$\frac{\Delta\rho_-}{\Delta t} = \frac{-1}{t}\left\{\frac{\sqrt{p'(\rho_-)} + \rho_+(u_- - u_+)/(\rho_+ - \rho_-)}{\rho_-^{-1}\sqrt{p'(\rho_-)} + \frac{1}{2}(p'(\rho_-))^{-1/2}p''(\rho_-)}\right\}. \tag{12}$$

At this point, we need the following lemma.

LEMMA. $\rho_+/\rho_- \to \infty$ *as* $\rho_- \to 0$.

Proof. We have, from the jump condition, $(u_+ - u_-)^2 = (p(\rho_-) - p(\rho_+))(\rho_1^{-1} - \rho_+^{-1})$, so that

$$(u_+ - u_-)^2 = \left(1 - \frac{\rho_-}{\rho_+}\right)\left(\frac{\rho_+}{\rho_-} - 1\right)p'(\tilde{\rho}), \tag{13}$$

where $\tilde{\rho} \in (\rho_-, \rho_+)$. Note that $\rho_+/\rho_- > 1$, and that $(u_+ - u_-)$ tends to the finite positive number $(u' - u_{-\infty})$ as ρ_- tends to zero. Now if ρ_+/ρ_- is bounded for a sequence ρ_+^i/ρ_-^i, where ρ_-^i tends to zero, then (13) shows that $p'(\tilde{\rho}_i)$ must be bounded away from zero; i.e., $p'(\tilde{\rho}_i) \geq k > 0$. Thus $p'' > 0$ implies that $p'(\rho) \geq k$ for all ρ in some interval, $0 < \rho \leq \delta$, $\delta > 0$. But this violates (4) and completes the proof.

It follows then, that the numerator of (12) tends to the constant $u' - u_{-\infty}$, as ρ_- tends to zero. If we set

$$\phi(\rho_-) = \rho_-^{-1}\sqrt{p'(\rho_-)} + \tfrac{1}{2}\left(p'(\rho_-)^{-1/2}p''(\rho_-)\right) \tag{14}$$

then (4) implies that $\int_0^\rho \phi(\rho_-)\,d\rho_-$ converges. Moreover, the differential equation for $\rho_-(t)$ is

$$-\frac{d\rho_-}{dt} = \frac{1}{t}\frac{C}{\phi(\rho_-)},$$

where $C = C(\rho_-) = \sqrt{p'(\rho_-)} + \rho_+(u_- - u_+)(\rho_+ - \rho_-)^{-1}$. Thus

$$\ln\frac{t}{t_0} = \int_{\rho_-}^{\rho_0} \frac{\phi(x)}{C(x)}\,dx, \tag{14}$$

and the finiteness of the integral as $\rho_- \to 0$ implies that $\rho_-(t)$ tends to zero in finite time. This is our desired result.

Observe that the strength of the shock $u_- - u_+$ appears in the denominator of the integrand in (14) so that weaker shocks actually survive for longer times; strong shocks penetrate the rarefaction wave more rapidly. Finally we note that if the initial data can be resolved into a finite number of S_- shocks and R_+ rarefaction waves, then only a finite number of S_- shock waves can meet the R_+ rarefaction wave which bounds the vacuum. Each such shock wave then must disappear in finite time.

As a second example, we consider the interaction of an R_+ rarefaction wave bounding the vacuum state, with an S_+ shock wave. The asymptotic behavior of the solution can be investigated using the methods in [9, 10].[3] In this case the asymptotic state is again a single R_+ rarefaction wave. To see this, we observe that each S_- shock wave, which is generated by the formation of compression waves (see [3, 11]) must disappear, as in the previous case, in finite time. The original S_+ shock wave is cancelled by the R_+ rarefaction wave, and it decays at the rate $t^{-1/2}$; see [10].

5. Counterexample to Glimm's Estimates near the Vacuum

We assume here that the initial data curve $(u_0(x), \rho_0(x))$, $-\infty < x < \infty$, lies in a bounded region Λ in the $u - \rho$ plane. Let Γ_2 (respectively, Γ_1), denote the R_+ (respectively R_-) rarefaction wave curve which bounds Λ on the left (respectively right), and let Ω denote the region between Γ_1 and Γ_2 (see Fig. 14). Our assumption (4) implies that Γ_1 and Γ_2 both intersect $\rho = 0$, so that Ω is a bounded region in the $u - \rho$ plane. In Lagrangian coordinates, the basic dependent variables are $v = \rho^{-1}$ and u. In these coordinates, it is well known ([4, 17]) that the corresponding region $\tilde{\Omega} = \{(u, v) : (u, v^{-1}) \in \Omega\}$ is an invariant region in the sense that if the initial data lies in $\tilde{\Omega}$, then so does the solution, for all $t > 0$. Now since the transformation $(u, v) \to (u, \rho)$ is bijective for $\rho \neq 0$, we see that Ω is also an invariant region. Note, however, that Ω is a bounded region, while $\tilde{\Omega}$ is not. Thus, the Glimm approximate solutions can be defined for all time, and these approximate solutions are uniformly bounded. We are unable to prove the convergence of the approximate solutions, however. The reason for this is due to the fact that even if we assume that the initial data is such that the invariant region Ω is small (and thus all waves are weak), the interaction of waves of strengths α and β may produce waves having strengths which differ from that of α and β by an amount greater than a quadratic term $O(1)\alpha\beta$. In other words, near the vacuum, waves do not interact linearly, modulo terms of the form $O(1)\alpha\beta$ and the Glimm estimates are no longer valid. We shall illustrate this explicitly by investigating the interaction of R_+ rarefaction waves with S_- shock waves; see Fig. 15. Let α and $\tilde{\alpha}$ denote the strengths of the R_+ rarefaction waves connecting states **1** to **2**, and **4** to **3**, respectively, and let β and $\tilde{\beta}$ denote, respectively, the strengths of the S_- shock waves connecting **2** to **3** and **1** to **4**. We will take $|\rho_1 - \rho_4| \ll |\rho_2 - \rho_1|$, and for simplicity, we will assume that states **4** and **1** are the vacuum states $\rho = 0$; see Fig. 16. We shall compare the strength of the S_- shock **2**–**3**, with that of **1**–**4**. We draw through **2** the horizontal line which meets the R_+ rarefaction curve at **5**.

[3] Note that in [9] it is not required that the solutions have small total variation.

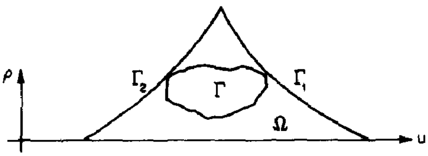

FIGURE 14

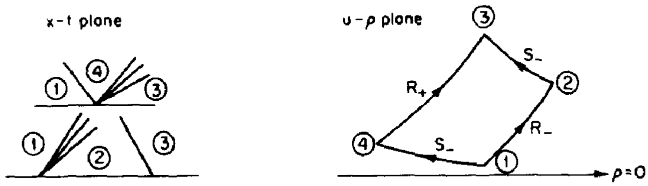

FIGURE 15

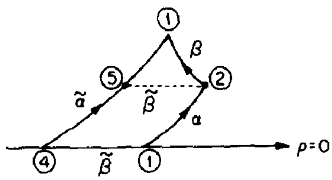

FIGURE 16

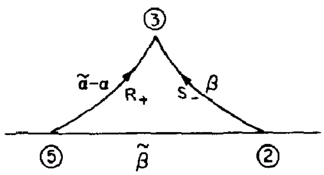

FIGURE 17

Then, because the curves **5–4** and **2–1** are congruent we have

$$\tilde{\alpha} - \alpha = (\mathbf{5}, \mathbf{3}),$$

$$\tilde{\beta} - \beta = (\mathbf{5}, \mathbf{2}) - (\mathbf{3}, \mathbf{2}).$$

Now keep **1** and **2** fixed, and move **3** toward **2**; i.e., let $\beta \to 0$. Then the region bounded by points **5**, **3** and **2** becomes almost an isosceles triangle since the S_- shock curve **2–3** gets close to an R_- rarefaction curve, and the R_- rarefaction curves is a reflection of the R_+ rarefaction curve about the line $u =$ constant; see Fig. 17. Thus $\tilde{\alpha} - \alpha$ gets close to β, a linear, rather than a quadratic term, $\alpha\beta$. Thus the Glimm estimates simply fail near the vacuum. For the other types of interactions, similar things occur, and thus no regular transformations of coordinates can cure this difficulty.

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