JOURNAL OF ALGEBRA 78, 181–214 (1982)

Near Solvable Signalizer Functors on Finite Groups

PATRICK PASCHAL MCBRIDE*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by W. Feit

Received November 4, 1981

1. INTRODUCTION

The object of this paper is to prove the following result.

THEOREM. *Suppose p is a prime, A is an elementary Abelian p-subgroup of a finite group G, $m(A) = 3$, and θ is a near solvable A-signalizer functor on G. Then θ is complete.*

Non-solvable signalizer functors were first treated by Gorenstein and Lyons (see [11]). They identified certain "unbalancing" problems in their work. These problems can be traced to the existence of certain nontrivial subgroups; if X is such a subgroup and θ is an A-signalizer functor then $C_X(A)$ is solvable. This situation occurs in the extreme when $\theta(C(A))$ is solvable. Our main theorem fixes on this case. It provides a means to pass from the solvable theorems, as treated in [6, 7, 9, 10] and culminating in [5], to general signalizer functor theorems (see [14]).

By [2, 5] it is sufficient to treat only odd primes in the main theorem. We shall assume in the sequel that p is a fixed odd prime. All groups treated are assumed to be finite. Notation for groups of Lie type agrees with [1], other notation is taken from [5, 6, 8, and 12]. For the convenience of the reader we shall repeat some of this notation. The notation of associated set of signalizers is altered to suit the problem.

DEFINITION. (1) The group G is *near p-solvable* means that G is a p'-group and any non-abelian simple section of G is isomorphic to $\mathscr{L}_2(2^p)$, $\mathscr{L}_2(3^p)$, $Sz(2^p)$, or $U_3((2^p)^2)$.

* Partially supported by NSF Grant MCS76–06626.

0021-8693/82/090181–34$02.00/0

(2) The statement G is *near A-solvable* means that A is an elementary abelian p-group acting on the near p-solvable group G, and that $C_G(A)$ is solvable.

(3) The statement "θ is an A-signalizer functor on G" means that A is an abelian r-subgroup of the group G for some prime r, and that for each $a \in A^{\#}$ there is defined an A-invariant r'-subgroup $\theta(C_G(a))$ of $C_G(a)$ such that

$$C_G(a) \cap \theta(C_G(b)) \subseteq \theta(C_G(a)) \qquad \text{for all } a, b \in A^{\#}. \tag{$*$}$$

The property $(*)$ is called *balance.* θ is said to be a *near solvable A-signalizer functor*, if in addition $\theta(C_G(a))$ is near A-solvable for all $a \in A^{\#}$.

(4) The *associated set of A-signalizers* is the set of all near A-solvable subgroups X of G having the property that $C_X(a) \subseteq \theta(C_G(a))$ for all $a \in A^{\#}$. It is denoted $И_\theta(A)$. The set of maximal elements of $И_\theta(A)$ under inclusion is denoted by $И_\theta^*(A)$.

(5) Let $\pi(\theta) = \bigcup_{a \in A^{\#}} \pi(\theta(C_G(a))$ and $|\theta| = \sum_{a \in A^{\#}} |\theta(C_G(a))|$.

(6) For $s \in \pi(\theta)$ let $И_\theta(A; s)$ be the set of all s-groups in $И_\theta(A)$, and let $И_\theta^*(A; s)$ be the set of maximal elements of $И_\theta(A; s)$. The elements of $И_\theta^*(A; s)$ are called *$S_s(A)$-subgroups* of G.

(7) We say θ is *complete* if G contains a unique maximal element of $И_\theta(A)$ under inclusion. The element is then denoted $\theta(G)$.

(8) We say θ is *locally complete* if for every non-identity element X of $И_\theta(A)$, $N_G(X)$ contains a group $\theta(N_G(X))$ which is the unique maximal element of $И_\theta(A)$ contained in $N_G(X)$. In this case we set $\theta(C_G(A)) = \theta(N_G(X)) \cap C_G(X)$.

(9) For every non-identity subgroup B of A, let

$$\theta(C_G(B)) = \bigcap_{b \in B^{\#}} \theta(C_G(b)).$$

(10) A group is *semi-simple* means that G is the direct product of its normal non-abelian simple groups. A group is *perfect* if it is its own derived subgroup. A group G is an *E-group* if G is perfect and $G/Z(G)$ is semi-simple. Given any group H, $E(H)$ is the unique maximal normal E-subgroup of H, $F(H)$ is the *fitting subgroup* of H, and $F^*(H) = E(H)\,F(H)$ is the generalized fitting subgroup of H.

(11) The solvable radical of a group G is the unique maximal solvable normal sugroup of G. It is denoted $\mathrm{Sol}(G)$.

(12) Let G be a group. The *components of G*, $\mathscr{L}(G)$, is the set of subnormal non-abelian simple subgroups of G. $\tilde{\mathscr{L}}(G) = \{G\}$ when G is

solvable. When G is nonsolvable $\tilde{\mathscr{L}}(G)$ is the set of all subgroups X of G which contain $\mathrm{Sol}(G)$ and satisfy $X/\mathrm{Sol}(G) \in \mathscr{L}(G/\mathrm{Sol}(G))$.

(13) $K(G) \supseteq \mathrm{Sol}(G)$ and $K(G)/\mathrm{Sol}(G) = E(G/\mathrm{Sol}(G))$.

(14) $\hat{K}(G) = \bigcap \{N_G(X) | X \in \tilde{\mathscr{L}}(G)\}$.

In definitions (15) to (24), A is a p-subgroup of a group G, θ is a near solvable A-signalizer functor on G, and $D = \theta(C_G(A))$.

(15) $\hat{\mathit{V}}_\theta(A) = \{X \in \mathit{V}_\theta(A) | X^D = X\}$.

(16) $\tilde{\mathit{V}}_\theta(A) = \{X \in \mathit{V}_\theta(A) | D \subseteq X\}$.

(17) $P(\theta)$ is the set of all pairs (X_1, X_2) satisfying:

(a) $X_i \subseteq \hat{\mathit{V}}_\theta(A)$ for $i = 1$ or 2.

(b) $X_2 \lhd X_1$ and X_1/X_2 is a chief factor of $X_1 DA$.

(c) $C_{X_1 D}(X_1/X_2) = X_2$.

(18) $D(\theta) = \{D \cap X_2 | (X_1, X_2) \in P(\theta)\}$.

(19) For each $Y \in D(\theta)$ and $X \in \tilde{\mathit{V}}_\theta(A)$

$$\begin{aligned}
P(\theta, Y) &= \{(X_1, X_2) \in P(\theta) | X_2 \cap D = Y\},\\
P(\theta, Y, X) &= \{(X_1, X_2) \in P(\theta, Y) | X_1 \subseteq X\},\\
C(\theta, Y) &= \{X \in \tilde{\mathit{V}}_\theta(A) | P(\theta, Y, X) \neq \varnothing\},\\
U(\theta, Y) &= \{Z \in \tilde{\mathit{V}}_\theta(A) | \langle Y^Z \rangle \cap D = Y\},\\
U(\theta, Y, X) &= \{Z \in U(\theta, Y) | Z \subseteq X\},\\
E(\theta, Y) &= \{Z \in \hat{\mathit{V}}_\theta(A) | Z \cap D = Y\},\\
E(\theta, Y, X) &= \{Z \in E(\theta, Y) | Z \subseteq X\}.
\end{aligned}$$

(20) For each $Y \in D(\theta)$ and $X \in \tilde{\mathit{V}}_\theta(A)$ define

$$\theta_Y(X) = \langle E(\theta, Y, X) \rangle.$$

(21) For each $Y \in D(\theta)$ and $X \in C(\theta, Y)$ define

$$\begin{aligned}
\theta_Y^u(X) &= \langle U(\theta, Y, X) \rangle,\\
\theta_Y^l(X) &= \langle X_2 | (X_1, X_2) \in P(\theta, Y, X) \rangle,\\
\theta_Y^m(X) &= \langle X_1 | (X_1, X_2) \in P(\theta, Y, X) \rangle.
\end{aligned}$$

(22) For each $Y \in D(\theta)$ define

$$\begin{aligned}
\theta_Y^l(G) &= \langle \theta_Y^l(X) | X \in C(\theta, Y) \rangle,\\
\theta_Y^m(G) &= \langle \theta_Y^m(X) | X \in C(\theta, Y) \rangle.
\end{aligned}$$

(23) For each $X \in \tilde{\mathcal{M}}_\theta(A)$

$$\theta_{\text{Sol}}(X) = \langle Z \in \hat{\mathcal{M}}_\theta(A) | Z \subseteq X; Z \text{ is solvable}\rangle.$$

(24) Suppose $\Gamma = \theta_Y^u$, θ_Y^m, θ_Y^l, θ_Y, or θ_{Sol}, and X is an A-invariant subgroup of G such that $\theta(X)$ and $\Gamma(\theta(X))$ are defined. Then we write $\Gamma(X) = \Gamma(\theta(X))$.

(25) Suppose X and N are subgroups of a group G, and $G = N \times C_G(N)$. Then $\text{Proj}_N(X)$ is the projection of X on N where projections are taken with respect to the pair $(N, C_G(N))$.

Glauberman conjectured that θ_{Sol} is a solvable signalizer functor whenever θ is a signalizer functor. We shall show this when θ is a near solvable A-signalizer functor (see Lemma 3.1). This subfunctor furnishes Frattini type arguments which simplify proofs (see Theorem 2.11(c)). In [14], θ_{Sol} is nested inside another subfunctor $\theta_{n.s.}$. Combining these ideas it can be seen that the above conjecture is valid in a large class of signalizer functors.

Remarks on the proof. The proof pivots on showing that θ_Y is a signalizer functor for all $Y \in D(\theta)$. Assume θ is a minimal counterexample. Then θ_Y is complete and θ is locally complete. Since $\theta_Y^l(G) \subseteq \theta_Y(G) \in \mathcal{M}_\theta(A)$ it follows first that $\theta_Y^l(G) \in \mathcal{M}_\theta(A)$ and then $\theta_Y^m(G) \in N_G(\theta_Y^l(G))$. By local completeness $\theta_Y^m(G) \in \mathcal{M}_\theta(A)$ or $\theta_Y^l(G) = 1$.

First suppose that $\theta_Y^m(G) \notin \mathcal{M}_\theta(A)$ for some $Y \in D(\theta)$. Then the structure of $A\theta(C_G(a))$ for all $a \in A^\#$ is obtained. The structure of $\theta(G)$ readily follows and leads to a contradiction.

Next suppose $\theta_Y^m(G) \in \mathcal{M}_\theta(A)$ for all $Y \in D(\theta)$. Then for any $X, Z \in \tilde{\mathcal{M}}_\theta(A)$, such that $X \cap Z$ is nonsolvable, there is a $Y \in D(\theta)$, which depends on $X \cap Z$, such that $\langle K(X), K(Z)\rangle \subseteq \langle \theta_Y^m(X), \theta_Y^m(Z)\rangle \subseteq \theta_Y^m(G) \in \mathcal{M}_\theta(A)$. It is then almost enough to obtain a non-solvable $W \in \tilde{\mathcal{M}}_\theta(A)$ satisfying: $K(W) \subseteq U \in \hat{\mathcal{M}}_\theta(A)$ implies $U \subseteq W$. Subgroups with such properties are treated in Section 4.

The principal idea, used in this paper and in [14], is illustrated in [5, Lemma 2.11 and Theorem 4.5]. This technique focuses on subfamilies of $\hat{\mathcal{M}}_\theta(A)$. In this paper we are keying on families each of whose members intersect $\theta(C_G(A))$ in a fixed subgroup.

2. Preliminary Lemmas

Lemma 2.1. *Suppose the abelian p-group A acts on the p'-group X. Then $X = \langle C_X(A_0) | A/A_0$ is cyclic$\rangle$.*

Proof. See [6, Lemma 2.1].

LEMMA 2.2 (Glauberman). *Suppose the π-group A acts on the π'-group K. Suppose K is generated by A-invariant subgroups $K_1, K_2, \ldots, K_n$, and $K_iK_j = K_jK_i$ for all $1 \leqslant i, j \leqslant n$. Then*

$$C_K(A) = C_{K_1}(A)\, C_{K_2}(A) \cdots C_{K_n}(A).$$

Proof. See [11, Lemma 2.1].

LEMMA 2.3. *Suppose θ is an A-signalizer on a group G, $P \in И_\theta(A; r)$,* and B is a non-cyclic subgroup of A. Then the following statements are equivalent:

(1) $P \in И_\theta^*(A; r)$,

(2) $C_P(b)$ is an S_r-subgroup of $\theta(C(b))$ for all $b \in B^\#$.

Proof. See [6, Lemma 3.2].

LEMMA 2.4. *Let G be a group and $\bar{G} = G/\mathrm{Sol}(G)$. Then the functors F^*, K, E, and* Sol *satisfy*:

(a) $\mathrm{Sol}(\bar{G}) = \bar{1}$.

(b) $C_G(F^*(G)) \subseteq F^*(G)$.

(c) $\overline{K(G)} = K(\bar{G}) = E(\bar{G}) = F^*(\bar{G})$ is semi-simple.

Proof. (a) follows directly from the definition of Sol. (b) is well known. (c) is an immediate consequence of (a) and the definition of F^*.

LEMMA 2.5. *Suppose the elementary abelian p-groups A acts on the p'-group G, $m(A) \geqslant 3$, and $C_G(a)$ is abelian for all $a \in A^\#$. Then G is abelian.*

Proof. $\theta(C_{GA}(a)) = C_G(a)$ for all $a \in A^\#$ is a solvable A-signalizer functor on GA. By Lemma 2.1 and [5], G is solvable. Let G/M be a chief A factor, and let $B = C_A(G/M)$. By induction we may suppose M is abelian. Since G/M is solvable, Lemma 2.1 implies that A/B is cyclic. Lemma 2.1 implies that $C_G(B)$ centralizes M. Hence $G = MC_M(B)$ is abelian.

LEMMA 2.6. *Suppose the group G acts faithfully on the set Ω, G has a Sylow r-subgroup S acting transitively on Ω, and $O^p(G) = O_{p'}(G)$. Then $G = S$.*

Proof. Let $a \in \Omega$. Then $G_aS = G$, whence

$$O^p(G) \subseteq \bigcap \{G_a \mid a \in \Omega\} = 1.$$

LEMMA 2.7. *Suppose the elementary abelian p-group A acts on the p'-group X. Suppose the outer automorphism group of each chief section of each characteristic section of X has cyclic sylow p-subgroups. Let B be a subgroup of A. Let* $W = \langle C_X(E) | E \times B = A \rangle$, *and* $Z = C_X(B)$. *Then*

(a) *If X is a chief XA factor, it follows that* $X = W$ *or* $X = Z$.

(b) $WZ = ZW = X$.

Proof. (a) The subgroups W, Z are unaffected if we replace (A, B) by $(A/D, BD/D)$ where $D = C_A(X)$. Hence we may first suppose that $C_A(X) = 1$, and then suppose that X is non-solvable. The hypothesis applies to $(C_X(a), A, C_W(a), C_Z(a))$ replacing (X, A, W, Z) whenever $a \in A^{\#}$ and $\langle a \rangle$ acts semi-regularly on $\mathscr{L}(X)$. By induction $C_X(V) \subseteq W$ or $C_X(V) \subseteq Z$ whenever V is a non-identity subgroup of A acting semi-regularly on $\mathscr{L}(X)$.

Suppose first that we can find $V_1, V_2, V_3 \in \mathscr{E}_1(A)$ all distinct and such that V_i acts semi-regularly on $\mathscr{L}(X)$. Then by permuting the indices we may suppose $\langle C_X(V_1), C_X(V_2) \rangle \subseteq L$ where $L = W$ or $L = Z$. However, $C_X(V_i)$ is a maximal A-invariant subgroup of X, and $C_X(V_1) \neq C_X(V_2)$. Hence $X = W$ or $X = Z$.

We may therefore suppose that there is at most one element of $\mathscr{E}_1(A)$ not acting semiregularly on $\mathscr{L}(X)$. We may suppose that $1 \neq A$ is cyclic. If $B = 1$, then $X = Z$. If $B = A$, then $X = W$. Hence (a) holds.

(b) Let X/Y be a chief XA section. By induction $Y = (Y \cap W)(Y \cap Z)$. By (a) applied to X/Y, it follows that $X = YZ$ or $X = WY$. Hence $X = WZ = ZW$.

Our next theorem is very important. It lists most of the common properties of simple near p-solvable groups needed to prove the main theorem.

THEOREM 2.8. *Suppose G is a non-abelian simple near p-solvable group. Let f be an automorphism of G of order p. Let* $C = C_G(f)$, $C_0 = F(C)$, $C_1 = C_C(C/C_0)$, *and* $M = N_G(F(C))$. *Then all of the following hold*:

(a) *f exists.*

(b) $\mathrm{Aut}(G)$ *has cyclic sylow p-subgroups.*

(c) $C_{\mathrm{Aut}(G)}(C)$ *is a p-group.*

(d) *M is the unique maximal subgroup of G containing* C_1.

(e) C_1, C *and M are frobenius groups with abelian frobenius kernels* $F(C)$, $F(C)$ *and* $F(M)$, *respectively.*

(f) $F(M) = C_G(F(C))$ *and* $M = F(M)\, C$.

(g) *Any p'-automorphism of G centralizing $M/F(M)$ is an inner automorphism induced by an element of M.*

(j) *$F(C)$ is the unique minimal normal subgroup of C.*

(k) *Let X be a proper C-invariant subgroup of G. Then $F(C) \subseteq X$ or $X \subseteq [F(M),f] = [M,f]$.*

(m) *Let X be a nilpotent C-invariant subgrouup of G. Then $X \subseteq F(M)$.*

(n) *The class of subgroups of G isomorphic to M is a conjugacy class of subgroups of G.*

(o) *$\pi(G) - \pi(M) \neq \varnothing$. Moreover, if $r \in \pi(G) - \pi(M)$, then G has an abelian sylow r-subgroup.*

Proof. (a) and (b) follow by [15; 16, Theorem 11]. Hence by (b) and sylow theorems we may suppose f is a field automorphism. Now a count (see [1, 9.4.10 and 14.3.2]) shows that $F(C)$ is a sylow subgroup of G. So (n) holds. The sylow r-subgroups of $SL(3, 2^p)$ are abelian for $r \neq 2$ or 3. Hence by [16, Theorem 9; Lemma 15.1.1], (o) holds. By (e), $F(M) = [F(M),f] \times F(C)$. Since $F(C)$ is a sylow subgroup of $F(M)$, (k) is a consequence of (f) and (j). Part (m) is a consequence of (e) and (k). So it remains to prove (c), (d),..., (j).

First suppose $G \cong L_2(3^p)$ or $U_3((2^p)^2)$. By [3, Sects. 8.4 and 8.5], C is a maximal subgroup of G and (d) holds. Hence $M = C$. Hence (e), (f) and (j) follow directly from the structure of C. By [15], $\mathrm{Aut}(G) = \mathrm{Inn}(G)\, C_{\mathrm{Aut}(G)}(f)$; so $N_{\mathrm{Aut}(G)}(C) = C_{\mathrm{Aut}(G)}(f)$. Moreover $O^p(C_{\mathrm{Aut}(G)}(\mathrm{f})) \cong \mathrm{Aut}(C)$. Both (c) and (g) follow directly from the structure of $\mathrm{Aut}(C)$.

Suppose then $G \cong L_2(2^p)$ or $Sz(2^p)$. By [15, 16], $O^p(\mathrm{Aut}(G)) = \mathrm{Inn}(G)$. Hence (c) is a consequence of (e), (g) is a consequence of (d), and (j) holds by inspection. So it is enough to verify (d), (e), and (f). The results for $G \cong L_2(2^p)$ are well known. The results for $G \cong Sz(2^p)$ are given by [16, Theorem 9].

LEMMA 2.9. *Suppose the abelian group A acts on the group $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose A acts on $\{G_1, G_2,..., G_n\}$, via the induced action of A on subgrouups. Then*

$$\mathrm{Proj}_{G_i}(C_G(A)) = C_{G_i}(N_A(G_i))$$

when projections are taken with respects to $\{G_1, G_2,..., G_n\}$.

Proof. Let $S = \{\langle G_i^A \rangle | i = 1, 2,..., n\}$. Then $C_G(A) = \times \{C_X(A) | X \in S\}$. Hence by induction we may suppose A acts transitively on $\{G_1, G_2,..., G_n\}$. Let $B = N_A(G_i)$. Since A is abelian and acts transitively on $\{G_1, G_2,..., G_n\}$, it follows that B is independent of i. Hence we may suppose first that $B = C_A(G)$ and then $B = 1$. So it suffices to treat the case when A acts regularly on $\{G_1, G_2,..., G_n\}$. This is straightforward.

THEOREM 2.10. *Suppose the elementary abelian p-group A acts on the near p-solvable non-abelian semisimple group G. Suppose $D = C_G(A)$ is solvable. Let $M = N_G(C_G(F(D)))$. For each $J \in \mathscr{L}(G)$, let $M_J = M \cap J$, $K_J = N_J(C_J(F(C_J(N_A(J)))))$, and $J^{\circledA} = \langle J^A \rangle \cap D$. Let X, Y be DA-invariant subgroups of G. Then all of the following hold:*

(a) $M = \times \{M_J | J \in \mathscr{L}(G)\} = \times \{K_J | J \in \mathscr{L}(G)\}$.

(b) *M is the unique maximal solvable subgroup of G containing D.*

(c) *$F(M)$ is abelian, and $F(M) = C_G(F(D))$.*

(d) *Suppose ϕ is an automorphism of G. Then there is an inner automorphism i of G such that $M^{\phi i} = M$.*

(e) *Suppose X is solvable and contains D. Then*

$$M = N_G(C_G(F(X))) \supseteq X.$$

(f) *If X is nilpotent, then $X \subseteq F(M)$.*

(g) *Suppose* $\mathrm{Proj}_J(X)$ *is nonsolvable. Then* $\mathrm{Proj}_J(X) = J$ and $J^{\circledA} \subseteq X$.

(h) *Suppose* $\mathrm{Proj}_J(X)$ *is not nilpotent. Then* $F(J^{\circledA}) \subseteq X$.

(j) *Suppose $F(J^{\circledA}) \nsubseteq X$. Then*

$$X \cap D \subseteq C_D(J^{\circledA}) = \times \{K^{\circledA} | K \in \mathscr{L}(G), K^{\circledA} \neq J^{\circledA}\}.$$

(k) *Let S be the set of all DA-invariant subgroups of G which intersect D trivially. Then $[F(M), A]$ is the unique maximal element of S under inclusion.*

(m) *Let $V = F(J^{\circledA})$. Suppose $X \lhd Y$, Y is solvable, $V \nsubseteq X$, but $V \subseteq Y$. Then* $\mathrm{Proj}_J(C_G(Y/X))$ *is abelian.*

Proof. Let $A_J = N_A(J)/C_A(J)$ and $C_J = C_J(A_J)$ for each $J \in \mathscr{L}(G)$. Since D is solvable, Theorem 2.8(b) implies that $A_J \cong Z_p$ for all J. Now $D = \times \{J^{\circledA} | J \in \mathscr{L}(G)\}$, and $J^{\circledA} \to \mathrm{Proj}_J(J^{\circledA})$ is an isomorphism; whence, by Lemma 2.9, $\mathrm{Proj}_J(D) = C_J(A_J) = C_J \cong J^{\circledA}$ and $\mathrm{Proj}_J(F(D)) = \mathrm{Proj}_J(F(J^{\circledA})) = F(C_J)$. Let $K = \times \{K_J | J \in \mathscr{L}(G)\}$. Theorem 2.8(e) implies that K is solvable and $F(K)$ is abelian. Now let X be a solvable DA-invariant subgroup of G which contains D. Then $\mathrm{Proj}_J(X)$ is a solvable $\mathrm{Proj}_J(D) = C_J$ invariant subgroup; hence $\mathrm{Proj}_J(X) \subseteq K_J$ by Theorem 2.8(d). In particular, $F(K) \cap X \subseteq F(X)$. Now $\mathrm{Proj}_J(F(X))$ is a nilpotent C_J invariant subgroup. By Theorem 2.8(m), $F(X) = X \cap F(K)$. Since $F(D) = D \cap F(K) \subseteq F(K)$, it follows that $1 \neq F(C_J) \subseteq \mathrm{Proj}_J(F(X)) \subseteq F(K_J)$ for all J. Theorem 2.8(e) implies that $F(K) = C_G(F(X))$. Theorem 2.8(d) implies that $K = N_G(F(K))$. We have shown (a), (b), (c), (e) and (f).

Next we show (h). So suppose $\mathrm{Proj}_J(X)$ is not nilpotent. Let $G_1 = \langle J^A \rangle$ and $X_1 = \mathrm{Proj}_{G_1}(X)$. If $F(J^{\circledast}) \subseteq X_1$, then

$$F(J^{\circledast}) = [F(J^{\circledast}), J^{\circledast}] \subseteq [X_1, J^{\circledast}] = [X, J^{\circledast}] \subseteq [X, D] \subseteq X.$$

Hence by induction $G = G_1$ and $D = J^{\circledast}$. Since $J^{\circledast} \cong C_J$, Theorem 2.8(j) implies that $F(J^{\circledast})$ is the unique minimal normal subgroup of $J^{\circledast} = D$. Hence $F(J^{\circledast}) \subseteq X$ or $D \cap X = 1$. By (f), $X \not\subseteq F(M)$. By Lemma 2.1, there is a hyperplane B of A such that $C_X(B) \not\subseteq F(M)$. Let $Z = C_X(B)$. By (f), Z is not nilpotent. Theorem 10.2.1 of [8] implies that $C_Z(A/B) = C_Z(A) = C_X(A) \neq 1$. So $D \cap X \neq 1$. Hence $F(J^{\circledast}) \subseteq X$ as required.

Next we show (g). So suppose $\mathrm{Proj}_J(X)$ is non-solvable. Let $G_1 = \langle J^A \rangle$ and $X_1 = X \cap G_1$. By (h), $\mathrm{Proj}_J(X_1) \supseteq \mathrm{Proj}_J(F(J^{\circledast})) \neq 1$. Theorem 2.8(d) implies that $J = \mathrm{Proj}_J(X) \trianglerighteq \mathrm{Proj}_J(X_1) \neq 1$. Hence $\mathrm{Proj}_J(X_1) = J$. Hence by induction we may assume $G = \langle J^A \rangle$, $D = J^{\circledast}$, and $\mathrm{Proj}_K(X) = K$ for all $K \in \mathscr{S}(G)$. We may suppose $C_A(G) = 1$. By Lemma 2.1 and Theorem 2.8(d), we may suppose that $C_A(X) = B$ is a hyperplane of A. Let $E = A \cap \hat{K}(GA)$. Theorem 2.8(b) implies that $Z_p \cong E$ and that $C_G(E)$ is solvable. So $E \times B = A$. In particular, B acts regularly on $\mathscr{S}(G)$. Hence $X \subseteq C_G(B) \cong J = \mathrm{Proj}_J(X)$ for any $J \in \mathscr{S}(G)$. So $X = C_G(B) \supseteq J^{\circledast}$.

To prove (j) we may suppose $X \subseteq D$, and $J^{\circledast} \cap X = 1$. Then $[X, J^{\circledast}] \subseteq X \cap J^{\circledast} = 1$ which proves (j).

Let S be as in (k). By (f) and (h), each $Z \in S$ satisfies $[Z, A] \subseteq [F(M), A]$. Part (c) and [8, Theorem 5.2.3] imply that $[F(M), A] \in S$. Hence (k) holds.

Theorem 2.8(n) implies (d). It remains to prove (m). Suppose X, Y, J, and V are as in part (m). We may and do assume $Y = XV$. Let $G_1 = \langle J^A \rangle$, $G_2 = C_G(G_1)$ and $X_1 = \mathrm{Proj}_{G_1}(X)$. As in (h) we get $V \cap X_1 = 1$. Since $C_G(XV/X) \subseteq C_G(X_1 V/X_1) = G_2 \times C_{G_1}(X_1 V/X_1)$, we may suppose $G_1 = G$. Then $X \cap D = 1$. By (f) and (h), $X \subseteq F(M)$. Let $T = C_G(XV/X)$ and $U = C_G(T)$. To complete (m), it suffices to show $C_M(U) \leqslant F(M)$. By (c) and Theorem 2.8(e), $F(M)$ is abelian and has order relatively prime to the order of $T/F(M)$. Since $X \neq XV$, it follows that $1 \neq C_{F(M)}(T/F(M)) = C_{F(M)}(T) \leqslant U$. So $1 \neq U = \times \{C_J(\mathrm{Proj}_J(T)) | J \in (G)\} = \times \{U \cap J | J \in \mathscr{S}(G)\}$. Since U is also A-invariant, it follows that $1 \neq U \cap J$ for any $J \in \mathscr{S}(G)$. Also $U = C_G(T) \leqslant C_G(F(M)) = F(M)$. So $U \cap F(M) \cap J \neq 1$ for any $J \in \mathscr{S}(G)$. By Theorem 2.8(e), $C_M(U) = \times \{C_{K_J}(U \cap K_J) \mid J \in \mathscr{S}(G)\} = \times \{F(K_J) \mid J \in \mathscr{S}(G)\} = F(M)$. We are done.

THEOREM 2.11. *Suppose the elementary abelian p-group A acts on the near p-solvable group G. Suppose $D = C_G(A)$ is solvable. Let X be any DA-invariant subgroup of G. Let $\mathscr{S}(X)$ be the set of all subgroups of X which are $(X \cap D)A$ invariant and solvable. Then all of the following hold:*

(a) *Suppose* G *is semi-simple. Then* $\langle \mathscr{S}(G)\rangle = N_G(C_G(F(D)))$ *is solvable.*

(b) $\langle \mathscr{S}(X)\rangle = \langle \mathscr{S}(G)\rangle \cap X$ *is solvable.*

(c) *Suppose* N *is a normal subgroup of* GA *in* G. *Then* $G = N(N_G(\langle \mathscr{S}(N)\rangle))$. *Moreover, if* $X \cap N$ *is solvable, then* $X \subseteq N_G(\langle \mathscr{S}(N)\rangle)$.

(d) $N_X(\langle \mathscr{S}(X)\rangle) = \langle \mathscr{S}(X)\rangle$.

Proof. Theorem 2.10(b) implies (a). Suppose N is a non-trivial normal subgroup of GA in G. By induction on $|G|$, (c) holds with respect to this N if $\mathrm{Sol}(N) \neq 1$. Suppose then $\mathrm{Sol}(N) = 1$. Let K be a minimal normal subgroup of GA in N. Let $M = N_K(C_K(F(D \cap K)))$. Theorem 2.10(d) implies that $G = KN_G(M)$ and $N = KN_K(M)$. Theorem 2.10(e) implies that $X \subseteq XD \subseteq N_G(M)$ if $X \cap K$ is solvable. Hence (c) follows by induction on $|G|$.

Next consider (b). D permutes $\mathscr{S}(X)$, whence DA normalizes $\langle \mathscr{S}(X)\rangle$. Hence it suffices to assume $\langle \mathscr{S}(X)\rangle$ is solvable, for all DA invariant $X \not\subseteq G$ and show $\langle \mathscr{S}(G)\rangle$ is solvable. We may suppose $G = \langle \mathscr{S}(G)\rangle$ and $\mathrm{Sol}(G) = 1$. Hence by (a) and (c), $G = 1$.

It remains to prove (d). By (b) it suffices to treat the case $1 = \langle \mathscr{S}(G)\rangle$. Then $\mathrm{Sol}(G) = D = 1$. By (a), $F^*(G) = 1$. Then $G = 1$ and (d) is trivially true.

LEMMA 2.12. *Suppose the elementary abelian p-group* A *acts on the near p-solvable group* G, $D = C_G(A)$ *is solvable,* $\mathrm{Sol}(G) = 1$, $E(G)$ *is a minimal normal subgroup of* G, *and* $G = E(G)\,D$. *Then there is a subgroup* B *of* A *such that*

(a) A/B *is cyclic and*

(b) $C_{E(G)}(B)$ *is a non-solvable minimal normal subgroup of* $AC_G(B) = ADC_{E(G)}(B)$. *Moreover* $\mathrm{Sol}(C_G(B)) = 1$.

Proof. We may and do suppose $C_A(K(G)) = 1$. Let $F = A \cap \hat{K}(GA)$ and let B be a complement for F in A. Since AD is transitive on $\mathscr{L}(G)$, it follows that $F = N_A(J) \cong Z_p$ for each $J \in \mathscr{L}(G)$. Hence B acts regularly on J^A for all $J \in \mathscr{L}(G)$. Hence $J \cong \langle J^A\rangle \cap C(B) \in \mathscr{L}(C_G(B))$ for all $J \in \mathscr{L}(G)$, and $C_{K(G)}(B) = \times\,\{\langle J^A\rangle \cap C(B) \mid J \in \mathscr{L}(G)\} = \times\,\mathscr{L}(C_G(B))$. Clearly, D acts transitively on $\mathscr{L}(C_G(B))$. So $E(C_G(B)) \circ\!\!\lhd AC_G(B) = ADC_{E(G)}(B)$. Let $S = \mathrm{Sol}(C_G(B))$. Let $K = \langle J^A\rangle \cap C(B)$ for some $J \in \mathscr{L}(G)$. Then S centralizes K. Hence S normalizes K. Hence S normalizes $C_{K(G)}(C_{K(G)}(K)) = \langle J^A\rangle$. Let $N = \bigcap\{N_{AG}(J^e) \mid e \in A\}$. Then, replacing (G, S, Ω) by $(AS/AS \cap N, A/A \cap N, J^A)$ in Lemma 2.6, it follows that S normalizes J. Hence S centralizes $\mathrm{Proj}_J(K) = J$. Since $J \in \mathscr{L}(G)$ was chosen arbitrarily, it follows that $S \subseteq C_G(K(G)) = 1$.

LEMMA 2.13. *Suppose the elementary abelian p-group A acts on the near p-solvable group G. Suppose $D = C_G(A)$ is solvable and* $\mathrm{Sol}(G) = 1 \neq G$. *Let $K = K(G)$. Let W be a perfect DA-invariant subgroup of K. Let $K_1 = C_K(W)$, $K_2 = C_K(K_1)$, and $D_i = D \cap K_i$. Then*

(a) $K = K_1 \times K_2$.

(b) $D \cap K = D_1 \times D_2$ *and* $D_2 = D \cap W$.

(c *If* $K_1 = 1$, *then* $C_G(W) = 1$.

(d) *Suppose Z_0, Z are DA-invariant subgroups of G, $Z_0 \lhd Z$, and $Z_0 \cap K \cap D = D_1$. Then Z normalizes K_1.*

Proof. Let $J \in \mathscr{L}(G)$. Theorem 2.10(g) implies that $\mathrm{Proj}_J(W) = 1$ or J. Hence (a) holds, and $D \cap K = D_1 \times D_2$. To complete (b) we may suppose $K_1 = 1$. Then Theorem 2.10(g) implies $D \cap K = \times \{J^{\circledA} \mid J \in \mathscr{L}(G)\} \subseteq W$. Hence (b) holds.

(c) Suppose $K_1 = 1$. Let $S = C_G(W)$. By (b) and Lemma 2.6, $S \subseteq \hat{K}(G)$. Hence S centralizes $\mathrm{Proj}_J(W) = J$ for all $J \in \mathscr{L}(G)$. So $S \subseteq C_G(K(G)) = 1$, proving (c).

(d) We take projections of subgroups of K with respect to internal direct products of K. Let N be the product of all components J of K satisfying $\mathrm{Proj}_J(Z_0 \cap K)$ is nilpotent. Theorem 2.10(b) implies that $N = K_2$. Hence K_2 and consequently K_1 admits Z.

LEMMA 2.14. *Suppose the elementary abelian p-group A acts on the near p-solvable semi-simple group X, $C_X(a)$ is solvable for some $a \in A^{\#}$, and W is a subgroup of $C_X(a)$ admitting $AC_X(a)$. Suppose $C_W(A) = 1$. Then $W = 1$.*

Proof. Let $J \in \mathscr{L}(X)$, $K = C_J(a)$, and $Z = \mathrm{Proj}_J(W)$. Then Z is a normal subgroup of K. Theorem 2.8e, j imply that $Z = 1$ or $F(K) \subseteq Z$, and $F(K) = [F(K), K]$. Suppose $F(K) \subseteq Z$. Then $F(K) \subseteq [Z, K] = [W, K] \subseteq W$. Hence $1 \neq \langle F(K)^A \rangle \cap D \subseteq W$. This is false, whence $\mathrm{Proj}_J(W) = 1$ for all $J \in \mathscr{L}(X)$. So $W = 1$ as required.

LEMMA 2.15. *Suppose G is a group and $K(G) \subseteq X \subseteq G$. Then $K(G) = K(X)$.*

Proof. We may and do suppose $\mathrm{Sol}(G) = 1$. Then $C_G(K(G)) = 1$. Now $[\mathrm{Sol}(X), K(G)] \subseteq \mathrm{Sol}(X) \cap K(G) \subseteq \mathrm{Sol}(K(G)) = \mathrm{Sol}(G) = 1$, whence $\mathrm{Sol}(X) = 1$. Hence $K(X) = K(G) \times C_{K(X)}(K(G)) = K(G)$.

LEMMA 2.16. *Suppose G is near A-solvable,* $\mathrm{Sol}(G) = 1 \neq G$. *Let $D = C_G(A)$, $K = K(G)$, and S be the unique maximal solvable DA-invariant* *subgroup of* G. *Let* $S_0 = S \cap K$, $D_0 = D \cap S_0 = D \cap K$, $D_1 = F(D_0)$, *and* $S_1 = F(S_0)$. *Finally let* $D_2 = C_{D_0}(D_0/D_1)$ *and* $S_2 = C_{S_0}(S_0/S_1)$. *Suppose* Z *is any* DA*-invariant subgroup of* G. *Then all of the following hold*:

(a) $S_0 \subseteq C_G(D_2 S_1/S_1) \subseteq \hat{K}(G)$.

(b) $S_2 = C_G(D_0 S_1/S_1)$,

(c) *If* $Z \cap K \leqslant S_1$, *then* $Z \leqslant S_2$,

(d) *If* $Z \cap K \cap D = 1$, *then* $Z \leqslant S_1$.

Proof. (a) Let $W = C_G(D_2 S_1/S_1)$. Then W is DA-invariant. Let $J \in \mathscr{L}(G)$ and $J^{\circledA} = \langle J^A \rangle \cap D$. Then W normalizes $J^{\circledA} S_2$. Hence W normalizes $C_K((C_K(J^{\circledA} S_2))^\infty) = \langle J^A \rangle$. By Lemma 2.6, W normalizes J. So (a) holds. Combine this with Theorem 2.8(g) to get (b).

(c) By Theorem 2.11(c), $Z \leqslant N_G(S_0)$. Hence $[Z, D_0] \leqslant Z \cap K \leqslant S_1$. Now (c) follows from (b).

(d) This follows from Theorem 2.10(f) and part (b).

3. Subfunctors

In this section G is a group, A is a non-identity elementary abelian p-subgroup of G, θ is a near solvable A-signalizer functor on G, and $D = \theta(C_G(A))$.

Lemma 3.1. θ_{Sol} *is a solvable* A*-signalizer functor on* G. *Moreover, if* $m(A) \geqslant 3$, *then* θ *is complete.*

Proof. Theorem 2.11(b) implies θ_{Sol} is a solvable A-signalizer functor on G. Now apply the main theorem of [5] to finish.

Theorem 3.2. *Suppose* $Y \in D(\theta)$ *and* $(X_1, X_2) \in P(\theta, Y)$. *Then there are subgroups* $Z_i \subseteq X_i$ *such that* $(Z_1, Z_2) \in P(\theta, Y)$, Z_2 *is solvable and* $A/C_A(Z_1/Z_2)$ *is cyclic.*

Proof. We may suppose $X_1 DA = G$. By Theorem 2.11(c, d) we may suppose X_2 is solvable. We may then reduce to $X_2 = 1$ and apply Lemma 2.12 to finish.

Theorem 3.3. *Let* $Y \in D(\theta)$, *and* X, $Z \in C(\theta, Y)$. *Then all of the following hold*:

(a) $\theta_Y^u(X) \in U(\theta, Y, X)$.

(b) $(\theta_Y^m(X), \theta_Y^l(X)) \in P(\theta, Y, X)$.

(c) $K(X) \subseteq \theta_Y^m(X)$.

(d) $\theta_Y(X) \in E(\theta, Y, X)$.

(e) $\theta_Y^l(X) \subset \theta_Y(X) \subset \theta_Y^m(X) \subset \theta_Y^u(X)$.

(f) $\theta_Y^u(X) \cap Z = X \cap \theta_Y^u(Z)$.

(g) $\theta_Y(X) \cap Z = X \cap \theta_Y(Z)$.

(h) *Suppose* $a, b \in A^{\#}, X = \theta(C_G(a))$, *and* $Z = \theta(C_G(b))$. *Then* $\theta_Y^l(X) \cap Z = X \cap \theta_Y^l(Z)$.

Proof. (a) By induction we may first suppose $G = XA$, then $X = \theta_Y^u(X)$, and finally

$$\text{whenever } N \lhd XA,\ N \subset X, \text{ and } N \cap D \subseteq Y, \text{ then } N = 1. \tag{3.1}$$

By Theorem 3.2, choose $(X_1, X_2) \in P(\theta, Y)$ with X_2 solvable. Suppose M is any normal subgroup of XA in X satisfying $M \cap X_1 \subseteq X_2$. Then $[M \cap D, X_1] \subseteq M \cap X_1$. Hence $M \cap D \subseteq C_D(X_1/X_2) = Y$. By (3.1), $M = 1$. In particular, $\mathrm{Sol}(X) = 1$ and $X_1 \cap K(X) \not\subseteq X_2$. Let $W = (X_1 \cap K(X))^{\infty}$. Then $X_1 = W \times X_2$ and W is a perfect DA-invariant subgroup of $K(X)$. Let $Z \in U(\theta, Y, X)$ and $Z_0 = \langle Y^Z \rangle$. By definition of $U(\theta, Y, X)$, $Z_0 \cap D = Y$. Hence

$$\begin{aligned} Z_0 \cap K(X) \cap D &= Y \cap K(X) = C_D(X_1/X_2) \cap K(X) \\ &= C_D(W) \cap K(X) = C_{K(X)}(W) \cap D. \end{aligned}$$

By Lemma 2.13(d), $C_{K(X)}(W)$ admits Z. Since $X = \theta_Y^u(X)$, it follows that $C_{K(X)}(W)$ is normal in G. But $D \cap C_{K(X)}(W) \subseteq Y$, whence $C_{K(X)}(W) = 1$. Lemma 2.13(c) implies $Y = D \cap X_2 \subseteq C_X(W) = 1$. Hence $X \in U(\theta, Y, X)$ proving (a).

(c) This is much the same as (a). We may suppose $G = XA$, $X = K(X)$ $\theta_Y^u(X)$, and $\mathrm{Sol}(X) = 1$. If $K(X) \cap D \subseteq Y$, then $(X_1 K(X), X_2 K(X)) \in P(\theta, Y, X)$. So we may suppose $K(X) \cap X_1 \not\subseteq X_2$. Then, as in (a) using Lemma 2.13, it follows that $(K(X)\, C_D(W), C_{K(X)D}(W)) \in P(\theta, Y, X)$ where $W = (X_1 \cap K(X))^{\infty}$.

(b), (d), and (e). By (a) we may suppose $X = \theta_Y^u(X) \in U(\theta, Y, X)$. We may again assume (3.1). In particular, $Y \subseteq \langle Y^{XA} \rangle = 1$. With (X_1, X_2) as before we have $[\mathrm{Sol}(X) \cap D, X_1] \subseteq X_2$. So $\mathrm{Sol}(G) = 1$. As in part (a), $X_1 = X_2 \times W$ where $W = (X_1 \cap K(X))^{\infty}$. Also as in part (a), $C_{K(X)}(W)$ is normal in G and intersects D trivially. By (3.1), $C_{K(X)}(W) = 1$. Lemma 2.13 implies $C_X(W) = C_X(K(X)) = 1$. Now $W = \times \{\langle J^A \rangle \cap W | J \in \mathscr{L}(X)\}$. Hence D acts transitively on $\{\langle J^A \rangle | J \in \mathscr{L}(X)\}$. Hence $K(X)$ is a minimal normal subgroup of $K(X)DA$ and $C_D(K(X)) = 1 = Y$. So $(K(X), 1) \in P(\theta, Y, X)$. Hence $(\theta_Y^m(X), \theta_Y(X)) = (K(X), 1) \in P(\theta, Y, X)$. Let $M = N_{K(X)}(C_{K(X)}(F(D \cap K(X))))$, and $R = [F(M), A]$. Let $T \in E(\theta, 1)$. Lemma 2.16(d) implies that $T \subseteq K(M)$. Theorem 2.10(k) implies that $\theta_Y(X) = R \in E(\theta, Y, X)$. This proves (b), (d), and (e).

(f), (g). These follow directly from the definitions and parts (a) and (d).

(h) Observe that this follows directly from the definitions and (b) if $X \cap Z \in C(\theta, Y)$. Hence we suppose $C_{\theta_Y^m(X)/\theta_Y^l(X)}(b)$ and $C_{\theta_Y^m(Z)/\theta_Y^l(Z)}(a)$ are both solvable. Let $M = \theta_Y^u(Z)$, $M_1 = \theta_Y^m(Z)$, $M_2 = \theta_Y^l(Z)$ and $\bar{M} = M/M_2$. Let $W = \theta_Y^u(X) \cap C(b)$ and $W_0 = \theta_Y^l(X) \cap C(b)$. By (f), $W = C_M(a)$. Since $W_0 \lhd W$ we have $\bar{W}_0 \lhd \bar{W} = C_{\bar{M}}(a)$ and $\bar{D} \cap \bar{W}_0 = \bar{Y} = \bar{1}$. In particular, $\bar{W}_0 \cap \bar{M}_1 \lhd AC_{\bar{M}_1}(a)$, $\bar{M}_1$ is semi-simple, and $C_{\bar{M}_1}(a)$ is solvable; so Lemma 2.14 implies $\bar{W}_0 \cap \bar{M}_1 = \bar{1}$. By (e), $\bar{W}_0 = 1$. Hence $\theta_Y^l(X) \cap Z = W_0 \subseteq X \cap \theta_Y^l(Z)$. The symmetric inclusion completes (h) and the theorem.

THEOREM 3.4. *Suppose* $(X_1, X_2) \in P(\theta)$. *Let* $D_i = D \cap X_i$. *Then* $F(D_1/D_2) = F(D/D_2)$ *is the unique minimal normal subgroup of* D/D_2.

Proof. By induction we may suppose $G = X_1DA$ and $X_2 = 1$. By Theorem 3.2 we may suppose $C_A(X_1)$ is a hyperplane of A. Consequently A normalizes each component, D acts transitively on the components, and $D_1 = \times\, \{C_J(A) | J \in \mathscr{L}(G)\}$. Theorem 2.8(j) implies that $F(D_1)$ is the unique minimal normal subgroup of D in D_1. Hence it suffices to show $F(D) = F(D_1)$. Lemma 2.16 (b) does this.

THEOREM 3.5. *Suppose* $D_2 \in D(\theta)$ *and* $X \in C(\theta, D_2)$. *Let* $D^* \supset D_2$ *be such that* $D^*/D_2 = F(D/D_2)$. *Suppose* N, L, R *are subgroups of* X *which satisfy*:

(a) $N, L,$ *and* R *are normal in* $\langle R, D, A\rangle = RDA$,

(b) $N \subset L \subset R$, *and* L/N *is a chief* RDA *factor*,

(c) AR *centralizes* L/N, *and*

(d) $R \cap D = D^*$, $D \cap N \subseteq D_2$, *and* $D^*N/N = (D_2N/N) \times (L/N)$.

Then it follows that R *has an* RDA *invariant subgroup* B *such that* $B \cap D = D_2$ *and* $BD^* = R$.

Proof. Suppose false. Choose a counterexample G of least possible order. Subject to this restriction choose one with L of least possible order. By Theorem 3.2 there is a pair $(X_1, X_2) \in P(\theta, D_2)$ with X_2 solvable. Fix such a pair with X_2 of least possible order.

First we observe some structure of G. The requirements are satisfied by $\langle X_1, R, D, A\rangle$. So $X = \langle X_1, R, D\rangle$ and $G = XA$. Suppose M is any normal subgroup of G in X such that $M \cap D \subseteq D_2$. If $M \neq 1$, a short argument assisted by Lemma 2.2 yields a subgroup B_1 of RM, which contains M, is normal in $RMDA$, intersects D in D_2, and satisfies $B_1 D^* = RM$. The subgroup $B_1 \cap R$ satisfies the conclusion. This is false, whence $M = 1$. Let V be any minimal normal subgroup of G in X. Let $W = (X_1 \cap V)^\infty$. Suppose $V \cap X_1 \subseteq X_2$. Then $V \cap D \subseteq C_D(X_1/X_2) = D_2$. This is false. So $V \cap X_1 \not\subseteq X_2$. Since $V \cap X_1 \lhd X_1$, it follows that $X_1 = X_2(V \cap X_1)$. Hence $X_1 = X_2 \times W = D_2 \times W$, and $W = X_1^\infty$. Since all minimal normal subgroups of G in X contain W, it follows that $K(X)$ is the unique minimal normal subgroup of G in X. Let $K = K(X)$. Now $X = KRD$.

Let $K_1 = C_K(W)$, $K_2 = C_K(K_1)$, $E = D \cap K_2$, and $E^* = F(E)$, $S_2 = C_K(E^*)$, and $S = N_K(S_2)$. We shall first show $K_1 = 1 = D_2$, then $L \subseteq S_2$, then $R \cap K \subseteq S_2$ is abelian, and finally $R \subseteq S_2$.

Since $N \cap K \cap D \leqslant D_2 \cap K \leqslant K_1$, Lemma 2.16(d) implies that $N \leqslant S_2 C_G(K_2)$. Hence $\mathrm{Proj}_J(N \cap K)$ is nilpotent if $J \in \mathscr{L}(K_2)$. Since $(D_2 \cap K) = C_{K_1}(A) \leqslant N$, it follows that $\mathrm{Proj}_J(N)$ is not nilpotent for any $J \in \mathscr{L}(K_1)$. Hence $K_1 \lhd G$ and $K_1 \cap D \leqslant N$. Hence $K_1 = 1$ and $D_2 = C_D(W) = C_D(K_2) = 1$.

Now $N \cap E = 1$, so $N \leqslant S_2$ by Lemma 2.16(d). Now E^* is the unique minimal normal subgroup of DA, and $N \cap D = 1$. Hence $L = NE^* = N \times E^* \leqslant S_2$.

Now apply Theorem 2.10(m) with L in the role of Y, N in the role of X, and V in the role of E^* to conclude $R \subseteq S_2$. In particular R is abelian. By [8, Theorem 5.2.3], $R = [R, A] \times C_R(A) = [R, A] \times E^*$. Then $[R, A]$ is obviously a suitable candidate for B, a contradiction.

THEOREM 3.6. *Suppose that $m(A) \geqslant 3$. Suppose $X \in \tilde{\mathrm{N}}_\theta(A)$, X_i are DA-invariant subgroups of X for $i = 1, 2$, and X_1/X_2 is a non-solvable chief X_1DA-factor. Let $Y \in D(\theta)$. Suppose that $X_2 \cap D \subseteq Y$. Then one of the following occurs:*

(a) $X_1 \cap D \subseteq Y$,

(b) $(X_1, X_2) \in P(\theta, Y)$.

Proof. Suppose false. Let G be a counterexample of least possible order. By Theorem 3.2 there is a hyperplane B_1 of A such that $\theta(C_G(B_1)) \in C(\theta, Y)$. After the fashion of Theorem 3.2 there is a hyperplane B of A such that $C_{X_1/X_2}(B)$ is a chief non-solvable $(C_{X_1}(B))\,DA$-factor. Were $C_X(B) \in C(\theta, Y)$, then $X \in C(\theta, Y)$ would hold. Hence we may and do suppose $B = C_A(X)$. Let $E = B \cap B_1$ and $W = \theta(C_G(E))$. Since $X \subseteq W \in C(\theta, Y)$, it follows that $G = WA$.

Fix $(Z_1, Z_2) \in C(\theta, Y)$ with Z_2 of least possible order. By Theorem 3.2, Z_2 is solvable. Now G has no nontrivial normal subgroups in W which intersect D in a subgroup of Y. This leads to: $K = K(W)$ is the unique minimal normal subgroup of G in W, $Z_2 = Y$, $Z_1 = Y \times (Z_1 \cap K)^\infty = Y \times Z_1^\infty$, and $W = KX_1D$. Suppose that $X_1 \cap K \subseteq X_2$. Then $[D \cap X_1, D \cap Z_1^\infty] \subseteq X_2 \cap D \cap Z_1^\infty \subseteq Y \cap Z_1^\infty = 1$. Hence $D \cap X_1 \subseteq C_D(D \cap Z_1^\infty) = Y$. This is false, whence $X_1 = X_2(X_1 \cap K)$. Now $D \cap ((D \cap X_1)(X_1 \cap K)) = D \cap X_1 \nsubseteq Y$. Hence we may suppose $X_1 = (D \cap X_1)(X_1 \cap K)$. In particular, $KD = W$. By Lemma 2.13, $(K, 1) = (\theta_Y^m(G), \theta_Y^l(G))$ and $Y = 1$. Now $X_1 \cap K$ is non-solvable. Since $\mathrm{Proj}_J(X_1 \cap K)$ admits $\mathrm{Proj}_J(D \cap K)$ for $J \in \mathscr{L}(W)$ and D is transitive on $\mathscr{L}(W)$, we have $\mathrm{Proj}_J(X_1 \cap K) = J$ for all $J \in \mathscr{L}(W)$ (see Theorem 2.8(d)). Hence $X_1 \cap K$ is semi-simple, $X_2 \cap K = 1$, and $X_2 \subseteq C_W(X_1 \cap K) = 1$. So $X_2 = 1$, and $(X_1, 1) \in P(\theta, Y)$, a contradiction.

THEOREM 3.7. *Assume the hypotheses of Theorem* 3.6. *Assume also that* $X \in C(\theta, Y)$. *Then*

(a) $X_1 \subseteq \theta_Y^m(X)$, *and*

(b) *either* $X_1 \cap D \subseteq Y$ *or* $(X_1 D \cap \theta_Y^m(X), X_2 D \cap \theta_Y^l(X)) \in P(\theta, Y)$.

Proof. Theorem 3.3(d, e) imply both conditions if $X_1 \cap D \subseteq Y$. *So suppose* $X_1 \cap D \nsubseteq Y$. By Theorem 3.6(b), $(X_1, X_2) \in P(\theta, Y)$. Now the result follows by Theorem 3.3(b).

Hypothesis A (applied to a vector (G, H, A, D, D_1, D_2) of groups):

A1: A is a p-group and H is near A-solvable.

A2: $G = HA$.

A3: $D = C_H(A)$.

A4: D_i are normal subgroups of D such that

(a) $D_2 \subset D_1$,

(b) D_1/D_2 is the unique minimal normal subgroup of AD/D_2 in D/D_2,

(c) $C_D(D_1/D_2) = D_1$.

A5: Suppose X_1 is any subgroup of H admitting DA, X_1/X_2 is a chief nonsolvable X_1DA-factor, and $X_2 \cap D \subseteq D_2$. Then $X_1 \cap D \subseteq D_2$.

A6: Suppose N, L, and R are subgroups of H which satisfy the following conditions:

(a) N, L, and R admit DA.

(b) $L \subseteq R$, and L/N is a chief RDA factor. Moreover, RA centralizes L/N.

(c) $R \cap D = D_1$ and $D \cap N \subseteq D_2$.

(d) $D_1N/N = (D_2N/N) \times (LN/N)$.

Then it follows that R has an RDA invariant subgroup B such that $B \cap D = D_2$ and $R = BD_1$.

THEOREM 3.8. *Suppose (G, H, A, D, D_1, D_2) satisfies hypothesis* A. *Define $\mathscr{S}(H) = \{M \mid M \subseteq H$; M is a DA-invariant; and $M \cap D = D_2\}$. Then $\mathscr{S}(H)$ has a unique maximal element under set inclusion.*

Proof. Let $\mathscr{S} = \mathscr{S}(H)$. Suppose by way of contradiction that the conclusion is false. Choose a counterexample G of least possible order. The hypotheses inherit to $(\langle \mathscr{S} \rangle DA, \langle \mathscr{S} \rangle D, A, D_1, D_2)$. Hence

$$G = \langle \mathscr{S} \rangle DA \tag{3.2}$$

and

$$\langle \mathscr{S} \rangle \neq 1. \tag{3.3}$$

Let N be a minimal normal subgroup of G contained in H. By (3.3) such an N exists. Suppose $D \cap N \subseteq D_2$. For each subgroup X of G let $\bar{X} = XN/N$. Hypothesis A inherits to $(\bar{G}, \bar{H}, \bar{A}, \bar{D}, D_1, \bar{D}_2)$. Write $\overline{\mathscr{S}(H)} = \{\bar{X} \mid X \in \mathscr{S}(H)\}$. By Lemma 2.2, $\overline{\mathscr{S}(H)} = \mathscr{S}(\bar{H})$. Hence $\langle \overline{\mathscr{S}(H)} \rangle \in \mathscr{S}(\bar{H})$. This is false. Hence

$$D \cap N \nsubseteq D_2. \tag{3.4}$$

Next suppose that N is nonsolvable. Then $N = N_1 \times N_2 \times \cdots \times N_k$ where each N_i is a minimal normal subgroup of NDA. By A5, $N_i \cap D \subseteq D_2$ for $1 \leqslant i \leqslant k$. Lemma 2.2 implies that $N \cap D \subseteq D_2$, against (3.4). Hence

$$N \text{ is solvable.} \tag{3.5}$$

By (3.4) and A4 there follows

$$D_1 = D_2(D_1 \cap N). \tag{3.6}$$

Suppose there is $M \in \mathscr{S}$ such that $NMDA = G$. Then $M \cap N \lhd G$. So $M \cap N = 1$ or N. By (3.6), $M \cap N = 1$. Also $MD_1 \cap N \lhd G$. So $MD_1 \cap N = N$ by (3.6). Hence $MDA = (MD_1) DA = NMDA = G$. Hence $M \lhd G$. By (3.4), $1 = M$. Hence $G = DA$. Hence $\mathscr{S} = \{1\}$. This is false, whence

$$NMDA \neq G \qquad \text{for any } M \in \mathscr{S}. \tag{3.7}$$

For each $M \in \mathscr{S}$ let M^* be the unique maximal element of $\mathscr{S}(NMD)$. This is well defined by (3.7). Next we show

$$M^* \cap N = D_2^* \cap N \qquad \text{for any } M \in \mathscr{S}. \tag{3.8}$$

Clearly, $D_2^* \subseteq M^*$ for any $M \in \mathscr{S}$. Hence $N \cap D_2^* \subseteq N \cap M^*$. Conversely, $M^* \cap N$ admits DA, whence $(M^* \cap N)\, D_2 \subseteq D_2^*$. So (3.8) holds. By (3.2) and (3.8), $D_2^* \cap N \lhd G$. Now (3.4) and (3.8) yield

$$M \cap N = 1 \qquad \text{for all } M \in \mathscr{S}. \tag{3.9}$$

Let $M \in \mathscr{S}$. Then $[M, D \cap N] \subseteq M \cap N = 1$ by (3.9). By (3.2) and (3.6), $1 \neq D_1 \cap N \lhd G$. Hence $N \times D_2 = D_1$ and $\langle \mathscr{S}(H) \rangle \subseteq C_H(N)$. Let $R = C_H(N)$. By A4, $R \cap D = D_1$. Applying A6, with (R, N, 1) in place of (R, L, N), yields a subgroup B normal in $RDA = G$ such that $BD_1 = R$ and $B \cap D = D_2$. By (3.4), $B = 1$. Hence $D_2 = 1$ and $G = RDA = D_1 DA = DA$. So $\mathscr{S} = \{1\}$, a contradiction.

LEMMA 3.9. *Suppose* (G, H, A, D, D_1, D_2) *satisfies hypotheses* A1, A2, A3, A4, *and* A5. *Suppose* E *is a subgroup of* A *of rank* 2 *such that* A6 *is satisfied by* ($C_G(e)$, $C_H(e)$, A, D, D_1, D_2) *for all* $e \in E^\#$. *Suppose further that if* $p = 3$, D_1/D_2 *is a* 5*-group. Then* (G, H, A, D, D_1, D_2) *satisfies hypothesis* A.

Proof. Suppose false and let G be a counterexample of least possible order. Then there are subgroups N, L, and R of H satisfying the conditions but not the conclusion of A6. Then (RDA, RD, A, D, D_1, D_2) is a counterexample to this lemma; so $G = RDA$.

Suppose $M \cap D \subseteq D_2$ for some normal subgroup M of G contained in H. Let $\bar{G} = G/M$. The conditions of the lemma apply to ($\bar{G}$, $\bar{H}$, $\bar{A}$, $\bar{D}$, $\bar{D}_1$, $\bar{D}_2$). Hence $M = 1$. In particular

$$N = 1 \qquad \text{and} \qquad D_1 = D_2 \times L = C_D(L). \tag{3.10}$$

A direct consequence of (3.10) is

$$L \text{ is a minimal normal subgroup of } G \text{ in } H. \tag{3.11}$$

By (3.10) we may and do assume

$$R = C_H(L). \tag{3.12}$$

Next let K be a normal extension of L in G, which is maximal subject to the condition that KD_1 be a proper subgroup of R. Since D_1 is proper in R such a K exists. The hypotheses apply to (KDA, KD, A, D, D_1, D_2). Since $KDA \neq G$, there is a subgroup B in KD_1 which is normal in KDA, and which satisfies $BD_1 = KD_1$ and $B \cap D = D_2$. Hence $KD_1 = BD_1 = B(D_2L) = BL = B \times L$. Hence $\Phi(K) \subseteq \Phi(KD_1) = \Phi(B \times L) = \Phi(B)$; hence $\Phi(K) \cap D \subseteq B \cap D = D_2$. Since $\Phi(K)$ is normal in G we must have $\Phi(K) = 1$. By (3.11), L is an r-group for some prime r. $O_{r'}(K) \cap D \subseteq O^r(D_1) \subseteq D_2$, whence $O_{r'}(K) = 1$. This implies

$$K \text{ is an elementary abelian } r\text{-group.} \tag{3.13}$$

Let K_1/K be a chief G factor in R. Then $R = K_1D_1$. Let $E_1 = C_E(K_1/K)$. Since $G = K_1DA$ it follows that $E_1 = C_E(G/K)$. So $[K, E_1]$ is normal in G. By (3.13), $[K, E_1] \cap D = 1$, whence $[K, E_1] = 1$. Hence $E_1 \subseteq Z(G) \cap E = 1$. In particular, K_1/K is non-solvable. Since $KC_G(e) \neq G$ for any $e \in E^{\#}$, hypothesis A applies to $(KC_G(e), KC_H(e), A, D, D_1, D_2)$ for any $e \in E^{\#}$. Then $KC_R(e) = U_e \times L$, where $U_e \lhd KC_G(e)$ and $U_e \cap D = D_2$ for all $e \in E^{\#}$. Let $W = [K, A]$ and $Z = C_K(A)$. Then $W = [K, A] \subseteq K \cap [L \times U_e, A] \subseteq K \cap U_e$. Hence $K \cap U_e = W \times (Z \cap U_e) = W \times (K \cap D_2)$ is independent of $e \in E^{\#}$. By Lemma 2.1, $W \times (K \cap D_2) \lhd G$. Hence

$$K = L. \tag{3.14}$$

By A5 and (3.14), K_1 is perfect. Let J be a component of K_1/L. Hence r is a prime divisor of the schur multiplier of J. Hence the conditions of the lemma imply that $p \geqslant 5$. There are only two possibilities: either $J \cong \mathscr{L}_2(3^p)$ and $r = 2$, or $J \cong U_3((2^p)^2)$ and $r = 3$. Let S be the unique maximal solvable subgroup of H containing D. Theorems 2.10(c, b) and 2.11(a, b), imply that S is well defined and that S contains a sylow r-subgroup of K_1. The conditions of the lemma apply to (SA, S, A, D, D_1, D_2). Since $SA \neq G$, hypothesis A6 applies and implies that L has a complement in $S \cap R$. Hence L has a complement in a sylow r-subgroup of K_1. By [4], AK_1 splits over L. Hence, $K_1 = K_1^{\infty} \times L$, a contradiction.

THEOREM 3.10. *Suppose $Y \in D(\theta)$ and $m(A) \geqslant 3$. Then θ_Y is an A-signalizer functor on G.*

Proof. It suffices to show $\theta_Y(C_G(a)) \in E(\theta, Y, \theta(C_G(a)))$ for all $a \in A^{\#}$. Theorem 3.3(d) implies this whenever $\theta(C_G(a)) \in C(\theta, Y)$. Let us fix $X = \theta(C_G(a)) \notin C(\theta, Y)$ and prove the result for this X. By Theorem 3.2 we may and do fix $(X_1, X_2) \in P(\theta, Y)$ such that $C_A(X_1/X_2)$ has a subgroup E of rank 2. Rename $Y = D \cap X_2 = D_2$. Let $D_0 = D \cap X_1$ and $D_1 \supset D_2$ satisfy $D_1/D_2 = F(D_0/D_2)$. By Theorem 3.4, (XA, X, A, D, D_1, D_2) satisfies hypotheses A1, A2, A3 and A4. By Theorem 3.6, X satisfies A5. By Theorem 3.5 $(C_{XA}(e), C_X(e), A, D, D_1, D_2)$ satisfies A6 for all $e \in E^{\#}$. Moreover, if $p = 3$, then $X_1/X_2 \cong nSz(8)$ whence $D_1/D_2 \cong F(n\,\mathrm{Frob}(20)) = nZ_5$. Theorem 3.8 and Lemma 3.9 yield the conclusion.

THEOREM 3.11. *Suppose $m(A)=3$ and θ_Y is complete. Then*

(a) $\theta_Y^l(G) \in \mathscr{N}_\theta(A)$, *and*

(b) $\theta_Y^m(G) \subseteq N_G(\theta_Y^l(G))$.

Proof. (a) This is a direct consequence of Theorem 3.3(e).

(b) Let $W=\theta_Y^l(G)$, $S=\{B \in \mathscr{E}_2(A) \mid \theta(C_G(B)) \in C(\theta, Y)\}$, and $T=\{a \in A^{\#} \mid \theta(C_G(a)) \in C(\theta, Y)\}$. Let $(X_1, X_2) \in C(\theta, Y)$. Let E be the largest subgroup of A which normalizes each component of X_1/X_2, and let $F=C_A(X_1/X_2)$. Let E_1 be a complement to F in E. Extend E_1 to a complement A_1 of F in A. Let $U=\{F \times Y \mid Y \text{ is a complement of } E_1 \text{ in } A_1\}$. Then $U \subseteq S$. Substituting (A_1, E_1) for (A, B) in Lemma 2.7 yields $X_1=\langle X_2, C_{X_1}(V) \mid V \in U\rangle$ and $X_2=\langle C_{X_2}(d) \mid d \in B^{\#}\rangle$ for any $B \in U$. There follows

$$X_1 \subseteq \langle W, \theta_Y^m(C_G(B)) \mid B \in U\rangle \qquad \text{for any} \quad (X_1, X_2) \in C(\theta, Y) \quad (3.15)$$

and

$$W=\langle \theta_Y^l(C_G(t)) \mid t \in T\rangle. \qquad (3.16)$$

Now let $B \in U$. Let $W_B=\langle \theta_Y^l(C_G(b)) \mid b \in B^{\#}\rangle$ and $X=\theta_Y^m(C_G(B))$. By (3.15) it suffices to show $X \subseteq N_G(W)$. We shall do this by showing $X \subseteq N_G(W_B)$ and $W_B=W$. Now $X \subseteq \theta_Y^m(C_G(b)) \subseteq N_G(\theta_Y^l(C_G(b)))$ for all $b \in B^{\#}$ (see Theorem 3.3(b)). Hence $X \subseteq N_G(W_B)$. Let $t \in T$. By Theorem 3.3(h), $\theta_Y^l(C_G(t)) \cap C_G(b) \subseteq W_B$ for any $b \in B^{\#}$. Hence by (3.16) and Lemma 2.1, it follows that $W \subseteq W_B$. Hence by (3.16), $W_B=W$. This completes the proof of Theorem 3.11.

4. A FAMILY OF SUBGROUPS

The goal of this section is to prove:

THEOREM 4.1. *Suppose θ is a locally complete near solvable A-signalizer functor on G. Suppose θ is not solvable. Then there is an $X \in \hat{\mathscr{N}}_\theta(A)$ which satisfies:*

(a) *X is non-solvable.*

(b) *Suppose Z, $U \in \hat{\mathscr{N}}_\theta(A)$ satisfies Z $\mathrm{Sol}(X)=K(X)$, and $Z \subseteq U$. Then Z $\mathrm{Sol}(U)=K(U)$.*

(c) *Suppose $K(X) \subseteq U \in \hat{\mathscr{N}}_\theta(A)$. Then $U \subseteq X$.*

(d) *Suppose Z, $T \in \hat{\mathscr{N}}_\theta(A)$, Z $\mathrm{Sol}(X)=K(X)$, T is solvable, and $\mathrm{Sol}(X) \subseteq T$. Then $\mathrm{Sol}(X)$ is the unique maximal subgroup of T normalized by Z.*

DEFINITION. Suppose G is a group. $\deg_p(G)$ is the least integer n for which G has a faithful permutation representation of degree n.

$$\begin{aligned} \deg(G) &= 0 && \text{if } G \text{ is solvable} \\ &= \deg_p(G/\mathrm{Sol}(G)) && \text{otherwise.} \end{aligned}$$

$\mathscr{B}(G)$ is the set of all subgroups X of G for which $K(X) = X$ holds. Let T be a subset of $\mathscr{B}(G)$.

$$\mathscr{B}_1(T) = \{X \in T \mid \deg(X) \geqslant \deg(Y) \text{ for any } Y \in T\},$$
$$\mathscr{B}_2(T) = \{X \in \mathscr{B}_1(T) \| X/\mathrm{Sol}(X)| \geqslant |Y/\mathrm{Sol}(Y)| \text{ for any } Y \in \mathscr{B}_1(T)\},$$
$$\mathscr{B}^*(T) = \{X \in \mathscr{B}_2(T) \| X| \geqslant |Y| \text{ for any } Y \in \mathscr{B}_2(T)\}.$$

We write $\mathscr{B}^*(G) = \mathscr{B}^*(\mathscr{B}(G))$. Suppose θ is an A-signalizer functor on G. Then $\mathscr{B}(\theta) = \bigcup \{\mathscr{B}(X) | X \in \hat{\mathfrak{N}}_\theta(A)\}$ and $\mathscr{B}^*(\theta) = \mathscr{B}^*(\mathscr{B}(\theta))$.

Hypothesis B. G is a simple nonabelian group. Let X be any perfect member of $\mathscr{B}(\mathrm{Aut}(G))$. Then $\deg(X) < \deg(G)$ if $X \not\subseteq \mathrm{Inn}(G)$.

Hypothesis C. G is a group. Each non-abelian simple section of G satisfies hypothesis B.

Let G be a permutation group on a set Ω. Let Δ be a subset of Ω and let S be a set of subsets of Ω. We define

$$\begin{aligned} G_\Delta &= \bigcap \{G_a | a \in \Delta\}, \\ G^\Delta &= \{g \in G | \, g(\Delta) = \Delta\}, \\ G_S &= \bigcap \{G^\Delta | \Delta \in S\}, \\ G^S &= \{g \in G | \, g(\Delta) \in S \text{ for all } \Delta \in S\}, \end{aligned}$$
$$G(S) = G^S/G_S \qquad \text{and} \qquad G(\Delta) = G^\Delta/G_\Delta.$$

We consider $G(S)$ and $G(\Delta)$ as permutation groups on S and Δ respectively in the natural way.

LEMMA 4.2. *Suppose $G = K(G)$ is a group. Then* $\deg_p(G) \geqslant \deg(G)$.

Proof. Suppose false. Choose a counterexample G of least possible order. Let Ω be a set of order $\deg_p(G)$ on which G acts faithfully.

First suppose G is not transitive on Ω. Let Ω be the disjoint union of non-empty sets Ω_1 and Ω_2, both of which admit G. Let $H_2 = G_{\Omega_1}$, and $H_1 = C_G(H_2 \,\mathrm{Sol}(G)/\mathrm{Sol}(G))$. Let $H_1^* = H_1/H_1 \cap H_2$, and $H = H_1^* \times H_2$. Then $H/\mathrm{Sol}(H) \cong G/\mathrm{Sol}(G)$. Hence $\deg(H) = \deg(G)$. Now $H_1^* \subseteq G(\Omega_1)$ and H_2 acts faithfully on Ω_2, whence $\deg(H_1^*) \leqslant \deg_p(H_1^*) \leqslant |\Omega_1|$ and $\deg(H_2) \leqslant$

$\deg_p(H_2) \leqslant |\Omega_2|$. Hence $\deg(G) = \deg(H) \leqslant \deg(H_1^*) + \deg(H_2) \leqslant |\Omega_1| + |\Omega_2| = \deg_p(G)$. This is false, whence G is transitive on Ω.

Next suppose $\mathrm{Sol}(G)$ acts transitively on Ω. Let $a \in \Omega$. Then $G = G_a \mathrm{Sol}(G)$, $G_a \neq G$, and $G_a/\mathrm{Sol}(G_a) \cong G/\mathrm{Sol}(G)$. Hence $\deg(G) = \deg(G_a) \leqslant \deg_p(G_a) < \deg_p(G)$. This is false. Let S be the set of orbits of $\mathrm{Sol}(G)$ on Ω. Then S is a system of imprimitivity for G.

Let $K = G_S$ and $Z = C_G(K/\mathrm{Sol}(G))$. Then $G/\mathrm{Sol}(G) \cong K/\mathrm{Sol}(K) \times Z/\mathrm{Sol}(Z)$. Now $Z/Z \cap K \subseteq G(S)$, whence $\deg(Z) = \deg(Z/Z \cap K) \leqslant |S|$. Let Δ and $\Gamma \in S$. Then $K_\Delta \lhd K$. Hence $(K_\Delta \mathrm{Sol}(G))^\infty = K_\Delta^\infty$. However, $K_\Delta \mathrm{Sol}(G) \lhd G$ and G is transitive on S. Hence $K_\Delta^\infty = (K_\Delta \mathrm{Sol}(G))^\infty = (K_\Gamma \mathrm{Sol}(G))^\infty = K_\Gamma^\infty$. Hence $K_\Delta^\infty \subseteq G_\Omega = 1$. Thus $K_\Delta \subseteq \mathrm{Sol}(K)$. So

$$\deg(K) = \deg(K/K_\Delta) \leqslant \deg_p(K/K_\Delta) \leqslant |\Delta|.$$

Hence

$$\begin{aligned} |\Delta||S| = \deg_p(G) < \deg(G) &\leqslant \deg(K) + \deg(Z) \\ &\leqslant |\Delta| + |S|. \end{aligned}$$

This is false since $2 \leqslant \min\{|\Delta|, |S|\}$. This completes the proof of Lemma 4.2.

LEMMA 4.3. *Suppose $G = K(G)$ is a group. Then*

$$\deg(G) = \sum \{\deg(J) | J \in \mathscr{L}(G/\mathrm{Sol}(G))\}.$$

Proof. We may and do assume $\mathrm{Sol}(G) = 1$ and G is not simple. Choose a set Ω of order $\deg(G)$ on which G acts faithfully. Suppose G acts primitively on Ω. Then $G = G_1 \times G_2$, $G_1 \cong G_2$ is simple, and $|\Omega| = |G_1|$. Let N be a maximal subgroup of G_1. Then $\deg(G) \leqslant 2\deg(G_1) \leqslant 2|G_1 : N| = |\Omega|(2/|N|)$. Hence $|N| \leqslant 2$. This is false.

Now suppose G is transitive on Ω. Let S be a system of imprimitivity for G. Let $K = G_S$, $Z = C_G(K)$, $\Delta \in S$, and $H = K_\Delta$. Then $G = K \times Z$. Now K is isomorphic to a subgroup of $G(S)$ and so $\deg(K) \leqslant |S|$. Since $H \lhd G$, G acts on the fixed points of H; so $H = 1$. Hence $K \cong K(\Delta)$. Hence $|\Omega| = |\Delta||S| = \deg(G) \leqslant \deg(K) + \deg(Z) \leqslant |\Delta| + |S|$. This is also false.

We have shown that G is not transitive. Let Ω be the disjoint union of nonempty sets Ω_1, Ω_2, both of which are fixed blocks of G. Let $H_2 = G_{\Omega_1}$ and $H_1 = C_G(H_2)$. Then $G = H_1 \times H_2$, and H_i acts faithfully on Ω_i. Hence $\deg(H_i) = |\Omega_i|$. The result follows by induction.

COROLLARY 4.4. *Suppose $G = K(G)$ is a group. Suppose $N \lhd G$. Then* $\deg(G) = \deg(N) + \deg(G/N)$.

Proof. Let $\bar{G} = G/\mathrm{Sol}(G)$. Then $(G/N)/\mathrm{Sol}(G/N) \cong \bar{G}/\bar{N}$, $N/\mathrm{Sol}(N) \cong \bar{N}$, and $\bar{G} \cong (\bar{G}/\bar{N}) \times \bar{N}$. Hence we may suppose $\mathrm{Sol}(G) = 1$. In this case the result is a direct consequence of Lemma 4.3.

LEMMA 4.5. *Suppose G satisfies hypothesis C. Suppose X is a perfect element of $\mathscr{B}_1(\mathscr{B}(G))$. Then $X \subseteq K(G)$.*

Proof. Suppose false. Choose a counterexample G of least possible order. Then $\mathrm{Sol}(G) = 1$, and $G = K(G)\,X$.

Let $K = K(G)$. Suppose $K = K_1 \times K_2$, $K_i \neq 1$, and $K_i \lhd G$. Then $|G/C_G(K_1)| < |G|$. Hence

$$\deg(X/C_X(K_1)) = \deg(XC_G(K_1)/C_G(K_1)) \leqslant \deg(K_1),$$

since $K_1 \cong K(G/C_G(K_1))$. By Corollary 4.4,

$$\begin{aligned}\deg(C_X(K_1)) &= \deg(X) - \deg(X/C_X(K_1)) \geqslant \deg(X) - \deg(K_1)\\ &= \deg(X) - (\deg K - \deg(K_2))\\ &= \deg(K_2) + (\deg(X) - \deg K) \geqslant \deg(K_2).\end{aligned}$$

Hence $(C_X(K_1))^{\infty} \subseteq K_2$. Hence $\deg(C_X(K_1)) = \deg(K_2)$. Hence $\deg(X/C_X(K_1)) = \deg(X) - \deg(K_2) \geqslant \deg(K_1)$. Hence X induces only inner automorphisms on K_1. By symmetry, X induces only inner automorphisms on K_2. Hence $X \subseteq K$. This is false. So X acts transitively on $\mathscr{L}(G)$.

Let $Y = \hat{K}(G) \cap X$. Then X/Y acts faithfully on $\mathscr{L}(G)$. So $\deg(X/Y) \leqslant |\mathscr{L}(G)|$. Let $J \in \mathscr{L}(G)$, and $W = C_Y(J)$. Then $W^{\infty} = (\mathrm{Sol}(X)\,Y)^{\infty} \lhd X$. Since X acts on the components of G centralized by W^{∞}, it follows that $W^{\infty} = 1$. Hence W is solvable. Hence $\deg(Y) = \deg(Y/W) = \deg(YC_{JY}(J)/C_{JY}(J)) \leqslant \deg(J)$ by Hypothesis C. By Corollary 4.4, $\deg(X) \leqslant |\mathscr{L}(G)| + \deg(J)$. Now $\deg(K) = |\mathscr{L}(G)| \deg(J) \geqslant \deg(X)$. Hence $|\mathscr{L}(G)| = 1$. Put differently, $K(G)$ is simple. Hypothesis C implies that $X \subseteq K(G)$, a contradiction.

THEOREM 4.6. *Suppose G satisfies Hypothesis* C. *Then*

(a) $K(G) = X\,\mathrm{Sol}(G)$ *for any* $X \in \mathscr{B}_2(\mathscr{B}(G))$, *and*

(b) $\mathscr{B}^*(G) = \{K(G)\}$.

Proof. (a) We may and do suppose $\mathrm{Sol}(G) = 1$. Let $X \in \mathscr{B}_2(\mathscr{B}(G))$. By Lemma 4.5, $X^{\infty} \subseteq K(G)$. Hence $X^{\infty} = K(G)$. Hence $X = \mathrm{Sol}(X) \times K(G) = K(G)$. This proves (a).

(b) This follows directly from (a).

THEOREM 4.7. *Suppose θ is a near solvable A-signalizer functor on G. Suppose θ is not solvable. Let $X \in \mathscr{B}^*(\theta)$. Then*

(a) *X is non-solvable.*

(b) *Suppose $Z, U \in \hat{M}_\theta(A)$, $Z\ \mathrm{Sol}(X) = X$, and $Z \subseteq U$. Then $Z\ \mathrm{Sol}(U) = K(U)$.*

(c) *Suppose $X \subseteq U \in \hat{M}_\theta(A)$. Then $K(U) = X$.*

Proof. (a) This is a direct consequence of θ being non-solvable.

(b), (c) Let Z, U be as in (b). Then $\deg(X) = \deg(Z) \leqslant \deg(K(U))$ by Theorem 4.6. By definition, $\deg(Z) \geqslant \deg(K(U))$. Hence $K(Z)^\infty \subseteq K(U)$ by Lemma 4.5. Now $|K(Z)^\infty/\mathrm{Sol}((K(Z))^\infty)| = |X/\mathrm{Sol}(X)| \leqslant |K(U)/\mathrm{Sol}(U)|$. Hence $K(U) \in \mathscr{B}_2(\mathscr{B}(\theta))$. Hence $Z \in \mathscr{B}_2(\mathscr{B}(U))$. Now (b) holds by Theorem 4.6(a). So suppose $X = Z$. Then $X \subseteq K(U)$, whence $K(U) \in \mathscr{B}^*(\theta)$. Hence $|X| = |K(U)|$ and (c) follows from (b).

Proof of Theorem 4.1. Let $W \in \mathscr{B}^*(\theta)$ and $X = \theta(N_G(W))$. By local completeness $X \in M_\theta(A)$. Theorem 4.7 implies X satisfies parts (a), (b), and (c) of Theorem 4.1. It remains to show (d). Let Z and T be as in part (d). Let M be the unique maximal subgroup of T normalized by Z. Since Z and T admit DA so does M. Hence $K(X) = \mathrm{Sol}(X)\, Z \subseteq MZ \in \hat{M}_\theta(A)$. By (c), $MZ \subseteq X$. By Theorem 4.7(c), $MZ = K(X)$. Hence $M = \mathrm{Sol}(X)$ as required.

5. THE MINIMAL COUNTEREXAMPLE

Henceforth in this paper we shall assume that the main theorem is false and that G is a counterexample of minimal order. Subject to this restriction, we assume that $|\theta|$ is minimal.

When convenient we shall write $H_B = \theta(C_G(B))$ if B is a non-trivial subgroup of A. We also write $H_a = H_{\langle a \rangle}$ for $a \in A$ and $D = H_A$.

THEOREM 5.1. *Suppose $Y \in D(\theta)$. All of the following hold:*

(a) *θ is non-solvable,*

(b) *θ is locally complete,*

(c) *$G = \langle M_\theta(A) \rangle A$,*

(d) *$Z(\langle M_\theta(A) \rangle) = 1$,*

(e) *θ_Y is complete, and*

(f) *either $\theta_Y^m(G) \in M_\theta(A)$, or $\theta_Y^l(G) = 1$.*

Proof. See [5] for (a). See [6, Lemmas 2.6(1) and 5.1] for (b) and (c). Theorems 3.10, 3.11, 3.3(e), and parts (a) (b) of this theorem yield (e) and (f). It remains to show (*d*).

Suppose (d) is false. Let $W = \langle И_\theta(A) \rangle$. Let Z_0 be a minimal normal subgroup of G contained in $Z(W)$. Then Z_0 is an r-group for some prime r. Suppose first that $r \neq p$. The procedure of [6, Lemma 2.6(2)], applied to G/Z_0, yields $Z_0 C_W(a) = Z_0 \theta(C_G(a))$ for all $a \in A$, and W is a p'-group. $Z_0 \cap \theta(C_G(a)) \lhd G$ for all $a \in A^\#$ and $Z_0 \notin И_\theta(A)$ whence $Z_0 \cap \theta(C_G(a)) = 1$ for all $a \in A^\#$. By (a) and Lemma 2.1, a sylow r-subgroup of W splits over Z_0. By [4], Z_0 has an A-invariant complement W_0. Let $\theta_0(C_G(a)) = \theta(C_G(a)) \cap W_0$. Now $\theta_0(C_G(a)) \lhd \theta(C_G(a))$, and θ_0 is complete. Hence for any $B \in \mathscr{E}_2(A)$, $\theta(C_G(B)) \subseteq N_G(\langle \theta_0(C_G(b)) | b \in B^\# \rangle) = N_G(\theta_0(W))$. So by (c), $\theta_0(G) \lhd G$. By (b), $\theta_0(G) = 1$. Hence θ is solvable, contrary to (a). Hence $r = p$. Since $C_{Z_0}(A) \neq 1$ it follows that $Z_0 \cong Z_p$ and $Z_0 \subseteq Z(G)$. *Then*

$$Z_0(\theta(C_G(a))) \cap C_G(\langle b, Z_0 \rangle / Z_0) = Z_0 \theta(C_G(a)) \cap C_G(b)$$
$$= Z_0 \theta(C_G(\langle a, b \rangle))$$

for $a, b \in A^\#$. The argument of [6, Lemma 2.6(2)] again applies, and yields $W = Z_0 \times O_{p'}(W)$. Hence $W = \langle И_\theta(A) \rangle \subseteq O^p(W) = O_{p'}(W) \neq W$, a contradiction.

THEOREM 5.2. *Suppose $X \in И_\theta(A)$. Then*

(a) *There is an $a \in A^\#$ such that $K(H_a) \not\subseteq X$.*

(b) *There is a $B \in \mathscr{E}_2(A)$ such that $K(H_B) \not\subseteq X$.*

Proof. Let $a \in A^\#$ and $a \in B \in \mathscr{E}_2(A)$. Then $C_G(B) \cap K(H_a) \subseteq K(H_B)$. Hence

$$K(H_a) = \langle K(H_a) \cap C(B) | a \in B \in \mathscr{E}_2(A) \rangle \subseteq \langle K(H_E) | E \in \mathscr{E}_2(A) \rangle.$$

Hence it suffices to prove (a).

Suppose that (a) is false. Choose $X \in И_\theta(A)$ such that $K(H_a) \subseteq X$ for all $a \in A^\#$. Let $B \in \mathscr{E}_2(A)$. By Lemma 2.15, $C_{K(X)}(b) \subseteq K(H_b)$ for all $b \in B^\#$. Let $W = \langle K(H_b) | b \in B^\# \rangle$. Then $K(X) \subseteq W \subseteq X$ and $H_B \subseteq N_G(K(W))$. By Lemma 2.15, $K(W) = K(X)$. Hence $H_B \subseteq N_G(K(X))$. Local completeness of θ now yields a contradiction.

THEOREM 5.3. $D(\theta) = \{1\}$ *and* $\theta_1^m(G) \notin И_\theta(A)$.

Proof. Let $S = \theta_{\mathrm{Sol}}(G)$. By Theorems 4.1 and 5.1 there is a subgroup W such that:

(a) $W \in \tilde{И}_\theta(A)$.

(b) W is non-solvable

(c) Suppose $K(W) = Z\,\mathrm{Sol}(W)$, $Z \subseteq \hat{\mathcal{U}}_\theta(A)$, and $Z \subseteq U \in \hat{\mathcal{U}}_\theta(A)$. Then $K(U) = Z\,\mathrm{Sol}(U)$.

(d) Suppose $K(W) \subseteq U \in \hat{\mathcal{U}}_\theta(A)$. Then $U \subseteq W$.

(e) Suppose $Z \in \hat{\mathcal{U}}_\theta(A)$ and $Z\,\mathrm{Sol}(W) = K(W)$. Then $\mathrm{Sol}(W)$ is the unique maximal subgroup of S normalized by Z.

Suppose $\theta_Y^m(G) \in \mathcal{U}_\theta(A)$ for all $Y \in D(\theta)$. Let $R = \{E \in \mathscr{E}_1(A) | H_E \cap W$ is nonsolvable$\}$ and $T = \mathscr{E}_1(A) - R$. Let $E \in R$. By (a), there is $(X_1, X_2) \in P(\theta)$ such that $X_1 \subseteq H_E \cap W$. Let $Y = D \cap X_2$. Then W, $H_E \in C(\theta, Y)$. Let $V = \theta_Y^m(G)$. By assumption and Theorem 3.3(c), $\langle K(W), K(H_E)\rangle \subseteq V \in \hat{\mathcal{U}}_\theta(A)$. By (d), $K(H_E) \subseteq V \subseteq W$. Hence

$$K(H_E) \subseteq W \qquad \text{for all} \quad E \in R. \tag{5.1}$$

There follows by Theorem 5.2(a)

$$T \neq \varnothing. \tag{5.2}$$

Let $A^* = \langle T\rangle$. By (5.2), $A^* \neq 1$. Each member of T fixes each component of $K(W)/\mathrm{Sol}(W)$. Hence $A^* \subseteq \hat{K}(WA)$. Hence $W = K(W)\,C_W(A^*)$. Since $C_W(A^*)$ is solvable there follows by (5.1),

$$W/K(W) \text{ is solvable and } K(H_E)^\infty \subseteq K(W) \text{ for all } E \in R. \tag{5.3}$$

For each $B \in \mathscr{E}_2(A)$ define

$$W_B = \langle (K(H_E))^\infty \,|\, E \in R \text{ and } E \subset B\rangle.$$

By Lemma 2.15 and (5.1), $K(C_W(E)) = K(H_E)$ for all $E \in R$. Hence by (5.3), $(K(H_E))^\infty = (C_{K(W)}(E))^\infty$ for all $E \in R$. Hence

$$W_B = \langle (K(W) \cap C(b))^\infty \,|\, b \in B^\#\rangle.$$

Hence by Lemma 2.7 applied to each B orbit of $\mathscr{L}(W/\mathrm{Sol}(W))$

$$W_B \in \hat{\mathcal{U}}_\theta(A) \qquad \text{and} \qquad W_B\,\mathrm{Sol}(W) = K(W) \qquad \text{for all} \quad B \in \mathscr{E}_2(A). \tag{5.4}$$

By the initial definition of W_B there follows

$$\theta(C_G(B)) \subseteq \theta(N_G(W_B)) \qquad \text{for all} \quad B \in \mathscr{E}_2(A). \tag{5.5}$$

By (e) and (5.5), $C_S(B) \subseteq N_G(\mathrm{Sol}(W))$ for all $B \in \mathscr{E}_2(A)$. Lemma 2.1 and Theorem 5.1(b) implies that $S \subseteq W$ if $\mathrm{Sol}(W) \neq 1$. By (5.5), (5.4), and local completeness we have,

$$\mathrm{Sol}(W) \neq 1.$$

Hence

$$S \subseteq W. \tag{5.7}$$

By Theorem 5.2(b) there is a $B \in \mathscr{E}_2(A)$ such that $K(H_B) \nsubseteq W$. Fix this B and let $L_B = \theta(N_G(W_B))$. By (5.5), $\langle H_B, W_B \rangle \subseteq L_B \in \tilde{\mathcal{U}}_\theta(A)$. Let L_B^∞/M be a chief factor of $L_B^\infty DA$. Let $Y = C_D(L_B^\infty/M)$, and $(X_1, X_2) = (L_B^\infty Y, MY)$. Then $(X_1, X_2) \in P(\theta)$ and $Y = X_2 \cap D$. Hence $X_1 \subseteq \theta_Y^m(G) \in \mathcal{U}_\theta(A)$. Let $D_1 = D \cap \mathrm{Sol}(W)$. Then by (c),

$$\begin{aligned} D_1 \subseteq C_D(K(W)/\mathrm{Sol}(W)) &= C_D(W_B/\mathrm{Sol}(W_B)) \\ &= C_D(K(L_B)/\mathrm{Sol}(L_B)) \subseteq Y. \end{aligned}$$

Hence $\mathrm{Sol}(W) \subseteq \mathrm{Sol}(W)\, Y \subseteq \theta_Y(G) \subseteq \theta_Y^m(G)$. By (d), $\theta_Y^m(G) \subseteq W$. Hence $(H_B)^\infty \subseteq (L_B)^\infty \subseteq X_1 \subseteq \theta_Y^m(G) \subseteq W$. By (5.7), $H_B = (H_B)^\infty\, (H_B \cap S) \subseteq W$, a contradiction. Hence

$$\theta_Y^m(G) \notin \mathcal{U}_\theta(A) \qquad \text{for some } Y \in D(\theta). \tag{5.8}$$

Fix $Y \in D(\theta)$ such that $\theta_Y^m(G) \notin \mathcal{U}_\theta(A)$. Theorem 5.1(f) implies that $\theta_Y^l(G) = 1$. In particular, $Y = 1$. Let $a \in A^\#$. Suppose $H_a \in C(\theta, 1)$, then $\theta_1^m(H_a) = K(H_a)$ is semi-simple. In particular, $(S \cap H_a)\, \theta_1^m(H_a)$ is a group. Define $\theta_1(C_G(a)) = (S \cap H_a)\, \theta_1^m(H_a)$. Let $b \in A$. Suppose $H_b \notin C(\theta, 1)$. Define $\theta_1(C_G(b)) = H_b \cap S$. An application of Lemma 2.2 and Theorem 3.3 shows that θ_1 is an A-signalizer functor. Since θ_1 cannot be complete, it follows that $|\theta| = |\theta_1|$. Hence $\theta = \theta_1$. This proves the result.

LEMMA 5.4. *Suppose X is a subgroup of G generated by some elements of $\mathcal{U}_\theta(A)$. Then either*

(i) *X contains every element of $\mathcal{U}_\theta(A)$ or*

(ii) *$X \in \mathcal{U}_\theta(A)$.*

Proof. See [5, Lemma 5.4].

6. THE STRUCTURE OF θ

We continue use of $S = \theta_{\mathrm{Sol}}(G)$. For the convenience of the reader, we summarize in Theorems 6.1 and 6.2 all the significant structural features of θ established in this section.

First we shall introduce some important new notation which we shall fix for the rest of the paper.

$K_B = K(H_B)$ for any $B \leqslant A$,
$R = \{B \mid 1 \neq B \subseteq A, H_B \in C(\theta, 1)\}$,
$R_i = \{B \in R \mid m(B) = i\}$ for $i = 1$ or 2,
$D_0 = D \cap K_B$ for any $B \in R$ (see Lemma 6.6),
$D_1 = F(D_0)$,
$D_2 = C_{D_0}(D_0/D_1)$,
$S_1 = \theta(C_G(D_1))$,
$S_2 = \theta(C_G(D_2 S_1/S_1))$,
$S_3 = \theta(C_G(D_0 S_1/S_1))$.

Finally, let

$$\theta^*(C_G(a)) = H_a^\infty(H_a \cap S_2) \qquad \text{for any} \quad a \in A^\#.$$

THEOREM 6.1. *There is a distinguished $E \in \mathscr{E}_1(A)$ and a simple group J such that all of the following hold*:

(a) *H_E is solvable.*

(b) *Let $f \in A - E$. Then $K_f \cong pJ$, $\langle E, f\rangle = A \cap \hat{K}(H_f A)$, $H_f = \hat{K}(H_f)$, and H_f/K_f is a $\{2, 3\}$-group.*

(c) *Suppose $f \in A - E$, $X \in И_\theta(A)$ and $K_f \subseteq X$. Then $X \subseteq H_f$.*

(d) *D_0 and D_2 are Frobenius groups with common Frobenius kernel D_1.*

(e) *Suppose $E \subset B \in \mathscr{E}_2(A)$. Let $E \times F = B$. Then $S_3 \cap H_B = \times\{J \cap S_3 \cap H_E \mid J \in \mathscr{L}(H_F)\} \cong pD_2$.*

THEOREM 6.2. *Let E be as in Theorem 6.1. There is an $r \in \pi(\theta)$ and an $S_r(A)$-subgroup V which satisfies*:

(a) *$V \nsubseteq S$,*

(b) *V is abelian,*

(c) *$1 \neq C_V(f)$ is a sylow r-subgroup of H_f for all $f \in A - E$.*

(d) *Let $f \in A - E$ and $F = \langle E, f\rangle$. Then*

$$\langle C_V(f), H_F \cap S_3\rangle = K_f.$$

LEMMA 6.3. *Suppose B is a non-trivial subgroup of A such that H_B is non-solvable. Then*

(a) *$H_B/K(H_B)$ is solvable, and*

(b) *$K(H_B)$ is the unique minimal normal subgroup of $K(H_B)\,DA$.*

Proof. Theorems 5.1(f) and 5.3 yield $\theta_1^l(G)=1$ and $D(\theta)=\{1\}$. The conclusion of the lemma is simply a more appropriate formulation of these facts. The details follow directly from Theorem 3.3(a, c).

LEMMA 6.4. *Suppose* $B \in R_2$. *Let* $W=\langle K(H_b) | b \in B^\#\rangle$. *Then* $W=\langle И_\theta(A)\rangle$.

Proof. Let $f \in A^\#$ such that H_f is non-solvable. By Lemma 6.3, $\theta_1^m(H_f)=K(H_f)=\langle (K(H_f)\cap C(b))^\infty | b \in B^\#\rangle \subseteq W$. Hence by Theorem 5.3, $W \notin И_\theta(A)$. Now Lemma 5.4 implies the conclusion.

Theorem 3.2 yields

LEMMA 6.5. *Suppose* $F \in R_1$. *Then there is a* $B \in R_2$ *which contains* F. *In particular,* $R_2 \neq \varnothing$.

LEMMA 6.6. D_0 *is well defined.*

Proof. Let E, $F \in R$. Write $E \sim F$ if and only if $D \cap K(H_E)=D\cap K(H_F)$. Then $\sim$ is an equivalence relation. Clearly $E\sim F$ if $E \subseteq F$. Hence by Lemma 6.5, R is an equivalence class.

LEMMA 6.7. *Suppose* $B \in R$. *Let* $H=H_B$, $K=K_B$, *and* $M=S\cap K$. *Then*

(a) $C_H(D_1)=F(M)$ *is abelian,*

(b) $M \subseteq C_H(D_2F(M)/F(M)) \subseteq \hat{K}(H)$, *and*

(c) $C_H(D_0F(M)/F(M)) \subseteq M$.

Proof. By Theorem 2.10(c), $C_K(D_1)=F(M)$. Now (a) follows by Lemma 2.16(c). Part (b) is a direct consequence of Lemma 2.16(a). Part (c) results from Lemma 2.16(b).

LEMMA 6.8. *Suppose* $B \in R$. *Let* $J \in \mathscr{L}(H_B)$. *Suppose* $J \not\cong L_2(3^p)$. *Then*

(a) *There is an involution in* D_0 *which acts fixed point freely on* D_1.

(b) *Suppose* t *is any involution which satisfies* (a). *Then* t *acts fixed point freely on* $S \cap K(H_B)$.

Proof. For each $K \in \mathscr{L}(H_B)$ let $K^{\circledA}=\langle K^A\rangle \cap D$. Let $A_1=A\cap \hat{K}(AH_B)$, and $A_0=C_A(K(H_B))$. Then $Z_p \cong A_1/A_0$ induces an automorphism group of order p on $K\in\mathscr{L}(H_B)$. By Lemma 2.9, $K^{\circledA}\cong C_K(A_1/A_0)$. $D_0=\times\{K^{\circledA} | K\in \mathscr{L}(H_B)\}$. By Theorem 2.8, an involution t of D_0 acts fixed point freely on D_1 if and only if $t=t_1t_2\cdots t_n$, $1\neq t_i\in U_i$ and $\{K^{\circledA} | K \in \mathscr{L}(H_B)\}=\{U_1, U_2,\ldots, U_n\}$. Hence it suffices to show that if d is an involution on $K^{\circledA}$, then $C(d)\cap\langle K^A\rangle \cap S=1$. This is implied by Lemma 2.9 and Theorem 2.8(e, f).

LEMMA 6.9. *Suppose $B \in R$. Let $J \in \mathscr{L}(H_B)$. Suppose $J \cong L_2(3^p)$. Then $F(K(H_B) \cap S)$ is the unique minimal normal subgroup of $(H_B \cap S)A$.*

Proof. In the proof of Theorem 2.8 we showed that $S \cap J \cong \mathrm{Alt}(4)$ the alternating group on four letters. By Lemma 6.3(b), $F(K(H_B) \cap S)$ is the unique minimal normal subgroup of $(H_B \cap S)A$ in $K(H_B)$. Now Lemma 6.7(a) implies the conclusion.

LEMMA 6.10. *Suppose $E, F \in R$. Suppose $J \in \mathscr{L}(H_E)$ and $K \in \mathscr{L}(H_F)$. Then $J \cong K$.*

Proof. Define an equivalence relation $\sim$ on R by: $U \sim Z$ if and only if H_U and H_Z have isomorphic components. Then $U \sim Z$ if $U \subseteq Z$. Lemma 6.5 implies that R is an equivalence class.

LEMMA 6.11. *Suppose $E \in R$. Then $S_1 \cap H_E = F(K(H_E) \cap S)$.*

Proof. This is a direct consequence of Lemma 2.16(c).

THEOREM 6.12. *S_1 is abelian.*

Proof. Suppose first that $J \cong L_2(3^p)$ whenever $E \in R$ and $J \in \mathscr{L}(H_E)$. By Lemma 6.5 choose $B \in R_2$. Now fix a minimal normal subgroup Z of SA in S. By Lemma 2.1, there is a $b \in B^{\#}$ such that $C_Z(b) \neq 1$. By Lemma 6.9, $C_Z(b) = F(K(H_b) \cap S)$. Hence $C_Z(B) \neq 1$. Then $C_Z(e) = F(K_e \cap S)$ for all $e \in B^{\#}$. Lemmas 2.1 and 6.11 imply $Z = S_1$. Hence S_1 is abelian.

By Lemma 6.10, we may suppose that $J \not\cong L_2(3^p)$ whenever $E \in R$ and $J \in \mathscr{L}(H_E)$. By Lemma 6.8(a) choose an involution $t \in D_0$ which acts fixed point freely on D_1. Then t acts on S_1. Let $U = C_{S_1}(t)$. Suppose $U \neq 1$. Fix $B \in R_2$. Then B normalizes U. Hence there is a $b \in B^{\#}$ such that $C_U(b) = V \neq 1$. By Lemma 6.11, $V \subseteq F(K_b \cap S) \cap C(t)$, contrary to Lemma 6.8(b). Hence $C_{S_1}(t) = 1$. So S_1 is abelian.

LEMMA 6.13. *Suppose $B \in R$. Then $K(H_B) \cap S \subseteq N(S_1)$.*

Proof. Lemma 6.11 and Theorem 6.12 imply that $S_1 = \theta(C_G(F(K(H_B) \cap S)))$. The result follows directly.

LEMMA 6.14. *Suppose $B \in R$. Then $K_B \cap S \subseteq H_B \cap S_2 \subseteq \hat{K}(H_B)$.*

Proof. This follows from Lemma 6.13 and Lemma 6.7(b).

THEOREM 6.15. *$\theta = \theta^*$.*

Proof. Let $a, b \in A^{\#}$, and $B = \langle a, b \rangle$. Suppose $B \in R$. Then by

Lemma 3.6(b), $K_a \cap C_G(b) \subseteq K_b$, whence $\theta^*(C_G(a)) \cap C_G(b) = (K_a \cap H_b)(H_a \cap S_2) \cap H_b) \subseteq K_b(H_b \cap S_2) = \theta^*(C_G(b))$.

Next suppose $B \notin R$. Then by Lemma 6.14, $\theta^*(C_G(a)) \cap C_G(b) = S_2 \cap H_B \subseteq \theta^*(C_G(b))$. Hence θ^* is an A-signalizer functor.

Lemma 6.4 yields that θ^* is not complete. Minimality of $|\theta|$, among incomplete signalizer functors, forces $\theta = \theta^*$, as required.

LEMMA 6.16. *Let $\langle a \rangle \in R_1$. Then H_a/K_a is a $\{2, 3\}$-group.*

Proof. If $J \not\cong U_3((2^p)^2)$, then $D_2 = D_0$ and $S_2 = S_3$. Then Lemma 6.7(c) implies that $H_a = K_a$. Hence suppose $J \cong U_3((2^p)^2)$. By [15], the outer automorphism group of J is isomorphic to $D_6 \times Z_p$. The result now follows by Lemma 6.11 and Theorem 6.15.

THEOREM 6.17. (a) *There is a unique $E \in \mathscr{E}_1(A) - R_1$.*

(b) *$\langle E, f \rangle = (\hat{K}(AH_f)) \cap A$ for all $f \in A - E$.*

Proof. Let $A^* = \langle \mathscr{E}_1(A) - R_1 \rangle$. Then $\langle f, A^* \rangle \subseteq \hat{K}(AH_f)$ for all $f \in A^\#$. Suppose first that $A^* = A$. Then Theorem 6.15 and Lemma 6.3(b) imply that K_F is simple for any $F \in R$. By Lemma 6.5, choose $B \in R_2$. Then $H_b = H_B$ for all $b \in B^\#$. But then $H_B = \langle И_\theta(A) \rangle$. This is false. Hence $A^* < A$. Suppose next that $m(A^*) = 2$. Then K_f is simple for any $f \in A - A^*$. Let $f \in A - A^*$, and extend $\langle f \rangle$ to $B \in R_2$ by Lemma 6.5. Then $K_b \subseteq K_B$ for all $b \in B - A^*$. Hence $K_{B \cap A^*} = \langle И_\theta(A) \rangle$. This is also false, whence A^* is cyclic.

Let $R^0 = \{F \in R_1 | K_F$ is simple$\}$ and $R^1 = R_1 - R^0$. Then since D is solvable, K_F has p-components for each $F \in R^1$. For each $F \in R$, let $F_C = C_A(H_F)$ and $F_N = \hat{K}(AH_F) \cap A$. Suppose $\langle R^1 \rangle < A$. Then by Lemma 6.5, there is a $B \in R_2$ such that $U = B \cap \langle R^1 \rangle$ is cyclic. Then $\langle H_b | b \in B^\# \rangle$ equals H_B if $U = 1$ and equals K_U otherwise. This is false. Hence $\langle R^1 \rangle = A$. Suppose that $R^0 \neq \varnothing$. Fix an $F \in R^0$ and $L \in R^1$ such that $L \subseteq A - F_C$. Then $S_3 \cap H_F \cap C(L) = D_2$. Since H_{LF} is solvable, $F \subseteq L_N$. Hence $S_3 \cap C(F) \cap H_L \cong pD_2$. Then balance yields $D_2 = H_F \cap C(L) \cap S_3 = C(F) \cap H_L \cong pD \neq 1$, a contradiction. Hence (b) holds.

It remains to show $R_1 \neq \mathscr{E}_1(A)$. Suppose by way of contradiction that $R_1 = \mathscr{E}_1(A)$. Fix F, $L \in R_1$ with $FL \in R_2$. Then $F \not\subseteq L_N$ and $L \not\subseteq F_N$. Let $B = F_N \cap L_N$. Then $F_N = B_N = L_N$, a contradiction. This completes the proof of Theorem 6.16.

LEMMA 6.18. *Let $B \in R$ and $X \in И_\theta(A)$. Suppose X is solvable and X admits K_B. Then $X = 1$.*

Proof. We may and do suppose $B \in R_2$. Let $e \in B^\#$. Then $C_X(e)$ admits K_B. Hence $C_X(e) \cap K_e$ admits K_B. Thus $C_X(e) \cap K_e = 1$. By Lemma 2.16, $C_X(e) = 1$. Hence $X = 1$.

Proof of Theorem 6.1. Lemma 6.16 and Theorem 6.17 yield Theorem 6.1(a, b). Parts (d) and (e) are direct consequences of (a), (b) and Lemma 6.6. It remains to show (c).

Let $f \in A - E$, and $K_f \subseteq X \in И_\theta(A)$. By Lemma 2.16, $K_f \subseteq K(X)$. If $K(X) = K_f$, then $f \in C_G(K(X)) \cap N_G(X) = C_G(X)$, whence $X \subseteq H_f$. It is therefore sufficient to show if $X = K(X)$, then $X = K_f$. So suppose $X = K(X)$. By Lemma 6.18, $\mathrm{Sol}(X) = 1$. Let $\langle f \rangle = F \subset B \in R_2$. Let $e \in B - F$. Now $K_B = H_F \cap C_G(e) \subseteq C_X(e)$. However, K_B is a maximal A-invariant subgroup of K_e. Hence $C_X(e) \cap K_e = K_B$ or K_e. By Lemma 6.4, $C_X(e) \cap K_e = K_B$ for some $e \in B - F$. Hence X is simple or $X \cong pK_B \cong K_f$. Thus X is simple or $X = K_f$. If X is simple, then $C_A(X) = B_1$ has rank at least 2 since X is near p-solvable. Thus $X = C_X(B \cap B_1) \cong K_B$. This is ridiculous. Hence $X = K_f$ as required.

Proof of Theorem 6.2. Assume the notation of Theorem 6.1. Then $\pi(H_f) = \pi(J)$ for all $f \in A - E$. By Theorem 2.8(o) and Theorem 2.10 there is an $r \in \pi(H_f) - \pi(H_f \cap S)$ for all $f \in A - E$. Let V be an $S_r(A)$-subgroup of G. Theorem 2.10(o) and Theorem 6.1(b) implies

$$H_f \cap V = K_f \cap V \text{ is abelian for all } f \in A - E. \tag{6.1}$$

Let $B \in R_2$. By choice of r it follows that $(V \cap H_E) \cap C_G(b) = 1$ for any $b \in B$. Hence $V \cap H_E = 1$; so Lemma 2.5 and (6.1) imply that V is abelian. Lemma 2.3 implies (c).

It remains to show (d). Let L be a component of H_f. By Theorems 2.8d and 2.10, $L \cap S$ is the unique maximal subgroup of L containing $L \cap S_2$. By (c) and the choice of r, $C_V(f) \cap L \nsubseteq L \cap S$. Hence $L \subseteq \langle C_V(f), K_f \cap S_3 \rangle$. This completes the proof of (d) and Theorem 6.2.

7. The Structure of G.

For the remainder of the paper, let $W = \langle И_\theta(A) \rangle$ and let E be the unique cyclic sugroup of A such that H_E is solvable.

Theorem 7.1. *Suppose B is a subgroup of A of rank 2 which contains E. Then W has subgroups W_i, $1 \leqslant i \leqslant p$, such that all of the following hold*:

(a) $W = W_1 \times W_2 \times \cdots \times W_p$.

(b) *A permutes $\{W_i \mid 1 \leqslant i \leqslant p\}$ transitively, and $B = N_A(W_i)$ for each i.*

(c) $K_b = \times \{W_i \cap K_b \mid 1 \leqslant i \leqslant p\}$ *for all* $b \in B - E$.

Proof. Theorem 6.1(d, e) imply that $H_B \cap S_3$ is the direct product of p unique indecomposable subgroups. Let $\mathscr{B}$ be the set of indecomposable subgroups. Hence for any $b \in B - E$,

$$\mathscr{B} = \{K \cap S_3 \cap H_E | K \in \mathscr{L}(H_b)\} \tag{7.1}$$

Let $r \in \pi(\theta)$ and V be an $S_r(A)$-subgroup of G which satisfies the conclusion of Theorem 6.2. For each $F \in \mathscr{B}$, let $V_F = \bigcap \{C_V(H) | H \in \mathscr{B}$, and $H \neq F\}$ and $W_F = \langle F, V_F \rangle$. Since V is abelian, there follows

$$[W_F, W_H] = 1 \qquad \text{if} \quad F \neq H. \tag{7.2}$$

Let $\hat{W} = \langle W_F | F \in \mathscr{B} \rangle$. Let $b \in B - E$, $K \in \mathscr{L}(H_b)$, and $F \subseteq C_K(E)$. By (7.1), $K \cap V \subseteq V_F$. Hence $C_V(b) \subseteq \hat{W}$ by Theorem 6.2(c). Theorem 6.2(d) implies that $K_b \subseteq \hat{W}$. Theorem 6.1(c) and Lemma 5.4 imply that $\hat{W} = W$. Theorem 5.1(d) and (7.2) yield that $W = \times \{W_F | F \in \mathscr{B}\}$. Now Theorem 6.2(d) yields (c).

It remains to prove (b). B fixes each element of $\mathscr{B}$ and normalizes V, whence B normalizes W_F for each $F \in \mathscr{B}$. The set $\mathscr{B}$ is acted on transitively by A, and A normalizes V. Hence A acts transitively on $\{W_F | F \in \mathscr{B}\}$. This completes the proof of Theorem 7.1.

THEOREM 7.2. *G does not exist.*

Proof. Let B be a complement of E in A. Let $B = F_1 F_2$ where each F_i is cyclic. Let $B_i = F_i E$ for $i = 1, 2$. By Theorem 7.1, there are sets $\mathscr{B}_j = \{W_i^j | 1 \leqslant i \leqslant p\}$, $j = 1$ or 2, such that $W = \times \mathscr{B}_j$, B_j normalizes each W_i^j, A is transitive on $\mathscr{B}_j$, and $K_{F_j} = \times \{W_i^j \cap K_{F_j} | 1 \leqslant i \leqslant p\}$. Let $W_{i,j} = W_j^1 \cap W_j^2$ for $1 \leqslant i, j \leqslant p$. Since F_i fixes each member of $\mathscr{B}_i$ and acts transitively on $\mathscr{B}_{3-i}$, it follows that B acts transitively on $\{W_{i,j} | 1 \leqslant i, j \leqslant p\} = \mathscr{B}$. Let $Z = \langle \mathscr{B} \rangle$. Then

$$[W, W] = [\langle \mathscr{B}_1 \rangle, \langle \mathscr{B}_2 \rangle] = \langle [W_i^1, W_j^2] | 1 \leqslant i, j \leqslant p > \subseteq Z.$$

By Theorem 6.4, W is generated by perfect subgroups, whence W is perfect. So $Z = W$. Clearly, $[W_{i,j}, W_{s,t}] = 1$ if $(i, j) \neq (s, t)$. Hence

$$W = \times \mathscr{B} \tag{7.3}$$

$$B \text{ acts regularly on } \mathscr{B} \tag{7.4}$$

Let $K = K_B$ and $K_{F_i} = K_i$ for $i = 1$ or 2. By Theorem 6.1, K is simple and $K_i \cong pK$. Let $W^* = \times \{\mathrm{Proj}_U(H) | U \in \mathscr{B}\}$. Then

$$W^* \cong p^2 K. \tag{7.5}$$

Let $W_i^* = W^* \cap W_i^1$. For subgroups U of W^* let $p_i\colon U \to W_i^*$ be the projection map of U into W_i^*. For subgroups V of W let $p^i\colon V \to W_i^1$ be the projection map of V into W_i^1. Since $K_1 = \times\, \{K_1 \cap W_i^1 \mid 1 \leqslant i \leqslant p\}$, and F_2 is transitive on $\mathscr{B}_1$ and $K = K_1 \cap C(F_2)$, it follows that $K_1 = \times\, \{p^i(K) \mid 1 \leqslant i \leqslant p\}$. Since F_2 is transitive on $\{W_i^* \mid 1 \leqslant i \leqslant p\}$ and $H \subseteq C_{W^*}(F_2)$, it follows that $K \cong p_i(K) \subseteq p^i(K) \cong K$ for $1 \leqslant i \leqslant p$. Hence $K_1 \subseteq W^*$. Similarly, $K_2 \subseteq W^*$. Lemma 5.4 and Theorem 6.1(c) now yield

$$W^* = W. \tag{7.6}$$

By (7.5) and (7.6), W is near p-solvable. Let $b \in B^{\#}$. By (7.4), (7.5), (7.6), and Theorem 6.1, it follows that $pK \cong K_b \subseteq C_W(b) \cong pK$. Hence $C_W'(b) = H_b$ for all $b \in B^{\#}$. Let $a \in A^{\#}$. Then

$$\begin{aligned} C_W(a) &= \langle C_W(a) \cap C(b) \mid b \in B^{\#} \rangle \\ &\subseteq \langle H_b \cap C_W(a) \mid b \in B^{\#} \rangle \subseteq H_a. \end{aligned}$$

Hence θ is complete, a contradiction. This contradiction completes the proof of the main theorem of this paper.

References

1. R. Carter, *Simple Groups of Lie Type*, Wiley, New York, 1972.
2. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
3. W. Feit, "The Current Situation in the Theory of Finite Simple Groups," Yale Univ. Press, New Haven, 1971.
4. W. Gaschutz, Zur Erweiterungs Theorie Englicher Gruppen, *J. Math.* **190** (1952), 93–107.
5. G. Glauberman, On solvable signalizer functors in finite groups, *Proc. London Math. Soc.* (3) **33** (1976), 1–27.
6. D. Goldschmidt, Solvable signalizer functors on finite groups, *J. Algebra* **21** (1972), 341–351.
7. D. Goldschmidt, 2-signalizer functors on finite groups, *J. Algebra* 21 (1972), 321–340.
8. D. Gorenstein, "Finite Groups," Harper and Row, New York, 1969.
9. D. Gorenstein, On the centralizers of involutions in finite groups I, II, *J. Algebra* II (1969), 243–277, **14** (1970), 350–372.
10. D. Gorenstein, The flatness of signalizer functors on finite groups, *J. Algebra* **13** (1969), 509–512.
11. D. Gorenstein and R. Lyons, Nonsolvable signalizer functors on finite groups, unpublished.
12. B. Huppert, "Endliche Gruppen 1," Springer-Verlag, New York, 1967.
13. P. Martineau, Elementary abelian fixed point free automorphism groups, *Quart. J. Math. Oxford Ser.* 2 **23** (1972), 205–212.
14. P.McBride, Nonsolvable signalizer functors on finite groups, in preparation.
15. R. Steinberg, Automorphisms of finite linear groups, *Canad. J. Math.* **12** (1960), 606–615.
16. M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* **75** (1962), 105–145.