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Infinite-Dimensional Lie Algebras, Theta Functions and Modular Forms*

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INTRODUCTION

0.1. From the time they were introduced in analysis by Jacobi [13] and in geometry by Riemann [37] the theta functions found numerous applications in various fields of mathematics. The present paper displays one

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more application—to the representation theory of affine Kač–Moody Lie algebras. The representation theory provides in turn (among other things) a number of new theta function identities. The simplest cases of these identities are collected in Section 5.5, which may be read independently of the rest of the paper.

One of the starting points of this paper is an interpretation by Looijenga of the Macdonald identities in terms of theta functions [27] and an observation made by one of the authors [18] by analogy with the "Monstrous game" [4] that most of the generating functions for multiplicities which appear in the representation theory of affine Lie algebras become q-series of modular forms when multiplied by a suitable power of q.

Our approach to the study of these modular forms is roughly as follows. We rewrite the character of a highest weight representation of an affine Lie algebra in terms of theta functions and the modular forms in question. Then, using classical functional equations for theta functions, we deduce transformation properties of our modular forms. Furthermore, using the "very strange" formula, we estimate the orders of the poles at the cusps. As a result, the theory of modular forms makes it possible to compute any of these modular forms. We do so in a number of interesting cases. Moreover, combining our transformation laws with a Tauberian theorem, we obtain the asymptotics of the multiplicities in question.

Another starting point of the paper is the work of one of the authors [34] on explicit formulas for Kostant's partition function. Using the results of this work we derive explicit formulas for (generalized) Kostant's partition function in the case of certain affine Lie algebras. This allows us to compute weight multiplicities directly, at least for the simplest affine Lie algebra $A_1^{(1)}$. Quite unexpectedly the corresponding generating series turn out to be intimately related to certain modular forms discovered by Hecke [9], which are associated to real quadratic fields.

0.2. First, we explain the sort of objects studied in the paper. For the sake of simplicity we concentrate here on the "non-twisted" affine Lie algebras.

Let $\bar{\mathfrak{g}}$ be a complex simple finite-dimensional Lie algebra of type $X(=A_l, B_l,...)$, $\phi(.,.)$ its Killing form. Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$, $\bar{\Delta}$ the set of roots of $\bar{\mathfrak{h}}$ in $\bar{\mathfrak{g}}$. Fix a set of positive roots $\bar{\Delta}_+$ in $\bar{\Delta}$; let $\alpha_1,..., \alpha_l$ be the simple roots, θ the highest root. Introduce the important integer $g := \phi(\theta, \theta)^{-1}$. It is more convenient to deal with the normalized bilinear form $(x, y) := 2g\phi(x, y)$, so that $(\theta, \theta) = 2$. We identify $\bar{\mathfrak{h}}$ with $\bar{\mathfrak{h}}^*$ via the form $(.,.)$. Let $\bar{W}$ be the Weyl group of $\bar{\mathfrak{g}}$ and let M be the lattice spanned by the set $\bar{W}(\theta)$. M is called the dual root lattice of $\bar{\mathfrak{g}}$. It is a positive-definite integral (with respect to $(.,.)$) lattice, which plays an important role in our considerations.

The affine Lie algebra $\mathfrak{g}$ associated to $\bar{\mathfrak{g}}$ may be constructed as follows [14, 31, 7,18]. Let $L = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in t, and set $L(\bar{\mathfrak{g}}) = L \otimes_{\mathbb{C}} \bar{\mathfrak{g}}$. This is an infinite-dimensional Lie algebra over $\mathbb{C}$; denote its bracket by $[\ ,\ ]_L$. For $x = \sum_i t^i \otimes x_i$, $y = \sum_j t^j \otimes y_j$, define $(x, y)_t = \sum_{i,j} t^{i+j}(x_i, y_j) \in L$. The function $\psi(x, y) = \mathrm{Res}_{t=0}(dx/dt, y)_t$ is a cocycle on $L(\bar{\mathfrak{g}})$ with values in $\mathbb{C}$, hence determines a central extension $\tilde{L}(\bar{\mathfrak{g}})$ of $L(\bar{\mathfrak{g}})$. Explicitly $\tilde{L}(\bar{\mathfrak{g}}) = L(\bar{\mathfrak{g}}) \oplus \mathbb{C}c$, where the bracket is given by

$$[x + \lambda c, y + \mu c] = [x, y]_L + \psi(x, y)c \ \ (x, y \in L(\bar{\mathfrak{g}});\ \lambda, \mu \in \mathbb{C}).$$

The *affine Lie algebra* $\mathfrak{g}$ of *type* $X^{(1)}$ is then obtained by adjoining to $\tilde{L}(\bar{\mathfrak{g}})$ a derivation d which acts on $L(\bar{\mathfrak{g}})$ as $t(d/dt)$ and kills c. ($\mathfrak{g}$ is the Kač–Moody algebra associated to the extended Cartan matrix of $\bar{\mathfrak{g}}$.)

We identify $\bar{\mathfrak{g}}$ with the subalgebra $1 \otimes \bar{\mathfrak{g}}$ of $\mathfrak{g}$. The commutative subalgebra $\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is called the *Cartan subalgebra* of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$ we denote its restriction to $\bar{\mathfrak{h}}$ by $\bar{\lambda}$. We identify $\{\lambda \in \mathfrak{h}^* | \lambda(c) = \lambda(d) = 0\}$ with $\bar{\mathfrak{h}}^*$ by $\lambda \mapsto \bar{\lambda}$.

Introduce the important elements δ, Λ_0 and α_0 of $\mathfrak{h}^*$ defined by

$$\delta|_{\bar{\mathfrak{h}} + \mathbb{C}c} = 0, \qquad \delta(d) = 1; \qquad \alpha_0 = \delta - \theta;$$
$$\Lambda_0|_{\bar{\mathfrak{h}} + \mathbb{C}d} = 0, \qquad \Lambda_0(c) = 1.$$

The elements $\alpha_1, ..., \alpha_l, \delta, \Lambda_0$ form a basis of $\mathfrak{h}^*$, and we extend the form $(.,.)$ from $\bar{\mathfrak{h}}^*$ to a symmetric bilinear form on $\mathfrak{h}^*$ by

$$\left(\sum_{i=1}^{l} \mathbb{C}\alpha_i, \mathbb{C}\delta + \mathbb{C}\Lambda_0\right) = 0; \quad (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0; \quad (\delta, \Lambda_0) = 1.$$

Writing $|\lambda|^2$ for (λ, λ), we define

$$P(\text{resp. } P_+) := \{\lambda \in \mathfrak{h}^* \mid 2(\lambda, \alpha_i)/|\alpha_i|^2 \in \mathbb{Z} \ (\text{resp. } \mathbb{Z}_+)$$
$$\text{for} \quad i = 0, ..., l\}; \qquad \bar{P} = \{\bar{\lambda} \mid \lambda \in P\}.$$

Let $\bar{\mathfrak{n}}_+$ be the maximal nilpotent subalgebra of $\bar{\mathfrak{g}}$ which corresponds to $\bar{\Delta}_+$, and let $\mathfrak{n}_+ \subset L(\bar{\mathfrak{g}}) \subset \mathfrak{g}$ be the preimage of $\bar{\mathfrak{n}}_+$ under the map $\mathbb{C}[t] \otimes \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ defined by sending t to 0.

Following [16], for each $\Lambda \in P_+$ define an irreducible $\mathfrak{g}$-*module with highest weight* Λ, denoted by $L(\Lambda)$, by the property that there exists a nonzero vector $v \in L(\Lambda)$ such that

$$\mathfrak{n}_+(v) = 0 \qquad \text{and} \qquad h(v) = \Lambda(h)v \qquad \text{for all} \quad h \in \mathfrak{h}.$$

The $\mathfrak{g}$-module $L(\Lambda)$ admits the *weight space decomposition* $L(\Lambda) = \bigoplus_\lambda L(\Lambda)_\lambda$, where for $\lambda \in \mathfrak{h}^*$,

$$L(\Lambda)_\lambda := \{v \in L(\Lambda) \mid h(v) = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$$

is a finite-dimensional subspace; its dimension is denoted by $\text{mult}_\Lambda(\lambda)$.

The central element c acts on $L(\Lambda)$ as multiplication by a non-negative integer $m = \Lambda(c) = (\Lambda, \delta)$ called the *level* of $L(\Lambda)$ [22]. Note that $m = 0$ if and only if $\dim L(\Lambda) = 1$; we assume in the sequel that $m > 0$.

For $\lambda \in \mathfrak{h}^*$, define a function $e(\lambda)$ on $\mathfrak{h}$ by $e(\lambda)(h) = \exp \lambda(h)$. Introduce the following domain in $\mathfrak{h}$:

$$Y = \{h \in \mathfrak{h} \mid \text{Re}\, \delta(h) > 0\}.$$

Now we can define the *character* of the $\mathfrak{g}$-module $L(\Lambda)$:

$$\text{ch}\, L(\Lambda) = \sum_\lambda \text{mult}_\Lambda(\lambda)\, e(\lambda).$$

This series converges absolutely uniformly on compact sets to an analytic function on Y.

We express $\text{ch}\, L(\Lambda)$ in terms of theta functions. For $\mu \in \bar{\mathfrak{h}}^*$ and a positive integer m, set

$$\Theta_{\mu,m} := \sum_{\gamma \in M + m^{-1}\mu} e(m(\Lambda_0 + \gamma - \tfrac{1}{2}|\gamma|^2 \delta)). \tag{0.1}$$

This series converges absolutely on Y to an analytic function. In coordinates

$$Y = \{-2\pi i(z + \tau d + tc) \mid z \in \bar{\mathfrak{h}};\, \tau, t \in \mathbb{C}, \text{Im}\, \tau > 0\},$$

one recognizes in (0.1) the classical theta function

$$\Theta_{\mu,m}(\tau, z, t) = e^{-2\pi i m t} \sum_{\gamma \in M + m^{-1}\mu} e^{\pi i m |\gamma|^2 \tau} e^{-2\pi i m \gamma(z)}.$$

Note also that in these coordinates,

$$e(-\delta) = q,$$

where as usual q stands for $e^{2\pi i \tau}$.

Define $\rho \in \mathfrak{h}^*$ by $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$, $i = 0, \ldots, l$, and $\rho(d) = 0$. Let $\bar{W} \subset GL(\bar{\mathfrak{h}}^*)$ be the Weyl group of the Lie algebra $\bar{\mathfrak{g}}$. For $\lambda \in P$ such that $\lambda(c) > 0$, define

$$A_\lambda = \sum_{w \in \bar{W}} (\det w)\, \Theta_{w(\lambda), \lambda(c)}.$$

Now it is not difficult to rewrite the character formula from [16] in terms of theta functions:

$$q^{s_\Lambda} \operatorname{ch} L(\Lambda) = A_{\Lambda+\rho}/A_\rho, \tag{0.2}$$

where

$$s_\Lambda = \frac{|\Lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g}.$$

Another more "elementary" formula expresses $\operatorname{ch} L(\Lambda)$ in terms of theta functions and the so-called *string functions* $c_\lambda^\Lambda (\lambda \in P)$, defined as follows. Put $s_\Lambda(\lambda) = s_\Lambda - |\lambda|^2/2m$ and define c_λ^Λ by [18, 22]:

$$c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{n \in \mathbf{C}} \operatorname{mult}_\Lambda(\lambda - n\delta)\, q^n.$$

This is a holomorphic function in τ on the upper half-plane. We have

$$c^\Lambda_{w(\lambda)+m\gamma+a\delta} = c^\Lambda_\lambda \qquad \text{for} \quad w \in \bar{W}, \gamma \in M \text{ and } a \in \mathbb{C}.$$

It follows that there are only a finite number of distinct string functions for a given module $L(\Lambda)$.

We have [18, 22]

$$q^{s_\Lambda} \operatorname{ch} L(\Lambda) = \sum_{\lambda \in P \bmod (mM + \mathbf{C}\delta)} c_\lambda^\Lambda \Theta_{\lambda,m}. \tag{0.3}$$

This formula provides information about the $\mathfrak{g}$-module $L(\Lambda)$ as soon as we can say something about the string functions. Our main tool for investigating the string functions is the following identity, which follows from (0.2) and (0.3):

$$A_{\Lambda+\rho}/A_\rho = \sum_{\lambda \in P \bmod (mM + \mathbf{C}\delta)} c_\lambda^\Lambda \Theta_{\lambda,m}. \tag{0.4}$$

Finally, recall the Euler function

$$\varphi(q) = \prod_{n \geq 1} (1 - q^n)$$

and the Dedekind η-function

$$\eta(\tau) = q^{1/24}\varphi(q).$$

0.3. Now we can state the main results of the first part of the paper.

THEOREM 1. *Let $L(\Lambda)$ be a $\mathfrak{g}$-module with highest weight $\Lambda \in P_+$ of level $m > 0$. Then*

(a)

$$c_\lambda^\Lambda(-\tau^{-1}) = (-i\tau)^{-l/2} \sum_{\substack{\Lambda' \in P_+ \bmod \mathbb{C}\delta \\ \lambda' \in P \bmod (mM + \mathbb{C}\delta) \\ \Lambda'(c) = \lambda'(c) = m}} b(\Lambda, \lambda; \Lambda', \lambda')\, c_{\lambda'}^{\Lambda'}(\tau),$$

where

$$b(\Lambda, \lambda; \Lambda', \lambda') = i^{|\bar{\Delta}_+|} |\bar{P}/M|^{-1}\, m^{-l/2}(m+g)^{-l/2}$$

$$\times \exp(2\pi i m^{-1}(\bar{\lambda}, \bar{\lambda}')) \sum_{w \in \bar{W}} (\det w) \exp(-2\pi i (m+g)^{-1}\, (\bar{\Lambda} + \bar{\rho}, w(\bar{\Lambda}' + \bar{\rho}))).$$

(b) $\eta(\tau)^{\dim \bar{\mathfrak{g}}} c_\lambda^\Lambda(\tau)$ *is a holomorphic modular cusp form of weight* $|\bar{\Delta}_+|$ *for the group* $\Gamma(Nm) \cap \Gamma(N(m+g))$ *with the trivial multiplier system, where* N *is the least positive integer such that* $\frac{1}{2}N|\mu|^2 \in \mathbb{Z}$ *for all* $\mu \in \bar{P}$.

(c) *If* $\operatorname{mult}_\Lambda(\lambda) > 0$, *then one has, as* $n \to +\infty$,

$$\operatorname{mult}_\Lambda(\lambda - n\delta) \sim c n^{-(1/4)(l+3)} e^{4\pi (an)^{1/2}},$$

where c *is a constant which depends only on* $\mathfrak{g}$ *and* Λ *(and is computed in the paper), and*

$$a = \frac{\dim \bar{\mathfrak{g}}}{24} \frac{m}{m+g}. \tag{0.5}$$

Theorem 1 appears in the paper as Theorem A(1a, 4), and Theorem B for the "non-twisted" affine Lie algebras.

Let us make some comments on the proof. For (a) we use the identity (0.4) and a functional equation for theta functions. In particular, we use the formula

$$A_\rho\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = (-i)^{|\bar{\Delta}_+|}(-i\tau)^{l/2} A_\rho(\tau, z, t).$$

The crucial point in the proof of (b) is the following inequality:

$$s_\Lambda(\lambda) \geqslant -a \qquad \text{if} \quad \operatorname{mult}_\Lambda(\lambda) > 0, \tag{0.6}$$

where a is given by (0.5).

In order to prove (0.6), we employ the following "very strange" formula. Let $\lambda \in \sum_{i=1}^l \mathbb{Q}\alpha_i$ lie in the fundamental alcove, i.e., $(\lambda, \alpha_i) \geqslant 0$, $i = 1,..., l$, $(\lambda, \theta) \leqslant 1$, and let n be a positive integer such that $n\lambda \in M$. Then $\sigma = \exp 2\pi i\lambda$ is an automorphism of $\bar{\mathfrak{g}}$ of order dividing n; let d_s be the multiplicity of the eigenvalue $\exp 2\pi i s/n$ of σ. We have

$$\frac{1}{2g}|\bar{\rho} - g\lambda|^2 = \frac{\dim \bar{\mathfrak{g}}}{24} - \frac{1}{4n^2}\sum_{j=1}^{n-1} j(n-j)\, d_j. \tag{0.7}$$

In a different form (0.7) is established in [17] for "rational" σ; the proof of (0.7) in the general case is essentially the same. For $\lambda = 0$, (0.7) turns into the celebrated "strange formula" of Freudenthal–de Vries:

$$\frac{1}{2g}|\bar{\rho}|^2 = \frac{\dim \bar{\mathfrak{g}}}{24}. \tag{0.8}$$

Formulas (0.7) and (0.8) yield the inequality

$$|\bar{\rho} - g\lambda|^2 \leqslant |\bar{\rho}|^2 \tag{0.9}$$

for any λ from the fundamental alcove which, in turn, implies (0.6).

Finally, for (c) we use a Tauberian theorem of Ingham. We need here the fact that $\mathrm{mult}_\Lambda(\lambda - n\delta)$ increases with n, which we prove using a Heisenberg Lie algebra. This is the only point in the main body of the paper where a representation-theoretical argument appears necessary. Of course, the representation-theoretical framework provides motivations and clarifies arguments. However, it remains an interesting open problem to find an interpretation of the results of the paper in this framework.

Using Theorem 1, we explicitly determine the string functions in many interesting cases. Let us demonstrate our method in the simplest case, that of representations of level 1 of the affine Lie algebras of type $A_l^{(1)}$, $D_l^{(1)}$ or $E_l^{(1)}$. It is easy to see that in this case all non-zero string functions c_λ^Λ for Λ of level 1 are equal to $c(\tau) = q^{-l/24} \sum_{n \geqslant 0} \mathrm{mult}_{\Lambda_0}(\Lambda_0 - n\delta)\, q^n$. Hence, by Theorem 1, the function $\eta(\tau)^l c(\tau)$ is $SL(2, \mathbb{Z})$-invariant. It is also holomorphic and has value 1 at $i\infty$; hence it is identically 1. So, we recover the result obtained in [17] by the method of "principal" specialization (and in [6] for $\mathfrak{g} = A_1^{(1)}$ by a straightforward computation):

$$\sum_{n \geqslant 0} \mathrm{mult}_{\Lambda_0}(\Lambda_0 - n\delta)\, q^n = \varphi(q)^{-l}.$$

In the case of $E_8^{(1)}$ this result is related to the "Monstrous game" [18].

We remark that in general the multiplicities apparently fail to be given by simple combinatorial functions such as the classical partition function $p(n)$. In this sense, the results of [6] and [17] appear not to generalize. To understand the string functions, it is necessary to replace the combinatorial point of view of $\varphi(q)^{-1}$ as the generating function for $p(n)$ by the realization that $\eta(\tau) = q^{1/24}\varphi(q)$ is a modular form.

0.4. The rest of the main results of the paper deal with the partition function for the affine Lie algebra $\mathfrak{g}$ of type $A_l^{(1)}$. The set Δ_+ of positive roots of $\mathfrak{g}$ consists of the roots $(n-1)\delta + \alpha$ and $n\delta - \alpha$ of multiplicity 1, and $n\delta$ of multiplicity l, where $\alpha \in \bar{\Delta}_+$ and $n \geqslant 1$. The partition function $K(\lambda)$ on $\mathfrak{h}^*$ (K

in honor of Kostant) is defined to be the number of representations of λ as a sum of positive roots (counting multiplicities).

Using the results of [34] on the Kostant partition function for A_l, we find an explicit formula for the partition function for $A_l^{(1)}$ (given by Theorem C). Here we state the result only for $A_1^{(1)}$.

THEOREM 2. *Let* $\mathfrak{g}$ *be of type* $A_1^{(1)}$, *and let* $p^{(3)}(n)$, $n \in \mathbb{Z}$, *be defined by* $\sum_n p^{(3)}(n)\, q^n = \varphi(q)^{-3}$. *Then for* $n_0, n_1 \in \mathbb{Z}$ *one has*

$$K(n_0\alpha_0 + n_1\alpha_1) = \sum_{k \geqslant 0} (-1)^k p^{(3)}((k+1)\, n_0 - kn_1 - \tfrac{1}{2}k(k+1)).$$

The importance of the partition function K lies in the fact that (as in the finite-dimensional case) the multiplicities which appear in representation theory may often be computed in terms of K.

Using Theorem 2, we compute the string functions for any highest weight module over $A_1^{(1)}$. The result, given by Theorem D, is

THEOREM 3. *Let* $L(\Lambda)$, $\Lambda \in P_+$, *be an* $A_1^{(1)}$-*module of level* $m > 0$, *and let* $\lambda \in P$, $c_\lambda^\Lambda \neq 0$. *Set*

$$a(\lambda) = {}^t\left(\frac{(\Lambda, \alpha_1) + 1}{2(m+2)}, \frac{(\lambda, \alpha_1)}{2m}\right) \in \mathbb{R}^2.$$

For $v = {}^t(x, y) \in \mathbb{R}^2$, *set* $\operatorname{sign} v = \operatorname{sign}(x)$ *and* $F(v) = (m+2)\, x^2 - my^2$. *Let* G_0 *be the subgroup of* $SL(2, \mathbb{R})$ *generated by the matrix* $\begin{pmatrix} m+1 & m \\ m+2 & m+1 \end{pmatrix}^2$. *Then*

$$\eta(\tau)^3\, c_\lambda^\Lambda(\tau) = \sum_{\substack{v \in \mathbb{Z}^2 + a(\lambda) \\ F(v) > 0 \\ v \bmod G_0}} (\operatorname{sign} v)\, e^{2\pi i \tau F(v)}. \tag{0.10}$$

The function (0.10) is a cusp-form of weight 1 of a type studied by Hecke [9]. Together with Theorem 3, identity (0.4) generates an intriguing series of identities for elliptic theta functions.

0.5. Here is a brief account of the contents of the paper. In Section I we present the basic facts about affine Lie algebras, starting with the general framework of Kač–Moody algebras $\mathfrak{g}(A)$. In Section 1.1 we recall the definition and properties of the invariant bilinear form, the root system, and the Weyl group W of $\mathfrak{g}(A)$. In Section 1.2 we recall the classification of affine Lie algebras and introduce their invariants l; k; a_i ($i = 0, \ldots, l$); h and g. In Sections 1.3 and 1.4 we give an explicit description of the invariant bilinear form and of the root system Δ of an affine Lie algebra. In Section 1.5 we introduce the new notion of the *adjacent root system* Δ', necessitated by technical complications appearing in the study of theta functions in the "twisted" case. We have $\Delta' = \Delta$ in the case $k = 1$. In Section 1.6 we introduce the lattices M and M' and describe the structure of the Weyl group of an affine Lie algebra. Note that formula (1.7) "explains" why theta functions appear in our considerations. In Section 1.7 we recall the realization of affine Lie algebras in terms of simple finite-dimensional Lie algebras.

In Section II we study the highest weight representations $L(\Lambda)$ and their characters. In Section 2.1 we work in the general framework of Kač–Moody algebras. New results here are the description of the region of convergence of $\mathrm{ch}_{L(\Lambda)}$ (Proposition 2.5), and separation of $W \ltimes 2\pi i Q^\vee$-orbits by the characters (Proposition 2.10). In Section 2.2 we describe the set of weights of a highest weight module over an affine Lie algebra and a convexity property of weight multiplicities (Proposition 2.12), using the fact that $\mathrm{mult}(\lambda - n\delta)$ increases with n (Proposition 2.11). In Section 2.3 we introduce the string functions and deduce the fundamental identity (2.18).

Section III gives the necessary information on theta functions, the modular group, and modular forms. The main result of Section 3.1 is Proposition 3.8 on the behavior of the Riemann theta function under the full modular group. In Section 3.2 we introduce and study the ring of theta functions. We give a basis for it and describe its multiplicative structure in this basis (Propositions 3.13 and 3.14). In Section 3.3 we briefly discuss modular forms and prove Lemma 3.20, which is used in the proof of the "very strange" formula.

In Section IV we apply the results of the theta function theory to affine Lie algebras. In Section 4.1 we adapt the general transformation laws for theta functions to our situation and deduce the transformation properties of the functions A_λ, which are anti-invariants of the Weyl group (Proposition 4.5). In Section 4.2 we find more explicit transformation laws for A_ρ (Proposition 4.6).

In Section 4.3 we use the transformation properties of some specializations of the function A_ρ to obtain a simple new proof of the "very strange" formula (Proposition 4.12). As a consequence, we obtain an estimate for $s_\Lambda(\lambda)$ (Proposition 4.14). We mention that the material of this section is related to η-function identities [29, 17, 44] and to the "Monstrous game" [4, 18].

In Sections 4.4 and 4.7 we deduce the main results partially stated above in Theorem 1. They concern the transformation properties of the string functions (Theorem A) and asymptotics of weight multiplicities (Theorem B).

In Section 4.5 we present results of the second author on the determinant and the inverse of the matrix of the string functions of a given level (Proposition 4.18 and formulas (4.20), (4.20.1, 2, 3); see [36] for more detail). In Section 4.6 we compute the string functions for all modules of level 1 over all affine Lie algebras, except $C_l^{(1)}$, and state a theorem of the second author unifying these results. We also express most of the string functions for $A_1^{(1)}$ as linear combinations of infinite products up to level 10, and for $A_2^{(1)}$ and $A_2^{(2)}$ up to level 3.

In some cases the transformation properties of A_λ are as nice as those of A_ρ. In Section 4.8 we study these A_λ and deduce interesting facts about three remarkable elements of a compact Lie group (cf. [25, 21]).

Finally, in Section 4.9 we outline an approach to the general restriction problem (Propositions 4.34 and 4.36).

In Section V, which is largely independent of the previous sections, we prove Theorems 2 and 3 above. We first find an explicit formula for the partition function for the affine Lie algebras of type $A_l^{(1)}$ (Theorem C and Section 5.2). In Section 5.4 we use Theorem C to compute the string functions directly in the case $A_1^{(1)}$ and unexpectedly encounter "indefinite" theta series (see Section 5.3 and Theorem D). Finally, in Section 5.5 we present explicit formulas which are special cases of our results.

Apart from the material cited above, the paper has four Appendixes, which are only indirectly related to the main body of the paper. In Appendix 1 (Section 1.8) we study the asymptotic behavior of root multiplicities in general Kač–Moody algebras. Appendix 2 (Section 2.4) is intimately related to Appendix 1. Here we study the structure of a highest weight module over an arbitrary Kač–Moody algebra and give an explicit description of the region of convergence of its characters. We included Appendix 3 (Section 2.5) in the paper only because of the mysterious coincidence of a constant involved in a cocycle (studied here) and the constant a (see formula (0.5)). Finally, in Appendix 4 (Section 4.10) the results of the second author on the independence of the fundamental characters are announced.

0.6. This paper represents work done by the authors primarily from August, 1979, to March, 1980. Theorems A (except for (2)), B, C, and D were proved during this period. The authors have subsequently discussed this work in several conferences, including the conference on Infinite-dimensional Lie algebras held in Oberwolfach in June, 1980. Some of the results of the paper were announced in [22]. Preprints of the paper were distributed in February 1982.

Notations and Conventions

(a) $\mathbb{Z}_+, \mathbb{Z}, \mathbb{Q}, \mathbb{R}_+, \mathbb{R}, \mathbb{C}$ denote the sets of non-negative integral, integral, rational, non-negative real, real and complex numbers, respectively. For a, $b \in \mathbb{C}$, we write $a \geqslant b$ if $a - b \in \mathbb{R}_+$.

(b) $\oplus$ and $\otimes$ denote direct sum and tensor product of vector spaces.

(c) $|S|$ denotes the cardinality of a set S.

(d) $|\lambda|^2$ stands for (λ, λ), where $(.,.)$ is a bilinear form.

(e) If Q is an abelian group and P a subgroup, Q mod P denotes a set of representatives of cosets of Q with respect to P.

(f) If $z \in \mathbb{C}$, $z \neq 0$, define $\log z$ by requiring that $e^{\log z} = z$ and $-\pi \leqslant \operatorname{Im} \log z < \pi$, and let $z^r = e^{r \log z}$ for all $r \in \mathbb{C}$.

(f′) For $\tau \in \mathbb{C}$, let q stand for $e^{2\pi i \tau}$, and more generally, let q^r stand for $e^{2\pi i r \tau}$. (This conflicts with (f), but should not cause confusion.)

(g) The topology of a real or complex finite-dimensional vector space is taken to be the metric topology.

(h) If V is a finite-dimensional real Euclidean space and L is a full lattice in V, we put $\operatorname{vol}(L) = \mu(V/L)$, where μ is the Euclidean measure on V.

(*i*) $U(\mathfrak{g})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{g}$.

(*j*) The base field is $\mathbb{C}$ unless otherwise specified.

I. Affine Lie Algebras and Root Systems

In Section I we present the necessary information about root systems of affine Lie algebras. We first outline the general framework of Kač–Moody Lie algebras, and then work out in detail the case of affine Lie algebras. Some proofs and details are omitted; they may be found in [14, 17] or in the book [50].

1.1. *Basic Facts about Kač–Moody Algebras*

(A) Let I be a finite set and let $A = (a_{ij})_{i,j \in I}$ be a *generalized Cartan matrix*, i.e., a matrix satisfying the following conditions: $a_{ii} = 2$ for all i; a_{ij} is a non-positive integer if $i \neq j$; $a_{ij} = 0$ implies $a_{ji} = 0$.

The matrix A is called *indecomposable* if I cannot be decomposed into a disjoint union of non-empty sets I_1 and I_2 such that $a_{ij} = 0$ for $i \in I_1, j \in I_2$.

The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix D such that DA is symmetric.

Let $\mathfrak{h}$ be a complex vector space of dimension $|I| + \operatorname{corank} A$. Then there exist linearly independent indexed sets

$$\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^* \qquad \text{and} \qquad \Pi^\vee = \{h_i\}_{i \in I} \subset \mathfrak{h},$$

such that $\alpha_j(h_i) = a_{ij}$. They are determined up to isomorphism by A. The α_i (resp. h_i) are called *simple roots* (resp. *dual simple roots*).

(B) Let $\mathfrak{g}(A)$ be the complex Lie algebra generated by $\mathfrak{h} \cup \{e_i, f_i\}_{i \in I}$ with defining relations

$$[e_i, f_j] = \delta_{ij} h_i \qquad \text{for} \quad i, j \in I;$$
$$[h, e_i] = \alpha_i(h)\, e_i, \qquad [h, f_i] = -\alpha_i(h) f_i \qquad \text{for} \quad i \in I,\ h \in \mathfrak{h};$$
$$[h, h'] = 0 \qquad \text{for} \quad h, h' \in \mathfrak{h};$$
$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \qquad \text{for} \quad i, j \in I,\ i \neq j.$$

The Lie algebra $\mathfrak{g}(A)$ is called a *Kač–Moody algebra* and A is called its *Cartan matrix*. The commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is called the *Cartan subalgebra.*

Let $\mathfrak{g}'(A)$ denote the derived algebra of $\mathfrak{g}(A)$. Then $\mathfrak{g}'(A)$ is generated by the elements e_i, f_i, $i \in I$, and we have $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$.

The center of $\mathfrak{g}(A)$ is $\mathfrak{c} := \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \text{ for all } i \in I\}$.

$\mathfrak{g}(A)$ decomposes into a direct sum of Kač–Moody algebras associated to the indecomposable components of A.

It has been established only recently that for an indecomposable A, any ideal of $\mathfrak{g}(A)$ either contains $\mathfrak{g}'(A)$ or is contained in $\mathfrak{c}$, provided that A is symmetrizable [8]. Since $\mathfrak{g}(A)$ is usually defined to be the quotient of our $\mathfrak{g}(A)$ by the sum of all ideals intersecting $\mathfrak{h}$ trivially, our definition and the usual one coincide for symmetrizable A.

(C) Fix a Kač–Moody algebra $\mathfrak{g}(A)$. Denoting by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}(A)$ generated by $\{e_i\}_{i \in I}$ (resp. $\{f_i\}_{i \in I}$), we obtain a vector space decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Furthermore, one has the *root space decomposition* of $\mathfrak{g}(A)$ with respect to $\mathfrak{h}$:

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

Here $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ and $\mathfrak{g}_0 = \mathfrak{h}$. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq (0)$, then α is called a *root* of *multiplicity* $\operatorname{mult} \alpha := \dim \mathfrak{g}_\alpha$ (which is always finite). Note that $\pm\alpha_i$ are roots of multiplicity 1 since $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$. Denote by Δ the set of all roots.

The $\mathbb{Z}$-span Q of the set Π is called the *root lattice*. For $\alpha = \sum_i k_i \alpha_i$, the number $\operatorname{ht} \alpha := \sum_i k_i$ is called the *height* of α. Let $Q_+ = \sum_i \mathbb{Z}_+ \alpha_i$, and introduce a partial order on $\mathfrak{h}^*$ by

$$\lambda \geqslant \mu \qquad \text{if} \quad \lambda - \mu \in Q_+.$$

Denote by $\Delta_+ = \Delta \cap Q_+$ the set of all *positive roots*. Then:

$$\Delta = \Delta_+ \cup (-\Delta_+).$$

(D) The Kač–Moody algebra $\mathfrak{g}({}^tA)$ is called *dual* to $\mathfrak{g}(A)$. We can (and will) identify the Cartan subalgebra $\mathfrak{h}^\vee$ of $\mathfrak{g}({}^tA)$ with $\mathfrak{h}^*$, so that the set of simple roots of $\mathfrak{g}({}^tA)$ (resp. dual simple roots) is identified with $\Pi^\vee$ (resp. Π). We will use freely the notions $Q^\vee$, $\Delta^\vee$, etc., which are defined in an obvious way.

(E) For $i \in I$, define the *fundamental reflection* $r_i \in GL(\mathfrak{h})$ by

$$r_i(h) = h - \alpha_i(h)\, h_i \qquad \text{for} \quad h \in \mathfrak{h}.$$

Note that r_i operates contragrediently on $\mathfrak{h}^*$ by $r_i(\alpha) = \alpha - \alpha(h_i)\,\alpha_i$.

The *Weyl group* W is the subgroup of $GL(\mathfrak{h})$ generated by the r_i, $i \in I$. Note that we can identify r_i with $r_i^\vee$ and W with $W^\vee$ via the contragredient action. One knows that for $i \in I$, $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ are locally nilpotent, and $\tilde{r}_i := (\exp \operatorname{ad} e_i)(\exp \operatorname{ad}(-f_i))(\exp \operatorname{ad} e_i) \in \operatorname{Aut} \mathfrak{g}(A)$ satisfies

$$\tilde{r}_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{r_i(\alpha)} \qquad \text{and} \qquad \tilde{r}_i|_{\mathfrak{h}} = r_i.$$

In particular, the root system Δ is W-invariant and, moreover, $\operatorname{mult} \alpha = \operatorname{mult} w(\alpha)$ for $w \in W$. It is easy to see that r_i permutes $\Delta_+ \backslash \{\alpha_i\}$.

(F) A root which is W-equivalent to a simple root is called *real*; a real root has multiplicity 1. Denote the set of all real roots by Δ^{re}. All other roots are called *imaginary*; the set of all imaginary roots is denoted by Δ^{im}. We put $\Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap \Delta_+$, $\Delta_+^{\mathrm{im}} = \Delta^{\mathrm{im}} \cap \Delta_+$. Then the sets Δ^{re} and Δ_+^{im} are W-invariant, and $\Delta^{\mathrm{re}} = \Delta_+^{\mathrm{re}} \cup (-\Delta_+^{\mathrm{re}})$, $\Delta^{\mathrm{im}} = \Delta_+^{\mathrm{im}} \cup (-\Delta_+^{\mathrm{im}})$.

Let $\alpha \in \Delta^{\mathrm{re}}$; then $w(\alpha) = \alpha_i \in \Pi$ for some $w \in W$ and $i \in I$, and we set

$$\alpha^\vee = w^{-1}(h_i) \in \mathfrak{h}.$$

$\alpha^\vee \in \Delta^\vee$ is called the *dual root* of α. This is well-defined by the following lemma.

LEMMA 1.1. *If $w(\alpha_i) = \alpha_j$ for some $w \in W$ and $i, j \in I$, then $w(h_i) = h_j$.*

Proof. $w = \tilde{w}|_{\mathfrak{h}}$ for some $\tilde{w}$ from the subgroup of $\operatorname{Aut} \mathfrak{g}$ generated by the $\tilde{r}_k$, $k \in I$. Applying $\tilde{w}$ to both sides of $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] = \mathbb{C}h_i$, we obtain $\mathbb{C}h_j = \mathbb{C}w(h_i)$. Since $w(\alpha_i)(w(h_i)) = \alpha_i(h_i) = 2$, we get $w(h_i) = h_j$. ∎

Now it is clear that the map ${}^\vee: \Delta^{\mathrm{re}} \to (\Delta^\vee)^{\mathrm{re}}$ defined by $\alpha \mapsto \alpha^\vee$ is a W-equivariant bijection which maps Π onto $\Pi^\vee$.

For $\alpha \in \Delta_+^{\mathrm{re}}$, we define $r_\alpha \in W$ by

$$r_\alpha(h) = h - \alpha(h)\, \alpha^\vee \qquad \text{for} \quad h \in \mathfrak{h},$$

so that $r_\alpha^2 = 1$, $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ for $\beta \in \mathfrak{h}^*$, $w r_\alpha w^{-1} = r_{w(\alpha)}$ for $w \in W$, and $r_{\alpha_i} = r_i$ for $i \in I$.

(G) Symmetrizability of the matrix A is a necessary and sufficient condition for the existence of a non-degenerate $\mathfrak{g}(A)$-invariant symmetric bilinear form $(,)$ on $\mathfrak{g}(A)$. The restriction of such a form to $\mathfrak{h}$ is non-degenerate and W-invariant. Conversely, any non-degenerate W-invariant symmetric bilinear form $(,)$ on $\mathfrak{h}$ can be uniquely extended to a non-degenerate $\mathfrak{g}(A)$-invariant symmetric bilinear form on $\mathfrak{g}(A)$.

For the reminder of (G), we assume that A is symmetrizable; we then can choose a non-degenerate invariant symmetric bilinear form $(,)$ on $\mathfrak{g}(A)$ such that (h_i, h_i) is positive rational for all $i \in I$. Such a form is called *standard.* We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using $(,)$. A root α is real if and only if $(\alpha, \alpha) > 0$ and is imaginary if and only if $(\alpha, \alpha) \leqslant 0$ [14]. A root α is called *isotropic* if $(\alpha, \alpha) = 0$.

Furthermore, for $\alpha \in \Delta^{\mathrm{re}}$,

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)},$$

$$r_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha,$$

and for $\alpha \in \Delta$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\alpha.$$

The last equation defines a non-degenerate pairing of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

Remark. If α is a root such that $(\alpha, \alpha) > 0$, i.e., α is real, then $\operatorname{mult}(\pm\alpha) = 1$ and $\operatorname{mult} n\alpha = 0$ for $n \neq \pm 1$. If $(\alpha, \alpha) \leqslant 0$, then $n\alpha$ is a root for any integer $n \neq 0$ [14]. Any isotropic root α is W-equivalent to an imaginary root of an affine Lie subalgebra [19] and hence $\operatorname{mult} \alpha$ can be found from Table M in Section 1.4; in particular, we have: $\operatorname{mult} \alpha < |I|$. The situation changes drastically when we pass to a non-isotropic imaginary root α. In this case $\bigoplus_{n>0} \mathfrak{g}_{n\alpha}$ is a free Lie algebra, $\operatorname{mult} n\alpha$ is a non-decreasing sequence, and moreover, $\lim_{n\to\infty}(\log \operatorname{mult}(n\alpha))/n$ exists and is positive. We prove these facts in Appendix 1 (Section 1.8).

(H) Set $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{R} \text{ for all } i \in I\}$. This is a W-stable real subspace of $\mathfrak{h}$. We define $\mathfrak{h}_{\mathbb{R}}^*$ similarly. Define the *fundamental chamber* $C \subset \mathfrak{h}_{\mathbb{R}}$ by

$$C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geqslant 0 \text{ for all } i \in I\}.$$

The set

$$X = \bigcup_{w \in W} w(C)$$

is called the *Tits cone*; each $w(C)$ is called a *chamber*. Define the *imaginary cone* Z to be the closure of the convex hull of $\{0\} \cup \Delta_+^{\mathrm{im}}$ [17].

Using Lemma 1.1, the usual proof of the "exchange condition" works in the framework of Kač–Moody algebras. We need only the following corollary of it.

LEMMA 1.2. *Let $w = r_{i_1} \cdots r_{i_s}$ be a reduced expression of $w \in W$ (i.e., w cannot be represented as a product of fewer than s fundamental reflections). Then $-w(\alpha_{i_s}) > 0$.*

Proof. Standard (see, e.g., [11, p. 50].) ∎

It follows that $^\vee$ maps Δ_+^{re} onto $(\Delta^\vee)_+^{\mathrm{re}}$. (In particular, for $i \in I$ and $\beta \in \Delta_+^{\mathrm{re}}$, we have $\alpha_i(\beta^\vee) \leqslant 0 \Leftrightarrow r_\beta(\alpha_i) \in \Delta_+^{\mathrm{re}} \Leftrightarrow r_\beta(h_i) \in (\Delta_+^\vee)^{\mathrm{re}} \Leftrightarrow \beta(h_i) \leqslant 0$. Hence, for $\alpha, \beta \in \Delta^{\mathrm{re}}$, we have $\alpha(\beta^\vee) \leqslant 0 \Leftrightarrow \beta(\alpha^\vee) \leqslant 0$.)

We now establish some important properties of the Tits cone (cf. [41] and [28]).

PROPOSITION 1.3. (a) *For any $h \in X$ the orbit $W(h)$ meets C in exactly one element.*

(b) *The stabilizer W_h of any $h \in C$ is generated by the fundamental reflections contained in it.*

(c) $C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid$ *for all* $w \in W$, $h - w(h) \in \sum_i \mathbb{R}_+ h_i\}$.

(d) $X = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(h) < 0$ *for only a finite number of* $\alpha \in \Delta_+\}$. *In particular, X is a convex cone. The same result holds with Δ_+ replaced by Δ_+^{re}.*

(e) *If $h \in X$, then $h \in$* Interior X *if and only if* $|W_h| < \infty$.

(f) $Z = \{\alpha \in \mathfrak{h}^* \mid \alpha(h)$ *is non-negative real for all* $h \in X\}$.

Proof. Let $w = r_{i_1} \cdots r_{i_s}$ be a reduced expression. Take $h \in C$ and suppose that $h' = w(h) \in C$. We have: $\alpha_{i_s}(h) \geqslant 0$, so that $w(\alpha_{i_s})(h') \geqslant 0$. But by Lemma 1.2, $w(\alpha_{i_s}) < 0$ and hence $w(\alpha_{i_s})(h') \leqslant 0$. Therefore, $w(\alpha_{i_s})(h') = 0$ and so $\alpha_{i_s}(h) = 0$. Hence, $r_{i_s}(h) = h$ and both (a) and (b) follow by induction on the length s of w. To prove (c), note that $C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid h - r_i(h) \in \mathbb{R}_+ h_i$ for all $i \in I\}$. Hence it suffices to show that for $h \in C$ and $w \in W$, $h - w(h) = \sum_i c_i h_i$, where all $c_i \geqslant 0$. This is proved by induction on the length s of a reduced expression $w = r_{i_1} \cdots r_{i_s}$. Indeed, for $s = 1$ it follows from the definition of C. For $s > 1$ we have: $h - w(h) = (h - r_{i_1} \cdots r_{i_{s-1}}(h)) + r_{i_1} \cdots r_{i_{s-1}}(h - r_{i_s}(h))$; using the inductive assumption and Lemma 1.2, applied to $\Delta^\vee$, (c) follows.

To prove (d) set $X' = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(h) < 0$ for only a finite number of $\alpha \in \Delta_+\}$, and for $h \in X'$ set $M_h = \{\alpha \in \Delta_+ \mid \alpha(h) < 0\}$. It is clear that $C \subset X'$ and that X' is stable under W. Hence $X' \supset X$. We prove that $h \in X'$ implies $h \in X$ by induction on $|M_h|$. If $|M_h| = 0$, then $h \in C \subset X$. Otherwise, $\alpha_i \in M_h$ for some $i \in I$. But then $M_{r_i(h)} = r_i(M_h \setminus \{\alpha_i\})$, so that $r_i(h) \in X$ by the inductive assumption. The same argument works for Δ_+^{re}, proving (d).

To prove (f) we may assume that the matrix A is indecomposable. Put $Z' = \{\alpha \in \mathfrak{h}^* \mid \alpha(h) \geqslant 0$ for all $h \in X\}$. We must show that $Z = Z'$. $Z \subset Z'$ since $\Delta_+^{\mathrm{im}} \subset Q_+$ is W-stable. If $|\Delta| < \infty$, then $\{0\} = Z' \subset Z$ by (d). If A is an affine matrix (cf. Section 1.2), then $Z' \subset Z$ follows from Proposition 1.9(a) below. In the remaining case, there exists $\gamma \in \Delta_+^{\mathrm{im}}$ such that $\gamma(h_i) < 0$ for all $i \in I$ [19]. Suppose $Z' \not\subset Z$. Since Z is a closed convex cone, we may choose $h' \in \mathfrak{h}$ such that $\alpha(h') \geqslant 0$ for all $\alpha \in Z$, but $\alpha(h') < 0$ for some $\alpha \in Z'$. We deduce that $\alpha(h') \geqslant 0$ for all $\alpha \in \Delta_+^{\mathrm{im}}$, but $h' \notin$ Closure X. Choose $h \in \mathfrak{h}$ near h' and $\varepsilon > 0$ such that $h \notin X$ but $\alpha_i(h) \geqslant \alpha_i(h') + \varepsilon$ for all $i \in I$. If $\beta \in \Delta_+^{\mathrm{re}}$, then $\gamma(\beta^\vee) \leqslant -1$, and $r_\beta(\gamma)(h) \geqslant \varepsilon \operatorname{ht} r_\beta(\gamma)$ since $r_\beta(\gamma) \in \Delta_+^{\mathrm{im}}$. We deduce that $\beta(h) \geqslant -\gamma(h) + \varepsilon \operatorname{ht} \beta$ for all $\beta \in \Delta_+^{\mathrm{re}}$, so that $h \in X$ by (d). This contradicts $h \notin X$, proving (f).

To prove (e), we may assume that $h \in C$. Then (e) follows from (b) and the following lemma applied to W_h. ∎

LEMMA 1.4. *The following conditions are equivalent*:

(i) $|W| < \infty$,

(ii) $X = \mathfrak{h}_{\mathbb{R}}$,

(iii) $|\Delta| < \infty$.

Proof. (i) ⇒ (ii) since for any $h \in \mathfrak{h}_{\mathbb{R}}$, each $h' \in W(h)$ with maximal $\operatorname{ht}^\vee(h' - h)$ lies in C. (ii) ⇒ (iii) by taking $h \in \mathfrak{h}_{\mathbb{R}}$ such that $\alpha_i(h) < 0$ for all $i \in I$, and applying Proposition 1.3(d). (iii) ⇒ (i) as Δ is W-invariant and any $w \in W$ leaving Δ pointwise fixed is the identity by Lemma 1.2. ∎

Finally, introduce the following important domain Y in $\mathfrak{h}$ [28]:

$$Y = \text{Interior } (X + i\mathfrak{h}_{\mathbb{R}}).$$

EXAMPLES. If A is an affine matrix, then

$$Y = \{h \in \mathfrak{h} \mid \operatorname{Re} \delta(h) > 0\}.$$

If A is a generalized Cartan matrix of hyperbolic type, then

$$Y \cup -Y = \{x + iy \mid x, y \in \mathfrak{h}_{\mathbb{R}}, (x, x) < 0\}.$$

1.2. *The Classification of Affine Lie Algebras*

One knows that $\dim \mathfrak{g}(A) < \infty$ if and only if A is of *finite type*, i.e., all principal minors of A are positive. In this case, $\mathfrak{g}(A)$ is semisimple; conversely, every finite-dimensional semisimple Lie algebra is of the form $\mathfrak{g}(A)$.

A generalized Cartan matrix A is called an *affine matrix*, and is said to be of *affine type*, if all its proper principal minors are positive, and $\det A = 0$. Note that A is an affine matrix if and only if ${}^tA$ is. One knows that any generalized Cartan matrix with all principal minors non-negative decomposes into a direct sum of matrices of finite and affine types. The corresponding Kač–Moody algebras are characterized by the property that root multiplicities are bounded.

The Kač–Moody algebra associated to an affine matrix is called an *affine Lie algebra*.

To each affine matrix $A = (a_{ij})_{i,j \in I}$ we assign a diagram $S(A)$ as follows. The set of vertices of $S(A)$ is $I = \{0, 1,..., l\}$[1], and if $i, j \in I$, $i \neq j$, the vertices i and j are connected by $a_{ij}a_{ji}$ lines; if $|a_{ij}| > |a_{ji}|$, these lines are equipped with an arrow, pointing toward the vertex i. One associates numerical marks a_i to the vertices i as follows: a_i, $i \in I$, are positive integers with greatest common divisor 1 such that for all $i \in I$, $\sum_{j \in I} a_j a_{ij} = 0$. The diagram $S(A)$ with $l+1$ vertices is called the *Dynkin diagram* of $\mathfrak{g}(A)$.

It happens that two affine Lie algebras are isomorphic if and only if they have isomorphic Dynkin diagrams. These diagrams are listed in Tables I, II, and III [14, 31]. The numerical marks are written beside the vertices. To the left of the Dynkin diagram of $\mathfrak{g}(A)$ in the tables is a symbol such as $A_1^{(1)}$, called the *type* of $\mathfrak{g}(A)$. The superscript of this symbol is k, the number of the table, which, along with l, is an important invariant of $\mathfrak{g}(A)$.

In the remainder of this paper we always assume that 0 is the leftmost vertex of the Dynkin diagram of $\mathfrak{g}(A)$ as shown in the tables. In particular, $a_0 = 1$ unless $\mathfrak{g}(A)$ is of type $A_{2l}^{(2)}$, when $a_0 = 2$.

We denote by $a_i^\vee$ the numerical marks of the diagram $S({}^tA)$, so that $\sum_i a_i^\vee a_{ij} = 0$ for all $j \in I$. Note that $a_0^\vee = 1$ in all cases.

The integers

$$h := \sum_{i \in I} a_i \qquad \text{and} \qquad g := \sum_{i \in I} a_i^\vee$$

are called the *Coxeter number* and the *dual Coxeter number*, respectively, of the affine Lie algebra $\mathfrak{g}(A)$. Note that when $k = 1$, h is the Coxeter number of the finite root system with Dynkin diagram $S(A) \backslash \{0\}$, and g is the inverse of the square of the length of a long root with respect to the Killing form for this root system (cf. 4.12.2, 3).

1.3. *The Normalized Invariant Form on an Affine Lie Algebra*

For the next four sections we fix an affine matrix A. Let $\mathfrak{g}(A)$ be the associated affine Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, etc.

[1] We take I of this form merely for convenience.

TABLE I

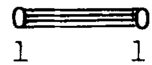
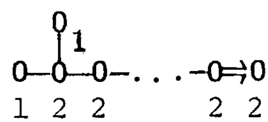
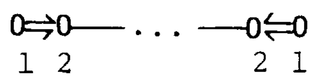
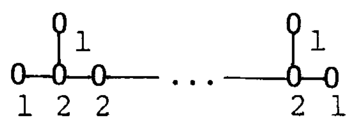
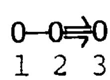
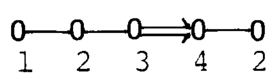
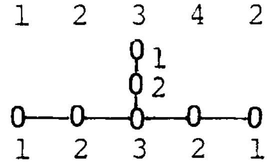
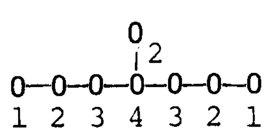
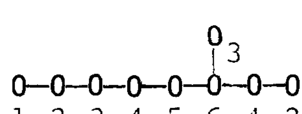
$A_1^{(1)}$	
$A_\ell^{(1)}$ $(\ell \geq 2)$	
$B_\ell^{(1)}$ $(\ell \geq 3)$	
$C_\ell^{(1)}$ $(\ell \geq 2)$	
$D_\ell^{(1)}$ $(\ell \geq 4)$	
$G_2^{(1)}$	
$F_4^{(1)}$	
$E_6^{(1)}$	
$E_7^{(1)}$	
$E_8^{(1)}$	

TABLE II

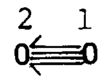
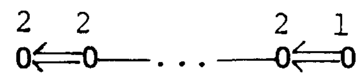
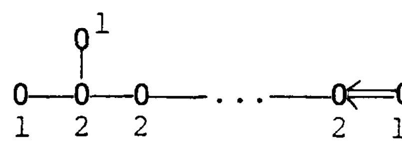
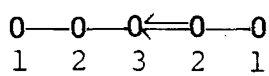
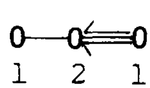
$A_2^{(2)}$	
$A_{2\ell}^{(2)}$ $(\ell \geq 2)$	
$A_{2\ell-1}^{(2)}$ $(\ell \geq 3)$	
$D_{\ell+1}^{(2)}$ $(\ell \geq 2)$	
$E_6^{(2)}$	

TABLE III

$D_4^{(3)}$	

Then the center of $\mathfrak{g}(A)$ is one-dimensional and is spanned by the *canonical central element*

$$c := \sum_{i \in I} a_i^{\vee} h_i.$$

Since $\alpha_i(c) = 0$ for all $i \in I$, c is fixed under the action of the Weyl group W.

Fix an element d of $\mathfrak{h}$ such that:

$$\alpha_i(d) = 0 \qquad \text{for} \quad i = 1, \ldots, l;\ \alpha_0(d) = 1.$$

Then $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathbb{C}d$. The elements $h_0, \ldots, h_l, d$ form a $\mathbb{C}$-basis of $\mathfrak{h}$. For $i \in I$, define $\Lambda_i \in \mathfrak{h}^*$ by

$$\Lambda_i(h_j) = \delta_{ij}; \qquad \Lambda_i(d) = 0.$$

The elements $\Lambda_0, \ldots, \Lambda_l$ are called *fundamental weights*. Note that $\Lambda_i(c) = a_i^\vee$ and that $\{\alpha_0, \ldots, \alpha_l, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$.

We define a bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ by

$$\begin{aligned} (\alpha_i, \alpha_j) &= a_i^{-1} a_i^\vee a_{ij} && (i, j \in I), \\ (\Lambda_0, \alpha_i) = (\alpha_i, \Lambda_0) &= 0 && \text{for} \quad i \neq 0, \\ (\Lambda_0, \alpha_0) = (\alpha_0, \Lambda_0) &= a_0^{-1}, && (\Lambda_0, \Lambda_0) = 0. \end{aligned}$$

It is easy to see that this form is symmetric, non-degenerate, and W-invariant (so that, in particular, A is symmetrizable). Hence, it induces a bilinear form on $\mathfrak{h}$, which we extend to a standard form $(\ ,\ )$ on the whole Lie algebra $\mathfrak{g}(A)$ (see Section 1.1(G)). The form $(\ ,\ )$ on $\mathfrak{g}$ defined above is called the *normalized standard form.*

Introduce the following two important elements of $\mathfrak{h}^*$:

$$\delta = \sum_{i \in I} a_i \alpha_i,$$

$$\theta = \delta - a_0 \alpha_0.$$

As $|\alpha_0|^2 = 2a_0^{-1}$, we have

$$|\theta|^2 = 2a_0,$$

$$\delta - \theta = a_0^\vee.$$

Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ via the normalized standard form, we obtain

$$\delta = c; \qquad \Lambda_0 = a_0^{-1} d; \qquad a_i \alpha_i = a_i^\vee \alpha_i^\vee \qquad \text{for} \quad i \in I.$$

Hence $(\delta, Q) = 0$ and δ is fixed by W.

Denote by $\bar{\mathfrak{h}}^*$ the linear span over $\mathbb{C}$ of $\alpha_1, \ldots, \alpha_l$. Then $\mathfrak{h}^*$ is the orthogonal direct sum of $\bar{\mathfrak{h}}^*$ and the two-dimensional space $\mathbb{C}\delta + \mathbb{C}\Lambda_0$. One knows that the restriction of $(\ ,\ )$ to $\bar{\mathfrak{h}}^*_{\mathbb{R}} := \sum_{i=1}^{l} \mathbb{R}\alpha_i$ (resp. $\sum_{i=0}^{l} \mathbb{R}\alpha_i$) is positive-definite (resp. positive-semidefinite with kernel $\mathbb{R}\delta$). Furthermore, one has

$$(\Lambda_0, \Lambda_0) = (\delta, \delta) = 0; \qquad (\Lambda_0, \delta) = 1.$$

For $\lambda \in \mathfrak{h}^*$, denote by $\bar{\lambda}$ the orthogonal projection of λ on $\bar{\mathfrak{h}}^*$. Then, if $\lambda \in \mathfrak{h}^*$ is such that $\lambda(c) \neq 0$, one has the following useful formula:

$$\lambda - \bar{\lambda} = \lambda(c) \Lambda_0 + (2\lambda(c))^{-1} (|\lambda|^2 - |\bar{\lambda}|^2) \delta. \tag{1.5}$$

We put $\bar{Q} = \{\bar{\alpha} \mid \alpha \in Q\}$, $\bar{Q}^\vee = \{\bar{\alpha} \mid \alpha \in Q^\vee\}$†. Note that:

$$\overline{Q^\vee} \subset \bar{Q} \text{ if } k = 1; \ \bar{Q} \subset \overline{Q^\vee} \text{ if } k \neq 1.$$

1.4. *An Explicit Description of the Root System of an Affine Lie Algebra*

Set $\bar{\Delta} = \Delta \cap \bar{\mathfrak{h}}^*$. Then $\bar{\Delta}$ is isomorphic to the root system of the finite-dimensional complex simple Lie algebra $\bar{\mathfrak{g}} := \bar{\mathfrak{h}} + \sum_{\alpha \in \bar{\Delta}} \mathfrak{g}_\alpha$, with Dynkin diagram that of $\mathfrak{g}(A)$ with the vertex 0 omitted. Thus $\bar{\Pi} := \{\alpha_1, ..., \alpha_l\}$ is a set of simple roots of $\bar{\Delta}$, and $\bar{\Delta}_+ := \bar{\Delta} \cap \Delta_+$ is the corresponding set of positive roots. Denote by $\bar{\Delta}^l$ and $\bar{\Delta}^s$ the sets of long and of short roots of $\bar{\Delta}$.

We shall reconstruct Δ, Δ^{im}, Δ^{re}, Δ_+, Π, θ, etc., from $\bar{\Delta}_+$, δ, k, a_0. The proof can be easily adduced from the explicit construction of affine Lie algebras in Section 1.7.

It is known that

$$\Delta^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}, n \neq 0\},$$

and that the multiplicity of an imaginary root $n\delta$ is l except in the following cases:

TABLE M

Type $X_n^{(k)}$	$A_{2l-1}^{(2)}$	$D_{l+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
n	odd	odd	odd	$\not\equiv 0 \bmod 3$
mult $n\delta$	$l-1$	1	2	1

Note that mult $n\delta = l$ if k divides $a_0 n$, and mult $n\delta = |\bar{\Pi} \cap \bar{\Delta}^s|$ otherwise.

Furthermore:

$$\Delta^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}\} \qquad \text{when} \quad k = 1,$$

$$\Delta^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}^s\} \cup \{\alpha + nk\delta \mid \alpha \in \bar{\Delta}^l\} \qquad \text{when} \quad a_0 k = 2 \text{ or } 3,$$

$$\Delta^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}^s\} \cup \{\alpha + 2n\delta \mid \alpha \in \bar{\Delta}^l\} \cup \{\tfrac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \bar{\Delta}^l\} \qquad \text{when} \quad a_0 k = 4.$$

Here n ranges over $\mathbb{Z}$.

The set Δ_+ of positive roots consists of those roots given above for which $n > 0$, and of $\bar{\Delta}_+$.

Note that θ is the highest root of $\bar{\Delta}_+$ when $a_0 k = 1$ or 4, and is the highest short root of $\bar{\Delta}_+$ when $a_0 k = 2$ or 3. We have

$$\Pi = \{\alpha_0 = a_0^{-1}(\delta - \theta), \alpha_1, ..., \alpha_l\}.$$

† For typographical reasons, $\bar{Q}^\vee$ has sometimes been rendered in the text as $\overline{Q^\vee}$, $\overline{\rho'}$ as $\bar{\rho}'$, etc.

Warning. $\bar{\Delta}$ is the orthogonal projection of Δ^{re} on $\bar{\mathfrak{h}}^*$ in all cases except for $A_{2l}^{(2)}$, when the projection of Δ^{re} on $\bar{\mathfrak{h}}^*$ is a non-reduced root system and $\bar{\Delta}$ is the associated reduced root system.

1.5. *The Adjacent Root System*

The facts presented in this section are easily deduced from Section 1.4.

We associate to the root system Δ the *adjacent root system* Δ', with multiplicities mult$'$, etc., as follows. If $a_0 k = 1$ or 4, then $\Delta' = \Delta$, mult $=$ mult$'$, etc. Otherwise,

$$\Delta'^{\text{im}} := \{k^{-1} n\delta \mid n \in \mathbb{Z}, n \neq 0\},$$

$$\text{mult}'\, k^{-1} n\delta := \begin{cases} \text{mult}\, k\delta & \text{if} \quad n \equiv 0 \pmod{k} \\ \text{mult}\, k\delta - \text{mult}\, \delta & \text{if} \quad n \not\equiv 0 \pmod{k}; \end{cases}$$

$$\begin{aligned} \Delta'^{\text{re}} :&= \{\alpha + n\delta \mid \alpha \in \bar{\Delta}^s, n \in \mathbb{Z}\} \\ &\quad \cup \{k^{-1}(\alpha + n\delta) \mid \alpha \in \bar{\Delta}^l, n \in \mathbb{Z}\} \\ &= \{2(\alpha, \alpha)^{-1}(\alpha + n\delta) \mid \alpha \in \bar{\Delta}, n \in \mathbb{Z}\}; \end{aligned}$$

$$\Delta' := \Delta'^{\text{im}} \cup \Delta'^{\text{re}};$$

$$\Delta'_+ := \Delta' \cap k^{-1} Q_+.$$

Set $k' = a_0^{-1} k$, so that $k' = 1$ if $a_0 k = 1$ or 4 and $k' = k$ otherwise.

Then Δ' is isomorphic to the root system associated to some affine Cartan matrix A'; furthermore, $\mathfrak{g}(A)$ and $\mathfrak{g}(A')$ are isomorphic unless $\mathfrak{g}(A)$ is of type $A_{2l-1}^{(2)}$ or $D_{l+1}^{(2)}$, when $\mathfrak{g}(A')$ is of type $D_{l+1}^{(2)}$ or $A_{2l-1}^{(2)}$, respectively. More precisely, there exists a linear isomorphism Φ from $\mathfrak{h}^*$ onto the dual of the Cartan subalgebra of $\mathfrak{g}(A')$ such that for all $\alpha \in \mathfrak{h}^*$, we have: $\dim \mathfrak{g}(A')_{\Phi(\alpha)} = \text{mult}'\, \alpha$ if $\alpha \neq 0$; $\Phi(\alpha) > 0$ if $\alpha \in \Delta'_+$; $|\Phi(\alpha)|^2 = k' |\alpha|^2$; $\Phi(\bar{\alpha}) = \overline{\Phi(\alpha)}$. Using Φ, we have notions Π', δ', θ', Q', W', T'; $\overline{Q'}$, $\overline{\Delta'}$, $\bar{\Delta}'_+$; α'_0, Λ'_0; etc.

Denote by $\tilde{\theta}$ (resp. $\tilde{\theta}'$) the highest root of $\bar{\Delta}_+$ (resp. $\bar{\Delta}'_+$). Then:

$$|\tilde{\theta}|^2 = 2k; \qquad \tilde{\theta}' = \theta; \; k'\delta' = \delta; \qquad k'\theta' = \tilde{\theta};$$

$$a_0 \alpha'_0 = \delta' - \theta'; \qquad \Lambda'_0 = \Lambda_0; \qquad \bar{Q}' = \bar{Q} + \overline{Q^\vee}.$$

Moreover, the dual Coxeter number of $\mathfrak{g}(A')$ is g (cf. (4.6.1)).

For $\alpha \in \Delta'^{\text{re}}$, put $\alpha^\vee = 2\alpha/|\alpha|^2$. Put $Q'^\vee = \sum_{\alpha \in \Pi'} \mathbb{Z}\alpha^\vee$, $\overline{Q'^\vee} = \{\bar{\alpha} \mid \alpha \in Q'^\vee\}$. Then $\overline{Q'^\vee} = \bar{Q} \cap \overline{Q^\vee}$. If $k' \neq 1$, we have: $Q' = Q^\vee + \mathbb{Z}\delta'$, $Q'^\vee = Q$, $\bar{\Delta}'_+ = \bar{\Delta}^\vee_+$.

Warning. The Dynkin diagram of $\mathfrak{g}(A')$ is found in Table k, not in Table k'.

1.6. *The Weyl Group of an Affine Lie Algebra and the Lattice M*

Let W be the Weyl group of the affine Lie algebra $\mathfrak{g}(A)$. Recall that W is generated by the fundamental reflections $r_0, r_1, \ldots, r_l$ (cf. Section 1.1(E)). Let

$\bar{W}$ be the subgroup of W generated by $r_1, \ldots, r_l$; this is the Weyl group of the finite root system $\bar{\Delta}$.

Denote by M the $\mathbb{Z}$-span of the set $\bar{W}(\theta^\vee)$ and by M' the $\mathbb{Z}$-span of the set $\bar{W}(\tilde{\theta}^\vee)$. Then it is easy to see that

$$M' \supset M \supset k'M'.$$

We have $M = \bar{Q} \cap \bar{Q}^\vee (= \bar{Q}^\vee$ if $k = 1$ and $= \bar{Q}$ if $k \neq 1)$ and $M' = \bar{Q}^\vee = \sum_{i=1}^{l} \mathbb{Z}\alpha_i^\vee$. Moreover,

$$M + \mathbb{Z}\delta = Q'^\vee \text{ and } M' + \mathbb{Z}\delta = Q^\vee.$$

For $t \in \mathbb{C}$ set $\mathfrak{h}_t^* = \{\lambda \in \mathfrak{h}^* \mid (\lambda, \delta) = t\}$, $\mathfrak{h}_{t,\mathbb{R}}^* = \mathfrak{h}_t^* \cap \mathfrak{h}_{\mathbb{R}}^*$. Note that $\mathfrak{h}_0^* = \sum_{i=0}^{l} \mathbb{C}\alpha_i$. Since the bilinear form $(\,,\,)$ is W-invariant and δ is fixed by W, the affine hyperplanes $\mathfrak{h}_t^*$ are W-invariant.

Consider the affine space $\mathfrak{h}_1^* \bmod \mathbb{C}\delta$. Since the action of W on $\mathfrak{h}_0^*$ is faithful by Lemma 1.2, its action on $\mathfrak{h}^*/\mathbb{C}\delta \simeq (\mathfrak{h}_0^*)^*$ and thus on $\mathfrak{h}_1^* \bmod \mathbb{C}\delta$ is also faithful. The latter action has the following simple geometrical meaning. We identify $\mathfrak{h}_1^* \bmod \mathbb{C}\delta$ with $\bar{\mathfrak{h}}^*$ by projection, thus obtaining an isomorphism from W onto a group W_{af} of affine transformations of $\bar{\mathfrak{h}}^*$. We denote this isomorphism by af: $W \to W_{\mathrm{af}}$, so that

$$\overline{w(\lambda)} = \mathrm{af}(w)(\bar{\lambda}) \qquad \text{for} \quad \lambda \in \mathfrak{h}_1^*.$$

The group W_{af} is called the *affine Weyl group*.

For $w \in \bar{W}$, we have: $\mathrm{af}(w) = w$. Furthermore:

$$\mathrm{af}(r_{\alpha_0})(\lambda) = r_\theta(\lambda) + \theta^\vee \qquad \text{for} \quad \lambda \in \bar{\mathfrak{h}}^*,$$

so that $\mathrm{af}(r_{\alpha_0})$ is a reflection in the hyperplane $\theta = 1$, i.e., in $\{\lambda \in \bar{\mathfrak{h}}^* \mid (\lambda, \theta) = 1\}$.

Since $r_\theta \in \bar{W}$, the group W is generated by $\bar{W}$ and the element $t_{\theta^\vee} := r_{\alpha_0} r_\theta$. We have $\mathrm{af}(t_{\theta^\vee})(\lambda) = \lambda + \theta^\vee$ for $\lambda \in \bar{\mathfrak{h}}^*$. For $\alpha = w(\theta^\vee)$, where $w \in \bar{W}$, set $t_\alpha := w t_{\theta^\vee} w^{-1}$, so that $\mathrm{af}(t_\alpha)(\lambda) = \lambda + \alpha$. Denote by T the subgroup of W generated by $\{t_{w(\theta^\vee)} \mid w \in \bar{W}\}$.

Then $T \cong \mathrm{af}(T)$ is an abelian normal subgroup of W, and we have the semidirect product decomposition:

$$W = \bar{W} \ltimes T.$$

Since M is the $\mathbb{Z}$-span of $\bar{W}(\theta^\vee)$, we have an isomorphism $\alpha \mapsto t_\alpha$ of M onto T defined by: $\mathrm{af}(t_\alpha)(\lambda) = \lambda + \alpha$.

Since $(\,,\,)$ is W-invariant, we have $|t_\alpha(\lambda)|^2 = |\lambda|^2$ for all $\lambda \in \mathfrak{h}^*$. From this we deduce the following formula, which is crucial in our considerations:

$$t_\alpha(\lambda) = \lambda + (\lambda, \delta)\alpha - (\tfrac{1}{2}(\lambda, \delta)\,|\alpha|^2 + (\alpha, \lambda))\delta \qquad \text{for} \quad \lambda \in \mathfrak{h}^*. \tag{1.6}$$

We extend the definition of t_α to arbitrary $\alpha \in \bar{\mathfrak{h}}^*$ by (1.6). Then one easily checks that t_α is linear and preserves $(,)$, $t_{\alpha+\beta} = t_\alpha t_\beta$, and $w t_\alpha w^{-1} = t_{w(\alpha)}$ for $w \in \bar{W}$.

It is sometimes convenient to use an equivalent formula for t_α, derived using (1.5), which holds for $\lambda \in \mathfrak{h}^*$ such that $m := \lambda(c) \neq 0$:

$$t_\alpha(\lambda) = m\Lambda_0 + \frac{1}{2m} |\lambda|^2 \delta + (\bar{\lambda} + m\alpha) - \frac{1}{2m} |\bar{\lambda} + m\alpha|^2 \delta. \tag{1.7}$$

Another useful pair of formulas, not depending on the normalization of the form, is

$$(\lambda, \delta) = \frac{|\theta|^2}{2a_0} \lambda(h_0) + (\bar{\lambda}, \theta); \qquad \lambda(c) = \sum_{i=0}^{l} a_i^\vee \lambda(h_i). \tag{1.8}$$

For the adjacent root system, we similarly define W' and the decomposition $W' = \bar{W} \ltimes T'$. We note that W and W' differ only in their *translation subgroups* T and T', and that T' consists of the translations t_α, $\alpha \in M'$.

Recall Section 1.1(H). We describe explicitly the Tits cone X and the domain Y.

PROPOSITION 1.9. (a) $X = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \delta(h) > 0\} \cup \mathbb{R}c$;

$$Y = \{h \in \mathfrak{h} \mid \operatorname{Re} \delta(h) > 0\}.$$

(b) *If $h \in C(=\{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geqslant 0$ for all $i \in I\})$, then W_h is generated by the fundamental reflections contained in it. If $h \in X$, then $W(h) \cap C$ has exactly one element. If $h \in Y$, then W_h is finite.*

Proof. (a) follows from Proposition 1.3(d) and the description of affine root systems in Section 1.4. (b) is a particular case of Proposition 1.3(a), (b), (e). ∎

Consider the (surjective) projection map π: $\mathfrak{h}^*_{1,\mathbb{R}} \to \bar{\mathfrak{h}}^*_{\mathbb{R}}$, and put $C_{\mathrm{af}} = \{\lambda \in \bar{\mathfrak{h}}^*_{\mathbb{R}} \mid (\lambda, \alpha_i) \geqslant 0$ for $1 \leqslant i \leqslant l$, and $(\lambda, \theta) \leqslant 1\}$. Then, identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ using $(,)$, we have $\pi^{-1}(C_{\mathrm{af}}) = C \cap \mathfrak{h}^*_{1\mathbb{R}}$. Since $\mathrm{af}(w) \circ \pi = \pi \circ w$ for all $w \in W$, we deduce from Proposition 1.9 that C_{af} is a fundamental domain for $W_{\mathrm{af}} = \mathrm{af}(W)$ on $\bar{\mathfrak{h}}^*_{\mathbb{R}}$; more precisely, using $W = \bar{W} \ltimes T$ we obtain:

PROPOSITION 1.10 (a) *Any point of $\bar{\mathfrak{h}}^*_{\mathbb{R}}$ is $\bar{W}$-equivalent* mod M *to a unique point of C_{af}.*

(b) *The stabilizer of any point of C_{af} under the action of $\bar{W}$ on $\bar{\mathfrak{h}}^*_{\mathbb{R}}/M$ is generated by its intersection with $\{r_\theta, r_{\alpha_1}, \ldots, r_{\alpha_l}\}$.*

Remark. The volume of the simplex C_{af} is $(l!\ \mathrm{vol}(\overline{Q^\vee}) \prod_{i\in I} a_i^\vee)^{-1}$, and the volume of a fundamental domain for af(T) on $\bar{\mathfrak{h}}_{\mathbb{R}}^*$ is vol(M). Since C_{af} is a fundamental domain for af(W) on $\bar{\mathfrak{h}}_{\mathbb{R}}^*$, a comparison yields

$$|\bar{W}| = |W/T| = l!\ |\bar{P}/M| \prod_{i\in I} a_i^\vee.$$

1.7. *Realizations of Affine Lie Algebras*

(A) Let $\mathfrak{p}$ be a complex reductive Lie algebra, i.e., a direct sum of a semisimple and an abelian Lie algebra. Consider the "loop algebra"

$$\tilde{\mathfrak{p}} := \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{p},$$

a complex Lie algebra with bracket $[\ ,\ ]_\sim$ given by

$$[t^m \otimes a, t^n \otimes b]_\sim = t^{m+n} \otimes [a, b].$$

Let B be a non-degenerate $\mathfrak{p}$-invariant symmetric bilinear form on $\mathfrak{p}$, and extend B to such a form on $\tilde{\mathfrak{p}}$ by

$$B(t^m \otimes a,\ t^n \otimes b) = \delta_{m,-n} B(a, b).$$

We define a Lie algebra

$$\hat{\mathfrak{p}}_B = \tilde{\mathfrak{p}} \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$$

by the following commutation relations:

$$[\hat{\mathfrak{p}}_B, c_0] = 0, \qquad [d_0, x] = t\frac{dx}{dt},$$

$$[x, y] = [x, y]_\sim + B([d_0, x], y)\, c_0$$

for $x, y \in \tilde{\mathfrak{p}}$. Then B extends from $\tilde{\mathfrak{p}}$ to a $\hat{\mathfrak{p}}_B$-invariant non-degenerate symmetric bilinear form on $\hat{\mathfrak{p}}_B$ by

$$B(x, c_0) = B(x, d_0) = 0 \qquad \text{for} \quad x \in \tilde{\mathfrak{p}},$$

$$B(c_0, c_0) = 0 = B(d_0, d_0), \qquad B(c_0, d_0) = 1.$$

Let σ be a finite-order automorphism of $\mathfrak{p}$ preserving B. Fix a positive integer N such that $\sigma^N = I$ and set $\varepsilon = \exp(2\pi i/N)$. Let $\mathfrak{p} = \bigoplus_s \mathfrak{p}_s$ be the corresponding $\mathbb{Z}/N\mathbb{Z}$-gradation, where $\mathfrak{p}_s = \{x \in \mathfrak{p} \mid \sigma(x) = \varepsilon^s x\}$. We extend σ to an automorphism of $\hat{\mathfrak{p}}_B$, preserving B and the subspace $\tilde{\mathfrak{p}}$, by

$$c_0 \mapsto c_0, \qquad d_0 \mapsto d_0,$$

$$t^m \otimes a \mapsto (\varepsilon^{-1} t)^m \otimes \sigma(a).$$

Let $\hat{\mathfrak{p}}_B(\sigma, N)$ and $\tilde{\mathfrak{p}}(\sigma, N)$ denote the subalgebras of $\hat{\mathfrak{p}}_B$ and $\tilde{\mathfrak{p}}$, respectively, fixed by this automorphism. Then: $\tilde{\mathfrak{p}}(\sigma, N) = \bigoplus_{s \in \mathbb{Z}} t^s \otimes \mathfrak{p}_{s \bmod N}$; $\hat{\mathfrak{p}}_B(\sigma, N) = \tilde{\mathfrak{p}}(\sigma, N) \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$; B restricts to a non-degenerate form on $\hat{\mathfrak{p}}_B(\sigma, N)$ and on $\tilde{\mathfrak{p}}(\sigma, N)$.

(B) Now we explain how the construction above gives explicit realizations of all affine Lie algebras (see [14, 17] or the book [50] for details). In the construction above, let $\mathfrak{p}$ be a simple Lie algebra of type X_n and let σ be a finite-order automorphism of $\mathfrak{p}$. Let $\mathfrak{h}_0^\sigma$ be a Cartan subalgebra of the fixed point set $\mathfrak{p}^\sigma$ of σ. Then the centralizer $\mathfrak{h}_0$ of $\mathfrak{h}_0^\sigma$ in $\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{p}$. Put

$$l_\mathfrak{p} = \dim \mathfrak{h}_0, \quad h_\mathfrak{p} = -1 + l_\mathfrak{p}^{-1} \dim \mathfrak{p}.$$

Fix a set of positive roots of $\mathfrak{p}$ with respect to $\mathfrak{h}_0$, let ρ_0 be half their sum and let θ_0 be the highest root. We normalize the invariant form B by $B(\theta_0, \theta_0) = 2$ and set

$$g_\mathfrak{p} = 1 + B(\rho_0, \theta_0).$$

Let k be the least positive integer such that σ^k is an inner automorphism of $\mathfrak{p}$ ($k = 1, 2$ or 3). Then we have:

PROPOSITION 1.11. *Let A be the affine matrix of type $X_n^{(k)}$. Then*

(a) *There exists an isomorphism F: $\mathfrak{g}(A) \simeq \hat{\mathfrak{p}}_B(\sigma, N)$ such that*:

(i) $$F(c) = Nc_0, \qquad F(\mathfrak{h}) = \mathfrak{h}_0^\sigma + \mathbb{C}c_0 + \mathbb{C}d_0,$$
$$[d_0, F(e_i)] = s_i F(e_i), \qquad [d_0, F(f_i)] = -s_i F(f_i)$$

for some non-negative integers s_i, $0 \leqslant i \leqslant l$, satisfying the relation

$$k \sum_{i=0}^{l} a_i s_i = N.$$

(ii) $k(h, h') = B(F(h), F(h'))$ *for* $h, h' \in \mathfrak{g}(A)$.

(b) *Define $\gamma \in \bar{\mathfrak{h}}^*$ by*

$$(\gamma, \alpha_i) = ks_i/N, \qquad 1 \leqslant i \leqslant l.$$

Then $F(t_\gamma(d)) = N^{-1}ka_0 d_0$, where t_γ is defined by (1.6) *and F is as in* (a).

(c) *Let h and g be the Coxeter number and the dual Coxeter number, respectively, of $\mathfrak{g}(A)$. Then: $g_\mathfrak{p} = g$; $h_\mathfrak{p} l_\mathfrak{p} = khl$; $h_\mathfrak{p} = h$ if $k = 1$, and $h_\mathfrak{p} = g$ if $k \neq 1$.*

(d) *Let* $\bar{\rho}' = \frac{1}{2}\sum_{\alpha \in \bar{\Delta}'_+} \alpha$. *Then* $(k/2g)\,|\bar{\rho}'|^2 = (1/24)\dim \mathfrak{p}$.

(e) *Let* $\bar{\rho} = \frac{1}{2}\sum_{\alpha \in \bar{\Delta}_+} \alpha$. *Then*

$$(2g(h_{\mathfrak{p}}+1))^{-1}\,|\bar{\rho}|^2 = \frac{1}{24}\,((k+1)l - l_{\mathfrak{p}}).$$

Moreover, the following "very strange" formula holds:

$$\frac{1}{2kg}\,|\bar{\rho} - g\gamma|^2 = \frac{1}{24}\dim \mathfrak{p} - \frac{1}{4N^2}\sum_{i=1}^{N-1} i(N-i)\dim \mathfrak{p}_i.$$

Proof. The "very strange" formula of (e) is just Proposition 4.12 (by the existence of $\bar{F}$ below). In a different form, it is proved in [17] for "rational" σ; the same proof, using Lemma 3.20, applies to any σ. We give a simpler proof in Section 4.3. The rest of (c), (d) and (e) may be checked case-by-case or deduced from the "very strange" formula. We omit this here.

We now proceed to construct F. For $x \in \mathfrak{g}'(A)$, write $\bar{x}$ for $x + \mathbb{C}c \in \mathfrak{g}'(A)/\mathbb{C}c$. In [15], an isomorphism $\bar{F}$ from $\mathfrak{g}'(A)/\mathbb{C}c$ onto $\tilde{\mathfrak{p}}(\sigma, N)$ is constructed, such that $\bar{F}(\bar{h}_i) \in \mathfrak{h}_0^\sigma$ for all $i \in I$ and such that for some non-negative integers s_i, $i \in I$, with $k\sum_{i \in I} a_i s_i = N$, we have $[d_0, \bar{F}(\bar{e}_i)] = s_i \bar{F}(\bar{e}_i)$ and $[d_0, \bar{F}(\bar{f}_i)] = -s_i \bar{F}(\bar{f}_i)$.

By formula (4.12.2) in Section IV (which is a consequence of (e)), we have

$$\sum_{\alpha \in \Delta^{\mathrm{re}} \bmod k\mathbb{Z}\delta} \alpha(h)^2 = 2kg\,|h|^2$$

for all $h \in \mathfrak{h}_0 := \sum_{i \in I} \mathbb{C}h_i$. On the other hand, it is well-known that

$$\operatorname{tr}(\operatorname{ad}_{\mathfrak{p}} h)^2 = 2g_{\mathfrak{p}} B(h, h)$$

for all $h \in \mathfrak{p}$. Since $g = g_{\mathfrak{p}}$ by (c), a comparison shows that

$$B(\bar{F}(\bar{h}), \bar{F}(\bar{h}')) = k(h, h')$$

for all $h, h' \in \mathfrak{h}_0$.

Let $\pi\colon \mathfrak{g}'(A) \to \mathfrak{g}'(A)/\mathbb{C}c$ be the canonical map and let γ be as in (b). Define a linear map $F\colon \mathfrak{g}(A) \to \hat{\mathfrak{p}}_B(\sigma, N)$ by requiring that F coincides with $\bar{F} \circ \pi$ on $\mathfrak{n}_+$ and on $\mathfrak{n}_-$, that $F(t_\gamma(d)) = N^{-1}ka_0 d_0$, and that $F(h_i) = \bar{F}(\bar{h}_i) + \frac{1}{2}ks_i\,|h_i|^2\,c_0$ for $i \in I$. Using the definition of $\mathfrak{g}(A)$ by generators and relations, it is easy to see that F is a homomorphism. Since $F(c) = Nc_0$ by an easy computation, F is an isomorphism, and it is easy to check that (a) and (b) hold. ∎

Remark. It is not difficult to show that the Coxeter number of Δ' is $(1 + k'^{-1})h_{\mathfrak{p}} - h$. This forces $g \in k'\mathbb{Z}$, and hence $g\overline{Q^\vee} \subset \bar{Q}$. Applied to $\Delta^\vee$, this gives $h\bar{Q} \subset \overline{Q^\vee}$.

1.8. *Appendix* 1: *On Asymptotics of Root Multiplicities*

Here we prove the following:

PROPOSITION 1.12. *Let* $\mathfrak{g}(A)$ *be a Kač–Moody algebra with a symmetrizable Cartan matrix* A, *let* $(,)$ *be a standard form on* $\mathfrak{h}^*$, *and let* $\alpha = \sum_i k_i \alpha_i$ *be a positive imaginary root of* $\mathfrak{g}(A)$. *Set* $\psi(\alpha) = \limsup_{n \to +\infty} n^{-1} \log \operatorname{mult}(n\alpha)$. *Then*:

(a) $\psi(\alpha) = \lim_{n \to +\infty} n^{-1} \log \operatorname{mult}(n\alpha)$; $\psi(\alpha) = \sup_{n \geqslant 1} n^{-1} \log \operatorname{mult}(n\alpha)$ *if* $(\alpha, \alpha) < 0$.

(b) $\psi(\alpha) = 0$ *if* $(\alpha, \alpha) = 0$, *and* $.48 < \psi(\alpha) \leqslant \operatorname{ht}(\alpha) \log \operatorname{ht}(\alpha) - \sum_i k_i \log k_i$ *if* $(\alpha, \alpha) < 0$ (*here* $0 \log 0$ *is interpreted as* 0).

(c) $\psi(n\alpha) = n\psi(\alpha)$ *for* $n > 0$; $\psi(w(\alpha)) = \psi(\alpha)$ *for* $w \in W$.

(d) $\psi(\alpha + \beta) \geqslant \psi(\alpha) + \psi(\beta)$ *if* $\alpha, \beta, \alpha + \beta \in \Delta_+^{\text{im}}$.

LEMMA 1.13. *Let* L *be a free abelian group on generators* $\beta_1, \ldots, \beta_r$, *let* $L_+ = \sum_i \mathbb{Z}_+ \beta_i$, *and let* $J = J_1 \cup \cdots \cup J_r$ *be a disjoint union of non-empty finite sets. Let* $\mathfrak{a} = \bigoplus_{\alpha \in L} \mathfrak{a}_\alpha$ *be a free Lie algebra on generators* e_j $(j \in J)$ *graded by* $\deg e_j = \beta_i$ *for* $j \in J_i$. *For* $\alpha = \sum_i k_i \beta_i \in L_+$, *set* $k = \sum_i k_i$ *and*

$$\psi_0(\alpha) = k \log k - \sum_i k_i \log(k_i / |J_i|).$$

Then one has for all $\alpha \in L_+ \backslash \{0\}$,

$$\lim_{n \to \infty} n^{-1} \log(1 + \dim \mathfrak{a}_{n\alpha}) = \psi_0(\alpha).$$

Proof. Since the universal enveloping algebra of $\mathfrak{a}$ is the free associative algebra on the e_j, we have

$$\prod_\alpha (1 - e^\alpha)^{-\dim \mathfrak{a}_\alpha} = \Bigl(1 - \sum_i |J_i| \, e^{\beta_i}\Bigr)^{-1}.$$

Take the logarithm of both sides and match the coefficients of e^α, obtaining

$$\sum_{n=1}^{\infty} n^{-1} \dim \mathfrak{a}_{\alpha/n} = \frac{(k-1)!}{\prod_i k_i!} \prod_i |J_i|^{k_i}.$$

Stirling's formula completes the proof. ∎

At this point, we need the construction of the Lie algebra $\mathfrak{g}'(A)$ associated to a (possibly infinite) symmetric matrix $A = (a_{ij})_{i,j \in I}$ over $\mathbb{C}$ (see [14, 20] or the book [50] for details). Let $\tilde{\mathfrak{g}}(A)$ be the Lie algebra with generators e_i, f_i, h_i ($i \in I$) and the following defining relations ($i, j \in I$):

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, e_j] = a_{ij} e_j,$$
$$[h_i, f_j] = -a_{ij} f_j, \qquad [h_i, h_j] = 0.$$

Let Q be a free abelian group on an indexed set $\Pi = \{\alpha_i\}_{i \in I}$. Setting $\deg e_i = -\deg f_i = \alpha_i$ ($i \in I$) defines a Q-gradation $\tilde{\mathfrak{g}}(A) = \bigoplus_\alpha \tilde{\mathfrak{g}}_\alpha$. Let $\mathfrak{r}$ be the sum of all graded ideals of $\tilde{\mathfrak{g}}(A)$ intersecting $\tilde{\mathfrak{g}}_0$ trivially. We set

$$\mathfrak{g}'(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}.$$

We have the induced Q-gradation $\mathfrak{g}'(A) = \bigoplus_\alpha \mathfrak{g}_\alpha$. Put $Q_+ = \sum_{i \in I} \mathbb{Z}_+ \alpha_i \subset Q$; for $\alpha \in Q_+ \backslash \{0\}$, we write $\alpha > 0$. Setting $\mathfrak{n}_\pm = \bigoplus_{\alpha > 0} \mathfrak{g}_{\pm\alpha}$ defines the decomposition $\mathfrak{g}'(A) = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+$ (direct sum of vector spaces). The center of $\mathfrak{g}'(A)$ lies in $\mathfrak{g}_0$.

Now let $\mathfrak{g}(A)$ be a Kač–Moody algebra with a symmetrizable Cartan matrix, $(\, ,\,)$ a standard form on $\mathfrak{g}(A)$.

LEMMA 1.14. *Let $L \subset \Delta_+$ satisfy*

(i) $\alpha, \beta \in L \Rightarrow (\alpha, \beta) < 0$,

(ii) $\alpha, \beta \in L, \alpha - \beta \in \Delta_+ \Rightarrow \alpha - \beta \in L$.

Let $\mathfrak{n}_L^+$ (resp. $\mathfrak{n}_L^-$) be the subalgebra of $\mathfrak{g}(A)$ generated by $\bigoplus_{\alpha \in L} \mathfrak{g}_\alpha$ (resp. $\bigoplus_{\alpha \in L} \mathfrak{g}_{-\alpha}$). Then $\mathfrak{n}_L^+$ is a free Lie algebra on a basis of the space $\mathfrak{n}_L^+ \cap [\mathfrak{n}_L^-, \mathfrak{n}_L^-]^\perp$.

Proof. We may assume that L is finite. By induction on $|L|$ we prove simultaneously using [20, Corollary 1] that:

(a) For each $\alpha \in L$, there exist bases $I_\alpha^\pm = \{x_{\alpha,\pm}^{(1)}, \ldots, x_{\alpha,\pm}^{(d_\alpha)}\}$ of $\mathfrak{g}_{\pm\alpha} \cap [\mathfrak{n}_L^\mp, \mathfrak{n}_L^\mp]^\perp$ which are dual under $(\, ,\,)$.

(b) Put $I^\pm = \bigcup_{\alpha \in L} I_\alpha^\pm$. Then $I^\pm$ generates $\mathfrak{n}_L^\pm$. Moreover, we have: $[x_{\alpha,+}^{(i)}, x_{\beta,-}^{(j)}] = \delta_{\alpha\beta} \delta_{ij} \alpha$.

(c) Put $B = (b_{ij})_{i,j \in I^+}$, where $b_{ij} = (\alpha, \beta)$ if $i \in I_\alpha^+$, $j \in I_\beta^+$. Then $\mathfrak{g}_L := \mathfrak{n}_L^- \oplus \sum_{\alpha \in L} \mathbb{C}\alpha \oplus \mathfrak{n}_L^+$ is isomorphic in the obvious way to a quotient of $\mathfrak{g}'(B)$ by a central ideal.

(d) $\mathfrak{n}_L^+$ and $\mathfrak{n}_L^-$ are non-degenerately paired by $(\, ,\,)$.

(e) The statement of the lemma. ∎

We recall some useful notions. For $\alpha = \sum k_i \alpha_i \in Q$, put $\operatorname{supp} \alpha = \{i \in I \mid k_i \neq 0\}$. We say that a subset J of I is *connected* if, whenever J is the disjoint union of J_0 and J_1, and $a_{ij} = 0$ for all $i \in J_0$ and $j \in J_1$, then $J_0 = \varnothing$ or $J_1 = \varnothing$. It is easy to see that if $\alpha \in \Delta$, then $\operatorname{supp} \alpha$ is connected.

LEMMA 1.15. *If $\alpha \in \Delta_+^{\text{im}}$ is non-isotropic and $\alpha(h_i) \leqslant 0$ for all $i \in I$, and if $\beta \in \Delta_+$ is such that $\alpha \geqslant \beta$, then* mult $\alpha \geqslant$ mult β.

Proof. Using the argument of [19, Lemma 1.6], one shows that $\beta = \alpha$ or else $\beta(h_i) < 0$ for some $i \in \text{supp}(\alpha - \beta)$. In the second case, $\beta + \alpha_i \in \Delta_+$, $\alpha \geqslant \beta + \alpha_i$ and $\text{mult}(\beta + \alpha_i) \geqslant \text{mult}(\beta)$. The result follows by downward induction on $\text{ht}(\beta)$. ∎

LEMMA 1.16. *If α, β, $\alpha + \beta \in \Delta_+^{\text{im}}$ and $(\beta, \beta) < 0$, then $(\alpha, \beta) < 0$.*

Proof. By Proposition 2.4(b), we may assume that $\beta(h_i) \leqslant 0$ $(i \in I)$, so that $(\alpha, \beta) \leqslant 0$. Suppose that $(\alpha, \beta) = 0$. Since $\text{supp}(\alpha + \beta)$ is connected, we deduce that $\text{supp}(\alpha) \subset \text{supp}(\beta)$. Choose $w \in W$ such that $w(\alpha)(h_i) \leqslant 0$ $(i \in I)$. Then as above, $\text{supp}\, w(\beta) \subset \text{supp}\, w(\alpha)$, so by Proposition 1.3(c) applied to $\Delta^\vee$, we obtain

$$\text{supp}\, \beta \subset \text{supp}\, w(\beta) \subset \text{supp}\, w(\alpha) \subset \text{supp}\, \alpha.$$

Hence $\text{supp}(\beta) = \text{supp}(\alpha)$, and since also $(\alpha, \beta) = 0$ and $\beta(h_i) \leqslant 0$ $(i \in I)$, we have $(\beta, \beta) = 0$, a contradiction. ∎

Proof of Proposition 1.12. If $(\alpha, \alpha) = 0$, then $1 \leqslant \text{mult}\, j\alpha < |I|$ for $j = 1, 2, \ldots$ (see Section 1.1(G)), so that (a) and (b) are clear. We now check (a) and (b) for $(\alpha, \alpha) < 0$. For a positive integer m, let $L = \{m\alpha\}$. Then using Lemmas 1.13 and 1.14 we have

$$\lim_{j \to +\infty} \inf((mj)^{-1} \log \text{mult}(mj\alpha)) \geqslant m^{-1} \log \text{mult}(m\alpha). \qquad (1.12.1)$$

But by Lemma 1.15,

$$\text{mult}((j+1)\alpha) \geqslant \text{mult}(j\alpha) \geqslant 1 \qquad \text{for} \quad j \geqslant 1.$$

It is easy to deduce from this that

$$\lim_{j \to +\infty} \inf (mj)^{-1} \log \text{mult}(mj\alpha) = \lim_{j \to +\infty} \inf j^{-1} \log \text{mult}(j\alpha).$$

From this and (1.12.1) we deduce (a).

Since mult $\alpha \geqslant 1$ and mult $2\alpha \geqslant 1$, a computation using Lemmas 1.13 and 1.14 gives $\psi(\alpha) > .48$. The rest of (b) follows from Lemma 1.13.

(c) is clear from (a).

Let α and β be as in (d). If α or β is isotropic, (d) follows from (b) and Lemma 1.15. If α and β are proportional, (d) follows from (c). Otherwise, by Lemma 1.16, Lemma 1.14 applies to $L := \{m\, \text{ht}(\beta)\alpha,\ m\, \text{ht}(\alpha)\beta\}$ for each positive integer m. Apply Lemma 1.13 to $\mathfrak{n}_L^+$, with $\beta_1 = m\, \text{ht}(\beta)\alpha$,

$\beta_2 = m\,\mathrm{ht}(\alpha)\beta$, $|J_1| = \mathrm{mult}\,\beta_1$, $|J_2| = \mathrm{mult}\,\beta_2$, $k_1 = \mathrm{ht}\,\alpha$, $k_2 = \mathrm{ht}\,\beta$, $\gamma = k_1\beta_1 + k_2\beta_2 = m(\mathrm{ht}\,\alpha)(\mathrm{ht}\,\beta)(\alpha+\beta)$. Then we obtain

$$\begin{aligned} m(\mathrm{ht}\,\alpha)(\mathrm{ht}\,\beta)\,\psi(\alpha+\beta) = \psi(\gamma) \geqslant \psi_0(\gamma) \\ &= \mathrm{ht}(\alpha+\beta)\log \mathrm{ht}(\alpha+\beta) - (\mathrm{ht}\,\alpha)\log \mathrm{ht}\,\alpha \\ &\quad - (\mathrm{ht}\,\beta)\log \mathrm{ht}\,\beta \\ &\quad + (\mathrm{ht}\,\alpha)\log \mathrm{mult}(m\,\mathrm{ht}(\beta)\alpha) \\ &\quad + (\mathrm{ht}\,\beta)\log \mathrm{mult}(m\,\mathrm{ht}(\alpha)\beta). \end{aligned}$$

We divide both sides by $m(\mathrm{ht}\,\alpha)(\mathrm{ht}\,\beta)$ and let $m \to \infty$, obtaining (d). ∎

Remarks. (a) If A is indecomposable, it follows from Proposition 1.12 that ψ extends uniquely to a concave function on the interior of the imaginary cone Z such that $\psi(t\alpha) = t\psi(\alpha)$ for $t > 0$.

(b) For a free Lie algebra $\mathfrak{a}$ on N generators $e_1,\ldots,e_N$ of linearly independent degrees $\alpha_1,\ldots,\alpha_N$, and $\alpha = \sum_i k_i\alpha_i$ with all $k_i > 0$, we have

$$\dim \mathfrak{a}_{n\alpha} \sim C(\alpha)\, n^{-(N+1)/2}\, e^{n\psi_0(\alpha)} \qquad \text{as} \quad n \to \infty,$$

where $C(\alpha) = (2\pi)^{(1-N)/2}\left(\sum_i k_i\right)^{-1/2}\prod_i k_i^{-1/2}$. This and other evidence suggests the following conjecture:

Under the hypotheses of Proposition 1.12, provided that A is indecomposable and α lies in the interior of the imaginary cone Z, there exists $C(\alpha) > 0$ such that

$$\mathrm{mult}(n\alpha) \sim C(\alpha)\, n^{-(|I|+1)/2} e^{n\psi(\alpha)} \qquad \text{as} \quad n \to \infty.$$

II. Highest Weight Representations

In Section II we describe the structure of the weight system of an irreducible highest weight representation with dominant integral highest weight of an affine Lie algebra and present the character formula obtained in [16]. Then we use the decomposition of the Weyl group to express the character of such a representation as a finite sum of classical theta functions with coefficients called string functions [18, 22]. This gives us the theta function identity (2.18), which is the basic fact for the theory of string functions which we develop in Section IV.

2.1. *Basic Facts about Irreducible Highest Weight Modules over Kač–Moody Algebras*

(A) Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kač–Moody algebra. Recall the decomposition

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (Section 1.1(C)). Then for any $\Lambda \in \mathfrak{h}^*$, there exists an irreducible $\mathfrak{g}(A)$-module $L(\Lambda)$, unique up to isomorphism, satisfying:

There exists a non-zero vector $v_\Lambda \in L(\Lambda)$ such that $\mathfrak{n}_+(v_\Lambda) = 0$ and $h(v_\Lambda) = \Lambda(h)\, v_\Lambda$ for all $h \in \mathfrak{h}$. (L1)

$L(\Lambda)$ is called the *irreducible highest weight module* with *highest weight* Λ [16].

We shall sometimes describe Λ by its *labels* $\Lambda(h_i)$, $i \in I$. If Λ, Λ' have the same labels, they may differ only off $\sum_i \mathbb{C}h_i$; however, then $L(\Lambda)$ and $L(\Lambda')$ are isomorphic as (irreducible) $\mathfrak{g}'(A)$-modules, and the actions of elements of $\mathfrak{g}(A)$ on them differ only by scalar operators. Note that

$$\dim L(\Lambda) = 1 \qquad \text{if and only if } \Lambda(h_i) = 0 \text{ for all } i \in I.$$

One has the *weight space decomposition* of $L(\Lambda)$ with respect to $\mathfrak{h}$:

$$L(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda,$$

where $L(\Lambda)_\lambda := \{v \in L(\Lambda) \mid h(v) = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$.

(B) Consider the formal expansion

$$\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\operatorname{mult}\alpha} = \sum_{\beta \in \mathfrak{h}^*} K(\beta)\, e^{-\beta}, \tag{2.1}$$

defining a function K on $\mathfrak{h}^*$ called the *partition function*. As $(1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots$, $K(\beta)$ is the number of partitions of β into a sum of positive roots, where each root is counted with its multiplicity. Since v_Λ is a cyclic vector for the $\mathfrak{n}_-$-module $L(\Lambda)$ (i.e., no proper $\mathfrak{n}_-$-submodule of $L(\Lambda)$ contains v_Λ), we find that

$$\dim L(\Lambda)_\lambda \leqslant K(\Lambda - \lambda) \qquad \text{for} \quad \lambda \in \mathfrak{h}^*. \tag{2.2}$$

In particular, $L(\Lambda)_\Lambda = \mathbb{C}v_\Lambda$ and $\dim L(\Lambda)_\lambda$ is finite for all $\lambda \in \mathfrak{h}^*$. If $L(\Lambda)_\lambda \neq 0$, then λ is called a *weight* of $L(\Lambda)$ of *multiplicity* $\dim L(\Lambda)_\lambda$; we write $\operatorname{mult}_\Lambda(\lambda) := \dim L(\Lambda)_\lambda$. We denote by $P(\Lambda)$ the set of weights of $L(\Lambda)$. It follows from the irreducibility of $L(\Lambda)$ that if $\lambda \in P(\Lambda)\backslash\{\Lambda\}$, then $e_i(L(\Lambda)_\lambda) \neq 0$ for some $i \in I$ and hence $\lambda + \alpha_i \in P(\Lambda)$ for some $i \in I$.

As v_Λ is a cyclic vector for the $\mathfrak{n}_-$-module $L(\Lambda)$, we have

$$P(\Lambda) \subset \Lambda - Q_+.$$

(C) For $\lambda \in \mathfrak{h}^*$, define a function e^λ on $\mathfrak{h}$ by $e^\lambda(h) = e^{\lambda(h)}$. We define the *character* $\operatorname{ch}_{L(\Lambda)}$ of $L(\Lambda)$ to be the function

$$h \mapsto \operatorname{ch}_{L(\Lambda)}(h) = \sum_{\lambda \in \mathfrak{h}^*} \operatorname{mult}_\Lambda(\lambda)\, e^{\lambda(h)}$$

defined on the set, which we denote by Y_Λ, of all $h \in \mathfrak{h}$ such that the series converges absolutely.

LEMMA 2.3. *Let $\Lambda \in \mathfrak{h}^*$. Then:*

(a) *Y_Λ is convex and contains the set of all $h \in \mathfrak{h}$ satisfying the following two conditions:*

(i) $\operatorname{Re} \alpha_i(h) > 0$ *for all* $i \in I$,

(ii) $\sum_{\alpha \in \Delta_+} (\operatorname{mult} \alpha) |e^{-\alpha(h)}| < \infty$.

(b) $Y_\Lambda \supset \{h \in \mathfrak{h} \mid \operatorname{Re} \alpha_i(h) > \log |I|$ *for all* $i \in I\}$.

(c) *If $\mathfrak{g}$ is an affine Lie algebra, then*

$$Y_\Lambda \supset \{h \in \mathfrak{h} \mid \operatorname{Re} \alpha_i(h) > 0 \text{ for all } i \in I\}.$$

Proof. The convexity is clear from the convexity of $|e^\lambda|$. From (2.2) we obtain, for $h \in \mathfrak{h}$,

$$\sum_{\lambda \in \mathfrak{h}^*} \operatorname{mult}_\Lambda(\lambda) |e^{\lambda(h)}| \leqslant |e^{\Lambda(h)}| \sum_{\beta \in Q_+} K(\beta) |e^{-\beta(h)}|.$$

But (2.1) implies, for h satisfying (i):

$$\sum_{\beta \in Q_+} K(\beta) |e^{-\beta(h)}| = \prod_{\alpha \in \Delta_+} (1 - |e^{-\alpha(h)}|)^{-\operatorname{mult} \alpha}.$$

This product converges if h satisfies (ii), which proves (a). (b) follows from (a) by the easy estimate $\operatorname{mult} \alpha \leqslant |I|^{\operatorname{ht} \alpha}$. Finally, (c) follows from (a) since for an affine Lie algebra root multiplicities are bounded by $|I|$. ∎

We remark that using the convexity of $|e^\lambda|$, the absolute convergence is uniform on compact subsets of the interior of Y_Λ, and hence $\operatorname{ch}_{L(\Lambda)}$ is holomorphic on the interior of Y_Λ.

(D) We call $\lambda \in \mathfrak{h}^*$ an *integral weight* if $\lambda(h_i)$ is integral for all $i \in I$; an integral weight λ is called *dominant* if $\lambda(h_i) \geqslant 0$ for all $i \in I$, and *regular dominant* if $\lambda(h_i) > 0$ for all $i \in I$. Let P (respectively, P_+, P_{++}) be the sets of integral (respectively, dominant integral, regular dominant integral) weights. Note that $P \supset Q$ and that any coset of P mod Q is W-invariant. Fix $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i \in I$. Note that $P_{++} = \rho + P_+$.

PROPOSITION 2.4. *Let $\Lambda \in P_+$ and let $\lambda, \mu \in P(\Lambda)$ be weights of the $\mathfrak{g}(A)$-module $L(\Lambda)$.*

(a) *Let $\alpha \in \Delta^{\mathrm{re}}$. If $\lambda(\alpha^\vee) < 0$, then for any non-zero $x \in \mathfrak{g}_\alpha$, the map $x^{-\lambda(\alpha^\vee)}: L(\Lambda)_\lambda \to L(\Lambda)_{r_\alpha(\lambda)}$ is an isomorphism.*

(b) *Let $w \in W$. Then there exist $\tilde{w} \in \operatorname{Aut} \mathfrak{g}$ and $\hat{w} \in GL(L(\Lambda))$ such that $\tilde{w}(\mathfrak{h}) = \mathfrak{h}$, $\tilde{w}|_{\mathfrak{h}} = w$, and $\hat{w}(x(v)) = \tilde{w}(x)(\hat{w}(v))$ for all $x \in \mathfrak{g}$ and $v \in L(\Lambda)$;*

in particular, $\tilde{w}(\mathfrak{g}_\alpha) = \mathfrak{g}_{w(\alpha)}$ *for all* $\alpha \in \mathfrak{h}^*$, *and* $\hat{w}(L(\Lambda)_\lambda) = L(\Lambda)_{w(\lambda)}$. *We may take* $\tilde{r}_i$ *(resp.* $\hat{r}_i$*) to be* $(\exp e_i)$ $(\exp -f_i)$ $(\exp e_i)$, *regarded as an operator on* $\mathfrak{g}$ *(resp.* $L(\Lambda)$*). Moreover,* $\Delta_+^{\mathrm{im}} \subset -W(P_+)$ *and* $P(\Lambda) \subset W(P_+)$.

(c) *λ lies in the convex hull of* $W(\Lambda)$.

(d) *Suppose that A is symmetrizable and* $(\, , \,)$ *is a standard form on* $\mathfrak{g}(A)$. *Then:*

(i) $(\lambda, \mu) \leqslant |\Lambda|^2$, *with equality if and only if* $\lambda = \mu \in W(\Lambda)$.

(ii) $|\lambda + \rho|^2 \leqslant |\Lambda + \rho|^2$, *with equality if and only if* $\lambda = \Lambda$.

(e) *The set of asymptotic rays for the set* $-P(\Lambda)$ *is contained in the imaginary cone Z.*

Proof. The proof of (a)–(d) requires some minor modifications of that for the classical finite-dimensional case. For (a) and (b) see [14] or the book [50].

(c) is proved by induction on $ht(\Lambda - \lambda)$. If $\lambda = \Lambda$, there is nothing to prove. If $\lambda \neq \Lambda$, choose $i \in I$ such that $\lambda + \alpha_i \in P(\Lambda)$, and let $\mu \in P(\Lambda)$ be such that $\mu = \lambda + s\alpha_i$, $s \geqslant 1$, $\mu + \alpha_i \notin P(\Lambda)$. Then μ (and hence also $r_i(\mu)$) lies in the convex hull of $W(\Lambda)$ by the inductive assumption. But λ lies in the interval $[\mu, r_i(\mu)]$ and hence λ also lies in the convex hull of $W(\Lambda)$.

Since both $P(\Lambda)$ and $(\, , \,)$ are W-invariant, we can assume by (b) that $\lambda \in P_+$ in the proof of $d(i)$. But $\beta := \Lambda - \lambda \in Q_+$ and $\beta_1 := \Lambda - \mu \in Q_+$, so: $(\Lambda, \Lambda) - (\lambda, \mu) = (\beta, \Lambda) + (\beta_1, \lambda) \geqslant 0$. If we have equality, then $(\Lambda, \beta) = (\lambda, \beta_1) = 0$. But $\Lambda \in P_+$, $\beta \in Q_+$, $(\Lambda, \beta) = 0$ and $\Lambda - \beta \in P(\Lambda)$ imply $\beta = 0$. Hence, $\Lambda = \lambda$ and so $(\Lambda, \beta_1) = 0$. By the same argument, $\beta_1 = 0$, proving d(i). For d(ii), we have

$$\begin{aligned}(\lambda + \rho, \lambda + \rho) &= (\lambda, \lambda) + 2(\lambda, \rho) + (\rho, \rho) \\ &\leqslant (\Lambda, \Lambda) + 2(\lambda, \rho) + (\rho, \rho) \\ &= (\Lambda + \rho, \Lambda + \rho) - 2(\beta, \rho) \leqslant (\Lambda + \rho, \Lambda + \rho),\end{aligned}$$

and the equality holds only if $\beta = 0$.

Since $P(\Lambda) \subset \bigcap_{w \in W} w(\Lambda - Q_+)$, (e) follows from Proposition 1.3(f). ∎

Now we can prove an important result about the region of convergence of $\mathrm{ch}_{L(\Lambda)}$.

PROPOSITION 2.5. *Let* $\Lambda \in P_+$ *and let* Y_Λ *be the region of absolute convergence of* $\mathrm{ch}_{L(\Lambda)}$. *Then:*

(a) *Y_Λ is a convex W-invariant set, which for any $y \in Y$ contains ty for sufficiently large $t \in \mathbb{R}_+$.*

(b) $Y_\Lambda \supset Y' :=$ Interior$\{h \in \mathfrak{h} \mid \sum_{\alpha \in \Delta_+} (\text{mult}\, \alpha) |e^{-\alpha(h)}| < \infty\}$.

(c) *If $\mathfrak{g}$ is an affine Lie algebra and* $\dim L(\Lambda) \neq 1$, *then*

$$Y_\Lambda = Y (= \{h \in \mathfrak{h} \mid \text{Re}\, \delta(h) > 0\}).$$

Proof. First, we prove (b). Set $C' = \{h \in \mathfrak{h} \mid \text{Re}\, \alpha_i(h) > 0 \text{ for all } i \in I\}$. Then by Lemma 2.3(a), $Y' \cap C' \subset Y_\Lambda$. It is clear that Y' is convex and W-invariant. It follows from Lemma 2.3(a) and Proposition 2.4(b) that the same is true for Y_Λ. Hence it remains only to show that the convex hull of $W(Y' \cap C')$ contains Y'. Indeed, obviously, $Y' \subset Y$. Since the union of the "walls," say $R := \{h \in \mathfrak{h} \mid \text{Re}\, \alpha(h) = 0 \text{ for some } \alpha \in \Delta^{\text{re}}\}$, is nowhere dense in $\mathfrak{h}$, and since Y' is open, Y' is contained in the convex hull of $Y' \backslash R = W(Y' \cap C')$. This proves (b).

Consider $Y'' := \{y \in Y \mid ty \in Y_\Lambda \text{ for sufficiently large } t \in \mathbb{R}_+\}$. Y'' is W-invariant and convex; it contains C' by Lemma 2.3(b). An argument as in the proof of (b) gives $Y'' = Y$, proving (a).

The inclusion $Y_\Lambda \supset Y$ in (c) follows from (b) by the structure of the root system. The reverse inclusion follows from Proposition 2.11(a) below. ∎

Remark. Suppose that A is indecomposable and symmetrizable, and $\Lambda \in P_+$ is such that $\Lambda(h_i) \neq 0$ for some $i \in I$. Then Y_Λ is open and:

$$Y_\Lambda = \left\{ h \in \mathfrak{h} \;\middle|\; \sum_{\alpha \in \Delta_+} (\text{mult}\, \alpha) |e^{-\alpha(h)}| < \infty \right\}.$$

We prove this fact in Appendix 2 (Section 2.4).

(E) Assume that the Cartan matrix A of $\mathfrak{g}$ is symmetrizable. Then one knows the following character and denominator formulas [16]:

$$\left(\sum_{w \in W} (\det w)\, e^{w(\rho)} \right) \text{ch}_{L(\Lambda)} = \sum_{w \in W} (\det w)\, e^{w(\Lambda + \rho)}, \tag{2.6}$$

$$\sum_{w \in W} (\det w)\, e^{w(\rho)} = e^{\rho} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}\, \alpha}. \tag{2.7}$$

Recall also the following multiplicity formula, which is a formal consequence of the latter two formulas [16]:

$$\text{mult}_\Lambda(\lambda) = \sum_{w \in W} (\det w)\, K(w(\Lambda + \rho) - (\lambda + \rho)). \tag{2.8}$$

Remark. Formula (2.6) implies the following "star" formula (which enables one to compute the multiplicities of weights inductively):

$$\sum_{w \in W} (\det w) \operatorname{mult}_\Lambda(\lambda + \rho - w(\rho)) = 0 \qquad \text{if} \quad \lambda \in P(\Lambda)\backslash\{\Lambda\}.$$

Indeed, we equate the coefficients of $e^{\lambda+\rho}$ and use the fact that $w(\lambda+\rho) \neq \Lambda + \rho$ for $w \in W$, since $(\Lambda - w(\lambda)) + (\rho - w(\rho)) \geqslant 0$, and $\rho = w(\rho) \Rightarrow w = 1$.

(F) We recall (in a modified version) a generalization of the Weyl complete reducibility theorem to the case of Kač–Moody algebras [17, Proposition 2.8].

PROPOSITION 2.9. *Let $\mathfrak{g}'$ be the derived algebra of a Kac–Moody algebra $\mathfrak{g}$ with symmetrizable Cartan matrix. Let V be a $\mathfrak{g}'$-module satisfying*:

(i) *If $v \in V$, then $\mathfrak{n}_+^k(v) = (0)$ for some $k > 0$.*

(ii) *If $v \in V$ and $i \in I$, then $f_i^k(v) = 0$ for some $k > 0$.*

Then V is isomorphic, as a $\mathfrak{g}'$-module, to a direct sum of irreducible $\mathfrak{g}'$-modules $L(\Lambda)$ with dominant integral highest weights Λ.

Proof. is a corrected and modified version of that in [17]. Recall the algebra gradation $U(\mathfrak{g}') = \oplus_{\beta \in Q} U(\mathfrak{g}')_\beta$, with $e_i \in U(\mathfrak{g}')_{\alpha_i}$ and $f_i \in U(\mathfrak{g}')_{-\alpha_i}$. Since I is finite, (i) implies:

If $v \in V$, then $U(\mathfrak{n}_+)_\beta v = (0)$ for all but a finite number of $\beta \in Q_+$. (2.9.1)

For $\lambda \in \mathfrak{h}'^*$, put $V_\lambda = \{v \in V \mid h(v) = \lambda(h)v$ for all $h \in \mathfrak{h}'\}$. For $i \in I$, e_i and f_i act locally-nilpotently on V by (i) and (ii), so that $\dim U(\mathbb{C}f_i + \mathbb{C}h_i + \mathbb{C}e_i)v < \infty$ for all $v \in V$ by Lemma 2.9.14 below. Since I is finite and $\mathbb{C}f_i + \mathbb{C}h_i + \mathbb{C}e_i \cong \mathfrak{sl}_2(\mathbb{C})$, a standard argument proves:

$$V = \bigoplus_{\lambda \in \mathfrak{h}'^*} V_\lambda. \tag{2.9.2}$$

If $\lambda \in \mathfrak{h}'^*$, $v \in V_\lambda$, $v \neq 0$ and $i \in I$, then $\lambda(h_i) \in \mathbb{Z}$, and moreover, $e_i^{-\lambda(h_i)}(v) \neq 0$ if $\lambda(h_i) < 0$. (2.9.3)

Our objective is to prove (2.9.13) below. Let (,) be a standard form on $\mathfrak{g}$, and let ν: $\mathfrak{h} \to \mathfrak{h}^*$ be the vector space isomorphism induced by (,). For $\beta \in Q$ and $\lambda \in \mathfrak{h}'^*$, put $\bar{\beta} = \beta|_{\mathfrak{h}'}$ and $F(\beta, \lambda) = \lambda(\nu^{-1}(\beta)) + \frac{1}{2}(\beta + 2\rho)(\nu^{-1}(\beta))$. Put $V^{\mathfrak{n}_+} = \{v \in V \mid \mathfrak{n}_+(v) = (0)\}$. We will need:

If $\lambda \in \mathfrak{h}'^*$, $v \in V_\lambda$, $i \in I$ and $e_i(v) \neq 0$, then $e_i^n(v) \neq 0$ and $F(n\alpha_i, \lambda) > 0$ for some positive integer n. (2.9.4)

If $\beta \in Q_+\backslash\{0\}$, $\lambda \in \mathfrak{h}'^*$, $V_\lambda \cap V^{\mathfrak{n}_+} \neq (0)$ and $V_{\lambda-\bar{\beta}} \cap V^{\mathfrak{n}_+} \neq (0)$, then $F(-\beta, \lambda) < 0$. (2.9.5)

The statement (2.9.4) follows from (2.9.3) by taking $n = 1$ if $\lambda(h_i) \geqslant 0$ and $n = -\lambda(h_i)$ if $\lambda(h_i) < 0$. If λ and β are as in (2.9.5), then $\lambda(\nu^{-1}(\beta)) \geqslant 0$ and $(\lambda - \bar{\beta})(\nu^{-1}(\beta)) \geqslant 0$ by (2.9.3), and $\rho(\nu^{-1}(\beta)) > 0$ since $\beta \in Q_+\backslash\{0\}$, proving (2.9.5). We now deduce:

If $\lambda \in \mathfrak{h}'^*$, $v \in V_\lambda$ and $v \notin V^{\mathfrak{n}_+}$, then there exists $\beta \in Q$ such that $U(\mathfrak{g}')_\beta v \cap V^{\mathfrak{n}_+} \neq (0)$ and $F(\beta, \lambda) \neq 0$. (2.9.6)

If $\beta \in Q_+\backslash\{0\}$, $\lambda \in \mathfrak{h}'^*$, $v \in V_\lambda \cap V^{\mathfrak{n}_+}$ and $U(\mathfrak{n}_-)_{-\beta} v \cap V^{\mathfrak{n}_+} \neq (0)$, then $F(-\beta, \lambda) \neq 0$. (2.9.7)

By (2.9.1), the statement (2.9.6) follows by repeated application of (2.9.4), using the identity $F(\beta, \lambda) + F(\beta', \lambda + \bar{\beta}) = F(\beta + \beta', \lambda)$. The statement (2.9.7) is immediate from (2.9.5).

For $\alpha \in \Delta_+$, choose bases $\{e_\alpha^{(i)}\}$ of $\mathfrak{g}_\alpha$ and $\{e_{-\alpha}^{(i)}\}$ of $\mathfrak{g}_{-\alpha}$, dual under (,). Following [16], we define the "partial Casimir operator" Ω_0 on V by

$$\Omega_0(v) = \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)}(e_\alpha^{(i)}(v)).$$

Ω_0 is well-defined by (2.9.1), and clearly commutes with $\mathfrak{h}'$ on V. For $a \in \mathbb{C}$, put

$$V^a = \{v \in V \mid (\Omega_0 - aI_V)^k v = 0 \text{ for some } k > 0\}.$$

We have, since V is $U(\mathfrak{g}')_0$-finite by (2.9.1) and (2.9.2), and $\mathbb{C}$ is algebraically closed:

$$V = \bigoplus_{a \in \mathbb{C}} V^a. \qquad \Omega_0 \text{ and } \mathfrak{h}' \text{ commute on } V. \tag{2.9.8}$$

We will need:

If V' is a $\mathfrak{g}'$-submodule of V, $v \in V$ and $\mathfrak{n}_+(v) \subset V'$, then $v \in V' + V^0$. (2.9.9)

If $\beta \in Q$, $\lambda \in \mathfrak{h}'^*$ and $a \in \mathbb{C}$, then $U(\mathfrak{g}')_\beta(V_\lambda \cap V^a) \subset V_{\lambda+\bar{\beta}} \cap V^{a-F(\beta,\lambda)}$. (2.9.10)

Indeed, let V' and v be as in (2.9.9), and suppose $v \notin V'$. By (2.9.8), choose a non-zero polynomial p such that $p(\Omega_0)v = 0$. Since $\Omega_0(v) \in U(\mathfrak{g}')\,\mathfrak{n}_+(v) \subset V'$ and hence $\Omega_0^r(v) \in V'$ for $r = 1, 2,\ldots$, $v \notin V'$ forces $p(0) = 0$. Write $p(X) = X^r q(X)$, where $r \geqslant 1$ and $q(0) \neq 0$. Then $q(0)v = (q(0)v - q(\Omega_0)(v)) + q(\Omega_0)(v) \in V' + V^0$, proving (2.9.9).

As in [16, 17], for $x = e_i$ or f_i we have on V: $x\Omega_0 - \Omega_0 x = [x, e_{-\alpha_i} e_{\alpha_i}]$. It is easy to deduce from this that, for $\beta \in Q$ and $u \in U(\mathfrak{g}')_\beta$, we have on V:

$$u\Omega_0 - \Omega_0 u = u(v^{-1}(\beta) + \tfrac{1}{2}(\beta + 2\rho)(v^{-1}(\beta))I_V).$$

Statement (2.9.10) follows.

We can now deduce:

$$V^{\mathfrak{n}_+} = V^0. \tag{2.9.11}$$

$$\mathfrak{n}_- U(\mathfrak{n}_-)V^{\mathfrak{n}_+} \subset \bigoplus_{a \neq 0} V^a. \tag{2.9.12}$$

$$V = V^{\mathfrak{n}_+} \oplus \mathfrak{n}_- U(\mathfrak{n}_-)V^{\mathfrak{n}_+}. \tag{2.9.13}$$

By (2.9.9) for $V' = (0)$, $V^{\mathfrak{n}_+} \subset V^0$. Using this, (2.9.6) and (2.9.10) imply that $V_\lambda \cap V^0 \subset V^{\mathfrak{n}_+}$ for all $\lambda \in \mathfrak{h}'^*$. Using (2.9.2) and (2.9.8), (2.9.11) follows. By (2.9.7), (2.9.10) and (2.9.11), $U(\mathfrak{n}_-)_{-\beta}(V^{\mathfrak{n}_+} \cap V_\lambda) \subset \oplus_{a \neq 0} V^a$ for all $\beta \in Q_+ \backslash \{0\}$ and $\lambda \in \mathfrak{h}'^*$. Using (2.9.2), the statement (2.9.12) follows. By (2.9.11) and (2.9.12), $V^{\mathfrak{n}_+} \cap \mathfrak{n}_- U(\mathfrak{n}_-)V^{\mathfrak{n}_+} = (0)$. Using (2.9.1), (2.9.9) applied to $V' = U(\mathfrak{g}')V^0$ gives $V = U(\mathfrak{g}')V^0$. Hence, by (2.9.11), $V = U(\mathfrak{g}')V^{\mathfrak{n}_+} = U(\mathfrak{n}_-)\, U(\mathfrak{h}' + \mathfrak{n}_+)\, V^{\mathfrak{n}_+} = U(\mathfrak{n}_-)\, V^{\mathfrak{n}_+} = V^{\mathfrak{n}_+} + \mathfrak{n}_- U(\mathfrak{n}_-)\, V^{\mathfrak{n}_+}$. This proves (2.9.13).

Finally, (2.9.13) and Lemma 2.9.16 below show that, if $\lambda \in \mathfrak{h}'^*$, $v \in V^{\mathfrak{n}_+} \cap V_\lambda$, $v \neq 0$, and if $\Lambda \in \mathfrak{h}^*$ satisfies $\Lambda(h_i) = \lambda(h_i)$, $i \in I$, then $U(\mathfrak{g}')v$ is isomorphic to the irreducible $\mathfrak{g}'$-module $L(\Lambda)$. Hence, by (2.9.2), V is isomorphic as a $\mathfrak{g}'$-module to a direct sum of modules $L(\Lambda)$; these Λ are dominant integral by (2.9.3). ∎

LEMMA 2.9.14. *Let $\mathfrak{a}$ be a Lie algebra over a field F of characteristic* 0, *and let V be an $\mathfrak{a}$-module. Then the span of $\{a \in \mathfrak{a} \mid$ for all $b \in \mathfrak{a}$ and $v \in V$, $(\operatorname{ad} a)^k\, b = 0$ and $a^k(v) = 0$ for some $k > 0\}$ is a subalgebra of $\mathfrak{a}$.*

Proof. If a, b lie in the set in question, then so does $(\exp(\operatorname{ad} ta))(b)$ for all $t \in F$. ∎

LEMMA 2.9.15. *Let $\mathfrak{a} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}_n$ be a $\mathbb{Z}$-graded Lie algebra over an arbitrary field, and put $\mathfrak{a}_\pm = \bigoplus_{n \geqslant 1} \mathfrak{a}_{\pm n}$. Let $\mathrm{Mod}_{\mathfrak{a}}^+$ be the category of all $\mathfrak{a}$-modules V satisfying $V = V^+ \oplus \mathfrak{a}_- U(\mathfrak{a}_-)V^+$, where $V^+ := \{v \in V \mid \mathfrak{a}_+(v) = (0)\}$, and all $\mathfrak{a}$-module homomorphisms. Let $\mathrm{Mod}_{\mathfrak{a}_0}$ be the category of all $\mathfrak{a}_0$-modules and all $\mathfrak{a}_0$-module homomorphisms. Define functors R: $\mathrm{Mod}_{\mathfrak{a}}^+ \to \mathrm{Mod}_{\mathfrak{a}_0}$ and L: $\mathrm{Mod}_{\mathfrak{a}_0} \to \mathrm{Mod}_{\mathfrak{a}}^+$ as follows. $R(V) = V^+$, $R(f)(v) = f(v)$. If V is an $\mathfrak{a}_0$-module, regard V as an $(\mathfrak{a}_0 + \mathfrak{a}_+)$-module with trivial $\mathfrak{a}_+$-action, form the $\mathfrak{a}$-module $M(V) = U(\mathfrak{a}) \otimes_{U(\mathfrak{a}_0 + \mathfrak{a}_+)} V$, the submodule $I(V) = \{m \in M(V) \mid U(\mathfrak{a})m \subset \mathfrak{a}_-(M(V))\}$, and the quotient module $L(V) = M(V)/I(V)$. For $m \in M(V)$, write $\bar{m}$ for $m + I(V) \in L(V)$. Define $L(f)$ by: $L(f)(\overline{u \otimes v}) = \overline{u \otimes f(v)}$. Then:*

(a) *t: $LR \simeq I$ and t^0: $I \simeq RL$ are natural equivalences, where, for V an object of* $\mathrm{Mod}_{\mathfrak{a}}^+$ *and V^0 an object of* $\mathrm{Mod}_{\mathfrak{a}_0}$, *$t_V$: $LR(V) \to^{\sim} V$ and $t^0_{V^0}$: $V^0 \to^{\sim} RL(V^0)$ are defined by $t_V(\overline{u \otimes v}) = u(v)$ and $t^0_{V^0}(v) = \overline{1 \otimes v}$.*

(b) *$\oplus$ is the coproduct in* $\mathrm{Mod}_{\mathfrak{a}}^+$ *and* $\mathrm{Mod}_{\mathfrak{a}_0}$, *and L and R preserve coproducts.*

(c) *If V is an object of* $\mathrm{Mod}_{\mathfrak{a}}^+$, *then V^+ is an essential $\mathfrak{a}_+$-submodule of V. If V is an irreducible $\mathfrak{a}_0$-module, then $L(V)$ is an irreducible $\mathfrak{a}$-module.*

The proof, which is not difficult, is left to the reader. ∎

Below, we prove the special case of the lemma which is used in the proof of Proposition 2.9:

LEMMA 2.9.16. *Keep the assumptions and notations of Lemma* 2.9.15, *and let V be an $\mathfrak{a}$-module satisfying $V = V^+ \oplus \mathfrak{a}_- U(\mathfrak{a}_-)V^+$. Then*:

(a) *V is isomorphic to $L(V^+)$.*

(b) *V is an irreducible $\mathfrak{a}$-module if V^+ is an irreducible $\mathfrak{a}_0$-module.*

Proof. Since $V = U(\mathfrak{a})V^+$, we have:

$$U(\mathfrak{a}_+)v \cap V^+ \neq (0) \qquad \text{if} \quad v \in V \text{ and } v \neq 0. \tag{2.9.17}$$

(b) follows from (2.9.17) and $V = U(\mathfrak{a})V^+$.

To prove (a), let ψ: $M(V^+) \to V$ be the surjective $\mathfrak{a}$-module homomorphism defined by $\psi(u \otimes v) = u(v)$. We must show that $\operatorname{Ker} \psi = I(V^+)$. If $v \in I(V^+)$, then $U(\mathfrak{a})\,\psi(v) = \psi(U(\mathfrak{a})(v)) \subset \psi(\mathfrak{a}_-(M(V^+))) = \mathfrak{a}_-(\psi(M(V^+))) = \mathfrak{a}_-(V)$, so that $(U(\mathfrak{a})\,\psi(v) \cap V^+) \subset (\mathfrak{a}_-(V) \cap V^+) = (0)$. Hence, by (2.9.17), $\psi(v) = 0$, proving that $I(V^+) \subset \operatorname{Ker} \psi$. Now let $U(\mathfrak{a}) = \bigoplus_{n \in \mathbb{Z}} U(\mathfrak{a})_n$ be the $\mathbb{Z}$-gradation of $U(\mathfrak{a})$ induced by $\mathfrak{a} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}_n$, and suppose that $v = v_0 + v_1 + \cdots + v_k \in \operatorname{Ker} \psi$, where $v_i \in U(\mathfrak{a})_{-i}(1 \otimes V^+)$. We have $U(\mathfrak{a})_k v_k = U(\mathfrak{a})_k v \subset (1 \otimes V^+) \cap \operatorname{Ker} \psi = (0)$, and hence $U(\mathfrak{a})v_k = \sum_{n<k} U(\mathfrak{a})_n v_k + \sum_{n \geqslant k} U(\mathfrak{a})_n v_k \subset \mathfrak{a}_-(M(V^+)) + (0)$, so that $v_k \in I(V^+)$. Hence, by an inductive argument, $v \in I(V^+)$, proving (a). ∎

Remarks. The proof of Proposition 2.9 also shows:

(1) The partial Casimir operator Ω_0 is diagonalizable on V, and its eigenvalues are positive rational.

(2) With the additional assumption that V is $\mathfrak{h}$-diagonalizable, Proposition 2.9 also holds for $\mathfrak{g}$.

(3) Proposition 2.9 holds over any field of characteristic zero.

(4) With the hypotheses (ii), (2.9.1) and (2.9.2), Proposition 2.9 holds for arbitrary index sets I.

(5) Let I be a set, $A=(a_{ij})_{i,j\in I}$ a matrix over a field, $\mathfrak{g}'(A)$ the associated Lie algebra (see Section 1.8). Then by Lemma 2.9.16, if V is a $\mathfrak{g}'(A)$-module satisfying (2.9.2) and (2.9.13), V is isomorphic to a direct sum of modules $L(\Lambda)$. Suppose, moreover, that A is real symmetric. Then the same conclusion follows from (2.9.1), (2.9.2), (2.9.4) and (2.9.5), where $F(\sum k_i\alpha_i, \lambda)$ in (2.9.4) and (2.9.5) means

$$\sum_i k_i\lambda(h_i)+\sum_i \tfrac{1}{2}k_i(k_i+1)a_{ii}+\tfrac{1}{2}\sum_{i\neq j} k_ik_ja_{ij}. \tag{2.9.18}$$

Remark. In addition to Proposition 2.4, one has the following description of $P(\Lambda)$ for $\Lambda\in P_+$. We call $\lambda\in P$ *Λ-non-degenerate* if either $\lambda=\Lambda$ or else $\lambda<\Lambda$ and for any connected component S of $\operatorname{supp}(\Lambda-\lambda)$ one has

$$S\cap\{i\mid \Lambda(h_i)\neq 0\}\neq\varnothing.$$

Then $P(\Lambda)=W\cdot\{\lambda\in P_+\mid \lambda \text{ is } \Lambda\text{-non-degenerate}\}$. Furthermore, $\lambda\in P(\Lambda)\Leftrightarrow$ $w(\lambda)$ is Λ-non-degenerate for all $w\in W\Leftrightarrow \Lambda-\lambda\in Q$ and $\lambda\in$ convex hull of $W(\Lambda)$. This is a generalization of Proposition 2.12 (a), (b) below; its proof is similar to that of Lemma 1.6 from [19]. It follows that Proposition 2.12(c) holds for arbitrary Kač–Moody algebras as well. (See [50] for details.)

(G) Let $Q^\vee$ act on $\mathfrak{h}$ by $h_j\cdot h=h+2\pi i h_j$. Then we have an action of $\tilde{W}:=W\ltimes Q^\vee$ on $\mathfrak{h}$.

PROPOSITION 2.10. *The* $\operatorname{ch}_{L(\Lambda)}$, $\Lambda\in P_+$, *separate the orbits of* $\tilde{W}$ *on* $Y':=\bigcap_{\Lambda\in P_+}Y_\Lambda$, *i.e., given* $h,h'\in Y'$, $\tilde{W}(h)=\tilde{W}(h')$ *if and only if* $\operatorname{ch}_{L(\Lambda)}(h)=\operatorname{ch}_{L(\Lambda)}(h')$ *for all* $\Lambda\in P_+$

Proof. Let h, $h'\in Y'$ be such that $\operatorname{ch}_{L(\Lambda)}(h)=\operatorname{ch}_{L(\Lambda)}(h')$ for all $\Lambda\in P_+$; we have to show that $\tilde{W}(h)=\tilde{W}(h')$ (the other implication is obvious). First we show that

$$\operatorname{ch}_{L(\Lambda)}(nh)=\operatorname{ch}_{L(\Lambda)}(nh') \qquad \text{for all} \quad \Lambda\in P_+ \text{ and } n=1,2,\ldots. \tag{2.10.1}$$

For that, set

$$F_k(h)=\operatorname{ch}_{\wedge^k L(\Lambda)}(h), \qquad G_k(h)=\operatorname{ch}_{L(\Lambda)}(kh).$$

(Here $\wedge^k L(\Lambda)$ is the kth exterior power of $L(\Lambda)$.)

Then it follows from the Newton identities for power sums that

$$\mathbb{Q}[F_1,\ldots,F_n]=\mathbb{Q}[G_1,\ldots,G_n]. \tag{2.10.2}$$

This implies (2.10.1).

Since $Y' \subset Y$, we may assume that $\operatorname{Re} \alpha_i(h) \geqslant 0$ and $\operatorname{Re} \alpha_i(h') \geqslant 0$ for all $i \in I$. Note that $\Delta_0 := \{\alpha \in \Delta \mid \operatorname{Re} \alpha(h) = 0\}$ is a finite root system; let Q_0 be the lattice spanned by Δ_0, $Q_{0+} := Q_0 \cap Q_+$. Then the sum

$$\chi_\Lambda := \sum_{\alpha \in Q_{0+}} \operatorname{mult}_\Lambda(\Lambda - \alpha)\, e^{-\alpha}$$

is finite and hence $\chi_\Lambda(rh)$ is an almost periodic function of $r \in \mathbb{R}$. We have

$$\lim_{n \to +\infty} \left(e^{-\Lambda(nh)} \operatorname{ch}_{L(\Lambda)}(nh) - \chi_\Lambda(nh)\right) = 0. \qquad (2.10.3)$$

Similarly, for h' we define χ'_Λ, etc., and have

$$\lim_{n \to +\infty} \left(e^{-\Lambda(nh')} \operatorname{ch}_{L(\Lambda)}(nh') - \chi'_\Lambda(nh')\right) = 0. \qquad (2.10.3')$$

It follows from (2.10.1)–(2.10.3′), and from the fact that $\chi_\Lambda(nh)$ and $\chi'_\Lambda(nh')$ are non-zero almost periodic functions, that for all $\Lambda \in P_+$ we have: $\operatorname{Re} \Lambda(h) = \operatorname{Re} \Lambda(h')$. Hence $i(h - h') \in \sum_i \mathbb{R} h_i$, $Q_{0+} = Q'_{0+}$, and

$$\sum_{\alpha \in Q_{0+}} \operatorname{mult}_\Lambda(\Lambda - \alpha)\, e^{i \operatorname{Im}(\Lambda - \alpha)(h)}$$
$$= \sum_{\alpha \in Q_{0+}} \operatorname{mult}_\Lambda(\Lambda - \alpha)\, e^{i \operatorname{Im}(\Lambda - \alpha)(h')}. \qquad (2.10.4)$$

But (2.10.4) is an equality of irreducible characters at two elements of a compact group. Now it remains to apply two facts about connected compact Lie groups: the irreducible characters separate the conjugacy classes, and a conjugacy class intersects a maximal torus in an orbit of the Weyl group. ∎

2.2. *Modules $L(\Lambda)$ over Affine Lie Algebras*

Now let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra, $L(\Lambda)$ an irreducible highest weight $\mathfrak{g}$-module.

We recall that the center of $\mathfrak{g}$ is spanned by $c = \sum_{i \in I} a_i^\vee h_i$, where the $a_i^\vee$ are positive integers. c operates on $L(\Lambda)$ by the scalar operator $\Lambda(c) I$. In particular, $\lambda(c) = \Lambda(c)$ for all $\lambda \in P(\Lambda)$. The number

$$\Lambda(c) = \sum_{i \in I} a_i^\vee \Lambda(h_i)$$

is called the *level* of Λ, or of the module $L(\Lambda)$ [22].

Define $\rho \in \mathfrak{h}^*$ by $\rho(h_i) = 1$, $i \in I$, and $\rho(d) = 0$. The level of ρ is the dual Coxeter number g. Note that $|\rho|^2 = |\bar{\rho}|^2$ and $\bar{\rho} = \frac{1}{2} \sum_{\alpha \in \bar{\Delta}_+} \alpha$.

Define $\rho' \in \mathfrak{h}^*$ by $(\rho', \alpha^\vee) = 1$ for all $\alpha \in \Pi'$, and $\rho'(d) = 0$. We shall see that $\rho'(c) = g$.

Define $\rho^\vee \in \mathfrak{h}^*$ by $(\rho^\vee, \alpha_i) = 1$, $i \in I$, and $\rho^\vee(d) = 0$, so that $\rho^\vee(c) = h$. Note that $\bar{\rho}' = \bar{\rho}$ if $k' = 1$, and $\bar{\rho}' = \overline{\rho^\vee}$ if $k' \neq 1$.

Let $\Lambda \in P_+$, i.e., the labels $\Lambda(h_0),\ldots, \Lambda(h_l)$ are non-negative integers. Then the level of Λ is zero if and only if $\Lambda(h_i) = 0$ for all $i \in I$, that is, if and only if $L(\Lambda)$ is one-dimensional. From now on, we assume that $\Lambda \in P_+$ and that Λ has a positive level m.

Proposition 2.4(a) describes the strings $P(\Lambda) \cap (\lambda + \mathbb{Z}\alpha)$ for a real root α. Now we consider the case of an imaginary root α.

PROPOSITION 2.11. *Let $\Lambda \in P_+$, $\Lambda(c) > 0$, and $\lambda \in P(\Lambda)$; let α be a positive imaginary root* ($=s\delta$, $s > 0$). *Then*:

(a) *The set of all $t \in \mathbb{Z}$ such that $\lambda - t\alpha \in P(\Lambda)$ is an interval $[-p, +\infty)$, where $p \geqslant 0$, and $t \mapsto \operatorname{mult}_\Lambda(\lambda - t\alpha)$ is a non-decreasing function on this interval.*

(b) *If $x \in \mathfrak{g}_{-\alpha}$, $x \neq 0$, then $x: L(\Lambda) \to L(\Lambda)$ is an injection.*

Proof. Fix $x \in \mathfrak{g}_{-\alpha}$, $x \neq 0$, and choose $y \in \mathfrak{g}_\alpha$ such that $[x, y] = c$ (cf. Section 1.1(G) and recall that δ is identified with c, via (,)). Suppose that $v \in L(\Lambda)_\lambda$ is such that $v \neq 0$ but $x(v) = 0$. Then by induction on n we obtain

$$xy^n(v) = \Lambda(c)\, ny^{n-1}(v) \qquad \text{for} \quad n \geqslant 1.$$

Indeed: $xy^n(v) = [x, y]\, y^{n-1}(v) + y(xy^{n-1}(v)) = cy^{n-1}(v) + y\Lambda(c)(n-1)$ $y^{n-2}(v) = \Lambda(c) ny^{n-1}(v)$. Hence, $y^n(v) \neq 0$ and so $L(\Lambda)_{\lambda + n\alpha} \neq 0$ for $n \geqslant 1$, which is impossible. This proves (b). (a) follows from (b). ∎

The proof of (a) and (b) of the following proposition is now essentially the same as in the finite-dimensional theory (cf. [3]).

PROPOSITION 2.12. *Let $\Lambda \in P_+$, $\Lambda(c) = m > 0$. Then*:

(a) $P(\Lambda) = W\{\lambda \in P_+ \mid \Lambda \geqslant \lambda\}$.

(b) *The following conditions on λ are equivalent*:

(i) $\lambda \in P(\Lambda)$;

(ii) $\Lambda - w(\lambda) \geqslant 0$ *for all* $w \in W$;

(iii) $\Lambda - \lambda \in Q$ *and λ lies in the convex hull of* $W(\Lambda)$.

(c) *If $\lambda - \lambda' \in Q$ and λ' lies in the convex hull of $W(\lambda)$, then* $\operatorname{mult}_\Lambda(\lambda') \geqslant \operatorname{mult}_\Lambda(\lambda)$.[2]

(d) *If $\lambda \in P(\Lambda)$, then $|\lambda|^2 \leqslant |\Lambda|^2$, i.e., $P(\Lambda)$ is contained in the paraboloid* $\{\lambda \in \mathfrak{h}^* \mid |\bar{\lambda}|^2 + 2a_0^{-1} m\lambda(d) \leqslant |\Lambda|^2;\ \lambda(c) = m\}$; *equality holds if*

[2] It seems that this result, whose proof is valid for an arbitrary Kač–Moody algebra, is new even in the finite-dimensional case.

and only if $\lambda \in W(\Lambda)$. *Furthermore,* $|\lambda+\rho|^2 \leqslant |\Lambda+\rho|^2$; *equality holds if and only if* $\lambda = \Lambda$.

Proof. The inclusion $\subset$ in (a) follows from Proposition 2.4(b). In order to check the other inclusion, we prove by induction on htα that if $\mu \in P_+$, $\alpha \in Q_+$ are such that $\mu + \alpha \in P(\Lambda)$, then $\mu \in P(\Lambda)$. For $\alpha \in \mathbb{Z}\delta$, this is clear by Proposition 2.11(a). If $\alpha = \sum k_i \alpha_i \notin \mathbb{Z}\delta$, then $(\alpha, \alpha) > 0$ and therefore there exists $i \in I$ such that $k_i > 0$ and $(\alpha, \alpha_i) > 0$. But then $(\mu + \alpha, \alpha_i) > 0$ and so, by Proposition 2.4(a)(i), $\mu + (\alpha - \alpha_i) \in P(\Lambda)$. Applying the inductive assumption, we get $\mu \in P(\Lambda)$. This completes the proof of (a).

Now we prove (b). The implication (ii) $\Rightarrow$ (i) follows from (a) by taking $\mu \in W(\lambda)$ with minimal ht$(\Lambda - \mu)$, so that $\mu \in P_+$. For the implication (iii) $\Rightarrow$ (ii) remark that $\Lambda - w(\Lambda) \geqslant 0$ for all $w \in W$. A λ from the convex hull of $W(\Lambda)$ can be written in the form $\lambda = \sum_{w \in W} c_w w(\Lambda)$, where c_w are non-negative real numbers such that all but a finite number of them are 0 and $\sum c_w = 1$. Hence, for each $w_0 \in W$ we have $\Lambda - w_0(\lambda) = \sum_w c_w(\Lambda - w_0 w(\Lambda))$, so that $\Lambda - w_0(\lambda) \geqslant 0$. Hence, (iii) $\Rightarrow$ (ii). The implication (i) $\Rightarrow$ (iii) follows from Proposition 2.4(c).

For (c) we can assume that $\lambda \in P_+$. Then by the equivalence of (i) and (iii) in (b) applied to $L(\lambda)$, $\lambda' \in P(\lambda)$. We prove (c) by induction on ht$(\lambda - \lambda')$. If $\lambda' = \lambda$, there is nothing to prove. Otherwise, $\lambda' + \alpha_i \in P(\lambda)$ for some $i \in I$. Let $s > 0$ be such that $\mu := \lambda' + s\alpha_i \in P(\lambda)$ but $\mu + \alpha_i \notin P(\lambda)$. Since, by (b), μ lies in the convex hull of $W(\lambda)$, we can apply the inductive assumption: $\text{mult}_\Lambda(\lambda) \leqslant \text{mult}_\Lambda(\mu)$. Since λ' lies in the interval $[\mu, r_i(\mu)]$, $\text{mult}_\Lambda(\mu) \leqslant \text{mult}_\Lambda(\lambda')$ by Proposition 2.4(a), proving (c).

(d) follows from Proposition 2.4(d). ∎

Denote by $\bar{P}$, $\bar{P}_+$, $\bar{P}_{++}$ the orthogonal projections on $\bar{\mathfrak{h}}^*$ of P, P_+, P_{++}, respectively. These are the integral, dominant integral, and regular dominant integral weights for the finite root system $\bar{\Delta}$. Similarly, using the map Φ of Section 1.5, we define P', $\bar{P}'$, etc.

In the following proposition we collect some technical facts which will be needed later.

PROPOSITION 2.13. (a) $\bar{P}' = \{\lambda \in \bar{\mathfrak{h}}^* \mid (\lambda, \alpha) \in \mathbb{Z}$ *for all* $\alpha \in M\}$; $\bar{P} = \{\lambda \in \bar{\mathfrak{h}}^* \mid (\lambda, \alpha) \in \mathbb{Z}$ *for all* $\alpha \in M'\}$.

(b) *Let* $\mu \in \mathfrak{h}^*$. *Then*: $\mu \in P'$ *if and only if* $\mu(c) \in \mathbb{Z}$ *and* $\bar{\mu} \in \bar{P}'$; $\mu \in P$ *if and only if* $\mu(c) \in \mathbb{Z}$ *and* $\bar{\mu} \in \bar{P}$.

(c) *Let* $\gamma \in M$, $\gamma' \in M'$. *Then*

$$(\gamma, \gamma') \in \mathbb{Z}, \qquad a_0 |\gamma|^2 \in 2\mathbb{Z}, \qquad k|\gamma'|^2 \in 2\mathbb{Z}.$$

Proof. (a) and (b) follow from $Q^\vee = M' + \mathbb{Z}\delta$ and $Q'^\vee = M + \mathbb{Z}\delta$. To prove (c), note that: $(M, M') \subset \mathbb{Z}$ since $M \subset \bar{Q}$ and $M' = \bar{Q}^\vee$; $a_0\theta^\vee = \theta \in M'$

and $k\tilde{\theta}^\vee = \tilde{\theta} \in M$; M (resp. M') is spanned over $\mathbb{Z}$ by $\bar{W}(\theta^\vee)$ (resp. $\bar{W}(\tilde{\theta}^\vee)$). ∎

2.3. *String Functions and Classical Theta Functions*

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra. Fix $\Lambda \in P_+$ of positive level $m = \Lambda(c)$.

A weight $\lambda \in P(\Lambda)$ of the $\mathfrak{g}$-module $L(\Lambda)$ such that $\lambda + \delta \notin P(\Lambda)$ is called a *maximal* weight of $L(\Lambda)$.[3] Denote by $\max(\Lambda)$ the set of all maximal weights of $L(\Lambda)$. It is clear that $\max(\Lambda)$ is a W-invariant set and hence, by Proposition 2.12(a), each maximal weight is W-equivalent to a (unique) dominant maximal weight. On the other hand, it follows from Proposition 2.11(a) that for any $\mu \in P(\Lambda)$ there exists a unique $\lambda \in \max(\Lambda)$ and a unique non-negative integer n such that $\mu = \lambda - n\delta$.

PROPOSITION 2.14. *Let $\Lambda \in P_+$ be a weight of level m. Then $\lambda \mapsto \bar{\lambda}$ defines a bijection from* $\max(\Lambda) \cap P_+$ *onto $mC_{\mathrm{af}} \cap (\bar{\Lambda} + \bar{Q})$. In particular, the set of dominant maximal weights of $L(\Lambda)$ is finite.*

Proof. Straightforward using Propositions 2.11 and 2.12. ∎

For $\lambda \in \max(\Lambda)$ introduce the generating function:

$$b_\lambda^\Lambda := \sum_{n=0}^{\infty} \operatorname{mult}_\Lambda(\lambda - n\delta)\, e^{-n\delta}.$$

This series converges absolutely on Y since it is majorized by $|e^{-\lambda}| \sum_{\mu \in P(\Lambda)} (\operatorname{mult} \mu)\, |e^\mu|$, which converges on Y by Proposition 2.5(c). Since $W_\lambda \cap T = \{1\}$ for $\lambda \in P(\Lambda)$, and since $b_{w(\lambda)}^\Lambda = b_\lambda^\Lambda$ for $w \in W$, we have

$$\operatorname{ch}_{L(\Lambda)} = \sum_{\lambda \in \max(\Lambda)} e^\lambda b_\lambda^\Lambda = \sum_{\substack{\lambda \in \max(\Lambda) \\ \lambda \bmod T}} \sum_{t \in T} e^{t(\lambda)} b_\lambda^\Lambda. \tag{2.15}$$

We proceed to rewrite character formulas (2.6) and (2.15) in terms of theta functions. For $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) > 0$, set

$$\Theta_\lambda = e^{-(|\lambda|^2/2\lambda(c))\delta} \sum_{t \in T} e^{t(\lambda)}. \tag{2.16}$$

Similarly, we define Θ'_λ by replacing T by T'.

Using (1.7), we obtain for $\lambda(c) = m$:

$$\Theta_\lambda = e^{m\Lambda_0} \sum_{\gamma \in M + m^{-1}\bar{\lambda}} e^{-(1/2)m|\gamma|^2\delta + m\gamma}, \tag{2.16.1}$$

[3] Note a discrepancy with [17, p. 128]—we do not require that μ be dominant.

which is a classical theta function (see Sections III and IV for details). It is clear that this series converges absolutely on Y and that Θ_λ depends only on $\lambda \bmod mM + \mathbb{C}\delta$.

Introduce the following number:

$$s_\Lambda := \frac{|\Lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g}.$$

For a weight $\lambda \in P(\Lambda)$ introduce the number

$$s_\Lambda(\lambda) := s_\Lambda - \frac{|\lambda|^2}{2m},$$

called the *characteristic* of λ. It is easy to see that $s_\Lambda(\lambda)$ is a rational number. It will be "responsible" for the leading term in the q-expansion of a modular form.

For $\lambda \in \max(\Lambda)$, set

$$c_\lambda^\Lambda := e^{-s_\Lambda(\lambda)\delta} \sum_{n \geqslant 0} \operatorname{mult}_\Lambda(\lambda - n\delta)\, e^{-n\delta}.$$

As we have seen, this series converges absolutely to a holomorphic function on Y. Furthermore, if $\lambda \in \mathfrak{h}^*$ is such that $\lambda - \mu \in \mathbb{C}\delta$ for some $\mu \in \max(\Lambda)$, then μ is uniquely determined (by Proposition 2.11(a)), and we set $c_\lambda^\Lambda = c_\mu^\Lambda$; if $(\lambda + \mathbb{C}\delta) \cap \max(\Lambda) = \varnothing$, we set $c_\lambda^\Lambda = 0$. The function c_λ^Λ is called the *string function* of $\lambda \in \mathfrak{h}^*$. Note that

$$c_{w(\lambda)}^\Lambda = c_\lambda^\Lambda \qquad \text{for} \quad w \in W,\ \lambda \in \mathfrak{h}^*.$$

Since $W = \bar{W} \ltimes T$, using (1.6) we deduce that

$$c_{w(\lambda)+m\gamma+a\delta}^\Lambda = c_\lambda^\Lambda \qquad \text{for} \quad \lambda \in \mathfrak{h}^*,\ w \in \bar{W},\ \gamma \in M,\ a \in \mathbb{C}. \tag{2.17}$$

Note also that c_λ^Λ depends only on $\Lambda \bmod \mathbb{C}\delta$.

Using $W = \bar{W} \ltimes T$, we combine (2.6) and (2.15) to obtain:

$$e^{-s_\Lambda\delta} \operatorname{ch}_{L(\Lambda)} = \frac{\sum_{w\in\bar{W}} (\det w)\, \Theta_{w(\Lambda+\rho)}}{\sum_{w\in\bar{W}} (\det w)\, \Theta_{w(\rho)}} = \sum_{\substack{\lambda \in P \bmod mM + \mathbb{C}\delta \\ \lambda(c) = m}} c_\lambda^\Lambda \Theta_\lambda. \tag{2.18}$$

We use this important identity in Section IV to study and compute the string functions.

2.4. *Appendix* 2: *On the Region of Convergence of* $\operatorname{ch}_{L(\Lambda)}$

Let $\mathfrak{g}(A)$ be a Kač–Moody algebra with symmetrizable Cartan matrix A, let $(\ ,\ )$ be a standard form on $\mathfrak{g}(A)$, and let $\Lambda \in P_+$. For $\alpha \in \Delta_+$, we set

$\mathfrak{n}_{\pm}^{(\alpha)} = \bigoplus_{j=1}^{\infty} \mathfrak{g}_{\pm j\alpha}$, $\mathfrak{g}^{(\alpha)} = \mathfrak{n}_{-}^{(\alpha)} \oplus \mathbb{C}\alpha \oplus \mathfrak{n}_{+}^{(\alpha)}$. One knows [14] that if $\alpha \in \Delta_{+}^{\text{re}}$, then $\mathfrak{g}^{(\alpha)} \simeq sl_2(\mathbb{C})$ and the module $L(\Lambda)$ restricted to $\mathfrak{g}^{(\alpha)}$ is a direct sum of finite-dimensional irreducible submodules. Furthermore, $\mathfrak{g}^{(\alpha)}$ is an infinite-dimensional Heisenberg algebra if $(\alpha, \alpha) = 0$; and $\mathfrak{n}_{\pm}^{(\alpha)}$ are free Lie algebras if $(\alpha, \alpha) < 0$ (by Lemma 1.14). In this appendix we study the restriction of $L(\Lambda)$ to $\mathfrak{g}^{(\alpha)}$ for $\alpha \in \Delta_{+}^{\text{im}}$ and deduce an explicit description of the region of convergence of $\text{ch}_{L(\Lambda)}$. The results are stated in the following two propositions.

PROPOSITION 2.19. *Let $\alpha \in \Delta_{+}^{\text{im}}$ and $\Lambda \in P_{+}$; introduce the following two subspaces of $L(\Lambda)$:*

$$L(\Lambda)_0^{(\alpha)} = \bigoplus_{\lambda:(\lambda,\alpha)=0} L(\Lambda)_\lambda, \qquad L(\Lambda)_{+}^{(\alpha)} = \bigoplus_{\lambda:(\lambda,\alpha)>0} L(\Lambda)_\lambda.$$

(a) *One has a direct sum of $\mathfrak{g}^{(\alpha)}$-modules:*

$$L(\Lambda) = L(\Lambda)_0^{(\alpha)} \oplus L(\Lambda)_{+}^{(\alpha)}. \tag{2.19.1}$$

(b) $L(\Lambda)_0^{(\alpha)} = \{x \in L(\Lambda) \mid \mathfrak{g}^{(\alpha)}(x) = 0\}$.

(c) *The $U(\mathfrak{n}_{-}^{(\alpha)})$-module $L(\Lambda)_{+}^{(\alpha)}$ is free on a basis of $\{v \in L(\Lambda)_{+}^{(\alpha)} \mid \mathfrak{n}_{+}^{(\alpha)}(v) = 0\}$.*

(d) *The $\mathfrak{g}^{(\alpha)}$-module $L(\Lambda)$ is completely reducible.*

PROPOSITION 2.20. *Suppose that A is indecomposable and that the $\mathfrak{g}(A)$-module $L(\Lambda)$, where $\Lambda \in P_{+}$, is not 1-dimensional. Let Y_Λ be the region of absolute convergence of $\text{ch}_{L(\Lambda)}$. Then Y_Λ is open and*

$$Y_\Lambda = \left\{ h \in \mathfrak{h} \;\middle|\; \sum_{\alpha \in \Delta_{+}} (\text{mult}\,\alpha)\, |e^{-\alpha(h)}| < \infty \right\}.$$

Now let $B = (b_{ij})_{i,j \in I}$ be an (in general infinite) symmetric matrix over $\mathbb{C}$, and let $\mathfrak{g}'(B)$ be the associated Lie algebra (see Section 1.8). We have the triangular decomposition $\mathfrak{g}'(B) = \mathfrak{n}_{-} \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_{+}$, where $\mathfrak{n}_{\pm} = \bigoplus_{\alpha \in Q_{+}\setminus\{0\}} \mathfrak{g}'(B)_{\pm\alpha}$.

LEMMA 2.21. *Suppose that all entries of B are non-positive real. Let V be a $\mathfrak{g}'(B)$-module satisfying the following conditions:*

(i) *$V = \bigoplus_{\lambda \in \mathfrak{g}_0^*} V_\lambda$, i.e., V is $\mathfrak{g}_0$-semisimple;*

(ii) *if $v \in V$, then $U(\mathfrak{n}_{+})_\beta v = 0$ for all but a finite number of $\beta \in Q_{+}$;*

(iii) *if $V_\lambda \neq 0$, then $\lambda(h_i) > 0$ for all $i \in I$.*

Then the module V is isomorphic to a direct sum of irreducible $\mathfrak{g}'(B)$-modules $L(\Lambda)$, which are free $U(\mathfrak{n}_{-})$-modules on one generator $v \in L(\Lambda)_\Lambda$.

Proof. By Remark (5) following Proposition 2.9, we must verify conditions (2.9.4) and (2.9.5) for the $\mathfrak{g}'(B)$-module V. By (iii), (2.9.4) holds with $n=1$. The definition (2.9.18) shows that if $\beta \in Q_+ \setminus \{0\}$ and $V_\lambda \neq 0$, then $F(-\beta, \lambda) < 0$ by (iii) and the conditions on B; this verifies (2.9.5). Hence, V is a direct sum of modules $L(\Lambda)$. Since the corresponding Verma modules are indecomposable and satisfy (i), (ii) and (iii), they are irreducible and hence coincide with the $L(\Lambda)$. ∎

Proof of Proposition 2.19. Since $\Delta_+^{\mathrm{im}} \subset -W(P_+)$ by Proposition 2.4(b), we may assume that $\alpha(h_i) \leqslant 0$ for all i.

Write $\alpha = \sum_{i \in I} c_i \alpha_i$, and put $I' = \{i \in I \mid c_i \neq 0\}$. Let $A' = (a_{ij})_{i,j \in I'}$ be the corresponding generalized Cartan matrix, so that we may regard $\mathfrak{g}(A')$ as a subalgebra of $\mathfrak{g}(A)$. (a) and (b) now follow from Proposition 2.9 applied to the $\mathfrak{g}(A')$-module $L(\Lambda)$.

In order to prove (c) and (d), note that $\mathfrak{g}^{(\alpha)}$ is isomorphic to a quotient by a central ideal of the Lie algebra $\mathfrak{g}'(B)$, where B is an (in general infinite) matrix whose entries are non-positive real numbers. This is clear when $(\alpha, \alpha) = 0$ because, by Proposition 1.11, $\mathfrak{g}^{(\alpha)}$ is an infinite Heisenberg algebra and hence is a quotient of $\mathfrak{g}'(B)$, $B = 0$, by a central ideal; when $(\alpha, \alpha) < 0$, we apply Lemma 1.14. Now we can apply Lemma 2.21 to the $\mathfrak{g}^{(\alpha)}$-module $L(\Lambda)_+^{(\alpha)}$. ∎

We are grateful to P. Slodowy for calling our attention to the following fact.

LEMMA 2.20.1. *Let $\Lambda \in P_+$. Then the region of absolute convergence of $\psi_\Lambda := \sum_{w \in W} (\det w)\, e^{w(\Lambda+\rho)}$ is Y.*

Proof. If $h \in \mathfrak{h}$ satisfies $\operatorname{Re} \alpha_i(h) > 0$ for all $i \in I$, then $e^{-(\Lambda+\rho)}\psi_\Lambda$ is majorized at h by the convergent series

$$\sum_{k \in \mathbb{Z}_+^l} \exp - \sum_i k_i \operatorname{Re} \alpha_i(h);$$

here we use Proposition 1.3(a), (b), (c). Since the region of absolute convergence of ψ_Λ is convex and W-invariant, we deduce that it contains Y.

On the other hand, suppose that $h \in \mathfrak{h} \setminus Y$. Then $\Delta_0 := \{\alpha \in \Delta_+^{\mathrm{re}} \mid \operatorname{Re} \alpha(h) \leqslant 0\}$ is infinite by Proposition 1.3(d), (e), and for any $\alpha \in \Delta_0$, $|e^{(r_\alpha(\Lambda+\rho))(h)}| \geqslant |e^{(\Lambda+\rho)(h)}|$. Hence ψ_Λ does not converge at h. ∎

Proof of Proposition 2.20. Put

$$Y' = \left\{ h \in \mathfrak{h} \;\middle|\; \sum_{\alpha \in \Delta_+} (\operatorname{mult} \alpha)\, |e^{-\alpha(h)}| < \infty \right\}.$$

By Proposition 2.5, it suffices to show that Y' is open and that $Y_\Lambda \subset Y'$.

Put $B_\rho = \prod_{\alpha \in \Delta_+^{\mathrm{im}}} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha}$. By (2.7) and Lemma 2.20.1, and by the extension theorem for holomorphic functions across sets of codimension two, B_ρ extends to a holomorphic function on Y. For $h \in$ Interior (X), consider the meromorphic function $f(t) := B_\rho(th)^{-1}$. By a standard property of Dirichlet series with positive coefficients [39, Chapter VI, Proposition 7], the set $\{t \in \mathbb{R} \mid th \in Y'\}$ of convergence of the series obtained by multiplying out $\prod_{\alpha \in \Delta_+^{\mathrm{im}}} (1 + e^{-t\alpha(h)} + e^{-2t\alpha(h)} + \cdots)^{\mathrm{mult}\,\alpha}$, which represents $f(t)$, is an open segment $(c, +\infty)$. Since Y' is convex, and since for any $h' \in Y$, there exists a $t' > 0$ such that $t'h' \in Y'$ by the argument proving Proposition 2.5, this shows that $Y' \cap \mathfrak{h}_{\mathbb{R}}$ is open in $\mathfrak{h}_{\mathbb{R}}$, so that $Y' = (Y' \cap \mathfrak{h}_{\mathbb{R}}) + i\mathfrak{h}_{\mathbb{R}}$ is open in $\mathfrak{h}$.

We now show that $Y_\Lambda \subset Y'$. If A is of finite type, there is nothing to prove. If A is of affine type, this is shown by Proposition 2.5(c). Otherwise, by [19, Proposition 1.3], there exists $\alpha \in \Delta_+^{\mathrm{im}}$ such that $\operatorname{supp} \alpha = I$ and $\alpha(h_i) < 0$ for all $i \in I$. But then $\mu := \Lambda - \alpha$ lies in $P(\Lambda)$ by Proposition 2.19(c). Moreover, by Proposition 2.19(c) and Proposition 2.4(a), for any non-zero $v \in L(\Lambda)_\mu$ the map $\psi: \mathfrak{n}_- \to L(\Lambda)$ defined by $\psi(y) = y(v)$ is an injection. This shows that $Y_\Lambda \subset Y'$. ∎

Remark. By Lemma 2.20.1 and by the character formula (2.6), for any $\Lambda \in P_+$ the function $\mathrm{ch}_{L(\Lambda)}$ extends to a meromorphic function on Y. Let A be indecomposable and not of affine or finite type. By the proof of Proposition 2.20 and by Proposition 1.12(b), for any $h \in Y \cap \mathfrak{h}_{\mathbb{R}}$ there exists a $t > 0$ such that for every $\Lambda \in P_+$ with $\dim L(\Lambda) \neq 1$, the meromorphic extension of $\mathrm{ch}_{L(\Lambda)}$ has a pole at th. The existence of such an "immovable" pole was first shown for the rank two hyperbolic case by A. Meurman.

Remark. (1) For an indecomposable symmetrizable Cartan matrix which is not of finite or affine type, let X° and Z° be the interiors of the dual convex cones $X \bmod \mathfrak{c}$ and Z, and define continuous W-invariant functions $\Phi: X^\circ \to (0, +\infty)$ and $\Psi: Z^\circ \to (0, +\infty)$ as follows:

$$\Phi(h) = \min\{t > 0 \mid B_\rho(t^{-1}h) = 0\},$$

where B_ρ is holomorphic on Y and satisfies

$$B_\rho \prod_{\alpha \in \Delta_+^{\mathrm{re}}} (1 - e^{-\alpha}) = \sum_{w \in W} (\det w) e^{w(\rho) - \rho};$$

$$\Psi(\alpha) = \min_{h \in X^\circ} \frac{\alpha(h)}{\Phi(h)}.$$

Then for $\alpha \in \Delta_+^{\mathrm{im}} \cap Z^\circ$, $\Psi(\alpha)$ coincides with $\psi(\alpha)$ $(= \sup_{n \geq 1} n^{-1} \log \operatorname{mult} n\alpha = \lim_{n \to \infty} n^{-1} \log \operatorname{mult} n\alpha)$ introduced in Proposition 1.12, Ψ is of class C^1, and

$$\Phi(h) = \min_{\alpha \in Z^\circ} \frac{\alpha(h)}{\Psi(\alpha)}.$$

Then the following conditions on $h \in X^{\circ}$ are equivalent:

(a) The zero of $B_{\rho}(th)$ at $t = \Phi(h)^{-1}$ is simple;

(b) Φ is real-analytic in a neighborhood of h;

(c) Ψ is real-analytic in a neighborhood of $Z_h := \{\alpha \in Z^{\circ} \mid \Psi(\alpha)\,\Phi(h) = \alpha(h) = 1\}$;

(d) $(\alpha, \alpha)\,\beta(h) > 2(\beta, \alpha)\,\alpha(h)$ for all $\alpha, \beta \in Z_h$;

(e) Z_h consists of a single element.

If the Cartan matrix is of hyperbolic type, it is easy to see that $\beta - 2((\beta, \alpha)/(\alpha, \alpha))\alpha \in -Z^{\circ}$ for all $\alpha, \beta \in Z^{\circ}$, verifying condition (d), so that Φ and Ψ are real-analytic, and condition (a) holds for all $h \in X^{\circ}$.

(2) Consider the Cartan matrix $\left(\begin{smallmatrix} 2 & -4 \\ -4 & 2 \end{smallmatrix}\right)$ of hyperbolic type, put $|m\alpha_1 + n\alpha_2|^2 = m^2 - 4mn + n^2$ and, for $\alpha \in \Delta_+^{\mathrm{im}}$, put

$$R(\alpha) = (-|\alpha|^2)^{-1/2}\, \psi(\alpha).$$

Then one can show that for $\alpha \in \Delta_+^{\mathrm{im}}$,

$$.9255989 \cdots = R(\alpha_1 + 2\alpha_2) \leqslant R(\alpha) \leqslant R(\alpha_1 + \alpha_2) = .9256000 \cdots,$$

so that $R(\alpha)$ is almost constant!

The results stated in this remark are due to the second author.

We note the following useful corollary of Propositions 2.19(c) and 2.4(a).

COROLLARY 2.20.2. *Let $\Lambda \in P_+$ and $\lambda \in P(\Lambda)$. Fix a non-zero vector $v \in L(\Lambda)_\lambda$ and set*

$$\mathfrak{n}_-^{\lambda} = \bigoplus_{\substack{\alpha \in \Delta_+ \\ (\lambda, \alpha) > 0}} \mathfrak{g}_{-\alpha}.$$

Define a map $\psi \colon \mathfrak{n}_- \to L(\Lambda)$ by $\psi(y) = y(v)$. Then ψ is injective on $\mathfrak{n}_-^{\lambda}$.

2.5. *Appendix* 3: *On the Segal Operators*

The material of this section has no apparent relevance to the rest of the paper. Nevertheless, we decided to place it here because of the unexpected mysterious coincidence of a constant in the cocycle below (formula (2.26)) on the one hand and a constant in the asymptotics of weight multiplicities (Theorem B in Section 4.7) on the other hand.

Let $\mathfrak{p}$ be a finite-dimensional complex simple Lie algebra. We keep the notations of Section 1.7. Let $\{u_i\}$ be a basis of $\mathfrak{p}$ and let $\{u^i\}$ be the dual basis with respect to the invariant symmetric bilinear form B.

LEMMA 2.22. (a) *The element* $\sum_i u_i \otimes u^i$ *of* $\mathfrak{p} \otimes \mathfrak{p}$ *is independent of the choice of the basis* $\{u_i\}$, *is symmetric and is killed by* $\mathfrak{p}$.

(b) $\sum_i (\operatorname{ad} u_i)(\operatorname{ad} u^i) = 2gI_{\mathfrak{p}}$.

Proof. (a) is obvious. (b) follows from the standard fact that the Casimir element $\sum_i u_i u^i$ acts by the scalar $B(\theta_0 + \rho_0, \theta_0 + \rho_0) - B(\rho_0, \rho_0) = 2g_{\mathfrak{p}}$ in the adjoint representation, along with $g_{\mathfrak{p}} = g$ from Proposition 1.11(c). ∎

Let N be a positive integer, let σ be an automorphism of the Lie algebra $\mathfrak{p}$ such that $\sigma^N = I$, and let $\mathfrak{p} = \bigoplus_s \mathfrak{p}_s$ be the corresponding $\mathbb{Z}/N\mathbb{Z}$-gradation. Set $d_s = \dim \mathfrak{p}_s$, $s \in \mathbb{Z}/N\mathbb{Z}$, and choose bases $u_{1,s}, \dots, u_{d_s,s}$ and $u^{1,s}, \dots, u^{d_s,s}$ of $\mathfrak{p}_s$ and $\mathfrak{p}_{-s}$, respectively, dual under B. We shall assume that $u_{i,-s} = u^{i,s}$, which is possible due to the symmetry of B.

Let $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}_B(\sigma, N)$ be the associated affine Lie algebra (cf. Section 1.7), $\hat{\mathfrak{p}}'$ its derived algebra. We shall write $x(n)$ for $t^n \otimes x \in \hat{\mathfrak{p}}'$. Recall that $[y(m), z(n)] = [y, z](m+n) + mB(y, z)\delta_{m,-n}c_0$ and $c = Nc_0$.

Define the following elements of the universal enveloping algebra $U(\hat{\mathfrak{p}}')$ of $\hat{\mathfrak{p}}'$ for $n, r \in \mathbb{Z}$:

$$S_n(r) = \sum_{i=1}^{d_r} u_{i,-r}(-r)\, u_{i,r}(nN + r),$$

$$T_n(r) = \sum_{j=1}^{N} S_n(r+j).$$

Fix $m, n \in \mathbb{Z}$ and $x \in \mathfrak{p}_m$. For $r \in \mathbb{Z}$ define the following auxiliary elements of $U(\hat{\mathfrak{p}}')$:

$$F(r) = \sum_{i=1}^{d_r} u_{i,-r}(-r)[u_{i,r}, x](nN + m + r),$$

$$G(r) = \sum_{j=1}^{N} F(r+j),$$

$$H(r) = \sum_{i=1}^{d_r} [u_{i,-r}, x](m - r)\, u_{i,r}(nN + r),$$

$$I(r) = \sum_{j=1}^{N} H(r+j).$$

We need the following technical lemma.

LEMMA 2.23. *For all* $r \in \mathbb{Z}$ *we have*:

(a) $T_n(r) = T_n(-(n+1)N - r - 1) - \delta_{n,0} \left(\sum_{j=1}^{N} (r+j)\, d_{r+j}\right) N^{-1}c$.

(b) $[T_n(r), x(m)] = G(r) + I(r) - \delta_r m x(nN + m) N^{-1}c$,

where $\delta_r = \sum_{j=1}^{N} (\delta_{m,r+j} + \delta_{-nN-m,r+j})$.

(c) $G(r) + I(r+m) = 0$.

(d) $I(r) = G(-(n+1)N - r - 1) - 2gx(nN+m)$ *if* $m \neq -nN$.

Proof. We have:

$$S_n(r) - S_n(-nN-r) = \sum_{i=1}^{d_r} [u_{i,-r}(-r), u_{i,r}(nN+r)]$$
$$= -\delta_{n,0} r d_r N^{-1} c + \sum_{i=1}^{d_r} [u_{i,-r}, u_{i,r}](nN).$$

Substituting $i = N + 1 - j$ in the definition of $T_n(-(n+1)N - r - 1)$ and using the equation above, we obtain

$$T_n(r) - T_n(-(n+1)N - r - 1) = \sum_{i=1}^{N} S_n(r+i) - S_n(-nN-r-i)$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{d_{r+i}} [u_{j,-r-i}, u_{j,r+i}](nN)$$
$$- \delta_{n,0} \left(\sum_{i=1}^{N} (r+i)\, d_{r+i} \right) N^{-1} c.$$

The first sum is zero by the skew-symmetry of the bracket, proving (a).

To prove (b), write

$$[S_n(r), x(m)] = \sum_{i=1}^{d_r} ([u_{i,-r}(-r), x(m)]\, u_{i,r}(nN+r)$$
$$+ u_{i,-r}(-r)[u_{i,r}(nN+r), x(m)])$$
$$= H(r) + F(r) - (\delta_{r,m} + \delta_{r,-nN-m})\, m x(nN+m)\, N^{-1} c$$

by the definition of the bracket in $\hat{\mathfrak{p}}$. A summation now proves (b).

To prove (c), note that $F(r) + H(r+m)$ is the image, under the linear map from $\mathfrak{p}_{-r} \otimes \mathfrak{p}_{m+r}$ to $U(\hat{\mathfrak{p}}')$ defined by $y \otimes z \mapsto y(-r)\, z(nN+m+r)$, of $\sum_{i=1}^{d_r} u_{i,-r} \otimes [u_{i,r}, x] + \sum_{i=1}^{d_{r+m}} [u_{i,-r-m}, x] \otimes u_{i,r+m}$, which vanishes by an application of the $\mathbb{Z}/N\mathbb{Z}$-grading to Lemma 2.22(a). A summation now proves (c).

To prove (d), suppose $m \neq -nN$, and write

$$F(r) - H(-nN-r) = \sum_{i=1}^{d_r} [u_{i,-r}(-r), [u_{i,r}, x](nN+m+r)]$$
$$= \sum_{i=1}^{d_r} [u_{i,-r}, [u_{i,r}, x]](nN+m).$$

Using Lemma 2.22(b), a summation proves (d). ∎

Now let V be a $\hat{\mathfrak{p}}'$-module such that for any $v \in V$, there exists n_0 such that $x(n)(v) = 0$ whenever $n \geqslant n_0$ and $x \in \mathfrak{p}_n$. Any module $L(\Lambda)$ clearly satisfies this condition. For $n \in \mathbb{Z}$, define an operator T_n on V by

$$T_n = \sum_{r \geqslant 0} T_n(r) + \sum_{r \geqslant 0} T_n(r - (n+1)N).$$

This operator is well-defined on V since each $v \in V$ is killed by all but a finite number of summands.

Remark. By Lemma 2.23(a),

$$T_n = \sum_{r \in \mathbb{Z}} T_n(r) \qquad \text{if} \quad n \neq 0.$$

If, in addition, $\mathfrak{p}_0$ is semisimple, we have

$$T_n = N \sum_{r \in \mathbb{Z}} S_n(r).$$

In the case $\sigma = I$, $N = 1$, the operators T_n were originally introduced in this form by G. Segal (unpublished). We mention also that this kind of construction is popular in the dual string theory (cf. [7]).

Now we can prove the main lemma.

LEMMA 2.24. $[T_n, x(m)] = -2(g+c)\, mx(nN+m)$.

Proof. First, suppose that $m \neq -nN$, so that we can use Lemma 2.23(d). Using Lemma 2.23(a) and (b), we have

$$\begin{aligned}[T_n, x(m)] &= \sum_{r \in \mathbb{Z}} [T_n(r), x(m)] \\ &= \sum_{r \in \mathbb{Z}} (G(r) + I(r) - \delta_r mx(nN+m)\, N^{-1} c) \\ &= \sum_{r \in \mathbb{Z}} (G(r) + I(r)) - 2mx(nN+m)c.\end{aligned}$$

To evaluate the sum, first suppose $m \geqslant 0$. Then for $A + B \geqslant 2m$, Lemma 2.23(c) and (d) gives

$$\begin{aligned}\sum_{r=-A}^{B} (G(r) + I(r)) &= \sum_{r=-A}^{-A+m-1} I(r) + \sum_{r=B-m+1}^{B} G(r) \\ &= \sum_{r=-A}^{-A+m-1} (G(-(n+1)N - r - 1) - 2gx(nN+m)) \\ &\quad + \sum_{r=B-m+1}^{B} G(r).\end{aligned}$$

Since $\lim_{r\to+\infty} G(r) = 0$, we obtain

$$\sum_{r\in\mathbb{Z}} (G(r)+I(r)) = \lim_{A,B\to+\infty} \sum_{r=-A}^{B} (G(r)+I(r))$$
$$= -2mgx(nN+m).$$

A similar argument gives the same result for $m<0$, proving the lemma in case $m \neq -nN$.

If $m=-nN$, we may assume $x=[y,z]$, where $y\in\mathfrak{p}_i$, $z\in\mathfrak{p}_{m-i}$ and $i\neq 0, -nN$ (since these elements span $\mathfrak{p}_m$). Then $x(m)\in[y(i), z(m-i)]+\mathbb{C}c$, and a calculation of

$$[T_n, x(m)] = [T_n, [y(i), z(m-i)]]$$

using the Jacobi identity proves the lemma in this case too. ∎

We next calculate $[T_n, T_m]$. Using Lemma 2.24, we have

$$[T_n, S_m(r)] = 2(g+c)(rS_{m+n}(r-nN) - (r+mN)\,S_{m+n}(r)).$$

A formal calculation using this gives, for $n \geqslant 0$,

$$\left[T_n, \sum_{s\geqslant r} T_m(s)\right] = 2(g+c)\,N(n-m) \sum_{s\geqslant r} T_{m+n}(s)$$
$$+ 2(g+c) \sum_{s=r-nN}^{r-1} \left(2s+1+nN-r-N\left[\frac{s-r}{N}\right]\right) T_{m+n}(s).$$

The definition of T_m now gives

$$[T_n, T_m] = 2(g+c)\,N(n-m)\left(\sum_{s\geqslant 0} T_{m+n}(s) + \sum_{s\geqslant -(m+1)N} T_{m+n}(s)\right)$$
$$+ 2(g+c) \sum_{s=-nN}^{-1} \left(2s+1+nN-N\left[\frac{s}{N}\right]\right) T_{m+n}(s)$$
$$+ 2(g+c) \sum_{s=-(m+n+1)N}^{-(m+1)N-1} \left(2s+1+nN-N\left[\frac{s}{N}\right]\right) T_{m+n}(s).$$

Reparametrize the third sum by $t=-(m+n+1)N-s-1$, and apply Lemma 2.23(a) to its summands. It then becomes

$$\sum_{t=-(m+n+1)N}^{-(m+1)N-1} \left(-(m+1)N-2t-1-N\left[\frac{-t-1}{N}\right]\right) T_{m+n}(t) - \delta_{m,-n} LN^{-1}c,$$

where

$$L = \sum_{s=-nN}^{-1} \left(2s + 1 + nN - N\left[\frac{s}{N}\right]\right) \sum_{j=1}^{N} (s+j)\, d_{s+j}.$$

Using the fact that d_r depends only on $r \bmod N$, we obtain

$$L = -\,2N^3 \left(\frac{\dim \mathfrak{p}}{12}(n^3 - n) - \frac{n}{2N^2}\left(\sum_{i=0}^{N} i(N-i)\, d_i\right)\right).$$

Substituting the expression above for the third sum and combining the four sums, we obtain (using a similar argument for $n < 0$):

LEMMA 2.25.

$$[T_n, T_{n'}] = -2(g+c)\, N(n'-n)\, T_{n+n'} + \delta_{n,-n'} 4N^2 K_n (g+c)c,$$

where

$$K_n = \frac{\dim \mathfrak{p}}{12}(n^3 - n) - \frac{n}{2N^2} \sum_{j=0}^{N} j(N-j)\, d_j. \quad ∎$$

Note that by Proposition 1.11(d) and (e), we have for γ defined by Proposition 1.11(b):

$$K_n = \frac{k}{g}\,|\bar{\rho}'|^2\, n(n^2-2) + \frac{1}{kg}\,|\bar{\rho} - g\gamma|^2\, n. \tag{2.26}$$

Finally, we include the Lie algebra $\hat{\mathfrak{p}}'$ as an ideal in a larger Lie algebra $\hat{\hat{\mathfrak{p}}}_a$, where $a \in \mathbb{C}$, as follows:

$$\hat{\hat{\mathfrak{p}}}_a = \hat{\mathfrak{p}}' + \sum_{n \in \mathbb{Z}} \mathbb{C} d_n,$$

$$[d_j, x(n)] = N^{-1} n x(n+Nj); \qquad [d_j, c] = 0, \qquad j \in \mathbb{Z};$$

$$[d_n, d_{n'}] = (n'-n)\, d_{n+n'} + a\delta_{n',-n} K_n c, \qquad n, n' \in \mathbb{Z}.$$

Note that d_j operates on $\tilde{\mathfrak{p}}$ as $N^{-1} t^{Nj+1}(d/dt)$.

Now we can state the result of the calculations above as follows:

PROPOSITION 2.27. *Assume that c operates on V as multiplication by a scalar $m \neq -g$; set $a = (m+g)^{-1}$. Then the representation of $\hat{\mathfrak{p}}'$ on V extends to a representation of $\hat{\hat{\mathfrak{p}}}_a$ on V by*

$$d_j \mapsto D_j := (-2N(m+g))^{-1}\, T_j, \qquad j \in \mathbb{Z}. \quad ∎$$

Remark. In the case $k = N = 1$, a version of Proposition 2.27 was stated in an unpublished manuscript of G. Segal; unfortunately, his calculation contained an error. A more explicit version of Proposition 2.27 is established in [7] and [23] for all level 1 modules $L(\Lambda)$ of $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$, and $B_l^{(1)}$.

III. Classical Theta Functions and Modular Forms

In Section III we present the necessary background on theta functions and modular forms (cf. the books [5, 12, 24, 26, 33, 50]).

3.1. *Transformation Properties of Theta Functions*

Let $U_{\mathbb{R}}$ be an l-dimensional real vector space, $\langle\, ,\rangle$ a positive-definite symmetric bilinear form on $U_{\mathbb{R}}$. Introduce the Heisenberg group $N_{\mathbb{R}} = U_{\mathbb{R}} \times U_{\mathbb{R}} \times \mathbb{R}$, with multiplication

$$(\alpha, \beta, t)(\alpha', \beta', t') = (\alpha + \alpha', \beta + \beta', t + t' + \tfrac{1}{2}[\langle \alpha', \beta\rangle - \langle \alpha, \beta'\rangle]).$$

It is useful to know for computations that

$$(\alpha, \beta, t)(\alpha', \beta', t')(\alpha, \beta, t)^{-1}(\alpha', \beta', t')^{-1} = (0, 0, \langle \alpha', \beta\rangle - \langle \alpha, \beta'\rangle).$$

Let $\mathscr{H}_+ = \{\tau = x + iy \mid x, y \in \mathbb{R}, y > 0\}$ be the Poincaré upper half-plane, and let $SL(2, \mathbb{R})$ act on $\mathscr{H}_+$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Introduce the metaplectic group

$$\mathrm{Mp}(2, \mathbb{R}) = \left\{ (A, j) \mid A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \right.$$
$$\left. j\colon \mathscr{H}_+ \to \mathbb{C} \text{ holomorphic}, j(\tau)^2 = c\tau + d \right\},$$

with multiplication $(A, j)(A', j') = (AA', j'')$, where $j''(\tau) = j(A'\tau) j'(\tau)$.

Let $\mathrm{Mp}(2, \mathbb{R})$ act on $N_{\mathbb{R}}$ by automorphisms

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j \right) \cdot (\alpha, \beta, t) = (a\alpha + b\beta, c\alpha + d\beta, t),$$

and form the semidirect product $G_{\mathbb{R}} = \mathrm{Mp}(2, \mathbb{R}) \ltimes N_{\mathbb{R}}$, with $gng^{-1} = g \cdot n$ for $g \in \mathrm{Mp}(2, \mathbb{R})$, $n \in N_{\mathbb{R}}$.

Let $U = U_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and extend $\langle\, ,\rangle$ to a symmetric $\mathbb{C}$-bilinear form on U.

Let $G_{\mathbb{R}}$ act on

$$Y := \mathscr{H}_+ \times U \times \mathbb{C}$$

by analytic maps:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right) \cdot (\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t + \frac{c}{2}\frac{\langle z, z\rangle}{c\tau + d}\right);$$

$$(\alpha, \beta, t_0) \cdot (\tau, z, t) = (\tau, z - \alpha + \tau\beta, t - \langle\beta, z\rangle - \tfrac{1}{2}\tau\langle\beta, \beta\rangle + \tfrac{1}{2}\langle\alpha, \beta\rangle + t_0).$$

This action is well-known in the theory of theta functions. We will often write $A \cdot (\tau, z, t)$ for $(A, j) \cdot (\tau, z, t)$. Note that

$$(\alpha, 0, 0) \cdot (\tau, z, t) = (\tau, z - \alpha, t);$$

$$(0, \beta, 0) \cdot (\tau, z, t) = (\tau, z + \tau\beta, t - \langle\beta, z\rangle - \tfrac{1}{2}\tau\langle\beta, \beta\rangle),$$

which is nothing else but the action (1.6); and

$$(0, 0, t_0) \cdot (\tau, z, t) = (\tau, z, t + t_0).$$

Now we define a right action of the group $G_{\mathbb{R}}$ on functions on Y by

$$f|_{(A,j)}(\tau, z, t) = j(\tau)^{-l} f(A \cdot (\tau, z, t)),$$

$$f|_n(\tau, z, t) = f(n \cdot (\tau, z, t)).$$

Fix a lattice L spanning $U_{\mathbb{R}}$ such that $\langle\gamma, \gamma'\rangle \in \mathbb{Z}$ for $\gamma, \gamma' \in L$. Let $L^* := \{\gamma \in U_{\mathbb{R}} \mid \langle\gamma, \alpha\rangle \in \mathbb{Z}$ for all $\alpha \in L\}$ be the dual lattice, so that $L \subset L^*$.

Let

$$N_{\mathbb{Z}} = \{(\alpha, \beta, t) \in N_{\mathbb{R}} \mid \alpha, \beta \in L, t + \tfrac{1}{2}\langle\alpha, \beta\rangle \in \mathbb{Z}\}.$$

This is a subgroup of $N_{\mathbb{R}} \subset G_{\mathbb{R}}$. Denote by $G_{\mathbb{Z}}$ the normalizer of $N_{\mathbb{Z}}$ in $G_{\mathbb{R}}$. It is easy to see that

$$G_{\mathbb{Z}} = \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right)(\alpha, \beta, t) \in G_{\mathbb{R}} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z});\right.$$

$$\left. bd\langle\gamma, \gamma\rangle \equiv 2\langle\alpha, \gamma\rangle \bmod 2\mathbb{Z},\ ac\langle\gamma, \gamma\rangle \equiv 2\langle\beta, \gamma\rangle \bmod 2\mathbb{Z} \text{ for all } \gamma \in L\right\}.$$

Now we introduce the space $\widetilde{Th}_1$ of all holomorphic functions f on the complex manifold Y such that

$$f|_n = f \text{ for all } n \in N_{\mathbb{Z}}, \qquad \text{and} \qquad f|_{(0,0,t)} = e^{-2\pi i t} f \text{ for all } t \in \mathbb{R}.$$

It is clear that the space $\widetilde{Th}_1$ is $G_{\mathbb{Z}}$-invariant.

Remark. One has the following geometric interpretation of the space $\widetilde{Th}_1$ of functions on the complex manifold Y. Consider the maps $Y \to^f \mathscr{H}_+ \times U \times \mathbb{C} \to^\pi \mathscr{H}_+ \times U$ defined by $f(\tau, z, t) = (\tau, z, e^{-2\pi i t})$, $\pi(\tau, z, t) = (\tau, z)$. The action of $G_{\mathbb{R}}$ on Y induces actions of $G_{\mathbb{R}}$ on $\mathscr{H}_+ \times U \times \mathbb{C}$ and $\mathscr{H}_+ \times U$ by analytic maps such that f and π are equivariant. Moreover, $\mathbb{Z} = \{(0, 0, t) \in N_{\mathbb{R}} | t \in \mathbb{Z}\}$ acts trivially on $\mathscr{H}_+ \times U \times \mathbb{C}$ and $\mathscr{H}_+ \times U$, and $\bar{N}_{\mathbb{Z}} := N_{\mathbb{Z}}/\mathbb{Z}$ acts freely on $\mathscr{H}_+ \times U$. We regard π in the obvious way as the bundle projection of a holomorphic line bundle. Then $\bar{N}_{\mathbb{Z}}$ acts by bundle morphisms, so that we obtain a holomorphic line bundle $\mathscr{L}$: $(\mathscr{H}_+ \times U \times \mathbb{C}) \bmod \bar{N}_{\mathbb{Z}} \to^{\bar{\pi}} (\mathscr{H}_+ \times U) \bmod \bar{N}_{\mathbb{Z}}$. Note that the fibers $U/(L + \tau L)$ of the map $(\tau, z) \bmod \bar{N}_{\mathbb{Z}} \mapsto \tau$ are abelian varieties, so that we may regard $\mathscr{L}$ as a bundle over a family of abelian varieties. Let $\sigma: \mathscr{H}_+ \times U \times \mathbb{C} \to (\mathscr{H}_+ \times U \times \mathbb{C}) \bmod \bar{N}_{\mathbb{Z}}$ be the canonical map. Then $\widetilde{Th}_1$ is the pullback to Y under $\sigma \circ f$ of the space of holomorphic sections of the line bundle $\mathscr{L}^{-1}$ dual to $\mathscr{L}$, regarded as functions on the total space of $\mathscr{L}$ which are linear on the fibers. Hence, $\widetilde{Th}_1$ is canonically identified with the space of holomorphic sections of the line bundle $\mathscr{L}^{-1}$ over the family of abelian varieties $U/(L + \tau L)$, $\tau \in \mathscr{H}_+$, in a way consistent with the action of $G_{\mathbb{Z}}$.

Define an $N_{\mathbb{R}}$-invariant measure $dn = d\alpha\, d\beta\, dt$ on the homogeneous space $N_{\mathbb{Z}} \backslash N_{\mathbb{R}}$, where $n = (\alpha, \beta, t) \in N_{\mathbb{R}}$. Then $d(gng^{-1}) = dn$ for $g \in G_{\mathbb{Z}}$.

For $f, f' \in \widetilde{Th}_1$, define the pairing $(f, f'): \mathscr{H}_+ \to \mathbb{C}$ by

$$(f, f')(\tau) = \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} (f|_n)(\tau, 0, 0)\overline{(f'|_n)(\tau, 0, 0)}\, dn.$$

PROPOSITION 3.1. *Let $f \in \widetilde{Th}_1$, $g = (A, j) \in \mathrm{Mp}(2, \mathbb{R})$, $n \in N_{\mathbb{R}}$, $gn \in G_{\mathbb{Z}}$. Then $f|_{gn} \in \widetilde{Th}_1$ and $\|f|_{gn}\|^2 (\tau) = |j(\tau)|^{-2l} \|f\|^2 (A\tau)$, where $\|f\|^2$ stands for (f, f).*

Proof. $f|_{gn} \in \widetilde{Th}_1$ since gn normalizes $N_{\mathbb{Z}}$ and centralizes $(0, 0, t)$. For $n' \in N_{\mathbb{Z}} \backslash N_{\mathbb{R}}$ and $n'' = (gn)\, n' n (gn)^{-1}$, one has $dn'' = dn'$ by previous remarks, so that:

$$\begin{aligned}\|f|_{gn}\|^2 (\tau) &= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} |(f|_{gnn'})(\tau, 0, 0)|^2\, dn' \\ &= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} |(f|_{n''g})(\tau, 0, 0)|^2\, dn'' \\ &= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} |j(\tau)^{-l} (f|_{n''})(A\tau, 0, 0)|^2\, dn'' \\ &= |j(\tau)|^{-2l} \|f\|^2 (A\tau). \quad \blacksquare\end{aligned}$$

Now we define the Riemann theta function:

$$\Theta^L(\tau, z, t) = e^{-2\pi i t} \sum_{\gamma \in L} \exp[\pi i \tau \langle \gamma, \gamma \rangle - 2\pi i \langle \gamma, z \rangle].$$

This series converges absolutely on Y to a holomorphic function. It is easy to check that $\Theta^L \in \widetilde{Th}_1$. For $\mu \in U$, set

$$\Theta^L_\mu = \Theta^L |_{(0, -\mu, 0)},$$

so that

$$\Theta^L_\mu(\tau, z, t) = e^{-2\pi i t} \sum_{\gamma \in L + \mu} \exp[\pi i \tau \langle \gamma, \gamma \rangle - 2\pi i \langle \gamma, z \rangle].$$

Θ^L_μ is called a *classical theta function* of *degree* 1 (with *characteristic* μ). It is clear that Θ^L_μ depends only on μ mod L. For $\mu, \mu' \in L^*$, we have $\Theta^L_\mu \in \widetilde{Th}_1$ and:

$$\begin{aligned} \Theta^L_\mu |_{(\mu', 0, 0)} &= (\exp 2\pi i \langle \mu, \mu' \rangle)\, \Theta^L_\mu, \\ \Theta^L_\mu |_{(0, \mu', 0)} &= \Theta^L_{\mu - \mu'}. \end{aligned} \tag{3.2}$$

For $\tau \in \mathscr{H}_+$, set $Y_\tau = \{\tau\} \times U \times \mathbb{C} \subset Y$.

PROPOSITION 3.3. (a) *For* $\mu, \mu' \in L^*$,

$$(\Theta^L_\mu, \Theta^L_{\mu'})(\tau) = \operatorname{vol}(L)\, \delta_{L+\mu, L+\mu'} (2 \operatorname{Im} \tau)^{-l/2}.$$

In particular, the functions Θ^L_μ, $\mu \in L^*$ mod L, *are linearly independent on* Y_τ *for each* $\tau \in \mathscr{H}_+$.

(b) *For any non-zero* $\tau_0 \in i\mathbb{R}_+$, *the functions* Θ^L_μ, $\mu \in L^*$ mod L, *are linearly independent on* $\{\tau_0\} \times L^* \times \{0\} \subset Y$.

Proof. Proposition 3.1 and (3.2) show that $(\Theta^L_\mu, \Theta^L_{\mu'})(\tau) = \delta_{L+\mu, L+\mu'} \|\Theta^L\|^2 (\tau)$. We compute:

$$\begin{aligned} \|\Theta^L\|^2 (\tau) &= \int_{N_\mathbb{Z} \backslash N_\mathbb{R}} |\Theta^L(\tau, -\alpha + \tau\beta, -\tfrac{1}{2}\tau\langle \beta, \beta \rangle + \tfrac{1}{2}\langle \alpha, \beta \rangle + t)|^2 \, d(\alpha, \beta, t) \\ &= \int_{U_\mathbb{R}/L} \int_{U_\mathbb{R}/L} \left| \sum_{\gamma \in L} \exp[\pi i \tau \langle \gamma - \beta, \gamma - \beta \rangle + 2\pi i \langle \alpha, \gamma \rangle] \right|^2 d\alpha \, d\beta \\ &= \left(\int_{U_\mathbb{R}/L} d\alpha \right) \int_{U_\mathbb{R}} |\exp \pi i \tau \langle \beta, \beta \rangle|^2 \, d\beta = \operatorname{vol}(L) (2 \operatorname{Im} \tau)^{-l/2}. \end{aligned}$$

This proves (a). In order to prove (b), note that for $z \in L^*$ and $\mu \in L^*$,

$$\Theta^L_\mu(\tau_0, z, 0) = e^{-2\pi i \langle \mu, z \rangle} \sum_{\gamma \in L + \mu,} e^{\pi i \langle \gamma, \gamma \rangle \tau_0}. \tag{3.3.1}$$

It is clear that the sum in (3.3.1) is positive for $\tau_0 \in i\mathbb{R}_+$. (b) now follows from the fact that the characters of the group L^*/L are linearly independent. ∎

The following result goes back to Jacobi.

PROPOSITION 3.4. *For $\mu \in L^*$, one has the following transformation law:*

$$\Theta_\mu^L\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{\langle z, z\rangle}{2\tau}\right) = |L^*/L|^{-1/2}(-i\tau)^{l/2}$$
$$\times \sum_{\mu' \in L^* \bmod L} [\exp -2\pi i \langle \mu, \mu' \rangle]\, \Theta_{\mu'}^L(\tau, z, t).$$

Proof. This follows from the Poisson summation formula (see, e.g., [39]). ∎

Let $Th_1 \subset \widetilde{Th}_1$ be the $\mathbb{C}$-span of the linearly independent set $\{\Theta_\mu^L \mid \mu \in L^* \bmod L\}$.

PROPOSITION 3.5. *The space Th_1 is invariant under $G_{\mathbb{Z}}$. Furthermore, the matrix of any $g \in G_{\mathbb{Z}}$ with respect to the basis $\{\Theta_\mu^L \mid \mu \in L^* \bmod L\}$ of Th_1 is unitary.*

Proof. It suffices to prove the first statement, since the second follows from it by Propositions 3.1 and 3.3.

By (3.2), Th_1 is invariant under $G_{\mathbb{Z}} \cap N_{\mathbb{R}}$. Therefore, if $g_k = (A_k, j_k)\, n_k \in G_{\mathbb{Z}}$, $k = 1, 2$, and the A_k generate $SL(2, \mathbb{Z})$, it suffices to show that Th_1 is invariant under the g_k. For $A_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, Proposition 3.4 shows this. For $A_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $j_2 = 1$, $n_2 = (\alpha, 0, 0)$, where $\alpha \in U_{\mathbb{R}}$ satisfies $\langle \gamma, \gamma \rangle \equiv 2\langle \alpha, \gamma \rangle \bmod 2\mathbb{Z}$ for all $\gamma \in L$,

$$\Theta_\mu^L|_{g_2} = [\exp \pi i \langle \mu, \mu + 2\alpha \rangle]\, \Theta_\mu^L$$

shows it. The proposition follows since the matrices A_1 and A_2 generate the group $SL(2, \mathbb{Z})$. ∎

In order to obtain a more general transformation law for theta functions, we need two further results.

PROPOSITION 3.6. *Suppose that $\langle \gamma, \gamma \rangle \in 2\mathbb{Z}$ for all $\gamma \in L$. Then:*

$$\sum_{\gamma \in L^* \bmod L} e^{\pi i \langle \gamma, \gamma \rangle} = e^{2\pi i l/8} \sqrt{|L^*/L|}.$$

This result of Milgram is proved in [30].

Let n be a positive integer. Taking $L=\mathbb{Z}$, with bilinear form $\langle m, m'\rangle = 4nmm'$, Proposition 3.6 gives

$$\sqrt{n} = e^{-\pi i/4} \sum_{\substack{m\in\mathbb{Z} \bmod 2n\mathbb{Z}\\ m\equiv n \bmod 2\mathbb{Z}}} e^{\pi i m^2/4n}.$$

Hence, $\sqrt{n}\in\mathbb{Q}(e^{2\pi i/4n})$ for all non-zero $n\in\mathbb{Z}$, and $\sqrt{n}\in\mathbb{Q}(e^{2\pi i/n})$ if $n\equiv 1 \bmod 4\mathbb{Z}$.

For relatively prime integers n and k with k odd or $n\equiv 1 \bmod 4\mathbb{Z}$, we now define the *extended Jacobi symbol* $(\frac{n}{k})=\pm 1$. If $n=0$, put $(\frac{0}{\pm 1})=\pm 1$. If $n\neq 0$, define an automorphism σ_k of the field $F=\bigcup_N \mathbb{Q}(e^{2\pi i/N})$, where N runs over all positive integers relatively prime to k, by $\sigma_k(e^{2\pi i/N})=e^{2\pi ik/N}$. Since $\sqrt{n}\in F$, we may define the extended Jacobi symbol by:

$$\sigma_k(\sqrt{n}) = \left(\frac{n}{k}\right)\sqrt{n}.$$

LEMMA 3.7. *Suppose that $\langle\gamma,\gamma\rangle\in 2\mathbb{Z}$ for all $\gamma\in L$. If $|L^*/L|$ is odd, then the rank l of L is even, and*

$$(-1)^{l/2}\,|L^*/L|\equiv 1 \bmod 4\mathbb{Z}.$$

Proof. Put $N=|L^*/L|$, so that $NL^*\subset L$, and hence $N\langle\gamma,\gamma\rangle\in\mathbb{Z}$ for all $\gamma\in L^*$. Then $e^{2\pi il/8}\sqrt{N}\in\mathbb{Q}(e^{2\pi i/N})$ by Proposition 3.6. Choose $\varepsilon=\pm 1$ such that $\varepsilon N\equiv 1 \bmod 4\mathbb{Z}$. Then $\sqrt{\varepsilon N}\in\mathbb{Q}(e^{2\pi i/N})$. Hence, $\sqrt{\varepsilon}\,e^{-2\pi il/8}\in\mathbb{Q}(e^{2\pi i/N})\cap\mathbb{Q}(e^{2\pi i/8})=\mathbb{Q}$, so that $\varepsilon=e^{2\pi il/4}$. ∎

The following proposition gives a transformation law for theta functions which is sufficient for our purpose.

PROPOSITION 3.8. *Let $A=(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix})\in SL(2,\mathbb{Z})$, $(A,j)\in \mathrm{Mp}(2,\mathbb{R})$. Let $\alpha_0,\beta_0\in U_{\mathbb{R}}$ satisfy*

$$bd\langle\beta,\beta\rangle\equiv 2\langle\alpha_0,\beta\rangle \bmod 2\mathbb{Z} \qquad \textit{for all}\quad \beta\in L,$$

$$ac\langle\alpha,\alpha\rangle\equiv 2\langle\alpha,\beta_0\rangle \bmod 2\mathbb{Z} \qquad \textit{for all}\quad \alpha\in L^* \textit{ such that } c\alpha\in L.$$

(Such α_0 and β_0 always exist.) Fix $t_0\in\mathbb{R}$ and set $g=(A,j)(\alpha_0,\beta_0,t_0)$, so that $g\in G_{\mathbb{Z}}$. Then there exists $v(g)\in\mathbb{C}$ such that:

(a) $\Theta^L|_g = v(g)\sum_{\substack{\alpha\in L^*\\ c\alpha \bmod L}} (\exp \pi i[cd\langle\alpha,\alpha\rangle + 2\langle\alpha, c\alpha_0+d\beta_0\rangle])\,\Theta^L_{c\alpha}$.

(b)(i) $|v(g)| = |(L+cL^*)/L|^{-1/2}$, $v(g^{-1})=\overline{v(g)}$;

(ii) *if* $b\,|\gamma|^2 \in 2\mathbb{Z}$ *for all* $\gamma \in L$, $c\,|\gamma|^2 \in 2\mathbb{Z}$ *for all* $\gamma \in L^*$, *and* $\alpha_0 = \beta_0 = 0$, $t_0 = 0$, *then*

$$v(g) = \left(\frac{(-1)^{l/2}\,|L^*/L|}{d}\right) \qquad \textit{for even } l,$$

$$v(g)j(\tau)^l = \left(\frac{c\,|L^*/L|}{d}\right) e^{\pi i(d-1)l/4}(c\tau + d)^{l/2} \qquad \textit{for odd } l.$$

Proof. By Proposition 3.5, we may write $\Theta^L\,|_g = \sum_{\mu \in L^* \bmod L} f(\mu)\,\Theta^L_\mu$. Let $\alpha \in L^*$. Formula (3.2) shows that if $n = (\alpha, 0, 0)$, then $\Theta^L\,|_n = \Theta^L$, so that $(\Theta^L\,|_g)\,|_{(g^{-1}ng)} = \Theta^L\,|_g$. Using (3.2), this gives

$$f(\mu + c\alpha) = (\exp \pi i[cd\langle\alpha, \alpha\rangle + 2d\langle\alpha, \mu\rangle + 2\langle\alpha, c\alpha_0 + d\beta_0\rangle])\,f(\mu). \quad (3.8.1)$$

If, in addition, $c\alpha \in L$, then $L + \mu = L + \mu + c\alpha$ gives $f(\mu + c\alpha) = f(\mu)$. Therefore, $f(\mu) = 0$ unless one has mod $2\mathbb{Z}$:

$$\begin{aligned} 0 &\equiv cd\langle\alpha, \alpha\rangle + 2d\langle\alpha, \mu\rangle + 2\langle\alpha, c\alpha_0 + d\beta_0\rangle \\ &\equiv cd\langle\alpha, \alpha\rangle + 2d\langle\alpha, \mu\rangle + bd\langle c\alpha, c\alpha\rangle + ac\langle d\alpha, d\alpha\rangle \\ &\equiv 2d\langle\alpha, \mu\rangle. \end{aligned}$$

Therefore, $f(\mu) = 0$ unless $\alpha \in L^*$ and $c\alpha \in L$ (or, equivalently, $\alpha \in (L + cL^*)^*$) imply that $d\langle\alpha, \mu\rangle \in \mathbb{Z}$. Therefore, $f(\mu) = 0$ unless $d\mu \in L + cL^*$. But also, $f(\mu) = 0$ unless $c\mu \in cL^*$. Since c and d are relatively prime, we find that $f(\mu) = 0$ unless $\mu \in L + cL^*$.

On the other hand, we obtain from (3.8.1):

$$f(c\alpha) = (\exp \pi i[cd\langle\alpha, \alpha\rangle + 2\langle\alpha, c\alpha_0 + d\beta_0\rangle])\,f(0).$$

Setting $v(g) = f(0)$, we obtain (a).

(b)(i) now follows from Proposition 3.5.

It is easy to check (b)(ii) for $c = 0$. Assuming $c \neq 0$, we now compute $v(g)$ in general in terms of Gauss sums. Note that:

$$v(g) = \lim_{\operatorname{Im}\tau \to +\infty} (\Theta^L\,|_g)(\tau, 0, 0). \quad (3.8.2)$$

Putting

$$\lambda = -\frac{1}{c(c\tau + d)}, \qquad t = t_0 - \frac{1}{2}\langle\alpha_0, \beta_0\rangle - \frac{d}{2c}\langle\beta_0, \beta_0\rangle,$$

one computes directly that:

$$(\Theta^L |_g)(\tau, 0, 0) = j(\tau)^{-l} e^{-2\pi i t} \times \sum_{\mu \in L \bmod cL} e^{\pi i a c^{-1}\langle \mu, \mu\rangle} e^{-2\pi i c^{-1}\langle \mu, \beta_0\rangle} \times \sum_{\gamma \in \mu + cL} e^{\pi i \lambda \langle \gamma - c\alpha_0 - d\beta_0, \gamma - c\alpha_0 - d\beta_0\rangle}. \tag{3.8.3}$$

Using Proposition 3.4, one obtains

$$\sum_{\gamma \in \mu + cL} e^{\pi i \lambda \langle \gamma - c\alpha_0 - d\beta_0, \gamma - c\alpha_0 - d\beta_0\rangle} = |L^*/L|^{-1/2} (i/c^2\lambda)^{l/2} \sum_{\gamma \in L^*} e^{-\pi i c^{-2}\lambda^{-1}\langle \gamma, \gamma\rangle} e^{-2\pi i c^{-1}\langle \gamma, \mu - c\alpha_0 - d\beta_0\rangle}. \tag{3.8.4}$$

Combining (3.8.2)–(3.8.4), we obtain

$$j(\tau)^l |(L + cL^*)/L|^{1/2} e^{2\pi i t_0} v(g) = (c\tau + d)^{l/2} (-i \operatorname{sign} c)^{l/2} e^{\pi i c^{-1}\langle \beta_0, c\alpha_0 + d\beta_0\rangle} \times |L/(L \cap cL^*)|^{-1/2} \sum_{\mu \in L \bmod (L \cap cL^*)} e^{\pi i c^{-1}\langle \mu, a\mu - 2\beta_0\rangle}. \tag{3.8.5}$$

In the situation of (b)(ii), the last sum may be evaluated by using Proposition 3.6:

$$\sum_{\mu \in L \bmod cL^*} e^{\pi i a c^{-1}\langle \mu, \mu\rangle} = \sigma_a(e^{2\pi i (\operatorname{sign} c) l/8} \sqrt{|L/cL^*|}). \tag{3.8.6}$$

For odd a, (3.8.5)–(3.8.6) yield

$$j(\tau)^l v(g) = (c\tau + d)^{l/2} e^{2\pi i (a-1) l/8} \left(\frac{c^l |L^*/L|}{a}\right).$$

For even a, Lemma 3.7 shows that l is even and $(-1)^{l/2} |L^*/L| \equiv 1 \bmod 4\mathbb{Z}$, so that (3.8.5)–(3.8.6) now yield

$$v(g) = \left(\frac{(-1)^{l/2} |L^*/L|}{a}\right).$$

Applying these formulas to find $v(g^{-1})$, and using $v(g)^{-1} = v(g^{-1})$ from (b)(i), we obtain (b)(ii). ∎

COROLLARY 3.9. *Let $g \in G_{\mathbb{Z}}$ be as in Proposition 3.8, and let $\mu \in L^*$.*

Then

$$\Theta^L_\mu|_g = v(g) \sum_{\substack{\alpha\in L^* \\ c\alpha \bmod L}} (\exp \pi i[cd\langle\alpha,\alpha\rangle + 2bc\langle\alpha,\mu\rangle + ab\langle\mu,\mu\rangle + 2\langle\mu, a\alpha_0 + b\beta_0\rangle + 2\langle\alpha, c\alpha_0 + d\beta_0\rangle])\, \Theta^L_{a\mu+c\alpha}.$$

Proof. Set $n=(0,-\mu,0)$, write $\Theta^L_\mu|_g = (\Theta^L|_n)|_g = (\Theta^L|_g)|_{(g^{-1}ng)}$ and compute using Proposition 3.8 and (3.2). ∎

3.2. *The Ring of Theta Functions*

For any $m\in\mathbb{Z}$, denote by $\widetilde{Th}_m$ the space of holomorphic functions f on Y such that

$$f|_n = f \text{ for all } n\in N_{\mathbb{Z}} \qquad \text{and} \qquad f|_{(0,0,t)} = e^{-2\pi imt} f \text{ for all } t\in\mathbb{R}.$$

Remark. The space $\widetilde{Th}_m$ is the space of holomorphic sections of the line bundle $\mathscr{L}^{-m}$, where $\mathscr{L}$ is the line bundle constructed in Section 3.1.

Let $\widetilde{Th} = \bigoplus_{m\in\mathbb{Z}} \widetilde{Th}_m$. This is, clearly, a $\mathbb{Z}$-graded algebra over the ring $\mathscr{O}(\mathscr{H}_+) = \widetilde{Th}_0$ of all holomorphic functions of $\tau\in\mathscr{H}_+$. $\widetilde{Th}$ is called the *ring of theta functions.*

For $\mu\in U$ and a positive integer m, set

$$\Theta^L_{\mu,m}(\tau,z,t) = e^{-2\pi imt} \sum_{\gamma\in L+m^{-1}\mu} \exp(\pi im\tau\langle\gamma,\gamma\rangle - 2\pi im\langle\gamma,z\rangle).$$

$\Theta^L_{\mu,m}$ is called a *classical theta function* of *degree m* (and *characteristic* μ). Then $\Theta^L_{\mu,m}$ depends only on μ mod mL, and for $\mu,\mu'\in L^*$, one has $\Theta^L_{\mu,m}\in\widetilde{Th}_m$ and

$$\Theta^L_{\mu,m}|_{(m^{-1}\mu',0,0)} = e^{2\pi im^{-1}\langle\mu,\mu'\rangle}\Theta^L_{\mu,m}, \tag{3.10}$$
$$\Theta^L_{\mu,m}|_{(0,m^{-1}\mu',0)} = \Theta^L_{\mu-\mu',m}.$$

Remark also that taking $L'=L$ and $\langle\ ,\ \rangle' = m\langle\ ,\ \rangle$, we have

$$\Theta^L_{\mu,m}(\tau,z,t) = \Theta^{L'}_{m^{-1}\mu}(\tau,z,mt). \tag{3.11}$$

LEMMA 3.12. *Let $\tau\in\mathscr{H}_+$. For $m\in\mathbb{Z}$, let Th^τ_m be the space of holomorphic functions f on Y_τ which are $N_{\mathbb{Z}}$-invariant and satisfy $f|_{(0,0,t)} = e^{-2\pi imt}f$ for all $t\in\mathbb{R}$. Then:*

(a) $Th^\tau_0 = \mathbb{C}$.

(b) $Th^\tau_m = (0)$ *for* $m<0$.

(c) $\{\Theta^L_{\mu,m}|_{Y_\tau}\}_{\mu\in L^* \bmod mL}$ *is a* $\mathbb{C}$*-basis of* Th^τ_m *for* $m>0$.

Proof. If $f \in Th_0^\tau$, then f is independent of t, is holomorphic, and is periodic in z with respect to $L+\tau L$. Since $U/L+\tau L$ is compact, f is constant. This proves (a).

Now suppose $m \in \mathbb{Z}$, $m \neq 0, f \in Th_m^\tau$. Using $f|_{(0,0,t)} = e^{-2\pi i m t} f, f|_{(\alpha,0,0)} = f$ for $\alpha \in L$, and f holomorphic, write

$$f(\tau, z, t) = e^{-2\pi i m t} \sum_{\gamma \in L^*} a(\gamma)\, e^{-2\pi i \langle \gamma, z\rangle}.$$

Using $f|_{(0,\beta,0)} = f$ for $\beta \in L$, we find that $a(\gamma)\, e^{-\pi i m^{-1} \tau \langle \gamma, \gamma\rangle}$ depends only on $\gamma \bmod mL$.

If m is negative and $f \neq 0$, this shows that the $|a(\gamma)|$ increase without bound, a contradiction (since Fourier coefficients must tend to 0). This proves (b). If m is positive, it shows that f is in the $\mathbb{C}$-span of $\{\Theta^L_{\mu,m}|_{Y_\tau} \mid \mu \in L^* \bmod mL\}$. These functions are linearly independent by Proposition 3.3(a) and (3.11), which proves (c). ∎

PROPOSITION 3.13. *The ring* $\widetilde{Th}$ *of theta functions is a free module over* $\mathcal{O}(\mathcal{H}_+)$ *with basis* $\{\Theta^L_{\mu,m} \mid m \in \mathbb{Z},\, m > 0,\, \mu \in L^* \bmod mL\} \cup \{1\}$.

Proof. This is immediate from Lemma 3.12. ∎

By Proposition 3.13, we may expand $\Theta^L_{\mu,m}\Theta^L_{\mu',m'}$ as a linear combination of the $\Theta^L_{\mu'',m+m'}$. The coefficients are given by:

PROPOSITION 3.14. *Let* $\mu_1, \mu_2 \in L^*$, $m_1, m_2 \in \mathbb{Z}$, $m_1, m_2 > 0$. *Then*

$$\Theta^L_{\mu_1,m_1}\Theta^L_{\mu_2,m_2} = \textstyle\sum_{\gamma \in L \bmod (m_1+m_2)L} d_\gamma \Theta^L_{\mu_1+\mu_2+m_1\gamma, m_1+m_2},$$

where

$$d_\gamma = \Theta^L_{m_2\mu_1 - m_1\mu_2 + m_1 m_2 \gamma, m_1 m_2 (m_1+m_2)}(\tau, 0, 0).$$

Proof. Write

$$\Theta^L_{\mu_i,m_i}(\tau, z, t) = e^{-2\pi i m_i t} \sum_{\gamma_i \in L} \exp(\pi i m_i^{-1} \tau \langle m_i\gamma_i + \mu_i, m_i\gamma_i + \mu_i\rangle$$
$$- 2\pi i \langle m_i\gamma_i + \mu_i, z\rangle).$$

Reparametrize the resulting sum for $\Theta^L_{\mu_1,m_1}\Theta^L_{\mu_2,m_2}$ by $\gamma = \gamma_1 - \gamma_2$, $\gamma' = m_1\gamma_1 + m_2\gamma_2$, and write the sum as $\sum_{\gamma \in L} \sum_{\gamma' \in m_1\gamma + (m_1+m_2)L}$, obtaining

$$\sum_{\gamma \in L} \Theta_{\mu_1+\mu_2+m_1\gamma, m_1+m_2}(\tau, z, t)$$
$$\times \exp\left(\pi i \tau \frac{m_1 m_2}{m_1+m_2} \langle \gamma + m_1^{-1}\mu_1 - m_2^{-1}\mu_2, \gamma + m_1^{-1}\mu_1 - m_2^{-1}\mu_2\rangle\right).$$

This gives the desired result. ∎

Let m be a positive integer. Denote by Th_m the $\mathbb{C}$-span of the functions $\Theta^L_{\mu,m}$, $\mu \in L^*$ mod mL. Using (3.11), we deduce from Propositions 3.4, 3.5, 3.8, and Corollary 3.9:

PROPOSITION 3.15. *Let $\mu \in L^*$. Then*

$$\Theta^L_{\mu,m}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t+\frac{\langle z,z\rangle}{2\tau}\right) = |L^*/mL|^{-1/2}(-i\tau)^{l/2} \times \sum_{\mu'\in L^* \bmod mL} \exp[-2\pi i m^{-1}\langle\mu,\mu'\rangle]\,\Theta^L_{\mu',m}(\tau,z,t).$$

PROPOSITION 3.16. *The group $G_{\mathbb{Z}}$ preserves the space Th_m. The matrix of any $g\in G_{\mathbb{Z}}$ with respect to the basis $\{\Theta^L_{\mu,m} \mid \mu\in L^*$ mod $mL\}$ of Th_m is unitary.*

PROPOSITION 3.17. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, $(A,j)\in \mathrm{Mp}(2,\mathbb{R})$, and choose $\alpha_0, \beta_0 \in U_{\mathbb{R}}$ satisfying*

$$mbd\langle\beta,\beta\rangle \equiv 2m\langle\alpha_0,\beta\rangle \bmod 2\mathbb{Z} \qquad \text{whenever} \quad \beta\in L; \tag{3.17.1}$$

$$mac\langle\alpha,\alpha\rangle \equiv 2m\langle\alpha,\beta_0\rangle \bmod 2\mathbb{Z} \qquad \text{whenever} \quad c\alpha\in L \text{ and } m\alpha\in L^*. \tag{3.17.2}$$

Fix $t_0\in\mathbb{R}$ and set $g=(A,j)(\alpha_0,\beta_0,t_0)$. Then there exists $v(m,g)\in\mathbb{C}$ such that:

(a) *For all $\mu\in L^*$,*

$$\Theta^L_{\mu,m}|_g = v(m,g) \sum_{\substack{\alpha\in L^* \\ c\alpha \bmod mL}} (\exp \pi i[m^{-1}cd\langle\alpha,\alpha\rangle + 2m^{-1}bc\langle\alpha,\mu\rangle + m^{-1}ab\langle\mu,\mu\rangle + 2\langle\mu, a\alpha_0+b\beta_0\rangle + 2\langle\alpha, c\alpha_0+d\beta_0\rangle])\,\Theta^L_{a\mu+c\alpha,m}.$$

(b)(i) $|v(m,g)| = |(mL+cL^*)/mL|^{-1/2}$, *and* $v(m,g^{-1}) = \overline{v(m,g)}$;

(ii) *if $mb|\gamma|^2\in 2\mathbb{Z}$ for all $\gamma\in L$, $m^{-1}c|\gamma|^2\in 2\mathbb{Z}$ for all $\gamma\in L^*$, and $\alpha_0=\beta_0=0$, $t_0=0$, then*

$$v(m,g) = \left(\frac{(-1)^{l/2}|L^*/L|}{d}\right) \qquad \text{for even } l;$$

$$v(m,g)\,j(\tau)^l = \left(\frac{cm|L^*/L|}{d}\right) e^{\pi i(d-1)l/4}(c\tau+d)^{l/2} \qquad \text{for odd } l.$$

Remark. Define a "Laplacian" D operating on holomorphic functions on Y as follows. Let $u_1, \ldots, u_l$ be an orthonormal basis of $U_{\mathbb{R}}$, and let

$$D = 2\frac{\partial}{\partial t}\frac{\partial}{\partial \tau} + \sum_{i=1}^{l} \left(\frac{\partial}{\partial u_i}\right)^2.$$

Then for $n \in N_{\mathbb{R}}$ and $A = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \mathrm{Mp}(2, \mathbb{R})$, we have

$$(DF)|_n = D(F|_n), \qquad (DF)|_A = (c\tau + d)^2 D(F|_A).$$

Moreover, $Th_m = \{F \in \widetilde{Th}_m \mid DF = 0\}$ for $m \neq 0$. This "explains" why Th_m is invariant under $G_{\mathbb{Z}}$, and allows one to prove most of the results of Section 3.1 without appeal to the explicit expressions for the Θ_μ^L. The details may be found in [50].

3.3. *Some Facts about Modular Forms*

In this section, we summarize information on modular forms which is either used in the sequel or makes it more intelligible.

Recall the action of the group $SL(2, \mathbb{R})$ on the Poincaré upper half-plane $\mathscr{H}_+$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

For $N \in \mathbb{Z}$, $N > 0$, define subgroups $\Gamma_0(N)$ and $\Gamma(N)$ of $SL(2, \mathbb{Z})$ by:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid b \equiv c \equiv 0,\, a \equiv d \equiv 1 \pmod{N} \right\}.$$

Then $\Gamma_0(N)$ and $\Gamma(N)$ are of finite index in $\Gamma(1) = SL(2, \mathbb{Z})$.

Fix a subgroup Γ of finite index in $\Gamma(1)$, a function $\chi: \Gamma \to \mathbb{C}$ with $|\chi(A)| = 1$ for all $A \in \Gamma$, and a real number k. Then a function $f: \mathscr{H}_+ \to \mathbb{C}$ is called a *modular form* of *weight* k and *multiplier system* χ *for* Γ if:

(i) f is holomorphic on $\mathscr{H}_+$;

(ii) if $A = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$ and $\tau \in \mathscr{H}_+$, then $f(A\tau) = \chi(A)(c\tau + d)^k f(\tau)$.

We sometimes suppress the mention of one or more of Γ, χ and k, and speak of "modular forms," etc.

Set $T = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. Then since Γ is of finite index in $\Gamma(1)$, $T^r \in \Gamma$ for some positive integer r. Suppose f satisfies (i) and (ii) above, and let $C \in \mathbb{R}$ satisfy $\chi(T^r) = e^{2\pi i C}$. Set $F(e^{2\pi i \tau/r}) = e^{-2\pi i C\tau/r} f(\tau)$. Then F is a well-defined holomorphic function on the punctured disk $0 < |z| < 1$. Hence, F has a

Laurent expansion $F(z)=\sum_{n\in\mathbb{Z}} a_n z^n$ converging absolutely for $0<|z|<1$. Therefore, we have the "Fourier expansion" or "q-expansion"

$$f(\tau)=\sum_{n\in\mathbb{Z}} a_n e^{2\pi i(n+C)\tau/r}=\sum_{n\in\mathbb{Z}} a_n q^{(n+C)/r} \qquad \text{for} \quad \tau\in\mathscr{H}_+,\ q=e^{2\pi i\tau}.$$

We call f *meromorphic* at $i\infty$ if $a_n=0$ for n sufficiently small, *holomorphic* at $i\infty$ if $a_n\neq 0$ implies $n+C\geqslant 0$, *vanishing* at $i\infty$ if $a_n\neq 0$ implies $n+C>0$. If f is holomorphic at $i\infty$, we say that the *value* of f at $i\infty$ is A_{-C} (interpreted as 0 if $C\notin\mathbb{Z}$). We say that f *vanishes to order m* at $i\infty$ if $a_n\neq 0$ implies $(n+C)/r\geqslant m$.

A *cusp* of Γ is an orbit of Γ on $\mathbb{Q}\cup\{i\infty\}$ under the action $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\tau=(a\tau+b)/(c\tau+d)$, where $a/0$ is interpreted as $i\infty$ for $a\in\mathbb{Q}$, $a\neq 0$. Then since $\Gamma(1)$ acts transitively on $\mathbb{Q}\cup\{i\infty\}$, the set of cusps of Γ is finite. Sometimes we speak of the cusp $\alpha\in\mathbb{Q}\cup\{i\infty\}$ of Γ; this means the orbit of α under Γ.

Let f satisfy (i) and (ii), and consider a cusp α of Γ. Let $B=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma(1)$ be such that $B(i\infty)=\alpha$. Then $f_0(\tau):=(c\tau+d)^{-k}f(B\tau)$ is a modular form of weight k and some multiplier system χ_0 for $B^{-1}\Gamma B$. We say that f is *meromorphic*, *holomorphic*, or *vanishes* at α if f_0 is meromorphic, holomorphic, or vanishes at $i\infty$.

A modular form of weight k and multiplier system χ for Γ is called a *meromorphic modular form*, a *holomorphic modular form*, or a *cusp form* if it is meromorphic, holomorphic, or vanishes at all cusps of Γ. If $f(\tau)$ is a modular form of weight k and if $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in GL(2,\mathbb{Q})$, $ad-bc>0$, then $f_0(\tau):=(c\tau+d)^{-k}f((a\tau+b)/(c\tau+d))$ is a modular form of weight k. Moreover, f_0 is meromorphic, holomorphic or a cusp form if f is.

We shall use the following facts in the sequel.

(a) $SL(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

(b) $\Gamma_0(2)$ is generated by $\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0\\ 2 & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} -1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

(c) If p is prime, then $\Gamma_0(p)$ has two cusps, 0 and $i\infty$.

(d) A holomorphic modular form of weight 0 is constant.

(e) Let

$$\eta(\tau)=e^{2\pi i\tau/24}\prod_{n\geqslant 1}(1-e^{2\pi i n\tau}), \qquad \tau\in\mathscr{H}_+,$$

be the Dedekind η-function. It is a cusp form of weight 1/2 and some multiplier system χ_η for $SL(2,\mathbb{Z})$. We have $\chi_\eta(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right))=e^{2\pi i/24}$, $\chi_\eta(\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right))=e^{-2\pi i/8}$.

(f) Let U, $\langle\ \rangle$, etc., be as in Sections 3.1 and 3.2. For $\alpha,\beta\in U_{\mathbb{R}}$, $\tau\in\mathscr{H}_+$, and F a function on $\mathscr{H}_+\times U\times\mathbb{C}$, define a function $F(\alpha,\beta;\cdot)$ on $\mathscr{H}_+$ by:

$$F(\alpha, \beta; \tau) = (F|_{(\alpha,\beta,0)})(\tau, 0, 0)$$
$$= F(\tau, -\alpha + \tau\beta, -\tfrac{1}{2}\langle\beta, -\alpha + \tau\beta\rangle). \tag{3.18}$$

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $(A, j) \in \mathrm{Mp}(2, \mathbb{R})$. Then

$$F(a\alpha + b\beta, c\alpha + d\beta; A\tau) = j(\tau)^l \, (F|_{(A,j)})(\alpha, \beta; \tau). \tag{3.19}$$

It follows that if F is a classical theta function and $\alpha, \beta \in \mathbb{Q}L$, then $F(\alpha, \beta; \tau)$ is a holomorphic modular form of weight $l/2$ for some $\Gamma(N)$ (see Proposition 3.17).

In particular, the "structure constants" d_γ in Proposition 3.14 are holomorphic modular forms.

(g) We will also need the following fact.

LEMMA 3.20. *Let $d_1, d_2, \ldots$ be a periodic sequence of real numbers with period N, such that $d_j = d_{N-j}$ for $j = 1, \ldots, N-1$; set $d = \sum_{j=1}^{N} d_j$. Then $f(\tau) = q^b \prod_{j=1}^{\infty} (1 - q^j)^{d_j}$, where $q = e^{2\pi i \tau}$, is a modular form if and only if*

$$b = \frac{dN}{24} - \frac{1}{4N} \sum_{j=1}^{N-1} j(N-j)\, d_j. \tag{3.20.1}$$

Proof. We use the Jacobi triple product identity (which is nothing else but formula (2.7) for $\mathfrak{g}$ of type $A_1^{(1)}$):

$$\prod_{j \geqslant 1} (1 - u^j v^j)(1 - u^j v^{j-1})(1 - u^{j-1} v^j) = \sum_{j \in \mathbb{Z}} (-1)^j \, u^{(1/2)j(j+1)} v^{(1/2)j(j-1)}.$$

In this identity we let $u = q^r$, $v = q^{N-r}$, obtaining

$$q^{(2r-N)^2/8N} \prod_{j \geqslant 1} (1 - q^{Nj})(1 - q^{Nj-(N-r)})(1 - q^{Nj-r})$$
$$= \sum_{j \in \mathbb{Z}} (-1)^j \, q^{(N/2)(j + (2r-N)/2N)^2}. \tag{3.20.2}$$

By (f), the left-hand side is a modular form if $r \in \mathbb{Z}$. An easy computation now shows that for b given by formula (3.20.1) the function $f(\tau)$ can be represented as a finite product of real powers of functions of the form (3.20.2) with $r \in \mathbb{Z}$ and a real power of $\eta(N\tau)$, and hence is a modular form. Conversely, if $f(\tau)$ is a modular form for some b, then q^a is a modular form, where a is the difference between b and the right-hand side of (3.20.1); it follows that $a = 0$. ∎

IV. The Theory of String Functions

Section IV is the heart of the paper. Using the transformation properties of theta functions, we establish transformation laws for the numerator and denominator of the character formula and for the string functions. This, together with the fact that a string function multiplied by a "standard" modular form is a cusp form (which is also proved here), allows one in principle to compute any string function. We do this in a number of interesting cases, including "most" of the representations of level 1. Furthermore, using a Tauberian theorem, we deduce from the transformation law the asymptotic behaviour of the multiplicities of the weights. At the end of Section IV we indicate how one can apply our technique to the general restriction problem.

4.1. *Theta Functions and Affine Lie Algebras*

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, $\alpha_0,\ldots,\alpha_l$ the simple roots. Recall the space $\bar{\mathfrak{h}}^* = \mathbb{C}\alpha_1 + \cdots + \mathbb{C}\alpha_l$ and the positive-definite symmetric bilinear form $(\,,\,)$ on $\bar{\mathfrak{h}}^*_{\mathbb{R}} = \mathbb{R}\alpha_1 + \cdots + \mathbb{R}\alpha_l$ (see Section 1.3). Recall the lattices $M \subset M'$ in $\bar{\mathfrak{h}}^*_{\mathbb{R}}$ (see Section 1.6) and note that by Proposition 2.13, $M^* = \bar{P}' \supset M'$ and $M'^* = \bar{P} \supset M$. Here and further on, given a lattice $L \subset \bar{\mathfrak{h}}^*_{\mathbb{R}}$, L^* denotes the dual lattice with respect to the symmetric bilinear form $(\,,\,)$.

Introduce coordinates on $Y = \{h \in \mathfrak{h} \mid \operatorname{Re}\delta(h) > 0\}$ as follows. Let $\mathscr{H}_+ := \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$ be the upper half-plane. For $\tau \in \mathscr{H}_+$, $z \in \bar{\mathfrak{h}}^*$ and $t \in \mathbb{C}$, define $h = (\tau, z, t) \in \mathfrak{h}$ by requiring that for all $\lambda \in \mathfrak{h}^*$,

$$\lambda(h) = -2\pi i(\lambda, \tau\Lambda_0 + z + t\delta).$$

This allows us to identify Y with the domain $\mathscr{H}_+ \times \bar{\mathfrak{h}}^* \times \mathbb{C}$. Then we are in the situation of Section III with $U_{\mathbb{R}} = \bar{\mathfrak{h}}^*_{\mathbb{R}}$, $L = M$, $\langle\,,\,\rangle = (\,,\,)$, and shall freely use related notions from that section.

We observe, in particular, that for $\lambda \in \mathfrak{h}^*$ such that $m := \lambda(c)$ is a positive integer, the functions Θ_λ and Θ'_λ (defined by (2.16)) are in the above coordinates nothing else but classical theta functions of degree m and characteristic $\bar{\lambda}$ (we use (2.16.1)):

$$\Theta_\lambda = \Theta^M_{\bar{\lambda},m}; \qquad \Theta'_\lambda = \Theta^{M'}_{\bar{\lambda},m}.$$

(Here $\Theta^{M'}_{\bar{\lambda},m}$ is defined as in Section 3.2.) Note that $\Theta'_\lambda = \sum_{t\in T' \bmod T} \Theta_{t(\lambda)} = \sum_{\gamma\in M' \bmod M} \Theta^M_{\bar{\lambda}+m\gamma,m}$.

Recall the Weyl groups $W = \bar{W} \ltimes T$ and $W' = \bar{W} \ltimes T'$, and the groups $N_{\mathbb{R}} \subset G_{\mathbb{R}}$, which act on Y (see Sections 1.6 and 3.1). We relate these actions. Recall that for $\alpha \in \bar{\mathfrak{h}}^*_{\mathbb{R}}$,

$$t_\alpha(\tau, z, t) = (0, \alpha, 0) \cdot (\tau, z, t). \tag{4.1}$$

We identify T and T' with subgroups of $N_{\mathbb{R}} \subset G_{\mathbb{R}}$ using (4.1). Let $\bar{W}$ act by automorphisms on $G_{\mathbb{R}}$ by

$$w \cdot g = g \qquad \text{for} \quad g \in \mathrm{Mp}(2, \mathbb{R}),$$
$$w \cdot (\alpha, \beta, t) = (w(\alpha), w(\beta), t) \qquad \text{for} \quad (\alpha, \beta, t) \in N_{\mathbb{R}}.$$

This defines a group $\bar{W} \ltimes G_{\mathbb{R}}$, and we identify W and W' with subgroups of $\bar{W} \ltimes G_{\mathbb{R}}$ using (4.1). Noting that $\bar{W}$ acts on Y by $w \cdot (\tau, z, t) = (\tau, w(z), t)$, we have an action of $\bar{W} \ltimes G_{\mathbb{R}}$ on Y. Moreover, setting

$$(f|_w)(\tau, z, t) = f(w \cdot (\tau, z, t)) \qquad \text{for} \quad w \in \bar{W},$$

the right action of $G_{\mathbb{R}}$ on functions on Y extends to one of $\bar{W} \ltimes G_{\mathbb{R}}$. We extend det: $\bar{W} \to \{\pm 1\}$ to a homomorphism det: $\bar{W} \ltimes N_{\mathbb{R}} \to \{\pm 1\}$ by requiring that $\det n = 1$ for $n \in N_{\mathbb{R}}$.

Let $Q^\vee$ act on Y by $h \mapsto h + 2\pi i\alpha$, $\alpha \in Q^\vee$. It is easy to check that this coincides with the action of $\{(\alpha, 0, t) \in N_{\mathbb{R}} \mid \alpha \in M', t \in \mathbb{Z}\}$ on Y, so that we can identify $Q^\vee$ with a subgroup of $N_{\mathbb{R}}$.

In the sequel, we shall use the subgroups $\tilde{T} = T \ltimes Q^\vee$, $\tilde{T}' = T' \ltimes Q'^\vee$, $\tilde{W} = W \ltimes Q^\vee$, and $\tilde{W}' = W' \ltimes Q'^\vee$ of the group $\bar{W} \ltimes N_{\mathbb{R}}$ (cf. Section 2.1(G)). Note that

$$\tilde{T} = \{(\alpha, \beta, t) \in N_{\mathbb{R}} \mid \alpha \in M', \beta \in M, t + \tfrac{1}{2}(\alpha, \beta) \in \mathbb{Z}\},$$
$$\tilde{T}' = \{(\alpha, \beta, t) \in N_{\mathbb{R}} \mid \alpha \in M, \beta \in M', t + \tfrac{1}{2}(\alpha, \beta) \in \mathbb{Z}\},$$
$$\tilde{W} = \bar{W} \ltimes \tilde{T}, \tilde{W}' = \bar{W} \ltimes \tilde{T}'.$$

We now define and study certain spaces of theta functions on Y. Recall the space $\widetilde{Th}_m$ of theta functions of degree m on Y defined in Section 3.2 and the space Th_m spanned over $\mathbb{C}$ by the $\Theta^M_{\mu,m}$, $\mu \in M^*$ mod mM.

Then $\tilde{W}$ and $\tilde{W}'$ preserve $\widetilde{Th}_m$ and Th_m. Put

$$Th^0_m = \{f \in Th_m \,|f|_u = f \text{ for all } u \in \tilde{T}\}.$$

Note that $Th^0_m = \{f \in Th_m \,|f|_{(\alpha,0,0)} = f \text{ for all } \alpha \in M'\}$. In particular, $Th^0_m = Th_m$ in the most interesting case $k = 1$.

Furthermore, put

$$Th^+_m = \{f \in Th_m \mid f|_w = f \text{ for all } w \in \tilde{W}\},$$
$$Th^-_m = \{f \in Th_m \mid f|_w = (\det w) f \text{ for all } w \in \tilde{W}\}.$$

These are the subspaces of $\tilde{W}$-invariants and -anti-invariants of the action of the group $\tilde{W}$ on the space Th^0_m. Similarly we define the spaces Th'^0_m, Th'^+_m and Th'^-_m, by taking $\tilde{T}'$ and $\tilde{W}'$ in place of $\tilde{T}$ and $\tilde{W}$.

Replacing Th_m by $\widetilde{Th}_m$ above, we have similar notions $\widetilde{Th}_m^0$, $\widetilde{Th}_m^\pm$, $\widetilde{Th}_m'^0$, $\widetilde{Th}_m'^\pm$. Since the actions of $\tilde{W}$ and $\tilde{W}'$ on $\widetilde{Th}_m$ commute with multiplication by elements of the ring $\mathscr{O}(\mathscr{H}_+)$ of holomorphic functions of $\tau \in \mathscr{H}_+$, Lemma 3.12 yields:

PROPOSITION 4.2. *Let m be a positive integer. Then* $f \otimes \Theta \mapsto f\Theta$ *defines an isomorphism* $\mathscr{O}(\mathscr{H}_+) \otimes_{\mathbb{C}} Th_m \simeq \widetilde{Th}_m$ *of* $\mathscr{O}(\mathscr{H}_+)$*-modules, and of* $\tilde{W}$*- and* $\tilde{W}'$*-modules. Moreover, for any* $\tau \in \mathscr{H}_+$ *and any non-zero* $f \in Th_m$, *the restriction of f to* $Y_\tau = \{\tau\} \times \bar{\mathfrak{h}}^* \times \mathbb{C}$ *is non-zero.*

We next give bases for $Th_m^\pm$, etc.

Note that $\Theta_\lambda|_w = \Theta_{w^{-1}(\lambda)}$ for all $w \in W$. For $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) > 0$, put (cf. (2.16))

$$A_\lambda := e^{-(|\lambda|^2/2\lambda(c))\delta} \sum_{w \in W} (\det w)\, e^{w(\lambda)};$$

$$A'_\lambda := e^{-(|\lambda|^2/2\lambda(c))\delta} \sum_{w \in W'} (\det w)\, e^{w(\lambda)};$$

$$S_\lambda := e^{-(|\lambda|^2/2\lambda(c))\delta} \sum_{\mu \in W(\lambda)} e^\mu; \qquad S'_\lambda := e^{-(|\lambda|^2/2\lambda(c))\delta} \sum_{\mu \in W'(\lambda)} e^\mu.$$

Recall that the function A_λ(up to a "non-essential" factor) appears in the character formula (2.6). The "non-essential" factor is introduced in order to express A_λ as a finite alternating sum of classical theta functions. We fix a positive integer m.

PROPOSITION 4.3. (a) *Let* $\lambda, \lambda' \in \mathfrak{h}^*$, *with* $\lambda(c) = \lambda'(c) > 0$.

(i) $\Theta_\lambda = \Theta_{\lambda'}$ *if and only if* $T(\lambda) \equiv T(\lambda') \bmod \mathbb{C}\delta$;

(ii) $S_\lambda = S_{\lambda'}$ *if and only if* $W(\lambda) \equiv W(\lambda') \bmod \mathbb{C}\delta$;

$$S_\lambda = |W_\lambda|^{-1} \sum_{w \in \bar{W}} \Theta_{w(\lambda)}; \qquad S'_\lambda = |W'_\lambda|^{-1} \sum_{w \in \bar{W}} \Theta'_{w(\lambda)}.$$

(iii) $A_\lambda = 0$ *if and only if* $(\lambda, \alpha) = 0$ *for some* $\alpha \in \Delta^{re}$; $A_\lambda = \pm A_{\lambda'}$ *if and only if* $W(\lambda) \equiv W(\lambda') \bmod \mathbb{C}\delta$ *or else* $A_\lambda = A_{\lambda'} = 0$;

$$A_\lambda = \sum_{w \in \bar{W}} (\det w)\, \Theta_{w(\lambda)} = \sum_{w \in W \bmod T} (\det w)\, \Theta_{w(\lambda)};$$

$$A'_\lambda = \sum_{w \in \bar{W}} (\det w)\, \Theta'_{w(\lambda)} = \sum_{w \in W' \bmod T} (\det w)\, \Theta_{w(\lambda)}.$$

(b) $\{\Theta_\lambda \mid \lambda \in P' \bmod T \bmod \mathbb{C}\delta, \lambda(c) = m\} = \{\Theta^M_{\mu,m} \mid \mu \in M^* \bmod mM\}$ *is a basis for* Th_m *(resp.* $\widetilde{Th}_m$*) over* $\mathbb{C}$ *(resp.* $\mathscr{O}(\mathscr{H}_+)$*).*

(c) $\{\Theta_\lambda \mid \lambda \in P \bmod T \bmod \mathbb{C}\delta, \lambda(c) = m\} = \{\Theta^M_{\mu,m} \mid \mu \in \bar{P} \bmod mM\}$ *is a basis for* Th^0_m *(resp.* $\widetilde{Th}^0_m$*) over* $\mathbb{C}$ *(resp.* $\mathcal{O}(\mathscr{H}_+)$*).*

(d) $\{S_\lambda \mid \lambda \in P_+ \bmod \mathbb{C}\delta, \lambda(c) = m\}$ *is a basis for* Th^+_m *(resp.* $\widetilde{Th}^+_m$*) over* $\mathbb{C}$ *(resp.* $\mathcal{O}(\mathscr{H}_+)$*).*

(e) $\{A_\lambda \mid \lambda \in P_{++} \bmod \mathbb{C}\delta, \lambda(c) = m\}$ *is a basis for* Th^-_m *(resp.* $\widetilde{Th}^-_m$*) over* $\mathbb{C}$ *(resp.* $\mathcal{O}(\mathscr{H}_+)$*).*

(f) *If* $\lambda \in P_+$ *and* $\lambda(c) > 0$, *then* $e^{-s_\lambda \delta} \mathrm{ch}_{L(\lambda)} = A_{\lambda+\rho}/A_\rho$.

(g) $\{A_{\lambda+\rho}/A_\rho \mid \lambda \in P_+ \bmod \mathbb{C}\delta, \lambda(c) = m\}$ *is a basis of* $\widetilde{Th}^+_m$ *over* $\mathcal{O}(\mathscr{H}_+)$. $\widetilde{Th}^-$ *is a free* $\widetilde{Th}^+$*-module on one generator* A_ρ.

Proof. The conditions for $\Theta_\lambda = \Theta_{\lambda'}$ and $S_\lambda = S_{\lambda'}$ are clear from the definitions. The conditions for $A_\lambda = 0$ and $A_\lambda = \pm A_{\lambda'}$ are clear from Proposition 1.9 and the definition of A_λ. The formulas for A_λ and S_λ follow from definition (2.16). (Similarly, the formulas for A'_λ and S'_λ are clear.) This proves (a).

(b) is immediate from Propositions 2.13 and 3.13, and $\Theta_\lambda = \Theta^M_{\bar{\lambda},m}$. (c) now follows from formula (3.10) and Proposition 2.13. By Proposition 1.9, P_+ is a fundamental domain for W on $\{\lambda \in P \mid \lambda(c) > 0\} \cup \mathbb{C}\delta$. This and (c) imply (d). Similarly, the non-zero elements of $\{A_\lambda \mid \lambda \in P_+ \bmod \mathbb{C}\delta,\ \lambda(c) = m\}$ form a basis of Th^-_m, which by (a) implies (e). (f) is another form of (2.18).

By (e) and (f), F/A_ρ is holomorphic for any $F \in \widetilde{Th}_-$. It follows that $\widetilde{Th}^-$ is a free $\widetilde{Th}^+$-module on one generator A_ρ. (g) now follows from (e) and the fact that $\lambda \mapsto \lambda + \rho$ defines a bijection from P_+ onto P_{++}. ∎

Note that corresponding to (c), (d), (e), (g) we also have bases for Th'^0_m, etc.

Remark. As in Section 3.1, we have a geometric interpretation of the spaces $\widetilde{Th}^0_m$ of functions on Y, and of the representation of $\bar{W} \cong \tilde{W}/\tilde{T}$ on $\widetilde{Th}^0_m$, for which $\widetilde{Th}^\pm_m$ are the subspaces of invariant and anti-invariant functions. Namely, replacing $N_{\mathbb{Z}}$ by $\tilde{T}$ in the Remark in Section 3.1, we obtain a holomorphic line bundle $\mathscr{L}$ over the family of abelian varieties $\bar{\mathfrak{h}}^*/(M' + \tau M)$, $\tau \in \mathscr{H}_+$. Moreover, $\widetilde{Th}^0_m$ is identified with the space of all holomorphic sections of the mth tensor power $\mathscr{L}^{-m}$ of the line bundle $\mathscr{L}^{-1}$ dual to $\mathscr{L}$, in a way consistent with the obvious action of $\bar{W}$ on $\mathscr{L}$. Similarly, one may realize $\widetilde{Th}'^0_m$ as the space of all holomorphic sections of a line bundle over the family of abelian varieties $\bar{\mathfrak{h}}^*/(M + \tau M')$, $\tau \in \mathscr{H}_+$. (Using $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, one sees that the line bundles for $\widetilde{Th}^0_m$ and $\widetilde{Th}'^0_m$ are actually isomorphic.)

Let G_0 denote the intersection of the normalizer of $\tilde{W}$ in $\bar{W} \ltimes G_{\mathbb{R}}$ with

$G_{\mathbb{R}}$; let $N_0 = N_{\mathbb{R}} \cap G_0$ and $\Gamma = \mathrm{Mp}(2, \mathbb{R}) \cap G_0$. Then it is easy to check that $N_0 = \{(\alpha, \beta, t) \in N_{\mathbb{R}} \mid \alpha \in \bar{Q}^*, \beta \in \bar{Q}'^*\}$ and

$$\Gamma = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j \right) \in \mathrm{Mp}(2, \mathbb{R}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \right.$$

$$\left. c \equiv 0 \bmod k'\mathbb{Z}, ac \equiv bd \equiv 0 \bmod a_0 \mathbb{Z} \right\}.$$

Note that $\bar{Q}' = \bar{Q}$ if $k' = 1$, $\bar{Q}' = \bar{Q}^\vee$ if $k \neq 1$ and $\bar{Q}' = \bar{Q} + \bar{Q}^\vee$ in all cases. It is easy to check that $G_0 \subset G_{\mathbb{Z}}$, and

$$G_0 = \Gamma \ltimes N_0 \qquad \text{unless} \quad a_0 = 2.$$

Note also that $\Gamma = \mathrm{Mp}(2, \mathbb{Z})$ if $k = 1$.

Put $S = ((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), \tau^{1/2}) \in G_{\mathbb{Z}}$.

PROPOSITION 4.4. *Fix a positive integer m. Then*

(a) *Th_m and Th_m^0 are invariant under G_0, the group G_0 commutes with $\bar{W}$ on Th_m^0, and the matrix of any $g \in G_0$ with respect to the basis $\{\Theta_{\mu,m}^M \mid \mu \in M^* \bmod mM\}$ of Th_m is unitary.*

(b) *S preserves Th_m, and its matrix with respect to the basis $\{\Theta_{\mu,m}^M \mid \mu \in M^* \bmod mM\}$ of Th_m is unitary. Moreover, S commutes with $\bar{W}$ and exchanges Th_m^0 and $Th_m'^0$.*

Proof. A computation verifies that $G_0 \subset G_{\mathbb{Z}}$, so that by Proposition 3.16, G_0 preserves Th_m and is unitary with respect to the given basis. Clearly, $gwg^{-1}w^{-1} \in \tilde{T}$ for all $g \in G_0$ and $w \in \tilde{W}$, so that G_0 preserves Th_m^0 and commutes with $\bar{W}$ on it. This proves (a). Similarly, $S \in G_{\mathbb{Z}}$ commutes with $\bar{W}$, and $S\tilde{T}S^{-1} = S^{-1}\tilde{T}S = \tilde{T}'$, which along with Proposition 3.16 proves (b). ∎

The action of S on Th_m is given explicitly by Proposition 3.15. For $B = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \Gamma$, the action of B on Th_m may be computed as follows. Choose $\beta \in \mathfrak{h}^*$ such that $\mathrm{mac}\,|\alpha|^2 \equiv 2(\alpha, \beta) \bmod 2\mathbb{Z}$ whenever $c\alpha \in M$ and $m\alpha \in M^*$. Then $g := B(0, m^{-1}\beta, 0)$ is as in Proposition 3.17 (as one checks using the formula above for Γ, $k'M' \subset M$ and Proposition 2.13(c)), so that the action of g on Th_m is given by Proposition 3.17. Note that $(\alpha, \beta) \in \mathbb{Z}$ whenever $\alpha \in M'$, since then $c\alpha \in M$, $\alpha \in M^*$ and $ac\,|\alpha|^2 \in 2\mathbb{Z}$ (as one checks using Proposition 2.13(c) and the formula above for Γ). Hence, $\beta \in \bar{P} = M'^* \subset M^*$, so that the action of $(0, m^{-1}\beta, 0)$ on Th_m is given by formula (3.10). The result of these calculations is given by Proposition 4.5(a) and (c) below.

PROPOSITION 4.5. *Let m be a positive integer and let $\mu \in M^* = \bar{P}' \supset \bar{P}$.*

(a) *Let $B = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \Gamma$. Choose $\beta \in \bar{P}$ such that $mac\,|\alpha|^2 \equiv 2(\alpha, \beta) \bmod 2\mathbb{Z}$ whenever $c\alpha \in M$ and $m\alpha \in M^*$. Let $\varepsilon = v(m, B(0, m^{-1}\beta, 0))$ be the complex number defined in Proposition 3.17. Then:*

$$\Theta^M_{\mu,m}|_B = \varepsilon \sum_{\substack{\alpha \in M^* \\ c\alpha \bmod mM}} (\exp \pi i m^{-1}[(b\mu + d\alpha, a\mu + c\alpha + 2\beta) - (\mu, \alpha)])\, \Theta^M_{a\mu + c\alpha + \beta, m}.$$

(b) *Choose $\alpha \in \bar{\mathfrak{h}}^*$ such that $|\gamma|^2 \equiv 2(\alpha, \gamma) \bmod 2\mathbb{Z}$ for all $\gamma \in M$. Then $\Theta^M_{\mu,m}(\tau + 1, z - \alpha, t) = e^{\pi i m^{-1}|\mu|^2} e^{2\pi i(\alpha,\mu)} \Theta^M_{\mu,m}(\tau, z, t)$.*

(c)
$$\Theta^M_{\mu,m}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = |M^*/mM|^{-1/2}(-i\tau)^{l/2} \times \sum_{\mu' \in M^* \bmod mM} [\exp -2\pi i m^{-1}(\mu, \mu')]\, \Theta^M_{\mu',m}(\tau, z, t).$$

(d) *Let $\lambda \in P$, $\lambda(c) = m$. Then:*

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = |M^*/mM|^{-1/2}(-i\tau)^{l/2} \times \sum_{\substack{\lambda' \in P'_{++} \bmod C\delta \\ \lambda'(c) = m}} \left(\sum_{w \in \bar{W}} (\det w) \exp -2\pi i m^{-1}(w(\bar{\lambda}), \bar{\lambda}')\right) \times A'_{\lambda'}(\tau, z, t).$$

Proof. After the preceding discussion, only (d) deserves comment. To prove it, note that

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) \in Th'^{-}_m$$

by Proposition 4.4(b). On the other hand, (c) gives

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = |M^*/mM|^{-1/2}(-i\tau)^{l/2} \times \sum_{\substack{w \in \bar{W} \\ \mu \in M^* \bmod mM}} (\det w)[\exp -2\pi i m^{-1}(w(\bar{\lambda}), \mu)]\, \Theta^M_{\mu,m}(\tau, z, t). \qquad (4.5.1)$$

Since $A'_{\lambda'} = \sum_{w \in W' \bmod T} (\det w)\, \Theta^M_{w(\lambda'),m}$ if $\lambda'(c) = m$, and since the $A'_{\lambda'}$ with $\lambda'(c) = m$ form a basis of Th'^{-}_m by Proposition 4.3(e) applied to the adjacent root system, one writes

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right)$$

as a linear combination of the $A'_{\lambda'}(\tau, z, t)$, picking out the coefficient of $A'_{\lambda'}(\tau, z, t)$ as that of $\Theta^M_{\lambda',m}(\tau, z, t)$ in (4.5.1). ∎

4.2. *Transformation Properties of the Denominator* A_ρ

Since we are interested mainly in the functions (cf. Proposition 4.3(f))

$$e^{-s_\Lambda \delta} \mathrm{ch}_{L(\Lambda)} = A_{\Lambda+\rho}/A_\rho,$$

we want more precise information on the transformation properties of A_ρ.

Define a holomorphic function $F(\tau)$ on $\mathscr{H}_+$ by

$$F(\tau) = e^{2\pi i(|\bar\rho|^2/2g)\tau} \prod_{\alpha \in \Delta_+ \setminus \bar\Delta_+} (1 - e^{2\pi i(\alpha, \Lambda_0)\tau})^{\mathrm{mult}\,\alpha}.$$

Using the results of Section 1.4, it is easy to express F in terms of the η-function. The result is given in Table F.

TABLE F

Type $X_n^{(k)}$	$F(\tau)$
$X_l^{(1)}$	$\eta(\tau)^{l(h+1)}$
$A_{2l}^{(2)}$	$\eta\left(\frac{\tau}{2}\right)^{2l} \eta(\tau)^{l(2l-3)}\, \eta(2\tau)^{2l}$
$A_{2l-1}^{(2)}$	$\eta(\tau)^{(l-1)(2l+1)}\, \eta(2\tau)^{2l+1}$
$D_{l+1}^{(2)}$	$\eta(\tau)^{2l+1}\, \eta(2\tau)^{(l-1)(2l+1)}$
$E_6^{(2)}$	$\eta(\tau)^{26}\, \eta(2\tau)^{26}$
$D_4^{(3)}$	$\eta(\tau)^7\, \eta(3\tau)^7$

Introduce the following notations:

$$D := l + 2\,|\bar\Delta_+|\ (=\dim \bar{\mathfrak{g}});$$

for $\Lambda \in P_+$ of level m, set

$$b(\Lambda) := |\bar P/(m+g)M|^{-1/2} \prod_{\alpha \in \bar\Delta'_+} 2 \sin \frac{\pi(\alpha, \bar\Lambda + \bar\rho)}{m+g};$$

for $\lambda \in \bar{P}'_+$, set

$$\chi'_\lambda(\exp y) := \frac{\sum_{w \in \bar{W}} (\det w)\, e^{(w(\lambda+\bar{\rho}'), y)}}{\sum_{w \in \bar{W}} (\det w)\, e^{(w(\bar{\rho}'), y)}} \qquad (y \in \bar{\mathfrak{h}}^*).$$

PROPOSITION 4.6.

(a)
$$A_\rho(\tau, z, t) = e^{-(|\bar{\rho}|^2/2g)\delta} e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\operatorname{mult} \alpha}$$
$$= \exp 2\pi i \left(\frac{|\bar{\rho}|^2}{2g} \tau - (\bar{\rho}, z) - gt \right)$$
$$\times \prod_{\alpha \in \Delta_+} (1 - e^{2\pi i(\alpha, z + \tau \Lambda_0)})^{\operatorname{mult} \alpha}.$$

(b) *For any $B = (B_0, j) \in \Gamma$, there exists a complex number $v_\rho(B)$ such that*

$$A_\rho|_B = v_\rho(B) A_\rho \qquad \textit{and} \qquad F(B_0\tau) = v_\rho(B) j(\tau)^D F(\tau).$$

(c)
$$A_\rho \left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z, z)}{2\tau} \right) = |M'/M|^{-1/2} (-i)^{|\bar{\Delta}_+|} (-i\tau)^{l/2} A'_{\rho'}(\tau, z, t).$$

(d) *Let $\Lambda \in P_+$ be of level m. Then*:

$$(A_{\Lambda+\rho}/A_\rho) \left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z, z)}{2\tau} \right)$$
$$= b(\Lambda) \sum_{\substack{\Lambda' \in P'_+ \bmod \mathbb{C}\delta \\ \Lambda'(c) = m}} \chi'_{\bar{\Lambda}'} \left(\exp \left(-2\pi i \frac{\bar{\Lambda} + \bar{\rho}}{m + g} \right) \right) (A'_{\Lambda'+\rho'}/A'_{\rho'})(\tau, z, t).$$

(e) *For $\alpha \in \bar{Q}^*$ and $\beta \in \bar{Q}'^*$, we have*

$$A_\rho|_{(\alpha,0,0)} = (-1)^{(\alpha, 2\bar{\rho})} A_\rho;$$
$$A_\rho|_{(0,\beta,0)} = (-1)^{(\beta, 2\bar{\rho}')} A_\rho.$$

Proof. (a) is immediate from formula (2.7), the definition of A_ρ, and the fact that $|\rho|^2 = |\bar{\rho}|^2$. We next prove:

$$Th_g^- = \mathbb{C}A_\rho \qquad \text{and} \qquad Th'^-_g = \mathbb{C}A'_{\rho'}. \qquad (4.6.1)$$

Let $g' = \rho'(c)$. Then by Proposition 4.3(e), $Th_g^- = \mathbb{C}A_\rho$ and $Th'^-_{g'} = \mathbb{C}A'_{\rho'}$, and moreover, $Th_m^- = (0)$ for $m < g$ and $Th'^-_m = (0)$ for $m < g'$. But by Proposition 4.4(b), $\dim Th_m^- = \dim Th'^-_m$ for all m. Hence,

$$g' = g,$$

proving (4.6.1).

Let $B=(B_0, j)\in \Gamma$. By (4.6.1) and Proposition 4.4(a), $A_\rho|_B = v_\rho(B) A_\rho$ for some $v_\rho(B)\in \mathbb{C}$. But (a) implies that

$$F(\tau)=\lim_{\substack{z\to 0\\ z \text{ regular}}} \left(A_\rho(\tau, z, 0)\Big/\prod_{\alpha\in\bar{\Delta}_+}(-2\pi i(\alpha, z))\right),$$

from which one deduces $F(B_0\tau)=v_\rho(B) j(\tau)^D F(\tau)$. This proves (b).

To prove (c), note that by (4.6.1) and Proposition 4.5(d) we have

$$A_\rho\left(-\frac{1}{\tau}, \frac{z}{\tau}, t+\frac{(z,z)}{2\tau}\right)=c(-i\tau)^{l/2} A'_{\rho'}(\tau, z, t),$$

where

$$\begin{aligned} c &= |M^*/gM|^{-1/2} \sum_{w\in\bar{W}} (\det w)\exp(-2\pi i g^{-1}(w(\bar{\rho}), \bar{\rho}')) \\ &= |M^*/gM|^{-1/2} \prod_{\alpha\in\bar{\Delta}'_+} (-2i \sin \pi g^{-1}(\alpha, \bar{\rho})) \end{aligned}$$

by the Weyl denominator formula. Therefore, $i^{|\bar{\Delta}_+|}c>0$ due to:

$$0<(\alpha,\lambda)<\lambda(c) \qquad \text{for all} \quad \alpha\in\bar{\Delta}'_+ \text{ and } \lambda\in P_{++}. \tag{4.6.2}$$

(Since θ is the highest root of $\bar{\Delta}'_+$, we have $0<(\alpha,\lambda)\leqslant(\theta,\lambda)=\lambda(c)-\lambda(h_0)<\lambda(c)$.) Finally, since the matrix of S with respect to the basis $\{\Theta^M_{\mu,g} \mid \mu\in M^* \bmod gM\}$ of Th_g is unitary (by Proposition 4.4(b)), we have

$$|c|^2=|W/T|/|W'/T|=|M'/M|^{-1}.$$

Hence, $c=(-i)^{|\bar{\Delta}_+|}\,|M'/M|^{-1/2}$, proving (c).

Finally, (d) is easily derived from (c) and Proposition 4.5(d). The first formula of (e) is clear. The second formula follows from (c) and the first formula for the adjacent root system using $A_\rho|_{(0,\beta,0)}=A_\rho|_{S(\beta,0,0)S^{-1}}$. ∎

Remark. Proposition 4.6(a) is another form of the Macdonald identities [29]. Proposition 4.6(c) for $k=1$ is due to Looijenga [27].

4.3. *Specializations of A_ρ and the "Very Strange" Formula*

For $y, z\in\bar{\mathfrak{h}}^*_{\mathbb{R}}$, we define the associated "specialization" $F_{y,z}(\tau)$ of A_ρ as follows. Let

$$\begin{aligned} \Delta^{y,z} &= \{\alpha\in\Delta \mid (\alpha, \Lambda_0+z)=0 \text{ and } (\alpha, y)\in\mathbb{Z}\}, \\ D_{y,z} &= l+|\Delta^{y,z}|. \end{aligned}$$

Note that $\Delta^{y,z}$ is the set of all roots α such that $\exp 2\pi i(\alpha, \tau(\Lambda_0 + z) - y) = 1$ for all $\tau \in \mathscr{H}_+$. Clearly, $\Delta^{y,z}$ is a (finite) reduced root system, and

$$\Delta_+^{y,z} := \{\alpha \in \Delta^{y,z} \mid \bar{\alpha} \in \bar{\Delta}_+ \text{ or } \tfrac{1}{2}\bar{\Delta}_+\}$$

is a set of positive roots. Motivated by the definition (3.18), we set

$$F_{y,z}(\tau) = \Big(A_\rho \prod_{\alpha \in \Delta_+^{y,z}} (1 - e^{-\alpha})^{-1}\Big)\Big(\tau, \tau z - y, -\frac{1}{2}(z, \tau z - y)\Big). \tag{4.7}$$

Using Proposition 4.6(a), we obtain

$$F_{y,z}(\tau) = (-1)^{|(-\Delta_+) \cap \Delta_+^{y,z}|} e^{\pi i(y, 2\bar{\rho} - gz)} e^{\pi i g^{-1}|\bar{\rho} - gz|^2 \tau}$$
$$\times \prod_{\alpha \in \Delta_+ \setminus \Delta^{y,z}} (1 - e^{2\pi i(\alpha, \tau(\Lambda_0 + z) - y)})^{\text{mult}\,\alpha}. \tag{4.8}$$

(The power of -1 is due to the fact that $\Delta_+^{y,z}$ need not lie in Δ_+.) In particular, $F_{y,z}$ is holomorphic on $\mathscr{H}_+$, and $F_{y,z}(\tau) \neq 0$ for all $\tau \in \mathscr{H}_+$. Note that $F_{0,0}(\tau)$ is the function $F(\tau)$ from Section 4.2. Similarly, we define $F'_{y,z}(\tau)$, taking $A'_{\rho'}$ in place of A_ρ and $\Delta'^{y,z}_+$ in place of $\Delta_+^{y,z}$.

Let $W^{y,z} \subset W$ be the Weyl group of $\Delta^{y,z}$, and let $\rho^{y,z} = \frac{1}{2}\sum_{\alpha \in \Delta_+^{y,z}} \alpha$. Then we have

$$F_{y,z}(\tau) = |W^{y,z}|^{-1} \sum_{w \in W} (\det w) \Big(\prod_{\alpha \in \Delta_+^{y,z}} \frac{(w(\rho), \alpha)}{(\rho^{y,z}, \alpha)}\Big)$$
$$\times e^{\pi i(y, 2\overline{w(\rho)} - gz)} e^{\pi i g^{-1}|\overline{w(\rho)} - gz|^2 \tau}. \tag{4.9}$$

This formula is essentially a special case of Proposition 4.34(d)(ii) in Section 4.9 (the proof is the same).

We can now prove:

Proposition 4.10. (a) *For $B = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \Gamma$ with $c \equiv 0 \bmod k\mathbb{Z}$, one has:*

$$F_{ay+bz, cy+dz}\Big(\frac{a\tau + b}{c\tau + d}\Big) = v_\rho(B) j(\tau)^{D_{y,z}} F_{y,z}(\tau).$$

(b) *$F_{y,z}(-1/\tau) = K(-i)^{D/2} \tau^{(1/2)D_{y,z}} F'_{z,-y}(\tau)$, where $K = 1$ if $k = 1$, $K = |M'/M|^{-1/2} \prod_{\alpha \in \Delta_+^{y,z}} (2/|\alpha|^2)$ if $k' \neq 1$, and $K = \prod_{\alpha \in S} (2/|\alpha|^2)$ if $a_0 = 2$, where $S = \{\alpha \in \Delta_+^{y,z} \mid \bar{\alpha} + (\alpha, y)\delta \notin \Delta\}$.*

(c) *If* $y_0 \in \bar{Q}^*$ *and* $z_0 \in \bar{Q}'^*$, *then*:

$$F_{y+y_0,z+z_0}(\tau) = (-1)^{(y_0,2\bar{\rho})+(z_0,2\bar{\rho}')} \times e^{\pi i g((y,z_0)-(y_0,z)+(y_0,z_0))} F_{y,z}(\tau).$$

(d) *If* $y, z \in \sum_{i=1}^{l} \mathbb{Q}\alpha_i$, *then* $F_{y,z}(\tau)$ *is a holomorphic modular form of weight* $\frac{1}{2}D_{y,z}$ *for* $\Gamma(N)$, *where* N *is the least positive integer divisible by* k *such that* $y, z \in N^{-1}\bar{Q}'^*$.

Proof. For $y' \to y$ and $z' \to z$ and generic y', z', we have

$$F_{y,z}(\tau) = \lim \frac{A_\rho(\tau, \tau z' - y', -\frac{1}{2}(z', \tau z' - y'))}{\prod_{\alpha \in \Delta_+^{y,z}}(-2\pi i(\alpha, \tau(z'-z) - (y'-y)))}. \tag{4.10.1}$$

Let $B = (B_0, j) \in \mathrm{Mp}(2, \mathbb{R})$, $B_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the transformation law (3.19) gives, setting $y_0 = ay + bz$, $z_0 = cy + dz$,

$$j(\tau)^{-D_{y_0,z_0}} F_{y_0,z_0}(B_0 \tau) = \lim \frac{(A_\rho|_B)(\tau, \tau z' - y', -\frac{1}{2}(z', \tau z' - y'))}{\prod_{\alpha \in \Delta_+^{y_0,z_0}}(-2\pi i(\alpha, \tau(z'-z) - (y'-y)))}. \tag{4.10.2}$$

If $B \in \Gamma$, then (4.10.2) and Proposition 4.6(b) give

$$j(\tau)^{-D_{y_0,z_0}} F_{y_0,z_0}(B_0 \tau) = \lim \frac{v_\rho(B) A_\rho(\tau, \tau z' - y', -\frac{1}{2}(z', \tau z' - y'))}{\prod_{\alpha \in \Delta_+^{y_0,z_0}}(-2\pi i(\alpha, \tau(z'-z) - (y'-y)))}. \tag{4.10.3}$$

Moreover, it is easy to check that if, in addition, $c \in k\mathbb{Z}$, then $\alpha \mapsto \bar{\alpha} - (\alpha, z_0)\delta$ defines a bijection of $\Delta_+^{y,z}$ onto $\Delta_+^{y_0,z_0}$. Hence, in this case we may replace $\Delta_+^{y_0,z_0}$ by $\Delta_+^{y,z}$ and D_{y_0,z_0} by $D_{y,z}$ in (4.10.3). (a) now follows from a comparison of (4.10.1) and (4.10.3).

The proof of (b) is similar, using Proposition 4.6(c) in place of Proposition 4.6(b), and using the facts that $\alpha \mapsto \alpha'$, where $\alpha' \in \Delta'$ and α' or $\alpha'^\vee$ equals $\bar{\alpha} + (\alpha, y)\delta$, defines a bijection of $\Delta_+^{y,z}$ onto $\Delta_+'^{z,-y}$, and that K in (b) is $|M'/M|^{-1/2} \prod_{\alpha \in \Delta_+^{y,z}} (\bar{\alpha}'/\bar{\alpha})$. (Note that $\alpha' = \bar{\alpha} + (\alpha, y)\delta$ if $k = 1$, and $\alpha'^\vee = \bar{\alpha} + (\alpha, y)\delta$ if $k' \neq 1$.)

To prove (c), note that:

$$\Delta^{y+y_0,z+z_0} = t_{z_0}(\Delta^{y,z}).$$

To see this, one only needs to show that t_{z_0} preserves Δ; but this follows from Proposition 4.6(e) and the product decomposition of A_ρ (cf. the proof of Proposition 4.27(a)). Now (c) follows by Proposition 4.6(e).

Except for the assertion about behavior at cusps, (d) is clear from (a), (c), and $\bar{Q}'^* \subset \bar{Q}^*$. Since each $F_{y,z}(\tau)$ is holomorphic at the cusp $i\infty$ by (4.9), we may show as in the proof of Theorem A(4) in Section 4.4 that $F_{y,z}(\tau)$ is holomorphic at all cusps. This proves (d). ∎

Remark. Some especially interesting specializations of A_ρ are suggested by the results of Section 4.8. Besides $F_{0,0}(\tau) = F(\tau)$, we also have

$$F_{-g^{-1}\bar{\rho}',0}(\tau) = (-i)^{|\bar{\Delta}_+|}\, |\bar{P}/gM|^{1/2} \sum_{t \in T} q^{|\overline{t(\rho)}|^2/2g};$$

if $k = 1$, then

$$F_{0,(h+1)^{-1}\bar{\rho}^\vee}(\tau) = \eta((h+1)^{-1}\tau)^l,$$
$$F_{-(h+1)^{-1}\bar{\rho}^\vee,0}(\tau) = (-i)^{|\bar{\Delta}_+|}(h+1)^{l/2}\,\eta((h+1)\tau))^l.$$

These, together with (4.8) and (4.9), give nice identities. The first identity is (in an equivalent form) given by Macdonald [29, (8.16)] in the case when $|\alpha|^2 = 2$ for all $\alpha \in \bar{\Delta}$, while the others are his specializations Θ and Ψ. The identities of Proposition 4.30(d) also have beautiful specializations.

Using Proposition 4.6(d), the same proof as that of Proposition 4.10 gives

PROPOSITION 4.11. *For $\Lambda \in P_+$ of level m and $y, z \in \sum_{i=1}^{l} \mathbb{Q}\alpha_i$, define:*

$$\Phi^{\Lambda}_{y,z}(\tau) = \frac{A_{\Lambda+\rho}}{A_\rho}\left(\tau, \tau z - y, -\frac{1}{2}(z, \tau z - y)\right).$$

Then $\Phi^{\Lambda}_{y,z}$ is a modular form of weight 0 *with the transformation law:*

$$\Phi^{\Lambda}_{y,z}\left(-\frac{1}{\tau}\right) = b(\Lambda) \sum_{\substack{\Lambda' \in P'_+ \bmod \mathbb{C}\delta \\ \Lambda'(c) = m}} \chi'_{\bar{\Lambda}'}\left[\exp\left(-2\pi i\,\frac{\bar{\Lambda}+\bar{\rho}}{m+g}\right)\right] \Phi'^{\Lambda'}_{z,-y}(\tau),$$

where $b(\Lambda)$ and $\chi'_{\bar{\Lambda}'}$ are as in Section 4.2.

Remark. Proposition 4.11 is related to the "Monstrous game" [4, 18]. Roughly speaking, it shows that the linear span of the "Thompson series" corresponding to certain gradations of modules of level m and all elements of given period from the Cartan subgroup is invariant with respect to $SL(2, \mathbb{Z})$.

A nice application of Proposition 4.10 is the "very strange" formula (cf. Proposition 1.11(e)). We recall this formula in a slightly different form.

PROPOSITION 4.12. *Let $\bar{s} = (s_0, \ldots, s_l)$ be a non-zero sequence of non-negative integers and let $\mathfrak{g}(A) = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j(\bar{s})$ be the $\mathbb{Z}$-gradation of the affine Lie algebra $\mathfrak{g}(A)$ defined by $\deg e_i = -\deg f_i = s_i$, $\deg \mathfrak{h} = 0$. Set $N = k \sum_{i=0}^{l} a_i s_i$, $b_j = \dim \mathfrak{g}_j(\bar{s})$, and $b = \sum_{j=1}^{N} b_j$. Define $z \in \bar{\mathfrak{h}}^*$ by*

$$(z, \alpha_i) = ks_i/N \qquad (i = 1, \ldots, l).$$

Then

$$\frac{1}{2kg}|\bar{\rho}-gz|^2=\frac{b}{24}-\frac{1}{4N^2}\sum_{j=1}^{N-1} j(N-j)\,b_j. \tag{4.12.1}$$

Proof. Note that $(z+\Lambda_0, \alpha_i)=ks_i/N$ for $i=0,\dots,l$. Hence, setting $q_1=e^{2\pi ik\tau/N}$, we obtain

$$F_{0,z}(\tau)=q_1^{(N/2kg)|\bar{\rho}-gz|^2}\prod_{j\geqslant 1}(1-q_1^j)^{b_j}.$$

Note also that the sequence $b_j, j\geqslant 1$, is periodic with period N, and $b_j=b_{N-j}$ for $0<j<N$, by the structure of the root system Δ. Now (4.12.1) follows by Lemma 3.20, since $F_{0,z}(\tau)$ is a modular form by Proposition 4.10(d). ∎

As a corollary to the "very strange" formula, we have for $z\in\bar{\mathfrak{h}}^*$:

$$2kgz=\sum_{\alpha\in\Delta^{\mathrm{re}}\bmod k\mathbb{Z}\delta}(z,\alpha)\bar{\alpha}, \tag{4.12.2}$$

$$2khz=\sum_{\alpha\in\Delta^{\mathrm{re}}\bmod k\mathbb{Z}\delta}(z,\alpha^\vee)\bar{\alpha}. \tag{4.12.3}$$

To prove (4.12.2), rewrite (4.12.1) as

$$\frac{1}{2kg}|\bar{\rho}-gz|^2=\frac{b}{24}-\frac{1}{4k^2}\sum_{0<\alpha<k\delta}\operatorname{mult}(\alpha)(\alpha,z+\Lambda_0)(k\delta-\alpha,z+\Lambda_0). \tag{4.12.4}$$

Equating the terms quadratic in z proves (4.12.2). Equation (4.12.3) is just (4.12.2) for $\Delta^\vee$.

Now we deduce from the "very strange" formula an important inequality. Let J be the set of all $j\in I$ such that $j=\sigma(0)$ for some automorphism σ of the Dynkin diagram.

PROPOSITION 4.13. *Let* $z\in C_{\mathrm{af}}$. *Then*:

$$|\bar{\rho}-gz|^2\leqslant|\bar{\rho}|^2. \tag{4.13.1}$$

Moreover, equality holds if and only if $z=\bar{\Lambda}_j$ *for some* $j\in J$.

Proof. Equality holds in the stated cases since, by Proposition 4.27(b), $\bar{W}_0\subset\bar{W}$ acts simply-transitively on $\{\bar{\rho}-g\bar{\Lambda}_j\,|\,j\in J\}$.

Put $\Lambda=\Lambda_0+z$, so that $(\Lambda,\alpha)\geqslant 0$ for all $\alpha\in\Delta_+$. Put $S=\{\alpha\in\Delta'^{\mathrm{re}}\bmod k\mathbb{Z}\delta'\,|\,k\bar{\alpha}>0\}$. Using the descriptions of Δ and Δ' in Sections 1.4 and 1.5, formula (4.12.4) gives $a_0g^{-1}(|\bar{\rho}|^2-|\bar{\rho}-gz|^2)=\sum_{\alpha\in S}(\bar{\alpha},\Lambda)\;(\delta-\bar{\alpha},\Lambda)$. Moreover, $k\bar{\alpha}>0$ and $k(\delta-\bar{\alpha})>0$ for all $\alpha\in S$, so that the summands on the right-hand side are non-negative, proving (4.13.1). If equality holds in (4.13.1), then these summands all vanish, which forces $z \in \overline{Q'^*}$; hence, by Proposition 4.27(b), $z = \bar{\Lambda}_j$ for some $j \in J$. ∎

Now it is easy to deduce an estimate for the characteristic $s_\Lambda(\lambda)$ of a weight λ of the module $L(\Lambda)$ (see Section 2.3).

PROPOSITION 4.14. *Let $\Lambda \in P_+$ be of positive level m and let $\lambda \in P(\Lambda)$. Then*

$$s_\Lambda(\lambda) \geqslant -\frac{|\bar{\rho}|^2}{2}\left(\frac{1}{g} - \frac{1}{m+g}\right). \tag{4.14.1}$$

Moreover, equality holds if and only if $\Lambda \in m\Lambda_j + \mathbb{C}\delta$ for some $j \in J$ and $w(\lambda) = \Lambda$ for some $w \in W$.

Proof. Since $(m\rho - g\Lambda)(c) = 0$ and hence $|m\rho - g\Lambda|^2 = |m\bar{\rho} - g\bar{\Lambda}|^2$, we have

$$2s_\Lambda(\lambda) = m^{-1}(|\Lambda|^2 - |\lambda|^2) - (g^{-1} - (m+g)^{-1})\,|\bar{\rho} - gm^{-1}\bar{\Lambda}|^2.$$

Proposition 4.14 now follows from Proposition 2.12(d) and (4.13.1). ∎

4.4. *Transformation Properties of the String Functions*

Recall the string functions c_λ^Λ defined in Section 2.3 for $\Lambda \in P_+$ such that $m := \Lambda(c) > 0$ and $\lambda \in \max(\Lambda)$ by

$$c_\lambda^\Lambda = e^{2\pi i s_\Lambda(\lambda)\tau} \sum_{n \geqslant 0} \operatorname{mult}_\Lambda(\lambda - n\delta)\, e^{2\pi i n\tau}.$$

This is a holomorphic function of $\tau \in \mathscr{H}_+$.

Technically, instead of using string functions, it is often more convenient to deal with the functions $c(\lambda, \mu, m)$ (λ, $\mu \in \bar{\mathfrak{h}}^*$, m a positive integer), called *virtual string functions*, defined as follows.

For $w \in W$, $\Lambda \in P_+$, $\Lambda(c) = m$ and $\nu \in \max(\Lambda)$, set

$$c(\overline{w(\Lambda+\rho)}, \bar{\nu}, m) := (\det w)\, c_\nu^\Lambda.$$

Put $c(\lambda, \mu, m) = 0$ if it is not already defined. It is easy to see that $c(\lambda, \mu, m)$ is well-defined for all λ, $\mu \in \bar{\mathfrak{h}}^*$. Note that $c(\lambda, \mu, m) = 0$ unless $\lambda \in \bar{P}$ and $\mu \in \lambda - \bar{\rho} + \bar{Q} \subset \bar{P}$.

Furthermore, since $\overline{t_\gamma(\Lambda+\rho)} = \bar{\Lambda} + \bar{\rho} + (m+g)\gamma$ and $\overline{t_\gamma(\nu)} = \bar{\nu} + m\gamma$ for $\nu \in P(\Lambda)$, the function $c(\lambda, \mu, m)$ depends only on $\lambda \bmod (m+g)M$ and $\mu \bmod mM$.

Note that for $w, w' \in \bar{W}$:

$$c(w(\lambda), w'(\mu), m) = (\det w)\, c(\lambda, \mu, m).$$

Similarly, we define $c'(\lambda, \mu, m)$ for the adjacent affine Lie algebra; this function depends only on $\lambda \bmod (m+g) M'$ and $\mu \bmod mM'$ (we use the fact that $\rho'(c) = \rho(c) = g$ by (4.6.1)).

Now we can rewrite the theta function identity (2.18) as follows:

$$A_\rho^{-1} \sum_{w \in \bar{W}} (\det w)\, \Theta^M_{w(\lambda), m+g} = \sum_{\mu \in M^* \bmod mM} c(\lambda, \mu, m)\, \Theta^{M}_{\mu, m} \tag{4.15}$$

for any $\lambda \in \bar{P} = M'^*$ and any positive integer m.

To verify (4.15), put $\lambda' = \lambda + (m+g)\Lambda_0$, so that by Propositions 1.9 and 2.13, there exists a unique $\lambda'' \in W(\lambda') \cap P_+$. If $\lambda'' \in P_{++}$, then by Proposition 2.13, (4.15) is just (2.18) for $\Lambda = \lambda'' - \rho$. Otherwise, all $c(\lambda, \mu, m)$ are 0 by definition, and since $r_i(\lambda'') = \lambda''$ for some $i \in I$, the sum on the left-hand side of (4.15) is $\pm e^{-|\lambda|^2 \delta / 2(m+g)} \sum_{w \in W} (\det w) e^{w(\lambda'')} = 0$. This verifies (4.15).

Similarly, for the adjacent root system we get

$$(A'_{\rho'})^{-1} \sum_{w \in \bar{W}} (\det w)\, \Theta^{M'}_{w(\lambda), m+g} = \sum_{\mu \in M^* \bmod mM'} c'(\lambda, \mu, m)\, \Theta^{M'}_{\mu, m} \tag{4.16}$$

for any $\lambda \in \bar{P}' = M^*$ and any positive integer m.

By Lemma 3.12, the string functions are characterized as functions of τ satisfying the theta function identity (4.15). This immediately implies that the string functions are modular forms [18]. Indeed, as we already mentioned, the string functions are holomorphic on $\mathscr{H}_+$. Furthermore, by Proposition 3.3(b),

$$\det(\Theta^M_{\mu, m}(\tau, v, 0))_{\mu, v \in M^* \bmod mM} \neq 0$$

for generic τ. Hence, (4.15) gives us a non-degenerate system of linear equations with indeterminates being the string functions, whose coefficients are modular forms. Therefore the string functions are modular forms of weight $-\frac{1}{2}l$. In this section we show that, moreover, the string functions are meromorphic modular forms, and we derive their transformation properties and estimate the orders of their poles at cusps.

First, we introduce some notations.

For $m \neq 0$, $B = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \mathrm{Mp}(2, \mathbb{R})$ and $\alpha, \mu, \beta \in \bar{\mathfrak{h}}^*$, set $f(B; \alpha, \mu; \beta; m) = \exp \pi i m^{-1} [(a\mu + c\alpha + 2\beta, b\mu + d\alpha) - (\mu, \alpha)]$.

Let N be the least positive integer such that $N|\gamma|^2 \in 2\mathbb{Z}$ for all $\gamma \in M^*$. N is found in the following Table N.

TABLE N

Type $X_n^{(k)}$	l	N
$A_l^{(1)}$	odd	$2(l+1)$
	even	$l+1$
$B_l^{(1)}, D_l^{(1)}, A_{2l-1}^{(2)}$	odd	8
	$\equiv 2 \bmod 4$	4
	$4 \mid l$	2
$C_l^{(1)}, E_7^{(1)}, D_{l+1}^{(2)}$		4
$E_6^{(1)}, G_2^{(1)}, D_4^{(3)}$		3
$F_4^{(1)}, A_{2l}^{(2)}, E_6^{(2)}$		2
$E_8^{(1)}$		1

Recall the group $\Gamma \subset \mathrm{Mp}(2, \mathbb{R})$ introduced in Section 4.1.
Now we are in a position to prove our first main result.

THEOREM A. *Let $\mathfrak{g}$ be an affine Lie algebra. Let m be a positive integer and let μ, $\mu' \in \bar{P}$. Then the virtual string functions have the following transformation properties.*

(1) (a)
$$c\left(\mu, \mu', m; -\frac{1}{\tau}\right) = |M^*/(m+g)M'|^{-1/2} \times |M^*/mM'|^{-1/2}\, |M'/M|^{1/2}\, i^{|\bar{\Delta}_+|}(-i\tau)^{-l/2} \times \sum_{\substack{\nu \in M^* \bmod (m+g)M' \\ \nu' \in M^* \bmod mM'}} (\exp 2\pi i[-(m+g)^{-1}(\mu,\nu) + m^{-1}(\mu',\nu')])\, c'(\nu, \nu', m; \tau).$$

(b) *If $a_0 = 1$, then*

$$c(\mu, \mu', m; \tau+1) = (\exp \pi i[(m+g)^{-1}|\mu|^2 - m^{-1}|\mu'|^2 - g^{-1}|\bar{\rho}|^2])\, c(\mu, \mu', m; \tau).$$

If $a_0 = 2$, then

$$c(\mu, \mu', m; \tau+1) = (\exp \pi i[(m+g)^{-1}|\mu|^2 - m^{-1}|\mu'|^2 - g^{-1}|\bar{\rho}|^2 + |\mu - \mu' - \bar{\rho}|^2])\, c(\mu, \mu', m; \tau).$$

(2) *Let* $B = ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), j) \in \Gamma$, *and choose* $\beta, \beta'' \in \bar{P}$ *satisfying*

$$(m+g)\, ac\, |\alpha|^2 \equiv 2(\alpha, \beta) \bmod 2\mathbb{Z} \qquad \text{if } c\alpha \in M,\ (m+g)\alpha \in M^*;$$

$$mcd\, |\alpha|^2 \equiv 2(\alpha, \beta'') \bmod 2\mathbb{Z} \qquad \text{if } c\alpha \in M,\ m\alpha \in M^*.$$

Set $\beta' = -a\beta''$,

$$\varepsilon = [\exp -\pi i m^{-1} ab\, |\beta''|^2] \times v(m+g, B(0, (m+g)^{-1}\beta, 0))\, v(m, B^{-1}(0, m^{-1}\beta'', 0)).$$

Then

$$F\left(\frac{a\tau+b}{c\tau+d}\right) c\left(\mu, \mu', m; \frac{a\tau+b}{c\tau+d}\right) = \varepsilon (c\tau+d)^{|\bar{\Delta}_+|} \sum_{\substack{\alpha \in M^* \\ c\alpha \bmod (m+g)M}} \sum_{\substack{\alpha' \in M^* \\ c\alpha' \bmod mM}} f(B; \alpha, \mu; \beta; m+g) \times \overline{f(B; \alpha', \mu'; \beta'; m)} F(\tau)\, c(a\mu + c\alpha + \beta, a\mu' + c\alpha' + \beta', m; \tau).$$

(3) *If* $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(Nm) \cap \Gamma_0(N(m+g))$ *and* $b \equiv 0 \bmod a_0$, *then*:

$$F\left(\frac{a\tau+b}{c\tau+d}\right) c\left(\mu, \mu', m; \frac{a\tau+b}{c\tau+d}\right) = \varepsilon \exp \pi i ab \left(\frac{|\mu|^2}{(m+g)} - \frac{|\mu'|^2}{m}\right) \times (c\tau+d)^{|\bar{\Delta}_+|} F(\tau)\, c(a\mu, a\mu', m; \tau),$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } l \text{ is even}, \\ \left(\dfrac{m(m+g)}{d}\right) & \text{if } l \text{ is odd}. \end{cases}$$

(4) *If* $a_0 = 1$ (*resp.* $=2$), *then* $F(\tau)\, c(\mu, \mu', m; \tau)$ (*resp.* $\eta(\tau)^{2l(l+1)} c(\mu, \mu', m; \tau)$) *is a cusp form of weight* $|\bar{\Delta}_+|$ (*resp.* $\frac{1}{2}l(2l+1)$) *for* $\Gamma(Nm) \cap \Gamma(N(m+g))$ *with trivial multiplier system* (*resp. multiplier system* $(\frac{c}{d})^l$).

(5) *The linear span of all string functions for all highest weight modules* $L(\Lambda)$, $\Lambda \in P_+$, *of level* m *is invariant under the projective right action of* $\Gamma_0(k')$ *defined by*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(\tau) = (c\tau+d)^{l/2} f\left(\frac{a\tau+b}{c\tau+d}\right).$$

Proof. Put $\varepsilon' = v(m+g, B(0,(m+g)^{-1}\beta, 0))$, $\varepsilon'' = v(m, B^{-1}(0, m^{-1}\beta'', 0))$. To prove (2) we apply $B \circ B^{-1} = I$ to the left-hand side of identity (4.15) and use Propositions 4.5(a) and 4.6(b). Using the fact that $\bar{W}$ commutes with B, we obtain, for $\mu \in \bar{P}$:

$$\sum_{\mu' \in M^* \bmod mM} c(\mu, \mu', m)\, \Theta^M_{\mu', m}$$

$$= \left(A_\rho^{-1} \sum_{w \in \bar{W}} (\det w)\, \Theta^M_{w(\mu), m+g}\right)\Bigg|_B\Bigg|_{B^{-1}}$$

$$= v_\rho(B)^{-1}\varepsilon'[j \circ B^{-1}]^{-l} \sum_{\substack{\alpha \in M^* \\ c\alpha \bmod (m+g)M}} f(B; \alpha, \mu; \beta; m+g)$$

$$\times \left(A_\rho^{-1} \sum_{w \in \bar{W}} (\det w)\, \Theta^M_{w(a\mu + c\alpha + \beta), m+g}\right)\Bigg|_{B^{-1}}.$$

Now we expand the resulting expression using (4.15) again, obtaining

$$v_\rho(B)^{-1}\varepsilon'[j \circ B^{-1}]^{-l} \sum_{\substack{\alpha \in M^* \\ c\alpha \bmod (m+g)M}} f(B; \alpha, \mu; \beta; m+g)$$

$$\times \left(\sum_{\mu'' \in M^* \bmod mM} c(a\mu + c\alpha + \beta, \mu'', m)\, \Theta^M_{\mu'', m}\right)\Bigg|_{B^{-1}}.$$

We apply Proposition 4.5(a) to expand $\Theta^M_{\mu'', m}|_{B^{-1}}$ in this expression obtaining

$$v_\rho(B)^{-1}\varepsilon'\varepsilon''[j \circ B^{-1}]^{-l} \sum_{\substack{\alpha \in M^* \\ c\alpha \bmod (m+g)M}} f(B; \alpha, \mu; \beta; m+g)$$

$$\times \sum_{\mu'' \in M^* \bmod mM} (c(a\mu + c\alpha + \beta, \mu'', m) \circ B^{-1})$$

$$\times \sum_{\substack{\alpha'' \in M^* \\ c\alpha'' \bmod mM}} f(B^{-1}; \alpha'', \mu''; \beta''; m)\, \Theta^M_{d\mu'' - c\alpha'' + \beta'', m}.$$

We replace, in the last expression, the summation

$$\sum_{\substack{\alpha'' \in M^* \\ c\alpha'' \bmod mM}}$$

by the summation

$$|M^*/(mM+cM^*)|^{-1} \sum_{\alpha''\in M^* \bmod mM}$$

Finally, we reparametrize the resulting sums by α', $\mu' \in M^* \bmod mM$, where $\alpha' = a\alpha'' - b\mu''$ and $\mu' = -c\alpha'' + d\mu'' + \beta''$, and use the fact that $f(B^{-1}; \alpha'', \mu''; \beta''; m) = [\exp -\pi i m^{-1} ab\,|\beta''|^2]\,\overline{f(B; \alpha', \mu'; \beta'; m)}$. Then we obtain the equation

$$\begin{aligned}\sum_{\mu'\in M^* \bmod mM} & c(\mu,\mu',m)\,\Theta^M_{\mu',m} \\ &= \varepsilon v_\rho(B)^{-1}[j\circ B^{-1}]^{-l}\,|M^*/(mM+cM^*)|^{-1} \\ &\quad\times \sum_{\substack{\alpha\in M^* \\ c\alpha \bmod (m+g)M}} f(B;\alpha,\mu;\beta;m+g) \\ &\quad\times \sum_{\alpha',\mu'\in M^* \bmod mM} \overline{f(B;\alpha',\mu';\beta';m)} \\ &\quad\times [c(a\mu+c\alpha+\beta, a\mu'+c\alpha'+\beta', m)\circ B^{-1}]\,\Theta^M_{\mu',m}.\end{aligned}$$

We match the coefficients of $\Theta^M_{\mu',m}$, $\mu' \in \bar{P}$, on both sides, as permitted by Proposition 4.2, and note that the summand depends only on $c\alpha' \bmod mM$. We obtain

$$\begin{aligned}c(\mu,\mu',m) &= \varepsilon v_\rho(B)^{-1}[j\circ B^{-1}]^{-l} \\ &\quad\times \sum_{\substack{\alpha,\alpha'\in M^* \\ c\alpha \bmod (m+g)M \\ c\alpha' \bmod mM}} f(B;\alpha,\mu;\beta;m+g)\,\overline{f(B;\alpha',\mu';\beta';m)} \\ &\quad\times [c(a\mu+c\alpha+\beta, a\mu'+c\alpha'+\beta', m)\circ B^{-1}].\end{aligned}$$

We multiply both sides by F, and then compose them with B. Recalling that by Proposition 4.6(b),

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = v_\rho(B)\,j(\tau)^D\,F(\tau),$$

we obtain (2).

To prove (3), choose $\beta = \beta'' = 0$ (which satisfy the hypothesis of (2)), so that by Proposition 3.17(b), $\varepsilon = v(m+g, B)\,v(m, B^{-1}) = v(m+g, B)\,\overline{v(m,B)}$ is as given in (3). This proves (3).

The proof of (1a) is similar to that of (2). We expand the left-hand side of identity (4.15) using Propositions 4.5(c) and 4.6(c), obtaining, for $\mu \in \bar{P}$:

$$\sum_{\mu' \in M^* \bmod mM} c(\mu, \mu', m; \tau)\, \Theta^M_{\mu',m}(\tau, z, t)$$
$$= i^{|\bar{\Delta}_+|}\, |M^*/(m+g)\, M'|^{-1/2}$$
$$\times \sum_{\mu' \in M^* \bmod (m+g)M'} \exp[-2\pi i (m+g)^{-1}(\mu, \mu')]$$
$$\times \left(A'^{-1}_{\rho'} \sum_{w \in \bar{W}} (\det w)\, \Theta^{M'}_{w(\mu'),m+g} \right) (S^{-1} \cdot (\tau, z, t)).$$

We expand the right-hand side using identity (4.16), obtaining

$$i^{|\bar{\Delta}_+|}\, |M^*/(m+g)\, M'|^{-1/2} \sum_{\mu' \in M^* \bmod (m+g)M'} \exp[-2\pi i (m+g)^{-1}(\mu, \mu')]$$
$$\times \sum_{\mu'' \in M^* \bmod mM} c'\left(\mu', \mu'', m; -\frac{1}{\tau}\right) \Theta^M_{\mu'',m}(S^{-1} \cdot (\tau, z, t)).$$

We transform this expression using Proposition 4.5(c), obtaining

$$i^{|\bar{\Delta}_+|}\, |M^*/(m+g)\, M'|^{-1/2} \sum_{\mu' \in M^* \bmod (m+g)M'} \exp[-2\pi i (m+g)^{-1}(\mu, \mu')]$$
$$\times \sum_{\mu'' \in M^* \bmod mM} c'\left(\mu', \mu'', m; -\frac{1}{\tau}\right) (-i\tau)^{l/2}\, |M^*/mM|^{-1/2}$$
$$\times \sum_{\mu''' \in M^* \bmod mM} \exp[2\pi i m^{-1}(\mu'', \mu''')]\, \Theta^M_{\mu''',m}(\tau, z, t).$$

Here we used that $\Theta^M_{-\nu,m}(\tau, -z, t) = \Theta^M_{\nu,m}(\tau, z, t)$.

We match the coefficients of $\Theta^M_{\mu',m}$, $\mu' \in \bar{P}$, to obtain

$$c\left(\mu, \mu', m; -\frac{1}{\tau}\right)$$
$$= |M^*/(m+g)\, M'|^{-1/2}\, |M^*/mM'|^{-1/2}\, |M'/M|^{1/2}\, i^{|\bar{\Delta}_+|}(-i\tau)^{-l/2}$$
$$\times \sum_{\substack{\nu \in M^* \bmod (m+g)M' \\ \nu' \in M^* \bmod mM'}} \exp 2\pi i[-(m+g)^{-1}(\mu, \nu) + m^{-1}(\mu', \nu')]$$
$$\times c'(\nu, \nu', m; \tau).$$

This proves (1a). The proof of (1b) is similar, using (4.15) and Proposition 4.5(b).

We now prove (5). If $a_0 = 2$, (5) is immediate from (1) since $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ generate $SL(2, \mathbb{Z})$. If $a_0 = 1$, (5) follows from (2).

Finally, we prove (4). By (3), $\psi(\tau) := F(\tau)\, c(\mu, \mu', m; \tau)$ is a modular form of weight $|\bar{\Delta}_+|$ for $\Gamma' := \Gamma(Nm) \cap \Gamma(N(m+g))$ and multiplier system ε given in (3). To compute $\varepsilon(A)$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$, one uses standard properties of the Jacobi symbol. If $a_0 = 1$, then $2 \mid l$ or $4 \mid N$ by inspection of Table N; these imply that $\varepsilon(A) = 1$. If $a_0 = 2$, then one finds that $\varepsilon(A) = (\frac{2}{d})^l$. Put $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. Then $\theta(\tau) = \eta(\tau/2)^{-2}\, \eta(\tau)^5\, \eta(2\tau)^{-2}$ by (3.20.2), so that $\eta(\tau)^{2l(l+1)} = F(\tau)\theta(\tau)^l$ if $a_0 = 2$. By Proposition 3.8b(ii), $\theta(A\tau) = (\frac{c}{d})(\frac{2}{d})(c\tau + d)^{1/2}\theta(\tau)$ if $a_0 = 2$. This suffices to prove (4), except for the assertion about behavior at the cusps.

Let $H(\tau) = F(\tau)$ if $a_0 = 1$, $H(\tau) = \eta(\tau)^{2l(l+1)}$ if $a_0 = 2$. Then $H(\tau)$ is a modular form for $\Gamma_0(k')$. Since $H(\tau)$ vanishes to order $|\bar{\rho}|^2/2g$ at the cusp $i\infty$ of $\Gamma_0(k')$, $\psi_0(\tau) := H(\tau)\, c(\mu, \mu', m; \tau)$ vanishes at the cusp $i\infty$ of Γ' by Proposition 4.14. Similarly, (1a) and Proposition 4.14 applied to the adjacent root system show that $\psi_0(\tau)$ vanishes at the cusp 0 of Γ'. If $k' = 1$ (resp. $k' \neq 1$), then the set of cusps of $\Gamma_0(k')$ is $\{i\infty\}$ (resp. $\{i\infty, 0\}$); therefore, every cusp of Γ' lies in the $\Gamma_0(k')$-orbit of $i\infty$ or of 0. This along with (5) proves (4). ∎

Remark. It is also possible to prove (1) and (2) of Theorem A by using Proposition 3.3(a).

4.5. *The Matrix of String Functions*

Given a positive integer m, let $P_+^{(m)} = \{\lambda \in P_+ \bmod \mathbb{C}\delta \mid \lambda(c) = m\}$. The string functions c_λ^Λ $(\Lambda, \lambda \in P_+^{(m)})$ are characterized as elements of $\mathscr{O}(\mathscr{H}_+)$ such that for $\Lambda \in P_+^{(m)}$, we have

$$A_\rho^{-1} A_{\Lambda+\rho} = \sum_{\lambda \in P_+^{(m)}} c_\lambda^\Lambda S_\lambda .$$

Thus we may regard $(c_\lambda^\Lambda)_{\Lambda, \lambda \in P_+^{(m)}}$ as the matrix of the $\mathscr{O}(\mathscr{H}_+)$-linear isomorphism from $\widetilde{Th}_{m+g}^-$ onto $\widetilde{Th}_m^+$ defined by $F \mapsto A_\rho^{-1} F$.

Introduce the function

$$G(\tau) = e^{2\pi i R\tau} \prod_{n \geq 1} (1 - e^{2\pi i n\tau})^{\operatorname{mult} n\delta}, \tag{4.17}$$

where $R = |\bar{\rho}|^2/2g(h+1)$ if $k = 1$ and $R = |\bar{\rho}|^2/2g(g+1)$ if $k \neq 1$. Using Proposition 1.11(a), (c), (e), we deduce that

$$G(\tau) = \eta(\tau)^l \qquad \text{if} \quad k' = 1,$$

$$G(\tau) = \prod_{i=1}^{l} \eta\left(\frac{1}{2}|\alpha_i|^2 \tau\right) \qquad \text{if} \quad k' \neq 1.$$

Explicitly, we have the following Table G.

TABLE G

Type $X_n^{(k)}$	$G(\tau)$
$X_l^{(1)}$ or $A_{2l}^{(2)}$	$\eta(\tau)^l$
$A_{2l-1}^{(2)}$	$\eta(\tau)^{l-1}\,\eta(2\tau)$
$D_{l+1}^{(2)}$	$\eta(\tau)\,\eta(2\tau)^{l-1}$
$E_6^{(2)}$	$\eta(\tau)^2\,\eta(2\tau)^2$
$D_4^{(3)}$	$\eta(\tau)\,\eta(3\tau)$

Let $G'(\tau)=e^{2\pi iR'\tau}\prod_{n\geqslant 1}(1-e^{2\pi in\tau/k'})^{\mathrm{mult}'\,n\delta'}$, where $R'=|\bar\rho'|^2/2g(h+1)$ if $k=1$ and $R'=|\bar\rho'|^2/2g(g+1)$ if $k\neq 1$.

Remarks. (1) $R'=n/24k$ if $\mathfrak{g}$ is of type $X_n^{(k)}$.

(2) $F(\tau)=G(\tau)^{h+1}$ if $k=1$ and $F(\tau)=G(\tau)^{g+1}$ if $k'\neq 1$.

PROPOSITION 4.18 [36]. *For any positive integer m,* $\det(c_\lambda^\Lambda)_{\Lambda,\lambda\in P_+^{(m)}}=G^{-|P_+^{(m)}|}$.

Proof. Put $b=|P_+^{(m)}|$, and let $H(\tau)$ be the determinant in question. For $\tau\in\mathscr{H}_+$, $\{A_{\Lambda+\rho}|_{Y_\tau}\}_{\Lambda\in P_+^{(m)}}$ is linearly independent by Propositions 4.2 and 4.3(e), so that $H(\tau)\neq 0$. For any $B\in\Gamma_0(k')$, there exist $(B,j)\in Mp(2,\mathbb{R})$ and $g\in G_0\cap(B,j)N_{\mathbb{R}}$. Hence, by Proposition 4.4(a), $H(\tau)$ is a modular form of weight $-\frac{1}{2}bl$ for $\Gamma_0(k')$. Since $G(\tau)$ is a modular form of weight $\frac{1}{2}l$ for $\Gamma_0(k')$ by inspection, we conclude that $A(\tau):=G(\tau)^b\,H(\tau)$ is a modular form of weight 0 for $\Gamma_0(k')$. We must show that $A(\tau)=1$.

Put $H'(\tau)=(\det c_\lambda'^\Lambda)_{\Lambda,\lambda\in P_+'^{(m)}}$. By Proposition 4.4(b), $H(-1/\tau)=C'\tau^{-(1/2)bl}H'(\tau)$ for some $C'\in\mathbb{C}$, and by inspection, $G(-1/\tau)=|M'/M|^{-1/2}(-i\tau)^{(1/2)l}\,G'(\tau)$, so that for some $C\in\mathbb{C}$,

$$A\left(-\frac{1}{\tau}\right)=CG'(\tau)^b\,H'(\tau).$$

Put $S=S(m,\mathfrak{g})=\sum_{\Lambda\in P_+^{(m)}}s_\Lambda(\Lambda)$. Since $s_\Lambda(\Lambda)<s_\Lambda(\lambda)$ for all $\lambda\in\max(\Lambda)\backslash W(\Lambda)$, $H(\tau)$ is of the form $q^S(1+\sum_{n>0}c_nq^n)$, so that $H(\tau)$ vanishes to order S at the cusp $i\infty$ of $\Gamma_0(k')$. Similarly, $H'(\tau)$ vanishes to order $S':=k'^{-1}S(m,\mathfrak{g}_{\mathrm{adj}})$ at the cusp $i\infty$ of $\Gamma(k')$, since the isomorphism Φ of Section 1.5 taking the adjacent root system Δ' to the root system of the adjacent affine Lie algebra $\mathfrak{g}_{\mathrm{adj}}$ multiplies lengths by $k'^{1/2}$. Hence, $A(\tau)$ vanishes to order $bR+S$ at the cusp $i\infty$ and to order $bR'+S'$ at the cusp 0 of $\Gamma_0(k')$.

If $\mathfrak{g}_{\mathrm{adj}}$ is isomorphic to $\mathfrak{g}$, then $S=k'S'$ and $R=k'R'$, so that $bR+S=k'(bR'+S')$. Since $i\infty$ and 0 are the only cusps of $\Gamma_0(k')$ by (c) of

Section 3.3, this shows that either $A(\tau)$ or $A(\tau)^{-1}$ is a holomorphic modular form of weight 0, so that $A(\tau)$ is constant by (d) of Section 3.3. From the expansion $A(\tau)=q^{bR+S}(1+\sum_{n>0}a_nq^n)$, we now conclude that $A(\tau)=1$, and that $b^{-1}S=-R$. If $\mathfrak{g}_{\text{adj}}$ is not isomorphic to $\mathfrak{g}$, then $\mathfrak{g}$ is of type $A_{2l-1}^{(2)}$ or $D_{l+1}^{(2)}$, and one checks directly that $b^{-1}S=-R$, so that again $A(\tau)=1$. ∎

Since $b^{-1}S=-R$, from the proof, we have:

COROLLARY 4.19. *Put $h_\mathfrak{p}=h$ if $k=1$ and $h_\mathfrak{p}=g$ if $k\neq 1$. Then, for any positive integer m,*

$$|P_+^{(m)}|^{-1}\sum_{\Lambda\in P_+^{(m)}} s_\Lambda(\Lambda)=-\frac{|\bar{\rho}|^2}{2g(h_\mathfrak{p}+1)}.$$

Remark. In the limit as $m\to\infty$, Corollary 4.19 asserts that the average value of the square of the distance from $\mu\in\bar{\mathfrak{h}}_\mathbb{R}^*$ to $\overline{W(\rho)}$ is $(h_\mathfrak{p}+1)^{-1}|\bar{\rho}|^2$.

We now consider the matrix $(d_\Lambda^\lambda)_{\lambda,\Lambda\in P_+^{(m)}}$ inverse to the matrix $(c_\lambda^\Lambda)_{\Lambda,\lambda\in P_+^{(m)}}$ of string functions. It is the matrix of the isomorphism $F\mapsto A_\rho F$ from $\widetilde{Th}_m^+$ onto $\widetilde{Th}_{m+g}^-$ defined by:

$$A_\rho S_\lambda=\sum_{\Lambda\in P_+^{(m)}} d_\Lambda^\lambda A_{\Lambda+\rho}.$$

For $\lambda\in P$, with $\lambda(c)=m>0$, write

$$A_\rho\Theta_\lambda=\sum_{\substack{\Lambda\in P \bmod \mathbb{C}\delta \bmod T\\ \Lambda(c)=m+g}} a_\Lambda^\lambda\Theta_\Lambda,$$

with $a_\Lambda^\lambda\in\mathcal{O}(\mathscr{H}_+)$, and match coefficients of $e^{-2\pi i(\bar{\Lambda},z)}$ to obtain

$$\begin{aligned}
a_\Lambda^\lambda&=\sum_{\substack{t\in T\\ w\in W\\ \overline{t(\lambda)}+\overline{w(\rho)}=\bar{\Lambda}}}(\det w)\,q^{|\overline{t(\lambda)}|^2/2m+|\overline{w(\rho)}|^2/2g-|\bar{\Lambda}|^2/2(m+g)}\\
&=\sum_{t\in T}\varepsilon(\bar{\Lambda}-\overline{t(\lambda)})\,q^{(m(m+g)/2g)|(m+g)^{-1}\bar{\Lambda}-m^{-1}\overline{t(\lambda)}|^2}\\
&=\sum_{t\in T}\varepsilon(\overline{t(\Lambda)}-\bar{\lambda})\,q^{(m(m+g)/2g)|(m+g)^{-1}\overline{t(\Lambda)}-m^{-1}\bar{\lambda}|^2},
\end{aligned}$$

where $\varepsilon(\mu)=\det w$ if $\mu=\overline{w(\rho)}$, and $\varepsilon(\mu)=0$ if $\mu\notin\overline{W(\rho)}$. Clearly, we have for λ, $\Lambda\in P_+^{(m)}$:

$$\begin{aligned}
d_\Lambda^\lambda&=\sum_{w\in W/TW_\lambda}a_{\Lambda+\rho}^{w(\lambda)}\\
&=\sum_{\mu\in W(\lambda)}\varepsilon(\bar{\Lambda}+\bar{\rho}-\bar{\mu})\,q^{(m(m+g)/2g)|(\bar{\Lambda}+\bar{\rho})/(m+g)-\bar{\mu}/m|^2}.
\end{aligned}\tag{4.20}$$

Fix $\beta \in M^*$. Then by the product formula for A_ρ, we have for each $\Lambda \in P$ with $\Lambda(c) = m > 0$:

$$\sum_{\substack{\lambda \in P \bmod \mathbb{C}\delta \bmod T \\ \lambda(c) = m}} e^{2\pi i m^{-1}(\beta, \bar{\lambda})} a^{\lambda}_{\Lambda+\rho}$$
$$= e^{2\pi i m^{-1}(\beta, \bar{\Lambda})} q^{(g(m+g)/2m)|g^{-1}\bar{\rho} - (m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$
$$\times \prod_{\alpha \in \Delta_+} (1 - e^{2\pi i m^{-1}(\alpha, \beta + \tau(\Lambda+\rho))})^{\text{mult}\,\alpha}. \tag{4.20.1}$$

Similarly, for each $\lambda \in P$ with $\lambda(c) = m > 0$, we have

$$\sum_{\substack{\Lambda+\rho \in P \bmod \mathbb{C}\delta \bmod T \\ (\Lambda+\rho)(c) = m+g}} e^{-2\pi i (m+g)^{-1}(\beta, \bar{\Lambda}+\bar{\rho})} a^{\lambda}_{\Lambda+\rho}$$
$$= e^{-2\pi i (m+g)^{-1}(\beta, \bar{\lambda}+\bar{\rho})} q^{(mg/2(m+g))|g^{-1}\bar{\rho} - m^{-1}\bar{\lambda}|^2}$$
$$\times \prod_{\alpha \in \Delta_+} (1 - e^{2\pi i (m+g)^{-1}(\alpha, \beta + \tau\lambda)})^{\text{mult}\,\alpha}. \tag{4.20.2}$$

These formulas show that the d^{λ}_{Λ} are closely related to specializations of A_ρ. If we take $\beta = 0$ in (4.20.1), we obtain for $\Lambda \in P^{(m)}_+$:

$$\sum_{\lambda \in P^{(m)}_+} d^{\lambda}_{\Lambda} = q^{(g(m+g)/2m)|g^{-1}\bar{\rho} - (m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$
$$\times \prod_{\alpha \in \Delta_+} (1 - q^{m^{-1}(\alpha, \Lambda+\rho)})^{\text{mult}\,\alpha}. \tag{4.20.3}$$

If $\max(\Lambda) = W(\Lambda)$, then $d^{\Lambda}_{\Lambda} = \sum_{\lambda \in P^{(m)}_+} d^{\lambda}_{\Lambda}$ and $c^{\Lambda}_{\Lambda} = (d^{\Lambda}_{\Lambda})^{-1}$, so that (4.20.3) gives many of the results of the next section.

Finally, we show that $d^{m\Lambda_0}_{\Lambda}$ is often given by a product. Suppose that $M = \bar{Q}$, which is the case if $k \neq 1$, or if $k = 1$ and all $\alpha \in \bar{\Delta}$ have the same length. Suppose that $\Lambda \in P$, $\Lambda(c) = m > 0$, and that $\bar{\Lambda} \in M$, which is the case if $d^{m\Lambda_0}_{\Lambda} \neq 0$. Finally, suppose that the greatest common divisor of m and g is 1. Choose $a, b \in \mathbb{Z}$ with $a(m+g) + bg = 1$. Then

$$\overline{T(\Lambda+\rho)} \cap \overline{W(\rho)} = a(m+g)\,\bar{W}(\bar{\rho}) + bg(\bar{\Lambda}+\bar{\rho}) + g(m+g)M$$
$$= (m+g)\,\overline{W(\rho)} + bg(\bar{\Lambda}+\bar{\rho}),$$

and we obtain, since $\varepsilon(a\mu) = \varepsilon(a\bar{\rho})\varepsilon(\mu)$ for all $\mu \in \bar{P}$:

$$d^{m\Lambda_0}_{\Lambda} = \varepsilon(a\bar{\rho}) \sum_{w \in W} (\det w)\, q^{(mg(m+g)/2)|g^{-1}\overline{w(\rho)} + b(m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$
$$= \varepsilon(a\bar{\rho})\, q^{(mg(m+g)/2)|g^{-1}\bar{\rho} + b(m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$
$$\times \prod_{\alpha \in \Delta_+} (1 - q^{m(\alpha, (m+g)\Lambda_0 - b(\bar{\Lambda}+\bar{\rho}))})^{\text{mult}\,\alpha}.$$

The results of this section are due to the second author. A detailed account of these and some other results will appear in [36].

4.6. *Explicit Computation of String Functions*

Recall that each $\Lambda \in P_+$ is labelled by a vector of integers $(\Lambda(h_0),..., \Lambda(h_l))$, so that $\Lambda = a_0^{-1}\Lambda(d)\delta + \sum_{i=0}^{l} \Lambda(h_i)\Lambda_i$, where the Λ_i are fundamental weights, and Λ has level $\operatorname{lev}(\Lambda) = \sum_{i=0}^{l} a_i^{\vee}\Lambda(h_i)$ (see Section 2.2). If $\Lambda(h_i) = N_i$, $\lambda(h_i) = n_i$, we often write $c^{N_0N_1\cdots}_{n_0n_1\cdots}$ for c^{Λ}_{λ}.

EXAMPLE 1. Suppose that $M = \bar{Q}$, that is, that $k \neq 1$ or else $k = 1$ and $\bar{A}$ has only one root length. Then Proposition 4.27(b) in Section 4.8 shows that $\{\Lambda \in P_+ \mid \Lambda(c) = 1\} = \{\Lambda_j \mid j \in J\} + \mathbb{C}\delta$ and that the Λ_j $(j \in J)$ are incongruent modulo $Q + \mathbb{C}\delta$. Therefore, all non-zero string functions c^{Λ}_{λ} for $\Lambda \in P_+$ of level 1 are equal to $c(\tau) := c^{\Lambda_0}_{\Lambda_0}(\tau)$, which has level 1 since $a_0^{\vee} = 1$. It follows from Proposition 4.18 that

$$c(\tau) = G(\tau)^{-1}.$$

In other words, for $\Lambda \in P_+$ of level 1 we have ([17], [18])

$$\sum_{n \geqslant 0} \operatorname{mult}_{\Lambda}(\Lambda - n\delta)\, q^n = \prod_{j \geqslant 1} (1 - q^j)^{-\operatorname{mult} j\delta}.$$

As a result, by (2.18), $\operatorname{ch}_{L(\Lambda)}$ can be written as follows:

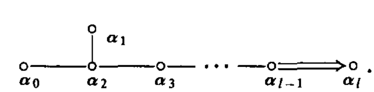
$$e^{-(1/2)|\Lambda|^2\delta}\operatorname{ch}_{L(\Lambda)} = \frac{\sum_{\gamma \in M + \bar{\Lambda}} e^{\Lambda_0 + \gamma - (1/2)|\gamma|^2\delta}}{\prod_{j \geqslant 1} (1 - e^{-j\delta})^{\operatorname{mult} j\delta}} \qquad \text{if} \quad \operatorname{lev}(\Lambda) = 1.$$

In particular, in the case $k = 1$ (i.e., for $\mathfrak{g}$ of type $A_l^{(1)}$, $D_l^{(1)}$, or $E_l^{(1)}$) we have

$$\operatorname{ch}_{L(\Lambda_0)} = \sum_{\gamma \in M} e^{\Lambda_0 + \gamma - (1/2)|\gamma|^2\delta} \Big/ \prod_{j \geqslant 1} (1 - e^{-j\delta})^l,$$

which is proved in [17, p. 131] by a more complicated method. Note also that these formulas are used in [7] for an explicit construction of the "basic" representation $L(\Lambda_0)$.

EXAMPLE 2. Let $\mathfrak{g}$ be of type $B_l^{(1)}$, $l \geqslant 3$, with simple roots numbered as in



Then all $\Lambda \in P_+$ mod $\mathbb{C}\delta$ of level 1 are Λ_0, Λ_1, and Λ_l; the maximal weights of the corresponding $L(\Lambda)$ are W-conjugates of either Λ, or $\Lambda_1 - \delta$ when

$\Lambda = \Lambda_0$, or Λ_0 when $\Lambda = \Lambda_1$. Hence, the non-zero string functions c_λ^Λ of level one are

$$c_{\Lambda_1}^{\Lambda_1} = c_{\Lambda_0}^{\Lambda_0},\ c_{\Lambda_0}^{\Lambda_1} = c_{\Lambda_1}^{\Lambda_0},\ c_{\Lambda_l}^{\Lambda_l}.$$

The initial powers of q in the expansions of the c_λ^Λ, i.e., the characteristics $s_\Lambda(\lambda)$ of λ, are

$$c_{\Lambda_1}^{\Lambda_1}\colon -\frac{2l+1}{48}; \qquad c_{\Lambda_0}^{\Lambda_1}\colon \frac{1}{2} - \frac{2l+1}{48}; \qquad c_{\Lambda_l}^{\Lambda_l}\colon \frac{1-l}{24}.$$

Set $A(\tau) = \eta(\tau)^{l+1}\,\eta(2\tau)^{-1}\,c_{\Lambda_l}^{\Lambda_l}(\tau)$. Then $A(\tau+1) = A(\tau)$, and using $c_{\Lambda_l}^{\Lambda_l}(-\tau^{-1}) = (-i\tau)^{-l/2}2^{-1/2}(c_{\Lambda_1}^{\Lambda_1} - c_{\Lambda_0}^{\Lambda_1})(\tau)$ from Theorem A(1), we find $A(-\tau^{-1}) = A(-(\tau+2)^{-1})$, so that

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = A(\tau) \qquad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since these generate $\Gamma_0(2)$,

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = A(\tau) \qquad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2).$$

From the leading terms above, one checks that $A(\tau)$ is holomorphic at the cusps $i\infty$ and 0 of $\Gamma_0(2)$. Hence $A(\tau) = \text{constant}$. The constant is 1 since $\text{mult}_{\Lambda_l}(\Lambda_l) = 1$. Therefore,

$$c_{\Lambda_l}^{\Lambda_l}(\tau) = \eta(\tau)^{-l-1}\,\eta(2\tau).$$

Replacing τ by $-1/\tau$ in this formula and using $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$, we obtain

$$c_{\Lambda_1}^{\Lambda_1}(\tau) - c_{\Lambda_0}^{\Lambda_1}(\tau) = \eta(\tau)^{-l-1}\,\eta(\tau/2).$$

Replacing τ by $\tau+1$ in this formula, we obtain

$$c_{\Lambda_1}^{\Lambda_1}(\tau) + c_{\Lambda_0}^{\Lambda_1}(\tau) = \eta(\tau/2)^{-1}\,\eta(\tau)^{2-l}\,\eta(2\tau)^{-1}.$$

These three equations determine all string functions of level 1 for $B_l^{(1)}$.

We can now deduce from (2.18) simple expressions for the characters of two $B_l^{(1)}$-modules:

$$\text{ch}_{L(\Lambda_l)} = \frac{\sum_{\gamma\in\bar{Q}} e^{\Lambda_l+\gamma-(1/2)|\gamma|^2\delta-(\gamma,\Lambda_l)\delta}}{\prod_{j\geq 1}(1-e^{-j\delta})^l(1-e^{-(2j-1)\delta})},$$

$$\text{ch}_{L(\Lambda_0)} + \text{ch}_{L(\Lambda_1-(1/2)\delta)} = \frac{\sum_{\gamma\in\bar{Q}} e^{\Lambda_0+\gamma-(1/2)|\gamma|^2\delta}}{\prod_{j\geq 1}(1-e^{-(1/2)j\delta})(1-e^{-j\delta})^{l-2}(1-e^{-2j\delta})}.$$

EXAMPLE 3. Let $\mathfrak{g}$ be of type $A_1^{(1)}$. For $m, n \in \mathbb{Z}$, define:

$$\varepsilon(m,n) = \exp 2\pi i \frac{m-n+2}{8} \qquad \text{for } n \text{ even},$$

$$\varepsilon(m,n) = \exp 2\pi i \frac{n+1}{4} \qquad \text{for } n \text{ odd}.$$

Then the following formulas hold:

level 1: $$c_{10}^{10} = \eta(\tau)^{-1},$$

level 2: $$c_{11}^{11} = \eta(\tau)^{-2}\,\eta(2\tau),$$

level 3: $$c_{21}^{21} = \eta(\tau)^{-2}\,q^{3/40} \prod_{n \not\equiv \pm 1(5)} (1-q^{3n}),$$

level 4: $$c_{22}^{40} = \eta(\tau)^{-2}\,\eta(6\tau)^{-1}\,\eta(12\tau)^2; \qquad c_{40}^{40} - c_{04}^{40} = \eta(2\tau)^{-1},$$

level 5: $$c_{rs}^{pq} = (-1)^p\,\eta(\tau)^{-3} \sum_{\substack{m \equiv r(5)\\ n \equiv 2p+2(7)\\ 7m^2+5n^2 \equiv 4(16)}} \varepsilon(m,n)\,q^{(7m^2+5n^2)/560}$$

(here $p+q=r+s=5$ and $p \equiv r \bmod 2$),

level 6: $$c_{51}^{33} = \eta(\tau)^{-3}\,\eta(2\tau)\,\eta(3\tau)\,\eta(6\tau)^{-1}\,\eta(12\tau),$$

$$c_{51}^{51} + c_{15}^{51} = \eta(\tau)^{-3}\,\eta(2\tau)\,\eta(6\tau)^2\,\eta(12\tau)^{-1},$$

$$c_{51}^{51} - c_{15}^{51} = \eta(\tau)^{-1},$$

level 7: $$c_{52}^{52} = \eta(\tau)^{-3}\,q^{3/28} \prod_{n \equiv 0, \pm 1(7)} (1-q^n) \prod_{n \not\equiv \pm 1(7)} (1-q^{3n}),$$

$$c_{70}^{52} = \eta(\tau)^{-3} \sum_{m \not\equiv n(2)} e^{2\pi i m/6} q^{(m^2+21n^2)/4},$$

level 8: $$c_{62}^{44} = \eta(\tau)^{-3}\,\eta(2\tau)\,\eta(10\tau),$$

$$c_{62}^{62} - c_{26}^{62} = \eta(\tau)^{-1}\,\eta(2\tau)^{-1}\,q^{1/10} \prod_{n \not\equiv \pm 1(5)} (1-q^{4n}),$$

level 9: $$c_{90}^{90} - c_{36}^{90} = \eta(\tau)^{-1}\,q^{-2/33} \prod_{n \equiv \pm 4, \pm 5(11)} (1-q^n)^{-1}$$

$$\times \prod_{n \equiv \pm 1, \pm 3(11)} (1-q^{3n})^{-1},$$

$$c_{90}^{90} + 2c_{36}^{90} = \eta(\tau)^{-3} \sum_{\substack{n \equiv 2(11)\\ 11m^2+n^2 \equiv 4(16)}} \varepsilon(m,n)\,q^{(11m^2+n^2)/176},$$

level 10:
$$c_{55}^{73} = \eta(\tau)^{-3}\,\eta(2\tau)\,\eta(5\tau)^{-1}\,\eta(10\tau)^2,$$

$$c_{91}^{55} = \eta(\tau)^{-3}\,q^{29/40} \prod_{n \not\equiv \pm 1(5)} (1-q^{2n}) \prod_{n \not\equiv \pm 2(5)} (1-q^{3n}),$$

$$c_{91}^{91} - c_{19}^{91} = \eta(\tau)^{-2}\,\eta(2\tau)\,q^{-1/15} \prod_{n \equiv \pm 1(5)} (1-q^{4n})^{-1},$$

$$c_{55}^{55} - 2c_{55}^{91} = \eta(\tau)^{-3}\,\tfrac{1}{2} \sum_{mn \equiv 7(10)} (-1)^{(m-n)/2}\,q^{(m^2+6n^2)/120},$$

$$c_{10,0}^{10,0} - c_{10,0}^{8,2} + c_{10,0}^{6,4} - c_{10,0}^{4,6} + c_{10,0}^{2,8} - c_{10,0}^{0,10}$$
$$= \eta(\tau)^{-3}\,\tfrac{1}{2} \sum_{mn \equiv 1(10)} (-1)^{(m-n)/2}\,q^{(2m^2+3n^2)/240}.$$

The method of proof is essentially that of Example 2. We use the transformation law for string functions under $SL(2, \mathbb{Z})$, together with the calculation of the first few terms in the q-expansions of the string functions using Theorem D from Section V. These computations and the fact that a modular form vanishing at the cusps to sufficiently high order is zero allow us to verify our formulas. The formulas were suggested by computations using q-expansions.

From these formulas and Theorem A we deduce the following additional ones which, together with knowledge of the fractional parts of the $s_\Lambda(\lambda)$ and the equality $c_{mn}^{MN} = c_{nm}^{NM}$, due to the outer automorphism of Δ_+, determine the string functions for the levels 1, 2, 3, 4, 5, and 6. One may do the same for levels 7, 8, 9 and 10, but we omit the result for brevity.

level 2:
$$c_{20}^{20} - c_{02}^{20} = \eta(\tau)^{-2}\,\eta(\tau/2),$$

level 3:
$$c_{12}^{30} = \eta(\tau)^{-2}\,q^{27/40} \prod_{n \not\equiv \pm 2(5)} (1-q^{3n}),$$

$$c_{30}^{30} - c_{12}^{30} = \eta(\tau)^{-2}\,q^{1/120} \prod_{n \not\equiv \pm 1(5)} (1-q^{n/3}),$$

$$c_{21}^{21} - c_{03}^{21} = \eta(\tau)^{-2}\,q^{3/40} \prod_{n \not\equiv \pm 2(5)} (1-q^{n/3}),$$

level 4:
$$c_{31}^{31} + c_{13}^{31} = \eta(\tau/2)^{-1},$$

$$c_{40}^{40} - 2c_{22}^{40} + c_{04}^{40} + 2c_{40}^{22} - 2c_{22}^{22} = \eta(\tau)^{-2}\,\eta(\tau/6)^{-1}\,\eta(\tau/12)^2,$$

level 6:
$$c_{33}^{33} - c_{51}^{33} = \eta(\tau/3)\,\eta(2\tau/3)^{-1}\,\eta(\tau)^{-3}\,\eta(4\tau/3)\,\eta(2\tau),$$

$$c_{51}^{51} - 2c_{33}^{51} + c_{15}^{51} = \eta(2\tau/3)^2\,\eta(\tau)^{-3}\,\eta(4\tau/3)^{-1}\,\eta(2\tau),$$

$$c_{60}^{60} + c_{42}^{60} - c_{24}^{60} - c_{06}^{60} + c_{60}^{42} + c_{42}^{42} - c_{24}^{42} - c_{06}^{42}$$
$$= \eta(\tau)^{-3}\,\eta(\tau/2)\,\eta(\tau/6)^2\,\eta(\tau/12)^{-1}.$$

EXAMPLE 4. Let $\mathfrak{g}$ be of type $A_2^{(1)}$. We determine the c_λ^Λ for Λ with $\operatorname{lev}(\Lambda) = 1, 2, 3$:

level 1:
$$c_{100}^{100} = \eta(\tau)^{-2},$$

level 2:
$$c_{110}^{110} = \eta(\tau)^{-4}\, \eta(2\tau)\, q^{1/20} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{2n}),$$

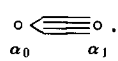
$$c_{011}^{200} = \eta(\tau)^{-4}\, \eta(2\tau)\, q^{9/20} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{2n}),$$

$$c_{200}^{200} - c_{011}^{200} = \eta(\tau)^{-4}\, \eta(\tau/2)\, q^{1/80} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{n/2}),$$

$$c_{110}^{110} - c_{002}^{110} = \eta(\tau)^{-4}\, \eta(\tau/2)\, q^{9/80} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{n/2}),$$

level 3:
$$c_{300}^{300} - c_{030}^{300} = \eta(\tau)^{-1}\, \eta(3\tau)^{-1},$$
$$c_{210}^{210} + c_{021}^{210} + c_{102}^{210} = \eta(\tau)^{-1}\, \eta(\tau/3)^{-1},$$
$$c_{111}^{111} = \eta(\tau)^{-6}\, \eta(2\tau)^3\, \eta(3\tau)^2\, \eta(6\tau)^{-1},$$
$$c_{300}^{300} - 3c_{111}^{300} + 2c_{030}^{300} + c_{111}^{111} - c_{300}^{111} = \eta(\tau)^{-6}\, \eta(\tau/2)^3\, \eta(\tau/3)^2\, \eta(\tau/6)^{-1}.$$

EXAMPLE 5. Let $\mathfrak{g}$ be of type $A_2^{(2)}$, with simple roots numbered as in



Then we have

level 1:
$$c_{10}^{10} = \eta(\tau)^{-1},$$

level 2:
$$c_{20}^{20} = \eta(\tau)^{-2}\, \eta(2\tau)\, \eta(4\tau)^{-1}\, q^{1/10} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{4n}),$$

$$c_{20}^{01} = \eta(\tau)^{-2}\, \eta(2\tau)\, \eta(4\tau)^{-1}\, q^{9/10} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{4n}),$$

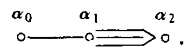
$$c_{20}^{20} + c_{01}^{20} = \eta(\tau)^{-2}\, \eta(\tau/2)\, \eta(\tau/4)^{-1}\, q^{1/160} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{n/4}),$$

$$c_{01}^{01} + c_{20}^{01} = \eta(\tau)^{-2}\, \eta(\tau/2)\, \eta(\tau/4)^{-1}\, q^{9/160} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{n/4}),$$

level 3:
$$c_{30}^{11} = 2\eta(\tau)^{-2}\, \eta(6\tau)^{-1}\, \eta(12\tau)^2,$$
$$c_{30}^{30} + 2c_{11}^{30} - c_{30}^{11} - 2c_{11}^{11} = \eta(\tau)^{-2}\, \eta(\tau/6)^{-1}\, \eta(\tau/12)^2.$$

Together with the initial powers $s_\Lambda(\lambda)$ of q, these determine the c^Λ_λ for Λ of levels 1, 2, 3.

EXAMPLE 6. Let $\mathfrak{g}$ be of type $G_2^{(1)}$, with simple roots numbered as in



Then the c^Λ_λ for $\operatorname{lev}(\Lambda) = 1$ are determined by:

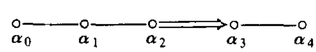
$$c^{\Lambda_2}_{\Lambda_2} = \eta(\tau)^{-3}\, q^{3/40} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{3n}),$$

$$c^{\Lambda_0}_{\Lambda_2} = \eta(\tau)^{-3}\, q^{27/40} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{3n}),$$

$$c^{\Lambda_0}_{\Lambda_0} - c^{\Lambda_0}_{\Lambda_2} = \eta(\tau)^{-3}\, q^{1/120} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{n/3}),$$

$$c^{\Lambda_2}_{\Lambda_2} - c^{\Lambda_2}_{\Lambda_0} = \eta(\tau)^{-3}\, q^{3/40} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{n/3}).$$

EXAMPLE 7. Let $\mathfrak{g}$ be of type $F_4^{(1)}$, with simple roots numbered as in



Then the c^Λ_λ for $\operatorname{lev}(\Lambda) = 1$ are determined by:

$$c^{\Lambda_4}_{\Lambda_4} = \eta(\tau)^{-6}\, \eta(2\tau)\, q^{1/20} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{2n}),$$

$$c^{\Lambda_0}_{\Lambda_4} = \eta(\tau)^{-6}\, \eta(2\tau)\, q^{9/20} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{2n}),$$

$$c^{\Lambda_0}_{\Lambda_0} - c^{\Lambda_0}_{\Lambda_4} = \eta(\tau)^{-6}\, \eta(\tau/2)\, q^{1/80} \prod_{n \not\equiv \pm 1 \pmod 5} (1 - q^{n/2}),$$

$$c^{\Lambda_4}_{\Lambda_4} - c^{\Lambda_4}_{\Lambda_0} = \eta(\tau)^{-6}\, \eta(\tau/2)\, q^{9/80} \prod_{n \not\equiv \pm 2 \pmod 5} (1 - q^{n/2}).$$

We now outline some methods of computing string functions (and therefore also characters).

(1) One may use explicit constructions of representations. For instance, Example 2 may be verified by using spin representations (cf. [23, 47]). In this connection, one may consider the restriction of a highest weight module

to a subalgebra (see, e.g., [23, Section 3c], which is related to the idea of "dual pairs").

Conversely, the results and methods of this paper (which rely on the character formula) may be applied to representation theory. For example, formula (4.20.3) verifies a conjecture from [48, p. 97]. Another example is the following result of the second author.

THEOREM [49]. *Let* $\mathfrak{g}$ *be an affine Lie algebra, and let* $\Lambda \in P_+$ *satisfy* $\Lambda(c) = 1$. *Put* $I^\# = \{i \in I \mid |\alpha_i|^2 < |\alpha_0|^2\}$, $Q^\# = \mathbb{Z}\delta + \sum_{i \in I^\#} \mathbb{Z}\alpha_i$, $\mathfrak{g}^\# = \bigoplus_{\alpha \in Q^\#} \mathfrak{g}_\alpha$. *Put* $T^\# = W^\# \cap T$, *where* $W^\#$ *is the subgroup of* W *generated by* $\{r_\alpha \mid \alpha \in \Delta^{\mathrm{re}} \cap Q^\#\}$. *Then we have the following decomposition of* $L(\Lambda)$ *into irreducible* $\mathfrak{g}^\#$*-submodules*:

$$L(\Lambda) = \bigoplus_{t \in T \bmod T^\#} t(L^\#(\Lambda)),$$

where $t(L^\#(\Lambda)) := \bigoplus_{\alpha \in Q^\#} L(\Lambda)_{t(\Lambda + \alpha)}$.

If $M = \bar{Q}$, so that $I^\# = \varnothing$, the Theorem is essentially Example 1, which was first verified in [17], [18]. Examples 2, 6 and 7 are essentially the cases $B_l^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$ of the Theorem. As in [7], it should be possible to construct the modules $L(\Lambda)$ of level one by using the Theorem.

The proof of the Theorem uses the asymptotics of weight multiplicities (Proposition 4.21).

(2) One may specialize the character formula. For example, if $\Lambda \in P_+$ and $\max(\Lambda) = W(\Lambda)$, then $c_\Lambda^\Lambda S_\Lambda A_\rho = A_{\Lambda+\rho}$, and c_Λ^Λ may be computed by using the "principal" specialization (cf. Section 4.7) as in [17].

(3) One may use the "star" formula of Section 2.1(E) (cf. [6]). This amounts to regarding the matrix C of string functions of given level as the inverse of the matrix D, as in Section 4.5. For example, Examples 1, 2, 6 and 7 may be verified using Proposition 4.18 and formula (4.20.3). This seems to be a good approach to the string functions of level one for $C_l^{(1)}$.

(4) One may use the partition function K, as in the proof of Theorem D.

(5) One may use the theory of theta functions and modular forms, as in Section IV. All of the examples above may be verified using this last approach.

4.7. *Asymptotics of Weight Multiplicities*

We shall determine the leading term in the Fourier expansion of $\tau^{l/2} c_\lambda^\Lambda(-1/\tau)$ at $\tau = i\infty$. Using a Tauberian theorem, this will yield an asymptotic formula for the weight multiplicities $\mathrm{mult}_\Lambda(\lambda - n\delta)$ of the $\mathfrak{g}$-module $L(\Lambda)$ as $n \to +\infty$.

We shall employ the important specialization:

$$-2\pi i\tau\rho = (g\tau, \tau\bar{\rho}, 0).$$

PROPOSITION 4.21. *Let $\Lambda \in \mathfrak{h}^*$ satisfy $m := \Lambda(c) > 0$. Then we have, uniformly for* $\operatorname{Im} \tau \to +\infty$:

$$\text{(a)} \quad \Theta_\Lambda(2\pi i\tau^{-1}\rho) \sim |M^*/M|^{-1/2} \left(\frac{-i\tau}{gm}\right)^{l/2};$$

$$\text{(b)} \quad A_\Lambda(2\pi i\tau^{-1}\rho) \sim |M'/M|^{-1/2} \left(\prod_{\alpha \in \bar{\Delta}'_+} 2\sin(\pi(\alpha, \bar{\Lambda})/m)\right) \left(\frac{-i\tau}{m}\right)^{l/2} \times e^{\pi i g^{-1} m^{-1} |\bar{\rho}'|^2 \tau};$$

$$\text{(c)} \quad c^\Lambda_\lambda\left(-\frac{1}{\tau}\right) \sim \operatorname{vol}(\bar{Q}) \left(\frac{g}{m(m+g)}\right)^{l/2} \prod_{\alpha \in \bar{\Delta}'_+} \frac{\sin(\pi(\alpha, \bar{\Lambda} + \bar{\rho})/(m+g))}{\sin(\pi(\alpha, \bar{\rho})/g)} \times (-i\tau)^{-l/2} e^{-\pi i (g^{-1} - (m+g)^{-1}) |\bar{\rho}'|^2 \tau},$$

if $\Lambda \in P_+$ and $\lambda \in \max(\Lambda)$.

Proof. An easy calculation gives

$$\Theta_\Lambda(2\pi i\tau^{-1}\rho) = \Theta^M(\tau', \tau'\mu, \tfrac{1}{2}\tau'(|g^{-1}\bar{\rho}|^2 - |\mu|^2)),$$

where $\tau' = -gm\tau^{-1}$ and $\mu = g^{-1}\bar{\rho} - m^{-1}\bar{\Lambda}$. Proposition 3.4 now gives (a).

To prove (b), put $\tilde{A}_\Lambda = e^{(|\Lambda|^2/2m)\delta} A_\Lambda = \sum_{w \in W} (\det w)\, e^{w(\Lambda)}$, so that we have the identity: $\tilde{A}_\Lambda(a\lambda) = \tilde{A}_\lambda(a\Lambda)$. Applying this with $\lambda = \rho$ and $a = 2\pi i\tau^{-1}$, we obtain

$$A_\Lambda(2\pi i\tau^{-1}\rho) = A_\rho\left(-m\tau^{-1}, -\tau^{-1}\bar{\Lambda}, -\frac{1}{2} m\tau^{-1}\left(\left|\frac{\bar{\rho}}{g}\right|^2 - \left|\frac{\bar{\Lambda}}{m}\right|^2\right)\right).$$

Applying Proposition 4.6(c), we obtain from this:

$$A_\Lambda(2\pi i\tau^{-1}\rho) = |M'/M|^{-1/2} (-i)^{|\bar{\Delta}_+|} \left(\frac{-i\tau}{m}\right)^{l/2} \times A'_{\rho'}\left(m^{-1}\tau, -m^{-1}\bar{\Lambda}, -\frac{1}{2} m\tau^{-1} |g^{-1}\bar{\rho}|^2\right).$$

(b) follows from this and the product expansion of $A'_{\rho'}$.

To prove (c), note that by Proposition 2.12 (or by a little thought), there exists $p \in \mathbb{Z}_+$ such that $\Lambda - p\delta$ lies in the convex hull of $W(\lambda)$. Then for all

$n \in \mathbb{Z}_+$, $\lambda - n\delta$ lies in the convex hull of $W(\Lambda - n\delta)$ and $\Lambda - (n+p)\delta$ lies in the convex hull of $W(\lambda - n\delta)$, so that by Proposition 2.12(c),

$$\operatorname{mult}_\Lambda(\Lambda - n\delta) \leqslant \operatorname{mult}_\Lambda(\lambda - n\delta) \leqslant \operatorname{mult}_\Lambda(\Lambda - (n+p)\delta).$$

This implies, since multiplicities are positive, that

$$c_\lambda^\Lambda(2\pi i\tau^{-1}\rho)/c_\Lambda^\Lambda(2\pi i\tau^{-1}\rho) \sim 1 \tag{4.21.1}$$

for $\tau \in i\mathbb{R}_+$, $\operatorname{Im}\tau \to +\infty$. Since $c_\lambda^\Lambda(\tau)$ and $c_\Lambda^\Lambda(\tau)$ are meromorphic modular forms, we deduce by considering q-expansions that (4.21.1) holds uniformly for $\operatorname{Im}\tau \to +\infty$.

Recall that by (2.18),

$$A_\rho^{-1}A_{\Lambda+\rho} = \sum_{\substack{\mu \in \Lambda + Q + \mathbb{C}\delta \\ \mu \bmod mM + \mathbb{C}\delta}}' c_\mu^\Lambda \Theta_\mu.$$

Evaluate both sides of this equation at $2\pi i\tau^{-1}\rho$ asymptotically for $\operatorname{Im}\tau \to +\infty$, using (a) and (b) and the fact that by (a) and (4.21.1), the $|\bar{Q}/mM|$ summands of the right-hand side are asymptotic to $c_\lambda^\Lambda \Theta_\lambda$. (c) now follows because

$$\operatorname{vol}\bar{Q} = |\bar{Q}/M|^{-1}\,|M^*/M|^{1/2}. \quad \blacksquare$$

Remark. Proposition 4.14 may be proved using Proposition 4.21(c).

Now we need the following special case of a Tauberian theorem of Ingham [43].

PROPOSITION 4.22. *Let $G\colon [0, +\infty) \to \mathbb{R}$ be a non-decreasing function. Suppose that there exist $c > 0$, $d \in \mathbb{R}$ and $N > 0$ such that for $s = \sigma + it$ within each fixed angle $|t| \leqslant r\sigma$, $0 < r < +\infty$, one has*

$$\int_0^\infty e^{-us}\,dG(u) \sim cs^{-d}e^{N/s}$$

uniformly for $s \to 0$. Then for $u \to +\infty$, one has

$$G(u) \sim \tfrac{1}{2}\pi^{-1/2}cN^{-(1/2)(d+1/2)}u^{(1/2)(d-1/2)}e^{2(Nu)^{1/2}}.$$

Now let $\Lambda \in P_+$, $\Lambda(c) = m > 0$, $\lambda \in \max(\Lambda)$. Define a function G: $[0, +\infty) \to \mathbb{R}$ by $G(0) = 0$, $G(u) = \operatorname{mult}_\Lambda(\lambda - [u]\delta)$ for $u > 0$.

Then Proposition 2.11 shows that G is non-decreasing. We have

$$\int_0^\infty e^{-us}\,dG(u) = (1 - e^{-s})\,e^{s_\Lambda(\lambda)s}\,c_\lambda^\Lambda\left(\frac{i}{2\pi}s\right).$$

Propositions 4.21(c) and 4.22 now combine to prove the following asymptotic formula for the weight multiplicities of the $\mathfrak{g}$-module $L(\Lambda)$:

THEOREM B. *Let $\mathfrak{g}$ be an affine Lie algebra. Let $\Lambda \in P_+$, $\Lambda(c) = m > 0$, $\lambda \in \max(\Lambda)$. Set:*

$$a = \frac{1}{2} |\bar{\rho}'|^2 \frac{m}{g(m+g)},$$
$$b = \mathrm{vol}(\bar{Q}) \left(\frac{g}{m(m+g)}\right)^{l/2} \prod_{\alpha \in \bar{\Delta}'_+} \frac{\sin(\pi(\alpha, \bar{\Lambda} + \bar{\rho})/(m+g))}{\sin(\pi(\alpha, \bar{\rho})/g)}. \tag{4.23}$$

Then for $n \to +\infty$, we have

$$\mathrm{mult}_\Lambda(\lambda - n\delta) \sim 2^{-1/2} a^{(1/4)(l+1)} b n^{-(1/4)(l+3)} e^{4\pi(an)^{1/2}}. \tag{4.24}$$

EXAMPLE. Let $\mathfrak{g}$ be of type $A_1^{(1)}$, let $\Lambda = (m-N)\Lambda_0 + N\Lambda_1 \in P_+$ be a weight of positive level m, and let $\lambda = (m-n)\Lambda_0 + n\Lambda_1$. Then $\lambda \in P(\Lambda) + \mathbb{Z}\delta$ if and only if $n \equiv N \bmod 2\mathbb{Z}$.

If $n \equiv N \bmod 2\mathbb{Z}$, then we have, as $j \to +\infty$,

$$\mathrm{mult}_\Lambda(\lambda - j\delta) \sim \frac{\sin(\pi((N+1)/(m+2)))}{2(m+2)j} \exp\left(\pi \left(\frac{2mj}{m+2}\right)^{1/2}\right). \tag{4.25}$$

For $m = 1$, we recover the asymptotics of the classical partition function p:

$$\mathrm{mult}_{\Lambda_0}(\Lambda_0 - j\delta) = p(j) \sim \frac{1}{4\sqrt{3}\, j} e^{\pi\sqrt{(2/3)j}}.$$

Remark. By Proposition 1.11(d), we have the following expression for the constant a defined by (4.23):

$$a = \frac{dm}{24k(m+g)}, \tag{4.26}$$

where d is the dimension of the simple Lie algebra of type X_n, such that $X_n^{(k)}$ is the type of the affine Lie algebra $\mathfrak{g}$. For example, if $X = A$, D or E, then

$$d = (g+1)n,$$

and we obtain in this case:

$$a = \frac{n}{24k} \quad \text{if} \quad m = 1.$$

On the other hand, Proposition 2.27 shows that, if $\Lambda \in \mathfrak{h}^*$ is of level $m \neq -g$ and a is defined by (4.23), then the representation of $\mathfrak{g}'$ on $L(\Lambda)$

gives rise to a projective representation of $\mathrm{Der}(\mathfrak{g}')$ on $L(\Lambda)$ in which the derivations d_n, $n \in \mathbb{Z}$ (cf. Section 2.5), map to operators D_n satisfying

$$[D_n, D_{n'}] = (n' - n)\, D_{n+n'} + \delta_{n',-n} 2kan^3 I_{L(\Lambda)}.$$

(The second term on the right-hand side has been simplified by adding a multiple of $I_{L(\Lambda)}$ to D_0.)

So, the same constant a appears in a completely different situation! We do not know an explanation of this coincidence.

4.8. *Three Remarkable A_λ and Three Remarkable Elements of a Compact Lie Group*

Both Proposition 4.6(c) and the characterization (4.6.1) of $\mathbb{C}A_\rho$ exploited in its proof will be generalized in Proposition 4.30. For this we need Proposition 4.27 below.

Recall from Section 4.1 the subgroup N_0 of $N_{\mathbb{R}}$ normalizing $\tilde{W}$, and put

$$T_0 = \{t_\gamma \mid \gamma \in \bar{Q}'^*\}.$$

Note that $T_0 = \{(0, \gamma, 0) \mid (0, \gamma, 0) \in N_0\}$ contains T and is abelian. Moreover, $twt^{-1}w^{-1} \in T$ for all $t \in T_0$ and $w \in W$, so that each coset of $\overline{Q'}^*$ mod M is $\bar{W}$-invariant. We extend the action af of W on $\bar{\mathfrak{h}}^*$ to the group $T_0 W$ by putting $\mathrm{af}(t_\gamma)\mu = \mu + \gamma$ for $\gamma \in \overline{Q'}^*$ and $\mu \in \bar{\mathfrak{h}}^*$.

Recall the set J of all $j \in I$ such that $j = \sigma(0)$ for some automorphism σ of the Dynkin diagram. We have (cf. [46]):

PROPOSITION 4.27. (a) *The following conditions on* $\gamma \in \bar{\mathfrak{h}}^*$ *are equivalent*:

(1) $t_\gamma \in T_0$.

(2) $A_\rho|_{t_\gamma} \in \mathbb{C}A_\rho$.

(3) $t_\gamma(\Delta) = \Delta$.

(b) *Let* $W_0 = T_0 W$, $W_0^+ = \{w \in W_0 \mid w(\Delta_+) = \Delta_+\}$, $\bar{W}_0 = \bar{W} \cap T_0 W_0^+$. *Then*: $W_0 = W_0^+ \ltimes W = \bar{W} \ltimes T_0$; $W_0^+ = t_{g^{-1}\bar{\rho}} \bar{W}_0 t_{g^{-1}\bar{\rho}}^{-1}$; *the set* $\bar{Q}'^* \cap C_{\mathrm{af}}$ *coincides with* $\{\bar{A}_j \mid j \in J\}$, *and* $\mathrm{af}(W_0^+)$ *acts simply-transitively on it.*

Proof. We first prove (a). (1) implies (2) by Proposition 4.4(a) and (4.6.1). Assume (2). Then Proposition 4.6(a) shows that for any $\alpha \in \Delta^{\mathrm{re}}$, there exist $\beta \in \Delta^{\mathrm{re}}$ and a positive integer m such that for all $h \in \mathfrak{h}$, $\alpha(h) = 2\pi i$ implies $\beta(t_\gamma h) = 2\pi i m$. But then $\beta = m t_\gamma \alpha$, and by a similar argument, $m' t_{-\gamma} \beta \in \Delta^{\mathrm{re}}$ for some positive integer m'. Hence $mm'\alpha \in \Delta^{\mathrm{re}}$. Since $\Delta^{\mathrm{re}} \cap \mathbb{Z}\alpha = \{\alpha, -\alpha\}$ we must have $mm' = 1$ and so $m = 1$. Hence, $t_\gamma \alpha = \beta \in \Delta^{\mathrm{re}}$. This proves that (2) implies (3). Assume (3). Then $(\gamma, \alpha_i)\delta =$

$\alpha_i - t_\gamma(\alpha_i) \in Q$, $0 \leqslant i \leqslant l$, so that $(\gamma, \alpha_i) \in \mathbb{Z}$, $0 \leqslant i \leqslant l$. Thus $\gamma \in \bar{Q}^*$. Since t_γ also preserves $\Delta^\vee$, a similar argument shows that $\gamma \in \bar{Q}^{\vee *}$. Hence, $\gamma \in \bar{Q}^* \cap \bar{Q}^{\vee *} = (\bar{Q} + \bar{Q}^\vee)^* = \bar{Q}'^*$, proving that (3) implies (1). This proves (a).

To prove (b), note that W and T_0 are normal subgroups of W_0. Since $T = W \cap T_0$ and $W = \bar{W} \ltimes T$, we obtain $W_0 = \bar{W} \ltimes T_0$. Moreover, the conjugation action of W_0 is trivial on T_0/T. By (a), W_0 preserves Δ and $W(\rho)$, so that $W_0^+ = \{w \in W_0 \mid w(\rho) = \rho\}$ and $W_0 = WW_0^+$. Since $W \cap W_0^+ = \{1\}$ by Lemma 1.2, we obtain $W_0 = W_0^+ \ltimes W$.

Define maps

$$W_0^+ \xrightarrow{A} C_{\mathrm{af}} \cap \bar{Q}'^* \xrightarrow{B} T_0/T \xrightarrow{C} \bar{W}_0 \xrightarrow{D} W_0^+$$

by

$$A(w_0) = \mathrm{af}(w_0)0, \qquad B(\gamma) = t_{-\gamma}T,$$

$$C(t_\gamma T) = \bar{w} \quad \text{if} \quad \bar{w} \in \bar{W} \cap t_\gamma T W_0^+, \qquad D(\bar{w}) = t_{g^{-1}\bar{\rho}} \bar{w} t_{g^{-1}\bar{\rho}}^{-1}.$$

A maps W_0^+ into $C_{\mathrm{af}} \cap \bar{Q}'^*$ since $\mathrm{af}(W_0^+)$ preserves C_{af} and $\bar{Q}'^*$. B is an injection since C_{af} is a fundamental domain for $\mathrm{af}(W)$. C is well-defined since $\bar{W} \cap TW_0^+ = \{1\}$ and $W_0 = TW_0^+ \bar{W}$. Since $W_0^+ \cap T_0 = \{1\}$ and W_0 acts trivially on T_0/T, C is an injective homomorphism. If $\bar{w} \in \bar{W}_0$, then clearly $D(\bar{w})\rho = \rho$, and moreover, $D(\bar{w}) = t_\gamma \bar{w}$, where $t_\gamma \in T_0$ since

$$g\gamma = \bar{\rho} - \bar{w}(\bar{\rho}) = \overline{\rho - \bar{w}(\rho)} \in \overline{\rho - T_0 W_0^+(\rho)}$$
$$= \overline{\rho - T_0(\rho)} = g\bar{Q}'^*.$$

Hence, $D(\bar{w}) \in W_0^+$. D is obviously an injective homomorphism.

If $w_0 \in W_0^+$, write $w_0 = t_\gamma \bar{w}$, where $t_\gamma \in T_0$ and $\bar{w} \in \bar{W}$. Using $w_0(\rho) = \rho$, one easily computes that $DCBA(w_0) = DCB(\gamma) = DC(t_{-\gamma}T) = D(\bar{w}) = w_0$. Thus $DCBA = I$, so that since B, C, and D are injective, A, B, C, and D are bijective.

Let $\mathscr{T} = \{\bar{\Lambda}_j \mid j \in J\}$. To prove (b), it remains to show that $\mathscr{T} = \bar{Q}'^* \cap C_{\mathrm{af}}$. Since $0 = \bar{\Lambda}_0 \in \mathscr{T} \cap (\bar{Q}'^* \cap C_{\mathrm{af}})$, since $\mathscr{T}$ is $\mathrm{af}(W_0^+)$-stable and since $\mathrm{af}(W_0^+)$ acts transitively on $\bar{Q}'^* \cap C_{\mathrm{af}}$, it suffices to show that $\mathscr{T} \subset \bar{Q}'^* \cap C_{\mathrm{af}}$. If $a_0 = 2$, then $\mathscr{T} = \{0\}$, so we may assume that $a_0 = 1$. If $j \in J$, then $a_j = a_0 = 1$ and $a_j^\vee = a_0^\vee = 1$, so that $\alpha_j = \alpha_j^\vee$ since $a_j\alpha_j = a_j^\vee \alpha_j^\vee$. Hence, if $j \neq 0$, $(\bar{\Lambda}_j, \theta) = (\bar{\Lambda}_j, \sum_{i=1}^l a_i^\vee \alpha_i^\vee) = a_j^\vee = 1$, and $(\bar{\Lambda}_j, \alpha_i) = (\bar{\Lambda}_j, \alpha_i^\vee) = \delta_{ij}$ for $1 \leqslant i \leqslant l$. This shows that $\bar{\Lambda}_j \in \bar{Q}'^* \cap C_{\mathrm{af}}$, so that we have shown $\mathscr{T} \subset \bar{Q}'^* \cap C_{\mathrm{af}}$. This proves (b). ∎

We note a consequence of (a):

$$g\bar{Q}'^* \subset \bar{Q} \qquad \text{and} \qquad h\bar{Q}'^* \subset \bar{Q}^\vee. \tag{4.28}$$

Indeed, $\gamma \in \bar{Q}'^*$ implies $A_\rho|_{t_\gamma} \in \mathbb{C}A_\rho$ by (a), so that $\bar{\rho} + g\gamma = \overline{t_\gamma(\rho)} \in \overline{t_\gamma(W(\rho))} = \overline{W(\rho)} \subset \bar{\rho} + \bar{Q}$. Hence, $g\bar{Q}'^* \subset \bar{Q}$. Similar reasoning applied to $\Delta^\vee$ gives $h\bar{Q}'^* \subset \bar{Q}^\vee$, proving (4.28).

Finally, we note a consequence of (b):

$$|J| = |\bar{P}'/\bar{Q}'| = |\bar{P}/\bar{Q}|. \tag{4.29}$$

Remarks. (1) It follows from Proposition 4.27(b) that: $\bar{W}_0$ acts simply-transitively on $\{\bar{\rho} - g\bar{\Lambda}_j \mid j \in J\}$; each coset of $\overline{Q'}^*$ mod M intersects C_{af} in a unique point; the abelian groups $\overline{Q'}^*/M$, T_0/T, $\bar{W}_0$, W_0^+ and W_0/W are canonically isomorphic.

(2) One can show that $J = \{i \in I \mid \delta - a_i\alpha_i \in \Delta\}$.

(3) One can check case-by-case that $\overline{Q}^\vee + g\bar{Q}^* = \bar{Q} \cap \bar{Q}^*$ if $k' = 1$, and $\bar{Q} + h\overline{Q}^{\vee *} = \overline{Q}^\vee \cap \overline{Q}^{\vee *}$ if $k \neq 1$.

We can now turn to:

PROPOSITION 4.30. (a) *For $\lambda \in P_{++}$, the following conditions on λ are equivalent*:

(1) *There exists a lattice $L \subset \bar{\mathfrak{h}}_{\mathbb{R}}^*$ such that*

$$\{\mu \in \lambda + L + \mathbb{C}\delta \mid W_\mu = \{1\}\} = W(\lambda) + \mathbb{C}\delta.$$

(2) *$\lambda + \mathbb{C}\delta$ contains a positive integral multiple of one of the elements*: ρ; $k\rho^\vee$; $\rho + \Lambda_j$ *for* $k \neq 1$ *and* $j \in J$; $\rho^\vee + \Lambda_j$ *for* $k = 1$ *and* $j \in J$; $2(\rho^\vee - \Lambda_0)$ *for* $a_0 = 2$.

(b) *Let $\lambda \in P_{++}$ have level m and let L' be a lattice satisfying condition* (1) *of* (a). *Then*:

$$\mathbb{C}A_\lambda = \{F \in Th_m^- \mid F|_{(\gamma,0,0)} = e^{2\pi i(\gamma,\bar{\lambda})}F \qquad \text{for all } \gamma \in L'^*\}.$$

Define a lattice L as follows: $L = n\bar{Q}^{\vee *}$ *for* $\lambda \in n\rho + \mathbb{C}\delta$; $L = n\bar{Q}^*$ *for* $\lambda \in n\rho^\vee + \mathbb{C}\delta$; $L = nM$ *for* $\lambda \in n(\rho + \Lambda_j) + \mathbb{C}\delta$ *or* $\lambda \in n(\rho^\vee + \Lambda_j) + \mathbb{C}\delta$; $L = n(\sum_{\alpha \in \bar{\Delta}} \mathbb{Z}\alpha)^*$ *for* $\lambda \in n(\rho^\vee - \Lambda_0) + \mathbb{C}\delta$. *Then λ and L satisfy condition* (1) *of* (a). *Let $\Lambda = \bar{\rho}' + m\Lambda_0$, let $W^\lambda = \bar{W} \ltimes \{t_\gamma \mid \gamma \in L^*\}$, and define a character ε of W^λ by*: $\varepsilon(wt_\gamma) = (\det w)\, e^{-2\pi i(\gamma,\bar{\lambda})}$. *Then the stabilizer of Λ in W^λ is trivial, and*:

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right)$$
$$= |L^*/M|^{-1/2}(-i)^{|\bar{\Delta}_+|}(-i\tau)^{l/2}\, e^{-(1/2m)|\Lambda|^2\delta} \sum_{w \in W^\lambda} \varepsilon(w)\, e^{w(\Lambda)}. \tag{4.30.1}$$

(c) *Let λ, m and L be as in* (b), *and define* $\varepsilon_\lambda : \bar{\lambda} + L \to \{-1, 0, 1\}$ *by*

$$\varepsilon_\lambda(\mu) = \det w \qquad \text{if} \quad \mu = \overline{w(\lambda)}, \ w \in W;$$

$$\varepsilon_\lambda(\mu) = 0 \qquad \text{if} \quad \mu \notin \overline{W(\lambda)}.$$

Then:

$$\varepsilon_\lambda(\mu) = |L/mM|^{-1/2} \prod_{\alpha \in \bar{\Delta}'_+} 2 \sin \frac{\pi(\alpha, \mu)}{m}.$$

In particular,

$$\prod_{\alpha \in \bar{\Delta}'_+} 2 \sin \frac{\pi(\bar{\lambda}, \alpha)}{m} = |L/mM|^{1/2}. \tag{4.30.2}$$

(d)(i) $A_{\rho^\vee} = e^{-(1/2h)|\bar{\rho}^\vee|^2 \delta} e^{\rho^\vee} \prod_{\alpha \in \Delta_+^\vee} (1 - e^{-\alpha})^{\operatorname{mult}^\vee \alpha}$.

(ii) *If* $k = 1$, *then*

$$A_{\rho^\vee + \Lambda_0} = e^{-(1/2(h+1))|\bar{\rho}^\vee|^2 \delta} e^{\rho^\vee} \left(\prod_{\alpha \in \Delta_+^{\mathrm{re}}} (1 - e^{-\alpha^\vee}) \right) \sum_{t \in T} e^{t\Lambda_0}.$$

(iii) *If* $k \neq 1$, *then*

$$A_{\rho + \Lambda_0} = e^{-(1/2(g+1))|\bar{\rho}|^2 \delta} e^{\rho} \left(\prod_{\alpha \in \Delta_+^{\mathrm{re}}} (1 - e^{-\alpha}) \right) \sum_{t \in T} e^{t\Lambda_0}.$$

For the proof we need the following:

LEMMA 4.31. *If* $\lambda + \mathbb{C}\delta = P_{++} \cap (\lambda + Q + \mathbb{C}\delta)$, *then* $\lambda + \mathbb{C}\delta$ *contains one of the elements*: ρ; $\rho + \Lambda_j$ *for* $M = \bar{Q}$ *and* $j \in J$; $k\rho^\vee$ *for* $D_{l+1}^{(2)}$ *and* $B_l^{(1)}$, $l \geqslant 2$; 2ρ *for* $A_1^{(1)}$.

Proof. Clearly, $\lambda \in P_{++}$ but $\lambda \pm \alpha \notin P_{++}$ whenever α is a dominant root for a subdiagram of the Dynkin diagram. A computation using this proves the lemma. ∎

Proof of Proposition 4.30. To prove (a), we first show that (1) implies (2). Assume (1); then we may take the lattice L in (1) to be $\sum_{w \in W} \mathbb{Z}(\overline{\lambda - w\lambda}) \subset \bar{Q}$ and we may assume that λ is not divisible in P_{++}, so that the greatest common divisor of the $n_i := \lambda(h_i)$, $i \in I$, is 1. $n_i \bar{\alpha}_i = \overline{\lambda - r_i(\lambda)} \in L$ for all $i \in I$, so that since L is $\bar{W}$-invariant and $\bar{\theta}$ is a long root of $\bar{\Delta}$, $n_i \bar{\theta} \in L$ for all $i \in I$. Hence, $k\bar{Q}^\vee = \sum_{w \in \bar{W}} \mathbb{Z} w(\bar{\theta}) \subset L$. We now assume $a_0 = 1$; the case $a_0 = 2$ is treated similarly. Since $L = \sum_{\alpha \in \bar{\Delta}} L \cap \mathbb{Z}\alpha$, L is $\bar{W}$-invariant, and $k\bar{Q}^\vee \subset L$, we have $L = \bar{Q}$ or $L = k\bar{Q}^\vee$. If $L = \bar{Q}$, then (2) holds by Lemma 4.31. If $L = k\bar{Q}^\vee$, then for all $i \in I$, $\bar{\alpha}_i^\vee$ is an indivisible element of $\bar{Q}^\vee$ and $(\lambda, \alpha_i) \bar{\alpha}_i^\vee = \overline{\lambda - r_i(\lambda)} \in L = k\bar{Q}^\vee$, so that $(\lambda, \alpha_i) \in k\mathbb{Z}$ for all $i \in I$. Hence in this case $\lambda \in kP_{++}^\vee$, and (2) follows from Lemma 4.31 applied to $\Delta^\vee$. Thus (1) implies (2).

We next verify (1) for the λ listed in (2) and the corresponding lattices L listed in (b), so that (2) implies (1), and moreover (1) holds for the λ and L in (b). It is easy to check that $\lambda + L + \mathbb{C}\delta$ is a W-invariant subset of $\{\mu \in P \mid \mu(c) = \lambda(c)\}$. Moreover P_{++} is a fundamental domain for W on $\{\mu \in P \mid \mu(c) > 0,\ W_\mu = \{1\}\}$. Hence, it suffices to check that $(\lambda + L + \mathbb{C}\delta) \cap P_{++} = \lambda + \mathbb{C}\delta$. For $\lambda = \rho$, $\{\mu \in P_{++} \mid \mu(c) = g\} = \rho + \mathbb{C}\delta$ verifies (1). If $M = \bar{Q}$ and $j \in J$, then $\{\mu \in P_{++} \mid \mu(c) = g + 1,\ \bar{\mu} \in \bar{\rho} + \bar{\Lambda}_j + M\} = \rho + \Lambda_j + \mathbb{C}\delta$ by Proposition 4.27(b), verifying (1). The same argument applied to $\Delta^\vee$ verifies (1) for $\lambda = k\rho^\vee$ and $\lambda = \rho^\vee + \Lambda_j$. The case $\lambda = 2(\rho^\vee - \Lambda_0)$ is left to the reader.

The characterization of $\mathbb{C}A_\lambda$ in (b) is clear.

We next check that for λ, L, W^λ, and Λ as in (b), the stabilizer of Λ in W^λ is trivial. This is equivalent to the assertion that $w \in \bar{W}$ and $\overline{\rho'} - w(\overline{\rho'}) \in \lambda(c)L^*$ imply $w = I$, which we proceed to check. If $\lambda = n\rho$, then $\lambda(c)L^* = gM'$, so that the assertion amounts to the fact that the stabilizer of $\rho' \in P'_{++}$ in W' is trivial. If $\lambda = nk\rho^\vee$, similar reasoning applied to $\Delta^\vee$ verifies the assertion. Suppose $k \neq 1$, $j \in J$, and $\lambda = n(\rho + \Lambda_j)$, so that $\lambda(c)L^* = (g+1)M^*$. If $\overline{\rho'} - w(\overline{\rho'}) \in (g+1)M^*$, then since $\overline{\rho'} - w(\overline{\rho'}) \in \overline{Q'} = \overline{Q^\vee} = M'$, and since $(g+1)M^* \cap M' = (g+1)M'$ by (4.28), we obtain $\overline{\Lambda_0 + \rho'} - w(\overline{\Lambda_0 + \rho'}) = \overline{\rho'} - w(\overline{\rho'}) \in (g+1)M'$; since the stabilizer of $\Lambda_0 + \rho' \in P'_{++}$ in W' is trivial, this forces $w = I$, verifying the assertion. If $k = 1$, $j \in J$ and $\lambda = n(\rho^\vee + \Lambda_j)$, similar reasoning applied to $\Delta^\vee$ verifies the assertion. The remaining case is left to the reader.

Put $A' = A_\lambda|_S$, and let ε be as in (b). Then the characterization of $\mathbb{C}A_\lambda$ in (b), together with $M' \subset L^*$, implies that

$$\mathbb{C}A' = \{F \in Th_m \mid F|_w = \varepsilon(w)F \text{ for all } w \in W^\lambda\}.$$

In particular, ε is a character of W^λ. Put

$$A'' = e^{-(1/2m)|\Lambda|^2\delta} \sum_{w \in W^\lambda} \varepsilon(w)\, e^{w(\Lambda)}.$$

Since $\Lambda \in P'$ and $mL^* \subset M^*$, we have $A'' \in Th_m$, and it is easy to check that $A''|_w = \varepsilon(w)A''$ for all $w \in W^\lambda$. Hence $A'' = CA'$ for some $C \in \mathbb{C}$. To find C we proceed as in the proof of Proposition 4.6(c), using the fact that the stabilizer of Λ in W^λ is trivial. We obtain $|C| = |L^*/M|^{1/2}$ and $C^{-1} = (-i)^{D/2}\ |M^*/mM|^{-1/2} \prod_{\alpha \in \bar{\Delta}'_+} 2\sin(\pi(\bar{\lambda}, \alpha)/m)$, so that by a comparison, (4.30.1) holds, proving (b), and (4.30.2) holds. It is easy to check that the function

$$\nu \mapsto \prod_{\alpha \in \bar{\Delta}'_+} (2 \sin \pi\nu(c)^{-1}(\alpha, \bar{\nu}))$$

is W-anti-invariant and vanishes precisely on the "walls" $(\nu, \alpha) = 0$, $\alpha \in \Delta$. Combining this with (4.30.2) and (1) for λ and L proves (c).

Finally, we prove (d). Part (d)(i) is just Proposition 4.6(a) applied to $\Delta^\vee$. Part (d)(iii) is immediate from Proposition 4.6(a) and Example 1 of Section 4.6. (Note that d(iii) holds whenever $M = \bar{Q}$.) Part (d)(ii) is (d)(iii) applied to $\Delta^\vee$. ▮

Remark. There are similar product expansions of A_λ for all λ from Proposition 4.30(a2).

Now we deduce a nice application to compact groups.

PROPOSITION 4.32. *Let G be a connected simply-connected compact Lie group with simple Lie algebra $\mathfrak{g}$, and let H be a Cartan subgroup of G with Lie algebra $\mathfrak{h}$. We identify $i\mathfrak{h}$ and $i\mathfrak{h}^*$ using the Killing form ϕ of $\mathfrak{g}$. Let W be the Weyl group, $P \subset i\mathfrak{h}^*$ the weight lattice, $\Delta \subset i\mathfrak{h}^*$ the root system, Δ_+ a set of positive roots, θ the highest root, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$, $\rho^\vee = \sum_{\alpha\in\Delta_+}(\alpha/\phi(\alpha,\alpha))$, $h = 1 + \phi(\theta,\rho^\vee)$ the Coxeter number of $\mathfrak{g}$, $g = \phi(\theta,\theta)^{-1}$.*

Consider the following subgroups of H:

$$\Gamma_\rho = \{\exp 4\pi i\lambda \mid \lambda \in P\}, \qquad \Gamma_h = \{a \in H \mid (\mathrm{Ad}\, a)^h = 1\},$$
$$\Gamma_{h+1} = \{a \in H \mid a^{h+1} = 1\}.$$

Define $\gamma_\rho \in \Gamma_\rho$, $\gamma_h \in \Gamma_h$ and $\gamma_{h+1} \in \Gamma_{h+1}$ by

$$\gamma_\rho = \exp 4\pi i\rho, \qquad \gamma_h = \exp 2\pi i h^{-1}\rho^\vee,$$
$$\gamma_{h+1} = \exp 2\pi i(h+1)^{-1}\, h\rho^\vee.$$

These elements of H are regular (i.e., have centralizer H). Moreover, we have:

(a) *Each regular element of Γ_ρ (resp. Γ_h or Γ_{h+1}) is W-conjugate to γ_ρ (resp. γ_h or γ_{h+1}).*

(b) *Let a be a regular element of H, Γ the subgroup of H generated by $W(a)$. Suppose that $W(a)$ is the set of regular elements of Γ. Then a is W-conjugate to one of γ_ρ, γ_h, γ_{h+1}.*

(c) *If $\Gamma = \Gamma_\rho$, Γ_h or Γ_{h+1}, and if a is a regular element of Γ, then:* $\det(I - \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(a)) = |\Gamma|$.

(d) *For any irreducible representation π of G over $\mathbb{C}$, $\mathrm{tr}\,\pi(\gamma_\rho)$, $\mathrm{tr}\,\pi(\gamma_h)$, and $\mathrm{tr}\,\pi(\gamma_{h+1})$ are 0, 1, or -1.*

(e) *The center of G is the set of all elements of G with maximal distance from the conjugacy class of γ_ρ in G in the invariant metric induced by $-\phi$.*

(f) *Let* vol G *be the volume of* G *under the invariant measure induced by* $-\phi$. *Then*:

$$(\mathrm{vol}\, G)^2 = (8\pi^2)^{\dim G} J(4\pi i\rho), \tag{4.32.1}$$

where J *is the Jacobian*[4] *of* $\exp\colon \mathfrak{g} \to G$.

Sketch of proof. Let X be the type of $\mathfrak{g}$, and let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $X^{(1)}$ (cf. Section 1.7). Let $Q^\vee \subset i\mathfrak{h}$ be the coroot lattice, so that $2\pi i Q^\vee = \mathrm{Ker}(\exp|_{\mathfrak{h}})$, and let $Q^* = \{x \in i\mathfrak{h} \mid \alpha(x) \in \mathbb{Z}$ for all $\alpha \in \Delta\}$. By Propositions 1.10 and 4.27(b) applied to $\hat{\mathfrak{g}}$, the cosets in $P/\frac{1}{2}Q^\vee$ (resp. $Q^*/hQ^\vee$, $Q^*/(h+1)Q^\vee$) with trivial stabilizer in W are those intersecting $W(\rho)$ (resp. $W(\rho^\vee)$, $W(\rho^\vee) + (h+1)Q^*$). Putting $\Gamma'_{h+1} = \{a \in H \mid \mathrm{Ad}(a)^{h+1} = 1\}$, it follows that the regular elements of Γ'_{h+1} are just the elements of $\mathrm{Center}(G)W(\exp 2\pi i(h+1)^{-1}\rho^\vee)$, and that the regular elements of Γ_ρ and Γ_h are just the elements of $W(\gamma_\rho)$ and $W(\gamma_h)$, respectively. Since $a \mapsto (a^{h+1}, a^{-h})$ defines the decomposition $\Gamma'_{h+1} = \mathrm{Center}(G) \times \Gamma_{h+1}$, (a) is now clear.

Let a and Γ be as in (b), and put $L = (1/(2\pi i))(\exp|_{\mathfrak{h}})^{-1}(\Gamma) \subset i\mathfrak{h}$. Let $a = \exp 2\pi i x$, where $x \in L$, so that the stabilizer in W of $x + Q^\vee$ in $L/Q^\vee$ is trivial. Then $x + \Lambda_0$ satisfies condition (1) of Proposition 4.30(a) for $\hat{\mathfrak{g}}$, so that a is W-conjugate to γ_ρ, γ_h or an element γ of $\mathrm{Center}(G)\gamma_{h+1}$. In the latter case, we have $\gamma_{h+1} = \gamma^{-h} \in \Gamma$. This proves (b).

For (c) we may assume by (a) that a is one of the elements γ_ρ, γ_h, or γ_{h+1}. Then (*c*) follows from (4.30.2). Part (d) follows from a comparison of the formulas in Propositions 4.5(d) and 4.30(b). Part (e) is immediate from Proposition 4.13.

To prove (f), let μ_G and μ_H be the invariant measures on G and H with total measure 1, inducing Euclidean measures $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{h}}$ on $\mathfrak{g}$ and $\mathfrak{h}$; we denote each of these measures by μ. Denote by μ' the invariant measures on G, H, $\mathfrak{g}$, and $\mathfrak{h}$ induced by $-\phi$. Then (c) implies

$$\mu'(H) = \mu'(\mathfrak{h}/2\pi i Q^\vee)$$
$$= (8\pi^2)^{(1/2)\dim H}|\Gamma_\rho|^{1/2} = (8\pi^2)^{(1/2)\dim H} \prod_{\alpha \in \Delta_+} 2 \sin 2\pi\phi(\alpha, \rho). \tag{4.32.2}$$

Let F be a continuous real-valued class function on G such that $F(\exp x) = e^{\pi\phi(x,x)}$ for $x \in \mathfrak{g}$ near 0, and $|F(g)| < 1$ for $g \in G$, $g \neq 1$.

Then we have asymptotically as $n \to +\infty$:

$$\int_G F^n \, d\mu' \sim \int_{\mathfrak{g}} e^{n\pi\phi(x,x)} \, d\mu'(x) = n^{-(1/2)\dim G}. \tag{4.32.3}$$

[4] $J(x) = \det((1 - e^{-\mathrm{ad}\, x})/\mathrm{ad}\, x)$, so that $J(4\pi i\rho) = \prod_{\alpha \in \Delta} (\sin 2\pi\phi(\rho, \alpha))/2\pi\phi(\rho, \alpha)$.

Moreover:

$$\int_G F^n\,d\mu = |W|^{-1}\int_H F^n \prod_{\alpha\in\Delta}(1-e^{\alpha})\,d\mu$$

$$\sim |W|^{-1}\int_{\mathfrak{h}} e^{n\pi\phi(h,h)} \prod_{\alpha\in\Delta}(1-e^{\alpha(h)})\,d\mu(h)$$

$$= \sum_{w\in W}(\det w)\int_{\mathfrak{h}} e^{n\pi\phi(h,h)}e^{(\rho-w\rho)(h)}\,d\mu(h)$$

$$= \sum_{w\in W}(\det w)\,e^{-(4\pi n)^{-1}\phi(\rho-w\rho,\rho-w\rho)}\int_{\mathfrak{h}} e^{n\pi\phi(h,h)}\,d\mu(h)$$

$$= \int_{\mathfrak{h}} e^{n\pi\phi(h,h)}\,d\mu(h)\prod_{\alpha\in\Delta_+}(1-e^{-(2\pi n)^{-1}\phi(\alpha,\rho)})$$

$$\sim n^{-(1/2)\dim G}\mu'(\mathfrak{h}/2\pi i Q^\vee)^{-1}\prod_{\alpha\in\Delta_+}\frac{\phi(\alpha,\rho)}{2\pi},$$

so that by (4.32.3) we have

$$\mu'(G)=\mu'(H)\prod_{\alpha\in\Delta_+}\frac{2\pi}{\phi(\alpha,\rho)}. \tag{4.32.4}$$

Combining (4.32.2) and (4.32.4), we obtain (4.32.1). ∎

Remarks. (a) Propositions 4.32(a), (c), (d) for Γ_ρ and Γ_h were originally proved in [25], and 4.32(d) for γ_{h+1} in [21] (note that σ_M in [21] should be replaced by its hth power). Proposition 4.32(f) is given in another form in [42]. Formula (4.32.4) is given in another form in [45].

(b) One can show that the lattices L of Proposition 4.30(b) are characterized by: (i) L is a lattice in $\bar{\mathfrak{h}}^*_{\mathbb{R}}$; (ii) $\lambda+L+\mathbb{C}\delta$ is W-stable; (iii) $\{\mu\in\lambda+L+\mathbb{C}\delta \mid W_\mu=\{1\}\}=W(\lambda)+\mathbb{C}\delta$; (iv) L is maximal with respect to (i), (ii) and (iii). Similarly, the groups Γ of Proposition 4.32(c) are characterized by: (i) Γ is a subgroup of H; (ii) Γ is W-stable; (iii) The regular elements of Γ form a single W-orbit; (iv) Γ is maximal with respect to (i), (ii) and (iii). This allows one to state Propositions 4.30(b, c) and 4.32(c, d) without reference to cases. It would be interesting to prove them without reference to cases.

4.9. *Restriction of a Highest Weight Module to a Subalgebra*

In this section, we describe the behavior of highest weight modules under restriction and deduce that certain "generalized string functions" are modular forms.

Recall the definition in Section 1.1 of Kač–Moody algebras. We modify it slightly as follows. Given a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, we require of the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ only that $\mathfrak{h}$ is finite-dimensional, that $\Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$ is linearly independent, that $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, and that $\alpha_j(h_i) = a_{ij}$ $(i, j \in I)$.

The Lie algebras thus defined are called *generalized Kač–Moody algebras*, and also *generalized affine Lie algebras* if the Cartan matrix A is a direct sum of affine Cartan matrices. This definition is convenient for certain applications, where a homomorphism of semisimple Lie algebras does not carry over to a homomorphism of the corresponding direct sums of affine Lie algebras.

Most notions from Sections 1.1 and 2.1 carry over to this context without difficulty. Moreover, we still have the character formula, the complete reducibility and the separation of $\tilde{W}$-orbits by characters (Propositions 2.9 and 2.10). An important difference is that Y may be empty. We define a *standard form* to be an invariant symmetric bilinear form $(\, , \,)$ on $\mathfrak{g}$ such that: (h_i, h_i) is positive rational for all $h_i \in \Pi^\vee$; $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha, \beta \in Q$ (= free abelian group on Π) and $\alpha + \beta \neq 0$ (this condition is redundant if $Y \neq \varnothing$).

Let $\mathfrak{g}$ and $\mathfrak{g}^\circ$ be generalized Kač–Moody algebras. We have notions of Cartan subalgebra $\mathfrak{h}$, set of simple roots Π, "complexified" Tits cone $Y \subset \mathfrak{h}$, imaginary cone $Z \subset \mathfrak{h}^*$, the domain $Y_\rho \subset Y$ where all $\operatorname{ch}_{L(\Lambda)}$ $(\Lambda \in P_+)$ converge, etc.; we have similar notions $\mathfrak{h}^\circ$, Π°, Y°, Z°, $Y^\circ_{\rho^\circ}$, etc., for $\mathfrak{g}^\circ$.

Let $\pi\colon \mathfrak{g}^\circ \to \mathfrak{g}$ be a homomorphism such that

$$\pi(\mathfrak{h}^\circ) \subset \mathfrak{h}; \qquad \pi(Y^\circ) \cap Y \neq \varnothing; \qquad \pm\Pi^\circ \cap \pi^*(Z) = \varnothing. \tag{4.33}$$

Here and further on, π^* denotes the pullback map from functions on $\mathfrak{h}$ to functions on $\mathfrak{h}^\circ$.

We first show that for $\Lambda \in P_+$, $L(\Lambda)$ is isomorphic as a $\mathfrak{g}^\circ$-module to a direct sum of modules $L^\circ(\mu)$, $\mu \in P^\circ_+$. $L(\Lambda)$ is $\mathfrak{h}^\circ$-diagonalizable since $\pi(\mathfrak{h}^\circ) \subset \mathfrak{h}$, so it suffices to verify conditions (i) and (ii) of Proposition 2.9 for the $\mathfrak{g}^\circ$-module $L(\Lambda)$. Suppose that f°_i is not locally nilpotent on $L(\Lambda)$. Choose $\lambda \in P(\Lambda)$ and $v \in L(\Lambda)_\lambda$ such that $\pi(f^\circ_i)^n v \neq 0$ for all $n \in \mathbb{Z}_+$. For $n \in \mathbb{Z}_+$, choose $\lambda_n \in P(\Lambda)$ such that the $L(\Lambda)_{\lambda_n}$-component of $\pi(f^\circ_i)^n v$ is non-zero, so that $\pi^*(\lambda_n) = \pi^*(\lambda) - n\alpha^\circ_i$. Since the sequence $n^{-1}\lambda_n$, $n \geqslant 1$, is bounded, it has a limit point β in $\mathfrak{h}^*$. Then $\pi^*(\beta) = -\alpha^\circ_i$, but $\beta \in -Z$ by Proposition 2.4(e). This contradicts $\Pi^\circ \cap \pi^*(Z) = \varnothing$. Hence, f°_i is locally nilpotent on $L(\Lambda)$, verifying (ii). Similarly, e°_i is locally nilpotent on $L(\Lambda)$.

To verify (i), choose $d \in Y^\circ$ such that $\pi(d) \in Y$ and $\alpha(d) \notin i\mathbb{R}$ for all $\alpha \in \Delta^\circ$, so that $d \in w(C^\circ) + i\mathfrak{h}^\circ_{\mathbb{R}}$ for some $w \in W^\circ$. Put $\mathfrak{m} = \bigoplus_{\alpha \in w(\Delta^\circ_+)} \mathfrak{g}^\circ_\alpha$, $\varepsilon = \min\{\operatorname{Re} w(\alpha^\circ_i)(d) \mid i \in I^\circ\} > 0$. Since $\pi(d) \in Y$, $\{\operatorname{Re} \lambda(\pi(d)) \mid \lambda \in P(\Lambda)\}$ is bounded above, say by B. If $v \in L(\Lambda)_\lambda$ and $N\varepsilon + \lambda(\pi(d)) > B$, then

$\pi(\mathfrak{m})^N v = (0)$, so that $\mathfrak{m}$ acts locally nilpotently on $L(\Lambda)$. Using the operators $(\exp e_i^\circ)(\exp -f_i^\circ)(\exp e_i^\circ)$ on $L(\Lambda)$ (cf. Proposition 2.4(b)), we deduce that $\mathfrak{n}_+^\circ$ acts locally nilpotently on $L(\Lambda)$. This verifies (i).

An argument similar to the one above verifying (ii) shows that the $\operatorname{ad}\pi(e_i^\circ)$ and $\operatorname{ad}\pi(f_i^\circ)$ are locally nilpotent.

Since $\operatorname{ch}_{L(\Lambda)}$ converges on $\pi(Y^\circ) \cap Y_\rho \neq \varnothing$, the weight spaces of $L(\Lambda)$ as an $\mathfrak{h}^\circ$-module are finite-dimensional. In particular, for $\Lambda \in P_+$ and $\mu \in P_+^\circ$, the multiplicity $\operatorname{mult}(\Lambda:\mu)$ of $L^\circ(\mu)$ in $L(\Lambda)$ is finite. For $\mu = w(\mu' + \rho^\circ) - \rho^\circ$, where $w \in W^\circ$ and $\mu' \in P_+^\circ$, set $\operatorname{mult}(\Lambda:\mu) = (\det w)\operatorname{mult}(\Lambda:\mu')$; for $\mu \in \mathfrak{h}^{\circ *}$, set $\operatorname{mult}(\Lambda:\mu) = 0$ if it is not already defined.

Put $\Delta_0 = \{\alpha \in \Delta \mid \pi^*(\alpha) = 0\}$. Then Δ_0 is a (finite) root system since $\pi(\mathfrak{h}^\circ) \cap Y \neq \varnothing$, and $\Delta_{0+} = \Delta_0 \cap \Delta_+^{\mathrm{re}}$ is a set of positive roots. Let $W_0 \subset W$ be the corresponding Weyl group, $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_{0+}} \alpha$, etc. Define a polynomial D_0 on $\mathfrak{h}^*$ by

$$D_0(\lambda) = \prod_{\alpha\in\Delta_{0+}} \lambda(\alpha^\vee)/\rho_0(\alpha^\vee).$$

PROPOSITION 4.34. *With the assumptions and notations above, we have*:

(a) $\pi(Q^{\circ\vee}) \subset Q^\vee$; $\pi(Z^{\circ\vee}) \subset Z^\vee$.

(b) *For every* $w^\circ \in W^\circ$, *there exists* $w \in W$ *such that* $\pi \circ w^\circ = w \circ \pi$ *on* $\mathfrak{h}^\circ$; *if* w° *is of order* 2, *we may take* w *to be of order dividing* 2.

(c) *If the kernel of* π *is contained in the center of* $\mathfrak{g}^\circ$, *then the pullback to* $\mathfrak{g}^\circ$ *of any standard form on* $\mathfrak{g}$ *is a standard form on* $\mathfrak{g}^\circ$.

(d) *For all* $\Lambda \in P_+$, *we have on* $Y_{\rho^\circ}^\circ \cap \pi^{-1}(Y_\rho) \neq \varnothing$:

(i) $$\sum_{\mu\in\mathfrak{h}^{\circ *}} \operatorname{mult}(\Lambda:\mu)\, e^\mu = \pi^*(\operatorname{ch}_{L(\Lambda)}) \prod_{\alpha\in\Delta_+^\circ} (1-e^{-\alpha})^{\operatorname{mult}^\circ \alpha}.$$

(ii) $$\sum_{w\in W_0\backslash W} (\det w)\, D_0(w(\Lambda+\rho))\, e^{\pi^*(w(\Lambda+\rho))} = \pi^*\left(\prod_{\alpha\in\Delta_{0+}} (1-e^{-\alpha})^{-1} \sum_{w\in W} (\det w)\, e^{w(\Lambda+\rho)}\right).$$

(iii) $$\pi^*\left(\prod_{\alpha\in\Delta_+\backslash\Delta_{0+}} (1-e^{-\alpha})^{\operatorname{mult}\alpha}\right) \sum_{\mu\in\mathfrak{h}^{\circ *}} \operatorname{mult}(\Lambda:\mu)\, e^\mu = \left(\sum_{w\in W_0\backslash W} (\det w)\, D_0(w(\Lambda+\rho))\, e^{\pi^*(w(\Lambda+\rho)-\rho)}\right) \times \prod_{\alpha\in\Delta_+^\circ} (1-e^{-\alpha})^{\operatorname{mult}^\circ \alpha}.$$

Proof. To prove (a), note that $\pi^*(W(P_+)) \subset W^\circ(P_+^\circ)$ since, for all $\Lambda \in P_+$, $L(\Lambda)$ as a $\mathfrak{g}^\circ$-module decomposes into a direct sum of $L^\circ(\mu)$ with $\mu \in P_+^\circ$, and moreover,

$$Z^{\vee}(\text{resp. } Q^{\vee}) = \{h \in \mathfrak{h} \mid \lambda(h) \geqslant 0 \text{ (resp. } \in \mathbb{Z}) \text{ for all } \lambda \in W(P_+)\},$$
$$Z^{\circ\vee}(\text{resp. } Q^{\circ\vee}) = \{h \in \mathfrak{h}^{\circ} \mid \lambda(h) \geqslant 0 \text{ (resp. } \in \mathbb{Z}) \text{ for all } \lambda \in W^{\circ}(P^{\circ}_+)\}.$$

To prove (b) and (c), we need the following lemma. We omit its standard proof.

LEMMA 4.35. *Let W be the Weyl group of a Kac–Moody algebra $\mathfrak{g}$. Let $w \in W$ be of order 2. Then there exist real roots $\beta_1,\dots,\beta_s$, satisfying $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ for $i \neq j$ such that $w = r_{\beta_1} \cdots r_{\beta_s}$. In particular, every standard form on $\mathfrak{g}$ is positive-definite on $\{h \in \mathfrak{h}_{\mathbb{R}} \mid w(h) = -h\} = \mathbb{R}\beta_1^{\vee} + \cdots + \mathbb{R}\beta_s^{\vee}$.*

We prove (b) only for a fundamental reflection $r_i^{\circ} \in W^{\circ}$; the proof in the general case is similar, using Lemma 4.35 applied to $\mathfrak{g}^{\circ}$. Recall from Section 1.1(E) the associated $\tilde{r}_i^{\circ} \in \operatorname{Aut}(\mathfrak{g}^{\circ})$, and let

$$\hat{r}_i^{\circ} = (\exp \operatorname{ad} \pi(e_i^{\circ}))(\exp \operatorname{ad} \pi(-f_i^{\circ}))(\exp \operatorname{ad} \pi(e_i^{\circ})) \in \operatorname{Aut}(\mathfrak{g}).$$

Then $\hat{r}_i^{\circ} \circ \pi = \pi \circ \tilde{r}_i^{\circ}$. Moreover, since $\hat{r}_i^{\circ}$ preserves the reductive Lie algebra $\mathfrak{g}_0 := \mathfrak{h} + \sum_{\alpha \in \Delta_0} \mathfrak{g}_{\alpha}$ and $\hat{r}_i^{\circ 2}$ is the identity on $\mathfrak{g}_0$, there exists g in the adjoint group of $\mathfrak{g}_0$ such that $g\hat{r}_i^{\circ}$ preserves $\mathfrak{h}$ and $w := (g\hat{r}_i^{\circ})|_{\mathfrak{h}}$ has order dividing 2. Then $w \circ \pi = \hat{r}_i^{\circ} \circ \pi$ on $\mathfrak{h}^{\circ}$, and using Proposition 2.10 we obtain $w \in W$. This proves (b).

To prove (c), let (,) be a standard form on $\mathfrak{g}$. If $h_i^{\circ} \in \Pi^{\circ\vee}$, then $\pi(h_i^{\circ}) \in Q^{\vee}$ by (a) and $w(\pi(h_i^{\circ})) = -\pi(h_i^{\circ})$ for some $w \in W$ of order 2 by (b). By Lemma 4.35, (,) is positive-definite on $\mathbb{R}\pi(h_i^{\circ})$, and $(\pi(h_i^{\circ}), \pi(h_i^{\circ}))$ is rational since $\pi(h_i^{\circ}) \in Q^{\vee}$. Since h_i° does not lie in the center of $\mathfrak{g}^{\circ}$, $\pi(h_i^{\circ}) \neq 0$ by the hypothesis of (c). Hence, $(\pi(h_i^{\circ}), \pi(h_i^{\circ}))$ is a positive rational number, proving (c).

Finally, we prove (d). Part (i) follows from the character formula for $\mathfrak{g}^{\circ}$. Part (iii) follows from (i), (ii), and the character formula for $\mathfrak{g}$. By the Weyl dimension formula, we have

$$\pi^* \left(\prod_{\alpha \in \Delta_{0+}} (1 - e^{-\alpha})^{-1} \sum_{w \in W_0} (\det w)\, e^{w(\lambda)} \right) = D_0(\lambda)\, e^{\pi^*(\lambda)}$$

for any $\lambda \in P$; (ii) follows from this. ∎

Remark. Let $R = \pi^*(\Delta_+) \setminus \{0\} \subset \mathfrak{h}^{\circ *}$, and suppose that R lies in an open half-space of $\mathfrak{h}^{\circ *}$, that the kernel of π lies in the center of $\mathfrak{g}^{\circ}$, and that $\pi(\mathfrak{n}^{\circ}_+) \subset \mathfrak{n}_+$. Let K_R be the partition function for R, where $\alpha \in R$ is counted with multiplicity

$$-\operatorname{mult}^{\circ}(\alpha) + \sum_{\substack{\beta \in \Delta_+ \\ \pi^*(\beta) = \alpha}} \operatorname{mult}(\beta).$$

Then an equivalent form of (d)(iii) is

$$\operatorname{mult}(\Lambda:\mu)=\sum_{w\in W_0\backslash W}(\det w)\,D_0(w(\Lambda+\rho))$$
$$\times K_R(\pi^*(w(\Lambda+\rho))-(\mu+\pi^*(\rho))),$$

valid for all $\Lambda\in P_+$ and $\mu\in\mathfrak{h}^{\circ *}$. This formula was obtained by G. Heckman in the finite-dimensional case, by a different method.

Now we turn to the case of a generalized affine Lie algebra $\mathfrak{g}$. Given $h\in\mathfrak{h}$ and $\lambda\in\mathfrak{h}^*$, we define the "symbol" $\lambda[h]\in\mathbb{C}$, which is a rational function of $\lambda(h_i)$ and $\alpha_i(h)$, $i\in I$, as follows. We extend the function $f\colon T\to\mathbb{C}$, defined by $f(t)=\lambda(h-t(h))$, in the obvious way to the Zariski closure $\bar{T}\simeq T\otimes_{\mathbb{Z}}\mathbb{C}$ of the translation group T in $\operatorname{End}_{\mathbb{C}}(\mathfrak{h})$. One can show that if $\lambda(c_i)\neq 0$ for all canonical central elements c_i and $\delta(h)\neq 0$ for all imaginary roots δ of $\mathfrak{g}$, then $f(t)$ has a unique critical point, say t_0. We let $\lambda[h]$ be the stationary value $f(t_0)$.

For example, if $\mathfrak{g}$ is an affine Lie algebra, then $f(t_\gamma)=\lambda(h-t_\gamma(h))$, where t_γ is defined by (1.6) for all $\gamma\in\bar{\mathfrak{h}}^*$, and we have, provided that $\lambda(c)\neq 0$ and $\delta(h)\neq 0$,

$$\lambda[h]=\lambda(h)-\frac{|\lambda|^2}{2(\lambda,\delta)}\,\delta(h)-\frac{|h|^2}{2(h,c)}\,\lambda(c),$$

a formula which is independent of the choice of the standard form (,).

We note the identities

$$w(\lambda)[w(h)]=\lambda[h]\qquad\text{and}\qquad \lambda(h)-\lambda[h]=\lambda(w(h))-\lambda[w(h)],$$

valid for all $w\in W$.

PROPOSITION 4.36. *Let $\mathfrak{g}^\circ$ and $\mathfrak{g}$ be generalized affine Lie algebras, and let $\pi\colon\mathfrak{g}^\circ\to\mathfrak{g}$ be a homomorphism such that $\pi(\mathfrak{h}^\circ)\subset\mathfrak{h}$. Suppose that $d\in Y^\circ$ is such that $\pi(d)\in Y$ and $\alpha(\pi(d))\in\mathbb{Q}$ for all $\alpha\in\Delta$. For $\Lambda\in P_+$ and $\mu\in P^\circ_+$, put*

$$e^\Lambda_\mu(\tau)=q^A\sum_{\substack{\mu'\in\mathfrak{h}^{\circ *}\\ \mu'(h_i^\circ)=\mu(h_i^\circ),\, i\in I^\circ}}\operatorname{mult}(\Lambda:\mu')\,q^{-\mu'(d)},$$

where

$$A=\Lambda(\pi(d))-(\Lambda+\rho)[\pi(d)]+\rho[\pi(d)]+(\mu+\rho^\circ)[d]-\rho^\circ[d].$$

Then:

(a) *If $\Lambda\in P_+$ and $\mu\in P^\circ_+$, then there exists a positive integer N such that $e^\Lambda_\mu(\tau)$ is a modular form of weight 0 and trivial multiplier system for $\Gamma(N)$.*

(b) *There exists a modular form $H(\tau)$ such that $H(\tau)\, e_{\mu}^{\Lambda}(\tau)$ is a cusp form for all $\Lambda \in P_+$ and $\mu \in P_+^{\circ}$.*

Proof. We first check that $\pm\Pi^{\circ} \cap \pi^*(Z) = \varnothing$, verifying (4.33), so that Proposition 4.34 applies. To see this, suppose that $\alpha_i^{\circ} \in \Pi^{\circ}$, $\delta \in Z$, $r \in \mathbb{C}$, and $r\alpha_i^{\circ} = \pi^*(\delta)$. Then $(\pi^*(\delta))(h_i^{\circ}) = \delta(\pi(h_i^{\circ})) = 0$ since $\pi(h_i^{\circ}) \in \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, so that $2r = r\alpha_i^{\circ}(h_i^{\circ}) = (\pi^*(\delta))(h_i^{\circ}) = 0$. Hence $r = 0$, showing that $\pm\Pi^{\circ} \cap \pi^*(Z) = \varnothing$.

Let $c_1, \ldots, c_n \in \mathfrak{h}$ and $c_1^{\circ}, \ldots, c_m^{\circ} \in \mathfrak{h}^{\circ}$ be the canonical central elements. Since $Q^{\vee} \cap Z^{\vee} = \sum_{j=1}^{n} \mathbb{Z}_+ c_j$ and $Q^{\circ\vee} \cap Z^{\circ\vee} = \sum_{k=1}^{m} \mathbb{Z}_+ c_k^{\circ}$, Proposition 4.34(a) shows that $\pi(c_k^{\circ}) \in \sum_{j=1}^{n} \mathbb{Z}_+ c_j$, $1 \leqslant k \leqslant m$. Clearly, we may assume that $\pi(c_k^{\circ}) \neq 0$, $1 \leqslant k \leqslant m$.

We next reformulate the character formula, the identities defining the c_{λ}^{Λ} and d_{Λ}^{λ}, and Proposition 4.34d(i) in terms of $\lambda[h]$. This reformulation is necessary since the theta functions A_{λ} and S_{λ} are not defined in our context.

For $\lambda \in P$ such that $\lambda(c_j) > 0$, $1 \leqslant j \leqslant n$, put $P_+^{(\lambda)} = \{\Lambda \in P_+ \mid \Lambda(c_j) = \lambda(c_j)$, $1 \leqslant j \leqslant n\}$, and define, for $\Lambda \in P_+^{(\lambda)}$ and $h \in Y$,

$$A_{\lambda}[h] = \sum_{w \in W} (\det w)\, e^{w(\lambda)[h]},$$

$$S_{\lambda}[h] = \sum_{\lambda' \in W(\lambda)} e^{\lambda'[h]},$$

$$c_{\lambda}^{\Lambda}[h] = e^{(\Lambda+\rho)[h] - \rho[h] - \lambda[h] - \Lambda[h]} \sum_{\lambda' \in \lambda + \sum \mathbb{C}\delta_k} \operatorname{mult}_{\Lambda}(\lambda')\, e^{\lambda'(h)},$$

$$d_{\Lambda}^{\lambda}[h] = \sum_{\lambda' \in W(\lambda)} \varepsilon(\Lambda + \rho - \lambda')\, e^{(\Lambda+\rho-\lambda')[h] + \lambda'[h] - (\Lambda+\rho)[h]},$$

where $\varepsilon(\mu) = \det w$ if $\mu(h_i) = w(\rho)(h_i)$, $i \in I$, for some $w \in W$, and $\varepsilon(\mu) = 0$ otherwise. Similarly, we define $A_{\mu}^{\circ}[h]$, etc., for $\mathfrak{g}^{\circ}$.

If $\mathfrak{g}$ is an affine Lie algebra, then $c_{\lambda}^{\Lambda}[h]$ and $d_{\Lambda}^{\lambda}[h]$ coincide with the corresponding functions defined earlier, while $A_{\lambda}[h]$ and $S_{\lambda}[h]$ differ from earlier versions by a simple exponential factor.

Then we have the identities:

$$e^{-\Lambda(h)} \operatorname{ch}_{L(\Lambda)}(h) = (e^{-(\Lambda+\rho)[h]} A_{\Lambda+\rho}[h]) / (e^{-\rho[h]} A_{\rho}[h]), \tag{4.36.1}$$

$$A_{\Lambda+\rho}[h] / A_{\rho}[h] = \sum_{\lambda \in P_+ \cap \max(\Lambda)} c_{\lambda}^{\Lambda}[h]\, S_{\lambda}[h], \tag{4.36.2}$$

$$A_{\rho}[h]\, S_{\lambda}[h] = \sum_{\substack{\Lambda \in P_+^{(\lambda)} \\ \Lambda \bmod (\Sigma \mathbb{C} h_i)^{\perp}}} d_{\Lambda}^{\lambda}[h]\, A_{\Lambda+\rho}[h], \tag{4.36.3}$$

$$\sum_{\mu \in P^{\circ}} \operatorname{mult}(\Lambda : \mu)\, e^{\mu(h)} = e^{-\rho^{\circ}[h]} A_{\rho^{\circ}}^{\circ}[h] \operatorname{ch}_{L(\Lambda)}(\pi(h)). \tag{4.36.4}$$

Now fix $\Lambda \in P_+$ and $\mu \in P^{\circ}_+$. We assume that $\Lambda(c_k) > 0$, $1 \leqslant k \leqslant n$, since the proposition reduces to this case. We shall derive a formula for $e^{\Lambda}_{\mu}(\tau)$, assuming that $\mu \in P^{\circ(\pi^*(\Lambda))}_+$ (since otherwise $e^{\Lambda}_{\mu}(\tau) = 0$).

Let $W_2 = \{w \in W \mid w \circ \pi = \pi \circ w^{\circ}$ on $\mathfrak{h}^{\circ}$ for some $w^{\circ} \in W^{\circ}\}$, $W_3 = \{w \in W \mid w \circ \pi = \pi$ on $\mathfrak{h}^{\circ}\}$. Then W_2 is a subgroup of W, W_3 is a finite (since $\pi(d) \in Y$) normal subgroup of W_2, and W_2/W_3 is canonically isomorphic to W° by Proposition 4.3(b). Define a function $(\ ,\ )$ on $T \times T$ by: $(t, t') = (\Lambda - t(\Lambda))(\pi(d) - t'(\pi(d)))$. It is easy to check using (1.6) that $(\ ,\ )$ is a positive-definite $\mathbb{Q}$-valued symmetric $\mathbb{Z}$-bilinear form on T. Let $T_2 = T \cap W_2$, $T_1 = \{t \in T \mid (t, t') = 0$ for all $t' \in T_2\}$. Clearly, T_1 is a subgroup of T and $|T/T_1T_2| < \infty$. It is easy to check that W_2 normalizes T_1 and $W_2 \cap T_1 = \{1\}$, so that W_2T_1 is a subgroup of W of finite index. We shall need the following:

LEMMA.

(a) $T_1 = \{t \in T \mid \Lambda(\pi(h) - t(\pi(h))) = 0$ *for all* $h \in \sum_{i \in I^{\circ}} \mathbb{C}h_i^{\circ}\}$.

(b) *If* $t \in T_1$ *and* $w \in W_2$, *then* $(\lambda - t(\lambda))(\pi(h) - w(\pi(h))) = 0$ *for all* $\lambda \in P(\Lambda)$ *and* $h \in \mathfrak{h}^{\circ}$.

Proof. If $w \in W_2$, choose $w^{\circ} \in W^{\circ}$ such that $w \circ \pi = \pi \circ w^{\circ}$ on $\mathfrak{h}^{\circ}$. Then for all $h \in \mathfrak{h}^{\circ}$, $t \in T$, and $\lambda \in \mathfrak{h}^*$, we have

$$(\lambda - t(\lambda))(\pi(h) - w(\pi(h))) = \lambda(\pi(h') - t^{-1}(\pi(h'))), \qquad \text{where} \quad h' = h - w^{\circ}(h). \tag{4.36.5}$$

(a) follows from (4.36.5) since, as one sees using (1.6), the span of the possible h' in (4.36.5) for $h = d$ and $w \in T_2$ is $\sum_{i \in I^{\circ}} \mathbb{C}h_i^{\circ}$. Note that in the characterization of T_1 in (a), the expression $\pi(h) - t(\pi(h))$ lies in $\sum_{k=1}^{n} \mathbb{C}c_k$ since $\pi(h) \in \sum_{i \in I} \mathbb{C}h_i$ by Proposition 4.34(a). Hence, if $\lambda \in P(\Lambda)$, then (a) holds with Λ replaced by λ. Along with (4.36.5), this proves (b). ∎

It follows from (b) of the lemma that if $\lambda \in P(\Lambda)$ and $h \in \mathfrak{h}^{\circ}$, then

$$wt(\lambda)[\pi(h)] = -\lambda[\pi(h)] + t(\lambda)[\pi(h)] + w(\lambda)[\pi(h)] \qquad \text{for all} \quad t \in T_1,\ w \in W_2. \tag{4.36.6}$$

Therefore,

$$\sum_{\substack{t \in T_1 \\ w \in W_2}} e^{wt(\lambda)[\pi(h)]} = e^{-\lambda[\pi(h)]} \left(\sum_{t \in T_1} e^{t(\lambda)[\pi(h)]} \right) \left(\sum_{w \in W_2} e^{w(\lambda)[\pi(h)]} \right). \tag{4.36.7}$$

Moreover, one easily computes using the isomorphism $W_2/W_3 \cong W^\circ$ that

$$\sum_{w \in W_2} e^{w(\lambda)[\pi(h)]} = |W_3|\,|W^\circ_{\pi^*(\lambda)}|\, e^{\lambda[\pi(h)] - \pi^*(\lambda)[h]} S^\circ_{\pi^*(\lambda)}[h]. \tag{4.36.8}$$

Choosing coset representatives $w_1, ..., w_s$ for $W_2 T_1 \backslash W$, it follows from (4.36.7) and (4.36.8) that

$$S_\lambda[\pi(h)] = \sum_{r=1}^{s} F^r_\lambda(h)\, S^\circ_{\pi^*(w_r(\lambda))}[h], \tag{4.36.9}$$

where

$$F^r_\lambda(h) = |W_\lambda|^{-1}\,|W_3|\,|W^\circ_{\pi^*(w_r(\lambda))}|\, e^{-\pi^*(w_r(\lambda))[h]} \sum_{t \in T_1} e^{tw_r(\lambda)[\pi(h)]}.$$

On the other hand, putting

$$A(h) = \Lambda(\pi(h)) - (\Lambda + \rho)[\pi(h)] + \rho[\pi(h)] - \rho^\circ[h],$$

it is immediate from (4.36.1), (4.36.2), and (4.36.4) that

$$\sum_{\mu \in P^\circ} \text{mult}(\Lambda : \mu)\, e^{\mu(h)} = e^{A(h)} \sum_{\lambda \in P_+ \cap \max(\Lambda)} c^\Lambda_\lambda[\pi(h)]\, A^\circ_{\rho^\circ}[h]\, S_\lambda[\pi(h)]. \tag{4.36.10}$$

Substituting (4.36.9) into (4.36.10) and using (4.36.3) applied to $\mathfrak{g}^\circ$, we obtain

$$\sum_{\mu \in P^\circ} \text{mult}(\Lambda : \mu)\, e^{\mu(h)} \tag{4.36.11}$$

$$= \sum_{\substack{\mu \in P^{\circ(\pi^*(\Lambda))}_+ \\ \mu \bmod (\sum \mathbb{C}h^\circ_i)^\perp}} e^{A(h) + (\mu + \rho^\circ)[h]} E^\Lambda_\mu(h)\, e^{-(\mu + \rho^\circ)[h]} A^\circ_{\mu + \rho^\circ}[h],$$

where

$$E^\Lambda_\mu(h) = \sum_{\lambda \in P_+ \cap \max(\Lambda)} \sum_{r=1}^{s} c^\Lambda_\lambda[\pi(h)]\, d^{\circ\pi^*(w_r(\lambda))}_\mu[h]\, F^r_\lambda(h).$$

It follows from (4.36.11) that for all $\mu \in P^{\circ(\pi^*(\Lambda))}_+$, we have

$$e^\Lambda_\mu(\tau) = E^\Lambda_\mu(-2\pi i \tau d). \tag{4.36.12}$$

To prove (a), we will show that the factors of the summands of $E^\Lambda_\mu(-2\pi i \tau d)$ are modular forms of appropriate weights. Fix $\lambda \in P_+ \cap \max(\Lambda)$ and $r \in \mathbb{Z}$, $1 \leqslant r \leqslant s$. It follows from Theorem A that $c^\Lambda_\lambda[-2\pi i \tau \pi(d)]$ is a modular form of weight $-\frac{1}{2}$ rank T for some $\Gamma(N)$. Remark (f) of Section 3.3

shows that $d_\mu^{\circ\pi^*(w_r(\lambda))}[-2\pi i\tau d]$ is a holomorphic modular form of weight $\frac{1}{2}$ rank T° for some $\Gamma(N)$. Remark (f) of Section 3.3 also shows that if S_1 is the stationary value of $tw_r(\lambda)[\pi(d)]$ for $t \in \bar{T}_1 := T_1 \otimes_{\mathbb{Z}} \mathbb{C}$, then

$$q^{S_1} \sum_{t \in T_1} q^{-tw_r(\lambda)[\pi(d)]}$$

is a holomorphic modular form of weight $\frac{1}{2}$ rank $T_1 = \frac{1}{2}(\text{rank } T - \text{rank } T_2) = \frac{1}{2}(\text{rank } T - \text{rank } T^\circ)$ for some $\Gamma(N)$. To show that $F_\lambda^r(-2\pi i\tau d)$ is a holomorphic modular form of weight $\frac{1}{2}(\text{rank } T - \text{rank } T^\circ)$, so that $E_\mu^\Lambda(-2\pi i\tau d)$ is a modular form of weight 0, we have to show that $S_1 = \pi^*(w_r(\lambda))[d]$. Let S and S_2 be the stationary values of $tw_r(\lambda)[\pi(d)]$ for $t \in \bar{T}$ and for $t \in \bar{T}_2 := T_2 \otimes_{\mathbb{Z}} \mathbb{C}$, respectively. Then (4.36.6) shows that

$$S = -w_r(\lambda)[\pi(d)] + S_1 + S_2.$$

On the other hand, $S = 0$ by the definition of $A[B]$, and using the definition of T_2, we easily obtain

$$S_2 = w_r(\lambda)[\pi(d)] - \pi^*(w_r(\lambda))[d].$$

Hence, $S_1 = \pi^*(w_r(\lambda))[d]$ as required. Except for the assertion about the multiplier system, whose proof is omitted, this proves (a). Since $d_\mu^{\circ\pi^*(w_r(\lambda))}[-2\pi i\tau d]$ and $F_\lambda^r(-2\pi i\tau d)$ are holomorphic modular forms, (b) follows from Theorem A(4). ∎

As an example, we consider in more detail the decomposition of the tensor product of two $\mathfrak{g}$-modules $L(\Lambda)$ and $L(\Lambda')$ where $\mathfrak{g}$ is an affine Lie algebra and $\Lambda, \Lambda' \in P_+$, using the diagonal inclusion $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$. Due to Proposition 2.9 this tensor product decomposes into a direct sum of modules $L(\Lambda'')$ where $\Lambda'' \in P_+$ and $\Lambda''(c) = \Lambda(c) + \Lambda'(c)$. Hence there exist non-negative integers $\text{mult}(\Lambda, \Lambda'; \Lambda'')$ such that

$$\text{ch}_{L(\Lambda)} \text{ch}_{L(\Lambda')} = \sum_{\Lambda'' \in P_+} \text{mult}(\Lambda, \Lambda'; \Lambda'')\, \text{ch}_{L(\Lambda'')}.$$

Introduce the generating functions

$$c(\Lambda, \Lambda'; \Lambda'') = \sum_{\lambda \in \Lambda'' + \mathbb{C}\delta} \text{mult}(\Lambda, \Lambda'; \lambda)\, e^{-(s_\Lambda + s_{\Lambda'} - s_\lambda)\delta}$$

(note that $s_{\lambda - n\delta} = s_\lambda - n$). Then as in Section 4.4, one can prove:

PROPOSITION 4.37. (a) *If $\Lambda, \Lambda', \Lambda'' \in P_+$, $\Lambda(c) = m > 0$, $\Lambda'(c) = m' > 0$ and $\Lambda''(c) = m + m'$, then $c(\Lambda, \Lambda'; \Lambda'')$ is a modular form of weight zero for $\Gamma(N(m + g)) \cap \Gamma(N(m' + g)) \cap \Gamma(N(m + m' + g))$. If $a_0 = 1$, then $F(\tau)\, c(\Lambda, \Lambda'; \Lambda'')$ is a cusp form; if $a_0 = 2$, then $\eta(\tau)^{l(2l+1)}\, c(\Lambda, \Lambda'; \Lambda'')$ is a cusp form.*

(b) *The span of the* $c(\Lambda, \Lambda'; \Lambda'')$ *with fixed* $\Lambda(c)$, $\Lambda'(c)$ *and* $\Lambda''(c)$ *is stable under* $\Gamma_0(k')$.

Proof. Only the assertion on behavior at the cusps deserves proof. For $\lambda \in \max(\Lambda)$, write

$$A_{\Lambda'+\rho} S_\lambda = \sum_{\Lambda''} e(\lambda, \Lambda'; \Lambda'') A_{\Lambda''+\rho},$$

where $e(\lambda, \Lambda'; \Lambda'') \in \mathcal{O}(\mathcal{H}_+)$. By Proposition 3.14 and Remark (f) in Section 3.3, the $e(\lambda, \Lambda'; \Lambda'')$ are holomorphic modular forms. Since

$$c(\Lambda, \Lambda'; \Lambda'') = \sum_\lambda c_\lambda^\Lambda e(\lambda, \Lambda'; \Lambda''),$$

Theorem A(4) completes the proof. ∎

Let $\Lambda \in P_+$. Then for $n \geqslant 0$, the symmetric group S_n acts on the nth tensor power $\otimes^n L(\Lambda)$ of $L(\Lambda)$. For any irreducible S_n-module V, the subspace $(\otimes^n L(\Lambda))_V$ of $\otimes^n L(\Lambda)$ transforming according to V is $\mathfrak{g}$-stable, hence decomposes into a direct sum of irreducible highest weight modules. We may then form generalized string functions for this decomposition, which are again modular forms of weight zero.

EXAMPLE. Take $n = 2$, so that S_n has two irreducible modules V_+ and V_- (V_+ is the trivial module). Write ch_+ and ch_- for the characters of the modules $(L(\Lambda) \otimes L(\Lambda))_{V_\pm}$. Then (cf. (2.10.2))

$$\mathrm{ch}_+ + \mathrm{ch}_- = \mathrm{ch}^2_{L(\Lambda)},$$
$$(\mathrm{ch}_+ - \mathrm{ch}_-)(h) = \mathrm{ch}_{L(\Lambda)}(2h).$$

In the case $\mathfrak{g} = A_1^{(1)}$ and $\Lambda = \Lambda_0$, a computation using these formulas yields the following illuminating result:

$$\mathrm{ch}_+ = \left(\sum_{k=0}^{\infty} a_{2k} e^{-k\delta} \right) \mathrm{ch}_{L(2\Lambda_0)},$$

$$\mathrm{ch}_- = \left(\sum_{k=0}^{\infty} a_{2k+1} e^{-k\delta} \right) \mathrm{ch}_{L(2\Lambda_1 - \delta)},$$

where

$$\sum_{k=0}^{\infty} a_k q^k = \prod_{k=1}^{\infty} (1 + q^{2k-1}).$$

(Note that in the related result [17, p. 134, Example (a)], the roles of a_{2k} and a_{2k+1} have been inadvertantly reversed.)

We now give another example, which shows that for restriction to a finite-dimensional subalgebra the situation is different—the generalized string functions are no longer modular forms.

Let $\mathfrak{g}$ be an affine Lie algebra, $\bar{\mathfrak{g}}$ the simple finite-dimensional subalgebra corresponding to $\bar{\Delta}$. Let $\Lambda \in P_+$, $\Lambda(d) = 0$, be of level $m > 0$, and consider the $\mathfrak{g}$-module $L(\Lambda)$ restricted to $\bar{\mathfrak{g}} \oplus \mathbb{C}d$ (direct sum of ideals). For $\lambda \in \bar{P}_+$, denote by $F(\lambda)$ the irreducible $\bar{\mathfrak{g}}$-module with highest weight λ. Let $L(\Lambda) = \oplus_{n \geqslant 0} L(\Lambda)_{-n}$ be the eigenspace decomposition with respect to $a_0^{-1}d$. Set

$$\phi_\lambda(q) = \sum_{n \geqslant 0} (\text{multiplicity of } F(\lambda) \text{ in } L(\Lambda)_{-n})\, q^n.$$

In order to calculate $\phi_\lambda(q)$, $q = e^{-\delta}$, recall that

$$q^{s_\Lambda} \operatorname{ch} L(\Lambda) = \sum_{\mu \in P \bmod mM + \mathbb{C}\delta} c_\mu^\Lambda \Theta_{\bar{\mu},m}^M, \tag{4.38}$$

where c_μ^Λ is a string function and

$$\Theta_{\bar{\mu},m}^M = e^{m\Lambda_0} \sum_{\gamma \in M + m^{-1}\bar{\mu}} q^{(1/2)m|\gamma|^2} e^{m\gamma}.$$

By the Weyl character formula, $\phi_\lambda(q)$ is the q-series, which is the coefficient of $e^{m\Lambda_0} e^{\lambda + \bar{\rho}}$ in $(\sum_{w \in \bar{W}} (\det w)\, e^{w(\bar{\rho})}) \operatorname{ch} L(\Lambda)$. Hence, the μth summand of (4.38) gives a contribution to $\phi_\lambda(q)$ if and only if $\bar{\mu} \equiv \lambda + \bar{\rho} - w(\bar{\rho}) \bmod mM$ for some $w \in \bar{W}$. But then $c_\mu^\Lambda = c_{\lambda + \bar{\rho} - w(\bar{\rho}) + m\Lambda_0}^\Lambda$, and we obtain

$$\phi_\lambda(q) = q^{-s_\Lambda} \sum_{w \in \bar{W}} (\det w)\, q^{(1/2m)|\lambda + \bar{\rho} - w(\bar{\rho})|^2} c_{\lambda + \bar{\rho} - w(\bar{\rho}) + m\Lambda_0}^\Lambda. \tag{4.39}$$

Now suppose that the type of $\mathfrak{g}$ is $X_n^{(k)}$, where $X = A$, D or E, and consider the "basic" $\mathfrak{g}$-module $L(\Lambda_0)$. Then there is a unique non-zero string function $c_{\Lambda_0}^{\Lambda_0} = q^{s_{\Lambda_0}} b(q)$ for $L(\Lambda_0)$, where

$$b(q) = \prod_{j \geqslant 1} (1 - q^j)^{-\operatorname{mult} j\delta}$$

(see Section 4.6, Example 1).

Hence, by (4.39), $\phi_\lambda(q) = 0$ if $\lambda \notin M$, and for $\lambda \in M$ we have

$$\begin{aligned}
\phi_\lambda(q) &= b(q)\, q^{(1/2)|\lambda|^2 + (\lambda + \bar{\rho}, \bar{\rho})} \sum_{w \in \bar{W}} (\det w)\, q^{-(\lambda + \bar{\rho}, w(\bar{\rho}))} \\
&= b(q)\, q^{(1/2)|\lambda|^2 + (\lambda + \bar{\rho}, \bar{\rho})} \prod_{\alpha \in \bar{\Delta}_+} (q^{-(1/2)(\lambda + \bar{\rho}, \alpha)} - q^{(1/2)(\lambda + \bar{\rho}, \alpha)}).
\end{aligned}$$

We thus we arrive at the following formula:

$$\phi_\lambda(q) = b(q)\, q^{|\lambda|^2/2} \prod_{\alpha \in \bar{\Delta}_+} (1 - q^{(\lambda + \bar{\rho}, \alpha)}) \qquad \text{if} \quad \lambda \in M \cap \bar{P}_+.$$

In the case $k=1$, this formula was deduced in [18] by a more complicated method.

4.10. *Appendix 4: On Independence of Fundamental Characters*

We state here a result of the second author, which is the precise version (b) of (a) below, which in turn is a theorem of I. Bernstein and O. Schwartzman [2]. The proof follows theirs in considering a Jacobian of theta functions; it will appear in [35].

For F a holomorphic function on $Y=\mathscr{H}_+ \times \bar{\mathfrak{h}} \times \mathbb{C} \subset \mathfrak{h}$, define the partial derivatives

$$\partial_i(F)(h) = \lim_{t\to 0} t^{-1}[F(h+th_i) - F(h)].$$

Then the Jacobian of $A_{\Lambda_j+\rho}/A_\rho$, $0 \leqslant j \leqslant l$, is

$$\mathscr{J} = \det(\partial_i(A_{\Lambda_j+\rho}/A_\rho))_{0\leqslant i,j\leqslant l}.$$

We have $\mathscr{J} \in \widetilde{Th}_g^-$, so that $\mathscr{J} = b(\tau) A_\rho$, where $b(\tau) \in \mathscr{O}(\mathscr{H}_+)$ (by Proposition 4.3(e)). Also, the $A_{\Lambda_i+\rho}/A_\rho$, $0 \leqslant i \leqslant l$, when restricted to Y_τ, generate $\widetilde{Th}^+|_{Y_\tau}$ if and only if $b(\tau) \neq 0$.

We have:

THEOREM 4.40. (a) [2] *For any* $\tau \in \mathscr{H}_+$, $\widetilde{Th}^+|_{Y_\tau}$ *is a polynomial algebra on* $l+1$ *generators* $\Theta_0,\ldots,\Theta_l$, *where* $\Theta_i \in \widetilde{Th}^+_{a_i^\vee}|_{Y_\tau}$.

(b) [35] *Suppose that* $\mathfrak{g}$ *is not of one of the types* $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_6^{(2)}$, $F_4^{(1)}$. *Then* $\widetilde{Th}^+$ *is a polynomial algebra over* $\mathscr{O}(\mathscr{H}_+)$ *on generators* $A_{\Lambda_i+\rho}/A_\rho$, $0 \leqslant i \leqslant l$. *Moreover,* $b(\tau)$ *does not vanish on* $\mathscr{H}_+$, *and is given up to a multiplicative constant in Table* J *below.*

Remark. The proof is case-by-case, and uses the theory of modular forms. We do not know if $b(\tau)$ vanishes on $\mathscr{H}_+$ for the excluded types.

In Table J we list the type $X_n^{(k)}$ of the algebra $\mathfrak{g}$, a positive integer M associated to $\mathfrak{g}$, and a product expansion of $b(\tau)$ valid up to a multiplicative constant. In the product expansions, F_r denotes the modular form

$$F_r = q^{(8M)^{-1}(M-2r)^2} \prod_{\substack{n\equiv 0 \bmod M\\ n\geqslant 1}} (1-q^n) \times \prod_{\substack{n\equiv r \bmod M\\ n\geqslant 1}} (1-q^n) \prod_{\substack{n\equiv -r \bmod M\\ n\geqslant 1}} (1-q^n).$$

TABLE J

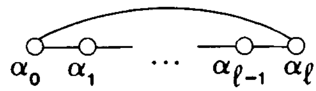
Type $X_n^{(k)}$	Modulus M	$b(\tau)$
$A_l^{(1)}, C_l^{(1)}$	1	$\eta(\tau)^l$
$B_l^{(1)}$	$2l+1$	$\eta(\tau)^{l-3} F_0^{-1} F_l^2$ times: $F_r^3 F_{r+1}$ if $l=2r$, $F_r F_{r+1}^3$ if $l=2r+1$.
$D_l^{(1)}$	l	$\eta(\tau)^{l-4}$ times: $F_r^3 F_{r+1}$ if $l=2r$, F_r^4 if $l=2r+1$.
$G_2^{(1)}$	9	$\eta(\tau) F_4$
$A_{2l}^{(2)}$	$2l+3$	$\eta(\tau)^{l-1} F_{l+1}$
$A_{2l-1}^{(2)}$	$2l+2$	$\eta(\tau)^{l-2} F_l F_{l+1}$
$D_{l+1}^{(2)}$	$2l+2$	$\eta(2\tau)^{l-2} F_l F_{l+1}$
$D_4^{(3)}$	36	$\eta(\tau)\,\eta(2\tau)^{-1}\,\eta(3\tau)^{-1}$ $\times\,\eta(4\tau)\,\eta(6\tau)^2\,\eta(9\tau)^2\,\eta(12\tau)^{-1}$ $\times\, q^{-49/72} \prod_{\substack{n\equiv 0,\pm 1 \bmod 9\\ n\geqslant 1}} (1-q^n)^{-1}$

V. The Partition Function and Hecke "Indefinite" Modular Forms

In Section V we find explicit formulas for the partition function K of the affine Lie algebra of type $A_l^{(1)}$ using methods developed in [34]. This allows us to compute the string functions directly using the multiplicity formula (2.8). In the simplest case, the affine Lie algebra of type $A_1^{(1)}$, these functions multiplied by the cube of the η-function turn out to be modular forms associated to indefinite binary quadratic forms. In conclusion we collect the main results of the paper in the case $A_1^{(1)}$, obtaining various identities for modular forms and elliptic theta functions.

5.1. *The Partition Function K for* $A_l^{(1)}$

Let $\mathfrak{g}$ be the affine Lie algebra of type $A_l^{(1)}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ the root system, Δ_+ the set of positive roots, $\{\alpha_0, \ldots, \alpha_l\}$ the set of simple roots, numbered as in



Let W be the Weyl group of $\mathfrak{g}$, generated by the reflections $r_{\alpha_0},\dots, r_{\alpha_l}$, which we denote as usual by $r_0,\dots, r_l$. Let $\bar{W}$ be the subgroup of W generated by $r_1,\dots, r_l$; we regard $\bar{W}$ as the Weyl group of the root system $\bar{\Delta}$ of type A_l. Let Q be the lattice in $\mathfrak{h}^*$ generated by $\alpha_0,\dots, \alpha_l$, and let $\bar{Q}$ be the sublattice generated by $\alpha_1,\dots, \alpha_l$. Define functions n_i, $0 \leqslant i \leqslant l$, on Q by

$$\alpha = n_0(\alpha)\,\alpha_0 + \cdots + n_l(\alpha)\,\alpha_l \qquad \text{for} \quad \alpha \in Q.$$

Note that $\{\alpha_0,\dots, \alpha_l, \rho\}$ is a basis of $\mathfrak{h}^*$. We define $\sigma \in GL(\mathfrak{h}^*)$ by

$$\sigma(\alpha_i) = \alpha_{i+1}, \qquad 0 \leqslant i < l;$$

$$\sigma(\alpha_l) = \alpha_0; \qquad \sigma(\rho) = \rho.$$

Then σ normalizes W, so that we may define the group $W_\sigma := \langle\sigma\rangle \ltimes W$, where $\langle\sigma\rangle$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by σ. Note that W_σ is the group W_0 of Proposition 4.27(b). Then Δ, Q, and $Q + \rho$ are W_σ-invariant. We define a "shifted" action of W_σ on Q by $w \,.\, \alpha = w(\alpha + \rho) - \rho$. This induces an action of W_σ on functions on Q by $(w \,.\, f)(\alpha) = f(w^{-1} \,.\, \alpha)$, and hence an action of the group ring $\mathbb{C}[W_\sigma]$ on these functions.

Recall the partition function K, defined on Q. Then on $\bar{Q}$, K is the usual Kostant partition function for $\bar{\Delta}_+$.

We introduce the following polynomial function on $\bar{Q}$ [34]:

$$F(\beta) = \sum_{\substack{\alpha \in \bar{Q} \\ n_l(\alpha) = 0}} K(\alpha) \prod_{r=1}^{l} \binom{n_r(\beta) - n_{r-1}(\beta) + l - r}{n_r(\alpha) - n_{r-1}(\alpha) + l - r}.$$

Here we define $\binom{n}{k} = n(n-1) \cdots (n-k+1)/k!$ for $k > 0$, $\binom{n}{0} = 1$, $\binom{n}{k} = 0$ for $k < 0$. Note that if $\alpha \in \bar{Q}$ gives a non-zero summand, then we must have $n_r(\alpha) \geqslant 0$ and $n_r(\alpha) - n_{r-1}(\alpha) + l - r \geqslant 0$ for $1 \leqslant r \leqslant l$. Since these and $n_l(\alpha) = 0$ imply $0 \leqslant n_r(\alpha) \leqslant \binom{l-r}{2}$ for $1 \leqslant r \leqslant l$, the sum defining F is actually finite.

For motivation, we mention that $F(\beta)$ coincides with the Kostant partition function for A_l on the set of all $\beta \in \bar{Q}$ satisfying $n_l(\beta) \geqslant n_{l-1}(\beta) \geqslant \cdots \geqslant n_1(\beta) \geqslant 0$ [34]. We set:

$$K' = (1 + r_l)(1 - r_{l-1} r_l) \cdots (1 - (-1)^l r_1 \cdots r_l) \,.\, K. \tag{5.1}$$

The crucial observation is that the function K' is "simpler" than the partition function K. So, we first give an effective algorithm for computing K' and then express K in terms of K'. For that we need the following two lemmas.

LEMMA 5.2 [34]. *If $\beta \in \bar{Q}$, then $K'(\beta) = F(\beta)$.*

Define a function K'' on Q by

$$\sum_{\gamma \in Q} K''(\gamma)\, e^{\gamma} = \prod_{\alpha \in \Delta_+ \setminus \bar{\Delta}_+} (1 - e^{\alpha})^{-\operatorname{mult} \alpha},$$

i.e., this is a partition function for which parts from $\bar{\Delta}_+$ are not permitted. Then clearly, $K(\beta) = \sum_{\alpha \in \bar{Q}} K(\alpha)\, K''(\beta - \alpha)$ for $\beta \in Q$. If $w \in \bar{W}$ and $\alpha \in Q$, then $K''(w(\alpha)) = K''(\alpha)$, so that:

$$\begin{aligned} K(w \,.\, \beta) &= \sum_{\alpha \in \bar{Q}} K(\alpha)\, K''(w \,.\, \beta - \alpha) \\ &= \sum_{\alpha \in \bar{Q}} K(\alpha)\, K''(w^{-1}(w \,.\, \beta - \alpha)) \\ &= \sum_{\alpha \in \bar{Q}} K(\alpha)\, K''(\beta - w^{-1} \,.\, \alpha) \\ &= \sum_{\alpha \in \bar{Q}} K(w \,.\, \alpha)\, K''(\beta - \alpha). \end{aligned}$$

Along with formula (5.1) and Lemma 5.2, this yields:

LEMMA 5.3. *For $\beta \in Q$, $K'(\beta) = \sum_{\alpha \in \bar{Q}} F(\alpha)\, K''(\beta - \alpha)$.*

For $k \in \mathbb{Z}_+$, $q \in \mathbb{C}$, $|q| < 1$, set $\sigma_k(q) = \sum_{n \geqslant 1} (\sum_{d \mid n} d^k)\, q^n$ and $\varphi(q) = \prod_{n \geqslant 1} (1 - q^n)$. Recall that both functions are intimately connected to classical modular forms, namely, setting $q = e^{2\pi i \tau}$, we have: $E_k(\tau) - 1 = \gamma_k \sigma_{2k-1}(q)$ for $k > 1$ and some constant γ_k, where $E_k(\tau)$ is the kth Eisenstein series (see, e.g., [39]), and $\eta(\tau) = q^{1/24} \varphi(q)$ is the Dedekind η-function.

Now we can give an algorithm for computing the function K'.

PROPOSITION 5.4. *There exists a polynomial R_l in $l + 2$ indeterminates such that for $\beta \in \bar{Q}$ and $|q| < 1$, one has*

$$\varphi(q)^{l(l+2)} \sum_{n \geqslant 0} K'(\beta + n\delta)\, q^n = R_l(n_1(\beta), \ldots, n_{l-1}(\beta), \sigma_1(q), \sigma_3(q), \sigma_5(q)).$$

Proof. We shall prove this as an equality of formal power series in q. The convergence is clear by the argument of Lemma 2.3.

For $\alpha \in \bar{\mathfrak{h}}^*$, regard e^{α} as a function on $\bar{\mathfrak{h}}$. Let D be the linear constant-coefficient differential operator on $\bar{\mathfrak{h}}$ such that $D(e^{\alpha}) = F(\alpha)\, e^{\alpha}$ for $\alpha \in \bar{\mathfrak{h}}^*$, so that $F(\alpha) = (D(e^{\alpha}))(0)$. (In other words, F is the symbol of the differential operator D.)

If $\beta \in \bar{Q}$, Lemma 5.3 gives

$$\sum_{n \geq 0} K'(\beta + n\delta)\, q^n = \left(D\left(e^{\beta} \sum_{\substack{\gamma \in \bar{Q} \\ n \geq 0}} K''(\gamma + n\delta)\, q^n e^{-\gamma}\right)\right)(0).$$

But from the description of the root system of an affine Lie algebra $\mathfrak{g}$ from Table I we obtain

$$\sum_{\substack{\gamma \in \bar{Q} \\ n \geq 0}} K''(\gamma + n\delta)\, q^n e^{-\gamma} = \varphi(q)^{-l} \prod_{\substack{\alpha \in \bar{\Delta} \\ n \geq 1}} (1 - q^n e^{\alpha})^{-1}.$$

Hence, we have, for $\beta \in \bar{Q}$,

$$\sum_{n \geq 0} K'(\beta + n\delta)\, q^n = \left(D\left(e^{\beta} \varphi(q)^{-l} \prod_{\substack{\alpha \in \bar{\Delta} \\ n \geq 1}} (1 - q^n e^{\alpha})^{-1}\right)\right)(0). \tag{5.4.1}$$

On the other hand, for $\alpha \in \bar{\mathfrak{h}}^*$ we have

$$\log \prod_{n \geq 1} (1 - q^n e^{\alpha})^{-1} = \sum_{n \geq 1} \sum_{r \geq 1} \frac{1}{r} (q^n e^{\alpha})^r = \sum_{n \geq 1} q^n \sum_{d \mid n} d^{-1} e^{d\alpha},$$

and in particular,

$$-\log \varphi(q) = \sum_{n \geq 1} q^n \sum_{d \mid n} d^{-1}.$$

Therefore we obtain

$$\prod_{n \geq 1} (1 - q^n e^{\alpha})^{-1} (1 - q^n e^{-\alpha})^{-1}$$
$$= \varphi(q)^{-2} \exp\left(\sum_{n \geq 1} q^n \sum_{d \mid n} d^{-1} (e^{(d/2)\alpha} - e^{-(d/2)\alpha})^2\right). \tag{5.4.2}$$

Introduce

$$G := \sum_{n \geq 1} q^n \sum_{d \mid n} d^{-1} \sum_{\alpha \in \bar{\Delta}_+} (e^{(d/2)\alpha} - e^{-(d/2)\alpha})^2.$$

Then (5.4.1) and (5.4.2) yield

$$\varphi(q)^{l(l+2)} \sum_{n \geq 0} K'(\beta + n\delta)\, q^n = (D(e^{\beta} \exp G))(0). \tag{5.4.3}$$

Let D' be the polynomial in the derivatives of f such that $D(\exp f) = D'(f) \exp f$. Then we obtain

$$(D(e^{\beta} \exp G))(0) = (D'(\beta + G))(0)$$
$$= R'(n_1(\beta),..., n_l(\beta), \sigma_1(q), \sigma_3(q), \sigma_5(q),...)$$

for some polynomial R'. This follows from the fact that G and its derivatives of odd order vanish at 0, and its derivatives of even order are multiples of the $\sigma_{2k-1}(q)$, $k \geqslant 1$, at 0. Since $F(\beta)$ does not involve $n_l(\beta)$, neither does R'. Now the Proposition follows from (5.4.3) and the well-known fact (see, e.g., [39]) that for $k \geqslant 2$, $\sigma_{2k-1}(q)$ is a polynomial in $\sigma_3(q)$ and $\sigma_5(q)$. ∎

For $1 \leqslant l \leqslant 4$, R_l is given below; in the formulas, we have set $m_i = m_i(\beta) = n_i(\beta + \bar{\rho}) = n_i(\beta) + \frac{1}{2}i(l - i + 1)$, $1 \leqslant i \leqslant l$.

$$R_1 = 1;$$

$$R_2 = m_1;$$

$$R_3 = \frac{1}{24}(3m_2 - 2m_1)(4m_1^2 - 1 + 24\sigma_1(q));$$

$$R_4 = \frac{1}{360}(m_1^3 - m_1)(3m_3 - 2m_2)(6m_1^2 - 15m_1 m_2 + 10m_2^2 - 4)$$
$$+ \frac{1}{6} m_1(3m_3 - 2m_2)(4m_1^2 - 6m_1 m_2 + 4m_2^2 - 3)\,\sigma_1(q)$$
$$+ \frac{1}{6} m_1(3m_3 - 2m_2)(\sigma_3(q) + 36\sigma_1(q)^2).$$

In order to recover the partition function K from the function K' we need three more lemmas. For $1 \leqslant i \leqslant l$, we introduce the following elements of W_σ:

$$w_i' = r_l r_{l-1} \cdots r_{l-i+1}; \qquad w_i = \sigma w_i'; \qquad t_i = w_i^{l-i+1}.$$

LEMMA 5.5.

(a) *If* $1 \leqslant i < j \leqslant l$, *then* $w_i' w_j = w_j w_i'$ *and* $t_i w_j = w_j t_i$.

(b) $t_i = t_{\bar{\Lambda}_{l-i+1}}$ *for* $i = 1,..., l$, *where* t_γ *is defined by formula* (1.6).

In particular, $t_i(\alpha_0) = \alpha_0 + \delta$, $t_i(\alpha_{l-i+1}) = \alpha_{l-i+1} - \delta$, *and* $t_i(\alpha_j) = \alpha_j$ *for* $j \neq 0$, $l - i + 1$.

The proof consists of a straightforward calculation, which we omit.

For $0 \leqslant i \leqslant l$, define functions $K^{(i)}$ on Q inductively by

$$K^{(0)} = K, \qquad K^{(i)} = (1 - (-1)^i\, w_i^{-1}) \,.\, K^{(i-1)} \qquad \text{for} \quad 1 \leqslant i \leqslant l.$$

LEMMA 5.6. *For* $0 \leqslant i \leqslant l$,

$$K^{(i)} = (1 + w_1'^{-1})(1 - w_2'^{-1}) \cdots (1 - (-1)^i w_i'^{-1}) . K.$$

In particular, $K^{(l)} = K'$.

Proof. The lemma is clear for $i = 0$. Suppose it is true for $i - 1$. Then since $(\sigma^{-1} . K)(\alpha) = K(\sigma . \alpha) = K(\sigma(\alpha)) = K(\alpha)$, we have $\sigma^{-1} . K = K$, so that $w_i^{-1} . K = w_i'^{-1} . K$. Using this, and $w_k' w_i = w_i w_k'$ for $1 \leqslant k < i$ from Lemma 5.5(a), we obtain

$$\begin{aligned} K^{(i)} &= (1 - (-1)^i w_i^{-1}) . K^{(i-1)} \\ &= (1 - (-1)^i w_i^{-1})(1 + w_1'^{-1}) \cdots (1 - (-1)^{i-1} w_{i-1}'^{-1}) . K \\ &= (1 + w_1'^{-1}) \cdots (1 - (-1)^{i-1} w_{i-1}'^{-1})(1 - (-1)^i w_i^{-1}) . K \\ &= (1 + w_1'^{-1}) \cdots (1 - (-1)^{i-1} w_{i-1}'^{-1})(1 - (-1)^i w_i'^{-1}) . K. \quad \blacksquare \end{aligned}$$

COROLLARY 5.7. *For* $0 \leqslant i \leqslant l$, *if* $\alpha \in Q$ *and* $n_0(\alpha) < 0$, *then* $K^{(i)}(\alpha) = 0$.

Proof. By Lemma 5.6, it suffices to note that for $w \in \bar{W}$ and $\alpha \in Q$ such that $n_0(\alpha) < 0$, we have $(w . K)(\alpha) = K(w^{-1} . \alpha) = 0$ since $n_0(w^{-1} . \alpha) = n_0(\alpha) < 0$. ∎

LEMMA 5.8. *For* $1 \leqslant i \leqslant l$ *and* $\alpha \in Q$:

(a) $$K^{(i-1)}(\alpha) = \sum_{n \geqslant 0} (-1)^{ni} K^{(i)}(w_i^n . \alpha) = - \sum_{n < 0} (-1)^{ni} K^{(i)}(w_i^n . \alpha).$$

(b) $$\sum_{n \in \mathbb{Z}} (-1)^{ni} K^{(i)}(w_i^n . \alpha) = 0.$$

(c) $$K^{(i-1)}(\alpha) = \sum_{k_i, \ldots, k_l \geqslant 0} (-1)^{ik_i + \cdots + lk_l} K^{(l)} (w_l^{k_l} \cdots w_i^{k_i}(\alpha + \rho) - \rho).$$

In each sum above, only a finite number of summands are non-zero.

Proof. For $k \in \mathbb{Z}^l$, $k = (k_1, \ldots, k_l)$, set $w(k) = w_l^{k_l} \cdots w_1^{k_1}$. Fix $\alpha \in Q$. We first show that $n_0(w(k) . \alpha) < 0$ for all but a finite number of $k \in \mathbb{Z}^l$, so that by Corollary 5.7, the sums in question are finite.

For $k = (k_1, \ldots, k_l) \in \mathbb{Z}^l$, write $k_s = (l - s + 1) q_s + r_s$ for $1 \leqslant s \leqslant l$, where $q_s, r_s \in \mathbb{Z}$ and $0 \leqslant r_s < l - s + 1$. Set $t(k) = t_l^{q_l} \cdots t_1^{q_1}$, $\phi(k) = w_l^{r_l} \cdots w_1^{r_1}$. Then by Lemma 5.5(a), $w(k) = t(k) \phi(k)$. Set $\gamma(k) = \sum_{s=1}^{l} q_s \bar{A}_{l-s+1}$. Then by Lemma 5.5(b), $t(k) = t_{\gamma(k)}$. For $|k| \to \infty$, $|\gamma(k)| \to \infty$ and $\phi(k)$ takes on only a finite number of distinct values. Since $(\rho, \delta) = l + 1 > 0$ and $n_0(\delta) = 1 > 0$, formula (1.7) shows that $n_0(w(k)\,.\,\alpha) < 0$ for all but a finite number of $k \in \mathbb{Z}^l$.

To prove (a), note that by definition,

$$K^{(i)}(\alpha) = K^{(i-1)}(\alpha) - (-1)^i K^{(i-1)}(w_i\,.\,\alpha),$$

so that

$$K^{(i-1)}(\alpha) = K^{(i)}(\alpha) + (-1)^i K^{(i-1)}(w_i\,.\,\alpha),$$

and hence, by induction on N,

$$K^{(i-1)}(\alpha) = \sum_{n=0}^{N-1} (-1)^{ni} K^{(i)}(w_i^n\,.\,\alpha) + (-1)^{Ni} K^{(i-1)}(w_i^N\,.\,\alpha).$$

Since $K^{(i)}(w_i^N\,.\,\alpha) = K^{(i-1)}(w_i^N\,.\,\alpha) = 0$ for N large, this proves the first part of (a). The second part follows similarly from

$$-K^{(i-1)}(\alpha) = (-1)^i K^{(i)}(w_i^{-1}\,.\,\alpha) - (-1)^i K^{(i-1)}(w_i^{-1}\,.\,\alpha),$$

and (b) is immediate from (a).

Finally, (c) follows from the first part of (a) by downward induction on i. ∎

Set $T_+ = \{t_l^{k_l} \cdots t_1^{k_1} \mid k_1,\ldots, k_l \in \mathbb{Z}_+\}$, $\Phi = \{w_l^{k_l} \cdots w_1^{k_1} \mid 0 \leqslant k_i \leqslant l-i$ for $1 \leqslant i \leqslant l\}$, and define a character χ of W_σ by $\chi(\sigma) = 1$, $\chi(w) = \det(w)$ for $w \in W$. Then since $K^{(l)} = K'$ by Lemma 5.6, Lemma 5.8(c) and its proof yield, for $i = 1$:

THEOREM C. *For the affine Lie algebra of type $A_l^{(1)}$, one has, for $\alpha \in Q$,*

$$K(\alpha) = \sum_{\substack{t \in T_+ \\ \phi \in \Phi}} \chi(t\phi)\, K'(t\phi(\alpha + \rho) - \rho).$$

Remark. One can show [34] that if $n_r(\beta) \geqslant n_{r-1}(\beta) + n_0(\beta)$ for $1 \leqslant r \leqslant l$, then $K(\beta) = K'(\beta)$. This, together with Lemma 5.2, implies a remark preceding Lemma 5.2.

5.2. *Formulas for K in Low Ranks*

Theorem C, along with Proposition 5.4, gives explicit formulas for the partition function K.

For example, let $l = 1$. Then

$$\Delta_+ = \{n_0\alpha_0 + n_1\alpha_1 \mid n_0, n_1 \in \mathbb{Z}_+, |n_0 - n_1| \leqslant 1\}\backslash\{0\};$$

$$\Delta_+^{\text{im}} = \{n\delta \mid n \in \mathbb{Z}, n \geqslant 1\}, \qquad \text{where} \quad \delta = \alpha_0 + \alpha_1;\ \bar{\Delta}_+ = \{\alpha_1\},$$

and $\operatorname{mult} \alpha = 1$ for all $\alpha \in \Delta$. Furthermore, we have: $\Phi = \{1\}$ and $T_+ = \{t^k \mid k \in \mathbb{Z}_+\}$, where $t(\alpha_0) = \alpha_0 + \delta$, $t(\alpha_1) = \alpha_1 - \delta$, $t(\rho) = \rho - \alpha_0$, $\chi(t) = -1$.

We also have: $R_1 = 1$ and therefore $K'(\beta) = p^{(3)}(n_0(\beta))$. Here $p^{(d)}(n)$ is defined by

$$\sum_{n \in \mathbb{Z}} p^{(d)}(n)\, q^n = \varphi(q)^{-d} \text{ for } |q| < 1,$$

i.e., $p^{(d)}(n)$ is the number of partitions of n in positive integral parts of d different "colors."

Theorem C and Lemma 5.8(b) now give:

PROPOSITION 5.9. *For the affine Lie algebra of type $A_1^{(1)}$, one has for n_0, $n_1 \in \mathbb{Z}$:*

(a) $$K(n_0\alpha_0 + n_1\alpha_1) = \sum_{k \geqslant 0} (-1)^k p^{(3)}\left((k+1)\, n_0 - kn_1 - \frac{k(k+1)}{2}\right),$$

(b) $$K(n_0\alpha_0 + n_1\alpha_1) = -\sum_{k<0} (-1)^k p^{(3)}\left((k+1)\, n_0 - kn_1 - \frac{k(k+1)}{2}\right).$$

Note that for use in computing $K(n_0\alpha_0 + n_1\alpha_1)$, (a) is best suited if $n_1 \geqslant n_0$, while (b) is best suited if $n_0 \geqslant n_1$.

Remark. Let $l = 1$. Then the proofs simplify considerably. One has: $F(\alpha) = 1 = K'(\alpha)$ for $\alpha \in \bar{Q}$, and so Lemma 5.2 is trivial. Lemma 5.3 and Proposition 5.4 mean that $K'(\beta) = \sum_{\alpha \in \bar{Q}} K''(\beta - \alpha) = p^{(3)}(n_0(\beta))$. These facts follow easily from the description of Δ_+ and $\bar{\Delta}_+$. Finally, $K'(\alpha) = K(\alpha) + K(r_1 . \alpha)$ and $K(\sigma . \alpha) = K(\alpha)$, so that setting $t = \sigma r_1$, we obtain the following form of Proposition 5.9(a):

$$K(\alpha) = \sum_{k \geqslant 0} (-1)^k K'(t^k . \alpha).$$

Now let $l = 2$. Then Theorem C and Proposition 5.4 give:

PROPOSITION 5.10. *For the affine Lie algebra of type $A_2^{(1)}$, one has, for $n_0, n_1, n_2 \in \mathbb{Z}$,*

$$\begin{aligned} &K(n_0\alpha_0 + n_1\alpha_1 + n_2\alpha_2) \\ &\quad = \sum_{k,l \geqslant 0} (n_1 - n_0 + 2k + l + 1) \\ &\quad \times p^{(8)}((k+l+1)\, n_0 - kn_1 - ln_2 - l(k+l+1) - k(k+1)) \\ &\quad - \sum_{k,l \geqslant 0} (n_2 - n_1 + 2k + l + 2) \\ &\quad \times p^{(8)}((l+1)\, n_0 + (k+1)\, n_1 - (k+l+1)\, n_2 \\ &\quad - (l+1)(k+l+1) - k(k+1)). \end{aligned}$$

Finally, we note that (see [34]) for any affine Lie algebra, there exist formulas for the partition function, derived using its "hidden" symmetries, similar to those given above. However, except for those for algebras of type $A_l^{(1)}$, they seem relatively intractable.

EXAMPLE. Consider the affine Lie algebra of type $A_2^{(2)}$, with simple roots numbered as in

and $\delta = 2\alpha_0 + \alpha_1$. Then the positive roots, all of multiplicity one, are given by

$$\alpha_0 + k\delta,\ (k+1)\delta,\ -\alpha_0 + (k+1)\delta,$$
$$\alpha_1 + 2k\delta,\ -\alpha_1 + 2(k+1)\delta, \qquad \text{where} \quad k \in \mathbb{Z}_+.$$

For $k \geqslant 1$, $e_1,\ldots, e_k \in \mathbb{Z}$, define a function $p^{(e_1\cdots e_k)}\colon \mathbb{Z} \to \mathbb{Z}$ by

$$\sum_{n\in\mathbb{Z}} p^{(e_1\cdots e_k)}(n)\, q^n = \prod_{n\geqslant 1} (1-q^n)^{-e_{n \bmod k}} \text{ for } |q| < 1.$$

For $\beta = n_0\alpha_0 + n_1\alpha_1 \in Q$, we have:

$$K(\beta) + K(r_1(\beta+\rho) - \rho) = p^{(2123)}(n_0),$$
$$K(\beta) + K(r_0(\beta+\rho) - \rho) = p^{(53)}(n_1).$$

Applying these formulas alternately, we obtain:

$$\begin{aligned}
K(n_0\alpha_0 + n_1\alpha_1)
&= \sum_{k\geqslant 0} p^{(2123)}((2k+1)\,n_0 - 4kn_1 - k(3k+2))\\
&\quad - \sum_{k\geqslant 0} p^{(53)}((k+1)\,n_0 - (2k+1)\,n_1 - \tfrac{1}{2}(k+1)(3k+2))\\
&= \sum_{k\geqslant 0} p^{(53)}(-kn_0 + (2k+1)\,n_1 - \tfrac{1}{2}k(3k+1))\\
&\quad - \sum_{k\geqslant 0} p^{(2123)}(-(2k+1)\,n_0 + 4(k+1)\,n_1 - (k+1)(3k+1)).
\end{aligned}$$

5.3. *Hecke "Indefinite" Modular Forms*

Let U be a two-dimensional real vector space, let L be a full lattice in U, and let B be an indefinite symmetric bilinear form on U such that $B(\gamma,\gamma)$ is an even non-zero integer for all non-zero $\gamma \in L$. Set $L^* = \{\gamma' \in U \mid B(\gamma,\gamma') \in \mathbb{Z} \text{ for all } \gamma \in L\}$. Let G_0 be the subgroup of the identity component of the orthogonal group of (U, B) preserving L and fixing L^*/L. Fix a factorization $B(\gamma, \gamma) = l_1(\gamma)\, l_2(\gamma)$, where l_1 and l_2 are real-linear, and set $\operatorname{sign}(\gamma) = \operatorname{sign} l_1(\gamma)$ for $l_1(\gamma) \neq 0$.

For $\mu \in L^*$, set

$$\theta_{L,\mu}(\tau) = \sum_{\substack{\gamma \in L+\mu \\ B(\gamma,\gamma)>0 \\ \gamma \bmod G_0}} \operatorname{sign}(\gamma)\, e^{\pi i \tau B(\gamma,\gamma)}.$$

This is a cusp form of weight 1. More precisely:

(1) $$\theta_{L,\mu}(\tau+1) = e^{\pi i B(\mu,\mu)} \theta_{L,\mu}(\tau),$$

$$\theta_{L,\mu}\left(-\frac{1}{\tau}\right) = -\frac{\tau}{\sqrt{|L^*/L|}} \sum_{\nu \in L^* \bmod L} e^{2\pi i B(\mu,\nu)} \theta_{L,\nu}(\tau).$$

The span of the $\theta_{L,\mu}$ is stable under $SL(2, \mathbb{Z})$.

(2) Let $N \in \mathbb{Z}$, $N > 0$ be such that $NB(\gamma, \gamma) \in 2\mathbb{Z}$ for all $\gamma \in L^*$. Then, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, there exists a $\varepsilon \in \mathbb{C}$ of absolute value 1 such that for all $\mu \in L^*$,

$$\theta_{L,\mu}\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(c\tau + d)(\exp \pi i ab B(\mu, \mu))\, \theta_{L,a\mu}(\tau).$$

These results are due essentially to Hecke [9], who treats by a general method the case where B is an even multiple of the norm on the integers of a real quadratic field. For this reason we call the functions $\theta_{L,\mu}$ *Hecke indefinite modular forms.*

For a treatment of this from the point of view of the Weil representation see [26], and from the point of view of the theory of Galois representations and Artin L-functions see [38].

5.4. *String Functions for $A_1^{(1)}$*

First let $\mathfrak{g}$ be of type $A_l^{(1)}$. Let $\Lambda \in P_+$ be of level $m = \Lambda(c) > 0$, and let λ be a maximal weight of $L(\Lambda)$. Recall the string function defined in Section 2.3:

$$c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{s \geqslant 0} \operatorname{mult}_\Lambda(\lambda - s\delta)\, q^s,$$

where $\tau \in \mathscr{H}_+$, $q = e^{2\pi i \tau}$, and $s_\Lambda(\lambda)$ is the characteristic of λ, which in our case can be computed by

$$s_\Lambda(\lambda) = n_0(\Lambda - \lambda) + \frac{|\bar{\Lambda} + \bar{\rho}|^2}{2(m + l + 1)} - \frac{|\bar{\lambda}|^2}{2m} - \frac{l(l+2)}{24}.$$

Substituting the expression for $\text{mult}_\Lambda(\lambda)$ given by (2.8) into the definition of c_λ^Λ, and rearranging using absolute convergence, we obtain

$$\varphi(q)^{l(l+2)}\, c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{w \in W} (\det w)\, S(w(\Lambda+\rho) - (\lambda+\rho)), \tag{5.11}$$

where

$$S(\beta) = \varphi(q)^{l(l+2)} \sum_{s \geqslant 0} K(\beta + s\delta)\, q^s. \tag{5.12}$$

Set $U = \bar{\mathfrak{h}}_{\mathbb{R}}^* \oplus \bar{\mathfrak{h}}_{\mathbb{R}}^*$, introduce the lattice $L = M \oplus M \subset U$, and define a quadratic form B on U by

$$B((\gamma;\gamma')) = (m+l+1)(\gamma,\gamma) - m(\gamma',\gamma').$$

Then $B((\gamma;\gamma')) \in 2\mathbb{Z}$ for $(\gamma;\gamma') \in L$. We will need the following formula:

$$\begin{aligned} &-n_0(t_\gamma(t_{\gamma'}(\Lambda+\rho)-\lambda)-\rho) \\ &\quad = -n_0(\Lambda-\lambda) - \frac{|\bar{\Lambda}+\bar{\rho}|^2}{2(m+l+1)} + \frac{|\bar{\lambda}|^2}{2m} \\ &\qquad + \frac{1}{2} B((\gamma+\gamma'+(m+l+1)^{-1}(\bar{\Lambda}+\bar{\rho}); \gamma + m^{-1}\bar{\lambda})). \end{aligned} \tag{5.13}$$

To prove this, rewrite the left-hand side as $n_0((\Lambda+\rho) - t_{\gamma+\gamma'}(\Lambda+\rho)) - n_0(\lambda - t_\gamma(\lambda)) - n_0(\Lambda-\lambda)$, and apply (1.7).

Now let $\mathfrak{g}$ be of type $A_1^{(1)}$. Then $\alpha := \alpha_1$, δ and ρ form a basis of $\mathfrak{h}^*$. The simple roots are $\alpha_0 = \delta - \alpha$ and $\alpha_1 = \alpha$, and the fundamental weights are $\Lambda_0 = \frac{1}{2}\rho - \frac{1}{4}\alpha$ and $\Lambda_1 = \frac{1}{2}\rho + \frac{1}{4}\alpha$. We have $\bar{\alpha} = \alpha$, $\bar{\delta} = 0$, $\bar{\rho} = \frac{1}{2}\alpha$, $\bar{\Lambda}_0 = 0$, $\bar{\Lambda}_1 = \frac{1}{2}\alpha$. The lattice M is $\mathbb{Z}\alpha$. The normalized symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ is defined by $(\alpha,\alpha) = 2$, $(\alpha,\rho) = 1$, $(\rho,\rho) = \frac{1}{2}$, $(\rho,\delta) = 2$, $(\delta,\delta) = (\delta,\alpha) = 0$.

Define $s, t \in GL(\mathfrak{h}^*)$ by

$$\begin{aligned} s(\alpha) &= -\alpha, & s(\delta) &= \delta, & s(\rho) &= \rho - \alpha, \\ t(\alpha) &= \alpha - \delta, & t(\delta) &= \delta, & t(\rho) &= \rho + \alpha - \delta. \end{aligned}$$

Then $t = t_{(1/2)\alpha}$, defined by (1.6). W_σ is the semidirect product of the finite Weyl group $\bar{W} = \{1, s\}$ and the free abelian normal subgroup generated by t. The subgroup W of W_σ is generated by s and t^2.

We have $L = \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha$ and $U = \mathbb{R}\alpha \oplus \mathbb{R}\alpha$. Hence, identifying L with $\mathbb{Z}^2$ and U with $\mathbb{R}^2$, we have

$$B((x,y)) = 2(m+2)\,x^2 - 2my^2.$$

Clearly, B does not vanish on $L\backslash\{0\}$.

The lattice dual to L with respect to B is

$$L^* = \frac{1}{2(m+2)}\mathbb{Z} \oplus \frac{1}{2m}\mathbb{Z}.$$

Define an element a of the identity component $SO_0(U)$ of the orthogonal group of (U, B) by

$$a((x, y)) = ((m+1)x + my, (m+2)x + (m+1)y).$$

Then a generates the subgroup G_0' of $SO_0(U)$ preserving L, and a^2 generates the subgroup G_0 of G_0' fixing L^*/L.

For $\mu \in L^* \bmod L$, we recall the Hecke indefinite modular form $\theta_{L,\mu}(\tau)$ defined in Section 5.3. We proceed to show that $\eta(\tau)^3 c_\lambda^\Lambda(\tau)$ is one of the $\theta_{L,\mu}(\tau)$.

Set $U^+ = \{u \in U \mid B(u) > 0\}$. Then $F := \{(x, y) \in \mathbb{R}^2 \mid -|x| < y \leqslant |x|\}$ is a fundamental domain for G_0' on U^+, and $F \cup a(F)$ is a fundamental domain for G_0 on U^+.

LEMMA 5.14. *Let $\mathfrak{g}$ be of type $A_1^{(1)}$, and let $\beta = n_0\alpha_0 + n_1\alpha_1 \in Q$ be such that either $n_0 \leqslant 0$ or $n_1 \leqslant 0$. Then:*

$$\text{(a)}\quad S(\beta) = \sum_{k \geqslant 0} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)},$$

$$\text{(b)}\quad S(\beta) = -\sum_{k<0} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)}.$$

Proof. Since $t = t_{(1/2)\alpha}$, (1.7) gives

$$n_0(t^k(\beta+\rho)-\rho) = (k+1)n_0 - kn_1 - k(k+1)/2.$$

In particular, $n_0(t^k(\beta+\rho)-\rho) = n_0(t^{k'}(\beta+\rho)-\rho)$ if $k + k' = 2n_0 - 2n_1 - 1$, so that

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)} = 0.$$

Hence, (a) and (b) of the lemma are equivalent.

Since $n_0 \leqslant 0$ or $n_1 \leqslant 0$, either $n_0 \leqslant 0$ and $n_0 \leqslant n_1$, or else $n_1 \leqslant 0$ and $n_1 \leqslant n_0$. If $n_0 \leqslant 0$ and $n_0 \leqslant n_1$, then $(k+1)n_0 - kn_1 - k(k+1)/2 \leqslant 0$ for $k \in \mathbb{Z}_+$, and Proposition 5.9(a) yields

$$\begin{aligned} S(\beta) &= \varphi(q)^3 \sum_{s\geqslant 0} K(\beta + s\delta) q^s \\ &= \varphi(q)^3 \sum_{s\geqslant 0}\sum_{k\geqslant 0} (-1)^k p^{(3)}\left((k+1)n_0 - kn_1 - \frac{k(k+1)}{2} + s\right) q^s \end{aligned}$$

$$= \sum_{k \geqslant 0} (-1)^k \varphi(q)^3 \sum_{s \geqslant 0} p^{(3)} \left((k+1)\, n_0 - kn_1 - \frac{k(k+1)}{2} + s\right) q^s$$

$$= \sum_{k \geqslant 0} (-1)^k q^{-(k+1)n_0 + kn_1 + k(k+1)/2}$$

$$= \sum_{k \geqslant 0} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)}.$$

This proves the lemma for $n_0 \leqslant 0$ and $n_0 \leqslant n_1$. If $n_1 \leqslant 0$ and $n_1 \leqslant n_0$, a similar argument using Proposition 5.9(b) gives the proof. ∎

Let $\Lambda \in P_+$, $m = \Lambda(c) > 0$, and let $\lambda \in P_+$, $\lambda \in \max(\Lambda)$. Then by (5.11), we have

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = q^{1/8 + s_\Lambda(\lambda)} \Big\{ \sum_{n \geqslant 0} S(t^{2n}(\Lambda+\rho) - (\lambda+\rho))$$
$$+ \sum_{n<0} S(t^{2n}(\Lambda+\rho) - (\lambda+\rho))$$
$$- \sum_{n>0} S(t^{2n}s(\Lambda+\rho) - (\lambda+\rho))$$
$$- \sum_{n \leqslant 0} S(t^{2n}s(\Lambda+\rho) - (\lambda+\rho)) \Big\}.$$

To the first and third sums we apply Lemma 5.14(a), and to the second and fourth sums we apply Lemma 5.14(b). This is allowed by the fact that $w(\Lambda+\rho) - (\lambda+\rho) - \delta \notin Q_+$ for all $w \in W$, which follows from Proposition 2.12(b). Then using formula (5.13), we obtain an *absolutely convergent* expansion:

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \Big(\sum_{\substack{k \geqslant 0 \\ n \geqslant 0}} - \sum_{\substack{k<0 \\ n<0}} \Big) (-1)^k$$
$$\times q^{(1/2)B(((k/2+n)\alpha + (m+2)^{-1}(\bar{\Lambda}+\bar{\rho}); (k/2)\alpha + m^{-1}\bar{\lambda}))}$$
$$- \Big(\sum_{\substack{k \geqslant 0 \\ n>0}} - \sum_{\substack{k<0 \\ n \leqslant 0}} \Big) (-1)^k$$
$$\times q^{(1/2)B(((k/2+n)\alpha - (m+2)^{-1}(\bar{\Lambda}+\bar{\rho}); (k/2)\alpha + m^{-1}\bar{\lambda}))}.$$

Apply $B((\gamma;\gamma')) = B((-\gamma;\gamma'))$ to the second summand, and combine to obtain

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \sum_{\substack{k,n \in (1/2)\mathbf{Z} \\ k \equiv n \bmod \mathbf{Z} \\ k \geqslant |n| \text{ or } -k > |n|}} (-1)^{2k} \operatorname{sign}(k + \tfrac{1}{4})$$
$$\times q^{(1/2)B((k\alpha + (m+2)^{-1}(\bar{\Lambda}+\bar{\rho}); n\alpha + m^{-1}\bar{\lambda}))}.$$

Break the sum into two parts, according as k is integral or half-integral, and to the second apply $B((\gamma;\gamma'))=B((-\gamma;\gamma'))$, obtaining

$$\eta(\tau)^3\, c^{\Lambda}_{\lambda}(\tau)=\sum_{\substack{k,n\in\mathbb{Z}\\ k\geqslant |n| \text{ or } -k>|n|}} \operatorname{sign}(k+\tfrac{1}{4})\times q^{(1/2)B((k\alpha+(m+2)^{-1}(\bar{\Lambda}+\bar{\rho});n\alpha+m^{-1}\bar{\lambda}))}$$

$$+\sum_{\substack{k,n\in\mathbb{Z}+1/2\\ k>|n| \text{ or } -k\geqslant |n|}} \operatorname{sign}(k-\tfrac{1}{4})\times q^{(1/2)B((k\alpha-(m+2)^{-1}(\bar{\Lambda}+\bar{\rho});n\alpha+m^{-1}\bar{\lambda}))}.$$

Write $(m+2)^{-1}(\bar{\Lambda}+\bar{\rho})=A\alpha$, $m^{-1}\bar{\lambda}=B\alpha$. Then $\frac{1}{2}>A>0$ and $\frac{1}{2}\geqslant B\geqslant 0$. Using these, and assuming $A\geqslant B$, we may combine the sums above to obtain

$$\eta(\tau)^3\, c^{\Lambda}_{\lambda}(\tau)=\sum \operatorname{sign}(x)\, q^{(1/2)B((x,y))},$$

where the sum ranges over

$$\{(x,y)\in F \mid (x,y)\equiv(A,B) \text{ or } (\tfrac{1}{2}-A,\tfrac{1}{2}+B) \bmod \mathbb{Z}^2\},$$

or equivalently, over

$$\{(x,y)\in F \mid (x,y)\equiv(A,B) \text{ or } a((A,B)) \bmod \mathbb{Z}^2\}.$$

Since $F\cup a(F)$ is a fundamental domain for G_0, we have obtained that

$$\eta(\tau)^3\, c^{\Lambda}_{\lambda}(\tau)=\theta_{L,((m+2)^{-1}(\bar{\Lambda}+\bar{\rho});m^{-1}\bar{\lambda})}$$

is a Hecke indefinite modular form.

It remains to remove the restriction $A\geqslant B$. Write $(m+2)^{-1}(\overline{\sigma(\Lambda)+\rho})=A'\alpha$, $m^{-1}\overline{\sigma(\lambda)}=B'\alpha$. Then $A+A'=B+B'=\frac{1}{2}$. Hence, if $A<B$, then $A'>B'$ and we have

$$\begin{aligned}\eta(\tau)^3\, c^{\Lambda}_{\lambda}(\tau)&=\eta(\tau)^3\, c^{\sigma(\Lambda)}_{\sigma(\lambda)}(\tau)\\ &=\theta_{L,(A'\alpha;B'\alpha)}(\tau)\\ &=\theta_{L,(\bar{\rho}-A\alpha;\bar{\rho}-B\alpha)}(\tau)\\ &=\theta_{L,(\bar{\rho}-A\alpha;-\bar{\rho}+B\alpha)}(\tau)\\ &=\theta_{L,a(A\alpha;B\alpha)}(\tau)\\ &=\theta_{L,(A\alpha;B\alpha)}(\tau).\end{aligned}$$

Thus $\eta(\tau)^3\, c^{\Lambda}_{\lambda}(\tau)=\theta_{L,((m+2)^{-1}(\bar{\Lambda}+\bar{\rho});m^{-1}\bar{\lambda})}(\tau)$ in all cases, and we have proved:

THEOREM D. *Let* $\mathfrak{g}$ *be of type* $A_1^{(1)}$. *Let* $\Lambda \in P_+$, $\Lambda(c) = m > 0$, *and let* $\lambda \in P_+$ *be a maximal weight of* $L(\Lambda)$. *Then*

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \theta_{L,((m+2)^{-1}(\bar{\Lambda}+\bar{\rho});m^{-1}\bar{\lambda})}(\tau)$$

is a Hecke indefinite modular form.

Remark. By Theorem D, the c_λ^Λ are closely related to the real quadratic fields $\mathbb{Q}(\sqrt{m(m+2)})$. Note that every real quadratic field is of this form.

5.5. *Some Applications*

In this final section we display various identities for modular forms and elliptic theta functions. Some of these formulas are classical, and many seem to be new. All of them appear naturally in the framework of the representation theory of the simplest affine Lie algebra $A_1^{(1)}$ and are very special cases of our general theory. This section can be read independently of the rest of the paper. As usual, q stands for $e^{2\pi i\tau}$ where $\operatorname{Im}\tau > 0$.

(a) Fix a positive integer m. For integers N and n with $N \equiv n \bmod 2$, put

$$\theta_n^N(\tau) = \sum_{\substack{(x,y)\in\mathbb{R}^2 \\ -|x| < y \leqslant |x| \\ (x,y) \text{ or } (1/2-x,1/2+y)\in((N+1)/2(m+2),n/2m)+\mathbb{Z}^2}} (\operatorname{sign} x)\, q^{(m+2)x^2 - my^2},$$

$$d_N^n(\tau) = \sum_{\substack{k\in\mathbb{Z} \\ k\equiv \pm n \bmod 2m}} (-1)^{(1/2)(k+N)}\, q^{(m(m+2)/8)(k/m+(N+1)/(m+2))^2}.$$

By Theorem A(4) and (5.15) below, the $\theta_n^N(\tau)$ are cusp forms for $\Gamma(4m)\cap\Gamma(4(m+2))$ with the trivial multiplier system. The $\theta_n^N(\tau)$ appear (in a different form) in Theorem D, which says that for $0 \leqslant N \leqslant m$ and $n \equiv N \bmod 2$,

$$c_{(m-n)\Lambda_0+n\Lambda_1}^{(m-N)\Lambda_0+N\Lambda_1}(\tau) = \eta(\tau)^{-3}\,\theta_n^N(\tau). \tag{5.15}$$

Here $\eta(\tau) = q^{1/24}\varphi(q)$, where $\varphi(q) = \prod_{k=1}^\infty (1-q^k)$.

From Propositions 2.12(b) and 2.19 and Theorem B we deduce that for $0 \leqslant n$, $N \leqslant m$, $n \equiv N \bmod 2$, we have

$$\eta(\tau)^{-3}\,\theta_n^N(\tau) = q^b(1 + b_1 q + b_2 q^2 + \cdots), \tag{5.16}$$

where

$$b = -\frac{1}{8} + \frac{(N+1)^2}{4(m+2)} - \frac{n^2}{4m} + \max\left(0, \frac{1}{2}(n-N)\right)$$

and

$$1 + b_1 q + b_2 q^2 + \cdots = \varphi(q)^{-1}(1 + c_1 q + \cdots), \tag{5.17}$$

where $c_i \geqslant 0$; moreover,

$$b_k \sim \frac{\sin \pi(N+1)/(m+2)}{2(m+2)k} e^{\pi \sqrt{(2m/(m+2))k}} \tag{5.18}$$

for $k \to \infty$.

Comparing (5.15) with the computation in Section 4.6 of string functions of low level, we derive a number of identities. Among the prettiest are (from $m=1$, $\theta_0^0(12\tau)$; $m=2$, $\theta_1^1(8\tau)$; $m=4$, $\theta_3^3(48\tau)-\theta_1^3(48\tau)$; $m=8$, $\theta_2^4(2\tau)$, respectively):

$$\eta(12\tau)^2 = \sum_{\substack{k,l \in \mathbb{Z} \\ k \geqslant 2|l|}} (-1)^{k+l} q^{[3(2k+1)^2-(6l+1)^2]/2}, \tag{5.19}$$

$$\eta(8\tau)\,\eta(16\tau) = \sum_{\substack{k,l \in \mathbb{Z} \\ k \geqslant 3|l|}} (-1)^k q^{(2k+1)^2-32l^2}, \tag{5.20}$$

$$\eta(24)\,\eta(96\tau) = \sum_{\substack{k,l \in \mathbb{Z} \\ 2k \geqslant l \geqslant 0}} (-1)^{l(l+1)/2} q^{8(3k+1)^2-3(2l+1)^2}(1-q^{24(2k+1)}), \tag{5.21}$$

$$\eta(4\tau)\,\eta(20\tau) = \sum_{\substack{k,l \in \mathbb{Z} \\ 2k \geqslant l \geqslant 0}} (-1)^k q^{[5(2k+1)^2-(2l+1)^2]/4}. \tag{5.22}$$

Identity (5.19) appears (in a different form) in Hecke [9]. Arithmetic properties of $\eta(8\tau)\,\eta(16\tau)$ were studied in detail in [32].

For $\varepsilon = 0$ or 1, introduce the matrices $C_\varepsilon = (\theta_n^N)$ and $D_\varepsilon = (d_N^n)$, where n, $N \in \{k \in \mathbb{Z} \mid k \equiv \varepsilon \bmod 2,\ 0 \leqslant k \leqslant m\}$. Then:

$$\begin{aligned} &\det C_0 = \det C_1 = \eta(\tau)^{m+1} && \text{if } m \text{ is odd;} \\ &\det C_0 = \eta(\tau)^{m+3}\,\eta(2\tau)^{-1} && \text{and} \\ &\det C_1 = \eta(\tau)^{m-1}\,\eta(2\tau) && \text{if } m \text{ is even.} \end{aligned} \tag{5.23}$$

$$C_0 D_0 = D_0 C_0 = \eta(\tau)^3 I; \qquad C_1 D_1 = D_1 C_1 = \eta(\tau)^3 I. \tag{5.24}$$

These formulas may be easily deduced from (5.15) and the results of Section 4.5 (see [36] for details). Note that (5.19) and (5.20) are the simplest cases of (5.23).

Introduce the following elliptic theta functions:

$$\Theta_{n,m}(\tau, z) = \sum_{k \in \mathbb{Z}+n/2m} q^{mk^2} e^{-2\pi i m k z},$$

$$A_{n,m}(\tau, z) = \Theta_{n,m}(\tau, z) - \Theta_{-n,m}(\tau, z) = -2i \sum_{k \in \mathbb{Z}+n/2m} q^{mk^2} \sin 2\pi m k z.$$

We have the following division formula:

$$\frac{A_{N+1,m+2}(\tau, z)}{A_{1,2}(\tau, z)} = \eta(\tau)^{-3} \sum_{\substack{0 \leqslant n < 2m \\ n \equiv N \bmod 2}} \theta_n^N(\tau)\, \Theta_{n,m}(\tau, z). \tag{5.25}$$

This formula is (by (5.15)) a special case of the "theta function identity" (2.18). Indeed, we have in the notation of Section 4.1:

$$\Theta_{n,m}(\tau, z) = \Theta_{m\Lambda_0 + n\bar{\rho}}(\tau, z\bar{\rho}, 0);$$
$$A_{n,m}(\tau, z) = A_{m\Lambda_0 + n\bar{\rho}}(\tau, z\bar{\rho}, 0).$$

Note also that the (particularly important) function $A_\rho(\tau, z\bar{\rho}, 0) = A_{1,2}(\tau, z)$ is, up to a constant factor, the Jacobi elliptic theta function $\vartheta_1(\tau, z)$:

$$A_\rho(\tau, z\bar{\rho}, 0) = -i\vartheta_1(\tau, z) = -2i \sum_{k=0}^{\infty} (-1)^k e^{\pi i \tau (k+1/2)^2} \sin((2k+1)\pi z).$$

(b) It is easy to see that Proposition 5.9 is equivalent to the following identity:

$$\prod_{k \geqslant 0} (1 - q^k z)^{-1}(1 - q^{k+1} z^{-1})^{-1} = \varphi(q)^{-2} \sum_{k \in \mathbb{Z}} (-1)^k \frac{q^{(1/2)k(k+1)}}{1 - q^k z}. \tag{5.26}$$

This expansion is valid for $|q| < 1$ whenever both sides are defined. Note that (5.26) is precisely the partial fraction decomposition in z of the left-hand side for fixed q. This expansion appears, in a different form, in [40, Section 486]; we thank G. Andrews for pointing out this reference to us.

References

1. G. E. Andrews, The theory of partitions, *in* "Encyclopedia of Mathematics," Vol. 2, 1976.
2. I. Bernstein and O. Schwartzman, Chevalley theorem for complex crystallographic Coxeter groups, *Functional Anal. Appl.* **12** (1978).
3. N. Bourbaki, "Groupes et algèbres de Lie," Chaps. I–III, 1972; Chaps. IV–VI, 1968; Hermann, Paris.
4. J. H. Conway and S. P. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308–339.
5. M. Eichler, "Introduction to the Theory of Algebraic Numbers and Functions," Academic Press, New York, 1966.
6. A. Feingold and J. Lepowsky, The Weyl–Kač character formula and power series identities, *Adv. in Math.* **29** (1978), 271–309.
7. I. B. Frenkel and V. G. Kač, Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.* **62** (1980), 23–66.
8. O. Gabber and V. G. Kač, On defining relations of certain infinite-dimensional Lie algebras, *Bull. Amer. Math. Soc.* **5** (1981), 185–189.
9. E. Hecke, Über einen neuen Zusammenhang zwischen elliptischen Modulfunktionen und

indefiniten quadratischen Formen, *in* "Mathematische Werke," pp. 418–427, Vandenhoeck & Ruprecht, Göttingen, 1959.
10. S. HELGASON, "Differential Geometry, Lie Groups and Symmetric Spaces," Academic Press, New York, 1980.
11. J. E. HUMPHREYS, "Introduction to Lie Algebras and Representation Theory," Springer-Verlag, New York/Berlin, 1972.
12. J. IGUSA, "Theta Functions," Springer-Verlag, New York/Berlin, 1972.
13. C. G. J. JACOBI, Fundamenta nova theoriae functionum ellipticarum (1829), *in* "Gesammelte Werke" (G. Reimer, Ed.), Vol. 1, pp. 49–239, (1881).
14. V. G. KAČ, Simple irreducible graded Lie algebras of finite growth, *Math. USSR-Izv.* **2** (1968), 1271–1311.
15. V. G. KAČ, Automorphisms of finite order of semisimple Lie algebras, *J. Funct. Anal. Appl.* **3** (1969), 252–254.
16. V. G. KAČ, Infinite-dimensional Lie algebras and Dedekind's η-function, *J. Funct. Anal. Appl.* **8** (1974), 68–70.
17. V. G. KAČ, Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula, *Adv. in Math.* **30** (1978), 85–136.
18. V. G. KAČ, An elucidation of "Infinite-dimensional algebras and the very strange formula." $E_8^{(1)}$ and the cube root of the modular invariant j, *Adv. in Math.* **35** (1980), 264–273.
19. V. G. KAČ, Infinite root systems, representations of graphs, and invariant theory, *Invent. Math.* **56** (1980), 57–92.
20. V. G. KAČ, On simplicity of certain infinite-dimensional Lie algebras, *Bull. Amer. Math. Soc.* **2** (1980), 311–314.
21. V. G. KAČ, Simple Lie groups and the Legendre symbol, *in* "Lecture Notes in Mathematics No. 848," pp. 110–123, Springer-Verlag, New York/Berlin, 1981.
22. V. G. KAČ AND D. H. PETERSON, Affine Lie algebras and Hecke modular forms, *Bull. Amer. Math. Soc.* **3** (1980), 1057–1061.
23. V. G. KAČ AND D. H. PETERSON, Spin and wedge representations of infinite-dimensional Lie algebras and groups, *Proc. Nat. Acad. Sci. U.S.A.* **78** (1981), 3308–3312.
24. M. KNOPP, "Modular Functions in Analytic Number Theory," Markham, Chicago, 1970.
25. B. KOSTANT, On Macdonald's η-function formula, the Laplacian and generalized exponents, *Adv. in Math.* **20** (1976), 179–212.
26. G. LION AND M. VERGNE, "The Weil representation, Maslov index and theta series," Birkhäuser, Basel, 1980.
27. E. LOOIJENGA, Root systems and elliptic curves, *Invent. Math.* **38** (1976), 17–32.
28. E. LOOIJENGA, Invariant theory for generalized root systems, *Invent. Math.* **61** (1980), 1–32.
29. I. MACDONALD, Affine root systems and Dedekind's η-function, *Invent. Math.* **15** (1972), 91–143.
30. J. MILNOR AND D. HUSEMOLLER, "Symmetric Bilinear Forms," Springer-Verlag, New York/Berlin, 1973.
31. R. V. MOODY, A new class of Lie algebras, *J. Algebra* **10** (1968), 211–230.
32. C. MORENO, The higher reciprocity laws: An example, *J. Number Theory* **12** (1980), 57–70.
33. D. MUMFORD, "Tata Lectures on Theta," Birkhäuser, Boston, 1982.
34. D. H. PETERSON, Kostant-type partition functions, to appear.
35. D. H. PETERSON, On independence of fundamental characters of certain infinite-dimensional groups, to appear.
36. D. H. PETERSON, An infinite class of identities connecting definite and indefinite quadratic forms, to appear.

37. B. Riemann, Theorie der Abelschen Functionen, *J. Reine Angew. Math.* **54** (1857), 115–155.
38. J.-P. Serre, Modular forms of weight one and Galois representations, *in* "Algebraic Number Fields (*L*-functions and Galois properties)," Symp. Durham, 1977.
39. J.-P. Serre, "Cours d'arithmetique," Presses Universitaires de France, Paris, 1970.
40. J. Tannery and J. Molk, "Éléments de la théorie des fonctions elliptiques," Paris, 1898.
41. E. B. Vinberg, Discrete linear groups generated by reflections, *Math. USSR-Izv.* **5** (1971), 1083–1119.
42. H. D. Fegan, The heat equation and modular forms, *J. Differential Geom.* **13** (1978), 589–602.
43. A. E. Ingham, A Tauberian theorem for partitions, *Ann. of Math.* **42** (1941), 1075–1090.
44. J. Lepowsky, Generalized Verma modules, loop space cohomology and Macdonald-type identities, *Ann. Sci. École Norm. Sup.* **12** (1979), 169–234.
45. I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and flag space cohomologies, *Funct. Anal. Appl.* **7** (1973).
46. D.-N. Verma, The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, *in* "Lie Groups and Their Representations," Wiley, New York/Toronto, 1971.
47. I. B. Frenkel, Spinor representations of affine Lie algebras, *Proc. Nat. Acad. Sci. U.S.A.* **77** (1980), 6303–6306.
48. I. B. Frenkel, Representations of affine Lie algebras, Hecke modular forms and Korteweg–de Vries type equations, *in* "Lecture Notes in Mathematics No. 933," Springer-Verlag, New York/Berlin, 1982.
49. D. H. Peterson, Level one modules of affine Lie algebras, to appear.
50. V. G. Kac, "Infinite Dimensional Lie Algebras," Birkhäuser, Boston, 1983.

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