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PRONY ESTIMATION OF AR PARAMETERS OF AN ARMA TIME SERIES

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The auto-covariance function of a white noise excited time series can be decomposed into the contributions of different modes, therefore having the same structure as that of the impulse response of a deterministic system. By matching the auto-covariance of the data with that of the ARMA model, the estimation of the characteristic roots of the system and the dispersion coefficients can be implemented using the Prony method, therefore the estimation of the AR parameters becomes a linear least squares problem. It is found that this estimate for the AR parameters of an ARMA model is identical to the asymptotically unbiased estimate using modified Yule-Walker equation for ARMA (n, $n-1$).

1. INTRODUCTION

The mixed autoregressive moving average (ARMA) model can be used to represent the dynamics of a mechanical system. It was shown by Bartlett [1] that the sampling of a continuous AR process of order n results in a discrete ARMA process of order n, $n-1$, i.e. ARMA (n, $n-1$). The example originally examined was the dynamics of a Brownian motion excited pendulum using discrete observations of the continuous system. Bartlett showed that the discrete time series that is covariance equivalent to the system

$$\frac{d^2x(t)}{dt^2}+2\zeta\omega_n\frac{dx(t)}{dt}+\omega_n^2=a(t)$$

is

$$x_t=\phi_1x_{t-1}+\phi_2x_{t-2}+a_t-\theta_1a_{t-1}$$

where a_t is a zero mean uncorrelated series. ϕ_i's are called the autoregressive (AR) parameters, and θ_i's the moving average (MA) parameter.

Generally, an ARMA (n, m) model is of the form

$$x_t=\phi_1x_{t-1}+\phi_2x_{t-2+}+\cdots+\phi_nx_{t-n}+a_t-\theta_1a_{t-1}-\cdots-\theta_ma_{t-m} \tag{1}$$

1.1. REPRESENTATION OF DYNAMICS

For the ARMA (n, m) model given in equation (1), its theoretical auto-covariance function can be expressed as [2]

$$\gamma_k=d_1\lambda_1^k+d_2\lambda_2^k+\cdots+d_n\lambda_n^k \tag{2}$$

where λ_i's are the characteristic roots of

$$1-\phi_1B-\cdots-\phi_nB^n=0 \tag{3}$$

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or

$$(1-\lambda_1 B)(1-\lambda_2 B)\cdots(1-\lambda_n B)=0 \tag{4}$$

and the d_i represents the contribution of the ith dynamic mode of the system λ_i to the variance γ_0, and $d_i/\gamma_0 \times 100\%$ is called the dispersion coefficient.

The estimate for the auto-covariance function is given by

$$\hat{\gamma}_k = \frac{1}{N} \sum_{t=k+1}^{N} (x_t - \bar{x})(x_{t-k} - \bar{x}) \tag{5}$$

where

$$\bar{x} = \frac{1}{N} \sum_{t=1}^{N} x_i$$

and N is the total number of data.

There are, of course, other methods of estimating the auto-covariance, but the estimator equation (5) has the attractive properties that its mean squared error is generally smaller than that of other estimators [3].

1.2. MODIFIED YULE-WALKER EQUATION

The estimation of the parameters of the ARMA model has been an active field in recent years. The maximum likelihood method provides good estimate for parameters of the ARMA model [4]. However, because of the severe non-linearity and local minimum problem, many sub-optimal estimation procedures have been suggested [5].

Gersch [6] showed that the modified Yule-Walker equation provides the asymptotically unbiased estimate for the AR parameters of an ARMA model. The estimator for the AR parameters of ARMA (n, m) model using modified Yule-Walker equation is given by

$$\begin{bmatrix} \hat{\gamma}_m & \hat{\gamma}_{m-1} & \cdots & \hat{\gamma}_{m-n+1} \\ \hat{\gamma}_{m+1} & \hat{\gamma}_m & \cdots & \hat{\gamma}_{m-n+2} \\ \vdots & & & \\ \hat{\gamma}_{m+n-1} & \hat{\gamma}_{m+n-2} & \cdots & \hat{\gamma}_m \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{Bmatrix} = \begin{Bmatrix} \hat{\gamma}_{m+1} \\ \hat{\gamma}_{m+2} \\ \vdots \\ \hat{\gamma}_{m+n} \end{Bmatrix}. \tag{6}$$

1.3. THE PRONY METHOD

The Prony method [7], a technique for modeling data of equally spaced samples by a linear combination of exponentials assumes the discrete time function of the form

$$z_k = \sum_{j=1}^{n} b_j z_j^k \tag{7}$$

for $k = 0, 1, \ldots, n-1$. It has found most of its application in transient analysis, such as finding resonant modes in experimental modal analysis. It has also been used to perform harmonic decomposition [5]. However, it has not been used for direct parameter estimation of an ARMA model.

In this paper, we develop the estimation for the AR parameters of an ARMA model using the Prony method, and show that the resulting estimate of the AR parameters are identical to that from modified Yule-Walker equation for ARMA $(n, n-1)$ model.

2. THE ESTIMATION PROBLEM

The model equation (7) assumed by the Prony method has the same structure as that of the auto-covariance function of the ARMA (n, m) model shown in equation (2). Equating the auto-covariance of the data at different lag with that of the model, we obtain the following system of equations:

$$\left.\begin{aligned}\hat{\gamma}_0 &= d_1 + d_2 + \cdots + d_n \\ \hat{\gamma}_1 &= d_1\lambda_1 + d_2\lambda_2 + \cdots + d_n\lambda_n \\ &\vdots \\ \hat{\gamma}_{2n-1} &= d_1\lambda_1^{2n-1} + d_2\lambda_2^{2n-1} + \cdots + d_n\lambda_n^{2n-1}\end{aligned}\right\}. \tag{8}$$

Multiplying both sides of the first equation of (8) by β_0, the second equation of (8) by $\beta_1, \ldots$, and the $n+1$th equation of (8) by β_n, we have

$$\left.\begin{aligned}\beta_0\hat{\gamma}_0 &= \beta_0 d_1 + \beta_0 d_2 + \cdots + \beta_0 d_n \\ \beta_1\hat{\gamma}_1 &= \beta_1 d_1\lambda_1 + \beta_1 d_2\lambda_2 + \cdots + \beta_1 d_n\lambda_n \\ &\vdots \\ \beta_n\hat{\gamma}_n &= \beta_n d_1\lambda_1^n + \beta_n d_2\lambda_2^n + \cdots + \beta_n d_n\lambda_n^n\end{aligned}\right\}. \tag{9}$$

Adding up the equations, we have

$$\sum_{j=0}^{n} \beta_j\hat{\gamma}_j = \sum_{i=1}^{n} d_i\left(\sum_{j=0}^{n} \beta_j\lambda_i^j\right). \tag{10}$$

The β_i's can be chosen as the coefficients of the following equation

$$\beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_n\lambda^n = 0 \tag{11}$$

with $\beta_n = 1$ and the roots $\lambda_1, \ldots, \lambda_n$.

The right-hand side of equation (10) is zero for any λ_i, and then equation (10) becomes

$$\sum_{j=0}^{n} \beta_j\hat{\gamma}_j = 0$$

i.e.

$$\beta_0\hat{\gamma}_0 + \beta_1\hat{\gamma}_1 + \cdots + \beta_{n-1}\hat{\gamma}_{n-1} = -\hat{\gamma}_n. \tag{12}$$

Repeating the process from equation (9) to equation (12) in a similar way, multiplying both sides of the second equation of (8) by β_0, the third equation of (8) by $\beta_1, \ldots$, and the $n+2$th equation of (8) by β_n, we have,

$$\beta_0\hat{\gamma}_1 + \beta_1\hat{\gamma}_2 + \cdots + \beta_{n-1}\hat{\gamma}_n = -\hat{\gamma}_{n+1}.$$

After repeating the above process n times, we have the following system of equations

$$\left.\begin{aligned}\beta_0\hat{\gamma}_0 + \beta_1\hat{\gamma}_1 + \cdots + \beta_{n-1}\hat{\gamma}_{n-1} &= -\hat{\gamma}_n \\ \beta_0\hat{\gamma}_1 + \beta_1\hat{\gamma}_2 + \cdots + \beta_{n-1}\hat{\gamma}_n &= -\hat{\gamma}_{n+1} \\ &\vdots \\ \beta_0\hat{\gamma}_{n-1} + \beta_1\hat{\gamma}_n + \cdots + \beta_{n-1}\hat{\gamma}_{2n-2} &= -\hat{\gamma}_{2n-1}\end{aligned}\right\}. \tag{13}$$

This system of equations can be solved using a linear least squares routine.

$$\hat{\Theta} = (X^TX)^{-1}(X^TY) \tag{14}$$

where "^{T}" denotes the transpose of a matrix,

$$X = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{n-1} \\ \hat{\gamma}_1 & \hat{\gamma}_2 & \cdots & \hat{\gamma}_n \\ \vdots & & & \\ \hat{\gamma}_{n-1} & \hat{\gamma}_n & \cdots & \hat{\gamma}_{2n-2} \end{bmatrix}.$$

$$Y = \begin{Bmatrix} -\hat{\gamma}_n \\ -\hat{\gamma}_{n+1} \\ \vdots \\ -\hat{\gamma}_{2n-1} \end{Bmatrix}$$

and Θ is the parameter vector,

$$\Theta = \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{Bmatrix}.$$

After identifying the parameters β_i's, the characteristic root λ_i's of the system can be calculated by solving the polynomial equation (11).

$$\beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1} + \lambda^n = 0.$$

From equation (8), we have

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} = \begin{Bmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_{n-1} \end{Bmatrix}. \tag{15}$$

Again, the above equation can be solved for d_i's using a linear least squares routine.

2.1. THE MEANING OF β_i

For an ARMA (n, m) model, the characteristic equation becomes

$$1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_n B^n = 0 \tag{16}$$

or

$$\lambda_i^n - \phi_1 \lambda_i^{n-1} - \cdots - \phi_{n-1}\lambda_i - \phi_n = 0. \tag{17}$$

Compare equations (17) and (11), we have,

$$\left.\begin{aligned} \phi_1 &= -\beta_{n-1} \\ \phi_2 &= -\beta_{n-2} \\ &\vdots \\ \phi_n &= -\beta_0 \end{aligned}\right\}. \tag{18}$$

That is, the least square estimate from equation (14) is directly related to the AR parameters of the ARMA model. The estimation of the AR parameters of the ARMA model becomes a linear least square problem.

Rewrite equation (13) using ϕ_i's instead of β_i's, we have

$$\begin{bmatrix} \hat{\gamma}_{n-1} & \hat{\gamma}_{n-2} & \cdots & \hat{\gamma}_0 \\ \hat{\gamma}_n & \hat{\gamma}_{n-1} & \cdots & \hat{\gamma}_1 \\ \vdots & & & \\ \hat{\gamma}_{2n-2} & \hat{\gamma}_{2n-3} & \cdots & \hat{\gamma}_{n-1} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{Bmatrix} = \begin{Bmatrix} \hat{\gamma} \\ \hat{\gamma}_{n+1} \\ \vdots \\ \hat{\gamma}_{2n-1} \end{Bmatrix}. \tag{19}$$

Comparing equation (6) with equation (19), we can see that equation (19) is identical to the asymptotically unbiased estimate using the modified Yule–Walker equation for the AR parameters of an ARMA $(n, n-1)$ model.

3. CONCLUSIONS

This paper provides another interpretation for the modified Yule–Walker estimation for the AR parameters of the ARMA model-Prony estimation using autocovariance function. The auto-covariance function of an autoregressive moving average time series is decomposed into the contributions of different modes. The estimation of the AR parameters of an ARMA model can be implemented using the Prony method, and this estimate for the AR parameters is identical to the asymptotically unbiased estimate of the AR parameters of an ARMA $(n, n-1)$ model using modified Yule–Walker equation.

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