Int. J. Mech. Sci. Vol. 32, No. 2, pp. 129–132, 1990
Printed in Great Britain.

0020-7403/90 \$3.00+.00

CONTACT PROBLEMS FOR THE THIN ELASTIC LAYER

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(*Received* 13 *September* 1989)

Abstract—Simple approximate solutions due to Jaffar and Johnson for the indentation by a rigid frictionless punch of a thin elastic layer on a rigid foundation are extended to the corresponding general three-dimensional problem. Results are given for the case where an incompressible layer is indented by an ellipsoidal punch and an analogy is demonstrated between the flat punch problem with arbitrary plan-form and the St Venant torsion problem.

1. INTRODUCTION

In a recent paper, Jaffar [1] used an elementary formulation due to Johnson [2] to obtain asymptotic results for the contact pressure between a frictionless axisymmetric rigid indenter and a thin elastic layer supported by a rigid foundation. The same technique can in fact be generalized to the arbitrary, three-dimensional problem for the thin elastic layer.

Following Jaffar, we consider the three limiting cases of (1) a frictionless unbonded layer, (2) a bonded layer for a compressible material ($\nu \neq 0.5$) and (3) a bonded incompressible layer ($\nu = 0.5$).

For simplicity, we shall follow Jaffar's notation wherever possible. We denote the contact surface of the layer by $z = 0$ and define a two-dimensional Cartesian coordinate system x_1, x_2 in the plane of the layer. The suffix notation and summation convention will be taken through $i = 1, 2$ only. In-plane displacements of the layer are denoted by the two-dimensional vector $\mathbf{u}$ with components u_1, u_2.

2. FRICTIONLESS UNBONDED LAYER

For this case, Johnson's approximation is to assume that plane sections within the layer remain plane, so that $\mathbf{u}$ is independent of z.

It follows that the in-plane components of strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

are also independent of z, whilst the only non-zero strain out of the plane is $\varepsilon_{zz} = -w/t$, where w is the local indentation and t is the layer thickness.

The in-plane stress components are therefore

$$\sigma_{ij} = \lambda\delta_{ij}(\varepsilon_{kk} - w/t) + 2\mu\varepsilon_{ij} \tag{2}$$

where λ, μ are Lamé's constants for the layer material.

Substituting into the in-plane equilibrium equation,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{3}$$

and writing the strain components in terms of the displacements, we obtain

$$(\lambda + \mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\lambda}{t}\frac{\partial w}{\partial x_i}. \tag{4}$$

We seek a particular solution of this equation in the form

$$u_i = \frac{\partial \phi}{\partial x_i}, \tag{5}$$

where ϕ is a scalar two-dimensional potential.

The governing equation for ϕ is then obtained by substituting (5) into (4), with the result

$$\frac{\partial^3 \phi}{\partial x_i \partial x_k \partial x_k} = \frac{\lambda}{(\lambda + 2\mu)t} \frac{\partial w}{\partial x_i} \tag{6}$$

a sufficiently general solution of which is

$$\frac{\partial^2 \phi}{\partial x_k \partial x_k} \equiv \nabla^2 \phi = \frac{\lambda w}{(\lambda + 2\mu)t}. \tag{7}$$

Finally, we recover the contact pressure $p(x_1, x_2)$ as

$$\begin{aligned} p(x_1, x_2) &= -\sigma_{zz} = -\lambda(\varepsilon_{kk} - w/t) + 2\mu w/t \\ &= -\lambda \nabla^2 \phi + (\lambda + 2\mu) w/t \\ &= \frac{2\mu w(x_1, x_2)}{(1-\nu)t} \end{aligned} \tag{8}$$

from equations (1, 5, 7).

The indentation w is the local interpenetration between the indenter and the layer in the undeformed state. Thus, if we assume that the contact area is identical with the indentation area (i.e. with the area in which $w > 0$), and if the indenter is smooth, we shall find that $w \to 0$ at the edge of the contact area and hence equation (8) defines a pressure which satisfies $p \to 0$ at the edge of the contact area. This therefore constitutes the general solution of the unbonded contact problem, despite the fact that it was developed as a particular solution through the assumption (5).

3. BONDED COMPRESSIBLE LAYER

For this case, Johnson's approximation involves the assumption $\varepsilon_{ij} = 0$. In other words, plane sections remain plane and do not move. It follows immediately that

$$\begin{aligned} p(x_1, x_2) &= -\sigma_{zz} = -(\lambda + 2\mu)\varepsilon_{zz} \\ &= (\lambda + 2\mu)\frac{w}{t} = \frac{E(1-\nu)w(x_1, x_2)}{(1+\nu)(1-2\nu)t}. \end{aligned} \tag{9}$$

4. BONDED INCOMPRESSIBLE LAYER

For this case, following Jaffar and Johnson, we approximate the in-plane displacement u_i by the quadratic expression

$$u_i = C_i(z^2 - t^2). \tag{10}$$

The equilibrium equation (3) can now only be satisfied in terms of force resultants and leads to the equation

$$\int_0^t \frac{\partial \sigma_{ij}}{\partial x_j} \mathrm{d}z + \tau_i = 0, \tag{11}$$

where τ_i is the component of shear stress in the i direction at the layer–substrate interface and is given by

$$\tau_i = \mu \left(\frac{\partial u_i}{\partial z} + \frac{\partial u_z}{\partial x_i} \right) = 2\mu t C_i \tag{12}$$

since $u_z = 0$ at the layer substrate interface.

The in-plane constitutive law for the incompressible material is

$$\sigma_{ij} = -A\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{13}$$

where $A(x_1, x_2)$ is a two-dimensional scalar potential, representing a state of hydrostatic compression.

By analogy with equation (5), we seek a solution for C as the gradient of a scalar potential ψ, i.e.

$$C_i = \frac{\partial \psi}{\partial x_i} \tag{14}$$

in which case, equations (1, 10–14) yield the governing equilibrium equation

$$-\frac{\partial A}{\partial x_i} - \frac{4\mu t^2}{3}\left(\frac{\partial^3 \psi}{\partial x_i \partial x_k \partial x_k}\right) + 2\mu \frac{\partial \psi}{\partial x_i} = 0. \tag{15}$$

Also, the incompressibility condition demands that

$$\int_0^t \varepsilon_{ii} \mathrm{d}z = -\int_0^t \varepsilon_{zz} \mathrm{d}z = w \tag{16}$$

and hence

$$-\frac{2t^3}{3}\frac{\partial C_i}{\partial x_i} = -\frac{2t^3}{3}\nabla^2 \psi = w. \tag{17}$$

We can use (17) to substitute for the second term in (15), from which we deduce that equation (15) will be satisfied if

$$A = 2\mu(\psi + w/t). \tag{18}$$

We can then recover the contact pressure as

$$p(x_1, x_2) = A + \frac{2\mu w}{t} = 2\mu\left(\psi + 2\frac{w}{t}\right)$$
$$= 2\mu\left(\psi - \frac{4t^2}{3}\nabla^2 \psi\right). \tag{19}$$

In this expression, the second term is small of order $(t/a)^2$ compared with the first, where a is a representative dimension of the contact area. It can therefore be neglected in the asymptotic solution, giving the simple expression

$$p = 2\mu\psi. \tag{20}$$

4.1. *Boundary conditions*

As before, we require that $p \to 0$ and hence $\psi \to 0$ at the edge of the contact area. However, we also must impose the condition $u_n = 0$ at the boundary and hence $\partial\psi/\partial n = 0$, where n is the normal to the boundary. This is a generalization of Johnson and Jaffar's criterion "that there should be no change in volume of the material under the indenter". This boundary condition is a consequence of the fact that the layer outside the contact region offers no restraint to lateral expansion.†

4.2. *Example*

As an example, we consider the indentation of the bonded incompressible layer by an ellipsoidal indenter, defined by the interpenetration function

$$w = v^* - Ax_1^2 - Bx_2^2. \tag{21}$$

Equation (17) shows that the function ψ must be a fourth-order polynomial in x_1, x_2 and the boundary conditions are clearly satisfied by the function

$$\psi = D\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right)^2, \tag{22}$$

where the contact area is an ellipse of semi-axes a, b.

†This is not exactly true, but it is easily demonstrated that the resistance is small in comparison with the prevailing contact pressure and hence that a more exact description of the boundary condition would involve only a second-order correction.

Substituting (21, 22) into (17) and equating polynomial coefficients, we obtain

$$v^* = \frac{8Dt^3}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

$$A = \frac{8Dt^3}{3a^2}\left(\frac{3}{a^2}+\frac{1}{b^2}\right)$$

$$B = \frac{8Dt^3}{3b^2}\left(\frac{3}{b^2}+\frac{1}{a^2}\right).$$

If v^*, A, B are given, these equations are sufficient to determine a, b, D—i.e. the semi-axes of the contact area and the multiplying constant in equation (22). The contact pressure can then be obtained from equations (20, 22).

Jaffar's results for the axisymmetric Hertz problem can be recovered by setting $A = B$ and the Johnson solution for the corresponding plane problem by setting $B = 0$.

As in the usual Hertzian indentation problem, we note that the ellipticity of the contact area differs from that of the interpretation function (21).

5. FLAT PUNCH PROBLEMS

We can also extend the above analysis to the case where the layer is indented by a flat rigid punch, in which case the contact area is determined by the plan-form of the punch.

For the compressible layer (bonded or unbonded), we must relax the condition that the pressure at the edge of contact area be zero and the resulting contact pressure will clearly be uniform throughout the contact area.

For the more interesting case of the incompressible bonded layer, we can argue as before that the surrounding material offers negligible restraint to lateral expansion, so that the hydrostatic component $A \to 0$ at the boundary. It therefore follows that the contact pressure still tends to zero at the boundary despite the sharp corner of the indenter. However, we clearly have to relax the "constant volume" assumption, since otherwise the punch would be unable to move. In practice a bulge of layer material will be produced just outside the contact region.

The problem of the bonded incompressible layer indented by a flat rigid punch of plan-form Ω therefore reduces to the determination of a function ψ satisfying the equation

$$\nabla^2\psi = -\frac{3v^*}{2t^3} \tag{23}$$

in Ω and equal to zero on the boundary of Ω, where v^* is the punch indentation.

This problem is formally equivalent to the St Venant torsion problem (see, e.g. [3]) and hence the solution to many problems can be written down. In particular, it can be shown that the total indenting force,

$$P = 2\mu \int_\Omega \psi \, d\Omega = \frac{3\mu\delta K}{4t^3}, \tag{24}$$

where K is the torsional stiffness of a bar cross-section Ω. Exact and approximate values of K for a wide variety of cross-sections are given in Table 20 of [4].

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