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ASYMPTOTICALLY RELIABLE SERIAL PRODUCTION LINES WITH A QUALITY CONTROL SYSTEM

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Abstract—A model of asymptotically reliable serial production lines with a quality control system is introduced. Its performance is analyzed in the case of a two machine, one buffer system.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Consider a manufacturing system defined by the following assumptions:

(i) The system consists of M machines, m_i, $i = 1, \dots, M$, arranged serially, and $M - 1$ buffers, B_i, separating each consecutive pair of machines, m_i and m_{i+1}.
(ii) The machines have identical cycle time T. The time axis is slotted with the slot duration T. Machines begin operating at the beginning of each time slot.
(iii) Each buffer is characterized by its capacity, N_i, $i = 1, \dots, M - 1$, where N_i is a positive integer.
(iv) Machine m_i is starved during a time slot if buffer B_{i-1} is empty at the beginning of this slot; m_i is blocked during a time slot if at the beginning of this time slot buffer B_i is full and machine m_{i+1} is either down or blocked. Machine m_1 is never starved, and machine m_M is never blocked.
(v) Machine m_i, being neither blocked nor starved, produces a part during a time slot with probability $q_i = 1 - \epsilon k_i$ and fails to do so with probability ϵk_i, $i = 1, \dots, M$, where $0 < \epsilon << 1$ and $k_i \geq 0$ is independent of ϵ. The k_i's are called the loss parameters.

Manufacturing systems defined by assumptions (i–v) are referred to as *asymptotically reliable serial production lines*. An asymptotic technique for their performance analysis and design has been developed in [1] and [2]. In many practical situations, however, serial production lines include a quality control system which identifies and discards some or all of the defective parts produced by the machines. To account for this possibility, introduce the following assumptions:

(vi) A non-defective part is made defective by machine m_i with probability ϵb_i, $i = 1, \dots, M$, where the $b_i \geq 0$ are independent of ϵ. The b_i's are referred to as the *quality loss parameters*. The defects introduced by machine m_i cannot be removed by the subsequent machines $m_{i+1}, \dots, m_M$.
(vii) A defective part can be identified as such and discarded from the system by the quality control devices, Q_i, placed between each two machines, m_i and m_{i+1}, and after the last machine. Each quality control device identifies a defective part with probability $0 \leq D_i \leq 1$, $i = 1, \dots, M$. The D_i's are referred to as the *measurement accuracy parameters*. The raw material is assumed to be free of defects.

Two different assumptions could be made with regard to the positions of the quality control devices between each two machines. Specifically,

(viii) Quality control devices are placed immediately after each machine and the defectives identified are removed from the system before being placed into the subsequent buffer. This is referred to as the *output quality control architecture*.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX (viii′) Quality control devices are placed immediately after each buffer. A discarded part is instantaneously replaced with another part from the buffer, if available. This is referred to as the *input quality control architecture.*

Manufacturing systems defined by (i–viii or viii') are called *asymptotically reliable serial production lines with a quality control system.* The performance of such lines can be described by the following characteristics:

(a) The line production rate, PR, i.e., the average, steady state number of parts produced, per cycle, by the line.
(b) The line production rate of non-defective parts, PR_{nd}, i.e., the average, steady state number of non-defective parts produced, per cycle, by the line.
(c) The line consumption rate, CR, i.e., the average, steady state number of parts utilized, per cycle, by the first machine.
(d) The line work-in-process, WIP, i.e., the average, steady state number of parts in the system.

Based on these performance characteristics, the **problem of analysis** of serial production lines with a quality control system can be formulated as follows: Given the parameters k_i, b_i, D_i, $i = 1,\dots,M$ and N_i, $i = 1,\dots,M-1$, find PR, PR_{nd}, CR and WIP as functions of these parameters.

The purpose of this paper is to give an asymptotic solution of this problem in the case of the simplest, two machine, one buffer system. Specifically, we show that the calculations of the performance characteristics of lines with a quality control system can be reduced to those for lines without the quality control devices but with the k_i's modified in compliance with the b_i's and D_i's. In addition, we show that the input and output quality control architectures are asymptotically equivalent to each other.

Although the quality control problem in manufacturing systems is an extremely popular topic (see, for instance, [3–5]), the model introduced above does not seem to have been studied in the current literature.

2. TWO MACHINES-ONE BUFFER LINES WITH QUALITY CONTROL SYSTEM

THEOREM 2.1. *The performance of production lines defined by (i–viii) with $M = 2$ can be characterized as follows:*
(a) The production rate is

$$PR = 1 - \epsilon \left[k_1 + d_1 + k_2 Q \left(\frac{k_1 + d_1}{k_2}, N_1 \right) + d_2 \right] + O(\epsilon^2), \tag{2.1}$$

where

$$Q(\alpha, N_1) = \frac{1-\alpha}{1-\alpha^{N_1}}, \qquad \alpha \geq 0, \quad N_1 \in \mathbb{Z}_+,$$
$$d_1 = b_1 D_1, \quad d_2 = [(1-D_1) b_1 + b_2] D_2. \tag{2.2}$$

(b) The production rate of non-defectives is:

$$PR_{nd} = 1 - \epsilon \left[k_1 + b_1 + k_2 Q \left(\frac{k_1 + d_1}{k_2}, N_1 \right) + b_2 \right] + O(\epsilon^2), \tag{2.3}$$

(c) The line consumption rate is:

$$CR = 1 - \epsilon \left[k_1 + k_2 Q \left(\frac{k_1 + d_1}{k_2}, N_1 \right) \right] + O(\epsilon^2), \tag{2.4}$$

(d) Finally, the work-in-process is:

$$WIP = \frac{1-(N_1+1)\,\alpha^{N_1} + N_1\,\alpha^{N_1+1}}{(1-\alpha)(1-\alpha^{N_1})} + O(\epsilon), \tag{2.5}$$

where

$$\alpha = \frac{k_2}{k_1+d_1}.$$

PROOF. Let $h(k)$, $k \geq 0$, equal the number of parts, at time k, in the buffer of a serial production line with a quality control system operating under assumptions (i–viii) with $M = 2$. Let $v_i(k)$ equal the probability that $h(k) \geq i$. Let $\overline{q_1}$ equal the probability that m_1 will produce a part into the buffer during a cycle, given that it is not blocked. Then $\overline{q_1}$ is equal to the probability that the first machine produces a part when not blocked or starved, and that the part is not made defective by m_1 and removed by the first quality control device. Therefore, $\overline{q_1} = (1-k_1\epsilon)(1-b_1\,D_1\epsilon)$, or $\overline{q_1} = 1-(k_1+d_1)\,\epsilon + O(\epsilon^2)$, where $d_1 = b_1 D_1$. The evolution equations for the probabilities $v_i(k)$, $1 \leq i \leq N_1$, $k \geq 0$ can be written as

$$\begin{aligned}
v_1(k+1) &= v_1(k) \quad + [1-v_1(k)]\,\overline{q_1} + [v_2(k) + (v_1(k)-v_2(k))\,\overline{q_1}]\,q_2 - v_1(k)\,q_2,\\
v_i(k+1) &= v_1(k) \quad + [v_{i-1}(k) - v_i(k)]\,\overline{q_1} + [v_{i+1}(k) + (v_i(k) - v_{i+1}(k))\,\overline{q_1}]\,q_2\\
&\qquad - [v_i(k) + (v_{i-1}(k) - v_i(k))\,\overline{q_1}]\,q_2 \qquad\qquad 2 \leq i \leq N_1 - 1,\\
v_{N_1}(k+1) &= v_{N_1}(k) + [v_{N_1-1}(k) - v_{N_1}(k)]\,\overline{q_1} + v_{N_1}(k)\,\overline{q_1}\,q_2\\
&\qquad - [v_{N_1}(k) + (v_{n_1-1}(k) - v_{N_1}(k))\,\overline{q_1}]\,q_2.
\end{aligned}$$

These equations represent an ergodic Markov chain, and will therefore converge to a stationary probability distribution. They are the same evolution equations obtained in [1], and their solution for the steady value of $v_1(k)$ yields

$$\begin{aligned}
v_1 = v_1(\infty) &= 1- \Big[(k_1 + d_1 + O(\epsilon))\, Q\,\Big(\frac{k_2}{k_1+d_1+O(\epsilon)}, N_1 \Big)\Big]\,\epsilon + O(\epsilon^2)\\
&= 1- \Big[(k_1+d_1)\, Q\,\Big(\frac{k_2}{k_1+d_1}, N_1 \Big)\Big]\,\epsilon + O(\epsilon^2),
\end{aligned}$$

where $Q(\alpha, N) = \frac{1-\alpha}{1-\alpha^N}$, $\alpha \geq 0+$, $N \in \mathbf{Z}_+$.

The production rate of the system is the probability that m_2 is not starved, m_2 produces a part, and the part is not defective and removed by the second quality control device. Thus,

$$\begin{aligned}
\mathrm{PR} &= v_1(1-k_2\,\epsilon)\,[1-(b_1\,(1-D_1)\,\epsilon + b_2\,\epsilon + O(\epsilon^2))\,D_2]\\
&= 1 - \Big[k_2 + d_2 + (k_1+d_1)\,Q\,\Big(\frac{k_2}{k_1+d_1}, N_1\Big)\Big]\,\epsilon + O(\epsilon^2),
\end{aligned}$$

where the substitution $d_2 = [b_1\,(1-D_1) + b_2]\,D_2$ has been made. Due to Lemma 2.1 of [1], the last equality can be rewritten as

$$\mathrm{PR} = 1 - \Big[k_1 + d_1 + k_2\,Q\,\Big(\frac{k_1+d_1}{k_2}, N_1\Big) + d_2\Big]\,\epsilon + O(\epsilon^2).$$

This establishes part (a) of the theorem.

The production rate of non-defective parts is given by the production rate multiplied by the fraction of the parts produced which are non-defective. A fraction $b = 1-(1-b_1\epsilon)(1-b_2\epsilon)$ of the parts consumed will be defective, and a fraction $d = 1-(1-b_1 D_1\epsilon)\,[1-(b_1(1-D_1)\epsilon + b_2\epsilon + O(\epsilon^2))D_2] = d_1\,\epsilon + d_2\,\epsilon + O(\epsilon^2)$ of the parts consumed will be removed by one of the quality control devices. The fraction of non-defective parts produced is then $1-(b-d)/(1-d) = 1-(b_1+b_2-d_1-d_2)\,\epsilon + O(\epsilon^2)$, from which follows part (b).

The steady state value of $v_{N_1}(k)$ from the evolution equations is, using [1],

$$v_{N_1} = v_{N_1}(\infty) = Q\left(\frac{k_1+d_1}{k_2}, N_1\right) + O(\epsilon).$$

The consumption rate can then be calculated as the probability that m_1 is not blocked, and m_1 does not fail,

$$\mathrm{CR} = (1 - v_{N_1} k_2 \epsilon)(1 - k_1 \epsilon) = 1 - \left[k_1 + k_2 Q\left(\frac{k_1+d_1}{k_2}, N_1\right)\right]\epsilon + O(\epsilon^2),$$

which establishes part (c).

The work-in-process is the expected value of $h(\infty)$. The steady state values of $v_j = v_j(\infty)$, $1 \le j \le N_1$, can be written, using [1], as

$$v_{N_1-j} = \frac{Q\left(\frac{k_1+d_1}{k_2}, N_1\right)}{Q\left(\frac{k_1+d_1}{k_2}, j+1\right)} + O(\epsilon),$$

from which it follows that

$$\mathrm{Prob}\,(h = i) = v_i - v_{i+1} = Q(\alpha, N_1)\,\alpha^{i-1},$$

where $\alpha = \frac{k_2}{k_1+d_1}$. Thus,

$$\begin{aligned} E[h] &= Q(\alpha, N_1)\,[1 + 2\alpha + 3\alpha^2 + \cdots + N\,\alpha^{n-1}] \\ &= Q(\alpha, N_1)\,\frac{1 - N_1\,\alpha^{N_1} - \alpha^{N_1} + N_1\,\alpha^{N_1+1}}{(1-\alpha)^2} \\ &= \frac{1 - (N_1+1)\,\alpha_1^{N_1} + N_1\,\alpha^{N_1+1}}{(1-\alpha)\,(1-\alpha^{N_1})} \end{aligned}$$

which establishes part (d). ∎

REMARKS.

1. If $b_i = 0$, $i = 1, \ldots, M$, Theorem 2.1 coincides with Theorem 2.1 of [1] where asymptotically reliable serial production lines with perfect quality machines have been considered. Thus, the introduction of the defects does not change the analytical structure of the performance characteristics, but rather leads to a modification of the first argument in the function Q.
2. From the point of view of the production rate, the line defined by (i–viii) is equivalent to a single, aggregated machine, characterized by the loss parameter k_L where, as it follows from (2.1),

$$k_L = k_1 + d_1 + k_2 Q\left(\frac{k_1+d_1}{k_2}, N_1\right) + d_2 + O(\epsilon).$$

3. From the point of view of the quality of parts produced, the line again can be characterized as a single machine with the quality loss parameter b_L, where $\epsilon\, b_k$ is the probability that a part produced by the line is defective. As it follows from (2.1) and (2.3),

$$b_L = b_1 + b_2 - d_1 - d_2 + O(\epsilon).$$

4. From the point of view of the parts removed as scrap, the line can be characterized by its scrap rate SR. As it follows from (2.1) and (2.4),

$$\mathrm{SR} = \epsilon\,(d_1 + d_2) + O(\epsilon^2). \tag{2.6}$$

5. Parameters d_i, defined in (2.2), have the following interpretation:

$$\text{Prob [a part produced by } m_i \text{ is removed by } Q_i] = \epsilon\, d_i + O(\epsilon^2), \quad i = 1, 2.$$

Thus, the d_i's can be referred to as the i-th *scrap parameters*. As it follows from (2.6), the corresponding *line scrap parameter*, d_L, is:

$$d_L = d_1 + d_2 + O(\epsilon).$$

To conclude, we compare the performance characteristics of lines utilizing the input and output quality control architectures. Let $P^I(k_i, N_i, b_i, D_i)$ denote any of the performance characteristics defined in (a), (b), and (c) of Section 1 in the line with the input quality control architecture. Let $P^O(k_i, N_i, b_i, D_i)$ denote the same characteristics in a line with the output quality control architecture. The following property holds:

THEOREM 2.2. *Under assumptions (i–viii) and (i–vii), (viii') with $M = 2$, P^O and P^I have identical leading order terms, i.e.,*

$$|P^O(k_i, N_i, b_i, D_i) - P^I(k_i, N_i, b_i, D_i)| \sim O(\epsilon^2).$$

PROOF. Let $h(k), k \geq 0$, equal the number of parts, at time k, in the buffer of a serial production line with a quality control system operating under the input quality control assumption with $M = 2$. Let $v_i(k)$ equal the pobability that $h(k) \geq i$. The evolution equations for the $v_i(k)$ can be written as

$$\begin{aligned}
v_1(k+1) &= v_1(k) \quad + (1 - v_1(k))\, q_1 - (v_1(k) - v_2(k))\,(1 - q_1)\, q_2 + O(\epsilon),\\
v_j(k+1) &= v_j(k) \quad + [v_{j-1}(k) - v_j(k)]\, q_1\,(1 - q_2) - [v_j(k) + v_{j+1}(k)]\,(1 - q_1)\, q_2\\
&\qquad - [v_j(k) + v_{j+1}(k)]\, q_1\, q_2\, d_1\, \epsilon + O(\epsilon^2), \qquad\qquad 2 \leq j \leq N_1 - 1,\\
v_{N_1}(k+1) &= v_{N_1}(k) + [v_{N_1-1}(k) - v_{N_1}(k)]\, q_1\,(1 - q_2) - v_{N_1}(k)\,(1 - q_1)\, q_2 - v_{N_1}\, q_1\, q_2\, d_1\, \epsilon,
\end{aligned}$$

where $d_1 = b_1 D_1$. These equations represent an ergodic Markov chain and therefore posses a unique equilibrium point. The equilibrium point is given by

$$v_{N_1-j} = \frac{Q\left(\frac{k_1+d_1}{k_2}, N_1\right)}{Q\left(\frac{k_1+d_1}{k_2}, j+1\right)} + O(\epsilon).$$

The production rate can be calculated as the probability that m_1 is not blocked and produces a part which is not removed by either quality control device. With d_1 and d_2 defined as in the proof of theorem 2.1,

$$\begin{aligned}
\text{PR}^I &= (1 - v_{N_1}\, k_2\, \epsilon)\,(1 - k_1\, \epsilon)\,(1 - d_1\, \epsilon)\\
&= 1 - \epsilon \left[k_1 + d_1 + k_2\, Q\left(\frac{k_1 + d_1}{k_2}, N_1 \right) + d_2 \right] + O(\epsilon^2).
\end{aligned}$$

Again, using Lemma 2.1 of [1], we write:

$$\text{PR}^I = 1 - \epsilon \left[k_2 + d_2 + (k_1 + d_1)\, Q\left(\frac{k_2}{k_1 + d_1}, N_1 \right)\right] + O(\epsilon^2).$$

Thus, $|\text{PR}^O - \text{PR}^I| \sim O(\epsilon^2)$.

The remaining performance characteristics can be calculated as in the proof to Theorem 2.1, and the asymptotic equalities then follow. ∎

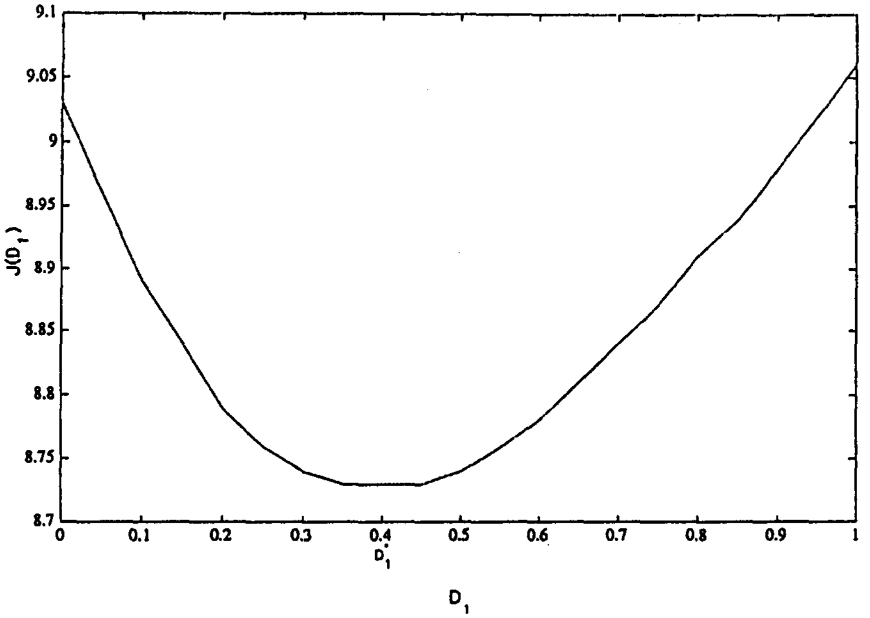

Figure 1.

3. CONCLUSIONS

The results of this paper can be utilized for analysis and design of asymptotically reliable serial production lines with quality control devices. As far as the analysis is concerned, the formulas obtained give the leading order terms for all line performance characteristics. In the problem of design, the results derived could be used to optimize the system behavior by an appropriate choice of its parameters. Indeed, assume that $D_2 = 1$ so that only perfect quality parts are produced by the line. Choose D_1^* so that $J(D_1) = k_L(D_1) + pD_1$ is minimized, where $p > 0$ is the price of measurement of Q_1. Figure 1 shows the behavior of $J(D_1)$ and the optimal value D_1^* for $k_1 = 1, k_2 = 2, b_1 = 3, b_2 = 4, N_1 = 5$ and $p = 1$.

Finally, from the results described, it follows that the quality control devices can be placed either according to the output or the input quality control architecture without affecting the leading order terms of all line performance characteristics.

REFERENCES

1. J.-T. Lim, S.M. Meerkov and F. Top, Homogeneous, asymptotically reliable serial production lines: Theory and case study, *IEEE Transactions on Automatic Control* **35** (5), 524–534 (May 1990).
2. S.M. Meerkov and F. Top, Asymptotically reliable serial production lines: Analysis, synthesis, and a case study, *Preprints of the 11th IFAC World Congress*, Tallinn, USSR, (August 1990).
3. E. Deming, *Out of the Crisis*, Massachusetts Institute of Technology, Cambridge, MA, (1986).
4. G. Taguchi, *System of Experimental Design: Engineering Methods to Optimize Quality and Minimize Cost*, UNIPUB/Kraus International Publications, Dearborm, Mich., (1987).
5. E.M. Goldratt and J. Cox, *The Goal*, North Rivers Press, (1986).