Even Cycles in Graphs

Joseph G. Conlon
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109-1109
E-mail: conlon@umich.edu

Revised July 11, 2003

Abstract: Let G be a 3-connected simple graph of minimum degree 4 on at least six vertices. The author proves the existence of an even cycle C in G such that $G - V(C)$ is connected and $G - E(C)$ is 2-connected. The result is related to previous results of Jackson, and Thomassen and Toft. Thomassen and Toft proved that G contains an induced cycle C such that both $G - V(C)$ and $G - E(C)$ is 2-connected. G does not in general contain an even cycle such that $G - V(C)$ is 2-connected.
J Graph Theory 45: 163–223, 2004

Keywords: *even cycle; 3-connected; removable*

1. INTRODUCTION

In this paper we shall be concerned with establishing the existence of an even cycle C in a simple graph G. We want the cycle C to have the property that if we remove the vertices $V(C)$ of C or the edges $E(C)$ of C from G, then the remaining graph is maximally connected. We prove the following theorem in this direction.

Theorem 1.1. *Let G be a connected simple graph of minimum degree* 4. *Then G contains an even cycle C such that $G - V(C)$ is non-empty and connected.*

The graph with the least number of vertices which satisfies the conditions of Theorem 1.1 is the complete graph on five vertices, K_5, which is also

3-connected. If we remove the vertices $V(C)$ of a cycle C of order 4 from K_5, then the remaining graph is just the vertex graph and hence is connected. If we remove the edges $E(C)$ of C then it is clear that $K_5 - E(C)$ is connected but not 2-connected. In contrast to this example, we have the following.

Theorem 1.2. *Let $G \neq K_5$ be a 3-connected simple graph of minimum degree 4. Then G contains an even cycle C such that $G - V(C)$ is non-empty and connected, and $G - E(C)$ is 2-connected. Further, there is no triangle in G consisting entirely of chords of C.*

Consider the graph G in Figure 1 below. Now G has 11 vertices and its edges are denoted by both solid and dashed lines. It is clear that G is 3-connected of minimum degree 4. The even cycles in G have orders 4, 6, and 8. If a cycle C has order 8, then $G - V(C)$ is disconnected. If C has order 6 or 4, then $G - V(C)$ is connected but not 2-connected. For any even cycle, $G - E(C)$ has a vertex of degree 2, whence $G - E(C)$ is not 3-connected. The dashed edges in Figure 1 give a cycle of order 6 satisfying the conclusions of Theorem 1.2. Theorem 1.2 is proved using an induction argument. It appears to be necessary to include as part of the induction hypothesis that one cannot form a triangle from chords of C. See Lemma 3.6.

Theorem 1.2 should be compared to the following theorems of Thomassen and Toft [8].

Theorem 1.3. *Let G be a 3-connected simple graph of minimum degree 4. Then G contains an induced cycle C such that $G - V(C)$ and $G - E(C)$ are 2-connected.*

Theorem 1.4. *Let G be a 2-connected simple graph of minimum degree 4. Then G contains an induced cycle C such that $G - V(C)$ is connected and $G - E(C)$ is 2-connected.*

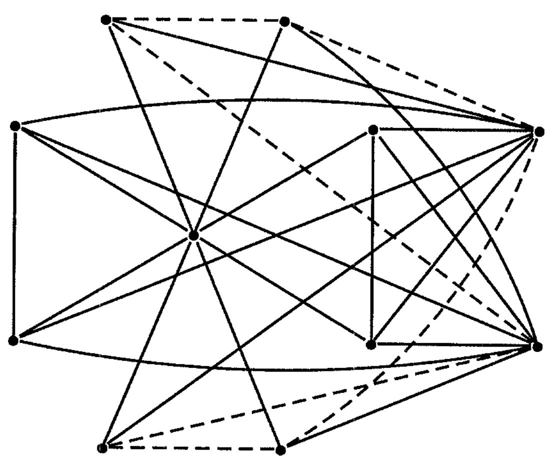

FIGURE 1.

Theorem 1.4 answers in the affirmative a conjecture of Hobbs that if G is a 2-connected simple graph of minimum degree 4, then G contains a cycle C such that $G - E(C)$ is 2-connected. Hobb's conjecture was first proved by Mader [7]. Later a stronger version of Mader's result was proved independently by Jackson [5]. More recently extensions of these results to matroids have been discussed by Lemos and Oxley [6].

In this paper we shall follow a similar strategy to that of [8] in order to prove Theorems 1.1 and 1.2. Thus in the proof of Theorem 1.1, we shall obtain the even cycle C as the even cycle which maximises the number of vertices of the largest connected component of $G - V(C)$. For Theorem 1.2 we shall also need to maximise the number of vertices of the largest block of $G - V(C)$.

The problem of the existence of even cycles in graphs does not seem to have been studied much. Theorems 1.8 and 4.13 of [2], as well as [3], give some information on the existence of even cycles, but do not address the question of connectivity when the cycle is removed from the graph. The article of Frank [4] discusses the issue of connectivity when cycles are removed from graphs, but does not relate that to whether the cycle is even or odd. A classic text which discusses graph connectivity is the book of Tutte [9]. For existence of cycles in graphs, the book of Voss [10] gives a comprehensive survey of the literature until 1990. Throughout the paper we shall use the terminology of the recent book of Bollobas [1].

In general, in the paper, an arc of a cycle or a path in a graph is called *even* (*odd*) if it has an even (odd) number of vertices. Hence an even path has an odd number of edges.

2. PROOF OF THEOREM 1.1

We first identify the block decomposition of a graph which contains no even cycles.

Lemma 2.1. *Suppose G is a connected graph which contains no even cycles. Then in the block decomposition of G each block is either a K_2 or an odd cycle.*

Proof. If a block B is not a K_2 then it must contain a cycle C which is necessarily odd. Suppose $V(B) \neq V(C)$ and $v \in V(B) - V(C)$ is adjacent to a vertex $w \in V(C)$. Let $w'' \neq w$ be another vertex of C. By Menger's theorem, there is a path in B from w'' to v, which does not contain the vertex w. Let w' be the vertex of C on this path closest to v and $P(w', v)$ the corresponding path from w' to v. Since C is odd it is clear, we can form an even cycle from one of the arcs $w'w$ of C joining w' to w, the edge wv and the path $P(w', v)$. We have a contradiction, whence $V(B) = V(C)$. If C has a chord then we can also form an even cycle with the chord and one of the arcs of C. We conclude that B is precisely an odd cycle. ∎

Corollary 2.1. *A graph G with minimum degree* 3 *contains an even cycle.*

Proof. Let B be an endblock of the block decomposition of a connected component of G. If $B = K_2$ then G has a vertex of degree 1. If B is an odd cycle then G has a vertex of degree 2. ■

We begin the proof of Theorem 1.1. By Corollary 2.1, there is an even cycle C' such that $G - V(C')$ is non-empty. Let C be an even cycle with the property that the largest connected component of $G - V(C)$ has the maximum number of vertices. Let H be this maximal component. We wish to show that $H = G - V(C)$. We shall prove this by contradiction, by assuming that $G - V(C)$ has a second connected component H'.

Lemma 2.2. *The component H' is a tree graph.*

Proof. Now H' cannot contain an even cycle. Otherwise the graph on the vertices $V(H) \cup V(C)$ is connected and is disjoint from an even cycle in G. Suppose then that H' contains no even cycles and that H attaches to a vertex $w \in V(C)$. Consider the endblocks in the block decomposition of H'. If an endblock is an odd cycle C', then C' contains two vertices a, b, which are not cutvertices of H'. Each of these vertices has degree at least 4 in G, whence a attaches to a vertex $a' \in V(C)$ with $a' \neq w$, and similarly b attaches to $b' \in V(C)$ with $b' \neq w$. It is possible for $a' = b'$. Let $a'b'$ be the arc of C, which does not contain w. Then one can form an even cycle with $a'b'$ and one of the arcs of C' joining a to b. This lies outside the connected graph on the vertices $V(H) \cup \{w\}$, contradicting the definition of H.

Then we can assume that all endblocks in the block decomposition of H' are K_2's. Suppose that one of the blocks of H' is a cycle C'. Then, two of the vertices of C', say a and b, are cutvertices of H'. There exist vertices a', b' of H' with the properties:

(a) a', b' have degree 1 in H';
(b) there is a path $P(a', a)$ in H' from a' to a and similarly a path $P(b', b)$ from b' to b;
(c) the paths $P(a', a)$ and $P(b', b)$ do not intersect.

Evidently a' attaches to a vertex $a'' \in V(C)$, $a'' \neq w$, and b' to $b'' \in V(C)$, $b'' \neq w$. Let $a''b''$ be the arc of C joining a'' to b'', which does not contain w. Then one can form an even cycle from $a''b''$, $P(a', a)$, $P(b', b)$ and one of the arcs of C' joining a to b . This cycle lies outside the graph on the vertices $V(H) \cup \{w\}$, again contradicting the definition of H. ■

Lemma 2.3. *Suppose H' is a tree graph and $w \in V(C)$. Then $G[V(H') \cup V(C) - \{w\}]$ contains an even cycle.*

Proof. Assume first that H' is a single vertex. Then H' attaches to three vertices $a, b, c \in V(C)$ different from w. We can assume wlog that arc ac of C joining a to c, which does not contain w, does include the vertex b. Let ab and

bc be the arcs of C joining a to b and b to c, respectively, which are contained in ac. Consider now the three cycles: C_1 consisting of the arc ac and H', C_2 consisting of the arc ab and H', C_3 consisting of the arc bc and H'. It is easy to see that one of C_1, C_2, C_3 is even.

Hence we may assume H' contains two vertices a, b of degree 1 in H'. Let ab be the path in H' joining a to b. Then a attaches to two vertices $a', a'' \in V(C)$ different from w and similarly b to vertices $b', b'' \in V(C)$ different from w. Let $a'a''$ be the arc in C joining a' to a'', which does not include w. We can assume $a'a''$ has an even number of vertices. Otherwise the cycle consisting of $a'a''$ and a is even. We assume a similar situation with b', b''. Consider now the arc $a'b'$ in C joining a' to b', which does not include w. If $a'b'$ is even then the arc $a''b'$ joining a'' to b' and not including w is odd. Conversely, if $a'b'$ is odd then $a''b'$ is even. Hence we can form an even cycle with ab and one of the arcs $a'b'$ or $a''b'$. ∎

***Proof of Theorem* 1.1.** This follows since Lemma 2.3 gives a contradiction to the definition of H. ∎

3. PROOF OF THEOREM 1.2—SETUP

We begin by applying the maximization technique introduced in Section 2 to 2-connectedness.

Lemma 3.1. *Let G be a 3-connected graph and C be an even cycle in G with the property that $G - V(C)$ contains the largest block outside an even cycle. If B^* is this largest block then every other block of $G - V(C)$ is either a K_2 or an odd cycle.*

Proof. Assuming $G - V(C)$ is not necessarily connected, let G^* denote the connected component of $G - V(C)$ containing B^*. The graph on the vertices $V(G^*) \cup V(C)$ is 2-connected. Hence if there is a block B not in G^*, it must by Lemma 2.1 be either a K_2 or an odd cycle. If $B \subset G^*$, we can make a similar argument. In fact, there is a cutvertex z such that $G^* = G_1^* \cup G_2^*$ with $V(G_1^*) \cap V(G_2^*) = \{z\}$ and $B^* \subset G_1^*$, $B \subset G_2^*$. Since the graph on the vertices $V(C) \cup V(G_1^*) - \{z\}$ is 2-connected, it follows again from Lemma 2.1 that B is either a K_2 or an odd cycle. ∎

We now do a further maximization of the cycle C of Lemma 3.1.

Lemma 3.2. *Suppose G is a 3-connected graph of minimum degree 4. Let C be an even cycle with the properties:*

(a) *$G - V(C)$ contains a largest block B^* outside an even cycle.*
(b) *The connected component of $G - V(C)$ containing B^* is the largest connected component containing B^* which lies outside an even cycle.*

Then $G - V(C)$ is connected.

Proof. Suppose H' is a second connected component of $G - V(C)$. Then from the arguments of Lemmas 2.2 and 2.3, we obtain a contradiction. ∎

Remark 1. *Note that the graph* $G - V(C)$ *of Lemma* 3.2 *could be a vertex graph.*

Proposition 3.1. *Let* $N > 5$ *be an integer and assume Theorem* 1.2 *holds for all graphs* G' *with* $|V(G')| = N - 2$. *Let* G *be a graph satisfying the conditions of Theorem* 1.2 *with* $|V(G)| = N$. *Suppose* C *is an even cycle in* G *satisfying* (a) *and* (b) *of Lemma* 3.2. *Then* $G - V(C)$ *is not a vertex graph. If* $G - V(C)$ *is* 2-*connected, then there is an even cycle* C' *in* G *such that* $G - V(C')$ *is non-empty and connected, and* $G - E(C')$ *is* 2-*connected. Further, a triangle cannot be formed entirely from chords of* C'.

We prove Proposition 3.1 in a series of lemmas.

Lemma 3.3. *Let* C *be the even cycle of Lemma* 3.2 *and suppose* C *has order larger than* 4. *Then* C *has at most one chord.*

Proof. Let x, y be two vertices of C, which are non-adjacent on the cycle C but for which the edge xy exists. Let $\text{arc}(x, y)$ denote a path from x to y on C. Since C is connected to $G - V(C)$ by at least three vertices, one can form a cycle C' from the edge xy and one of the arcs, $\text{arc}(x, y)$, such that $G - V(C')$ is connected. If $\text{arc}(x, y)$ is even we have a contradiction to (b) of Lemma 3.2. We conclude that $\text{arc}(x, y)$ is odd.

Suppose now there is a third vertex z of C, non-adjacent to x on C, for which the edge xz also exists. Then an arc, $\text{arc}(x, z)$, must be odd. Let $\text{arc}(y, z)$ be the path from y to z on C, which does not include x. We may then choose $\text{arc}(x, y)$ and $\text{arc}(x, z)$ so that C is the union of $\text{arc}(x, y)$, $\text{arc}(y, z)$ and $\text{arc}(x, z)$. Evidently the cycle C' consisting of $\text{arc}(y, z)$ and the edges xy and xz is even. It follows again from (b) of Lemma 3.2 that no vertex of $V(C) - V(C')$ attaches to $G - V(C)$. Hence at least three vertices of C' attach to $G - V(C)$.

Let w be an internal vertex of $\text{arc}(x, y)$, i.e., $w \neq x, y$, and v be a vertex of $\text{arc}(x, z)$, non-adjacent to w on C, such that the edge wv exists. As before, $\text{arc}(w, v)$ must be odd. Suppose that $v \neq z$ and let C'' be the cycle consisting of $\text{arc}(w, y) \subset \text{arc}(x, y)$, arc $(x, v) \subset \text{arc}(x, z)$, together with the edges xy and wv. Then C'' is even and $V(C'') \cap V(C') = \{x, y\}$. Since three vertices of C' attach to $G - V(C)$, we obtain a contradiction to (b) of Lemma 3.2. We get a similar contradiction if $v \neq x$. We conclude that w does not attach to any vertex of $\text{arc}(x, z)$.

Suppose next that v is a vertex of $\text{arc}(y, z)$, non-adjacent to w on C, such that the edge wv exists. Let C''_1 be the cycle consisting of arc $(w, x) \subset \text{arc}(x, y)$, $\text{arc}(y, v) \subset \text{arc}(y, z)$, together with the edges xy and wv. Let C''_2 be the cycle consisting of $\text{arc}(w, x) \subset \text{arc}(x, y)$, $\text{arc}(v, z) \subset \text{arc}(y, z)$, together with the edges xz and wv. It is easy to see as before that C''_1 and C''_2 are even cycles. Further, one has $V(C') \cap V(C''_1) \cap V(C''_2) = \{x, v\}$. Hence for either $i = 1$ or 2, $V(C') - V(C''_i)$ has a vertex which attaches to $G - V(C)$. Again we obtain a contradiction.

Suppose finally that v is a vertex of $\text{arc}(x, y)$ such that v is non-adjacent on C to w and the edge wv exists. Assume v is the nearest of w and v to y on C. Let C'' be the cycle consisting of $\text{arc}(x, w) \subset \text{arc}(x, y)$, $\text{arc}(v, y) \subset \text{arc}(x, y)$, together with the edges xy and wv. Then C'' is even and $V(C'') \cap V(C') = \{x, y\}$. We have a contradiction as previously.

We conclude then that w has degree 2 in the graph on $V(C)$. Since we have already seen that w cannot attach to $G - V(C)$, we have a contradiction. Hence every vertex of C has at most one chord, whence every vertex of C attaches to $G - V(C)$.

Suppose now that there are four distinct vertices of C, x, y, w, v such that x and y are non-adjacent on C but the edge xy exists and similarly w and v are non-adjacent but the edge wv exists. Suppose first that w and v are on different arcs, $\text{arc}(x, y)$. Since C has order larger than 4, we can assume wlog that w is not adjacent on C to y. Let C' be the cycle consisting of $\text{arc}(x, w) \subset \text{arc}(x, y)$, $\text{arc}(v, y) \subset \text{arc}(x, y)$, together with the edges xy and wv. Since $\text{arc}(x, y)$ and $\text{arc}(w, v)$ are odd, it follows that C' is an even cycle. Since w is not adjacent on C to y, one also has that $V(C) - V(C')$ is non-empty. We have already observed that every vertex of C attaches to $G - V(C)$, whence we obtain a contradiction. Next assume w, v are on the same arc, $\text{arc}(x, y)$, with w closest to x on C. Let C' be the cycle consisting of $\text{arc}(x, w) \subset \text{arc}(x, y)$, $\text{arc}(v, y) \subset \text{arc}(x, y)$, together with the edges xy and vw. Then C' is an even cycle and $V(C) - V(C')$ is non-empty. We obtain a contradiction as before. ∎

Corollary 3.1. *Let C be the even cycle of Lemma* 3.2 *and suppose C has order larger than* 4. *Then $G - V(C)$ has at least two vertices.*

Proof. By Lemma 3.3, there is a vertex of C, which has degree 2 in the graph on $V(C)$. Since this vertex has minimum degree 4, $G - V(C)$ has at least two vertices. ∎

Lemma 3.4. *Let C be the even cycle of Lemma* 3.2 *and suppose C has order larger than* 4. *If $G - V(C)$ is* 2*-connected, then there is an even cycle C' in G such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is* 2*-connected. Further, one cannot form a triangle entirely from chords of C'.*

Proof. Suppose first that C has no chords. Then every vertex of C attaches to two vertices of $G - V(C)$. Since $G - V(C)$ is 2-connected, it follows that $G - E(C)$ is 2-connected. Assume next that C has exactly one chord with vertices $x, y \in V(C)$. We may also assume as before that $\text{arc}(x, y)$ is odd. Now x attaches to a vertex $x' \in G - V(C)$ and y to a vertex $y' \in G - V(C)$. If $x' \neq y'$ then it is clear that $G - E(C)$ is 2-connected. Suppose now $x' = y'$ and let C' be the cycle consisting of the smaller of the arcs, $\text{arc}(x, y)$, and the vertex x'. Evidently C' is even and $G - V(C')$ is connected. Consider now $G - E(C')$. We need to show that there are two disjoint paths in $G - E(C')$ linking x to $G - V(C)$, and similarly with y. Since C has at least six vertices we have $|V(C)- V(\mathrm{arc}(x,y))| \geq 3$. Let w be the vertex on $C-$ arc(x,y) adjacent to x and v be the vertex adjacent to y, whence $w \neq v$. Then w attaches to a vertex w' of $G-V(C)$ and v to a vertex v' of $G-V(C)$ with $v' \neq w'$. Hence x can be connected to $G-V(C)$ by the disjoint paths xww' and $xyvv'$ in $G-E(C')$. Since C has just one chord, it follows that one cannot form a triangle from chords of C'. ■

Lemma 3.5. *Let C be the even cycle of Lemma* 3.2 *and suppose C has order* 4 *with either no chords or* 2 *chords. If $|V(G)| \geq 6$ and $G-V(C)$ is* 2*-connected, then there is a cycle C' of order* 4 *in G such that $G-V(C')$ is non-empty and connected, and $G-E(C')$ is* 2*-connected.*

Proof. If C has no chords we can argue as in Lemma 3.4 to conclude $G-E(C)$ is 2-connected. Suppose now C has two chords in which case the graph on $V(C)$ is K_4. Consider first the case when $G-V(C)=K_2$ with vertices A and B. Now A attaches to at least three vertices of C so let us denote by a the vertex of C to which A may not attach. Similarly denote by b the vertex of C to which B may not attach. Observe that both a and b must attach to one of the vertices A or B. It follows easily that $G-E(C)$ is 2-connected.

Next consider the case when $G-V(C)$ has at least three vertices. Since G is 3-connected and the graph on $V(C)$ is K_4, we can assume wlog that the vertices of C are a,b,d,f in order on C, and that a attaches to $A \in G-V(C), b$ to $B \in G-V(C)$, and d to $D \in G-V(C)$ with A,B,D distinct. Now f must attach also to a vertex $F \in G-V(C)$. If $F \neq B$, then $G-E(C)$ is 2-connected. Suppose $F=B$ and let C' be the cycle with edges ad,df,fb,ba. Then $G-E(C')$ is 2-connected. ■

Lemma 3.6. *Let $N>5$ be an integer and assume Theorem* 1.2 *holds for all graphs G' with $|V(G')|=N-2$. Let G be a* 3*-connected graph of minimum degree* 4 *with $|V(G)|=N$. Let C be the even cycle of Lemma* 3.2 *and suppose C has order* 4 *with one chord. If $G-V(C)$ is* 2*-connected, then there is an even cycle C' in G such that $G-V(C')$ is non-empty and connected, and $G-E(C')$ is* 2*-connected. Further, one cannot form a triangle in G entirely from chords of C'.*

Proof. As in Lemma 3.5, let the vertices of C be a,b,d,f, in order on C, with the edge ad being the unique chord of C. Consider first the case when $G-V(C)=K_2$. Then both vertices of the K_2 attach to b and f. One vertex of the K_2 attaches to a and the other to d. We conclude $G-E(C)$ is 2-connected. Next consider the case when $G-V(C)$ is 2-connected with at least three vertices. Suppose a attaches to $A \in G-V(C)$ and d to $D \in G-V(C)$. If $D \neq A$ then it is clear $G-E(C)$ is 2-connected so we shall assume $D=A$.

We consider the possible vertices of attachment of b and f in $G-V(C)$. Since G is 3-connected, b attaches to $B \in G-V(C)$ and f to $F \in G-V(C)$ with

A, B, F distinct. Suppose now one of b or f, say b, also attach to A. Then we may form a cycle C' of order 4 consisting of the edges Aa, ad, db, bA. Now $G - V(C')$ is connected and $G - E(C')$ is 2-connected. Hence we can assume that neither b nor f attaches to A.

Let us next suppose $G - V(C) = K_3$, whence the vertices are A, B, F as above, with A attaching to a and d in C while B and F both attach to b and f in C. Let C' be the cycle of order 4 consisting of the edges Aa, ab, bB, BA. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected.

Then, we can assume that $G - V(C)$ has at least four vertices and that there are distinct vertices A, B, F of $G - V(C)$ with the properties:

(1) a and d attach to A, b attaches to B, and f to F;
(2) A is the only vertex of attachment of a and d in $G - V(C)$;
(3) b and f attach to at least two vertices of $G - V(C)$ but do not attach to A.

Let G'' be the graph obtained from G by contracting f to a and d to b. One sees that $G'' \neq K_5$ is 3-connected of minimum degree 4, and $|V(G'')| = N - 2$. Hence by our assumption in the statement of the Lemma, there is an even cycle C'' in G'' with the property that $G'' - V(C'')$ is non-empty and connected, and $G'' - E(C'')$ is 2-connected. Further, one cannot form a triangle in G'' entirely from chords of C''. Let a'' be the vertex of G'' corresponding to the contraction of the edge af in G, and similarly let b'' be the vertex of G'' corresponding to the contraction of bd. If $V(C'')$ does not contain a'' or b'', then let C' be the cycle in G identical to C'' in G''. It is clear that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.Since there is a one-one correspondence between chords of C' and chords of C'', it follows that one cannot form a triangle in G entirely from chords of C'.

Next suppose $V(C'')$ contains a'' but not b''. Consider the two vertices of C'', which are adjacent to a''. If neither of these vertices is A then we define the cycle C' in G as the cycle identical to C'' in G'' but with the vertex $a'' \in G''$ replaced by $f \in G$. Then $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected. Observe that every chord of C' corresponds to a chord of C''. If $A \notin V(C'')$ then there is a one-one correspondence between chords of C' and chords of C''. If $A \in V(C'')$ the chord $a''A$ belongs to C'' but not to C'. We conclude a triangle cannot be formed from the chords of C'.

Suppose now A is adjacent to a'' on C'' and $b'' \notin V(C'')$. We can assume wlog that the other vertex adjacent to a'' on C'' is F. We define the cycle C' in G as identical to C'' outside the arc $Fa''A$, but with the arc $Fa''A$ replaced by the arc $FfdaA$ in G. Note that C' has two more vertices than C'' and hence is even. We see again that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected. Observe that C' contains the chords fa and dA, neither of which correspond to a chord of C''. Now fa is the only chord of C' with vertex a, and dA is the only chord of C' with vertex d. Hence a triangle consisting of chords of C' does not contain either of the chords *fa* or *dA*. It follows that a triangle cannot be formed from chords of C'.

Suppose now $V(C'')$ contains both a'' and b'' and a'', b'' are adjacent as vertices of C''. Assume A is the other adjacent vertex of either a'' or b'', say a''. We define the cycle C' in G as the cycle identical to C'' in G'', but with the arc $Aa''b''$ replaced by the arc *Aafdb*. It is easy to see that both $G - V(C')$ and $G - E(C')$ are connected. The graph $G - E(C')$ is also 2-connected. To see that $G - E(C') - \{a\}$ is connected, note that we use the fact that $G'' - E(C'') - \{b''\}$ is connected. Observe that C' contains the chords dA, ab, and ad. Arguing as in the previous paragraph, we see that a triangle in G, consisting of chords of C', does not contain any of the chords dA, ab, or ad. Hence a triangle cannot be formed from the chords of C'.

Suppose next that a'', b'' are adjacent vertices of C'' but neither is adjacent to A. We may assume therefore that F is the other vertex adjacent to a'' on C'' and B the other vertex adjacent to b''. To obtain C' from C'', we replace the arc $Fa''b''B$ on C'' by the arc *FfdabB* on C'. It is easy to see that $G - V(C')$ is non-empty and connected while $G - E(C')$ is 2-connected. Now C' contains the chords *fa* and *db*, which do not correspond to chords of C''. Again we see that a triangle in G consisting of chords of C' does not contain either of the chords *fa* or *db*. We conclude a triangle cannot be formed from chords of C'.

Finally, we suppose $V(C'')$ contains both a'' and b'' but a'', b'' are not adjacent as vertices of C''. Suppose $A \notin V(C'')$. We obtain C' from C'' by replacing a'' by f and b'' by b. Since $A \notin V(C'')$ it follows that $G - V(C')$ is non-empty and connected. It is easy to see that $G - E(C')$ is 2-connected. Since there is a one-one correspondence between chords of C' and chords of C'', apart from the chord $a''b''$, it is not possible to form a triangle from chords of C'.

Suppose $A \in V(C'')$. Then A must be adjacent on C'' to either a'' or b''. Otherwise the triangle with edges $Aa'', a''b''$, $b''A$ consists of chords of C''. Suppose A is adjacent on C'' to both a'' and b''. Then we obtain C' from C'' by replacing the arc $a''Ab''$ by the arc *faAdb*. It is clear that $G - V(C')$ is non-empty and connected while $G - E(C')$ is 2-connected. The chords of C', excepting the chords *ad* and *fd*, are in one-one correspondence to the chords of C''. Hence it is not possible to form a triangle from chords of C'.

The last case we need to consider is where $a'', b'', A \in V(C'')$ and A is adjacent on C'' to a'' but not to b''. Then we obtain C' from C'' by replacing b'' by b and, in the case of a'', acting as when $V(C'')$ was assumed to contain a'' and not b''. Thus, we replace the arc $Fa''A$ in C'' by the arc *FfdaA* in C'. We conclude as before that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected. It is also clear that one cannot form a triangle in G from chords of C'. ■

***Proof of Proposition* 3.1.** If C has order larger than 4, then the result follows from Lemmas 3.3, 3.4 and Corollary 3.1. If C has order 4 then $G - V(C)$ is not a vertex graph since $|V(G)| \geq 6$. In that case, the result follows from Lemmas 3.5, and 3.6. ■

4. $G - V(C)$ SEPARABLE IMPLIES C OF ORDER 4 OR 6

Our goal in this section is the prove the following.

Proposition 4.1. *Suppose G is a* 3*-connected graph of minimum degree* 4 *and C is an even cycle satisfying* (a) *and* (b) *of Lemma* 3.2. *If $G - V(C)$ is separable then $|V(C)| = 4$ or* 6.

To prove Proposition 4.1 we consider the block decomposition of $G - V(C)$. Now $G - V(C)$ has at least two endblocks, one of which may be the largest block B^* of Lemma 3.2. We consider an endblock other than B^*. By Lemma 3.1, this block is either a K_2 or an odd cycle. Let a_1 be a vertex of this block, which is not a cutvertex of $G - V(C)$. Since a_1 has degree 4 in G, a_1 must have at least two vertices of attachment a_2, a_3 in C.

Lemma 4.1. *Suppose the arc joining a_2 to a_3 in C is odd. Then $|V(C)| = 4$.*

Proof. Observe that there are two vertices of C, which attach to vertices of $G - V(C)$ other than the vertex a_1. This follows from the 3-connectedness of G. Denote these vertices by d, f. Suppose d does not coincide with either a_2 or a_3 and let arc(a_2, a_3) be the arc of C joining a_2 to a_3, which does not contain d. Let C' be the cycle consisting of a_1 and arc(a_2, a_3). Since C' is even, it follows from (b) of Lemma 3.2 that $V(C) - V(\text{arc}(a_2, a_3)) = \{d\}$. We conclude that if neither d nor f coincides with a_2 or a_3 then $|V(C)| = 4$.

Assume now that $d = a_2,\ f \neq a_3$, and $|V(C)| > 4$. We have then from the above argument that if arc(a_2, a_3) is the arc of C joining a_2 to a_3, which does not contain f then $V(C) - V(\text{arc}(a_2, a_3)) = \{f\}$. Suppose first that no internal vertex of arc(a_2, a_3) attaches to a vertex of $G - V(C)$. Since $|V(C)| > 4$, the arc, arc(a_2, a_3), has at least five vertices. Consider the graph H on the vertices of arc(a_2, a_3). Then H is connected, $|V(H)| \geq 5$, and all but two vertices of H have degree at least 3 (since they could attach to f). It follows from Lemma 2.1 that H contains an even cycle. Since f attaches to $G - V(C)$ we obtain a contradiction to (b) of Lemma 3.2. Suppose next that an internal vertex of arc(a_2, a_3) does attach to a vertex of $G - V(C)$. From previous argument, we can assume this vertex must be a_1. Moreover, we can similarly argue that there is precisely one internal vertex of arc(a_2, a_3), which attaches to a_1, say the vertex a_4, and the arc, arc$(a_2, a_4) \subset$ arc(a_2, a_3) is even. Let H be the graph on the vertices of arc(a_2, a_3). Then H is connected with $|V(H)| \geq 5$. The vertices a_2, a_3 have degree at least 1 in H and a_4 has degree at least 2. All other vertices have degree at least 3. Observe also that H contains two paths, from a_4 to a_2 and from a_4 to a_3, which contain an even number of vertices and are disjoint except for the common vertex a_4. It follows by Lemma 2.1 that H contains an even cycle. Note that the evenness of the paths a_4 to a_2 and a_4 to a_3 is required to rule out the configuration in Figure 2. Again, we have a contradiction to (b) of Lemma 3.2 since f attaches to $G - V(C)$.

We can assume now that $d = a_2,\ f = a_3,\ |V(C)| > 4$. Since G is 3-connected, there is a third vertex a_4 of C, which attaches to a_1. Let arc(a_2, a_3) be the arc of C

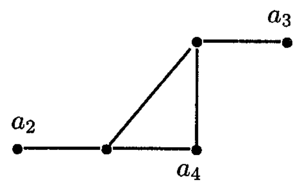

FIGURE 2.

joining a_2 to a_3, which contains a_4. Thus we must have that $\mathrm{arc}(a_2, a_4) \subset \mathrm{arc}(a_2, a_3)$ is even, and that a_4 is the only internal vertex of $\mathrm{arc}(a_2, a_3)$, which attaches to $G - V(C)$. Suppose now there is a vertex a_5 of $V(C) - V(\mathrm{arc}(a_2, a_3))$, which attaches to a_1. Then the arc joining a_2 to a_5 must be even, in order not to contradict (b) of Lemma 3.2. In that case, the arc joining a_4 to a_5 is odd. Since $d, f \neq a_4, a_5$ we conclude C has order 4. Suppose finally that no vertex of $V(C) - V(\mathrm{arc}(a_2, a_3))$ attaches to $G - V(C)$. Let H be the graph on the vertices $V(C) - \{a_2\}$. Then H is connected, $|V(H)| \geq 5$, and all but two vertices of H have degree at least 3. It follows from Lemma 2.1 that H contains an even cycle. Since a_2 attaches to $G - V(C)$, we have a contradiction to (b) of Lemma 3.2. ■

Corollary 4.1. *Suppose $G - V(C)$ is separable and has an endblock which is a K_2. Then C has order* 4.

Proof. The endblock, which is a K_2, has a vertex a_1 of degree 1 in $G - V(C)$. Hence a_1 has three vertices of attachment in C. Of these, we can choose two of them, a_2, a_3 such that the arc of C joining a_2 to a_3 is odd. ■

Lemma 4.2. *Suppose $G - V(C)$ is separable and has at least two endblocks different from the largest block B^* of Lemma* 3.2. *Then C has order* 4.

Proof. By Corollary 4.1 we may assume that two of the endblocks are odd cycles. We consider one of these odd cycles, and let a_1, a_1' be adjacent vertices of the cycle, neither of which is a cutvertex of $G - V(C)$. It follows that both a_1, a_1' have degree 2 in $G - V(C)$, whence they have each two vertices of attachment in C. By Lemma 4.1 we can assume there are vertices a_2, a_3 of C such that the arc of C joining a_2 to a_3 is even and a_1 attaches to a_2 while a_1' attaches to a_3. It is clear that the cycle consisting of $\mathrm{arc}(a_2, a_3)$ and the edge $a_1 a_1'$ is even. We can make similar assumptions about the second endblock, which is an odd cycle. Hence this cycle has adjacent vertices b_1, b_1', which are not cutvertices of $G - V(C)$. The vertex b_1 attaches to the vertex b_2 of C, while b_1' attaches to b_3 in C where the arc of C joining b_2 to b_3 is even.

We consider the situation where neither b_2 nor b_3 coincides with a_2 or a_3. Suppose first that b_2 and b_3 are on the same arc of C joining a_2 to a_3. Let arc(b_2, b_3) be the arc joining b_2 to b_3, which does not include a_2, a_3. We can form the even cycle C' consisting of b_1, b_1' and arc(b_2, b_3), which yields a contradiction to (b) of Lemma 3.2 unless $V(C) - V(\text{arc}(b_2, b_3)) = \{a_2, a_3\}$. Arguing similarly for a_1, a_1', a_2, a_3, we conclude C has order 4. Next suppose b_2 and b_3 are on different arcs of C joining a_2 to a_3. Arguing as above, we conclude that C has order 6. We may also assume that b_2 is adjacent to a_2 on C and b_3 adjacent to a_3. Consider now the largest block B^* of Lemma 3.2 and assume B^* has at most two cutvertices in $G - V(C)$. Since G is 3-connected there is a vertex v of C, which attaches to a vertex of B^* that is not a cutvertex of $G - V(C)$. Suppose now $v \neq a_2, a_3$ and is on the same arc of C joining a_2 to a_3, which contains b_2. Let C' be the even cycle consisting of a_1, a_1' and the arc, arc(a_2, a_3), which does not contain b_2. Let $b \in V(B^*)$ be the cutvertex of $G - V(C)$ whose removal disconnects B^* from the vertex b_1. Then there is a path $P(b, b_1)$ in $G - V(C)$ from b to b_1, which intersects B^* only in the vertex b and does not contain the vertices a_1, a_1'. Let B^{**} be the graph on the vertices $V(B^*) \cup V(P(b, b_1)) \cup \{v, b_2\}$. Then B^{**} is 2-connected and is disjoint from the even cycle C'. This contradicts (a) of Lemma 3.2. Let us assume now that B^* has at least three cutvertices in $G - V(C)$. Let b be as before and $a \in V(B^*)$ be the cutvertex of $G - V(C)$, whose removal disconnects B^* from the vertex a_1. Then there is an endblock B of $G - V(C)$, which remains connected to B^* if either a or b is removed from $G - V(C)$. Since there is a vertex v of C, which attaches to B we can argue as above to obtain a contradiction to (a) of Lemma 3.2.

We consider next the situation where $b_2 = a_2$ but $b_3 \neq a_3$. Let arc(a_2, a_3) be the arc of C joining a_2 to a_3, which does not contain b_3. We can form the even cycle C' consisting of a_1, a_1', and arc(a_2, a_3). Hence by Lemma 3.2(b), we must have $V(C) - V(arc(a_2, a_3)) = \{b_3, v\}$, and b_3 is adjacent to b_2 (since the arc joining b_2 to b_3 is even). Now by forming the cycle b_1, b_2, b_3, b_1' we see that Lemma 3.2(b) implies C has order 4.

Finally we consider the situation where $b_2 = a_2,\ b_3 = a_3$. We first show that we may assume the only vertices of attachment of $a_1,\ a_1',\ b_1,\ b_1'$ in C are $b_2 = a_2,\ b_3 = a_3$. To see this note that both b_1 and b_1' have two vertices of attachment in C. Since b_1 attaches to b_2, let us assume the other vertex of attachment is b_2' in C. By Lemma 4.1 we can assume the arc of C joining b_2 to b_2' is even. Suppose that $b_2' \neq b_3$. Now b_1' has a second vertex of attachment in C other than b_3. We denote it by b_3' and assume the arc of C joining b_3 to b_3' is even. It follows that the arc of C joining b_2' to b_3' is even. Since $b_2' \neq a_2, a_3$ this puts us back in the cases already considered. We conclude that C has order 4.

Let us assume now that all of the vertices $a_1,\ a_1',\ b_1,\ b_1'$ attach to both a_2 and a_3 and to no other vertices of C. Let H be the graph on $V(C) - \{a_2, a_3\}$. Since G is 3-connected it follows that H is connected to $G - V(C)$ and a_1, a_1' are not vertices of attachment of H in $G - V(C)$. Let C' be the cycle of order 4 consisting of the edges $a_1a_2,\ a_2a_1',\ a_1'a_3,\ a_3a_1$. Since the graph on $V(G) \cup V(H) - \{a_1, a_1', a_2, a_3\}$ is connected, we conclude from (b) of Lemma 3.2 that C has order 4. ■

Lemma 4.3. *Suppose $G - V(C)$ is separable and has two endblocks, one of the endblocks being the largest block B^* of Lemma 3.2 and the other endblock an odd cycle. Then C has order 4 or 6.*

Proof. Let a_1, a_1' be adjacent vertices of the endblock of $G - V(C)$, which is an odd cycle and assume neither a_1 nor a_1' is a cutvertex of $G - V(C)$. As in Lemma 4.2 we may assume that a_1 attaches to a vertex a_2 of C and a_1' to a vertex a_3, such that the arc of C joining a_2 to a_3 is even. Since G is 3-connected, there are two vertices d,f of C, which attach to two vertices of B^* different from the cutvertex of B^* in $G - V(C)$. Suppose first d,f do not coincide with either a_2 or a_3. If d and f are on different arcs of C joining a_2 to a_3 then it follows from (b) of Lemma 3.2 that C has order 6. If d and f are on the same arc of C joining a_2 to a_3, let us denote by arc(d,f) the arc of C joining d to f, which does not include a_2 or a_3. Let arc(a_2, a_3) be the arc of C joining a_2 to a_3, which does not include d,f. Then we may form the even cycle C' from a_1, a_1' and arc(a_2, a_3). Let B^{**} be the graph on $V(B^*) \cup V(\text{arc}(d,f))$. Then B^{**} is 2-connected and lies outside the even cycle C'. This contradicts (a) of Lemma 3.2.

We assume $d = a_2$, $f \neq a_3$. Let arc(a_2, a_3) be the arc of C joining a_2 to a_3, which does not include f. It follows by (b) of Lemma 3.2 that $V(C) - V(\text{arc}(a_2, a_3)) = \{f, v\}$. Consider now an internal vertex w of arc(a_2, a_3). If w attaches to a vertex of $G - V(C)$ other than a_1, a_1' then by (b) of Lemma 3.2, we may conclude C has order 6. If w attaches to a_1 then arc$(a_2, w) \subset$ arc(a_2, a_3) is even or else C has order 4. Let us assume now C has order larger than 6. By arguing as in Lemma 4.1, it follows that there is at most one internal vertex a_4 of arc(a_2, a_3), which attaches to a vertex of $G - V(C)$. Let H be the graph on the vertices $V(\text{arc}(a_2, a_3)) \cup \{v\}$. Thus H is connected and has at least seven vertices. At most two vertices, a_2 or a_3, and v have degree one in the graph, and all but the four vertices a_2, a_3, a_4, v have degree at least three (since any vertex can attach to f). There is also a path in H, which goes through every vertex of H. This rules out the configuration in Figure 3. We conclude from Lemma 2.1 that H must contain an even cycle. Since f attaches to $G - V(C)$, this contradicts (b) of Lemma 3.2. We conclude C has order 4 or 6.

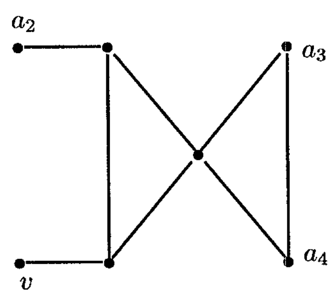

FIGURE 3.

Finally we assume $d = a_2, f = a_3$. Suppose there is a vertex $f' \in V(C) - \{a_2, a_3\}$, which attaches to a vertex of $G - V(C)$ other than a_1, a_1'. Then we may argue as in the previous paragraph to conclude C has order 4 or 6. Suppose next that there is a vertex $a_4 \in V(C) - \{a_2, a_3\}$, which attaches to a_1. Then we may assume the arc of C joining a_2 to a_4 is even. Consider the second vertex of attachment a_4' of a_1' in C. We may assume the arc of C joining a_4 to a_4' is even. Since $f \neq a_4, a_4'$, this puts us in the situation dealt with in the previous paragraph. We conclude that if C has order larger than 6, then a_2, a_3 are the vertices of attachment in C of a_1, a_1'. Further, no vertex of $V(C) - \{a_2, a_3\}$ attaches to $G - V(C)$. This contradicts the 3-connectedness of G. ■

***Proof of Proposition* 4.1.** This follows from the previous lemmas. Note that if C has order 6 then $G - V(C)$ has just two endblocks, one of which is B^*. ■

5. *G*−*V*(*C*) SEPARABLE, *C* OF ORDER 6

In this section we prove the following.

Proposition 5.1. *Suppose G is a* 3*-connected graph of minimum degree* 4 *and C is a cycle of order* 6 *satisfying* (a) *and* (b) *of Lemma* 3.2. *If $G - V(C)$ is separable then $G - E(C)$ is* 2*-connected.*

To prove Proposition 5.1 we first observe from Lemmas 4.2 and 4.3 that $G - V(C)$ has exactly two endblocks, one of which is the largest block B^* of Lemma 3.2. By Corollary 4.1 the other endblock is an odd cycle. Let a_1, a_1' be adjacent vertices of the cycle, neither of which is a cutvertex of $G - V(C)$. Then both a_1 and a_1' have two vertices of attachment in C.

Lemma 5.1. *Suppose a_1 attaches to a vertex $a_2 \in V(C)$ and a_1' to $a_3 \in V(C)$. Then a_2 and a_3 are not adjacent on C.*

Proof. Let us assume a_2 and a_3 are adjacent on C. Arguing as in Lemma 4.3 we see that there are two vertices d, f of C, which attach to two vertices of B^* different from the cutvertex of B^* in $G - V(C)$. If any vertex in $V(C) - \{a_2, a_3\}$ attaches to a vertex of $G - V(C)$ other than a_1, a_1' then we have a contradiction to (b) of Lemma 3.2, since we can form the cycle of order 4 with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$. We conclude that $d = a_2, f = a_3$. Since G is 3-connected we may also conclude that there is at least one vertex of $V(C) - \{a_2, a_3\}$ which attaches to a_1 or a_1'. Suppose the vertex is a_2' and that it attaches to a_1. In view of Lemma 4.1 the arc of C joining a_2 to a_2' is even. Since the arc of C joining a_3 to a_2' is odd, it follows again from Lemma 4.1 that the second vertex of attachment of a_1' in C cannot be a_2'. Suppose it is a_3' and let us assume $a_3' \neq a_2$. Let $\mathrm{arc}(a_2', a_3')$ be the arc of C, which does not contain a_2, a_3. Then $\mathrm{arc}(a_2', a_3')$ is even, whence one can form an even cycle C' consisting of a_1, a_1' and $\mathrm{arc}(a_2', a_3')$. Since C' is disjoint from d, f, we may argue as in Lemma 4.3 to obtain a contradiction to (a) of Lemma 3.2.

Assume now that $a_3' = a_2$. If a_2' is adjacent to a_2, then by forming the cycle of order 4 consisting of the edges $a_1a_2', a_2'a_2, a_2a_1', a_1'a_1$, we obtain a contradiction to (b) of Lemma 3.2. We conclude that the only vertices of attachment of a_1, a_1' in C are a_2, a_3, a_2'. The vertex a_1 attaches to a_2, a_2' and the arc joining a_2 to a_2' has order 4. The vertex a_1' attaches to a_2 and a_3 with a_2, a_3 adjacent. We can denote the vertices of C in order as $a_2, a_3, a_4, a_2', a_5, a_6$. We have already seen that a_4, a_5, a_6 have no vertices of attachment in $G - V(C)$. Suppose now the edge a_2a_4 exists. Let C' be the cycle of order 4 consisting of the edges $a_1'a_2, a_2a_4, a_4a_3, a_3a_1'$. Since a_2' attaches to a_1 it is clear that we have a contradiction to (b) of Lemma 3.2. Since the vertex a_4 has minimum degree 4 in G we conclude that the edge a_4a_6 must exist. Hence we can form the cycle C' of order 4 consisting of the edges $a_6a_5, a_5a_2', a_2'a_4, a_4a_6$. Since C' is disjoint from d,f we can argue again as in Lemma 4.3 to obtain a contradiction to (a) of Lemma 3.2. ■

Lemma 5.2. *The vertices a_1, a_1' have precisely 4 vertices of attachment in C.*

Proof. Note that by Lemma 4.1 the vertices a_1, a_1' have at most four vertices of attachment in C. By Lemma 5.1 we can assume that a_1 attaches to a vertex a_2 of C, a_1' to a vertex a_3 of C and the arc joining a_2 to a_3 has order 4. Suppose now a_1 also attaches to a vertex $a_2' \neq a_3$ of C. Then by Lemma 4.1 the vertices a_2, a_2' are adjacent on C. If a_1' attaches to a_2 then we have a contradiction to Lemma 5.1. If a_1' attaches to a_2' then we have a contradiction to Lemma 4.1. We conclude that a_1, a_1' have either four vertices of attachment in C or two vertices of attachment.

Assume now a_1, a_1' have two vertices of attachment in C, whence a_1, a_1' attach to both a_2, a_3. Let C' be the cycle of order 4 consisting of the edges a_1a_2, a_2a_1', $a_1'a_3, a_3a_1$. Observe now that since G is 3-connected the graph on $V(G) - V(C')$ is connected. Hence we have a contradiction to (b) of Lemma 3.2. ■

From Lemmas 5.1 and 5.2, we see that a_1 attaches to two vertices a_2, a_2' of C and a_1' to two different vertices a_3, a_3'. The arc joining a_2 to a_3 has order 4, a_2' is adjacent on C to a_2, a_3' is adjacent on C to a_3, and the arc joining a_2' to a_3' has order 4. We have also seen that there are vertices d,f of C, which attach to two vertices of B^* different from the cutvertex of B^* in $G - V(C)$. We then have the following.

Lemma 5.3. *Suppose the* arc *of C joining d to f has order* 4. *Then $G - E(C)$ is 2-connected.*

Proof. Let us assume that d is not a vertex of attachment in C of a_1, a_1'. Then wlog, we may write the vertices of C in order as $a_2, d, a_3', a_3, f, a_2'$. Suppose now d has a second vertex of attachment in $G - V(C)$. This vertex of attachment cannot be a_1 or a_1'. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_2', a_2'f, fa_3$, $a_3a_1', a_1'a_1$. By arguing as in Lemma 4.3 we see we can construct a block containing B^* and d, which is disjoint from C'. This contradicts (a) of Lemma 3.2. Suppose next that the edge da_3 exists. Let C' be the cycle consisting of the edges $a_1'a_3, a_3d, da_3', a_3'a_1'$. Since f attaches to a vertex of $G - V(C)$ other than a_1' we have a contradiction to (b) of Lemma 3.2. Suppose the edge da_2' exists. Let C' be the cycle consisting of the edges $a_1a_2, a_2d, da_2', a_2'a_1$. Since f attaches to a vertex of $G - V(C)$ other than a_1 we have again a contradiction to (b) of Lemma 3.2. Since d has minimum degree 4 in G, we conclude that the edge df must exist.

Consider next the vertex a_2. Suppose the edge a_2f exists. Let C' be the cycle consisting of the edges $a_1a_2, a_2f, fa_2', a_2'a_1$. Since d attaches to a vertex of $G - V(C)$ other than a_1, we have a contradiction to (b) of Lemma 3.2. Next suppose the edge a_2a_3 exists. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$. Since d and f attach to vertices of $G - V(C)$ other than a_1, a_1', we obtain a contradiction to (b) of Lemma 3.2. A similar argument rules out the possibility of an edge a_2a_3'. We conclude that, since a_2 has minimum degree 4, it must attach to a vertex of $G - V(C)$ other than a_1, a_1'. If this vertex of attachment is different from the vertex of attachment of d in B^* then we can construct a block containing B^* and the vertices a_2, d of C. This block is disjoint from the cycle C' consisting of the edges a_1a_2', $a_2'f$, fa_3, a_3a_3', $a_3'a_1'$, $a_1'a_1$. We have therefore a contradiction to (a) of Lemma 3.2. It follows that a_2 and d have the same vertex of attachment in $G - V(C)$ and it is a non-cutvertex of B^*. A similar argument shows that a_3' has a vertex of attachment in B^*, which is the same as the vertex of attachment of d. By symmetry, we can make similar conclusions about the vertices a_2', f, a_3.

Next we wish to show that $G - V(C)$ has just two blocks. One of the blocks is the largest block B^* and the other is an odd cycle, which we shall show must be a triangle. Suppose first that the odd cycle containing a_1, a_1' is not a triangle. Then we may assume wlog that there is a vertex a_1'' of the cycle adjacent to a_1', which is not a cutvertex of $G - V(C)$. Hence a_1'' has two vertices of attachment in C. It is easy to see that no matter which vertex of attachment a_1'' has in C one can form a cycle of order 4 with either a_1, a_1', a_1'' and one vertex of C or a_1', a_1'' and two adjacent vertices of C. Since this yields a contradiction to (b) of Lemma 3.2 we conclude the odd cycle containing a_1, a_1' is a triangle. Let a_1'' be the third vertex of this triangle and assume $a_1'' \notin B^*$. If a_1'' has a vertex of attachment in C then we can argue as previously to obtain a contradiction to (a) of Lemma 3.2. Hence a_1'' is the cutvertex of two odd cycles, one of which is the triangle containing a_1, a_1'. Let a_1''' be a vertex of the other odd cycle, adjacent to a_1'', which is not a cutvertex of $G - V(C)$. Since a_1''' has a vertex of attachment in C we can argue as before to obtain a contradiction to Lemma 3.2. If a_1''' attaches to d or f then we obtain a contradiction to (a) of Lemma 3.2. If a_1''' attaches to any other vertex of C we obtain a contradiction to (b) of Lemma 3.2. We conclude that $a_1'' \in B^*$. Hence the graph of G must be as in Figure 4. The cycle C is denoted by the dashed lines. It is easy to see that $G - E(C)$ is 2-connected.

The other possibility for the positions of d,f in C is $d = a_2$, $f = a_3$. We denote the vertices of C in order as a_2, a_2', a_4, a_3, a_3', a_5. We consider the vertex a_4. If the edge a_4a_2 exists we get a contradiction to (b) of Lemma 3.2. Similarly if the edge a_4a_3' exists we have a contradiction. Hence a_4 has a vertex of attachment in $G - V(C)$. By Lemma 4.1 it cannot be a_1 or a_1'. Hence it must be the same vertex of attachment of f in B^*. Otherwise we have a contradiction to (a) of Lemma 3.2.

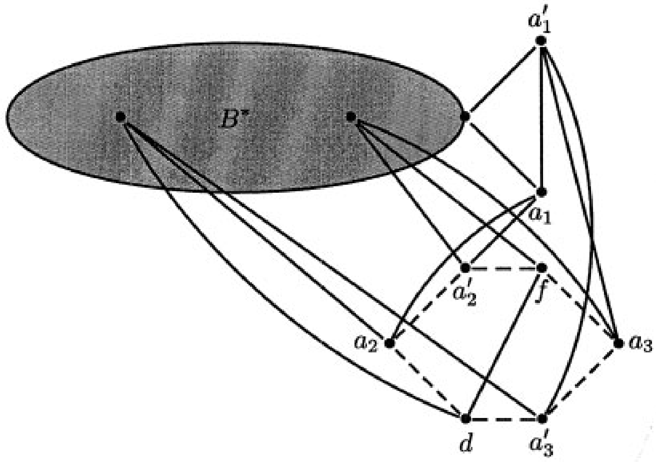

FIGURE 4.

Similarly a_5 has the same vertex of attachment in B^* as d. This now puts us in the case we already considered. ∎

Lemma 5.4. *Suppose an arc of C joining d to f has order* 3 *or* 2. *Then $G - E(C)$ is* 2*-connected.*

Proof. We assume first that an arc of C joining d to f has order 3 and that not both d,f are vertices of attachment of a_1, a_1' in C. We can assume wlog that $f = a_3$ and the vertices of C in order are $a_2, d, a_3', a_3, a_4, a_2'$. Consider the vertex a_4. If either of the edges a_4a_2 or a_4a_3' exist then we can obtain a contradiction to (b) of Lemma 3.2. Hence a_4 has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. To avoid a contradiction to (a) of Lemma 3.2 we see this vertex of attachment must be the same as the vertex of attachment of f in B^*. This puts us into the situation considered in Lemma 5.3.

Let us assume next that the arc of C joining d to f has order 2. To avoid a contradiction to (a) of Lemma 3.2 we can assume that $d = a_2$, $f = a_2'$. We write the vertices of C in order as $a_2, a_4, a_3', a_3, a_5, a_2'$. Consider now the vertex a_4. If the edge a_4a_3 exists we can form the cycle C' consisting of the edges $a_1'a_3, a_3a_4, a_4a_3', a_3'a_1'$. This contradicts (b) of Lemma 3.2 since a_2 attaches to $a_1 \in G - V(C')$. If the edge a_4a_2' exists we form the cycle C' consisting of the edges $a_1a_2', a_2'a_4, a_4a_2, a_2a_1$. This again gives a contradiction to (b) of Lemma 3.2 since a_3 attaches to $a_1' \in G - V(C')$. We conclude a_4 has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. In order to avoid a contradiction to (a) of Lemma 3.2, we see that this vertex of attachment must be the same as the vertex of attachment of d in B^*. This puts us into the situation discussed in the previous paragraph.

Finally we assume that an arc of C joining d to f has order 3 and that both d,f are vertices of attachment of a_1, a_1' in C. Then we can assume wlog that

$d = a_2'$, $f = a_3$. We write the vertices of C in order as $a_2, a_4, a_3', a_3, a_5, a_2'$. Consider now the vertex a_5. If either of the edges a_5a_2 or a_5a_3' exist, we obtain a contradiction to (b) of Lemma 3.2 as in the previous paragraph. Hence a_5 has a vertex of attachment in $G - V(C)$, which is different from a_1, a_1'. To avoid a contradiction to (a) of Lemma 3.2, a_5 can have only one vertex of attachment and it must be the same as the vertex of attachment of d in B^*. Similarly it must be the same as the vertex of attachment of f in B^*. Since the vertices of attachment of d and f in B^* are different, we have a contradiction. ■

***Proof of Proposition* 5.1.** This follows from the previous lemmas. ■

6. *B** WITH ONE CUTVERTEX IN *G*−*V*(*C*), *C* OF ORDER 4

In this section we prove the following.

Proposition 6.1. *Suppose G is a 3-connected graph of minimum degree 4 and the cycle C of Lemma 3.2 has order 4. If B^* has exactly one cutvertex in $G - V(C)$ then there is a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Let a denote the cutvertex of B^* in $G - V(C)$. Since G is 3-connected there are two vertices d, f of C, which attach to distinct vertices of B^*. We may assume that the vertex of attachment for d is different from a and the vertex of attachment for f is also different from a, unless B^* happens to be a K_2.

Since B^* has a cutvertex in $G - V(C)$, there are other endblocks of $G - V(C)$ apart from B^*. We have already seen in Lemma 3.1 that these endblocks must be either an odd cycle or K_2. We first assume that $G - V(C)$ has an endblock, which is a K_2. We denote by a_1 the vertex of the K_2, which has degree 1 in $G - V(C)$. Since a_1 has degree at least 4 in G, it has at least three vertices of attachment in C. We denote these vertices of attachment, in order on C, by a_2, a_3, a_4. There is a fourth vertex of C, which we denote by α. The vertex α may or may not attach to a_1.

Lemma 6.1. *Suppose $G - V(C)$ has just two blocks, B^* and a K_2. Then there is a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Proof. Note that the second vertex of the K_2 is the cutvertex a of B^* in $G - V(C)$. We consider the situation where all four vertices of C attach to a_1. Let us first assume that C has no chords, whence every vertex of C attaches to B^*. Suppose now $d = a_2$. Let C' be the cycle consisting of the edges $a_1a_3, a_3a_2, a_2\alpha, \alpha a_1$. Then $G - V(C')$ is the union of B^* and the vertex a_4. Since a_4 attaches to B^*, we conclude $G - V(C')$ is connected. One can also easily see that $G - E(C')$ is 2-connected. Next we assume that C has exactly one chord, which we can take to be the edge a_2a_4. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_4, a_4\alpha, \alpha a_1$. Thus $G - V(C')$ is the union of B^* and the vertex a_3. Since

a_3 has degree at least 4, it has an edge to B^*, whence $G - V(C')$ is connected. Suppose now $d = \alpha$. Then we can form the path consisting of the edges $aa_1, a_1a_4, a_4a_3, a_3a_2, a_2\alpha, \alpha d^*$, where $d^* \neq a$ is the vertex of attachment of d in B^*. None of these edges are edges of the cycle C'. We conclude that $G - E(C')$ is 2-connected. The alternative to $d = \alpha$ is that α attaches to the cutvertex a of B^*. In that case, d must be one of the vertices a_2, a_3, a_4. It is easy to see that no matter which vertex d is, the graph $G - E(C')$ is 2-connected. Finally we assume that C has two chords, whence the graph on the vertices of C is K_4. Hence we may form a cycle C' on the vertices of C such that d and f are adjacent on C'. It is clear that $G - E(C')$ is 2-connected.

We consider the situation where the vertex α of C does not attach to a_1. Suppose α has two edges to B^* and let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Then $G - V(C')$ is 2-connected and strictly larger than B^*, yielding a contradiction to (a) of Lemma 3.2. We conclude that the edge αa_3 must exist. Suppose the edge a_2a_4 does not exist. Then the vertices a_2, a_4, α attach to B^*. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Since α attaches to B^*, it follows that $G - V(C')$ is connected. One can also see that $G - E(C')$ is 2-connected by considering the possible positions of $d \in V(C)$. Finally we assume C has two chords. If d and f are adjacent on C then it is easy to see that $G - E(C)$ is 2-connected. Suppose then d and f are non-adjacent on C with $d = \alpha$, whence $f = a_3$. Let C' be the cycle consisting of the edges a_1a_2, a_2a_4, a_4a_3, a_3a_1. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. One similarly sees that if $d = a_3$, $f = \alpha$, then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. The only alternative left is to suppose $d = a_2$, $f = a_4$ and α attaches to $a \in B^*$. Let C' be the cycle consisting of the edges a_1a_2, a_2a_3, a_3a_4, a_4a_1. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. ■

Lemma 6.2. *Suppose $G - V(C)$ has an endblock which is a K_2 and that a_1 is the vertex of the K_2 which has degree 1 in $G - V(C)$. Assume all four vertices of C attach to a_1. Then there is a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Proof. Suppose first that $G - V(C)$ has a second endblock which is a K_2 and a_1' is its vertex of degree 1 in $G - V(C)$. Then a_1' attaches to three vertices of C. Suppose one of these vertices is d. Let C' be the cycle with vertices a_1 and the three vertices of C different from d. Since d attaches to a vertex of B^* different from its cutvertex in $G - V(C)$, it follows that $G - V(C')$ has a block which strictly contains B^*, contradicting (a) of Lemma 3.2. If d is not a vertex of attachment for a_1' in C then we can form the cycle C'' consisting of a_1' and the three vertices of attachment of a_1' in C. Again we see that $G - V(C'')$ has a block strictly containing B^*. We conclude that $G - V(C)$ has only one endblock, which is a K_2.

Next suppose $G - V(C)$ has an endblock, which is an odd cycle. Let a_1', a_1'' be adjacent vertices of the cycle, which are not cutvertices of $G - V(C)$. Note in this case that B^* cannot be a K_2, whence d,f attach to distinct vertices of B^* different from a. If a_1' or a_1'' attach to d or f then we can repeat the argument of the previous paragraph to obtain a contradiction. Hence both a_1', a_1'' attach to the vertices of C different from d,f. In that case we can form a cycle C' of order 4 from the vertices a_1', a_1'' and the vertices of C different from d,f. Since a_1 attaches to d, we again get a contradiction to (a) of Lemma 3.2. We conclude $G - V(C)$ does not have an endblock, which is an odd cycle.

We are reduced to the situation where $G - V(C)$ has just two endblocks, one of which is B^* and the other the K_2 with a_1 as vertex of degree 1 in $G - V(C)$. Let a_1' be the second vertex of the K_2. If $a_1' = a$ we are in the situation of Lemma 6.1 so we shall assume $a_1' \neq a$. Now a_1' is a cutvertex in $G - V(C)$ of another block. We shall show this block is also a K_2 and its second vertex is $a \in B^*$. To see this, we proceed by contradiction. Let $a_1'' \neq a$ be a vertex of the block adjacent to a_1'. Then a_1'' has a vertex of attachment in C. Suppose a_1'' attaches to d or f, and let C' be the cycle consisting of a_1, and the three vertices of C different from the vertex of attachment of a_1''. Then $G - V(C')$ has a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If a_1'' attaches to neither d nor f, then let C' be the cycle consisting of a_1, a_1', a_1'' and the vertex of attachment of a_1'' in C. Again $G - V(C')$ has a block strictly larger than B^*, yielding a contradiction.

We are now in the situation where $G - V(C)$ consists of three blocks, namely B^* and the two K_2's, aa_1' and $a_1'a_1$. Evidently a_1' has two vertices of attachment in C. If either of these coincides with d or f, we get a contradiction as in the previous paragraph. Hence they do not coincide with d or f. If d and f are adjacent on C, let C' be the cycle consisting of a_1, a_1' and the two vertices of C different from d,f. Then $G - V(C')$ contains a block larger than B^*, a contradiction. Hence d and f are non-adjacent on C, whence we get the situation in Figure 5. It is clear that $G - E(C)$ is 2-connected. ■

Lemma 6.3. *Suppose $G - V(C)$ has an endblock which is a K_2. If C has two chords then $G - V(C)$ has just two blocks.*

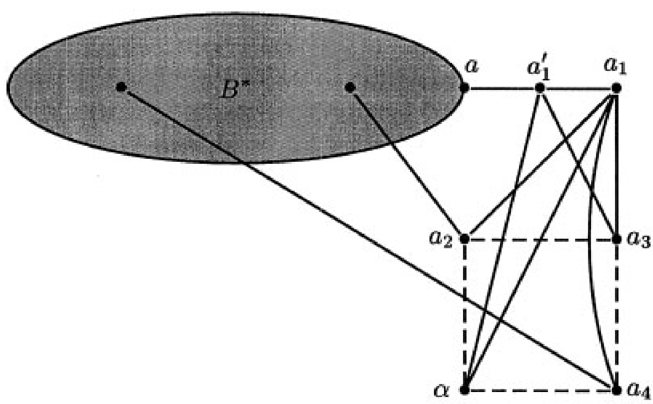

FIGURE 5.

Proof. We first show that $G - V(C)$ has only two endblocks, B^* and the K_2 with a_1 as vertex. To see this, suppose there is a third endblock which is a K_2 and let a_1' be its vertex of degree 1 in $G - V(C)$. Then a_1' has three vertices of attachment in C, whence it must attach to d or f, say d. One can construct a cycle C' from a_1 and the three vertices of C other than d. It is clear that $G - V(C')$ contains a block strictly larger than B^*, yielding a contradiction to (a) of Lemma 3.2. We conclude the third endblock cannot be a K_2. Suppose then that it is an odd cycle. Let a_1', a_1'' be adjacent vertices of the odd cycle, which are not cutvertices of $G - V(C)$. If a_1' or a_1'' attach to d or f then we can argue as above to obtain a contradiction to (a) of Lemma 3.2. Hence a_1' and a_1'' both attach to the two vertices of C different from d,f. Let C' be the cycle with vertices a_1', a_1'' and the two vertices of C other than d,f. It is clear that $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. We conclude that $G - V(C)$ has only two endblocks.

Finally let us assume the second vertex of the K_2 which is an endblock of $G - V(C)$ is $a_1' \neq a$. Now a_1' is a cutvertex for exactly two blocks. Suppose the other block is also a K_2. Then a_1' has two vertices of attachment in C. If a_1' attaches to d or f, we can argue as in the previous paragraph to obtain a contradiction. Hence a_1' attaches to two vertices v, w of C different from d,f. One of these vertices, say v, must attach to a_1. Let C' be the cycle consisting of the edges $a_1v, vw, wa_1', a_1'a_1$. Note that since the graph on $V(C)$ is K_4 the edges vw and df exist. It is clear therefore that $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. Suppose the other block with cutvertex a_1' is an odd cycle. Then there is a vertex a_1'' of this cycle, adjacent to a_1', which is not a cutvertex of $G - V(C)$. Hence a_1'' has two vertices of attachment in C. We can assume as above that neither of these vertices of attachment is d or f. Hence a_1'' attaches to a vertex v of C, different from d and f, which is also a vertex of attachment for a_1. Let C' be the cycle consisting of the edges $a_1v, va_1'', a_1''a_1', a_1'a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, whence we have a contradiction. We conclude the second vertex of the endblock K_2 is a. ■

Lemma 6.4. *Suppose $G - V(C)$ has an endblock which is a K_2 and the vertex a_1 of degree 1 in $G - V(C)$ does not attach to a vertex α of C. Suppose further that α is not the vertex of a chord of C. Then d and f must coincide with a_2 and a_4.*

Proof. We first observe that d or f cannot coincide with α. To see this, let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Now α has two edges to $G - V(C')$. If one of these attaches to a vertex of B^* other than a, then there is a block of $G - V(C')$ strictly larger than B^*, contradicting (a) of Lemma 3.2.

Next we show that d and f cannot be adjacent vertices of C. We proceed by contradiction. Hence let us assume $d = a_2,\ f = a_3$. We first show that the only endblocks of $G - V(C)$ are B^* and the K_2 with a_1 as vertex. To see this suppose there is a third endblock which is a K_2 and its vertex of degree 1 in $G - V(C)$ is

a_1'. Suppose a_1' attaches to a_3. Let C' be the cycle consisting of the edges $a_1a_2, a_2\alpha, \alpha a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence the vertices of attachment of a_1' are a_2, α, a_4. We can now form the cycle C' consisting of the edges $a_1'a_2, a_2\alpha, \alpha a_4$, a_4a_1'. Again we see that $G - V(C')$ contains a block strictly larger than B^*, a contradiction. We conclude $G - V(C)$ does not have a third endblock, which is a K_2. Suppose there is a third endblock which is an odd cycle. Let a_1', a_1'' be adjacent vertices of the cycle, which are not cutvertices of $G - V(C)$. As in the previous argument we see that neither a_1' nor a_1'' can attach to a_3. Now one can see that it is possible to form an even cycle C' from four of the vertices $a_1', a_1'', a_2, \alpha, a_4$. In that case, $G - V(C')$ contains a block strictly larger than B^*, a contradiction.

We are now in the situation where the only endblocks of $G - V(C)$ are B^* and the K_2 with a_1 as vertex. We have already observed that α has two edges to $G - V(C)$ and that the only vertex of B^* it can attach to is the cutvertex a of B^* in $G - V(C)$. Hence α attaches to two vertices a_1', a_1'' of $G - V(C)$ with $a_1' \notin B^*$ and $a_1'' = a$ if $a_1'' \in B^*$. Furthermore, there are paths $P(a_1', a_1)$ from a_1' to a_1 and $P(a_1'', a)$ from a_1'' to a in $G - V(C)$, which do not intersect and only intersect B^* in the vertex a. Suppose the path $P(a_1', a_1)$ has an even number of vertices, and let C' be the even cycle consisting of the edges $a_4\alpha, a_4a_1, \alpha a_1'$ and the path $P(a_1', a_1)$. Then $G - V(C')$ contains a block strictly larger than B^*, a contradiction. Hence $P(a_1', a_1)$ has an odd number of vertices. We show that α is the only vertex of attachment of a_1' in C. To see this, first note that a_3 cannot be a vertex of attachment for a_1' by the argument of the previous paragraph. Suppose now a_2 is a vertex of attachment for a_1'. Let C' be the cycle consisting of the edges a_2a_1, a_2a_1' and the path $P(a_1', a_1)$, whence C' is an even cycle. Let $f^* \neq a$ be the vertex of attachment of f in B^*. Since the path consisting of the edges $f^*a_3,\ a_3a_4,\ a_4\alpha,\ \alpha a_1''$ and $P(a_1'', a)$ is disjoint from C', it follows that $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. A similar argument yields a contradiction if a_1' attaches to a_4.

We are now in the situation where the path $P(a_1', a_1)$ has an odd number of vertices and the only vertex of attachment in C for a_1' is α. It follows that a_1' is a cutvertex of $G - V(C)$ and one of the two blocks with cutvertex a_1' is an odd cycle. Let a_1''' be a vertex of this cycle adjacent to a_1', which is not a cutvertex of $G - V(C)$. Then a_1''' has two vertices of attachment in C. If a_1''' attaches to a_3 then we get a contradiction as in the previous paragraph. If a_1''' attaches to α then we again get a contradiction since there is a path from a_1''' to a_1, which has an even number of vertices. We conclude that a_1''' attaches to both a_2 and a_4. Let C' be the cycle consisting of the edges $\alpha a_4,\ a_4a_1''',\ a_1'''a_1',\ a_1'\alpha$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude that d and f are not adjacent vertices of C and neither of them coincides with α. ■

Lemma 6.5. *Suppose $G - V(C)$ has an endblock which is a K_2 and the vertex a_1 of degree 1 in $G - V(C)$ does not attach to a vertex α of C. Suppose further that α is not the vertex of a chord of C. Then*

(1) *There are two endblocks of* $G - V(C)$ *which are* K_2*'s. Suppose* a_1' *is the vertex of degree* 1 *in* $G - V(C)$ *of the second* K_2*. Then the vertices of attachment of* a_1' *in* C *are* a_2, α, a_4.
(2) *Any endblock of* $G - V(C)$ *other than* B^* *which is an odd cycle is a triangle. The vertices of attachment in* C *of the non-cutvertices of the triangle are precisely* a_2, a_4.
(3) $G - V(C)$ *has at most two cutvertices. If* a *is the only cutvertex then* a_3, α *attach to* a.
(4) *If* $G - V(C)$ *has a second cutvertex* a_1''*, then* a_1'' *attaches to both* α *and* a_3 *in* C*. The only blocks of* $G - V(C)$ *with cutvertex* a *are* B^* *and the* K_2, $a_1''a$ *with vertices* a_1'' *and* a.

Proof. By Lemma 6.4 we can assume that $d = a_2, f = a_4$. Further, if α attaches to a vertex of B^* it must be the cutvertex a of B^* in $G - V(C)$.

We show first that there is at most one other endblock which is a K_2 and that the vertices of attachment of its vertex a_1' of degree 1 in $G - V(C)$ satisfy (1) in the statement of the lemma. To see this, suppose such a K_2 exists and a_3 is a vertex of attachment for a_1'. Then a_1' also attaches to either a_2 or a_4, say a_2. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_1', a_1'a_3, a_3a_1$. Since α does not attach to a_1, it attaches to a vertex of $G - V(C')$. Hence $G - V(C')$ contains a block strictly larger than B^*, a contradiction. We conclude the vertices of attachment of a_1' in C are precisely a_2, α, a_4. A similar argument shows there can be no further endblocks which are K_2's.

Next we show that if there is an endblock other than B^*, which is an odd cycle, it must be a triangle. Further, the vertices of attachment in C of the non-cutvertices of the triangle satisfy (2) in the statement of the lemma. To see this, let a_1', a_1'' be adjacent vertices of the odd cycle, which are not cutvertices of $G - V(C)$. Suppose a_2 is not a vertex of attachment for either a_1' or a_1''. Then we can form an even cycle C' from four of the vertices $a_1', a_1'', a_3, a_4, \alpha$. Since $d = a_2$, $G - V(C')$ contains a block strictly larger than B^*, a contradiction. We conclude a_2 is a vertex of attachment for a_1' or a_1'' and similarly also a_4. It is not hard to see now that a_2, a_4 are the only vertices of attachment in C of a_1', a_1'', and the cycle containing a_1', a_1'' is a triangle.

We consider the cutvertices of $G - V(C)$. Let us suppose $G - V(C)$ has more than one cutvertex. Then there is a cutvertex $a_1'' \notin B^*$ with the property that the connected components of $G - V(C) - \{a_1''\}$ consist of the component containing B^*, and components $B - \{a_1''\}$, where B is an endblock of $G - V(C)$ with cutvertex a_1''. Suppose one of the endblocks B with cutvertex a_1'' is a triangle. Then we can form a cycle C' of order 4 from the vertices of the triangle and a_2. To avoid a contradiction to (a) of Lemma 3.2, it follows that both α and a_3 attach to a_1''. Further, there are two endblocks, of $G - V(C)$ which are K_2's and their cutvertex in $G - V(C)$ is a_1''.

Suppose none of the endblocks B with cutvertex a_1'' are triangles. Consider the situation where there is just one endblock B with cutvertex a_1''. Then B is a K_2, which we can assume wlog is the K_2, $a_1''a_1$. Further, a_1'' has a vertex of attachment β in C. If $\beta = a_3$, we let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_1'', a_1''a_1$. Since α has two vertices of attachment in $G - V(C)$, it follows that $G - V(C')$ contains a block strictly larger than B^*. Since this contradicts (a) of Lemma 3.2, we conclude a_3 does not attach to a_1''. Hence a_3 attaches to a vertex of $G - V(C) - \{a_1, a_1''\}$. Let C' be the cycle with vertices a_1, a_1'', β, and a vertex of C adjacent to β, which is also a vertex of attachment for a_1. Evidently $V(C')$ does not contain both a_2, a_4, or both α, a_3. Hence $G - V(C')$ contains a block larger than B^*, contradicting (a) of Lemma 3.2. We conclude there cannot be just one endblock B with cutvertex a_1''. It follows that the $K_2, a_1'a_1''$ exists. We may therefore form the cycle C' with edges $a_2a_1, a_1a_1'', a_1''a_1', a_1'a_2$. Since $G - V(C')$ cannot contain a block larger than B^*, we see that both α and a_3 attach to a_1''.

We have then that the endblocks B of $G - V(C)$ with cutvertex a_1'' consist of triangles and the K_2's $a_1''a_1$, $a_1''a_1'$. Further a_1'' attaches to both α and a_3. We can also see that there are exactly two vertices of attachment of $V(C)$ in the connected component of $G - V(C) - \{a_1''\}$ containing B^*. These are the vertices of attachment in B^* of $d = a_2$ and $f = a_4$. It follows that the blocks of $G - V(C)$ are B^*, the K_2's $aa_1'', a_1''a_1, a_1''a_1'$ and triangles with cutvertex a_1'' in $G - V(C)$. We have proved (1) to (4) in the case $G - V(C)$ has more than one cutvertex.

We finally deal with the situation where $G - V(C)$ has just one cutvertex a. Since the vertices of attachment in C for the non-cutvertices of blocks, which are triangles are a_3, a_4, we conclude that a second block which is a K_2 must exist. This K_2 is $a_1'a$ and a_1' attaches to α. Since α, a_3 have degree at least 4, they must also attach to $a \in B^*$. We have again verified (1) to (4). ■

Corollary 6.1. *Suppose $G - V(C)$ has an endblock which is a K_2 and the vertex a_1 of degree 1 in $G - V(C)$ does not attach to a vertex α of C. Suppose further that α is not the vertex of a chord of C. Then there is cycle C' of order 4 in G such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Proof. This follows from Lemma 6.5. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_1', a_1'a_4, a_4a_1$. It follows from Lemma 6.5 that $G - V(C')$ is connected and $G - E(C')$ is 2-connected. We have illustrated this in Figure 6 in the case $G - V(C)$ has two cutvertices. The cycle C' has dashed edges. ■

Lemma 6.6. *Suppose $G - V(C)$ has an endblock which is a K_2 and the vertex a_1 of degree 1 in $G - V(C)$ does not attach to a vertex α of C. Suppose further that C has exactly one chord and α is a vertex of this chord. If the vertices d,f of C do not coincide with a_2, a_4 then $G - V(C)$ consists of two blocks.*

Proof. We first consider the case where d,f are adjacent and one of them, say d, coincides with α. We may assume then that $d = \alpha$,$f = a_2$. Consider a vertex β of $G - V(C)$, which has the property $\beta \neq a_1$, $\beta \notin B^*$. Then β can attach only to a_3 or a_4 in C. Otherwise we obtain a contradiction to (a) of Lemma 3.2. It follows that any endblock of $G - V(C)$, other than the K_2 with vertex a_1 and B^*, must be

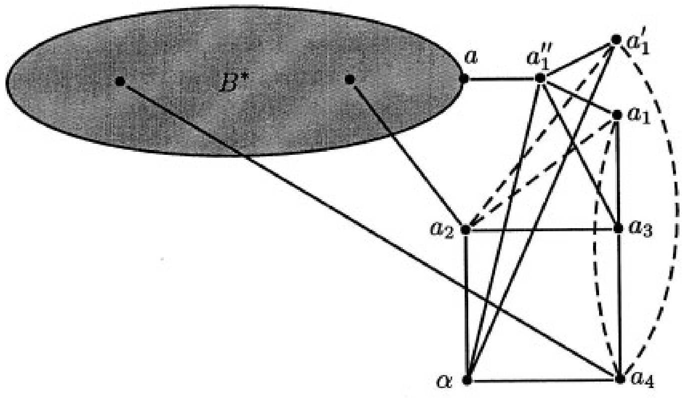

FIGURE 6.

an odd cycle. Let a_1', a_1'' be adjacent vertices of such an odd cycle, which are not cutvertices of $G - V(C)$. Then a_1', a_1'' attach to both a_3 and a_4. Hence we can form the cycle C' of order 4 consisting of the edges $a_1'a_3, a_3a_4, a_4a_1'', a_1''a_1'$. Evidently $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2.

We conclude the only endblocks of $G - V(C)$ are B^* and the K_2 with vertex a_1. Suppose now the second vertex of this K_2 is $a_1' \notin B^*$. Then a_1' must have a vertex of attachment in C, which must be either a_3 or a_4. Hence we can form a cycle C' with vertices a_1, a_1', a_3, a_4. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the second vertex of the K_2 is $a \in B^*$, whence $G - V(C)$ has two blocks.

Next we consider the case where d,f are adjacent but neither coincides with α. Hence we may assume $d = a_2$, $f = a_3$. By the argument of the previous paragraph, we see that a vertex β of $G - V(C), \beta \neq a_1,\ \beta \notin B^*$ can attach only to a_4 or α in C. Hence any endblock of $G - V(C)$, other than the K_2 with vertex a_1 and B^*, must be an odd cycle. Arguing as before, we see that the only endblocks of $G - V(C)$ are B^* and the K_2 with vertex a_1. Suppose now the second vertex of this K_2 is $a_1' \notin B^*$. Then a_1' has a vertex of attachment in C, which must be either α or a_4. If α is a vertex of attachment, then we may argue as in the previous paragraph to obtain a contradiction to (a) of Lemma 3.2. Hence a_1' attaches to a_4 and does not attach to α. Since a_1' attaches only to a_4 in C the second block with cutvertex a_1' is an odd cycle. Let a_1'' be a vertex of this cycle adjacent to a_1', which is not a cutvertex of $G - V(C)$. Then a_1'' attaches to both α and a_4. Let C' be the cycle consisting of the edges $a_4a_1, a_1a_1', a_1'a_1'', a_1''a_4$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. In all cases we have a contradiction, whence $G - V(C)$ consists of two blocks.

Finally we consider the case where d,f are non-adjacent, whence we may assume $d = \alpha$, $f = a_3$. As previously we see that a vertex β of $G - V(C)$, $\beta \neq a_1$, $\beta \notin B^*$ can attach only to a_2 or a_4 in C. Hence any endblock of $G - V(C)$, other than the K_2 with vertex a_1 and B^*, must be an odd cycle.

Suppose there is an endblock which is an odd cycle. Let a_1', a_1'' be adjacent vertices of the cycle, which are not cutvertices of $G - V(C)$. Then a_1', a_1'' attach to both a_2 and a_4. Hence we may form the cycle C' consisting of the edges $a_1'a_2, a_2a_1'', a_1''a_4, a_4a_1'$. Since the edge αa_3 exists, $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the only endblocks of $G - V(C)$ are B^* and the K_2 with vertex a_1. Let $a_1' \notin B^*$ be the second vertex of this K_2. Then a_1' has a vertex of attachment in C, which must be either a_2 or a_4. Suppose a_1' attaches to both a_2 and a_4. Let C' be the cycle with edges $a_1a_2, a_2a_1', a_1'a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Next suppose a_1' attaches to a_2 but does not attach to a_4. Then a_4 attaches to a vertex of $G - V(C)$ other than a_1, a_1'. Suppose d attaches to a vertex $d^* \in B^*$. If the vertex of attachment of a_4 is different from d^* then we can obtain a contradiction to (a) of Lemma 3.2. If the vertex of attachment of a_4 is d^*, then this puts us in the situation of adjacent d,f dealt with in the previous paragraphs. We conclude again in this case that $G - V(C)$ contains two blocks. ■

Lemma 6.7. *Suppose $G - V(C)$ has an endblock which is a K_2, and the vertex a_1 of degree 1 in $G - V(C)$ does not attach to a vertex α of C. Suppose further that C has exactly one chord and α is a vertex of this chord. If the vertices d,f of C coincide with a_2, a_4 then there exists a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Proof. Arguing as in the previous lemma, we see that a vertex β of $G - V(C)$, $\beta \neq a_1$, $\beta \notin B^*$ can attach only to α or a_3 in C. We can similarly see that the only endblocks of $G - V(C)$ are B^* and the K_2 with vertex a_1. Let a_1' be the second vertex of this K_2, and suppose $a_1' \notin B^*$. Then a_1' is the cutvertex of two blocks. One of these is the K_2 a_1a_1'. Let us assume the other is an odd cycle. Let a_1'' be a vertex of this cycle adjacent to a_1' which is not a cutvertex of $G - V(C)$. Then a_1'' attaches to both α and a_3. Let C' be the cycle consisting of the edges $a_3a_1, a_1a_1', a_1'a_1'', a_1''a_3$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude a_1' is the cutvertex of two blocks, both of which are K_2's. Hence a_1' attaches to both a_3 and α in C.

Consider now the second K_2 which has cutvertex a_1' in $G - V(C)$. One of its vertices is a_1' and denote the other vertex a_1''. Suppose $a_1'' \notin B^*$, whence a_1'' has a vertex of attachment in C. Suppose a_3 is this vertex of attachment, and let C' be the cycle consisting of the edges $a_3a_1, a_1a_1', a_1'a_1'', a_1''a_3$, Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. The other possibility for the vertex of attachment in C of a_1'' is α. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction.

We conclude that $G - V(C)$ consists of at most three blocks. One of these is B^* and the others are K_2's. If $G - V(C)$ has just two blocks, the conclusion of the lemma follows from Lemma 6.1. Suppose then that $G - V(C)$ has three blocks with the endblock, which is a K_2 having vertices a_1 of degree 1 in $G - V(C)$ and a_1' of degree 2 in $G - V(C)$. Now a_1 attaches to a_2, a_3, a_4 in C and a_1' to a_3, α. Furthermore, the diagonal $a_3\alpha$ exists. Let C' be the cycle consisting of the edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. It is clear that $G - V(C')$ is connected and $G - E(C')$ is 2-connected. We illustrate this in Figure 7.

Corollary 6.2. *Suppose G satisfies the conditions of Proposition* 6.1 *and $G - V(C)$ has an endblock which is a K_2. Then there is a cycle C' in G of order* 4 *such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is* 2*-connected.*

Proof. This follows from the previous lemmas. ■

To finish the proof of Proposition 6.1, we may now assume that all endblocks of $G - V(C)$ other than B^* are odd cycles. We denote by a_1, a_1' two adjacent vertices of an endblock which is an odd cycle, neither of which is a cutvertex of $G - V(C)$. Hence both a_1 and a_1' have two vertices of attachment in C. There are two possibilties for these vertices of attachment:

(1) Both a_1 and a_1' have precisely two vertices of attachment in C and these are identical for a_1 and a_1'. Furthermore, the vertices of attachment are non-adjacent on C.
(2) There are two adjacent vertices of C which form a cycle of order 4 together with the vertices a_1, a_1'.

In case (1) we denote the vertices of attachment of a_1, a_1' in C by a_2, a_3. In case (2) we denote by a_2, a_3 the vertices of C such that the cycle with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$ exists. Now C has two other vertices, which we denote α, β with α being the vertex of C adjacent to a_2. We continue to denote by d, f two vertices of C which have edges to distinct vertices of B^* different from the cutvertex a of B^* in $G - V(C)$.

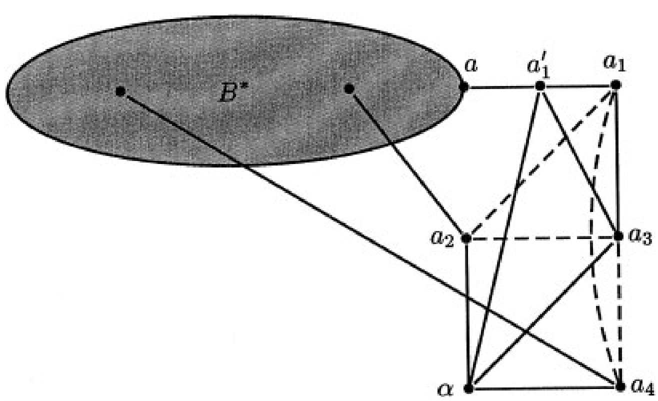

FIGURE 7.

Lemma 6.8. *Suppose all endblocks of $G - V(C)$ other than B^* are odd cycles. Let a_1, a_1' be two adjacent vertices of an endblock which is an odd cycle and such that neither a_1 nor a_1' is a cutvertex of $G - V(C)$. If a_2 and a_3 are non-adjacent on C then there is a cycle C' of order 4 in C such that $G - V(C')$ is connected and $G - E(C')$ is 2-connected.*

Proof. Since a_2 and a_3 are non-adjacent we are in case (1) above. Suppose first that d and f are non-adjacent and do not coincide with a_2, a_3. Now d and f do not attach to a_1 or a_1'. Hence either d attaches to a second vertex of $G - V(C)$ different from a_1, a_1' or the edge df exists. In either case we obtain a contradiction to (a) of Lemma 3.2. Suppose next that $d = a_2$, $f = a_3$. We show the odd cycle containing the vertices a_1, a_1' is a triangle. To see this, assume the cycle is not a triangle, whence we can suppose there is a vertex a_1'' of the cycle, adjacent to a_1' on the cycle, which is not a cutvertex of $G - V(C)$. Hence a_1'' has two vertices of attachment in C. Suppose a_1'' attaches to α. Let C' be the cycle with edges $a_1'a_2, a_2\alpha, \alpha a_1'', a_1''a_1'$. Then $G - V(C')$ contains a block which includes B^* and a_3, contradicting (a) of Lemma 3.2. Since we may make a similar argument if a_1'' attaches to β we conclude that the vertices of attachment in C of a_1'' are precisely a_2, a_3. Since α and β have degree at least 4 in G, they must attach to a vertex of $G - V(C)$ other than a_1, a_1', a_1''. Suppose d attaches to the vertex d^* in B^*. By forming the cycle C' consisting of the edges $a_3a_1, a_1a_1', a_1'a_1'', a_1''a_3$ we see that d^* can be the only vertex of attachment for α and β in $G - V(C)$. Otherwise we get a contradiction to (a) of Lemma 3.2. Since a similar argument implies that the vertex of attachment f^* of f in B^* is the only vertex of attachment for α and β in $G - V(C)$, we arrive at a contradiction. We conclude the odd cycle containing the vertices a_1, a_1' is a triangle. It is not difficult to see further that all endblocks other than B^* are triangles and the vertices of attachment in C of their non-cutvertices are precisely a_2, a_3.

We show that the cutvertices of all endblocks other than B^* are identical. To see this assume for contradiction this is not the case. Then there exists a cutvertex $a_1'' \notin B^*$ with the property that the components of $G - V(C) - \{a_1''\}$ consist of the component containing B^* and $B - \{a_1''\}$ for endblocks B of $G - V(C)$ with cutvertex a_1''. Let one of the endblocks B be the triangle with vertices a_1, a_1', a_1'' and C' be the cycle with edges $a_2a_1, a_1a_1'', a_1''a_1', a_1'a_2$. Since $G - V(C)$ contains an endblock triangle with cutvertex different from a_1'' it follows that $G - V(C)$ contains a block containing B^* and a_3, contradicting (a) of Lemma 3.2. We conclude that all endblocks of $G - V(C)$ other than B^* have the same cut-vertex a_1''.

Suppose $a_1'' \notin B^*$. Then the blocks of $G - V(C)$ consist of B^*, the K_2, aa_1'' and the endblock triangles with cut vertex a_1''. Otherwise there is a vertex of the component of $G - V(C) - \{a_1''\}$ containing B^*, which attaches to a vertex of C. Further, this vertex does not lie in B^*. By considering the cycle C' with vertices a_1, a_1', a_1'' and one of the vertices a_2 or a_3, we see $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Consider next the vertices of attachment in $G - V(C)$ of α. If α attaches to a vertex of B^*, then we can obtain a contradiction to (a) of Lemma 3.2. Hence α attaches to a_1'' and the diagonal $\alpha\beta$ exists. Similarly β attaches to a_1''. Let C' be the cycle consisting of the edges $a_1a_2, a_2\alpha, \alpha a_3, a_3a_1$. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. We illustrate this in Figure 8.

Next we assume $a_1'' = a \in B^*$. It follows from the previous paragraph that all endblocks are triangles with cutvertex a in $G - V(C)$. Further, the vertices of attachment in C of the non-cutvertices of these triangles are precisely a_2 and a_3. Hence α and β can attach only outside C to vertices of B^*. If α attaches to two vertices of B^* then we have a contradiction to (a) of Lemma 3.2. In fact if C' is the cycle with edges $a_1a_2, a_2\beta, \beta a_3, a_3a_1$ then $G - V(C')$ contains a block strictly larger than B^*. We conclude that the edge $\alpha\beta$ exists and that α and β attach to the same vertex of B^*. Note this last assertion is deduced by considering the cycle C' consisting of the edges $a_1a_2, a_2a_1', a_1'a_3, a_3a_1$. The situation we are in is illustrated in Figure 9. We have assumed that α and β attach to the cutvertex a of B^* in $G - V(C)$. Again it is clear that if C' is the cycle consisting of the edges $a_1a_2, a_2\alpha, \alpha a_3, a_3a_1$, then $G - V(C')$ is connected and $G - E(C')$ is 2-connected.

Finally we consider the situation where d and f are adjacent on C. We can assume wlog that $d = a_2, f = \alpha$. Now α does not attach to a_1 or a_1'. Hence if α attaches to two vertices of $G - V(C)$, we can obtain a contradiction to (a) of Lemma 3.2. It follows that the edge $\alpha\beta$ exists and that α and β have the same vertex of attachment in $G - V(C)$. Arguing as previously, we see that the odd cycle containing the vertices a_1, a_1' is a triangle and its third vertex is $a \in B^*$. It is also easy to see that all endblocks are triangles with cutvertex $a \in B^*$. Furthermore, the vertices of attachment in C of the non-cutvertices are precisely a_2 and a_3. Let C' be the cycle with edges, $a_1a_2, a_2\alpha, \alpha a_3, a_3a_1$. We see that $G - V(C')$ is connected and $G - E(C')$ is 2-connected. This is illustrated in Figure 10.

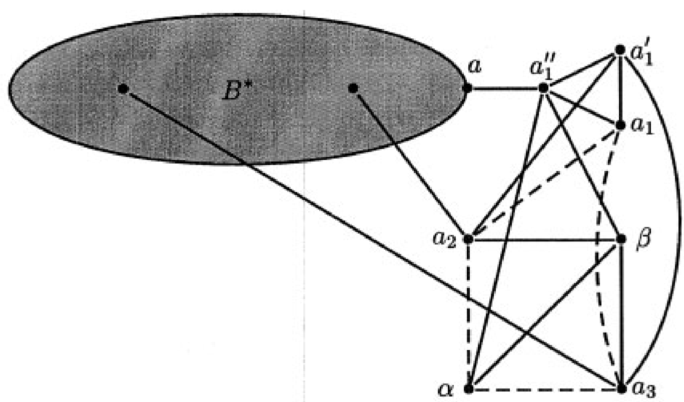

FIGURE 8.

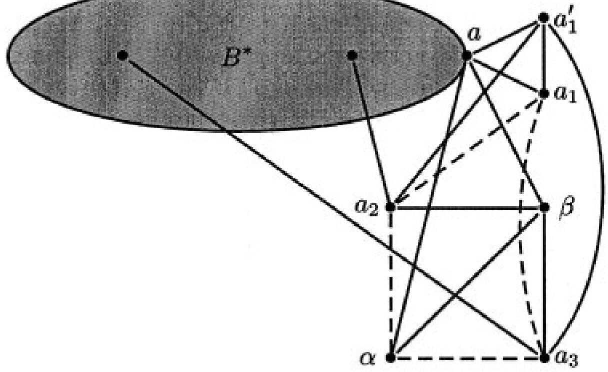

FIGURE 9.

Lemma 6.9. *Suppose all endblocks of $G - V(C)$ other than B^* are odd cycles. Let a_1 be a vertex of an endblock which is an odd cycle and assume a_1 is not a cutvertex of $G - V(C)$. Then a_1 has exactly two vertices of attachment in C.*

Proof. Evidently a_1 has at least two vertices of attachment in C. Let us first suppose a_1 has four vertices of attachment in C, whence a_1 attaches to every vertex of C. Let a'_1 be a vertex of the endblock odd cycle adjacent to a_1, which is also not a cutvertex of $G - V(C)$. If a'_1 attaches to d or f in C then we obtain a contradiction to (a) of Lemma 3.2. Hence a'_1 must attach to the two vertices of C different from d,f. In that case we also get a contradiction to (a) of Lemma 3.2. In fact if d,f are adjacent on C, we can construct the cycle C' with vertices a_1, a'_1 and the two vertices of C distinct from d,f. Then $G - V(C')$ contains a block strictly larger than B^*. If d,f are non-adjacent on C we can construct a cycle C' with vertices a'_1, f and the two vertices of attachment of a'_1 in C. Again $G - V(C')$ contains a block strictly larger than B^*. We conclude that a_1 cannot have four vertices of attachment in C.

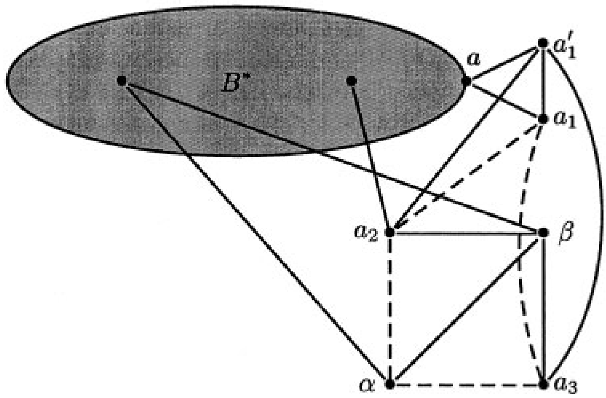

FIGURE 10.

Suppose a_1 has three vertices of attachment in C. We denote these vertices in order on C by a_2, a_3, a_4 and denote by α the remaining vertex of C, which does not attach to a_1. As in the previous paragraph, let a_1' be a vertex on the odd cycle adjacent to a_1, which is not a cutvertex of $G - V(C)$. Suppose that a_1' attaches to α. Then d,f cannot coincide with α or we would have a contradiction to (a) of Lemma 3.2. Assume a_3 is the second vertex of attachment for a_1' in C. Then d,f cannot coincide with a_3 either, whence we may assume $d = a_2$, $f = a_4$. Let C' be the cycle with edges $a_1'\alpha, \alpha a_4, a_4a_3, a_3a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_3 cannot be the second vertex of attachment for a_1' in C. We can assume then that a_2 is the second vertex of attachment for a_1'. If d and f are adjacent on C, it is easy to see we get a contradiction to (a) of Lemma 3.2. Hence we may assume $d = a_2$, $f = a_4$. Suppose now the diagonal $a_3\alpha$ exists. Let C' be the cycle consisting of the edges $a_1'\alpha, \alpha a_3, a_3a_2, a_2a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence the diagonal $a_3\alpha$ does not exist, in which case α attaches to a vertex of $G - V(C)$ different from a_1, a_1'. We have already seen that if α attaches to a vertex of B^*, it has to be the cutvertex of B^* in $G - V(C)$. In particular, it cannot be the same as the vertex of attachment in B^* of $f = a_4$. Let C' be the cycle consisting of the edges $a_1'a_2, a_2a_3, a_3a_1, a_1a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction to (a) of Lemma 3.2.

We conclude now that a_1' does not attach to α. Then α attaches to a vertex of $G - V(C)$ different from a_1, a_1'. Suppose first $\alpha = d$. Then, to avoid a contradiction to (a) of Lemma 3.2, α can have only one vertex of attachment in $G - V(C)$. Hence the diagonal $a_3\alpha$ exists. Consider now the position of f on C. If f is adjacent to α, then one can construct a cycle C' with vertices a_1, a_1' and the two vertices of C different from d,f. In that case, $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence we must have $f = a_3$. Observe that a_1' cannot attach to a_3. If it did, we could form the cycle C' from the edges $a_1a_2, a_2\alpha, \alpha a_4, a_4a_1$, and $G - V(C')$ contains a block strictly larger than B^*. We conclude the vertices of attachment in C of a_1' are a_2, a_4. In that case, let C' be the cycle with the edges $a_1a_2, a_2a_1', a_1'a_4, a_4a_1$. Since the diagonal $a_3\alpha$ exists, $G - V(C')$ contains a block strictly larger than B^*, contracting (a) of Lemma 3.2.

We can assume now that if the vertex of attachment of α in $G - V(C)$ lies in B^*, it is the cutvertex of B^* in $G - V(C)$. Hence one of d,f coincides with one of a_2, a_4, say $d = a_2$. In that case, we can form a cycle C' with vertices a_1, a_1', a_3, a_4, since a_1' attaches to a_3 or a_4. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction to (a) of Lemma 3.2. It follows that a_1 cannot have three vertices of attachment in C. ■

Lemma 6.10. *Suppose all endblocks of $G - V(C)$ other than B^* are odd cycles. Let a_1, a_1' be two adjacent vertices of an endblock which is an odd cycle, and such that neither a_1 nor a_1' is a cutvertex of $G - V(C)$. Suppose a_1 attaches to a vertex*

a_2 of C and a_1' to a vertex a_3 of C, and that a_2, a_3 are adjacent on C. If d, f are not adjacent on C, then $G - E(C)$ is 2-connected.

Proof. Let α, β be the remaining two vertices of C different from a_2, a_3, with α adjacent to a_2. We can assume wlog that $d = a_2$, $f = \beta$. By Lemma 6.9 we have that a_1 has exactly two vertices of attachment in C, one of which is a_2. Similarly a_1' has two vertices of attachment, one of which is a_3. Suppose that a_1' attaches to α. Let C' be the cycle consisting of the edges $a_1'a_3, a_3\beta, \beta\alpha, \alpha a_1'$. Since $d = a_2$, the graph $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_1' cannot attach to α. Suppose next that a_1 attaches to β. Then α has a vertex of attachment in $G - V(C)$, which is different from a_1, a_1'. Let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$. To avoid a contradiction to (a) of Lemma 3.2, we must have that the vertex of attachment of α in $G - V(C)$ is the same as the vertex of attachment in B^* of $f = \beta$. By similar argument, we see the vertex of attachment of α must also be the same as the vertex of attachment in B^* of $d = a_2$. Since the vertices of attachment in B^* of d, f are different, we have a contradiction.

We conclude the vertices of attachment of a_1 in C are adjacent, and similarly with a_1'. Suppose now that a_3 is the second vertex of attachment in C for a_1 and β is the second vertex of attachment for a_1'. By arguing as in the previous paragraph, we see that the vertex of attachment of α in $G - V(C)$ must coincide with the vertex of attachment in B^* of d and also of f, a contradiction. Suppose next that α is the second vertex of attachment in C for a_1, and a_2 is the second vertex of attachment for a_1'. Observe that since β cannot have two vertices of attachment in $G - V(C)$, the diagonal $a_2\beta$ exists. If the diagonal αa_3 exists, let C' be the cycle consisting of the edges $a_1\alpha, \alpha a_3, a_3a_1', a_1'a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the edge αa_3 does not exist, whence α has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. By considering the cycle C' with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$, we see that this vertex of attachment for α must be the same as the vertex of attachment for f in B^*. Otherwise we have a contradiction to (a) of Lemma 3.2. Now let C' be the cycle with edges $a_1'a_2, a_2\beta, \beta a_3, a_3a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction.

We are left with two possibilities for the second vertices of attachment for a_1, a_1' in C:

(1) a_1 attaches to a_3, a_1' attaches to a_2;
(2) a_1 attaches to α, a_1' attaches to β.

Neither of (1), (2) lead to contradictions, but they do yield a situation where the cycle C satisfies the conclusion of the lemma. We first consider (1).

In case (1) we see that the diagonals αa_3 and $a_2\beta$ exist and that α, β have both exactly one vertex of attachment in $G - V(C)$, which is the vertex of attachment for $f = \beta \in B^*$. Otherwise we would have a contradiction to (a) of Lemma 3.2. Next we show that the odd cycle containing the vertices a_1, a_1' is a triangle with cutvertex $a \in B^*$. To see this let a_1'' be a vertex of the cycle adjacent to a_1 or a_1', which is not a cutvertex of $G - V(C)$. Arguing as before, we see that the vertices of attachment in C of a_1'' must be precisely a_2, a_3. Hence we can form the cycle C' consisting of the edges $a_3a_1, a_1a_1', a_1'a_1'', a_1''a_3$, and $G - V(C')$ contains a block strictly larger than B^*, a contradiction to (a) of Lemma 3.2. We conclude the odd cycle is a triangle. A similar argument shows its cutvertex in $G - V(C)$ is $a \in B^*$. It is not difficult to see further that every endblock of $G - V(C)$, other than B^*, is a triangle with cutvertex $a \in B^*$. Futhermore, the vertices of attachment in C of the non-cutvertices of the triangle are a_2, a_3. It follows now that $G - E(C)$ is 2-connected. We illustrate this in Figure 11.

In case (2) we first see that the diagonal αa_3 cannot exist. If it did, we could form the cycle C' consisting of the edges $a_1\alpha, \alpha a_3, a_3a_2, a_2a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, a contradiction to (a) of Lemma 3.2. Hence α has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. By previous argument, we see this vertex of attachment must coincide with the vertex of attachment in B^* of $f = \beta$. Now, arguing as earlier in the paragraph, we see the diagonal $a_2\beta$ cannot exist. We also can see that a_3 has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. This vertex of attachment must coincide with the vertex of attachment in B^* of $d = a_2$. We have now accounted for all edges from C to $G - V(C)$. If another edge existed we would have a contradiction to (a) of Lemma 3.2. It follows that $G - V(C)$ consists of two blocks, B^* and a triangle containing the vertices a_1, a_1'. The third vertex of the triangle is $a \in B^*$. It is easy to see that $G - E(C)$ is 2-connected, as illustrated in Figure 12. ■

Lemma 6.11. *Suppose all endblocks of $G - V(C)$ other than B^* are odd cycles. Let a_1, a_1' be two adjacent vertices of an endblock which is an odd cycle, and such that neither a_1 nor a_1' is a cutvertex of $G - V(C)$. Suppose a_1 attaches to a vertex a_2 of C and a_1' to a vertex a_3 of C, and that a_2, a_3 are adjacent on C. If d,f are adjacent on C, then $G - E(C)$ is 2-connected.*

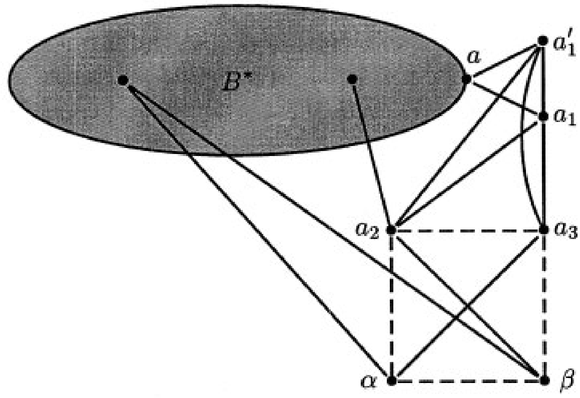

FIGURE 11.

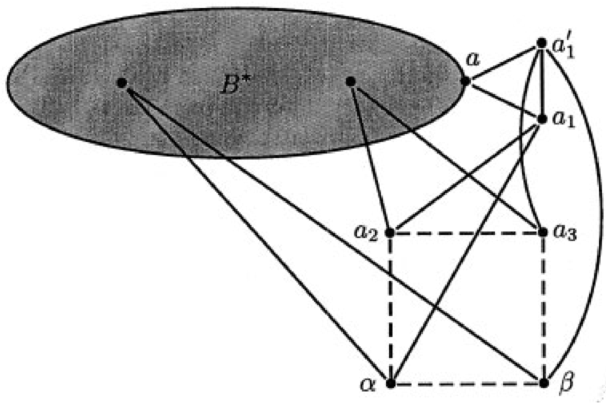

FIGURE 12.

Proof. There are two possible configurations for d,f on C, (1) $d = a_2$, $f = \alpha$, (2) $d = a_2$, $f = a_3$. We consider (1) first. Suppose a_1' attaches to α, and let C' be the cycle consisting of the edges $a_1'a_3, a_3\beta, \beta\alpha, \alpha a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Suppose a_1 attaches to β, and let C' be the cycle consisting of the edges $a_1\beta, \beta a_3, a_3a_1', a_1'a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. If a_1 attaches to a_3 and a_1' to β we similarly get a contradiction. Hence the only possibilities for the second vertices of attachment for a_1, a_1' in C are:

(I) a_1 attaches to a_3, a_1' to a_2,
(II) a_1 attaches to α, a_1' to β,
(III) a_1 attaches to α, a_1' to a_2.

We consider the three cases, first (I). Now β must have a vertex of attachment in $G - V(C)$ different from a_1, a_1'. To avoid a contradiction to (a) of Lemma 3.2 this vertex of attachment must be the same as the vertex of attachment of $f = \alpha$ in B^*. This puts us in the situation of non-adjacent d,f dealt with in the previous lemma. For case (II) we show first that the edge $a_2\beta$ cannot exist. If it did, we could form the cycle C' consisting of the edges $a_1'\beta, \beta a_2, a_2a_3, a_3a_1'$, and $G - V(C')$ then contains a block strictly larger than B^*, a contradiction to (a) of Lemma 3.2. Since the edge $a_2\beta$ does not exist, β has a vertex of attachment in $G - V(C)$ different from a_1, a_1'. To avoid a contradiction to (a) of Lemma 3.2, we see that it must be the same as the vertex of attachment of $f = \alpha$ in B^*. This again puts us in the situation of non-adjacent d,f. For case (III) we see that β must have a vertex of attachment in $G - V(C)$ different from a_1, a_1'. As before, this vertex of attachment must be the same as the vertex of attachment of $f = \alpha$ in B^*. Hence we are in the situation of non-adjacent d,f again.

Finally we consider the second possible configuration of d,f, (2) $d = a_2$, $f = a_3$. Suppose a_1 attaches to β and let C' be the cycle with edges a_1a_2,

$a_2\alpha, \alpha\beta, \beta a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_1 cannot attach to β, and similarly a'_1 cannot attach to α. Suppose now a_1 attaches to α and a'_1 to β. Then we may form the cycle C' with edges $a_1\alpha, \alpha\beta, \beta a'_1, a'_1 a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction. Hence the only possibilities for the second vertices of attachment for a_1, a'_1 in C are:

(I) a_1 attaches to a_3, a'_1 to β,
(II) a_1 attaches to α, a'_1 to a_2,
(III) a_1 attaches to a_3, a'_1 to a_2.

Cases (I) and (II) are similar, so we shall just deal with (I). In that case, α must attach to a vertex of $G - V(C)$ different from a_1, a'_1. To avoid a contradiction to (a) of Lemma 3.2, this vertex of attachment must be the same as the vertex of attachment of $d = a_2$ in B^*. This puts us once more in the situation of non-adjacent d,f dealt with in Lemma 6.10.

We are left to deal with case (III). We first show that the odd cycle containing the vertices a_1, a'_1 is a triangle. To see this, suppose the odd cycle is not a triangle, in which case there is a vertex a''_1 adjacent to a_1 or a'_1 which is not cutvertex of $G - V(C)$. We can assume wlog that a''_1 is adjacent to a'_1. If a''_1 attaches to α then we can form the cycle C' with edges $a'_1 a_2, a_2\alpha, \alpha a''_1, a''_1 a'_1$. Evidently $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Since we get a similar contradiction if a''_1 attaches to β, we conclude a''_1 attaches to both a_2 and a_3. Further, α attaches to a vertex of $G - V(C)$ different from a_1, a'_1, a''_1. By forming the cycle C' with edges $a_3 a_1, a_1 a'_1, a'_1 a''_1, a''_1 a_3$, we see the vertex of attachment of α in $G - V(C)$ must be the same as the vertex of attachment of $d = a_2$ in B^*. Otherwise we contradict (a) of Lemma 3.2. By forming the cycle C' with edges $a_2 a_1, a_1 a'_1, a'_1 a''_1, a''_1 a_2$, we see the vertex of attachment of α in $G - V(C)$ must be the same as the vertex of attachment of $f = a_3$ in B^*. Since the vertices of attachment of d,f are different, we have a contradiction. We conclude the odd cycle containing the vertices a_1, a'_1 is a triangle. It is not difficult to see further that every endblock of $G - V(C)$, other than B^*, is a triangle. The vertices of attachment in C of the non-cutvertices of the triangle are precisely a_2, a_3.

Let a''_1 be a vertex of attachment in $G - V(C)$ of α, and suppose $a''_1 \notin B^*$. If a''_1 is not the cutvertex of every endblock of $G - V(C)$ other than B^*, then there is an endblock $B \neq B^*$ such that a''_1 is in the same component of $G - V(C) - V(B)$ as B^*. From the previous paragraph, we see that we can form a cycle C' from the vertices of B and a_3. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude that a''_1 is the cutvertex of every endblock of $G - V(C)$ other than B^*. In particular, the third vertex of the triangle with vertices a_1, a'_1 is a''_1. Note that if a vertex of $G - V(C)$ which does not belong to an endblock (including B^*) attaches to C, then we can obtain a contradiction to (a) of Lemma 3.2. Hence the blocks of $G - V(C)$ are B^*, the K_2, aa''_1 with $a \in B^*$, and triangles with cutvertex a''_1 in $G - V(C)$. Further, a''_1 is the only vertex of attachment for α in $G - V(C)$, and hence the diagonal αa_3 exists. Similarly, a_1'' is the only vertex of attachment for β in $G - V(C)$, and hence the diagonal βa_2 exists. It is easy to see now that $G - E(C)$ is 2-connected. We illustrate this in Figure 13.

Suppose finally that α attaches to a vertex of B^*. If this vertex is different from the cutvertex a of B^* in $G - V(C)$, then we are in a situation dealt with previously in this lemma or in Lemma 6.10. Hence we may assume α attaches to $a \in B^*$. In that case it is easy to see that the cutvertex of every endblock of $G - V(C)$ is a. Further, the diagonal αa_3 exists. Similarly β attaches to a and the diagonal βa_2 exists. We see as above that $G - E(C)$ is 2-connected, as illustrated in Figure 14. ■

Corollary 6.3. *Suppose G satisfies the conditions of Proposition* 6.1 *and all endblocks of* $G - V(C)$ *other than* B^* *are odd cycles. Then there is a cycle* C' *in G of order* 4 *such that* $G - V(C')$ *is non-empty and connected, and* $G - E(C')$ *is* 2-*connected.*

Proof. This follows from Lemmas 6.8 through 6.11. ■

***Proof of Proposition* 6.1.** This follows from Corollaries 6.2, 6.3. ■

7. *B** WITH TWO CUTVERTICES IN *G*−*V*(*C*), *C* OF ORDER 4

Proposition 7.1. *Suppose G is a* 3-*connected graph of minimum degree* 4 *and the cycle C of Lemma* 3.2 *has order* 4. *If* B^* *has two cutvertices in* $G - V(C)$ *then there is a cycle* C' *in G of order* 4 *such that* $G - V(C')$ *is non-empty and connected, and* $G - E(C')$ *is* 2-*connected.*

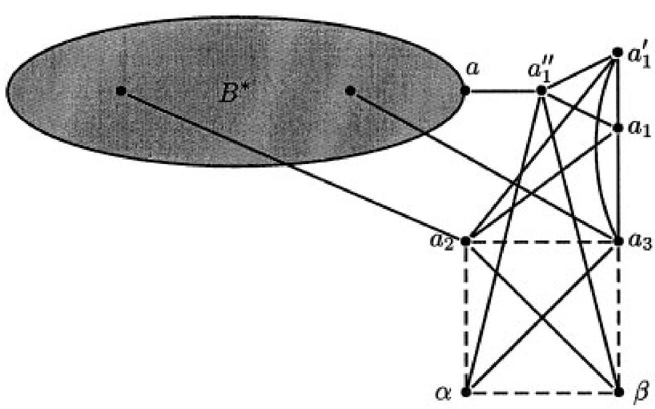

FIGURE 13.

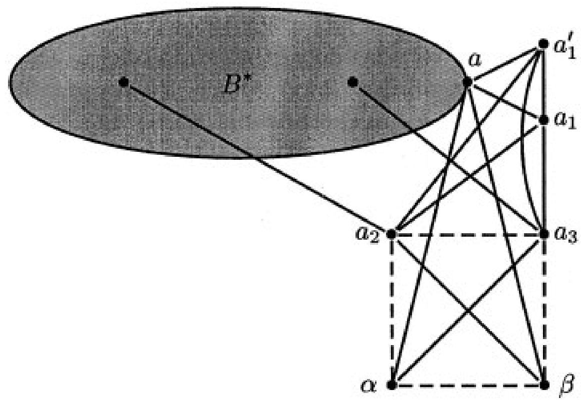

FIGURE 14.

Let a, b denote the cut vertices in $G - V(C)$ of B^*. Since G is 3-connected, there is a vertex d of C which attaches to a vertex of B^* different from a, b, unless B^* is a K_2. We shall first assume that $G - V(C)$ has an endblock, which is a K_2. Let a_1 be its vertex of degree 1 in $G - V(C)$. We can assume wlog that removal of a from $G - V(C)$ disconnects a_1 from B^*. Now a_1 has three vertices of attachment in C, which we denote by order on C as a_2, a_3, a_4. We denote by α the fourth vertex of C.

Lemma 7.1. *Suppose $G - V(C)$ has two endblocks which are K_2's and let b_1 be the vertex of degree 1 in $G - V(C)$ of the second K_2. Suppose further that removal of a from $G - V(C)$ does not disconnect b_1 from B^*. Then a, b are the only cutvertices of $G - V(C)$. The blocks of $G - V(C)$ are B^*, the two K_2's a_1a, b_1b and triangles. The vertices of attachment in C of the non-cutvertices of the triangles are precisely a_2, a_4.*

Proof. If b_1 attaches to both a_3 and α then, no matter what the position of d on C one can can form a cycle C' of order 4 such that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude that b_1 attaches to a_2 and a_4, and that a_1, b_1 have exactly three vertices of attachment in C. Further, d does not coincide with the vertex of attachment of a_1, b_1 different from a_2, a_4. In particular $d \neq a_3$. If $d = \alpha$ then a_3 is the third vertex of attachment in C of b_1.

We show that $G - V(C)$ has no further endblocks, which are K_2's. Suppose for contradiction there is a third endblock which is a K_2 and its vertex of degree 1 in $G - V(C)$ is c_1. We see as in the previous paragraph that c_1 attaches to a_2 and a_4. Consider now the third vertex of attachment for c_1. If c_1 attaches to a_3 then we can form the cycle C' with edges $a_1a_2, a_2c_1, c_1a_3, a_3a_1$. If d coincides with α or a_4 then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If $d = a_2$ we also obtain a contradiction by considering the cycle C' with edges $a_1a_4, a_4c_1, c_1a_3, a_3a_1$. Suppose c_1 attaches to α. Then by arguing as before we see that $d \neq a_2, a_4$ whence $d = \alpha$. In that case let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction to (a) of Lemma 3.2. We conclude there are only two endblocks which are K_2's.

We show next that endblocks of $G - V(C)$ which are odd cycles are triangles. Further, the vertices of attachment in C of the non-cutvertices of the triangle are precisely a_2, a_4. To see this, let c_1, c_1' be adjacent vertices of an endblock which is an odd cycle and assume they are not cutvertices of $G - V(C)$. Suppose first that $d = \alpha$, whence a_3 is the third vertex of attachment for b_1 in C. It is easy to see that if c_1 or c_1' attaches to α or to a_3 we will obtain a contradiction to (a) of Lemma 3.2. We conclude the vertices of attachment in C of c_1, c_1' are precisely a_2, a_4. Suppose next that $d = a_2$ and α is the third vertex of attachment for b_1 in C. Assume that c_1, c_1' are connected to B^* in $G - V(C) - \{b\}$. Then if c_1 or c_1' attach to α in C, we obtain a contradiction to (a) of Lemma 3.2. Hence the vertices of attachment in C of c_1, c_1' can only be a_2, a_3, a_4. If a_3 is a vertex of attachment then we can form a cycle C' of order 4 with the vertices c_1, c_1', a_3 and one of the vertices a_2 or a_4. It is clear that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude again the vertices of attachment of c_1, c_1' in C are precisely a_2, a_4. If c_1, c_1' are not connected to B^* in $G - V(C) - \{b\}$, then they are connected to B^* in $G - V(C) - \{a\}$. A similar argument implies then the vertices of attachment of c_1, c_1' in C are a_2, a_4. Suppose finally that $d = a_2$ and b_1 attaches to a_3. If c_1 or c_1' attach to a_3 then we can argue as previously to obtain a contradiction to (a) of Lemma 3.2. Hence the vertices of attachment in C of c_1, c_1' can only be a_2, α, a_4. If α is a vertex of attachment, then we can form a cycle C' with vertices c_1, c_1', α and one of the vertices a_2, a_4. Since $G - V(C')$ contains a block strictly larger than B^* we have a contradiction to (a) of Lemma 3.2. We conclude the vertices of attachment in C of c_1, c_1' are precisely a_2, a_4. In all cases then, c_1, c_1' attach to a_2, a_4 in C. It is easy to further conclude now that the odd cycle containing c_1, c_1' as vertices is a triangle.

We show that the only cutvertices of $G - V(C)$ are a, b. We proceed to contradiction by assuming there are more cutvertices. Let $a' \notin B^*$ be a cutvertex with the property that the connected components of $G - V(C) - \{a'\}$ consist of the component containing B^* and $B - \{a'\}$ for endblocks B with cutvertex a' in $G - V(C)$. We can assume wlog that a' is not the second vertex of the K_2 with vertex b_1. Suppose there is an endblock B with cutvertex a' which is a triangle. Let C' be the cycle with vertices consisting of the three vertices of B and the vertex a_2 or a_4 different from d. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. The alternative is that a' is the second vertex of the K_2 with vertex a_1, and a' has a vertex of attachment in C. Suppose that $d = \alpha$. If a' attaches to α then we see that if C' is the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$ then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a' must attach to one of a_2, a_3, a_4. In that case, let C' be the cycle with vertices a_1, a', the vertex of attachment in C of a' and an adjacent vertex not equal to α. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. We conclude that $d \neq \alpha$ whence we may assume $d = a_2$. If a' attaches to a_3, a_4 or α we form the cycle C' with vertices a_1, a', the vertex of attachment in C of a' and an adjacent vertex not equal to a_2. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence $d = a_2$ and a_2 is the only vertex of attachment in C of a'.

We conclude a contradiction by showing that a' has at least two vertices of attachment in C. To see this first observe that a' is the cutvertex of just two blocks of $G - V(C)$, one of which is the K_2, a_1a'. Further, every endblock of $G - V(C)$, which is a triangle, must have cutvertex a or b in $G - V(C)$. If a' has just one vertex of attachment in C then the second block with cutvertex a' must be an odd cycle. Let a'' be a vertex of this cycle adjacent to a' which is not a cutvertex of $G - V(C)$. Then a'' has two vertices of attachment in C. Suppose a'' attaches to a_3. Let C' be the cycle with edges $a_3a_1, a_1a', a'a'', a''a_3$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Since a similar argument can be made if a'' attaches to a_4, it follows that a'' attaches to both a_2 and α. Now let C' be the cycle of order 6 with edges $a_3a_1, a_1a', a'a'', a''\alpha$, $\alpha a_4, a_4a_3$. Then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. We conclude that a' has more than one vertex of attachment in C.

It follows now that a, b are the only cutvertices of $G - V(C)$. Hence the blocks of $G - V(C)$ are B^*, the two K_2's a_1a, b_1b, and triangles with cutvertex a or b. ■

Lemma 7.2. *Suppose $G - V(C)$ has two endblocks which are K_2's and let b_1 be the vertex of degree 1 in $G - V(C)$ of the second K_2. Suppose further the removal of a from $G - V(C)$ does not disconnect b_1 from B^*. Then there is a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected and $G - E(C')$ is 2-connected.*

Proof. We suppose first that there is an endblock of $G - V(C)$ which is a triangle. Let c_1, c_1' be the non-cutvertices of the triangle and a be the third vertex. Let C' be the cycle with edges $c_1a_2, a_2\alpha, \alpha a_4, a_4c_1$. If b_1 attaches to a_3 then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence the three vertices of attachment in C are a_2, α, a_4. Let C' be the cycle with edges $c_1a_2, a_2c_1', c_1'a_4, a_4c_1$. If the diagonal $a_3\alpha$ exists then $G - V(C')$ contains a block strictly larger than B^*, a contradiction. We conclude the edge $a_3\alpha$ does not exist. Hence a_3 has a vertex of attachment in $G - V(C)$ different from a_1. We have already seen this vertex of attachment cannot be b_1. Let C' be the cycle with edges $b_1a_2, a_2\alpha, \alpha a_4, a_4b_1$. Since $G - V(C')$ cannot contain a block larger than B^*, we see that the second vertex of attachment in $G - V(C)$ of a_3 must be a. Similarly we see that α attaches to b. The situation is illustrated in Figure 15. Let C' be the cycle with edges $a_2b_1, b_1a_4, a_4a_1, a_1a_2$. Then $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.

We may assume now that the blocks of $G - V(C)$ are B^* and the two K_2's a_1a, b_1b. If b_1 attaches to a_3, then α attaches to at least two vertices of B^*, or

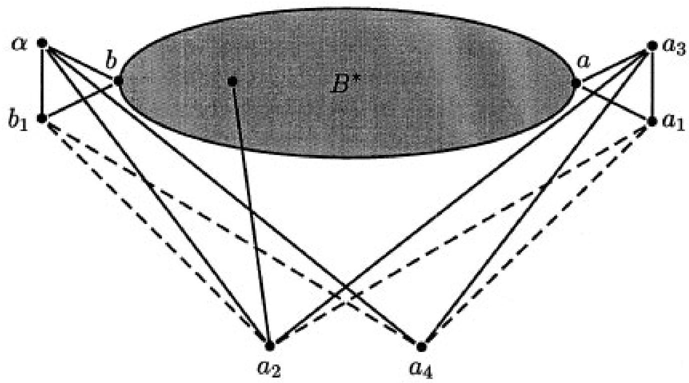

FIGURE 15.

the edge $a_3\alpha$ exists and α attaches to at least one vertex of B^*. If α attaches to two vertices of B^*, let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If the edge $a_3\alpha$ exists, then α also attaches to a vertex of B^*. In that case it is easy to see that $G - E(C)$ is 2-connected. The situation is illustrated in Figure 16.

The final situation we need to deal with is when b_1 attaches to α, in which case we may assume $d = a_2$. If the diagonal αa_3 exists, then $G - E(C)$ is 2-connected so we shall assume the edge αa_3 does not exist. Hence α has a second vertex of attachment in $G - V(C)$. This vertex of attachment must be b. Otherwise we can obtain a contradiction to (a) of Lemma 3.2. Similarly a_3 has a second vertex of attachment in $G - V(C)$, which must be a. We are in a situation similar to that illustrated in Figure 15. Hence if C' is the cycle with edges $a_2b_1, b_1a_4, a_4a_1, a_1a_2$ then $G - E(C')$ is 2-connected. ■

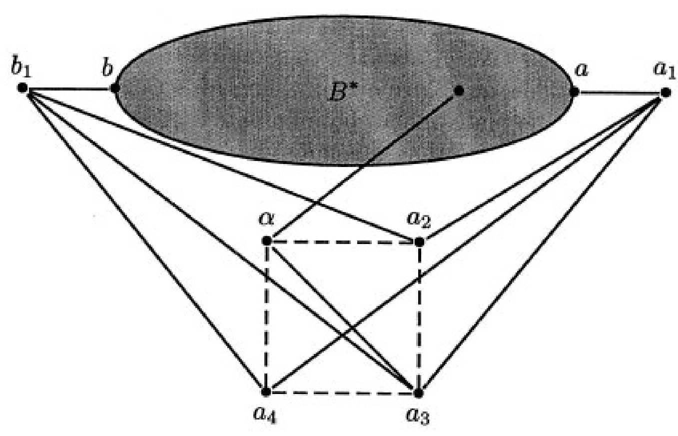

FIGURE 16.

Next we consider the situation where all endblocks of $G - V(C)$ connected to B^* through b are odd cycles. We continue to assume that $G - V(C)$ has an endblock which is a K_2 connected to B^* through a.

Lemma 7.3. *Suppose $G - V(C)$ has an endblock which is a K_2, connected to B^* through a, and denote by a_1 its vertex of degree 1 in $G - V(C)$. Suppose further that all endblocks of $G - V(C)$ connected to B^* through b are odd cycles. Let b_1, b_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. Then*

(1) *a_1 has three vertices of attachment in C, which we denote by a_2, a_3, a_4 in order on C,*
(2) *b_1, b_1' have exactly two vertices of attachment in C,*
(3) *If the vertices of attachment of b_1, b_1' are non-adjacent, on C then they coincide with a_2, a_4.*
(4) *If the vertices of attachment of b_1, b_1' are adjacent on C, one of them must be the fourth vertex α of C.*

Proof. To prove (1) assume that a_1 attaches to all four vertices of C. We consider the position of d on C. If d attaches to b_1 or b_1' then we form the cycle C' with vertices a_1 and the three vertices of C different from d. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. Hence d is not a vertex of attachment for b_1, b_1'. In that case we can form a cycle C' of order 4 from the vertices b_1, b_1' and the three vertices of C different from d. Again we have that $G - V(C')$ contains a block strictly larger than B^*, a contradiction. Hence (1) holds.

We prove (2). Assume first that there are four vertices of attachment of b_1, b_1' in C. If $d = \alpha$ or $d = a_3$ then we easily obtain a contradiction to (a) of Lemma 3.2. Hence we may assume wlog that $d = a_2$. One can see that it is possible to form a cycle C' of order 4 from b_1, b_1' and the vertices a_3, a_4, α of C. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. It follows that b_1, b_1' have at most three vertices of attachment in C. Suppose b_1, b_1' have exactly three vertices of attachment in C. Hence one of these vertices attaches to both b_1 and b_1'. Suppose the vertex is α. If a_3 is a vertex of attachment for b_1 or b_1', then we may argue as in Lemma 7.1 to obtain a contradiction to (a) of Lemma 3.2. Hence we may assume that b_1 attaches to a_2 and b_1' to a_4. It is easy to see now that no matter where d lies on C, we obtain a contradiction to (a) of Lemma 3.2. A similar argument can be made if the common vertex of attachment for b_1 and b_1' is a_3. We may therefore assume that the common vertex of attachment for b_1 and b_1' is a_2. Suppose now b_1 attaches to a_4, whence b_1' must attach to α or a_3. Again we see that no matter where d is located on C, we obtain a contradiction to (a) of Lemma 3.2. Thus we may further assume that b_1 attaches to a_3 and b_1' to α. In that case we must also have $d = a_2$. Since α has degree at least 4 in G, either the edge αa_3 exists or α

attaches to a vertex of $G - V(C)$ different from b_1'. Suppose the edge αa_3 exists. Let C' be the cycle with edges $a_1a_3, a_3\alpha, \alpha a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude that α has a second vertex of attachment α' in $G - V(C)$ and $\alpha' \neq a_1, b_1, b_1'$. Suppose α' is not disconnected in $G - V(C)$ from B^* by removal of b. Let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. The alternative is that α' is not disconnected in $G - V(C)$ from B^* by removal of a. In that case, let C' be the cycle with edges $a_2b_1', b_1'b_1, b_1a_3, a_3a_2$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. It follows that if b_1, b_1' have three vertices of attachment in C we have a contradiction. Hence (2) holds.

We turn to the proof of (3) and (4). Now (3) follows by the argument of Lemma 7.1 so we consider (4). We proceed to contradiction by assuming the vertices of attachment in C of b_1, b_1' are precisely a_2 and a_3. It is clear by previous argument that in this case we must have $d = a_2$. We show first that the only endblock of $G - V(C)$ which is disconnected from B^* by removal of a is the K_2 with vertex a_1. To see this, suppose there is a second such endblock and it is a K_2 with vertex a_1' of degree 1 in $G - V(C)$. Then a_1' has three vertices of attachment in C. If both the edges $a_1'a_3$ and $a_1'a_4$ exist, we can form the cycle C' with edges $a_1a_3, a_3a_1', a_1'a_4, a_4a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we obtain a contradiction to (a) of Lemma 3.2. Hence the edges $\alpha a_1'$ and a_2a_1' must exist. Suppose a_3 also attaches to a_1', whence we may form the cycle C' with edges $a_1'\alpha, \alpha a_4, a_4a_3, a_3a_1'$. Now $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. The alternative is that a_4 attaches to a_1'. In that case, let C' be the cycle with edges $a_1'a_2, a_2\alpha, \alpha a_4, a_4a_1'$. Again $G - V(C')$ contains a block strictly larger than B^*, yielding a contradiction. We conclude the only endblock of $G - V(C)$ which is a K_2 and is disconnected from B^* by removal of a is the K_2 with vertex a_1.

Suppose next that there is an endblock of $G - V(C)$ which is an odd cycle and is disconnected from B^* by removal of a. Let a_1', a_1'' be adjacent vertices of this odd cycle, neither of which is a cutvertex of $G - V(C)$. If a_1' or a_1'' attaches to a_3, then let C' be the cycle with edges $a_1a_2, a_2\alpha, \alpha a_4, a_4a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction. It follows that the vertices of attachment in C of a_1', a_1'' are among a_2, α, a_4. In that case we can form a cycle C' of order 4 from the vertices $a_1', a_1'', a_2, \alpha, a_4$. Since $G - V(C')$ contains a block which includes B^* and a_3, we have a contradiction to (a) of Lemma 3.2 again. We conclude the only endblock of $G - V(C)$ disconnected from B^* by removal of a is the K_2 with vertex a_1.

We consider next the vertices of attachment in $G - V(C)$ of α. If the diagonal αa_3 exists we can form the cycle C' with edges $a_1a_3, a_3\alpha, \alpha a_4, a_4a_1$. Since $d = a_2$ and b_1 also attaches to a_2, $G - V(C')$ contains a block which includes B^* and a_2, contradicting (a) of Lemma 3.2. Hence the diagaonal αa_3 does not exist, whence α has two vertices of attachment in $G - V(C)$. Let α' be a vertex of attachment for α satisfying $\alpha' \neq a$. If α' is not disconnected in $G - V(C)$ from B^* by removal of a, then we form the cycle C' with edges $b_1a_2, a_2a_3, a_3b'_1, b'_1b_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. Hence we may assume α' is disconnected in $G - V(C)$ from B^* by removal of a. Since the only endblock of $G - V(C)$ disconnected from B^* by removal of a is the K_2 with vertex a_1, there is a path $P(\alpha', a_1)$ in $G - V(C)$ from α' to a_1, which does not go through a. Let C' be the cycle with edges $\alpha\alpha'$, $P(\alpha', a_1)$, $a_4\alpha$, and either final edge a_1a_4, or final two edges a_1a_3, a_3a_4, whichever makes C' even. Since $d = a_2$, it follows that $G - V(C')$ contains a block which includes B^* and a_2, contradicting (a) of Lemma 3.2. We conclude that (4) holds. ∎

Lemma 7.4. *Suppose $G - V(C)$ has an endblock which is a K_2, connected to B^* through a, and denote by a_1 its vertex of degree 1 in $G - V(C)$. Suppose further that all endblocks of $G - V(C)$ connected to B^* through b are odd cycles. Let b_1, b'_1 be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. If the vertices of attachment in C of b_1, b'_1 are adjacent, then there exists a cycle C' of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

Proof. Denoting by a_2, a_3, a_4 the vertices of attachment in C of a_1 and α the fourth vertex of C, we may assume by Lemma 7.3 that b_1, b'_1 attach to precisely a_2, α in C. To avoid a contradiction to (a) of Lemma 3.2 we must also have that $d = a_2$. Suppose the edge αa_3 exists. Then we may form the cycle C' with edges $a_1a_3, a_3\alpha, \alpha a_4, a_4a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude the edge αa_3 does not exist, whence a_3 attaches to a vertex of $G - V(C)$ other than a_1.

We show that an endblock of $G - V(C)$, which is disconnected from B^* by removal of a, is either the K_2 with vertex a_1 or a triangle with cutvertex a. Further, the vertices of attachment in C of the non-cutvertices of the triangle are precisely a_2 and a_4. To see this, let us first suppose there is a second K_2 which is an endblock of $G - V(C)$ and is disconnected from B^* by removal of a. Let a'_1 be its vertex of degree 1 in $G - V(C)$. If a'_1 attaches to α then we obtain a contradiction to (a) of Lemma 3.2 by considering the cycle C' with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Hence a'_1 attaches to a_2, a_3, a_4. We can then form the cycle C' with edges $a_1a_3, a_3a'_1, a'_1a_4, a_4a_1$. Since $d = a_2$ it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the only endblock of $G - V(C)$, which is a K_2 that is disconnected from B^* by removal of a, is the K_2 with vertex a_1.

Next suppose there is an endblock of $G - V(C)$ which is an odd cycle that is disconnected from B^* by removal of a. Let a'_1, a''_1 be adjacent vertices of this odd cycle which are not cutvertices of $G - V(C)$. We see as in the previous paragraph that α is not a vertex of attachment in C of a'_1, a''_1. Suppose a_3 is a vertex of attachment for a'_1. Then either a_2 or a_4 is a vertex of attachment for a''_1. Hence we can form a cycle C' with vertices a'_1, a''_1, a_3 and one of the vertices a_2, a_4. It is easy to see that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the vertices of attachment in C of a_1', a_1'' are precisely a_2, a_4. It is easy to see further that the odd cycle with vertices a_1', a_1'' is a triangle. Otherwise we can obtain a contradiction to (a) of Lemma 3.2. Let a_1''' be the third vertex of the triangle, whence a_1''' is a cutvertex of $G - V(C)$. If $a_1''' \neq a$ we form the cycle C' with edges $a_4a_1', a_1'a_1''', a_1'''a_1'', a_1''a_4$. Since $d = a_2$ it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude $a_1''' = a$.

It is not difficult now to see that the second vertex of the K_2 with vertex a_1, is also a. To see this, let a_1' be the second vertex and suppose $a_1' \neq a$. Then, by virtue of the previous paragraph, a_1' is the cutvertex of exactly two blocks of $G - V(C)$. Hence a_1' attaches to a vertex of C. To avoid a contradiction to (a) of Lemma 3.2, we see that a_1' can attach only to a_2. Hence the second block with cutvertex a_1' is an odd cycle. Let a_1'' be a vertex of this odd cycle adjacent to a_1' which is not a cutvertex of $G - V(C)$. Then a_1'' has two vertices of attachment in C, whence a_1'' attaches to a vertex of C other than a_2. If a_1'' attaches to α, then we get a contradiction to (a) of Lemma 3.2 as before. If a_1'' attaches to a_3 or a_4, we can form a cycle C' with vertices a_1, a_1', a_1'' and fourth vertex a_3 or a_4, whichever attaches to a_1''. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude that the blocks of $G - V(C)$ connected to B^* through a are the K_2, a_1a, and triangles with cutvertex a. The vertices of attachment in C of the non-cutvertices of the triangles are precisely a_2, a_4.

We consider next the blocks of $G - V(C)$ connected to B^* through b. Consider first the endblock with adjacent vertices b_1, b_1' in the statement of the lemma. Now a_2, α are the vertices of attachment in C of b_1, b_1' and $d = a_2$. By arguing as in the previous paragraph, it is easy to see that the odd cycle containing the vertices b_1, b_1' is a triangle with cutvertex b in $G - V(C)$. It follows further, that any endblock of $G - V(C)$ which is an odd cycle and is connected to B^* through b is a triangle. The vertices of attachment in C of the non-cutvertices of the triangle are precisely a_2, α and the cutvertex of the triangle is b. Otherwise we can obtain a contradiction to (a) of Lemma 3.2. We conclude the blocks of $G - V(C)$ connected to B^* through b are triangles with cutvertex b. The vertices of attachment in C of the non-cutvertices of the triangles are a_2, α.

We have already seen that the edge αa_3 does not exist. In view of our analysis of the block structure of $G - V(C)$ it is easy to see then that a_3 attaches to $a \in B^*$. Since the vertex a_4 has degree at least 4, either the diagonal a_2a_4 exists or a_4 has a vertex of attachment in $G - V(C)$ different from a_1. To avoid a contradiction to (a) of Lemma 3.2, this vertex of attachment must be a. Let C' be the cycle with edges $a_2b_1, b_1b_1', b_1'\alpha, \alpha a_2$. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. This is illustrated in Figure 17. ∎

Lemma 7.5. *Suppose $G - V(C)$ has an endblock which is a K_2 connected to B^* through a, and denote by a_1 its vertex of degree 1 in $G - V(C)$. Suppose further*

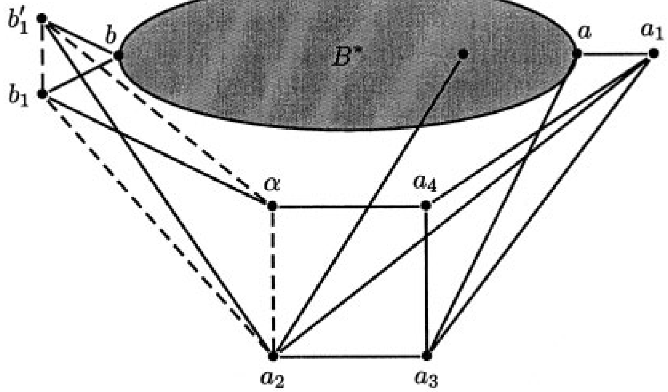

FIGURE 17.

that all endblocks of $G - V(C)$ connected to B^* through b are odd cycles. Let b_1, b_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. If the vertices of attachment in C of b_1, b_1' are non-adjacent, then there exists a cycle C' of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.

Proof. Using the notation of the previous lemmas, we see that a_2, a_4 are the vertices of attachment in C of b_1, b_1'. Suppose now $d = a_3$, and let C' be the cycle with edges $b_1a_2, a_2\alpha, \alpha a_4, a_4b_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude that the only vertex of B^* that a_3 can attach to is a. Suppose next $\alpha = d$. If the edge αa_3 exists, we form the cycle C' with edges $b_1a_2, a_2b_1', b_1'a_4, a_4b_1$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence we may assume that the edge αa_3 does not exist, whence α has two vertices of attachment in $G - V(C)$. Since b_1, b_1' are not vertices of attachment for α, we can argue as above to obtain a contradiction to (a) of Lemma 3.2. We conclude that $d \neq \alpha, a_3$, whence we may assume wlog $d = a_2$. We can show now that the edge αa_3 cannot exist. If the edge did exist, we could form the cycle C' with edges $a_1a_3, a_3\alpha, \alpha a_4, a_4a_1$. Since $d = a_2$ it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude $d = a_2$ and the edge αa_3 does not exist.

We consider now the endblocks of $G - V(C)$ which are connected to B^* through a. Suppose one of these is a second K_2 with vertex $a_1' \neq a_1$ of degree 1 in $G - V(C)$. Suppose a_1' attaches to a_3 in C. Then a_1' attaches to a_2 or a_4. Let us suppose a_1' attaches to a_4 and form the cycle C' with edges $a_1'a_3, a_3a_1, a_1a_4, a_4a_1'$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_1' does not attach to a_4, whence it attaches to both a_2 and α. Letting C' be the cycle with edges $a_1'a_3, a_3a_2, a_2\alpha, \alpha a_1'$, we see that $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. We conclude the vertices of attachment in C of a_1' are precisely a_2, α, a_4. It also follows that there are at most two endblocks of $G - V(C)$ connected to B^* through a which are K_2's.

Consider next, an endblock of $G - V(C)$ connected to B^* through a, which is an odd cycle. Let a_1', a_1'' be two adjacent vertices of the odd cycle which are not cutvertices of $G - V(C)$. We show the vertices of attachment in C of a_1', a_1'' are precisely a_2, a_4. To see this, let us suppose first that a_2 is not a vertex of attachment for a_1' or a_1''. Then we can form a cycle C' of order 4 from the vertices $a_1', a_1'', \alpha, a_4, a_3$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude a_2 is a vertex of attachment of a_1' or a_1''. Similarly a_4 is also a vertex of attachment for a_1' or a_1''. Suppose a_1' attaches to α. Then a_1'' does not attach to a_2 or a_4. Otherwise we have a contradiction to (a) of Lemma 3.2. Hence a_1'' attaches to both α and a_3. In that case we can form the cycle C' with edges $a_1''a_3, a_3a_4, a_4\alpha, \alpha a_1''$ and $G - V(C')$ contains a block strictly larger than B^*. Since this contradicts (a) of Lemma 3.2, we conclude that a_1' does not attach to α. By similar argument we see neither a_1' nor a_1'' attach to α or a_3, whence a_2, a_4 are the vertices of attachment in C of a_1', a_1''.

We investigate the blocks of $G - V(C)$ which are connected to B^* through b. First consider the endblock containing the vertices b_1, b_1'. It is easy to see that the vertices of attachment in C of any vertex of this odd cycle, which is not a cutvertex in $G - V(C)$, are precisely a_2, a_4. From this we conclude the odd cycle is a triangle and its cutvertex in $G - V(C)$ is b. Now we are assuming that any endblock of $G - V(C)$ connected to B^* through b is an odd cycle. One can see that the vertices of attachment in C of the non-cutvertices of the odd cycle are precisely a_2 and a_4. Further, the odd cycle is a triangle with cutvertex b in $G - V(C)$. We conclude the blocks of $G - V(C)$ connected to B^* through b are triangles with cutvertex b in $G - V(C)$. The vertices of attachment in C of the non-cutvertices of these triangles are precisely a_2, a_4.

We investigate the blocks of $G - V(C)$ which are connected to B^* through a. By virtue of the previous paragraphs it is easy to see that any such endblock, which is an odd cycle, is a triangle with cutvertex $a \in B^*$. Next let a_1'' be a cutvertex of $G - V(C)$ with the property that $a_1'' \notin B^*$, a_1'' is connected to B^* through a, and the connected components of $G - V(C) - \{a_1''\}$ consist of the component containing B^* and $B - \{a_1''\}$ for endblocks B of $G - V(C)$. Now the only endblocks with cutvertex a_1'' are K_2's. Suppose a_1'' is the second vertex of two such K_2's, whence the edges a_1a_1'' and $a_1'a_1''$ exist. Further, a_2, α, a_4 are the vertices of attachment in C of a_1'. Let C' be the cycle with edges $a_4a_1, a_1a_1'', a_1''a_1', a_1'a_4$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude a_1'' is the cutvertex of exactly two blocks, one of which is an endblock that is a K_2. We shall assume this endblock is the K_2 with vertex a_1 of degree 1 in $G - V(C)$. Consider now the vertices of attachment in C of a_1''. If a_1'' attaches to any of a_3, a_4, α, we obtain a contradiction to (a) of Lemma 3.2. Hence a_2 is the only possible vertex of attachment in C for a_1''. Since a_1'' is the cutvertex of just two endblocks, it follows that the second endblock is an odd cycle and that a_1'' attaches to a_2. We can therefore form the cycle C' with edges $a_1a_3, a_3a_2, a_2a_1'', a_1''a_1$. Now α has two vertices of attachment in $G - V(C)$, whence we can assume that one of these is $\alpha' \neq b$. By the previous paragraph we see that α' is not disconnected in $G - V(C)$ from B^* by removal of b. It follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude a_1'' cannot exist. Hence the blocks of $G - V(C)$, which are connected to B^* through a are triangles with cutvertex a, the K_2, a_1a, and possibly the K_2, $a_1'a$. The vertices of attachment in C of the non-cutvertices of the triangles are precisely a_2, a_4. If α attaches to both a and b then we have a contradiction to (a) of Lemma 3.2. Hence the K_2, $a_1'a$ must exist and α, a_3 attach to a. Let C' be the cycle with edges $a_2a_1, a_1a_4, a_4a_1', a_1'a_2$. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. The situation is illustrated in Figure 18. ■

To complete the proof of Proposition 7.1, we need to deal with the situation where all endblocks of $G - V(C)$ are odd cycles.

Lemma 7.6. *Suppose all endblocks of $G - V(C)$ are odd cycles. Let a_1, a_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. Then a_1, a_1' have precisely two vertices of attachment in C.*

Proof. Suppose a_1, a_1' are connected to B^* through a. Consider an endblock of $G - V(C)$ connected to B^* through b. Let b_1, b_1' be adjacent vertices of $G - V(C)$. There is a vertex d of C, which attaches to a vertex of B^* different from a, b. Suppose one of a_1, a_1' attaches to d, but d is not a vertex of attachment for b_1 or b_1'. We can then form a cycle C' of order 4 from the vertices b_1, b_1' and

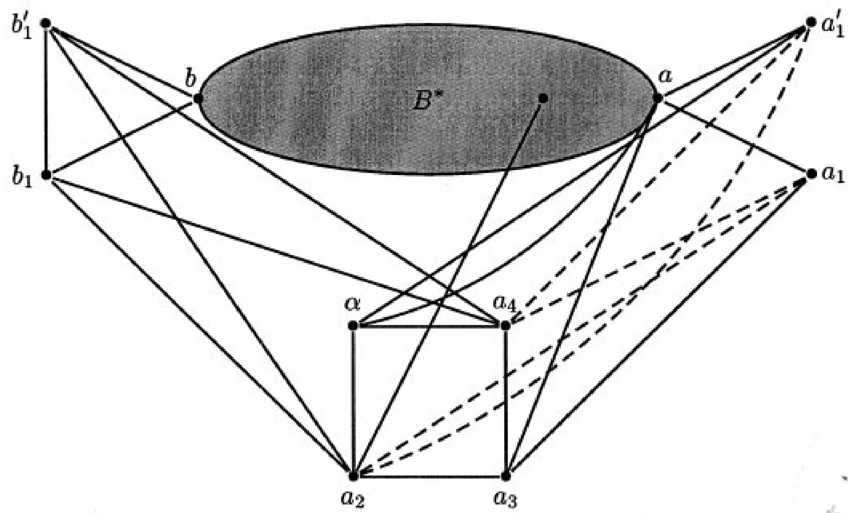

FIGURE 18.

the three vertices of C different from d. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude that there are two possibilities:

(1) One of a_1, a'_1 and one of b_1, b'_1 attaches to d,
(2) d attaches to none of the vertices a_1, a'_1, b_1, b'_1.

We consider first (1). If all vertices of C attach to either a_1 or a'_1, then we can form a cycle C' of order 4 from a_1, a'_1 and the three vertices of C different from d. Since d attaches to b_1 or b'_1, it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_1, a'_1 have at most three vertices of attachment in C, and similarly with b_1, b'_1. Suppose a_1, a'_1 have three vertices of attachment in C. One of these is d and we also have that d is a vertex of attachment for b_1 or b'_1. If d is not a vertex of attachment for both a_1 and a'_1, then we can form a cycle C' of order 4 from a_1, a'_1 and the three vertices of C different from d. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. Hence d attaches to both a_1 and a'_1. We consider the two other vertices of attachment in C of a_1, a'_1. It is easy to obtain a contradiction to (a) of Lemma 3.2 except in the situation where a_1 attaches to a vertex of C adjacent to d, and a'_1 attaches to the other vertex of C adjacent to d. Assume now that b_1, b'_1 have also three vertices of attachment in C. Then, by arguing as above we see that both b_1 and b'_1 attach to d, b_1 attaches to a vertex of C adjacent to d, and b'_1 attaches to the other vertex of C adjacent to d. The vertex of C not adjacent to d does not then attach to any of a_1, a'_1, b_1, b'_1, whence it has a vertex of attachment in $G - V(C)$ different from these. It is clear we can form a cycle C' of order 4 with vertices d, a vertex of C adjacent to d, and one of the pairs a_1, a'_1 or b_1, b'_1 such that $G - V(C')$ contains a block strictly larger than B^*. We have a contradiction to (a) of Lemma 3.2. Hence b_1, b'_1 must have only two vertices of attachment in C, one of which is d.

We denote the vertices of C in order by a_2, d, a_3, α. We are in the situation where d is a vertex of attachment for the four vertices a_1, a'_1, b_1, b'_1, and a_1 attaches to a_2, while a'_1 attaches to a_3. The vertices b_1, b'_1 have just one other vertex of attachment in C. Observe that the diagonal a_2a_3 cannot exist. If it did, we could form the cycle C' with edges $a_1a_2, a_2a_3, a_3a'_1, a'_1a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. Suppose α is the second vertex of attachment in C for b_1, b'_1. We consider the vertex of attachment a_3^* in $G - V(C)$ of a_3, which is different from a'_1, a_1. If $a_3^* = a$ or is disconnected from B^* by removal of a, then we form the cycle C' with edges $a_1a_2, a_2d, da'_1, a'_1a_1$. In that case, $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If $a_3^* \neq a$ and is still connected to B^* on removal of a, we form the cycle C' with edges $b_1d, da_2, a_2\alpha, \alpha b_1$. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude α cannot be the second vertex of attachment in C for b_1, b'_1. We may therefore assume that a_3 is the second vertex of attachment in C for b_1, b_1'. In that case, α has a vertex of attachment α^* in $G - V(C)$ different from a_1, a_1', b_1, b_1'. If $\alpha^* = a$ or is disconnected from B^* in $G - V(C)$ by removal of a, we form the cycle C' with edges $a_1a_2, a_2d, da_1', a_1'a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. The alternative is that $\alpha^* \neq a$ and α^* continues to be connected to B^* on removal of a from $G - V(C)$. In that case, let C' be the cycle with edges $b_1d, da_3, a_3b_1', b_1'b_1$. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude that in case (1), a_1, a_1' can have only two vertices of attachment in C.

We turn now to case (2). Then b_1 or b_1' has a vertex of attachment β in C which is adjacent on C to d. If a_1, a_1' have three vertices of attachment in C, we can form a cycle C' of order 4 from a_1, a_1' and the two vertices of C different from d, β. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude then that in case (2), a_1, a_1' have just two vertices of attachment in C. ■

Lemma 7.7. *Suppose all endblocks of $G - V(C)$ are odd cycles. Let a_1, a_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. If the vertices of attachment in C of a_1, a_1' are adjacent, then $G - E(C)$ is 2-connected.*

Proof. We assume a_1, a_1' are connected to B^* in $G - V(C)$ through a. Let a_2, a_3 be the vertices of attachment in C of a_1, a_1', and denote by α, β the other two vertices of C, with α adjacent to a_2. We consider an endblock of $G - V(C)$ connected to B^* through b. Let b_1, b_1' be adjacent vertices of this odd cycle, which are not cutvertices of $G - V(C)$. By Lemma 7.6, b_1, b_1' have two vertices of attachment in C. Suppose these vertices of attachment coincide with α, β. Then, no matter where d is located on C, there is a cycle C' of order 4 such that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We have therefore that a_2 or a_3 is a vertex of attachment of b_1, b_1'.

We consider the structure of endblocks of $G - V(C)$ connected to B^* through a. First suppose the odd cycle containing the vertices a_1, a_1' is not a triangle, whence we may assume there is a vertex a_1'' adjacent to a_1', which is not a cutvertex of $G - V(C)$. Since a_1' attaches to a_2, a_3, it follows from Lemma 7.6 that a_2, a_3 are also the vertices of attachment in C of a_1''. Hence we can form cycles C' of order 4 with vertices a_1, a_1', a_1'' and fourth vertex a_2 or a_3. For one of these cycles, $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the odd cycle with vertices a_1, a_1' is a triangle. It is not difficult to argue further that the cutvertex in $G - V(C)$ of the triangle is a. We also have that all endblocks of $G - V(C)$ connected to B^* through a are triangles with cutvertex a in $G - V(C)$. Further, the vertices of attachment in C of the non-cutvertices are precisely a_2, a_3. To see this, suppose there is a second endblock of $G - V(C)$ connected to B^* through a. Let a_1'', a_1''' be the vertices of this triangle which are not cutvertices of $G - V(C)$. If the vertices of attachment in C of a_1'', a_1''' are α, β, then it is easy to obtain a contradiction to (a) of Lemma 3.2 by considering the position of d on C. Let us assume that a_3, β are the vertices of attachment for a_1'', a_1'''. To avoid a contradiction to (a) of Lemma 3.2, we must have $d = a_3$ in this case. If a_2 or β are vertices of attachment for b_1, b_1', then we can obtain a contradiction to (a) of Lemma 3.2. It follows that a_3 and α are the vertices of attachment for b_1, b_1'. In that case we also obtain a contradiction to (a) of Lemma 3.2. We conclude the vertices of attachment in C for a_1'', a_1''' cannot be a_3, β. A similar argument rules out a_2, β as vertices of attachment for a_1'', a_1''', whence a_2, a_3 must be the vertices of attachment.

We consider the structure of endblocks of $G - V(C)$ connected to B^* through b. We see as in the previous paragraph that all such endblocks are triangles with cutvertex b in $G - V(C)$. We show that the vertices of attachment in C of the non-cutvertices of these triangles are all identical. If the vertices of attachment in C for b_1, b_1' are adjacent, we can argue exactly as in the previous paragraph to see this. Hence we can assume the vertices of attachment in C for b_1, b_1' are precisely a_2 and β. To avoid a contradiction to (a) of Lemma 3.2, we must then have $d = a_2$. Suppose now there is a second endblock of $G - V(C)$ connected to B^* through b. Let b_1'', b_1''' be the vertices of this triangle different from b. We need to show that a_2, β are the vertices of attachment in C for b_1'', b_1'''. If a_3 is a vertex of attachment, we obtain a contradiction to (a) of Lemma 3.2. If a_2 and α are the vertices of attachment for b_1'', b_1''', we can form the even cycle C' with edges $b_1''a_2, a_2\alpha, \alpha b_1''', b_1'''b_1''$. It is clear that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Since we can make a similar argument if α, β are the vertices of attachment for b_1'', b_1''', we conclude the vertices of attachment for b_1'', b_1''' must be a_2, β. Hence the vertices of attachment in C of the non-cutvertices of the endblock triangles are all identical.

We have shown that all endblocks of $G - V(C)$ are triangles with cutvertex a or b. The vertices of attachment in C for non-cutvertices of endblock triangles connected to B^* through a are precisely a_2, a_3. The vertices of attachment in C for non-cutvertices of endblock triangles connected to B^* through b are all identical. If α, β are these vertices of attachment, we have a contradiction to (a) of Lemma 3.2. Hence there are three possibilities:

(1) vertices of attachment are a_2, a_3,
(2) vertices of attachment are a_2, β,
(3) vertices of attachment are a_2, α.

In case (1), to avoid a contradiction to (a) of Lemma 3.2, we see that the diagonals of C, αa_3 and βa_2, must exist. Further, α, β attach to the same vertex of B^*. It is easy to see now that $G - E(C)$ is 2-connected. This is illustrated in Figure 19.

In case (2) we obtain a contradiction to (a) of Lemma 3.2. To see this, first observe that to avoid a contradiction, α can have only one vertex of attachment in $G - V(C)$. Hence the diagonal αa_3 exists. Suppose now α does not attach to a,

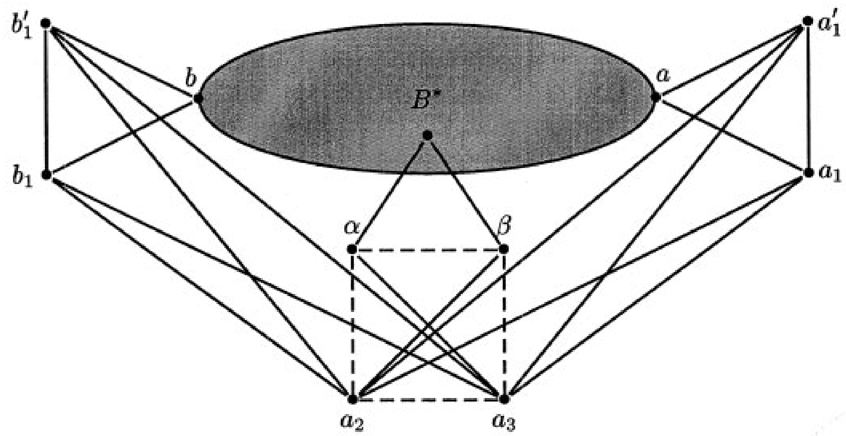

FIGURE 19.

whence it attaches to some other vertex of B^*. Let C' be the cycle with edges $b_1a_2, a_2b_1', b_1'\beta, \beta b_1$. Then $G - V(C')$ contains a block which includes B^*, α, and a_3, contradicting (a) of Lemma 3.2. Alternatively, suppose α attaches to a. If C' is the cycle with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$, then $G - V(C')$ contains a block strictly larger than B^*, again a contradiction. We conclude the situation (2) is not possible.

In case (3) we also obtain a contradiction to (a) of Lemma 3.2. First observe that to avoid a contradiction, we must have $d = a_2$. Further, the edge βa_2 must exist and β has precisely one vertex of attachment in $G - V(C)$. We can assume wlog that this vertex of attachment is a. Now it is clear that if C' is the cycle with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$ then $G - V(C')$ contains a block larger than B^*. ■

Lemma 7.8. *Suppose all endblocks of $G - V(C)$ are odd cycles. Let a_1, a_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. If the vertices of attachment in C of a_1, a_1' are non-adjacent, then there is a cycle C' of order 4 such that $G - V(C')$ is connected and $G - E(C')$ is 2-connected.*

Proof. We assume a_1, a_1' are connected to B^* in $G - V(C)$ through a. Let a_2, a_3 be the vertices of attachment in C of a_1, a_1', and denote by α, β the remaining two vertices of C. We consider an endblock of $G - V(C)$ connected to B^* through b. Let b_1, b_1' be adjacent vertices of this odd cycle which are not cutvertices of $G - V(C)$. By Lemma 7.6, b_1, b_1' have two vertices of attachment in C. If these vertices of attachment are adjacent, then the result follows from Lemma 7.7. If α, β are the vertices of attachment for b_1, b_1', then one can obtain a contradiction to (a) of Lemma 3.2 no matter where d is located on C. Hence we may assume a_2, a_3 are the vertices of attachment in C for b_1, b_1'. It is not difficult to see further that all endblocks of $G - V(C)$ are triangles with cutvertex a or b. The vertices of attachment in C for their non-cutvertices are precisely a_2, a_3.

We consider the 2 edges with vertex α, which do not belong to the cycle C. If both edges attach to vertices of $G - V(C)$—which must lie in B^* – we have a contradiction to (a) of Lemma 3.2. Hence the diagonal $\alpha\beta$ exists and α has just one vertex of attachment in $G - V(C)$. Similarly β has one vertex of attachment in $G - V(C')$ and it coincides with the vertex of attachment for α. Let C' be the cycle with edges $a_1a_2, a_2b_1, b_1a_3, a_3a_1$. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. This is illustrated in Figure 20. ■

Proof of Proposition 7.1. This follows from the previous lemmas. ■

8. PROOF OF THEOREM 1.2—COMPLETION

We complete the proof of Theorem 1.2 with the following.

Proposition 8.1. *Suppose G is a 3-connected graph of minimum degree 4 and the cycle C of Lemma 3.2 has order 4. If B^* has at least three cutvertices in $G - V(C)$, then there is a cycle C' in G of order 4 such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is 2-connected.*

To begin the proof of Proposition 8.1, we establish some properties of the endblocks of $G - V(C)$.

Lemma 8.1. *There are at most two cutvertices of B^* in $G - V(C)$ with the property that there exists an endblock of $G - V(C)$ which is a K_2, connected to B^* in $G - V(C)$ through the cutvertex.*

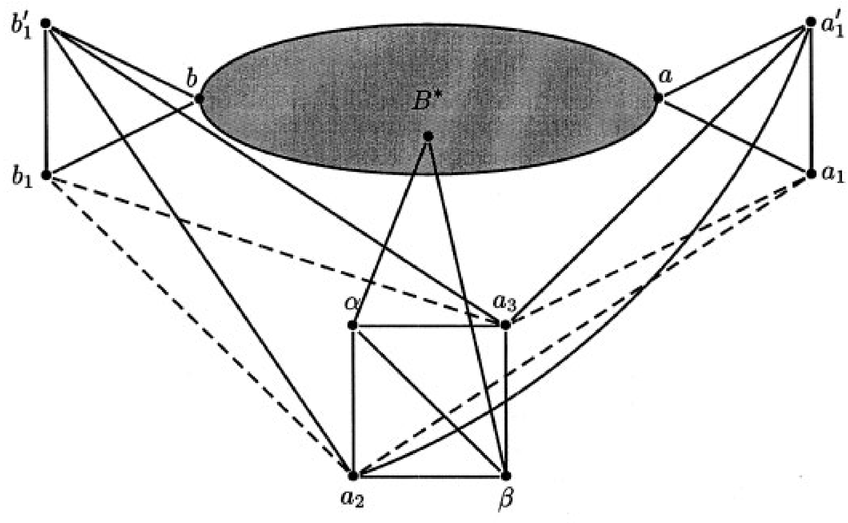

FIGURE 20.

Proof. Suppose B^* has cutvertices a, b, c in $G - V(C)$ and that there are three endblocks which are K_2's with vertices a_1, b_1, c_1 of degree 1 in $G - V(C)$. Suppose further that a_1 is connected to B^* in $G - V(C)$ through the cutvertex a, b_1 to B^* through b, and c_1 to B^* through c . Let a_2, a_3, a_4 be vertices of attachment for a_1 in C, in order on C, and α be the fourth vertex of C. If both b_1 and c_1 attach to α, then the cycle C' with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$ has the property that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We have a similar contradiction if both b_1 and c_1 attach to a_3. Hence both b_1 and c_1 attach to a_2 and a_4, and one of b_1, c_1 attaches to a_3, say c_1. In that case, we again obtain a contradiction to (a) of Lemma 3.2 by considering the cycle C' with edges $b_1a_2, a_2\alpha, \alpha a_4, a_4b_1$. We conclude there are not more than two cutvertices with the property in the statement of the lemma. ■

Lemma 8.2. *Suppose $G - V(C)$ has an endblock which is a K_2 and a_1 is its vertex of degree* 1 *in $G - V(C)$. Then a_1 has exactly three vertices of attachment in C.*

Proof. Let us assume a_1 is connected to B^* in $G - V(C)$ through the cutvertex $a \in B^*$ and that a_1 attaches to all four vertices of C. Let $b, c \in B^*$ be two other cutvertices of $G - V(C)$. Assume there is a K_2 which is an endblock of $G - V(C)$ and its vertex b_1 of degree 1 in $G - V(C)$ is connected to B^* through b. By Lemma 8.1, all endblocks of $G - V(C)$ connected to B^* through c are odd cycles. Let c_1, c_1' be adjacent vertices of such an odd cycle, which are not cutvertices of $G - V(C)$. Now c_1, c_1' have at least two vertices of attachment in C, and b_1 has at least three. It follows that there is a vertex α of C which attaches to b_1 and one of the vertices c_1, c_1'. Let C' be the cycle with vertices a_1 and the three vertices of C different from α. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2.

We conclude all endblocks of $G - V(C)$ connected to B^* through b are odd cycles. Let b_1, b_1' be adjacent vertices of such an odd cycle which are not cutvertices of $G - V(C)$. If the pairs b_1, b_1' and c_1, c_1' have a common vertex of attachment α in C, we can argue as before to obtain a contradiction to (a) of Lemma 3.2. Hence their vertices of attachment are disjoint. In that case we can form a cycle C' of order 4 with b_1, b_1', the two vertices of attachment in C of b_1, b_1', and at most one other vertex of C. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2.

Since in all cases we have a contradiction, we conclude that a_1 cannot have 4 vertices of attachment in C.

Lemma 8.3. *Suppose $G - V(C)$ has an endblock which is an odd cycle, and a_1, a_1' are adjacent vertices of this cycle which are not cutvertices of $G - V(C)$. Then a_1, a_1' have exactly the same two vertices of attachment in C.*

Proof. Suppose a_1 is connected to B^* in $G - V(C)$ through the cutvertex $a \in B^*$. Let $b, c \in B^*$ be two other cutvertices of $G - V(C)$. Assume there is a K_2

with vertex b_1 of degree 1 in $G - V(C)$ which is connected to B^* in $G - V(C)$ through b. Similarly assume there is a K_2 with vertex c_1 of degree 1 in $G - V(C)$ connected to B^* in $G - V(C)$ through c. By Lemma 8.2, c_1 has three vertices of attachment in C which we may write, in order on C, as c_2, c_3, c_4. Let α be the fourth vertex of C. Suppose α is not a vertex of attachment for b_1, whence c_2, c_3, c_4 are the vertices of attachment in C for b_1. If α is not a vertex of attachment for a_1, a_1', then we can form a cycle C' of order 4 out of the vertices a_1, a_1', two of the vertices c_2, c_3, c_4 and α. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. Hence α is a vertex of attachment for a_1, a_1'. In that case we can form a cycle C' of order 4 out of the vertices a_1, a_1', α and at most 2 other vertices of C. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2.

We conclude α is a vertex of attachment for b_1. Let us assume now that c_2, c_3 are the other vertices of attachment in C for b_1. It is easy to see that if either α, c_2 or c_4 is a vertex of attachment for a_1, a_1', then we can form a cycle C' such that $G - V(C')$ contains a block strictly larger than B^*. It follows that c_3 is the only possible vertex of attachment in C for a_1, a_1'. Since this again gives a contradiction, we conclude that the vertices of attachment in C for b_1 are α, c_2, c_4. We consider now the vertices of attachment in C for a_1, a_1'. If α is a vertex of attachment for a_1, a_1' then, by considering the cycle C' with edges $c_1c_2, c_2c_3, c_3c_4, c_4c_1$, we obtain a contradiction to (a) of Lemma 3.2. Making a similar argument for c_3, we conclude that a_1, a_1' have precisely two vertices of attachment in C, namely c_2, c_4.

Next we consider the situation where there is an odd cycle which is an endblock, connected to B^* through b. Let b_1, b_1' be adjacent vertices of this odd cycle, which are not cutvertices of $G - V(C)$. We continue to assume the K_2 with vertex c_1 of degree 1 in $G - V(C)$, connected through c to B^*, exists. Suppose a_1 or a_1' attach to all four vertices of C. There is a vertex β of C which attaches to c_1 and one of b_1, b_1'. Since we can form a cycle C' of order 4 from a_1, a_1' and the three vertices of C different from β, we obtain a contradiction to (a) of Lemma 3.2. We conclude a_1, a_1' have at most three vertices of attachment in C. Suppose there are exactly three vertices of attachment. If α is not one of these vertices of attachment, we can form a cycle C' of order 4 from a_1, a_1' and either of the pairs c_2, c_3 or c_3, c_4. Since one of α, c_2, c_4 attaches to b_1 or b_1', we again obtain a contradiction to (a) of Lemma 3.2. Hence α is a vertex of attachment for a_1, a_1'. Note that α cannot be a vertex of attachment for b_1, b_1' since we can form the cycle C' of order 4 with edges, $c_1c_2, c_2c_3, c_3c_4, c_4c_1$. Also c_3 cannot be a vertex of attachment for b_1, b_1' since we can form the cycle C' of order 4 with the vertices a_1, a_1', α and either the vertex c_2 or c_4. Hence c_2 and c_4 are the vertices of attachment in C for b_1, b_1'. In that case, for the cycle C' just constructed, $G - V(C')$ still contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude a_1, a_1' cannot have three vertices of attachment in C.

Finally we consider the situation where there is also an odd cycle which is an endblock, connected to B^* through c. Let c_1, c_1' be adjacent vertices of this odd cycle, which are not cutvertices of $G - V(C)$. Suppose first that all vertices of C attach to a_1 or a_1'. If there is a vertex α of C which attaches to one of b_1, b_1' and one of c_1, c_1', then we can form a cycle C' of order 4 from the vertices a_1, a_1' and the three vertices of C different from α. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. The alternative to this is that b_1, b_1' have exactly two vertices of attachment in C, and similarly with c_1, c_1'. Further, these vertices of attachment are disjoint. It is easy to obtain a contradiction to (a) of Lemma 3.2 in this case also. We conclude that all vertices of C cannot attach to a_1 or a_1'.

Suppose now a_1, a_1' have exactly three vertices of attachment in C, which we denote by a_2, a_3, a_4, in order on C. Let α be the fourth vertex of C. Note that we can form cycles C' of order 4 from the vertices a_1, a_1' and either the pair a_2, a_3 or a_3, a_4. Since $G - V(C')$ cannot contain a block larger than B^*, it follows that only a_3 can be a vertex of attachment for both pairs b_1, b_1' and c_1, c_1'. Suppose α is not a vertex of attachment for any of b_1, b_1', c_1, c_1'. Then we may assume that a_2, a_3 are the vertices of attachment for b_1, b_1', and a_3, a_4 the vertices of attachment for c_1, c_1'. In that case, let C' be the cycle with edges $b_1a_2, a_2a_3, a_3b_1', b_1'b_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2. It follows that α is a vertex of attachment for one of b_1, b_1', c_1, c_1', say b_1. If b_1, b_1' have three vertices of attachment in C, they must be either α, a_2, a_3 or α, a_4, a_3. Otherwise one of α, a_2, a_4 would be a vertex of attachment for both pairs b_1, b_1' and c_1, c_1'. Assume that α, a_2, a_3 are the vertices of attachment for b_1, b_1'. Then the vertices of attachment in C for c_1, c_1' must be precisely a_3, a_4. In that case, let C' be the cycle with edges $c_1a_3, a_3a_4, a_4c_1', c_1'c_1$. Since $G - V(C')$ contains a block larger than B^* we have a contradiction to (a) of Lemma 3.2. We conclude b_1, b_1' can have only two vertices of attachment in C, one of which is α. Suppose a_3 is the second vertex of attachment for b_1, b_1'. Then either a_2 or a_4 is a vertex of attachment for c_1, c_1', say a_2. We form the cycle C' with edges $b_1\alpha, \alpha a_4, a_4a_3, a_3b_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we again obtain a contradiction to (a) of Lemma 3.2. We may therefore assume that a_2 is the second vertex of attachment in C for b_1, b_1', whence a_3, a_4 must be the vertices of attachment in C for c_1, c_1'. By forming the cycle C' with edges $b_1\alpha, \alpha a_2, a_2b_1', b_1'b_1$, we obtain a contradiction to (a) of Lemma 3.2. It follows that a_1, a_1' cannot have three vertices of attachment in C. ■

Lemma 8.4. (1) *All endblocks of $G - V(C)$ which are odd cycles are triangles.* (2) *All cutvertices of $G - V(C)$ lie in B^*.*

Proof. To see (1), we consider an endblock which is an odd cycle, and let a_1, a_1' be adjacent vertices of this cycle which are not cutvertices of $G - V(C)$. By Lemma 8.3, a_1 and a_1' have the same two vertices of attachment in C. If the cycle is not a triangle then there is a vertex a_1'' of it adjacent to a_1' which is also not a cutvertex of $G - V(C)$. Again by Lemma 8.3, a_1'' has the same two vertices of attachment in C as a_1, a_1'. If α is one of these vertices of attachment, we can form the cycle C' with edges $\alpha a_1, a_1a_1', a_1'a_1'', a_1''\alpha$. Since B^* has at least three cutvertices in $G - V(C)$, it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude the endblock odd cycle containing the vertices a_1, a_1' is a triangle. It is not difficult to further conclude that the cutvertex of this triangle in $G - V(C)$ lies in B^*.

To finish the proof of (2), we need to show that any endblock of $G - V(C)$, which is a K_2, has cutvertex in B^*. Let $a_1'' \notin B^*$ be a cutvertex of $G - V(C)$ with the property that the connected components of $G - V(C) - \{a_1''\}$ consist of the component containing B^* and components $B - \{a_1''\}$ for endblocks B with cutvertex a_1'' in $G - V(C)$. By the previous paragraph, the blocks B must be K_2's. Let a_1 be the vertex of degree 1 in $G - V(C)$ of one such K_2. Suppose there is a second K_2 with vertex a_1' of degree 1 in $G - V(C)$. Then a_1, a_1' have a common vertex of attachment α in C. We may therefore form the cycle C' with edges $\alpha a_1, a_1a_1'', a_1''a_1', a_1'\alpha$. Since B^* has at least three cutvertices in $G - V(C)$, it follows that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude there is only one endblock of $G - V(C)$ with cutvertex a_1'', namely the K_2, a_1a_1''.

The vertex a_1'' is the cutvertex of exactly two blocks, one of which is the K_2, a_1a_1''. Hence a_1'' attaches to a vertex of C. Let a_2, a_3, a_4 be the vertices of attachment for a_1 in C, in order on C, and α be the fourth vertex of C. Suppose a_1 is connected to B^* in $G - V(C)$ through the cutvertex $a \in B^*$, and $b, c \in B^*$ are two other cutvertices of $G - V(C)$. Let b_1 be the vertex of an endblock of $G - V(C)$, which is not a cutvertex of $G - V(C)$, and is connected to B^* in $G - V(C)$ through b. Similarly, let c_1 be a corresponding vertex of an endblock connected to B^* through c. Then b_1 and c_1 have a minimum of two vertices of attachment in C. Suppose now a_1'' attaches to α, whence we can form cycles C' of order 4 from a_1, a_1'' and either of the pairs α, a_2 or α, a_4. Since $G - V(C')$ cannot contain a block strictly larger than B^*, we can assume that α, a_2 are the vertices of attachment for b_1, and α, a_4 the vertices of attachment for c_1. In that case, let C' be the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Then $G - V(C')$ contains a block larger than B^*, contradicting (a) of Lemma 3.2. We conclude a_1'' does not attach to α. A similar argument shows a_1'' does not attach to a_3.

We may assume now that a_1'' attaches to a_2. Suppose b_1 is the vertex of degree 1 in $G - V(C)$ of a K_2. Then, by the argument of Lemma 8.3, b_1 attaches to a_2, α, a_4 and c_1 to a_2, a_4. Let C' be the cycle with edges $a_2a_1'', a_1''a_1, a_1a_3, a_3a_2$. Then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. It follows that b_1, c_1 are the vertices of endblocks which are triangles. Each of b_1, c_1 have two vertices of attachment in C. If the vertices of attachment of b_1 and c_1 are disjoint, it is easy to obtain a contradiction to (a) of Lemma 3.2. Hence b_1 and c_1 have a common vertex of attachment in C. If α is the common vertex, we form the cycle C' with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$. Since $G - V(C')$ contains a block strictly larger than B^*, we have a contradiction to (a) of Lemma 3.2 again. A similar argument rules out a_3 as a common vertex of attachment for b_1, c_1. If a_4 is the common vertex of attachment, we form the cycle

C' with edges $a_2a_1'', a_1''a_1, a_1a_3, a_3a_2$. Again $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_2 is the common vertex of attachment for b_1, c_1. Further, either α or a_3 must be a vertex of attachment for one of b_1 or c_1. Let us assume b_1 attaches to α, whence c_1 attaches to a_3 or a_4. Hence if C' is the cycle with vertices a_2, α, b_1 and the vertex of $G - V(C)$ adjacent to b_1 which is not a cutvertex of $G - V(C)$, then $G - V(C')$ contains a block larger than B^*, again a contradiction. Since in the other cases we also have a contradiction, we conclude a_1'' cannot exist. ■

Lemma 8.5. *Suppose all endblocks of $G - V(C)$ are odd cycles. Then there exists a cycle C' of order* 4 *such that $G - V(C')$ is non-empty and connected, and $G - E(C')$ is* 2*-connected.*

Proof. Let $a, b, c \in B^*$ be three cutvertices of $G - V(C)$. Then there exists an endblock of $G - V(C)$ which is a triangle and a is one of its vertices. Let a_1, a_1' be the other two vertices of the triangle. Similarly there exist triangles with vertices b, b_1, b_1' and c, c_1, c_1'. Let a_2, a_3 be the vertices of attachment in C for a_1, a_1' and α, β be the remaining two vertices of C, with α adjacent to a_2. Suppose α, β are the vertices of attachment in C for b_1, b_1'. If c_1 attaches to α, we form a cycle C' of order 4 out of the five vertices $a_1, a_1', a_2, a_3, \beta$. Since $G - V(C')$ contains a block strictly larger than B^* we have a contradiction to (a) of Lemma 3.2. Arguing similarly if c_1 attaches to other vertices of C we conclude that a_2 or a_3 must be a vertex of attachment in C for b_1, b_1'. Assume a_2 is a vertex of attachment for b_1, b_1' but a_3 is not. Suppose first that a_2 and a_3 are not adjacent, in which case we may assume wlog that α is the second vertex of attachment in C for b_1, b_1'. Now c_1 attaches to a vertex of C different from a_2. If it attaches to α, we form the cycle C' with edges $a_1a_2, a_2\beta, \beta a_3, a_3a_1$. It is clear $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If c_1 attaches to a_3 or β, we form the cycle C' with edges $b_1a_2, a_2\alpha, \alpha b_1', b_1'b_1$. Again $G - V(C')$ contains a block larger than B^*, contradicting (a) of Lemma 3.2. We conclude that a_2, a_3 are adjacent. Now we need to separately consider α and β as being the second vertex of attachment in C for b_1, b_1'. In both cases we obtain a contradiction to (a) of Lemma 3.2. It follows that a_3 is the second vertex of attachment in C for b_1, b_1'.

From the previous paragraph, we see that the non-cutvertices of all endblocks of $G - V(C)$ have the same two vertices of attachment a_2, a_3 in C. Suppose now a_2, a_3 are adjacent on C. If the edge αa_3 does not exist, then α has two edges to B^*. Hence if C' is the cycle with edges $a_1a_2, a_2a_3, a_3a_1', a_1'a_1$, then $G - V(C')$ contains a block larger than B^*, contradicting (a) of Lemma 3.2. We conclude the edge αa_3 exists, and similarly the edge βa_2. Further, α, β must attach to the same vertex of B^*. It is easy to see now that $G - E(C)$ is 2-connected. This has already been illustrated in Figure 19.

In the case when a_2, a_3 are not adjacent, then it is easy to see that the diagonal $\alpha\beta$ exists, and α, β attach to the same vertex of B^*. Let C' be the cycle with edges $a_1a_2, a_2b_1, b_1a_3, a_3a_1$. Then $G - V(C')$ is connected and $G - E(C')$ is 2-connected. This has already been illustrated in Figure 20.

Lemma 8.6. *Suppose $G - V(C)$ has an endblock which is a K_2 and a_1 is its vertex of degree 1 in $G - V(C)$. Let a_2, a_3, a_4 be its vertices of attachment in C, in order on C, and α be the fourth vertex of C. Then*

(1) *$G - V(C)$ has exactly two endblocks which are K_2's. The vertices of attachment in C of the non-cutvertex of the second K_2 are a_2, α, a_4.*
(2) *The vertices of attachment in C of a non-cutvertex of an endblock which is an odd cycle are a_2, a_4.*

Proof. Let us suppose first that $G - V(C)$ has a second endblock which is a K_2, and b_1 is its vertex of degree 1 in $G - V(C)$. Assume further that a_1 is connected to B^* in $G - V(C)$ through $a \in B^*$, and b_1 is connected to B^* through $b \in B^*$ with $b \neq a$. Let $c \in B^*$ be a third cutvertex of $G - V(C)$. By Lemma 8.1, an endblock of $G - V(C)$ with cutvertex c is an odd cycle, whence it is a triangle. Let c_1 be a non-cutvertex of this triangle. By the argument of Lemma 8.3, we see that a_2, α, a_4 are the vertices of attachment in C for b_1, and a_2, a_4 are the vertices of attachment for c_1. Suppose now there is another endblock which is a K_2, connected to B^* in $G - V(C)$ through a. Again by the argument of Lemma 8.3, the vertices of attachment in C of its non-cutvertex a_1' are a_2, a_3, a_4. Hence we may form the cycle C' of order 4 with edges $a_2a_1, a_1a_3, a_3a_1', a_1'a_2$. Since $G - V(C)$ contains a block larger than B^*, we have a contradiction to (a) of Lemma 3.2. We conclude there are only two endblocks of $G - V(C)$ which are K_2's. Suppose next there is an endblock which is a triangle, connected to B^* in $G - V(C)$ through a. Let a_1', a_1'' be the non-cutvertices of the triangle. If α, a_3 are the vertices of attachment in C for a_1', a_1'' we can form the cycle C' with edges $a_1'\alpha, \alpha a_2, a_2a_3, a_3a_1'$. Since c_1 and a_1 attach to a_4, we see that $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We may therefore assume a_2 is a vertex of attachment for a_1', a_1''. If α or a_3 is the second vertex of attachment for a_1', a_1'', we easily obtain a contradiction to (a) of Lemma 3.2. Hence a_2, a_4 are the vertices of attachment in C for a_1', a_1''. We have proved (1) and (2) in the case when the K_2 with vertices b_1 and b exists.

Suppose next that $G - V(C)$ has a second endblock which is a K_2 and a_1' is its vertex of degree 1 in $G - V(C)$. We assume now that both a_1 and a_1' are connected to B^* in $G - V(C)$ through a. Let $b, c \in B^*$ be two other cutvertices of $G - V(C)$. From the previous paragraph, all endblocks of $G - V(C)$ with cutvertex b or c are triangles. Let b_1, b_1' be non-cutvertices of an endblock triangle with cutvertex b. Similarly, let c_1, c_1' be the corresponding vertices of a triangle with cutvertex c. If α or a_3 is a vertex of attachment for both b_1 and c_1, then we can obtain a contradiction to (a) of Lemma 3.2. Hence we may assume that b_1 attaches to a_2. If b_1 also attaches to α, we can form the cycle C' with edges $a_2b_1, b_1b_1', b_1'\alpha, \alpha a_2$. Since c_1 does not attach to α, it must attach to a_3 or a_4. Hence $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of

Lemma 3.2. It follows that α is not a vertex of attachment for b_1, and similarly neither is a_3. We conclude a_2, a_4 are the vertices of attachment in C for b_1. Consider now the vertices of attachment in C for c_1. If c_1 attaches to a_3, we obtain a contradiction to (a) of Lemma 3.2. Hence we may assume wlog that c_1 attaches to a_2. Arguing as before, we see that if c_1 attaches to α or a_3, then this gives a contradiction to (a) of Lemma 3.2. We conclude that a_2, a_4 are also the vertices of attachment in C for c_1, c_1'.

We consider the vertices of attachment in C for a_1'. Suppose α is not a vertex of attachment for a_1', whence a_2, a_3, a_4 are its vertices of attachment. We can therefore form the cycle C' with edges $a_2a_1, a_1a_3, a_3a_1', a_1'a_2$, and $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. Hence a_1' attaches to α. If a_1' also attaches to a_3, we can form the cycle C' with edges $a_1'\alpha, \alpha a_2, a_2a_3, a_3a_1'$. In that case, $G - V(C')$ contains a block larger than B^*, again a contradiction to (a) of Lemma 3.2. We conclude that a_2, α, a_4 are the vertices of attachment in C for a_1'. We can also see from the argument of Lemma 8.5 that if there is an endblock triangle with cutvertex a, then a_2, a_4 are the vertices of attachment in C for its vertices different from a. We have proved (1) and (2) in the case when $G - V(C)$ has at least two endblocks which are K_2's.

We are left to deal with the case where there is just one endblock of $G - V(C)$ which is a K_2, namely the K_2 with vertices a_1, a. By the argument of the previous paragraphs, we see that a_2, a_4 are the vertices of attachment in C for a non-cutvertex of an endblock of $G - V(C)$ which is a triangle. If the edge αa_3 does not exist, then α has two edges to B^*. Hence, if C' is the cycle with edges $a_1a_2, a_2a_3, a_3a_4, a_4a_1$, then $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. If the edge αa_3 does exist, then let C' be the cycle with edges $a_1a_3, a_3\alpha, \alpha a_2, a_2a_1$. Now $G - V(C')$ contains a block strictly larger than B^*, contradicting (a) of Lemma 3.2. We conclude $G - V(C)$ cannot have just one endblock which is a K_2. ■

Proof of Proposition 8.1. This follows from the previous lemmas. Note that the situation of Lemma 8.6 is similar to that illustrated in Figure 15. ■

ACKNOWLEDGMENT

The author thanks Andreas Blass for helpful conversations.

REFERENCES

[1] B. Bollobas, Modern Graph Theory, Springer, New York, 1998, MR 99h:05001.

[2] J. Bondy, Basic graph theory: Paths and circuits, Handbook of Combinatorics, Vol. 1, R. Graham, M. Grotschel, and L. Lovasz, Editors, Elsevier, Amsterdam, 1995, pp. 3–110. MR 97a:05129.

[3] J. Bondy and A. Vince, Cycles in a graph whose lengths differ by one or two, J Graph Theory 27 (1998), 11–15. MR 98m:05107.

[4] A. Frank, Connectivity and network flows, Handbook of Combinatorics, Vol. 1, R. Graham, M. Grotschel, L. Lovasz, Editors, Elsevier, Amsterdam, 1995, pp. 111–177. MR 97i:05084.

[5] B. Jackson, Removable cycles in 2-connected graphs of minimum degree at least 4, J Lond Math Soc 21 (1980), 385–392. MR 81h:05089.

[6] M. Lemos and J. Oxley, On removable circuits in graphs and matroids, J Graph Theory 30 (1999), 51–66. MR 99m:05094.

[7] W. Mader, Kreuzungfreie a,b-Wege in endlichen Graphen, Abh Math Sem Univ Hamburg 42 (1974), 187–204. MR 54#2525.

[8] C. Thomassen and B. Toft, Non-separating induced cycles in graphs, J Combin Theory Ser B 31 (1981), 199–224. MR 82m:05062.

[9] W. Tutte, Graph Theory, With a foreword by J. A. Nash-Williams, Reprint of the 1984 original, Encyclopedia of Mathematics and its Applications, 21, Cambridge University Press, Cambridge, 2001. MR 1813436.

[10] H. Voss, Cycles and bridges in graphs, Mathematics and its Applications, 49, Kluwer, Berlin, 1991. MR 92m:05118.