

Note

**Natural Convection Flow of a Non-Newtonian
Fluid Between Two Vertical Flat Plates**

By

K. R. Rajagopal, Pittsburgh, Pennsylvania, and T. Y. Na, Dearborn, Michigan

With 2 Figures

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Summary

The natural convection of a homogeneous incompressible fluid of grade three is investigated between two infinite parallel vertical plates. The effect of the non-Newtonian nature of fluid on the skin friction and heat transfer are studied.

1. Introduction

In this note we consider the natural convection of a non-Newtonian fluid, namely the Rivlin-Ericksen fluid of grade three, between two infinite parallel vertical flat plates. The stress in such a fluid is related to the motion in the following manner (cf. Truesdell and Noll [1]):

$$\begin{aligned} \mathbf{T} = & -p\mathbf{l} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 \\ & + \beta_2[\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1] + \beta_3(\text{tr}\mathbf{A}_1^2)\mathbf{A}_1 \end{aligned} \quad (1)$$

where μ is the coefficient of viscosity, α_1 and α_2 material moduli popularly referred to as normal stress moduli. The kinematical tensors \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 are defined through (cf. Rivlin and Ericksen [2]):

$$\mathbf{A}_1 = \text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T, \quad (2)$$

and

$$\mathbf{A}_n = \frac{d}{dt} \mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1}, \quad n = 1, 2, \quad (3)$$

where $\frac{d}{dt}$ denotes material time differentiation, and $\mathbf{L} = \text{grad } \mathbf{v}$.

The above model contains as a subclass the classical linearly viscous Newtonian fluid (when all material moduli except μ are zero) and the class of fluids of grade two (when the β 's are zero).

The natural convection problem between vertical flat plates for a certain class of non-Newtonian fluids has been carried out by Bruce and Na [3]. Other laminar natural convection problems involving heat transfer have been studied and we refer the reader to [4] for details of the same. However, in these problems a complete thermodynamic analysis of the constitutive functions have not been carried out. For the model (1), a detailed thermodynamic study has been carried out in [5]. If the model is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality (which is usually considered as an interpretation of the second law of thermodynamics) and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, then

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0. \quad (4)$$

Thus in the case of a thermodynamically compatible fluid of third grade (1) reduces to

$$\mathbf{T} = -p\mathbf{l} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(\text{tr } \mathbf{A}_1^2)\mathbf{A}_1. \quad (5)$$

For the problem under consideration, a fluid represented by (5), whose material coefficients satisfy (4), is in between two vertical flat plates a distance ' $2b$ ' apart. The walls at $x = +b$ and $x = -b$ are held at constant temperatures θ_2 and θ_1 respectively, where $\theta_1 > \theta_2$. This difference in temperature causes the fluid near the wall at $x = -b$ to rise and the fluid near the wall at $x = b$ to fall. In the next section we shall determine the velocity profile due to this flow.

2. Analysis

For the problem in question, we shall seek velocity and temperature fields of the form

$$\mathbf{v} = v(x)\mathbf{j}, \quad \theta = \theta(x). \quad (6.1, 2)$$

It follows from (5), (6) and the balance of linear momentum, that

$$(2\alpha_1 + \alpha_2) \frac{d}{dx} \left(\frac{dv}{dx} \right)^2 = \frac{\partial p}{\partial x}, \quad (7.1)$$

$$\mu \frac{d^2v}{dx^2} + 6\beta_3 \left(\frac{dv}{dx} \right)^2 \frac{d^2v}{dx^2} - \rho_0 [1 - \gamma(\theta - \theta_m)] g = \frac{\partial p}{\partial y}, \quad (7.2)$$

$$0 = \frac{\partial p}{\partial z}. \quad (7.3)$$

In deriving the Eqs. (7.1, 2, 3) the usual Boussinesq law is assumed for the body force, i.e.,

$$\rho \mathbf{b} = -\rho_0 [1 - \gamma(\theta - \theta_m)] g \mathbf{j},$$

where g denotes gravity, γ is the coefficient of thermal expansion and ρ_0 is a constant and θ_m a reference temperature which we shall pick as $\theta_m = \frac{1}{2} (\theta_1 + \theta_2)$. Defining a modified pressure through

$$\hat{p} = p - (2\alpha_1 + \alpha_2) \left(\frac{dv}{dx} \right)^2, \tag{8}$$

(7.1, 2, 3) can be re-written as

$$0 = \frac{\partial \hat{p}}{\partial x}, \tag{9.1}$$

$$\mu \frac{d^2 v}{dx^2} + 6\beta_3 \left(\frac{dv}{dx} \right)^2 \frac{d^2 v}{dx^2} - \rho_0 [1 - \gamma(\theta - \theta_m)] g = \frac{\partial \hat{p}}{\partial y}, \tag{9.2}$$

$$0 = \frac{\partial \hat{p}}{\partial z}. \tag{9.3}$$

Equations (9.1, 2, 3) imply that $\frac{\partial \hat{p}}{\partial y}$ is at most a constant. On appropriately extending the usual approximations, the equation of motion reduces to

$$\mu \frac{d^2 v}{dx^2} + 6\beta_3 \left(\frac{dv}{dx} \right)^2 \frac{d^2 v}{dx^2} + \rho_0 \gamma (\theta - \theta_m) g = 0. \tag{10}$$

We now proceed to derive the energy equation appropriate for the problem under consideration. We start with the energy equation

$$\rho \frac{d\varepsilon}{dt} = \mathbf{T} \cdot \mathbf{L} - \text{div } \mathbf{q} + \rho r, \tag{11}$$

where ε is the specific internal energy, \mathbf{L} is the gradient of velocity, \mathbf{q} is the heat flux vector and r the radiant heating. It follows from (1) that

$$\mathbf{T} \cdot \mathbf{L} = \frac{\mu}{2} |\mathbf{A}_1|^2 + \frac{\alpha_1}{4} \frac{d}{dt} |\mathbf{A}_1|^2 + \frac{(\alpha_1 + \alpha_2)}{2} \text{tr } \mathbf{A}_1^3 + \frac{\beta_3}{2} |\mathbf{A}_1|^4. \tag{12}$$

For the problem under consideration in virtue of (6.1), $\mathbf{T} \cdot \mathbf{L}$ reduces to

$$\mathbf{T} \cdot \mathbf{L} = \mu \left(\frac{dv}{dx} \right)^2 + 2\beta_3 \left(\frac{dv}{dx} \right)^4. \tag{13}$$

It has been shown in [5] that if the model (1) is to be compatible with thermodynamics then the specific Helmholtz free energy which characterizes the fluid has to take the form

$$\psi = \psi(\theta, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \bar{\psi}(\theta, \mathbf{A}_1) = \bar{\psi}(\theta, 0) + \frac{\alpha_1}{4\varrho} |\mathbf{A}_1|^2, \quad (14)$$

and further the specific entropy is defined through

$$\eta = -\bar{\psi}_\theta, \quad (15)$$

where the subscript denotes partial differentiation with respect to that variable. Since the specific internal energy is related to the specific Helmholtz free energy through

$$\varepsilon = \psi + \theta\eta, \quad (16)$$

it follows from (14), (15), (16) that

$$\varepsilon = \hat{\phi}(\theta) + \frac{\alpha_1}{4\varrho} |\mathbf{A}_1|^2 - \theta\hat{\phi}_\theta, \quad (17)$$

where

$$\hat{\phi}(\theta) \equiv \bar{\psi}(\theta, 0). \quad (18)$$

Thus,

$$\varrho \frac{d\varepsilon}{dt} = \varrho \left\{ \frac{d}{dt} (\hat{\phi}(\theta) - \theta\hat{\phi}_\theta) + \frac{\alpha_1}{4\varrho} \frac{d}{dt} |\mathbf{A}_1|^2 \right\}. \quad (19)$$

Next, note that (17) implies that

$$\varepsilon_\theta = \frac{d}{d\theta} (\hat{\phi} - \theta\hat{\phi}_\theta) = -\theta\hat{\phi}_{\theta\theta} \equiv c, \quad (20)$$

where c is called the specific heat. Thus

$$\frac{d\varepsilon}{dt} = c \frac{d\theta}{dt} = 0, \quad (21)$$

by virtue of the assumed form of the temperature field (6.2). Thus, the balance of energy (11), Eqs. (13) and (21) imply that

$$\mu \left(\frac{dv}{dx} \right)^2 + 2\beta_3 \left(\frac{dv}{dx} \right)^4 - \operatorname{div} \mathbf{q} + \varrho r = 0. \quad (22)$$

We shall assume that the heat flux vector \mathbf{q} satisfies Fourier's law with a thermal conductivity constant k , i.e.,

$$\mathbf{q} = -k \operatorname{grad} \theta.$$

Then, (6.2) implies that

$$\operatorname{div} \mathbf{q} = -k \frac{d^2\theta}{dx^2}. \tag{23}$$

Thus, if one ignores the radiant heating, (22) and (23) yield

$$\mu \left(\frac{dv}{dx}\right)^2 + 2\beta_3 \left(\frac{dv}{dx}\right)^4 + k \frac{d^2\theta}{dx^2} = 0. \tag{24}$$

In the next section we shall solve the system of Eqs. (10) and (24). The equations are coupled and highly non-linear. The appropriate boundary conditions are

$$v = 0, \quad \theta = \theta_1 \quad \text{at} \quad x = -b \tag{25.1}$$

$$v = 0, \quad \theta = \theta_2 \quad \text{at} \quad x = +b. \tag{25.2}$$

3. Solution

Let us introduce non-dimensional parameters

$$\bar{v} = \frac{v}{V_0}, \quad \bar{x} = \frac{x}{b}, \quad \bar{\theta} = \frac{\theta - \theta_m}{\theta_1 - \theta_2}, \tag{26}$$

where V_0 is some reference velocity. Then, Eqs. (10) and (24) can be re-written as

$$\frac{d^2\bar{v}}{d\bar{x}^2} + \frac{6\beta_3 V_0^2}{\mu b^2} \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 \frac{d^2\bar{v}}{d\bar{x}^2} + \frac{\rho_0 \gamma b^2}{\mu V_0} g(\theta_1 - \theta_2) \bar{\theta} = 0, \tag{27}$$

and

$$\frac{d^2\bar{\theta}}{d\bar{x}^2} + \frac{\mu V_0^2}{k(\theta_1 - \theta_2)} \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 + 2\beta_3 \frac{V_0^4}{b^2 k(\theta_1 - \theta_2)} \left(\frac{d\bar{v}}{d\bar{x}}\right)^4 = 0. \tag{28}$$

Let us select

$$V_0 = \frac{\rho_0 b^2 (\theta_1 - \theta_2) \gamma g}{\mu},$$

then (27) and (28) can be further simplified to

$$\frac{d^2\bar{v}}{d\bar{x}^2} + 6\delta \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 \frac{d^2\bar{v}}{d\bar{x}^2} + \bar{\theta} = 0, \tag{29}$$

$$\frac{d^2\bar{\theta}}{d\bar{x}^2} + E \cdot (Pr) \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 + 2\delta E \cdot (Pr) \left(\frac{d\bar{v}}{d\bar{x}}\right)^4 = 0, \tag{30}$$

where

$$E \equiv \frac{V_0^2}{c(\theta_1 - \theta_2)}, \quad Pr = \frac{\mu c}{k},$$

and

$$\delta = \frac{6\beta_3 V_0^2}{\mu b^2},$$

where c is the specific heat of the fluid. The appropriate boundary conditions are

$$\bar{v} = 0, \quad \bar{\theta} = \frac{1}{2} \quad \text{at } x = -b \quad (31)$$

$$\bar{v} = 0, \quad \bar{\theta} = -\frac{1}{2} \quad \text{at } x = +b. \quad (32)$$

The Eqs. (29)–(32) have been solved numerically.

The skin friction S on the plate at $x = -b$ is directly proportional to $\frac{d\bar{v}}{d\bar{x}}$,

$$S \sim \frac{d\bar{v}}{d\bar{x}} (-1),$$

and the heat transfer h is directly proportional to $\frac{d\bar{\theta}}{d\bar{x}}$,

$$h \sim \frac{d\bar{\theta}}{d\bar{x}} (-1).$$

Representative values of $\frac{d\bar{v}}{d\bar{x}} (-1)$ and $\frac{d\bar{\theta}}{d\bar{x}} (-1)$ have been provided for various values of E , Pr and δ . The analysis seems to indicate that an increase in δ holding E and Pr fixed increase the heat transfer slightly but decreases the skin friction. On the other hand, an increase in E holding δ and Pr fixed tends to increase the skin friction while decreasing the heat transfer. Similarly an increase in Pr holding E and δ fixed increases the skin friction and decreases the heat transfer.

Table 1. Variation of $\frac{d\bar{v}}{d\bar{x}} (-1)$ and $\frac{d\bar{\theta}}{d\bar{x}} (-1)$

γ	E	Pr	$\frac{d\bar{v}}{d\bar{x}} (-1)$	$\frac{d\bar{\theta}}{d\bar{x}} (-1)$
0.50	1.0	1.0	0.1628	-0.4966
1.0	1.0	1.0	0.1592	-0.4966
1.0	2.0	1.0	0.1593	-0.4932
1.0	4.0	1.0	0.1596	-0.4863
2	1.0	0.1	0.1532	-0.4997
2	1.0	1.0	0.1533	-0.4967

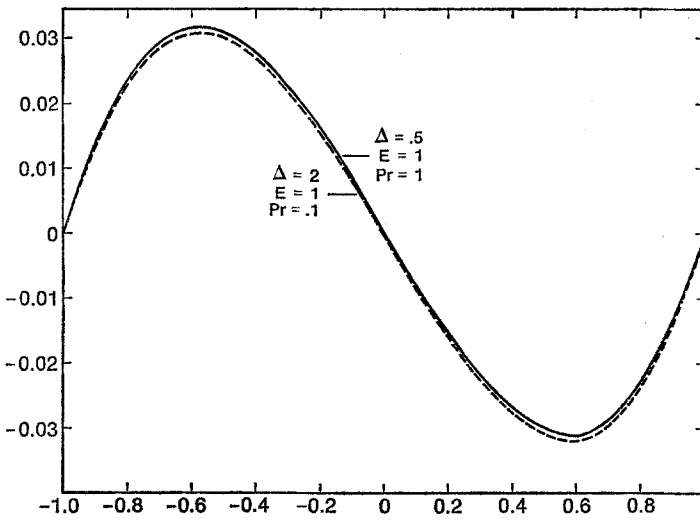


Fig. 1. Variation of velocity

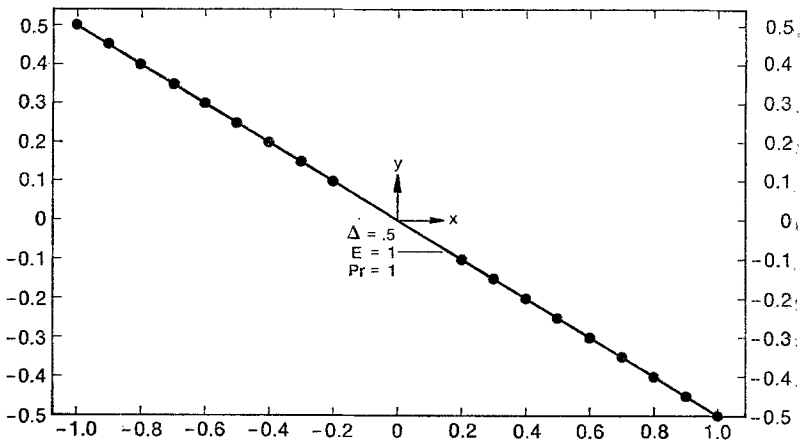


Fig. 2. Variation of temperature

Fig. 1 indicates the variation of the velocity profile \bar{v} with E , Pr , and δ . Fig. 2 depicts the temperature field $\bar{\theta}$. It is found that for the values of the parameters which have been considered, $\bar{\theta}$ varies linearly.

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Prof. K. R. Rajagopal
Department of Mechanical Engineering
University of Pittsburgh
Pittsburgh, PA 15261, U.S.A.

Dr. T. Y. Na
Department of Mechanical Engineering
University of Michigan
Deerborn Campus
Dearborn, MI 48128, U.S.A.