Arch. Rational Mech. Anal. 143 (1998) 105–193. © Springer-Verlag 1998

# *A Melnikov Method for Homoclinic Orbits with Many Pulses*

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*Communicated by* K. KIRCHGÄSSNER

## Contents



## Abstract

We present an extension of the Melnikov method which can be used for ascertaining the existence of homoclinic and heteroclinic orbits with many pulses in a class of near-integrable systems. The Melnikov function in this situation is the sum of the usual Melnikov functions evaluated with some appropriate phase delays. We show that a nonfolding condition which involves the logarithmic derivative of the Melnikov function must be satisfied in addition to the usual transversality conditions in order for homoclinic orbits with more than one pulse to exist.

## 1. Introduction

The intricate properties of homoclinic tangles were first known to POINCARÉ [52]. Later works of BIRKHOFF [5], CARTWRIGHT and LITTLEWOOD [8, 43, 44, 9], LEVINSON [42], SMALE [66], and MOSER [50] culminated in the discovery that homoclinic tangles are associated with a Smale horseshoe map, whereby the presence of infinitely many different periodic and aperiodic motions in a small neighborhood of the tangle can be established. In many applications, the existence of a homoclinic tangle can be ascertained by a simple regular perturbation method named after MELNIKOV [47]. Versions of this method [3, 10, 18, 19, 24–26, 41, 51, 57, 58, 74] can also be used to prove the existence of other types of homoclinic and heteroclinic orbits in the context of near-integrable systems, i.e., systems that are small perturbations of completely integrable Hamiltonian systems. In each case under investigation, the Melnikov method selects the survivors under perturbation from a surface of homoclinic or heteroclinic orbits. All these orbits make one excursion away from some hyperbolic equilibrium point, periodic orbit, or invariant torus, and then return to it in infinite time.

A closer inspection of a homoclinic tangle reveals that it also contains other types of homoclinic orbits, namely, those that make more than one excursion away from their target. Such orbits are said to consist of many pulses. For most of the usual tangles associated with periodically forced planar systems, the existence of multi-pulse orbits follows almost immediately from the topology of the tangle [50]. For certain orbits homoclinic to equilibria, this existence follows from simple return-map considerations [63–65, 73, 17] However, until recently, no systematic perturbation theory in the spirit of the Melnikov method was available that would establish the existence of homoclinic and heteroclinic orbits with many pulses in large classes of near-integrable systems. In this paper, we make a step in the direction of establishing just such a theory.

We present a general method for finding homoclinic and heteroclinic orbits that make several consecutive *fast* excursions away from a set of hyperbolic manifolds by constructing an extension of the Melnikov method. At the origin of our method are the ideas introduced in [6] to describe a slow manifold in an atmospheric model. We have developed these ideas into a systematic theory applicable to a large class of systems with several nonhyperbolic degrees of freedom. In particular, this theory applies to near-integrable dissipative as well as conservative systems, and to normally hyperbolic manifolds which support fast or slow dynamics. We reduce the search for multi-pulse homoclinic excursions to that of finding nondegenerate zeros of a function, $M_k(\varepsilon, I, \theta_0, \mu)$, of certain parameters $\varepsilon, I, \theta_0, \mu$, which we call the *$k$-pulse Melnikov function*. This function is computed by a recursion procedure from the ordinary 1-pulse Melnikov function, and depends on the small perturbation parameter $\varepsilon$, which is at variance with the usual Melnikov method and is peculiar to the general case of fast dynamics on the hyperbolic manifold. Moreover, the dependence on $\varepsilon$ is through a logarithmic function, which makes the calculation of the asymptotics in the small $\varepsilon$ limit particularly delicate.

Our approach consists in tracking the evolution of the global unstable manifold of the hyperbolic manifold. After setting up a fixed neighborhood of the hyperbolic manifold, this is done by alternately following the unstable manifold outside and inside this neighborhood. We call these two alternate stages global and local tracking, respectively. In essence, our construction amounts to a rigorous matching technique, and consists mainly of careful estimates. The global tracking is similar to the well-known technique of Melnikov, but even here we can detect complications as the global unstable manifold has passed near the hyperbolic manifold several times and deformed significantly. Closeness estimates presented in Sections 4 and 5 must be used to guarantee the integrity of the leading term of the distance measurement. The local tracking is performed under a special coordinate system adapted to the hyperbolic structure. Such a normal form is implicitly contained in [13–16] and used extensively in [28–30, 32, 31, 72, 70, 71]. Our version of this normal form is a special adaptation to the Hamiltonian setting of the unperturbed vector field. The local tracking gives distance estimates for trajectories near the hyperbolic manifold. Furthermore, under the action of the center vector field, a trajectory passing near the hyperbolic manifold switches "allegiance" from one unperturbed trajectory to another. The new unperturbed orbit is selected by an estimate of the phase difference between these two unperturbed trajectories, which is one of the main objectives of the local tracking. Finally, the $k$-pulse Melnikov function is constructed by collecting these global and local estimates.

The local tracking is the most delicate part of the estimating process, and we had to develop a special technique for this tracking, which we present in Section 5. This technique was first introduced in [71], and its differential-geometric tools are similar to those used in the derivation of the "Exchange Lemma" in [28–31, 70, 72]. Nevertheless, there are significant differences in the geometry of the manifolds described by the Exchange Lemma and by our tracking technique, which is stated in Lemma 1. In particular, the Exchange Lemma describes the evolution of orbits that lie on a manifold which intersects transversely either the stable or the unstable manifold of a hyperbolic manifold. On the other hand, our tracking technique describes the complementary situation in which orbits evolve on a manifold that is not allowed to intersect either the stable or the unstable manifold of a hyperbolic manifold, yet these orbits enter a very small neighborhood of this hyperbolic manifold. We thus find that, if the dynamics on the hyperbolic manifold is fast, the tracked manifold can develop sharp folds as it flies by the hyperbolic manifold in this small neighborhood. We show that the position of these folds can be located by the $k$-pulse Melnikov function. However, once a fold develops along an orbit, it becomes an obstacle to the further tracking of the manifold nearby and hence to the calculation of the subsequent multi-pulse Melnikov functions along this orbit. This is a new development, previously unknown either in the theory of the Melnikov method or the theory of multi-pulse orbits, and leads to a new "nonfolding condition," expressed in terms of the multi-pulse Melnikov function, which the multi-pulse homoclinic orbits must satisfy.

Prior to this paper, multi-pulse homoclinic and heteroclinic orbits, not necessarily associated with homoclinic tangles or perturbation theory, have been found in various applications by very different methods, of which we mention here only a small sample. Multi-pulse homoclinic orbits and the associated chaotic dynamics in a singularly perturbed three-dimensional model of excitable membranes are described in [69]. In [53–56], variational methods are used to find single-pulse and multi-pulse homoclinic and heteroclinic orbits in periodically time-dependent Hamiltonian systems defined on multi-dimensional tori. In [1, 2], traveling-wave solutions to certain semi-parabolic systems are constructed whose wave profiles are represented by 2-pulse and 3-pulse homoclinic orbits.

Multi-pulse homoclinic and heteroclinic orbits have also recently been found with methods closely resembling the Melnikov method. In particular, a method to find 2-pulse orbits based on the so called "whisker map" was developed for planar nonautonomous systems [59], and gives a result similar to our 2-pulse Melnikov function. In systems with two degrees of freedom, homoclinic and heteroclinic orbits that make several consecutive fast excursions away from normally hyperbolic manifolds were discovered in [6, 23, 48]. The dynamics on these manifolds is fast in [6, 48], while in [23] the flow on the hyperbolic manifold is perturbed away from a manifold of equilibrium points, and thereby sustains only slow dynamics.

In a different direction, a procedure that uses the Melnikov method in the process of finding homoclinic and heteroclinic orbits with a different kind of pulses, which we henceforth call "bumps," was developed in [32] for a special class of singularly perturbed near-integrable problems with internal resonances. These multi-bump homoclinic and heteroclinic orbits make several fast excursions away from the slow hyperbolic manifolds, which are interspersed with slow segments that are close to the hyperbolic manifolds themselves.

We briefly mention applications of our extended Melnikov method to two specific problems: multi-pulse orbits in a model of the atmospheric slow manifold and orbits homoclinic to resonance bands. These problems are chosen to illustrate the different regimes in which our method can be applied, namely, the case in which the dynamics on the hyperbolic manifold is fast, as in the former, or slow, as in the latter. As mentioned above, the investigation [6] of the atmospheric slow manifold model provided the main ideas and impetus for the development of the present paper. It is within the setup of this problem that the full power of our new method is needed. In particular, the dynamics on the hyperbolic manifold in this case is fast, and the equations describing the slow manifold model can only be cast in the most general form discussed in Section 10. Moreover, $k$-pulse homoclinic orbits in this example exhibit another characteristic trait predicted for a large subclass of systems by our general theory, which is that cascades of $(k+1)$-pulse homoclinic orbits, and therefore of $(k+l)$-pulse homoclinic orbits for any positive integer $l$, accumulate on a $k$-pulse homoclinic orbit. As opposed to [6], where the construction of the $k$-pulse homoclinic orbits depends very crucially on the symmetry of the slow manifold model and on the zeros of a very specific bifurcation function, the treatment in the present paper highlights this construction as almost automatic within a much more general and conceptually simpler framework of our new method.

In the problem of orbits homoclinic to resonance bands, we use the new Melnikov method for extending the results of [37] and [36] to cover homoclinic and heteroclinic orbits with several consecutive fast pulses rather than just one. This is a typical singular perturbation problem in which there are two different time scales, and the dynamics on the hyperbolic manifold is slow. Homoclinic and heteroclinic orbits are constructed by concatenating pieces of slow-time orbits on the hyperbolic manifold and fast-time heteroclinic orbits off of this manifold. Since the motion along the hyperbolic manifold is slow in this problem, our theory simplifies considerably due to the facts that the $k$-pulse Melnikov function does not depend on the small parameter $\varepsilon$, and that the nonfolding condition is automatically satisfied and thus not needed. The $k$-pulse Melnikov function in this case becomes identical to the energy-phase function of [23], which, however, was derived in an entirely different fashion. Moreover, there is a conceptual difference between our approach and that of [23], which is similar to the one between the Melnikov and the Poincaré-Arnold methods in that the geometric interpretation of a signed distance measured along the normal to a homoclinic manifold replaces the estimate of the change of energy computed along unperturbed homoclinic orbits. On a more technical level, the construction of the energy-phase function developed in [23] crucially employs the details of the geometry that depends on the dynamics along the hyperbolic manifold being slow, while our derivation of the $k$-pulse Melnikov function avoids these details entirely at the price of the more delicate local estimates near the hyperbolic manifold. Finally, we proceed in this particular case of orbits homoclinic to resonance bands to combine our results on the $k$-pulse Melnikov function with those of [32], and produce criteria for ascertaining the existence of homoclinic and heteroclinic orbits with several groups of consecutive pulses, interspersed with long segments that are close to the slow hyperbolic manifolds.

This paper is organized as follows. In Section 2 we discuss the setup of the problem. In Section 3 we state our main result. In Section 4 we put the system near the hyperbolic manifold in a normal form. In Section 5 we present the nonfolding condition and the resulting closeness estimates between the tangent spaces of the local unstable manifold of the normally hyperbolic manifold and the tangent spaces of manifolds of appropriate dimensions that fly near the normally hyperbolic manifold. In Section 6, we compute the distance between a manifold that exits a small neighborhood of the hyperbolic manifold and the unstable manifold of the hyperbolic manifold in terms of their distance at the exit time. In Section 7 we compute the distance between a manifold that flies through a small neighborhood of the hyperbolic manifold and the unstable manifold of the hyperbolic manifold in terms of the distance between the first manifold and the stable manifold of the hyperbolic manifold. In Section 8 we compute estimates for the phase differences experienced by trajectories flying close to the hyperbolic manifold and coming back to it after a pulse. In Section 9 we prove our main result. In Section 10 we discuss some important extensions of this result to the case of multiple normally hyperbolic manifolds connected by heteroclinic manifolds, and to the case of more that one nonhyperbolic degree of freedom. In Sections 11 through 13, we discuss the applications to multi-pulse orbits in a model of an atmospheric slow manifold and orbits homoclinic to resonance bands. Finally, conclusions are presented in Section 14.

## 2. The Setup

The systems we study have the form

$$\dot{x} = JD_x H(x, I) + \varepsilon g^x(x, I, \theta, \mu, \varepsilon), \tag{2.1a}$$

$$\dot{I} = \varepsilon g^I(x, I, \theta, \mu, \varepsilon), \tag{2.1b}$$

$$\dot{\theta} = \Omega(x, I) + \varepsilon g^\theta(x, I, \theta, \mu, \varepsilon), \tag{2.1c}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $I \in \mathbb{R}$, and $\theta \in S^1$. Furthermore, $D_x$ denotes the partial derivatives with respect to $x$, $D_I$ is the partial derivative with respect to $I$, $\mu \in \mathbb{R}$ is a real parameter, $\varepsilon \ll 1$ is a small parameter, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We let $\langle \cdot, \cdot \rangle$ denote the usual Euclidean inner product in $\mathbb{R}^n$, where $n$ is the dimension of the vectors in the arguments, and denote by $\| \cdot \|$ the induced Euclidean norm, as well as the corresponding matrix norm.

In the special case of a purely Hamiltonian perturbation, which arises when the entire system is derived from the Hamiltonian

$$\hat{H}(x, I, \theta, \mu, \varepsilon) = H(x, I) + \varepsilon H_1(x, I, \theta, \mu, \varepsilon), \tag{2.2}$$

the system (2.1) has the form

$$\dot{x} = JD_x H(x, I) + \varepsilon JD_x H_1(x, I, \theta, \mu, \varepsilon), \tag{2.3a}$$

$$\dot{I} = -\varepsilon D_\theta H_1(x, I, \theta, \mu, \varepsilon), \tag{2.3b}$$

$$\dot{\theta} = D_I H(x, I) + \varepsilon D_I H_1(x, I, \theta, \mu, \varepsilon). \tag{2.3c}$$

The unperturbed system is obtained by setting $\varepsilon = 0$ in the equations (2.1):

$$\dot{x} = JD_x H(x, I), \tag{2.4a}$$

$$\dot{I} = 0, \tag{2.4b}$$

$$\dot{\theta} = \Omega(x, I). \tag{2.4c}$$

Equation (2.4a) is a one-parameter family of Hamiltonian systems in the variable $x$ and can be analyzed independently of $\theta$. Once equation (2.4a) has been solved, equation (2.4c) can be solved by quadrature.

We now make two assumptions about the unperturbed equations (2.4). The first assumption concerns their smoothness.

**Assumption 1.** *The unperturbed Hamiltonian $H(x, I)$ is a real-analytic function of its arguments.*

This assumption is purely technical. In particular, it allows us to make the normal form estimates in the rest of the paper a bit less cumbersome. In fact, with appropriate changes in our estimates, we could use the results of [4, 11, 61, 67, 68] to get rid of this assumption altogether, and replace it by the requirement that $H(x, I)$ has a finite number $k$ of partial derivatives, $k > 2$.

The second assumption introduces the presence of homoclinic orbits in the phase space of equations (2.4):

**Assumption 2.** *For every $I$ with $I_1 < I < I_2$, equation* (2.4a) *possesses a hyperbolic equilibrium $x = X(I)$, which varies continuously with $I$, and whose stable and unstable manifolds, $W^s(X(I))$ and $W^u(X(I))$, intersect along a homoclinic orbit $W(X(I))$ connecting the equilibrium at $x = X(I)$ to itself.*

Because of the hyperbolicity of the equilibrium $x = X(I)$, the Jacobian $JD_x^2 H(X(I), I)$ must have a pair of nonzero real eigenvalues, say $\pm\lambda(I)$. Moreover, the implicit function theorem for analytic functions immediately implies that the vector $X(I)$ depends on the variable $I$ analytically. Since the system (2.4a) is autonomous, all the solutions on the homoclinic orbit $W(X(I))$ have a representation of the form $x^h(t - t_0, I)$, and a consistent parametrization of individual orbits in the manifold $W(X(I))$ is obtained by setting $t_0 = 0$ and varying $t$.

In the full four-dimensional $(x, I, \theta)$-phase space of the system (2.4), each equilibrium $X(I)$ corresponds to a periodic orbit $O^I$ parametrized by the solution

$$x = X(I), \quad I = I, \quad \theta = \Omega(X(I), I)t + \theta_0. \tag{2.5}$$

Each of these periodic orbits possesses two-dimensional stable and unstable manifolds, $W^s(O^I)$ and $W^u(O^I)$, that are the Cartesian products of the stable and unstable manifolds $W^s(X(I))$ and $W^u(X(I))$ of the equilibrium $X(I)$ and the angle $\theta$. The existence of the homoclinic manifolds $W(X(I))$ implies that the manifolds $W^s(O^I)$ and $W^u(O^I)$ coincide along a two-dimensional homoclinic manifold $W(O^I)$.

Taking the union of the orbits $O^I$ over all $I_1 < I < I_2$, we obtain a two-dimensional invariant annulus, which we denote by $\mathscr{M}$; see Figure 2.1. The annulus $\mathscr{M}$ possesses three-dimensional stable and unstable manifolds, $W^s(\mathscr{M})$ and $W^u(\mathscr{M})$, which intersect along the three-dimensional homoclinic manifold $W(\mathscr{M})$. All these manifolds are the unions of the manifolds $W^s(O^I)$, $W^u(O^I)$, and $W(O^I)$ along the interval $I_1 < I < I_2$. The homoclinic manifold $W(\mathscr{M})$ is parametrized by $t$, $I$, and $\theta_0$ in the solutions

$$x = x^h(t, I), \tag{2.6a}$$

$$I = I, \tag{2.6b}$$

$$\theta = \theta^h(t, I) + \theta_0 = \int_0^t \Omega(x^h(s, I), I)ds + \theta_0. \tag{2.6c}$$

The homoclinic manifold $W(\mathscr{M})$ can also be represented implicitly by the equation

$$H(x, I) - H(X(I), I) = 0, \tag{2.7}$$

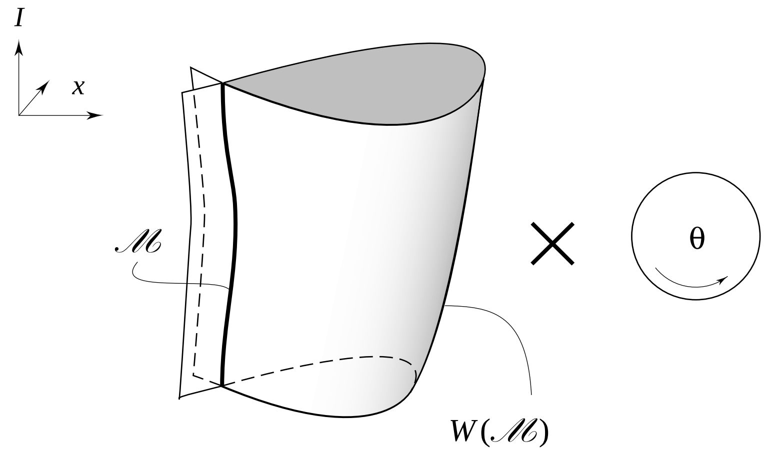


Fig. 2.1. The invariant annulus $\mathcal{M}$ and its three-dimensional homoclinic manifold $W(\mathcal{M})$ are the Cartesian product of a circle with a curve segment filled with equilibria, and its two-dimensional homoclinic manifold.

which holds on the annulus $\mathcal{M}$ at $x = X(I)$ and, hence, also on the homoclinic manifold $W(\mathcal{M})$.

As mentioned above, orbits (2.6) are homoclinic to the periodic orbits (2.5). In the course of the flight along such a homoclinic orbit, the asymptotic phase changes by the amount

$$\Delta\theta(I) = \int_{-\infty}^{\infty} \Big[ \Omega(x^h(s, I), I) - \Omega(X(I), I) \Big] \, ds. \tag{2.8}$$

This integral converges because $x^h(t, I)$ converges to $X(I)$ exponentially fast as $t \to \pm\infty$.

## 3. The Main Result

A substantial part of the hyperbolic structure introduced in the previous section persists for small positive $\varepsilon$. In particular, persistence results from [13–15] show that the unperturbed annulus $\mathcal{M}$ persists together with its *local stable and unstable manifolds*, $W^s_{\text{loc}}(\mathcal{M})$ and $W^u_{\text{loc}}(\mathcal{M})$, that is, the connected pieces of the stable and unstable manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ that are contained in some small enough neighborhood of $\mathcal{M}$ and intersect along $\mathcal{M}$. This ensures the existence of an $\mathcal{O}(\varepsilon)$ close, non-unique, locally invariant annulus $\mathcal{M}_\varepsilon$ and its local stable and unstable manifolds $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$ and $W^u_{\text{loc}}(\mathcal{M}_\varepsilon)$, which are $\mathcal{O}(\varepsilon)$ close to the local manifolds $W^s_{\text{loc}}(\mathcal{M})$ and $W^u_{\text{loc}}(\mathcal{M})$. The tangent spaces of the respective manifolds are also $\mathcal{O}(\varepsilon)$-close. Local invariance of the annulus $\mathcal{M}_\varepsilon$ reflects the fact that $\mathcal{M}_\varepsilon$ may leak phase points through its boundary, and is also responsible for the nonuniqueness of $\mathcal{M}_\varepsilon$. This nonuniqueness does not present any major difficulties, since all the copies of the annulus $\mathcal{M}_\varepsilon$ must contain all the invariant sets that are contained in any one of them, thus rendering unique the objects of our interest, homoclinic and heteroclinic orbits; see [37]. In the case of equations (2.3), when the perturbation is also derived from a Hamiltonian, the annulus $\mathcal{M}_\varepsilon$ and its stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ can be made invariant, and therefore also unique; see [36].

We obtain the global stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ of the annulus $\mathcal{M}_\varepsilon$ by evolving initial conditions in the local manifolds $W^s_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ and $W^u_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ in backward and forward time, respectively; see, for instance, [19] or [74]. The manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ are also only locally invariant. The annulus $\mathcal{M}_\varepsilon$ and the manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ vary smoothly with $\varepsilon$, and collapse onto their unperturbed counterparts as $\varepsilon \to 0$. In particular, the perturbed annulus $\mathcal{M}_\varepsilon$ can be written as a graph over the $I$ and $\theta$ variables in the form

$$x = X_\varepsilon(I, \theta, \mu, \varepsilon) \tag{3.1}$$

for some smooth function $X_\varepsilon(I, \theta, \mu, \varepsilon)$ with $X_0(I, \theta, \mu, 0) = X(I)$.

For what is to follow, we need several definitions. First, we define the Melnikov function, $M(I, \theta_0, \mu)$, which is given by the integral

$$M(I, \theta_0, \mu) = \int_{-\infty}^{\infty} \big\langle \boldsymbol{n}(\mathscr{P}^h(t)), \boldsymbol{g}(\mathscr{P}^h(t), \mu, 0)\big\rangle \, dt, \tag{3.2}$$

where

$$\begin{aligned} \boldsymbol{n} &= (D_x H(x, I), D_I H(x, I) - D_I H(X(I), I), 0)\,, \\ \boldsymbol{g} &= \big(g^x(x, I, \theta, \mu, 0), g^I(x, I, \theta, \mu, 0), g^\theta(x, I, \theta, \mu, 0)\big), \qquad (3.3) \\ \mathscr{P}^h(t) &= \big(x^h(t, I), I, \theta^h(t, I) + \theta_0\big), \end{aligned}$$

is any unperturbed homoclinic orbit given by equation (2.6) along which the integrand is evaluated; see, for instance, [74]. The vector $\boldsymbol{n}$ is the normal to the homoclinic manifold $W(\mathcal{M})$, and is computed from equation (2.7) as the gradient of its left-hand side.

Second, we define the signature $\sigma$ of the normal $\boldsymbol{n}$ by

$$\begin{aligned} \sigma &= \lim_{t\to+\infty} \frac{\big\langle \boldsymbol{n}(\mathscr{P}^h(t)), \dot{\mathscr{P}}^h(-t)\big\rangle}{||D_x H(x^h(t, I), I)||\;||D_x H(x^h(-t, I), I)||} \\ &= \lim_{t\to+\infty} \frac{\big\langle D_x H(x^h(t, I), I), J D_x H(x^h(-t, I), I)\big\rangle}{||D_x H(x^h(t, I), I)||\;||D_x H(x^h(-t, I), I)||}\,. \end{aligned} \tag{3.4}$$

Therefore $\sigma$ is positive if the normal $\boldsymbol{n}$ to the unperturbed homoclinic manifold $W(\mathcal{M})$ points in the direction of the unperturbed flow on the unstable manifold $W^u(\mathcal{M})$ at a point $(X(I), I, \theta)$ in $\mathcal{M}$, and negative otherwise. Notice that $\sigma \neq 0$ due to the transversality of the intersection between the manifolds $W^s(X(I))$ and $W^u(X(I))$ at the equilibria $X(I)$.

We also define the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$, $k = 1, 2, \ldots$, as

$$M_k(\varepsilon, I, \theta_0, \mu) = \sum_{j=0}^{k-1} M(I, j\Delta\theta(I) + \mathscr{T}_j(\varepsilon, I, \theta_0, \mu) + \theta_0, \mu), \tag{3.5}$$

where

$$\mathscr{T}_j(\varepsilon, I, \theta_0, \mu) = \frac{\Omega(X(I), I)}{\lambda(I)} \sum_{r=1}^{j} \log \left| \frac{\varsigma(I)}{\varepsilon M_r(\varepsilon, I, \theta_0, \mu)} \right|, \tag{3.6}$$

for $j = 1, \ldots, k-1$ and $\mathscr{T}_0(\varepsilon, I, \theta_0, \mu) = 0$. Thus, the 1-pulse Melnikov function $M_1(\varepsilon, I, \theta_0, \mu)$ coincides with the standard Melnikov function $M(I, \theta_0, \mu)$ given by equation (3.2). The function $\varsigma(I)$ is defined by the Jacobian of the vector field (2.4a) at the equilibrium point $x = X(I)$ as

$$\varsigma(I) = \frac{2(\lambda(I))^2 |A_2(I)| f_+(I) f_-(I)}{\sqrt{\left[\big(A_2(I)\big)^2 + \big(\lambda(I) - A_0(I)\big)^2\right]\left[\big(A_2(I)\big)^2 + \big(\lambda(I) + A_0(I)\big)^2\right]}}, \tag{3.7}$$

with

$$A_0(I) = D_{x_1} D_{x_2} H(X(I), I), \quad A_1(I) = D_{x_1}^2 H(X(I), I),$$
$$A_2(I) = D_{x_2}^2 H(X(I), I),$$
$$f_+(I) = \lim_{t\to+\infty} \frac{1}{\lambda(I)} e^{\lambda(I)t} \|D_x H(x^h(t, I), I)\|,$$
$$f_-(I) = \lim_{t\to-\infty} \frac{1}{\lambda(I)} e^{-\lambda(I)t} \|D_x H(x^h(t, I), I)\|\,. \tag{3.8}$$

We remark that if for some $I$ the frequency $\omega(I) = \Omega(X(I), I)$ vanishes, that is, if the periodic orbit corresponding to that $I$ degenerates into a circle of equilibria, the contribution from the functions $\mathscr{T}_j(\varepsilon, I, \theta_0, \mu)$ vanishes identically.

We are now ready to state our main result:

**Theorem 1.** *For some integer $k$, some constant $B > 0$ independent of $\varepsilon$, some $I = \bar{I}$, some $\mu = \bar{\mu}$, and all sufficiently small $\varepsilon > 0$ let there exist a function $\theta_0 = \bar{\theta}_0(\varepsilon)$, such that the following conditions are satisfied:*

1. *The $k$-pulse Melnikov function has a simple zero in $\theta_0$, i.e., $M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) = 0$, and $|D_{\theta_0} M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu})| > B$.*
2. *$M_i(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) \nRightarrow 0$ for all $i = 1, \ldots, k-1$, $k > 1$, and is positive if the signature $\sigma$ of the normal **n** is positive, and negative if $\sigma$ is negative.*
3. *For all $i = 1, \ldots, k-1$, $k > 1$,*

$$\left| \frac{1 - \dfrac{\Omega(X(\bar{I}), \bar{I})}{\lambda(\bar{I})} D_{\theta_0} \log \left|M_1 M_2 \ldots M_i\right| \big(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}\big)}{1 - \dfrac{\Omega(X(\bar{I}), \bar{I})}{\lambda(\bar{I})} D_{\theta_0} \log \left|M_1 M_2 \ldots M_{i-1}\right| \big(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}\big)} \right| > B, \tag{3.9}$$

*where the denominator in* (3.9) *is defined to be* 1 *when* $i = 1$.

*Then for all $I$ close to $\bar{I}$, all $\mu$ close to $\bar{\mu}$, and all sufficiently small $\varepsilon$, there exists a two-dimensional intersection surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ along which the stable and unstable manifolds $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ intersect transversely at an angle of size $\mathscr{O}(\varepsilon)$. Moreover, outside of a small neighborhood of the annulus $\mathscr{M}_\varepsilon$, the surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ is $\mathscr{O}(\varepsilon)$-close to the surface spanned by the union of orbits* (2.6) *selected by the phase angles*

$$\theta_0 = \hat{\theta}_0(\varepsilon, I, \mu) + j\Delta\theta(I) + \mathscr{T}_j(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu)$$

*for $j = 0, \ldots, k-1$, where the triple $(I, \hat{\theta}_0(\varepsilon, I, \mu), \mu)$ identically satisfies the equation*

$$M_k(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu) = 0$$

*in some neighborhood of $I = \bar{I}$ and $\mu = \bar{\mu}$, and $\hat{\theta}_0(\varepsilon, \bar{I}, \bar{\mu}) = \bar{\theta}_0(\varepsilon)$.*

The next five sections will be devoted to proving this theorem. Its extensions are presented in Section 10.

We make four remarks concerning the usage of the theorem. First, if the region enclosed by the unperturbed homoclinic manifold $W(\mathscr{M})$ is convex, the sign choice dictated by $\sigma$ in the second condition of Theorem 1 is equivalent to requiring that the sign of $M_i(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) \neq 0$, $i = 1, \ldots, k-1$, $k > 1$, be positive if the normal $\boldsymbol{n}$ to the unperturbed homoclinic manifold $W(\mathscr{M})$ points into the region enclosed by this manifold, and negative if the normal points out of this region.

Second, in the general case $\Omega(X(\bar{I}), \bar{I}) \neq 0$ the $k$-pulse Melnikov function $M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu})$ does not have a limit as $\varepsilon \to 0$, and likewise the intersection surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ does not collapse onto a limiting surface spanned by unperturbed orbits. This is in contrast to the special case of resonance, namely, $\Omega(X(\bar{I}), \bar{I}) = 0$, where the $k$-pulse Melnikov function is independent of $\varepsilon$ and, as discussed in Section 12, the surfaces $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ do collapse onto well-defined limiting surfaces. The limit process from the general case to the case of resonance is somewhat delicate, and the details are given in Section 12.

Our third remark concerns the calculation of the zeros of the $k$-pulse Melnikov function for the specific case of a single angle $\theta \in S^1$ and a homoclinic orbit when $\Omega(X(I), I) \neq 0$. Namely, although the $k$-pulse Melnikov function constitutes the leading term of the distance between the stable and unstable manifolds, $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$ only for $\varepsilon$ sufficiently small, it should be clear from its definition that it *can be* defined for large $\varepsilon$, say $\varepsilon = \mathscr{O}(1)$. Because of its periodicity in the argument $\theta_0$ for this single-angle case, the $k$-pulse Melnikov function enjoys the property that

$$M_k(\varepsilon_n, I, \theta_0, \mu) = M_k(\varepsilon, I, \theta_0, \mu), \quad \varepsilon_n = \varepsilon \exp\left(-2n\pi \frac{\lambda(I)}{\Omega(X(I), I)}\right) \tag{3.10}$$

for any $I$, $\theta_0$, $\mu$, and any integer $n$. This property is easily proved by induction together with the formula

$$\mathscr{T}_k(\varepsilon_n, I, \theta_0, \mu) = \mathscr{T}_k(\varepsilon, I, \theta_0, \mu) + 2kn\pi. \tag{3.11}$$

From (3.10), it follows that we have full knowledge of the $k$-pulse Melnikov function for any $\varepsilon > 0$ after we have analyzed it for all $\varepsilon$ inside some compact interval, say from $\exp\left(-2\pi\lambda(\bar{I})/\Omega(X(\bar{I}),\bar{I})\right)$ to 1. In particular, if the assumptions of Theorem 1 are satisfied on this interval, they are satisfied for all $\varepsilon > 0$. Furthermore, we can see that the simple-zero requirement in the first condition of Theorem 1 and the nonfolding condition (3.9) can be replaced by the simpler conditions

$$D_{\theta_0} M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) \neq 0,$$

$$\frac{\Omega(X(\bar{I}),\bar{I})}{\lambda(\bar{I})} D_{\theta_0} \log\left|M_1 M_2 \ldots M_i\right|\left(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}\right) \neq 1,$$

respectively, for all $\varepsilon$ in this compact interval.

Our fourth remark concerns some further interesting consequences of Theorem 1. Suppose the Melnikov function $M(I, \theta_0, \mu)$ given by equation (3.2) changes sign as $\theta_0$ varies over a period in $S^1$, and assume that a zero $\hat{\theta}_0(\varepsilon, I, \mu) \equiv \bar{\theta}_0^{(k)}$ of the $k$-pulse Melnikov function given by (3.5) has been determined. Then, for the $(k+1)$-pulse Melnikov function the $\theta$-argument diverges as $\theta_0 \to \bar{\theta}_0^{(k)}$ owing to the vanishing of $M_k(\varepsilon, I, \theta_0, \mu)$ in the definition (3.6) of the angle $\mathscr{T}_k$ (where $M_k(\varepsilon, I, \theta_0, \mu)$ appears as the argument of a logarithm). Since the $k$-pulse Melnikov function is periodic in the $\theta$-argument for all $k$, $M_{k+1}(\varepsilon, I, \theta_0, \mu)$ oscillates wildly for $\theta_0$ in a neighborhood of $\bar{\theta}_0^{(k)}$. Hence, there exists an infinite sequence of zeros $\bar{\theta}_0^{(k+1)}$ of $M_{k+1}(\varepsilon, I, \theta_0, \mu)$, which accumulate on $\bar{\theta}_0^{(k)}$ from both sides. This argument can of course be iterated for any of the zeros $\bar{\theta}_0^{(k+1)}$, leading to an analogous conclusion for $M_{k+2}(\varepsilon, I, \theta_0, \mu)$ and so on. This suggests that whenever the existence of a certain $k$-pulse homoclinic orbit has been established, an entire (infinite) cascade of higher pulse homoclinic orbits in a neighborhood of this $k$-pulse homoclinic should also exist. However, our estimates in the proof of Theorem 1 do not allow us to treat the limit $\bar{\theta}_0^{(k+1)} \to \bar{\theta}_0^{(k)}$ and conclude that an *infinite* number of $(k+1)$-pulse homoclinic orbits exist. A version of the estimates probably can be established so that this limit can be treated, but we leave this to future work. Notice that by reducing the size of $\varepsilon$, Theorem 1 can be applied to an increasingly large number of the zeros $\bar{\theta}_0^{(k+1)}$, and the existence of corresponding $(k+1)$-pulse homoclinic orbits can be established. It is also interesting to note that once a $k$-pulse homoclinic orbit has been detected, orbits with higher numbers of pulses can be constructed by using the Exchange Lemma [31]. The analysis of the relation between these orbits and the $(k+l)$-pulse orbits ($l > 0$) determined by the $(k+l)$-pulse Melnikov function will be relegated to future work.

## 4. The Normal Form

In this section we describe a coordinate change that brings equations (2.1) locally near the annulus $\mathscr{M}_\varepsilon$ into a normal form, closely related to Fenichel normal form [16]. We use this normal form to derive estimates that allow us to track orbits as they enter and leave a small neighborhood of the annulus $\mathscr{M}_\varepsilon$. These estimates are derived in Proposition 4.2, and further refined in Sections 5 to 10. Our normal form is a more restrictive case of the Fenichel normal form, and is needed in order to successfully carry out the proof of Proposition 7.1. This proposition is crucial for connecting a certain distance estimate at the entrance of an orbit into the small neighborhood of the annulus $\mathscr{M}_\varepsilon$, with a similar distance estimate at the exit of the neighborhood.

The coordinate change leading to our normal form is described in

**Proposition 4.1.** *In a neighborhood of the perturbed annulus $\mathscr{M}_\varepsilon$, there exist smooth local coordinates $a_\varepsilon = a_\varepsilon(x, I, \theta, \mu, \varepsilon)$, $b_\varepsilon = b_\varepsilon(x, I, \theta, \mu, \varepsilon)$, $I_\varepsilon = I_\varepsilon(x, I, \theta, \mu, \varepsilon)$, and $\psi_\varepsilon = \psi_\varepsilon(x, I, \theta, \mu, \varepsilon)$, with $a_\varepsilon, b_\varepsilon, I_\varepsilon \in \mathbb{R}$ and $\psi_\varepsilon \in S^1$, such that, at $\varepsilon = 0$, $a_0 \equiv a = a(x, I)$, $b_0 \equiv b = b(x, I)$, and equations* (2.1) *can be written in the normal form*

$$\dot{a}_\varepsilon = [\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) + \varepsilon f_a(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon)]\, a_\varepsilon, \tag{4.1a}$$

$$\dot{b}_\varepsilon = [-\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) + \varepsilon f_b(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon)]\, b_\varepsilon, \tag{4.1b}$$

$$\dot{I}_\varepsilon = \varepsilon\, [f_I(I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon) + g_I(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon) a_\varepsilon b_\varepsilon]\,, \tag{4.1c}$$

$$\begin{aligned}\dot{\psi}_\varepsilon = \omega(I_\varepsilon) + \Psi(a_\varepsilon, b_\varepsilon, I_\varepsilon) a_\varepsilon b_\varepsilon + \varepsilon \big[ f_\psi(I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon) \\ + g_\psi(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon) a_\varepsilon b_\varepsilon \big],\end{aligned} \tag{4.1d}$$

*where $\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) = D_{a_\varepsilon b_\varepsilon} K(a_\varepsilon b_\varepsilon, I_\varepsilon)$ for some analytic function $K(a_\varepsilon b_\varepsilon, I_\varepsilon)$, and $\omega(I) = \Omega(X(I), I)$. The numbers $\pm\lambda(0, I)$ are, with a slight abuse of notation, the two eigenvalues of the linearization of system* (2.4a) *about the equilibrium $x = X(I)$, i.e., $\lambda(0, I) = \lambda(I)$. In these local coordinates, the annulus $\mathscr{M}_\varepsilon$ is defined by $a_\varepsilon = b_\varepsilon = 0$, and its local stable and unstable manifolds $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ and $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ are defined by $a_\varepsilon = 0$ and $b_\varepsilon = 0$, respectively.*

**Proof.** We begin by recalling the structure of the unperturbed system (2.4), in particular the fact that the equation for the $x$ coordinate is decoupled from the rest of the system, and that the $\theta$ coordinate can be obtained by a quadrature. This structure suggests that we first transform the $x$ coordinate in a small neighborhood of the annulus $\mathscr{M}$, that is, in a small neighborhood of the equilibrium at $x = X(I)$ of the equation (2.4a). We translate this equilibrium to the origin by using the analytic (canonical) transformation $u = x - X(I)$, $\phi = \theta - \langle u, J \frac{dX(I)}{dI} \rangle$.

Using the results of [49] and [60] (see also [62, Secs. 16, 17]), we can then make another analytic canonical coordinate change which replaces $u$ by the two coordinates $a(u, I)$ and $b(u, I)$ such that the Hamiltonian $H(x, I) = H(u + X(I), I)$ becomes $K(ab, I) = F(I) + \lambda(0, I)ab + \mathscr{O}((ab)^2)$ for $F(I) = H(X(I), I)$ and some analytic function $K$ of two real variables. Thus, equations (2.4) may be rewritten as

$$\dot{a} = \lambda(ab, I)a, \tag{4.2a}$$

$$\dot{b} = -\lambda(ab, I)b, \tag{4.2b}$$

$$\dot{I} = 0, \tag{4.2c}$$

$$\dot{\phi} = \omega(I) + \Theta_a(a, b, I)a + \Theta_b(a, b, I)b, \tag{4.2d}$$

where $\lambda(ab, I) = D_{ab}K(ab, I)$, and $\omega(I) = \Omega(X(I), I)$. The form of equation (4.2d) follows from the fact that $\dot{\phi} = \omega(I)$ for $a = b = 0$. Clearly, $ab$ is a conserved quantity for equations (4.2).

We replace the angle $\phi$ by another angle $\psi$ such that $\psi - \omega(I)t$ is constant along orbits in the hyperplanes $a = 0$ and $b = 0$, which implies that system (4.2) becomes

$$\dot{a} = \lambda(ab, I)a, \tag{4.3a}$$

$$\dot{b} = -\lambda(ab, I)b, \tag{4.3b}$$

$$\dot{I} = 0, \tag{4.3c}$$

$$\dot{\psi} = \omega(I) + \Psi(a, b, I)ab. \tag{4.3d}$$

This is done in a smooth fashion in the following way. On the hyperplane $b = 0$, equations (4.2a) and (4.2d) combine into the equation

$$\frac{d(\phi - \omega(I)t)}{da} = \frac{\Theta_a(a, 0, I)}{\lambda(0, I)},$$

so that

$$\phi = \omega(I)t + \phi_0 + \frac{1}{\lambda(0, I)} \int_0^a \Theta_a(s, 0, I)\, ds.$$

Likewise, on the hyperplane $a = 0$, we obtain the equation

$$\phi = \omega(I)t + \phi_0 - \frac{1}{\lambda(0, I)} \int_0^b \Theta_b(0, s, I)\, ds.$$

If we take

$$\psi = \phi - \frac{1}{\lambda(0, I)} \left[ \int_0^a \Theta_a(s, 0, I)\, ds - \int_0^b \Theta_b(0, s, I)\, ds \right] \equiv \mathscr{G}(x, I), \tag{4.4}$$

the angle $\psi$ satisfies equation (4.3d).

For small nonzero $\varepsilon$, the results of [14] and [15] imply that we can choose in a smooth fashion new coordinates

$$a_\varepsilon = a_\varepsilon(a, b, I, \psi, \mu, \varepsilon) = a + \mathscr{O}(\varepsilon),$$

$$b_\varepsilon = b_\varepsilon(a, b, I, \psi, \mu, \varepsilon) = b + \mathscr{O}(\varepsilon),$$

$$I_\varepsilon = I_\varepsilon(a, b, h, \psi, \mu, \varepsilon) = I + \mathscr{O}(\varepsilon),$$

$$\psi_\varepsilon = \psi_\varepsilon(a, b, I, \psi, \mu, \varepsilon) = \psi + \mathscr{O}(\varepsilon),$$

such that the annulus $\mathcal{M}_\varepsilon$ is at $a_\varepsilon = b_\varepsilon = 0$, that the local stable and unstable manifolds $W^s_{\rm loc}(\mathcal{M}_\varepsilon)$ and $W^u_{\rm loc}(\mathcal{M}_\varepsilon)$ are at $a_\varepsilon = 0$ and $b_\varepsilon = 0$, respectively, and that the equations describing the dynamics of the $I_\varepsilon$ and $\psi_\varepsilon$ variables in the hyperplanes $a_\varepsilon = 0$ and $b_\varepsilon = 0$ are still independent of the $a_\varepsilon$ and $b_\varepsilon$ variables. These requirements imply equations (4.1). $\square$

Equations (4.1) are a special case of the Fenichel normal form; see [14–16] and also [23, 32, 70].

We now define the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$ of the annulus $\mathcal{M}_\varepsilon$ to be

$$U_\delta(\mathcal{M}_\varepsilon) = \Big\{(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)\ \Big|\ |a_\varepsilon| < \delta, |b_\varepsilon| < \delta, I_1 < I_\varepsilon < I_2\Big\},$$

and proceed to prove some important estimates that will be used throughout this paper:

**Proposition 4.2.** *Let the trajectory $\boldsymbol{q}_\varepsilon(t) = (a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t), \psi_\varepsilon(t))$ enter the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$ at $t = 0$ at a distance $c\varepsilon^\alpha$ from the stable manifold $a_\varepsilon = 0$ of the annulus $\mathcal{M}_\varepsilon$, where $\alpha$ and $c$ are some positive numbers. Then there exist positive constants $C$, $D$, $E$, $P$, $Q$, and $R$, which only depend on $\alpha$ and $c$ such that while this trajectory stays in the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$ the following estimates hold:*

1. *For small enough $\varepsilon$, the time of flight $T$ through the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$ of the trajectory $\boldsymbol{q}_\varepsilon(t)$ satisfies the double inequality*

$$\begin{aligned}\frac{\alpha\log\frac{1}{\varepsilon} - \log\frac{c}{\delta}}{\lambda(0, I_\varepsilon(0))}\left[1 - C\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)\right] &< T \\ &< \frac{\alpha\log\frac{1}{\varepsilon} - \log\frac{c}{\delta}}{\lambda(0, I_\varepsilon(0))}\left[1 + C\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)\right].\end{aligned} \tag{4.5}$$

2. *Let $0 < s < t < T$. Then the $a_\varepsilon$ and $b_\varepsilon$ coordinates of the trajectory $\boldsymbol{q}_\varepsilon(t)$ at times $t$ and $s$ satisfy the double inequalities*

$$\begin{aligned}|a_\varepsilon(s)|\ \exp\Big(\big[\lambda(0, I_\varepsilon(0)) - E\big(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log(1/\varepsilon)\big)\big](t - s)\Big) &< |a_\varepsilon(t)| \\ < |a_\varepsilon(s)|\ \exp\Big(\big[\lambda(0, I_\varepsilon(0)) + E\big(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log(1/\varepsilon)\big)\big](t - s)\Big),\end{aligned} \tag{4.6a}$$

$$\begin{aligned}|b_\varepsilon(s)|\ \exp\Big(\big[-\lambda(0, I_\varepsilon(0)) + E\big(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log(1/\varepsilon)\big)\big](t - s)\Big) &< |b_\varepsilon(t)| \\ < |b_\varepsilon(s)|\ \exp\Big(\big[-\lambda(0, I_\varepsilon(0)) - E\big(\varepsilon^{\alpha - D\varepsilon} + \varepsilon\log(1/\varepsilon)\big)\big](t - s)\Big).\end{aligned} \tag{4.6b}$$

3. *For $0 < t < T$, the $I$ coordinate of the trajectory $\boldsymbol{q}_\varepsilon(t)$ satisfies the inequality*

$$|I_\varepsilon(t) - I_\varepsilon(0)| < P\varepsilon\log\frac{1}{\varepsilon}. \tag{4.7}$$

4. *For $0 < t < T$, the $\psi$ coordinate of the trajectory $\boldsymbol{q}_\varepsilon(t)$ satisfies the inequality*

$$|\psi_\varepsilon(t) - \psi_\varepsilon(0) - \omega(I_\varepsilon)t| < Q\varepsilon\left(\log\frac{1}{\varepsilon}\right)^2 + R\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon}. \tag{4.8}$$

We remark that $b_\varepsilon(0) = \delta$ and $a_\varepsilon(T) = \delta$ in this proposition.

**Proof of Proposition 4.2.** Let $\kappa$ be any number that satisfies the inequality

$$\kappa < \min\left\{\lambda(ab, I) \mid |a| < \delta, |b| < \delta, I_1 \leqq I \leqq I_2\right\}.$$

Then the expression $\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) + \varepsilon f_a(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon)$ in equation (4.1a) can be bounded from below by $\kappa$ for small enough $\varepsilon$, so that $|a_\varepsilon(t)|$ satisfies the exponential estimate $|a_\varepsilon(t)| > |a_\varepsilon(0)|e^{\kappa t}$. Since $|a_\varepsilon(0)| = c\varepsilon^\alpha$, we must have the estimate $T = \mathscr{O}(\log(1/\varepsilon))$. After taking the supremum on the right-hand side of equation (4.1c), we immediately conclude that the inequality (4.7) holds, which proves part 3 of the proposition.

Multiplying equation (4.1a) by $b_\varepsilon$, equation (4.1b) by $a_\varepsilon$, and adding, we conclude that

$$(a_\varepsilon b_\varepsilon)^{\dot{}} = \varepsilon f(a_\varepsilon, b_\varepsilon, I_\varepsilon, \phi_\varepsilon, \mu, \varepsilon) a_\varepsilon b_\varepsilon,$$

for some smooth function $f(a_\varepsilon, b_\varepsilon, I_\varepsilon, \phi_\varepsilon, \mu, \varepsilon)$. Taking the supremum over this function, using the fact that $T = \mathscr{O}(\log(1/\varepsilon))$, and integrating, we obtain the inequality

$$|a_\varepsilon(0)|\,|b_\varepsilon(0)|\varepsilon^{D\varepsilon} < |a_\varepsilon(t)|\,|b_\varepsilon(t)| < |a_\varepsilon(0)|\,|b_\varepsilon(0)|\varepsilon^{-D\varepsilon} \tag{4.9}$$

for $0 < t < T$. Using equations $|a_\varepsilon(0)| = c\varepsilon^\alpha$ and $|b_\varepsilon(0)| = \delta$, we conclude from (4.9) that

$$|a_\varepsilon(t)|\,|b_\varepsilon(t)| < c\delta\varepsilon^{\alpha - D\varepsilon}. \tag{4.10}$$

Refining our first estimate using inequalities (4.7) and (4.10), we conclude that

$$\begin{aligned}\lambda(0, I_\varepsilon(0)) - E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right) &< \lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) + \varepsilon f_a(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon)\\ &< \lambda(0, I_\varepsilon(0)) + E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right),\end{aligned}$$

$$\begin{aligned}-\lambda(0, I_\varepsilon(0)) - E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right) &< -\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon) + \varepsilon f_b(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon, \mu, \varepsilon)\\ &< -\lambda(0, I_\varepsilon(0)) + E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)\end{aligned}$$

along the entire piece of trajectory $\boldsymbol{q}_\varepsilon(t)$ that is inside the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$. These two inequalities prove the estimates (4.6), and thus part 2 of the proposition.

We now let $s = 0$ and $t = T$ in the first inequality in formula (4.6a), together with the equations $|a_\varepsilon(0)| = c\varepsilon^\alpha$ for some positive constant $c$ and $|a_\varepsilon(T)| = \delta$, to show that

$$\begin{aligned}T &< \frac{\alpha\log\frac{1}{\varepsilon} - \log\frac{c}{\delta}}{\lambda(0, I_\varepsilon(0)) - E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)}\\ &< \frac{\alpha\log\frac{1}{\varepsilon} - \log\frac{c}{\delta}}{\lambda(0, I_\varepsilon(0))}\left[1 + C\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)\right],\end{aligned}$$

which holds if $\varepsilon$ is small enough. Similarly, we let $s = 0$ and $t = T$ in the second inequality in formula (4.6a) to show that

$$T > \frac{\alpha \log \frac{1}{\varepsilon} - \log \frac{c}{\delta}}{\lambda(0, I_\varepsilon(0))} \left[ 1 - C \left( \varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon} \right) \right].$$

This proves part 1 of the proposition.

To prove part 4 of the proposition, we set up the inequality

$$\begin{aligned}
&|\psi_\varepsilon(t) - \psi_\varepsilon(0) - \omega(I_\varepsilon(0))t| \\
&\quad < \int_0^T |\omega(I_\varepsilon(t)) - \omega(I_\varepsilon(0))|\, dt \\
&\qquad + \int_0^T |\Psi(a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t))|\, |a_\varepsilon(t)|\, |b_\varepsilon(t)|\, dt \\
&\qquad + \varepsilon \int_0^T | f_\psi(I_\varepsilon(t), \psi_\varepsilon(t), \mu, \varepsilon) \\
&\qquad + g_\psi(a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t), \psi_\varepsilon(t), \mu, \varepsilon) a_\varepsilon(t) b_\varepsilon(t)|\, dt,
\end{aligned} \tag{4.11}$$

which follows from equation (4.1d). The first term on the right-hand side of this inequality can be estimated to be of size $\mathscr{O}(\varepsilon(\log(1/\varepsilon))^2)$ by using the inequality (4.7) and the second inequality in formula (4.5) without the term $-\log(c/\delta)$. Taking suprema over the functions involved in the integrand of the last term, we likewise conclude that this term is of size $\mathscr{O}(\varepsilon \log(1/\varepsilon))$. The first and the last term together can therefore be bounded above by the expression $Q\varepsilon(\log(1/\varepsilon))^2$.

The middle term on the right-hand side of the inequality (4.11) is estimated by using formula (4.10), taking the supremum of the function $\Psi(a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t))$ over the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$, integrating in time, and omitting the term $-\log(c/\delta)$ in the second inequality (4.5). We conclude that

$$\int_0^T |\Psi(a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t))|\, |a_\varepsilon(t)|\, |b_\varepsilon(t)|\, dt < R\varepsilon^{\alpha - D\varepsilon} \log \frac{1}{\varepsilon}.$$

Combined with the upper bound $Q\varepsilon(\log(1/\varepsilon))^2$ for the first and third terms in the inequality (4.11), this estimate yields formula (4.8), and thus proves part 4 of the proposition. □

## 5. The Closeness of Tangent Spaces

In the following sections we will compute the distance between the stable and unstable manifolds $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ by an extension of the Melnikov method. In particular, we will follow the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ as it winds in and out of a small neighborhood of the annulus $\mathscr{M}_\varepsilon$, and measure its distance from the stable manifold $W^s(\mathscr{M}_\varepsilon)$. In order not to confuse this winding piece of the unstable manifold with a local piece of this manifold, we denote it by $\mathscr{L}$. In this section, we derive a closeness estimate between the tangent spaces of the manifolds $\mathscr{L}$ and $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ and the tangent spaces of the manifolds $\mathscr{L}$ and $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ at the points where the manifold $\mathscr{L}$ enters and exits the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$, respectively (see Figure 5.1).

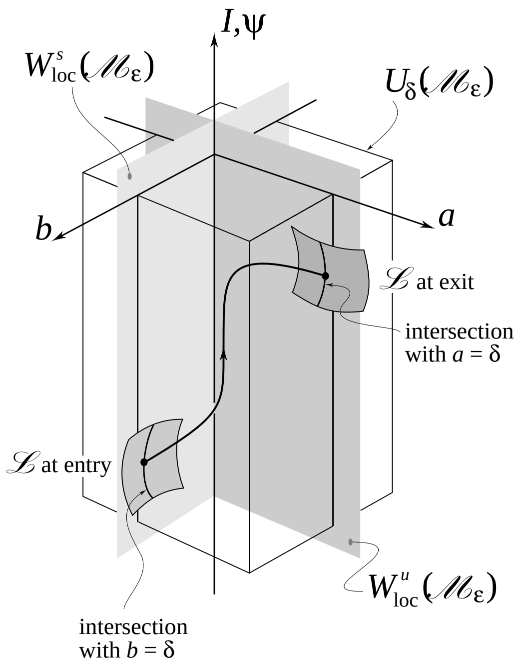


Fig. 5.1. A portion of the manifold $\mathcal{L}$ as it enters and leaves the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$.

We begin by deriving the equations of variation along a trajectory $\boldsymbol{q}_\varepsilon(t)$ from the equations (4.1). Here the trajectory $\boldsymbol{q}_\varepsilon(t)$ is as in Proposition 4.2. Throughout this section, we omit arguments of functions to prevent formulas from becoming unwieldy. In particular, we write $\lambda(a_\varepsilon b_\varepsilon, I_\varepsilon)$ simply as $\lambda$. The equations of variation are

$$d\dot{a}_\varepsilon = (\lambda + \varepsilon f_a) da_\varepsilon + a_\varepsilon \left( D_{\zeta_\varepsilon} \lambda d\zeta_\varepsilon + D_{I_\varepsilon} \lambda dI_\varepsilon + \varepsilon \langle \nabla f_a, d\boldsymbol{X} \rangle \right), \tag{5.1a}$$

$$d\dot{b}_\varepsilon = (-\lambda + \varepsilon f_b) db_\varepsilon + b_\varepsilon \left( -D_{\zeta_\varepsilon} \lambda d\zeta_\varepsilon - D_{I_\varepsilon} \lambda dI_\varepsilon + \varepsilon \langle \nabla f_b, d\boldsymbol{X} \rangle \right), \tag{5.1b}$$

$$d\dot{I}_\varepsilon = \varepsilon \left( D_{I_\varepsilon} f_I dI_\varepsilon + D_{\psi_\varepsilon} f_I d\psi_\varepsilon + g_I d\zeta_\varepsilon + \zeta_\varepsilon \langle \nabla g_I, d\boldsymbol{X} \rangle \right), \tag{5.1c}$$

$$\begin{aligned} d\dot{\psi}_\varepsilon &= D_{I_\varepsilon} \omega dI_\varepsilon + \Psi d\zeta_\varepsilon + \zeta_\varepsilon \left( D_{a_\varepsilon} \Psi da_\varepsilon + D_{b_\varepsilon} \Psi db_\varepsilon + D_{I_\varepsilon} \Psi dI_\varepsilon \right) \\ &\quad + \varepsilon \left( D_{I_\varepsilon} f_\psi dI_\varepsilon + D_{\psi_\varepsilon} f_\psi d\psi_\varepsilon + g_\psi d\zeta_\varepsilon + \zeta_\varepsilon \langle \nabla g_\psi, d\boldsymbol{X} \rangle \right), \end{aligned} \tag{5.1d}$$

where $\zeta_\varepsilon = a_\varepsilon b_\varepsilon$ and $d\boldsymbol{X} = (da_\varepsilon, db_\varepsilon, dI_\varepsilon, d\psi_\varepsilon)^T$. Instead of working directly with these equations, we utilize certain structures possessed by these equations. A similar strategy was used in [71] to study an atmospheric system which we will describe in Section 11.

**Proposition 5.1.** *Let* $U = da_\varepsilon$, $V = db_\varepsilon$, *and* $\boldsymbol{W} = (W_1, W_2)^T = (dI_\varepsilon, d\psi_\varepsilon)^T$. *Then equations* (5.1) *assume the form*

$$\dot{U} = (\lambda(0, I_\varepsilon(0)) + \varphi_1)U + \varphi_3 V + \boldsymbol{A}^T \boldsymbol{W}, \tag{5.2a}$$

$$\dot{V} = (-\lambda(0, I_\varepsilon(0)) + \varphi_2)V + \varphi_4 U + \boldsymbol{B}^T \boldsymbol{W}, \tag{5.2b}$$

$$\dot{\boldsymbol{W}} = (\Theta_0 + \Theta_1)\boldsymbol{W} + U\boldsymbol{H}_1 + V\boldsymbol{H}_2, \tag{5.2c}$$

*where* $\varphi_1(t, \varepsilon), \ldots, \varphi_4(t, \varepsilon)$ *are scalar functions,* $\boldsymbol{A}(t, \varepsilon)$, $\boldsymbol{B}(t, \varepsilon)$, $\boldsymbol{H}_1(t, \varepsilon)$, $\boldsymbol{H}_2(t, \varepsilon)$ *are vectors in* $\mathbb{R}^2$, $\Theta_0(\varepsilon)$, $\Theta_1(t, \varepsilon)$ *are* $2 \times 2$ *matrices, and there exist positive constants* $C_1, \ldots, C_9$, *each independent of* $\delta$ *and* $\varepsilon$, *satisfying*

$$|\varphi_1| \leqq C_1 \left( \varepsilon + |a_\varepsilon(t)||b_\varepsilon(t)| + \varepsilon \log \frac{1}{\varepsilon} \right), \tag{5.3a}$$

$$|\varphi_2| \leqq C_2 \left( \varepsilon + |a_\varepsilon(t)||b_\varepsilon(t)| + \varepsilon \log \frac{1}{\varepsilon} \right), \tag{5.3b}$$

$$|\varphi_3| \leqq C_3 |a_\varepsilon(t)|, \tag{5.3c}$$

$$|\varphi_4| \leqq C_4 |b_\varepsilon(t)|, \tag{5.3d}$$

$$\|\boldsymbol{A}\| \leqq C_5 |a_\varepsilon(t)|, \tag{5.3e}$$

$$\|\boldsymbol{B}\| \leqq C_6 |b_\varepsilon(t)|, \tag{5.3f}$$

$$\|\boldsymbol{H}_1\| \leqq C_7 |b_\varepsilon(t)|, \tag{5.3g}$$

$$\|\boldsymbol{H}_2\| \leqq C_8 |a_\varepsilon(t)|, \tag{5.3h}$$

$$\Theta_0 = \begin{pmatrix} 0 & 0 \\ D_{I_\varepsilon}\omega(I_\varepsilon(0)) & 0 \end{pmatrix}, \tag{5.3i}$$

$$\|\Theta_1\| \leqq C_9 \left( \varepsilon + \varepsilon \log \frac{1}{\varepsilon} + |a_\varepsilon(t)||b_\varepsilon(t)| \right). \tag{5.3j}$$

**Proof.** We identify the terms in equations (5.2) with those in equations (5.1). First, we have

$$\varphi_1 = \lambda - \lambda(0, I_\varepsilon(0)) + \varepsilon f_a + a_\varepsilon b_\varepsilon D_{\zeta_\varepsilon} \lambda + \varepsilon a_\varepsilon D_{a_\varepsilon} f_a.$$

Using the estimate (4.7) in Proposition 4.2, we see that

$$\lambda - \lambda(0, I_\varepsilon(0)) = \mathcal{O}\left( |a_\varepsilon||b_\varepsilon| + \varepsilon \log \frac{1}{\varepsilon} \right).$$

The rest of the terms are clearly of the size $\mathcal{O}(\varepsilon + |a_\varepsilon||b_\varepsilon|)$. The bound for $|\varphi_2|$ is similar. The bounds for $|\varphi_3|$, $|\varphi_4|$, $\|\boldsymbol{A}\|$, $\|\boldsymbol{B}\|$, $\|\boldsymbol{H}_1\|$, $\|\boldsymbol{H}_2\|$ are also straightforward. Finally, we identify

$$\Theta_1 = \begin{pmatrix} \varepsilon\left(D_{I_\varepsilon} f_I + \zeta_\varepsilon D_{I_\varepsilon} g_I\right) & \varepsilon\left(D_{\psi_\varepsilon} f_I + \zeta_\varepsilon D_{\psi_\varepsilon} g_I\right) \\ D_{I_\varepsilon}\omega(I_\varepsilon(t)) - D_{I_\varepsilon}\omega(I_\varepsilon(0)) + \zeta_\varepsilon D_{I_\varepsilon}\Psi + \varepsilon\left(D_{I_\varepsilon} f_\psi + \zeta_\varepsilon D_{I_\varepsilon} g_\psi\right) & \varepsilon\left(D_{\psi_\varepsilon} f_\psi + \zeta_\varepsilon D_{\psi_\varepsilon} g_\psi\right) \end{pmatrix}.$$

The same kind of consideration as for $|\varphi_1|$ gives the desired bound for $\|\Theta_1\|$. □

We now prove a technical result. Let $\boldsymbol{v}_0 = (U_0, V_0, \boldsymbol{W}_0^T)^T$ be a tangent vector of the manifold $\mathscr{L}$ at the point $\boldsymbol{q}_\varepsilon(0)$ where the trajectory $\boldsymbol{q}_\varepsilon(t)$ on $\mathscr{L}$ enters the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$. Assume that the inequalities

$$|U_0| \leqq M_1\varepsilon^\alpha, \qquad |V_0| \leqq M_2\varepsilon^\alpha, \qquad \|\boldsymbol{W}_0\| \leqq M_3 \tag{5.4}$$

hold, where $M_1$, $M_2$ and $M_3$ are positive constants independent of $\varepsilon$ and $\frac{1}{2} < \alpha \leqq 1$. Then we can show

**Proposition 5.2.** *Under the assumptions* (5.4) *on the initial conditions, for* $\beta > 0$ *sufficiently small and* $0 < \kappa < \beta\lambda(0, I_\varepsilon(0))/2\alpha$*, the estimates*

$$|U(t)| \leqq \varepsilon^{\alpha-3\beta} e^{[\lambda(0,I_\varepsilon(0))+\kappa]t}, \tag{5.5a}$$

$$|V(t)| \leqq \varepsilon^{\alpha-5\beta}, \tag{5.5b}$$

$$\|\boldsymbol{W}(t)\| \leqq \varepsilon^{-\beta} \tag{5.5c}$$

*hold, where* $0 \leqq t \leqq T$ *and* $\varepsilon$ *is sufficiently small.*

**Proof.** By the initial conditions (5.4), the bounds (5.5) are satisfied for $0 \leqq t \leqq T_0$, where $T_0 \leqq T$ is some small time, and for $\varepsilon$ sufficiently small. Our technique consists of assuming the inequalities (5.5) for $0 \leqq t \leqq T_0$ and deriving *new* bounds for the functions $|U(t)|$, $|V(t)|$ and $\|\boldsymbol{W}(t)\|$ in the interval $0 \leqq t \leqq T_0$. Furthermore, these bounds show that the inequalities (5.5) are strict. It follows that we can extend the validity of (5.5) beyond the interval $0 \leqq t \leqq T_0$, say to an interval $0 \leqq t \leqq T_1$, where $T_0 < T_1 \leqq T$. The same argument applies to the new interval $0 \leqq t \leqq T_1$, however, and we can use an elementary connectedness argument to show that (5.5) holds on the whole interval $0 \leqq t \leqq T$.

Accordingly, we now assume that the estimates (5.5) hold for $0 \leqq t \leqq T_0$. For $\varepsilon$ sufficiently small, we have the estimates

$$E\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right) < \kappa, \quad C\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right) < \frac{1}{2},$$

where $C$, $D$, $E$ are constants in Proposition 4.2. We shrink $\varepsilon$ further as we go along. Under these conditions, we obtain from the estimates (4.5), (4.6a), (4.6b) the further estimates

$$\left(\frac{1}{\varepsilon^\alpha}\frac{\delta\sqrt{e}}{c}\right)^{\frac{1}{\lambda(0,I_\varepsilon(0))}} < e^T < \left(\frac{1}{\varepsilon^\alpha}\frac{\delta e\sqrt{e}}{c}\right)^{\frac{1}{\lambda(0,I_\varepsilon(0))}}, \tag{5.6}$$

$$\delta e^{-[\lambda(0,I_\varepsilon(0))+\kappa](T-t)} < |a_\varepsilon(t)| < \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa](T-t)}, \tag{5.7a}$$

$$\delta e^{-[\lambda(0,I_\varepsilon(0))+\kappa]t} < |b_\varepsilon(t)| < \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t}. \tag{5.7b}$$

By further shrinking $\varepsilon$ if necessary, we may also assume that

$$\kappa > C_1 \left( \varepsilon + \delta^2 e^{-[\lambda(0, I_\varepsilon(0)) - \kappa]T} + \varepsilon \log \frac{1}{\varepsilon} \right),$$

so that $\lambda(0, I_\varepsilon(0)) + \varphi_1 \leqq \lambda(0, I_\varepsilon(0)) + \kappa$. Applying the Gronwall inequality and the estimate (5.7a) to equation (5.2a), we have, for $0 \leqq t \leqq T_0$,

$$\begin{aligned} |U(t)| &\leqq |U_0| e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t} \\ &\quad + \int_0^t e^{[\lambda(0, I_\varepsilon(0)) + \kappa](t-\tau)} (C_3 \varepsilon^{\alpha - 5\beta} + C_5 \varepsilon^{-\beta}) \delta e^{-[\lambda(0, I_\varepsilon(0)) - \kappa](T-\tau)} d\tau \\ &\leqq M_1 \varepsilon^\alpha e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t} \\ &\quad + (C_3 + C_5) \varepsilon^{-\beta} \delta e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t} e^{-[\lambda(0, I_\varepsilon(0)) - \kappa]T} \int_0^t e^{-2\kappa\tau} d\tau. \end{aligned} \tag{5.8}$$

Applying the estimate (5.6) gives

$$\begin{aligned} |U(t)| &\leqq M_1 \varepsilon^\alpha e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t} + \frac{C_3 + C_5}{2\kappa} \varepsilon^{-\beta} \delta e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t} \left( \frac{\varepsilon^\alpha c}{\delta \sqrt{e}} \right)^{\frac{\lambda(0, I_\varepsilon(0)) - \kappa}{\lambda(0, I_\varepsilon(0))}} \\ &\leqq \left[ M_1 \varepsilon^{3\beta} + \frac{C_3 + C_5}{2\kappa} \delta \left( \frac{c}{\delta \sqrt{e}} \right)^{\frac{\lambda(0, I_\varepsilon(0)) - \kappa}{\lambda(0, I_\varepsilon(0))}} \varepsilon^\beta \right] \varepsilon^{\alpha - 3\beta} e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t}, \end{aligned} \tag{5.9}$$

where we have used the assumption that $\beta > 2\alpha\kappa / \lambda(0, I_\varepsilon(0))$, so that

$$\varepsilon^{\frac{\alpha[\lambda(0, I_\varepsilon(0)) - \kappa]}{\lambda(0, I_\varepsilon(0))}} = \varepsilon^{\alpha - \frac{\alpha\kappa}{\lambda(0, I_\varepsilon(0))}} \leqq \varepsilon^{\alpha - \frac{\beta}{2}} \leqq \varepsilon^{\alpha - \beta}.$$

By choosing $\varepsilon$ sufficiently small, we have the estimate

$$M_1 \varepsilon^{3\beta} + \frac{C_3 + C_5}{2\kappa} \delta \left( \frac{c}{\delta \sqrt{e}} \right)^{\frac{\lambda(0, I_\varepsilon(0)) - \kappa}{\lambda(0, I_\varepsilon(0))}} \varepsilon^\beta < 1.$$

The estimate (5.9) now becomes

$$|U(t)| < \varepsilon^{\alpha - 3\beta} e^{[\lambda(0, I_\varepsilon(0)) + \kappa]t}.$$

Next, we derive a new estimate for $|V(t)|$. Again by taking $\varepsilon$ sufficiently small, we can assume that

$$\kappa > C_2 \left( \varepsilon + \delta^2 e^{-[\lambda(0, I_\varepsilon(0)) - \kappa]T} + \varepsilon \log \frac{1}{\varepsilon} \right),$$

so that $-\lambda(0, I_\varepsilon(0)) + \varphi_2 \leqq -\lambda(0, I_\varepsilon(0)) + \kappa$. Applying the Gronwall inequality and the estimate (5.7b) to equation (5.2b), we obtain

$$
\begin{aligned}
|V(t)| &\leqq |V_0| e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t} \\
&\quad + \int_0^t e^{-[\lambda(0,I_\varepsilon(0))-\kappa](t-\tau)} (C_4 \varepsilon^{\alpha-3\beta} e^{[\lambda(0,I_\varepsilon(0))+\kappa]\tau} \\
&\quad + C_6 \varepsilon^{-\beta}) \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa]\tau} d\tau \\
&\leqq M_2 \varepsilon^\alpha e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t} \\
&\quad + \delta C_4 \varepsilon^{\alpha-3\beta} e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t} \frac{e^{[\lambda(0,I_\varepsilon(0))+\kappa]t}}{\lambda(0,I_\varepsilon(0))+\kappa} + \delta C_6 \varepsilon^{-\beta} t e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t} \\
&\leqq M_2 \varepsilon^\alpha e^{-[\lambda(0,I_\varepsilon(0))-\kappa]t} \\
&\quad + \delta \left( C_4 \varepsilon^{\alpha-3\beta} \frac{e^{2\kappa T}}{\lambda(0,I_\varepsilon(0))+\kappa} + C_6 \varepsilon^{-\beta} T e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T} \right).
\end{aligned} \tag{5.10}
$$

Using the estimate (5.6) and $2\kappa < \beta\lambda(0, I_\varepsilon(0))/\alpha$, we have inequalities

$$
e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T} \leqq \left(\frac{c}{\delta\sqrt{e}}\right)^{\frac{\lambda(0,I_\varepsilon(0))-\kappa}{\lambda(0,I_\varepsilon(0))}} \varepsilon^{\alpha-\beta}, \quad e^{2\kappa T} \leqq \left(\frac{\delta e\sqrt{e}}{c}\right)^{\frac{\beta}{\alpha}} \varepsilon^{-\beta}, \quad T \leqq \varepsilon^{-\beta}
$$

if $\varepsilon$ is sufficiently small. It follows that

$$
\begin{aligned}
|V(t)| \leqq \Bigg( & M_2 \varepsilon^{5\beta} + \delta \tfrac{C_4}{\lambda(0,I_\varepsilon(0))+\kappa} \left(\tfrac{\delta e\sqrt{e}}{c}\right)^{\frac{\beta}{\alpha}} \varepsilon^\beta \\
& + \delta C_6 \left(\tfrac{c}{\delta\sqrt{e}}\right)^{\frac{\lambda(0,I_\varepsilon(0))-\kappa}{\lambda(0,I_\varepsilon(0))}} \varepsilon^{2\beta} \Bigg) \varepsilon^{\alpha-5\beta} < \varepsilon^{\alpha-5\beta},
\end{aligned} \tag{5.11}
$$

if $\varepsilon$ is sufficiently small.

Finally, we derive the estimate for the component $\|\boldsymbol{W}(t)\|$. First, we define the new variable $\boldsymbol{Z} = e^{-t\Theta_0}\boldsymbol{W}$. Then equation (5.2c) takes the form

$$
\dot{\boldsymbol{Z}} = e^{-t\Theta_0} \Theta_1 e^{t\Theta_0} \boldsymbol{Z} + U e^{-t\Theta_0} \boldsymbol{H}_1 + V e^{-t\Theta_0} \boldsymbol{H}_2. \tag{5.12}
$$

It is easy to verify that

$$
e^{t\Theta_0} = \begin{pmatrix} 1 & 0 \\ t D_{I_\varepsilon} \omega(I_\varepsilon(0)) & 1 \end{pmatrix},
$$

and so $\|e^{\pm t\Theta_0}\| \leqq 1 + C_{10} t$, where $t \geqq 0$ and $C_{10}$ is some constant independent of $\varepsilon$. By the estimates (5.7), we see that $|a_\varepsilon(t)||b_\varepsilon(t)| \leqq \delta^2 e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T}$. Now fix $\eta > 0$ so that

$$
\eta < \min\left( \frac{\beta}{3\alpha\lambda(0, I_\varepsilon(0))}, 2\kappa \right).
$$

Then

$$
\|e^{-t\Theta_0} \Theta_1 e^{t\Theta_0}\| \leqq (1 + C_{10} T)^2 C_9 \left( \varepsilon + \varepsilon \log \frac{1}{\varepsilon} + \delta^2 e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T} \right) \leqq \eta
$$

if $\varepsilon$ is sufficiently small. We apply the Gronwall inequality and estimates (5.7) to equation (5.12), and obtain

$$\begin{aligned}\|\boldsymbol{Z}(t)\| &\leqq \|\boldsymbol{W}_0\|e^{\eta t} + \int_0^t e^{\eta(t-\tau)}(1+C_{10}T)\\ &\qquad \Big(C_7\delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa]\tau}\varepsilon^{\alpha-3\beta}e^{[\lambda(0,I_\varepsilon(0))+\kappa]\tau}\\ &\qquad + C_8\delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa](T-\tau)}\varepsilon^{\alpha-5\beta}\Big)d\tau \qquad (5.13)\\ &\leqq M_3e^{\eta t} + \delta(1+C_{10}T)e^{\eta t}(C_7\varepsilon^{\alpha-3\beta}K_1\\ &\qquad + C_8\varepsilon^{\alpha-5\beta}e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T}K_2),\end{aligned}$$

where $K_1 = \int_0^t e^{(2\kappa-\eta)\tau}d\tau$ and $K_2 = \int_0^t e^{(\lambda(0,I_\varepsilon(0))-\kappa-\eta)\tau}d\tau$. It is easy to check that

$$K_1 \leqq \frac{e^{(2\kappa-\eta)T}}{2\kappa-\eta}, \quad K_2 \leqq \frac{e^{(\lambda(0,I_\varepsilon(0))-\kappa-\eta)T}}{\lambda(0,I_\varepsilon(0))-\kappa-\eta}.$$

It then follows that

$$\begin{aligned}\|\boldsymbol{Z}(t)\| &\leqq M_3e^{\eta t} + \delta(1+C_{10}T)e^{\eta t}\\ &\quad\times\left(C_7\varepsilon^{\alpha-3\beta}\frac{e^{(2\kappa-\eta)T}}{2\kappa-\eta} + C_8\varepsilon^{\alpha-5\beta}\frac{e^{-\eta T}}{\lambda(0,I_\varepsilon(0))-\kappa-\eta}\right) \qquad (5.14)\\ &\leqq (M_3+1)e^{\eta t},\end{aligned}$$

provided $\varepsilon$ is sufficiently small. Finally,

$$\|\boldsymbol{W}(t)\| \leqq (1+C_{10}t)\|\boldsymbol{Z}(t)\| \leqq (M_3+1)(1+C_{10}T)e^{\eta T} \leqq (M_3+1)\varepsilon^{-\frac{2}{3}\beta} < \varepsilon^{-\beta},$$

if $\varepsilon$ is sufficiently small.

We now complete the argument: Let $\mathscr{S}$ be the set of all times $T_0$ satisfying $0 \leqq T_0 \leqq T$ such that the inequalities (5.5) hold for $0 \leqq t \leqq T_0$. Since the inequalities (5.5) are not strict, the set $\mathscr{S}$ is closed in the interval $0 \leqq t \leqq T$. The above calculation shows that $\mathscr{S}$ is non-empty and open in the interval $0 \leqq t \leqq T$. Since the interval $0 \leqq t \leqq T$ is connected, the set $\mathscr{S}$ must be this whole interval. □

We now improve our knowledge of the components $dI_\varepsilon(t)$ and $d\psi_\varepsilon(t)$.

**Proposition 5.3.** *For $\beta > 0$ sufficiently small,*

$$\begin{pmatrix} dI_\varepsilon(t)\\ d\psi_\varepsilon(t)\end{pmatrix} = \begin{pmatrix} 1 & 0\\ tD_{I_\varepsilon}\omega(I_\varepsilon(0)) & 1\end{pmatrix}\begin{pmatrix} dI_\varepsilon(0)\\ d\psi_\varepsilon(0)\end{pmatrix} + \mathscr{O}(\varepsilon^{\alpha-6\beta}), \qquad (5.15)$$

*where $0 \leqq t \leqq T$ and $\varepsilon$ is sufficiently small.*

**Proof.** Let $\boldsymbol{W}_0 = (dI_\varepsilon(0), d\psi_\varepsilon(0))^T$, and define the function $\boldsymbol{P}(t)$ in $\mathbb{R}^2$ by the equations

$$\dot{\boldsymbol{P}} = \Theta_0\boldsymbol{P}, \qquad (5.16a)$$

$$\boldsymbol{P}(0) = \boldsymbol{W}_0. \qquad (5.16b)$$

Subtracting equation (5.16a) from equation (5.2c), we obtain the equation

$$\dot{\boldsymbol{W}} - \dot{\boldsymbol{P}} = \Theta_0(\boldsymbol{W} - \boldsymbol{P}) + \Theta_1 \boldsymbol{W} + U\boldsymbol{H}_1 + V\boldsymbol{H}_2. \tag{5.17}$$

Applying the Gronwall inequality to equation (5.17) and using Proposition 5.2 and estimates in Proposition 5.1, we have

$$\begin{aligned}
&\|\boldsymbol{W}(t) - \boldsymbol{P}(t)\| \\
&\quad = \left\| \int_0^t e^{(t-\tau)\Theta_0} \left(\Theta_1(\tau)\boldsymbol{W}(\tau) + U(\tau)\boldsymbol{H}_1(\tau) + V(\tau)\boldsymbol{H}_2(\tau)\right) d\tau \right\| \\
&\quad \leqq \int_0^t [1 + C_{10}(t-\tau)] \Bigg[ C_9 \left(\varepsilon + \varepsilon \log\frac{1}{\varepsilon} + \delta^2 e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T}\right)\varepsilon^{-\beta} \\
&\qquad + \varepsilon^{\alpha-3\beta} e^{[\lambda(0,I_\varepsilon(0))+\kappa]\tau} C_7 \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa]\tau} \qquad\qquad (5.18)\\
&\qquad + \varepsilon^{\alpha-5\beta} C_8 \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa](T-\tau)} \Bigg] d\tau \\
&\quad \leqq (1 + C_{10}T) \left[ C_9 \varepsilon^{\alpha-2\beta} T + \frac{\delta C_7}{2\kappa}\varepsilon^{\alpha-4\beta} + \frac{\delta C_8}{\lambda(0, I_\varepsilon(0)) - \kappa}\varepsilon^{\alpha-5\beta} \right] \\
&\quad \leqq \varepsilon^{\alpha-6\beta}
\end{aligned}$$

if $\varepsilon$ is sufficiently small. The proof is now completed by noting that $\boldsymbol{P}(t) = e^{t\Theta_0}\boldsymbol{W}_0$ and

$$e^{t\Theta_0} = \begin{pmatrix} 1 & 0 \\ tD_{I_\varepsilon}\omega(I_\varepsilon(0)) & 1 \end{pmatrix}. \quad \square$$

Recall that we are interested in the evolution of the tangent space of the manifold $\mathscr{L}$ along the trajectory $\boldsymbol{q}_\varepsilon(t)$ as this trajectory passes through the small neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the hyperbolic annulus $\mathscr{M}_\varepsilon$. The tangent space at the point $\boldsymbol{q}_\varepsilon(0)$, where $\boldsymbol{q}_\varepsilon(t)$ enters $U_\delta(\mathscr{M}_\varepsilon)$, is spanned by three tangent vectors, one of which can always be chosen to be the vector field itself. For the other two vectors, we can make use of the fact that the tangent space of the manifold $\mathscr{L}$ is $\mathscr{O}(\varepsilon^\alpha)$-close to the local stable manifold near the point $\boldsymbol{q}_\varepsilon(0)$. This gives two other tangent vectors at $\boldsymbol{q}_\varepsilon(0)$, which are of the form

$$\boldsymbol{v}_I(0) = \begin{pmatrix} \mathscr{O}(\varepsilon^\alpha) \\ \mathscr{O}(\varepsilon^\alpha) \\ 1 + \mathscr{O}(\varepsilon^\alpha) \\ \mathscr{O}(\varepsilon^\alpha) \end{pmatrix}, \quad \boldsymbol{v}_\psi(0) = \begin{pmatrix} \mathscr{O}(\varepsilon^\alpha) \\ \mathscr{O}(\varepsilon^\alpha) \\ \mathscr{O}(\varepsilon^\alpha) \\ 1 + \mathscr{O}(\varepsilon^\alpha) \end{pmatrix}. \tag{5.19a,b}$$

We also write $\boldsymbol{v}_t(0)$ for the vector field at $\boldsymbol{q}_\varepsilon(0)$, and write $\boldsymbol{v}_t(t)$, $\boldsymbol{v}_I(t)$, and $\boldsymbol{v}_\psi(t)$ for the solutions of the variational equations (5.1) with initial conditions $\boldsymbol{v}_t(0)$, $\boldsymbol{v}_I(0)$, and $\boldsymbol{v}_\psi(0)$ respectively.

At the point $\boldsymbol{q}_\varepsilon(T)$, where the trajectory $\boldsymbol{q}_\varepsilon(t)$ exits the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$, the vector $\boldsymbol{v}_t(T)$ is simply the vector field at $\boldsymbol{q}_\varepsilon(T)$. Using equation $a_\varepsilon(T) = (\operatorname{sign} a_\varepsilon(T))\,\delta$ and the estimate $|b_\varepsilon(T)| \leqq \delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa]T}$, we obtain

$$\boldsymbol{v}_t(T) = \begin{pmatrix} (\operatorname{sign} a_\varepsilon(T))\, \lambda(0, I_\varepsilon(0))\delta + \mathscr{O}(\varepsilon^{\alpha-\beta}) \\ \mathscr{O}(\varepsilon^{\alpha-\beta}) \\ \mathscr{O}(\varepsilon) \\ \omega(I_\varepsilon(0)) + \mathscr{O}(\varepsilon^{\alpha-\beta}) \end{pmatrix}. \tag{5.20}$$

Using Propositions 5.2 and 5.3, we also obtain

$$\boldsymbol{v}_I(T) = \begin{pmatrix} \mathscr{O}(\varepsilon^{-4\beta}) \\ \mathscr{O}(\varepsilon^{\alpha-5\beta}) \\ 1 + \mathscr{O}(\varepsilon^{\alpha-6\beta}) \\ D_{I_\varepsilon}\omega(I_\varepsilon(0)) \cdot T + \mathscr{O}(\varepsilon^{\alpha-6\beta}) \end{pmatrix}, \tag{5.21a}$$

$$\boldsymbol{v}_\psi(T) = \begin{pmatrix} \mathscr{O}(\varepsilon^{-4\beta}) \\ \mathscr{O}(\varepsilon^{\alpha-5\beta}) \\ \mathscr{O}(\varepsilon^{\alpha-6\beta}) \\ 1 + \mathscr{O}(\varepsilon^{\alpha-6\beta}) \end{pmatrix}. \tag{5.21b}$$

In order to conclude that the tangent space of the manifold $\mathscr{L}$ at the point $\boldsymbol{q}_\varepsilon(T)$ is close to that of the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$, we need to know the leading term of the $da_\varepsilon$-component of the vector $\boldsymbol{v}_\psi(T)$. Write

$$a_\varepsilon(0) = A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon). \tag{5.22}$$

The Fenichel coordinate $a_\varepsilon(0)$ of the initial point $\boldsymbol{q}_\varepsilon(0)$ depends on the variables $I_\varepsilon$ and $\psi_\varepsilon$ through some function $A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)$ for the following reason. Near the point $\boldsymbol{q}_\varepsilon(0)$, we assume that the manifold $\mathscr{L}$ is $\mathscr{O}(\varepsilon^\alpha)$-close to the manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$, and that the corresponding tangent spaces are also close. In the Fenichel coordinates, the manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ is given by the equation $a_\varepsilon = 0$ and parametrized by the coordinates $b_\varepsilon$, $I_\varepsilon$, and $\psi_\varepsilon$. Since the point $\boldsymbol{q}_\varepsilon(0)$ is chosen among points with $|b_\varepsilon| = \delta$, the coordinate $a_\varepsilon(0)$ depends on the remaining variables $I_\varepsilon$ and $\psi_\varepsilon$. From equations (4.5) and (5.22), we have the following leading term expansion for $e^{\lambda(0,I_\varepsilon(0))T}$,

$$e^{\lambda(0,I_\varepsilon(0))T} = \frac{\delta}{|a_\varepsilon(0)|}\left(1 + \mathscr{O}\left(\log\frac{\delta}{|a_\varepsilon(0)|}\left(\varepsilon^{\alpha-D\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}\right)\right)\right). \tag{5.23}$$

Now for the vector $\boldsymbol{v}_\psi(0)$, we have $dI_\varepsilon(0) = \mathscr{O}(\varepsilon^\alpha)$. Also, from equation (5.22), we obtain the equation

$$da_\varepsilon(0) = D_{\psi_\varepsilon} a_\varepsilon(0) = D_{\psi_\varepsilon} A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon). \tag{5.24}$$

We have to obtain the leading term of the differential $da_\varepsilon(T)$ for the vector $\boldsymbol{v}_\psi(T)$. To do so, we go back to equation (5.1a). For the present purpose, write this equation as

$$\begin{aligned} d\dot{a}_\varepsilon &= \lambda da_\varepsilon + \left(\varepsilon f_a + a_\varepsilon b_\varepsilon D_{\zeta_\varepsilon}\lambda\right) da_\varepsilon + a_\varepsilon^2 D_{\zeta_\varepsilon}\lambda db_\varepsilon \\ &\quad + a_\varepsilon D_{I_\varepsilon}\lambda dI_\varepsilon + \varepsilon a_\varepsilon \langle \nabla f_a, d\boldsymbol{X}\rangle. \end{aligned} \tag{5.25}$$

Using Proposition 5.2 and equation (5.1c), we easily show that $|dI_\varepsilon(t)| = \mathscr{O}(\varepsilon^\alpha) + \mathscr{O}(\varepsilon^{1-6\beta}) = \mathscr{O}(\varepsilon^{\alpha-6\beta})$. We apply this fact and Proposition 5.2 to equation (5.25) to see that equation (5.25) takes the form

$$d\dot{a}_\varepsilon = [\lambda(0, I_\varepsilon(0)) + \chi_1(t)]da_\varepsilon + \chi_2(t), \tag{5.26}$$

where $|\chi_1(t)| = \mathscr{O}(\varepsilon^{\alpha-\beta})$ and $|\chi_2(t)| = \mathscr{O}(\varepsilon^{\alpha-6\beta}|a_\varepsilon(t)|)$. Direct integration gives

$$\begin{aligned} da_\varepsilon(T) = da_\varepsilon(0)e^{\int_0^T \lambda(0,I_\varepsilon(0))+\chi_1(\tau)d\tau} \\ + \int_0^T e^{\int_\tau^T \lambda(0,I_\varepsilon(0))+\chi_1(s)ds}\chi_2(\tau)d\tau. \end{aligned} \tag{5.27}$$

The second term above can be estimated in the usual way. Choose $\varepsilon$ sufficiently small so that $\lambda(0, I_\varepsilon(0)) + \chi_1(t) \leqq \lambda(0, I_\varepsilon(0)) + \kappa$. Then the second term is bounded above by the expression

$$\begin{aligned} \mathscr{O}\left(\varepsilon^{\alpha-6\beta}\int_0^T e^{[\lambda(0,I_\varepsilon(0))+\kappa](T-\tau)}\delta e^{-[\lambda(0,I_\varepsilon(0))-\kappa](T-\tau)}d\tau\right) &= \mathscr{O}\left(\varepsilon^{\alpha-6\beta}\frac{e^{2\kappa T}}{2\kappa}\right) \\ &= \mathscr{O}(\varepsilon^{\alpha-7\beta}). \end{aligned}$$

The first term in equation (5.27) is $da_\varepsilon(0)e^{\lambda(0,I_\varepsilon(0))T}\left(1 + \mathscr{O}(\varepsilon^{\alpha-2\beta})\right)$. Using equations (5.23) and (5.24), we finally obtain

$$da_\varepsilon(T) = \delta\,\frac{D_{\psi_\varepsilon}A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon)}{|A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon)|}\left(1 + \mathscr{O}(\varepsilon^{\alpha-2\beta})\right) + \mathscr{O}(\varepsilon^{\alpha-7\beta}). \tag{5.28}$$

We can now show our closeness result in

**Lemma 1.** *Let the manifold $\mathscr{L}$ be $\mathscr{O}(\varepsilon^\alpha)$-close to the local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ at the point $\boldsymbol{q}_\varepsilon(0) = (a_\varepsilon(0), \delta, I_\varepsilon(0), \psi_\varepsilon(0))$, where $a_\varepsilon(0) = A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon) = \mathscr{O}(\varepsilon^\alpha)$, and let the corresponding tangent spaces also be $\mathscr{O}(\varepsilon^\alpha)$-close. Let $\boldsymbol{q}_\varepsilon(t) = (a_\varepsilon(t), b_\varepsilon(t), I_\varepsilon(t), \psi_\varepsilon(t))$ be the trajectory passing through the point $\boldsymbol{q}_\varepsilon(0)$, and let $T$ be the time when this trajectory exits the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$. Let there exist a constant $B > 0$, independent of $\varepsilon$, such that the inequality*

$$\left|\lambda(0, I_\varepsilon(0)) - \omega(I_\varepsilon(0))\frac{D_{\psi_\varepsilon}A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon)}{A_\varepsilon(I_\varepsilon(0), \psi_\varepsilon(0), \varepsilon)}\right| > B \tag{5.29}$$

*holds. Then the tangent space of the manifold $\mathscr{L}$ at the point $\boldsymbol{q}_\varepsilon(T)$ is $\mathscr{O}(\varepsilon^{\alpha-\beta})$-close to the tangent space of the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$, where the number $\beta$ can be made arbitrarily small.*

**Proof.** We consider the linear subspace in $\mathbb{R}^4$ formed by the vectors $\boldsymbol{v}_t(T)$, $\boldsymbol{v}_I(T)$, $\boldsymbol{v}_\psi(T)$. By equations (5.20), (5.21) and (5.28), we see that the matrix formed by these vectors has the form

$$\begin{pmatrix} (\operatorname{sign} a_\varepsilon(T))\lambda(0, I_\varepsilon(0))\delta + \mathscr{O}(\varepsilon^{\alpha-\beta}) & \mathscr{O}(\varepsilon^{-4\beta}) & \frac{\delta}{|a_\varepsilon(0)|} D_{\psi_\varepsilon} a_\varepsilon(0)\left(1+\mathscr{O}(\varepsilon^{\alpha-2\beta})\right) + \mathscr{O}(\varepsilon^{\alpha-7\beta}) \\ \mathscr{O}(\varepsilon^{\alpha-\beta}) & \mathscr{O}(\varepsilon^{\alpha-5\beta}) & \mathscr{O}(\varepsilon^{\alpha-5\beta}) \\ \mathscr{O}(\varepsilon) & 1+\mathscr{O}(\varepsilon^{\alpha-6\beta}) & \mathscr{O}(\varepsilon^{\alpha-6\beta}) \\ \omega(I_\varepsilon(0)) + \mathscr{O}(\varepsilon^{\alpha-\beta}) & D_{I_\varepsilon}\omega(I_\varepsilon(0))\cdot T + \mathscr{O}(\varepsilon^{\alpha-6\beta}) & 1+\mathscr{O}(\varepsilon^{\alpha-6\beta}) \end{pmatrix}. \tag{5.30}$$

The quantity $db_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon$ is defined as the subdeterminant obtained by removing the first row. Geometrically, $db_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon$ is the (signed) volume of the projection of the parallelepiped formed by $\boldsymbol{v}_t(T)$, $\boldsymbol{v}_I(T)$ and $\boldsymbol{v}_\psi(T)$ onto the $(b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$-plane. The other projections $da_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon$, $da_\varepsilon \wedge db_\varepsilon \wedge dI_\varepsilon$, and $da_\varepsilon \wedge db_\varepsilon \wedge d\psi_\varepsilon$ are defined similarly. (These projections are known to differential geometers as Plücker coordinates on the Grassmannian; see [31].) To show that the linear subspace spanned by the vectors $\boldsymbol{v}_t(T)$, $\boldsymbol{v}_I(T)$ and $\boldsymbol{v}_\psi(T)$ is close to the subspace $b_\varepsilon = 0$, it suffices to prove that $|da_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon|$ dominates

$$\left\{|db_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon|^2 + |da_\varepsilon \wedge db_\varepsilon \wedge dI_\varepsilon|^2 + |da_\varepsilon \wedge db_\varepsilon \wedge d\psi_\varepsilon|^2\right\}^{1/2} \tag{5.31}$$

for $\varepsilon$ sufficiently small. It is clear from the matrix (5.30) that the expression (5.31) is of order $\mathscr{O}(\varepsilon^{\alpha-9\beta})$. On the other hand, the projection $da_\varepsilon \wedge dI_\varepsilon \wedge d\psi_\varepsilon$ is the determinant

$$\begin{vmatrix} (\operatorname{sign} a_\varepsilon(T))\lambda(0, I_\varepsilon(0))\delta + \mathscr{O}(\varepsilon^{\alpha-\beta}) & \mathscr{O}(\varepsilon^{-4\beta}) & \frac{\delta}{|a_\varepsilon(0)|} D_{\psi_\varepsilon} a_\varepsilon(0)\left(1+\mathscr{O}(\varepsilon^{\alpha-2\beta})\right) + \mathscr{O}(\varepsilon^{\alpha-7\beta}) \\ \mathscr{O}(\varepsilon) & 1+\mathscr{O}(\varepsilon^{\alpha-6\beta}) & \mathscr{O}(\varepsilon^{\alpha-6\beta}) \\ \omega(I_\varepsilon(0)) + \mathscr{O}(\varepsilon^{\alpha-\beta}) & D_{I_\varepsilon}\omega(I_\varepsilon(0))\cdot T + \mathscr{O}(\varepsilon^{\alpha-6\beta}) & 1+\mathscr{O}(\varepsilon^{\alpha-6\beta}) \end{vmatrix},$$

which, upon expansion, is

$$\delta\left((\operatorname{sign} a_\varepsilon(T))\lambda(0, I_\varepsilon(0)) - \omega(I_\varepsilon(0))\frac{D_{\psi_\varepsilon} a_\varepsilon(0)}{|a_\varepsilon(0)|}\left(1+\mathscr{O}(\varepsilon^{\alpha-2\beta})\right)\right) + \mathscr{O}(\varepsilon^{\alpha-10\beta}). \tag{5.32}$$

Since $a_\varepsilon(t)$ does not change sign during the passage, we must have

$$\operatorname{sign} a_\varepsilon(T) = \operatorname{sign} a_\varepsilon(0) = \frac{a_\varepsilon(0)}{|a_\varepsilon(0)|}. \tag{5.33}$$

Expression (5.32) now becomes

$$\delta\frac{a_\varepsilon(0)}{|a_\varepsilon(0)|}\left(\lambda(0, I_\varepsilon(0)) - \omega(I_\varepsilon(0))\frac{D_{\psi_\varepsilon} a_\varepsilon(0)}{|a_\varepsilon(0)|}\left(1+\mathscr{O}(\varepsilon^{\alpha-2\beta})\right)\right) + \mathscr{O}(\varepsilon^{\alpha-10\beta}).$$

Condition (5.29) guarantees that this coordinate is of order $\mathscr{O}(1)$, and the desired conclusion on closeness follows. □

Condition (5.29) reflects the geometric fact that the manifold $\mathscr{L}$ may emerge from the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ having folds due to the nearly linear motion of the angle $\psi_\varepsilon$, which can be immediately verified for simple linear examples (see Figure 5.2). If $\omega(I_\varepsilon(0)) = 0$, condition (5.29) is automatically satisfied, that is, no folds can develop in the case of slow dynamics on the normally hyperbolic manifold $\mathscr{M}_\varepsilon$.

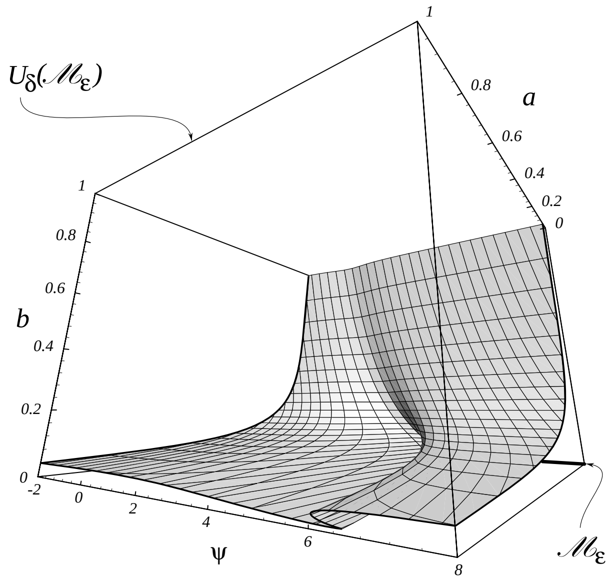


Fig. 5.2. An example of the folds that the manifold $\mathscr{L}$ can develop as it flies by $\mathscr{M}_\varepsilon$ in $a-\psi-b$ coordinates. The surface is actually generated by integration of a linear equation, given by the vector field $\dot{a}=\lambda a$, $\dot{b}=-\lambda b$, $\dot{\psi}=\omega+\psi_0$, with $\lambda=1$, $\omega=\pi/3$, $b(0)=1$, $a(0)=\epsilon(1-\frac{4}{5}\operatorname{sech}\psi_0)$, and $\epsilon=0.2$. This demonstrates how the creation of sharp folds in $\mathscr{L}$ is governed by the linearized dynamics in the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$.

The method of proof used in Lemma 1 can be adapted easily to the case of a multi-dimensional action and angle variable. In general, if $I\in\mathbb{R}^m$, $\theta\in\mathbb{R}^n$, the manifold $\mathscr{L}$ we are tracking is of dimension $m+n+1$. In exactly the same manner as above, we can follow tangent vectors complementary to the vector field, of which $m$ vectors are along the $I$ directions, and $n$ are along the $\theta$ directions. All the estimates go through verbatim to give the following higher-dimensional generalization of condition (5.29):

$$\left|\lambda(0,I_\varepsilon(0))-\left\langle\omega(I_\varepsilon(0)),\frac{D_{\psi_\varepsilon}A_\varepsilon(I_\varepsilon(0),\psi_\varepsilon(0),\varepsilon)}{|A_\varepsilon(I_\varepsilon(0),\psi_\varepsilon(0),\varepsilon)|}\right\rangle\right|>B. \tag{5.34}$$

## 6. Distance Estimates Along the Pulses

Recall the manifold $\mathscr{L}$ defined at the beginning of the previous section as the winding piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$. As stated there, the manifold $\mathscr{L}$ returns several times to the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ and then leaves it again. Upon leaving $U_\delta(\mathscr{M}_\varepsilon)$, the manifold $\mathscr{L}$ follows the local piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$. In this section, we compute the distance between these two manifolds in terms of their distance upon exiting the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$. For simplicity, we carry out the calculation only for the case when system (2.4) is Hamiltonian. The ideas involved in the general case are the same.

We introduce streamlined notation $\boldsymbol{q}(t) = (x(t), I(t), \theta(t))$, and rewrite equation (2.1) as

$$\dot{\boldsymbol{q}} = J\nabla H(\boldsymbol{q}) + \varepsilon \boldsymbol{g}(\boldsymbol{q}) + \mathcal{O}(\varepsilon^2), \tag{6.1}$$

where $\nabla = (D_x, D_I, D_\theta)$, $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{q}, 0)$ and $J$ now stands for the $4 \times 4$ matrix

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Consider some point $\boldsymbol{p}^h(0)$ on the unperturbed homoclinic manifold $W(\mathcal{M})$, where this manifold leaves the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$. Attach the normal $\boldsymbol{n}(\boldsymbol{p}^h(0))$, given by formula (3.3), at this point. This normal intersects the manifold $\mathcal{L}$ in at least one point and the local unstable manifold $W^u_{\text{loc}}(\mathcal{M}_\varepsilon)$ in precisely one point, say $\boldsymbol{q}^u(0)$. From among the intersection points of the normal $\boldsymbol{n}(\boldsymbol{p}^h(0))$ with the manifold $\mathcal{L}$ we choose the point $\boldsymbol{q}^l(0)$ to be the point whose backwards-time trajectory takes the least amount of time flying along $\mathcal{L}$ to reach some portion of the local unstable manifold $W^u_{\text{loc}}(\mathcal{M}_\varepsilon)$. We assume that the points $\boldsymbol{q}^l(0)$ and $\boldsymbol{q}^u(0)$ are at a distance $c\varepsilon^\alpha$ away from each other for some positive $c$ and $\alpha > \frac{1}{2}$. We consider now the trajectories $\boldsymbol{p}^h(t)$, $\boldsymbol{q}^l(t)$ and $\boldsymbol{q}^u(t)$ passing through the points $\boldsymbol{p}^h(0)$, $\boldsymbol{q}^l(0)$, and $\boldsymbol{q}^u(0)$. We require that the point $\boldsymbol{p}^h(0)$ be chosen in such a way that the trajectory $\boldsymbol{q}^l(t)$ does not lie near a fold of the manifold $\mathcal{L}$. This can be verified a posteriori by requiring that the nonfolding condition (3.9) is satisfied for the unperturbed trajectories approximating $\boldsymbol{q}^l(t)$. Note that the trajectory $\boldsymbol{p}^h(t)$ evolves under the unperturbed dynamics, and is therefore of the form (2.6), i.e., $\boldsymbol{p}^h(t) = (x^h(t - t_0, I), I, \theta^h(t - t_0, I) + \theta_0)$, while the trajectories $\boldsymbol{q}^l(t)$ and $\boldsymbol{q}^u(t)$ evolve under the perturbed dynamics. By Gronwall-type estimates, the trajectories $\boldsymbol{q}^l(t)$ and $\boldsymbol{q}^u(t)$ are a distance $\mathcal{O}(\varepsilon^\alpha)$ apart for all finite times $t$.

Any perturbed trajectory $\boldsymbol{q}(t)$ can be Taylor expanded as

$$\boldsymbol{q}(t) = \boldsymbol{p}(t) + \varepsilon \boldsymbol{r}(t) + \mathcal{O}(\varepsilon^2), \tag{6.2}$$

where $\boldsymbol{p}(t)$ is some unperturbed trajectory. This expansion is valid for finite values of time $t$. The first correction $\boldsymbol{r}(t)$ satisfies the first variational equation

$$\dot{\boldsymbol{r}} = J\nabla^2 H(\boldsymbol{p})\boldsymbol{r} + \boldsymbol{g}(\boldsymbol{p}). \tag{6.3}$$

In particular, for the trajectories $\boldsymbol{q}^l(t)$ and $\boldsymbol{q}^u(t)$, expansion (6.2) yields

$$\boldsymbol{q}^l(t) = \boldsymbol{p}^l(t) + \varepsilon \boldsymbol{r}^l(t) + \mathcal{O}(\varepsilon^2), \tag{6.4a}$$

$$\boldsymbol{q}^u(t) = \boldsymbol{p}^u(t) + \varepsilon \boldsymbol{r}^u(t) + \mathcal{O}(\varepsilon^2) = \boldsymbol{p}^h(t) + \varepsilon \boldsymbol{r}^h(t) + \mathcal{O}(\varepsilon^2), \tag{6.4b}$$

since we can set $\boldsymbol{p}^u(t) \equiv \boldsymbol{p}^h(t)$, and therefore rename $\boldsymbol{r}^u(t) \equiv \boldsymbol{r}^h(t)$. The distance between the unperturbed trajectories $\boldsymbol{p}^l(t)$ and $\boldsymbol{p}^h(t)$ is of order $\mathcal{O}(\varepsilon + \varepsilon^\alpha)$.

The validity of the Taylor expansions (6.4a) and (6.4b), and Lemma 1 show that the signed distance between the points at which the normal $\boldsymbol{n}(\boldsymbol{p}^h(t))$ pierces the manifolds $\mathcal{L}$ and $W^u(\mathcal{M}_\varepsilon)$ is equal to

$$d^{l,u}(\boldsymbol{p}^h(t)) = \frac{\langle \boldsymbol{q}^l(t) - \boldsymbol{q}^u(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^h(t))\|} + \mathscr{O}((\varepsilon^\alpha + \varepsilon)^{2-\beta}),$$

where $\beta$ can be taken as small as we please, provided that $\boldsymbol{q}^l(t)$ stays away from any folds of the manifold $\mathscr{L}$ (see Figure 6.1). Using expansions (6.4a) and (6.4b), we obtain the formula

$$d^{l,u}(\boldsymbol{p}^h(t)) = \frac{\langle \boldsymbol{p}^l(t) - \boldsymbol{p}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^h(t))\|} + \varepsilon \frac{\langle \boldsymbol{r}^l(t) - \boldsymbol{r}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^h(t))\|} + \mathscr{O}((\varepsilon^\alpha + \varepsilon)^{2-\beta}). \tag{6.5}$$

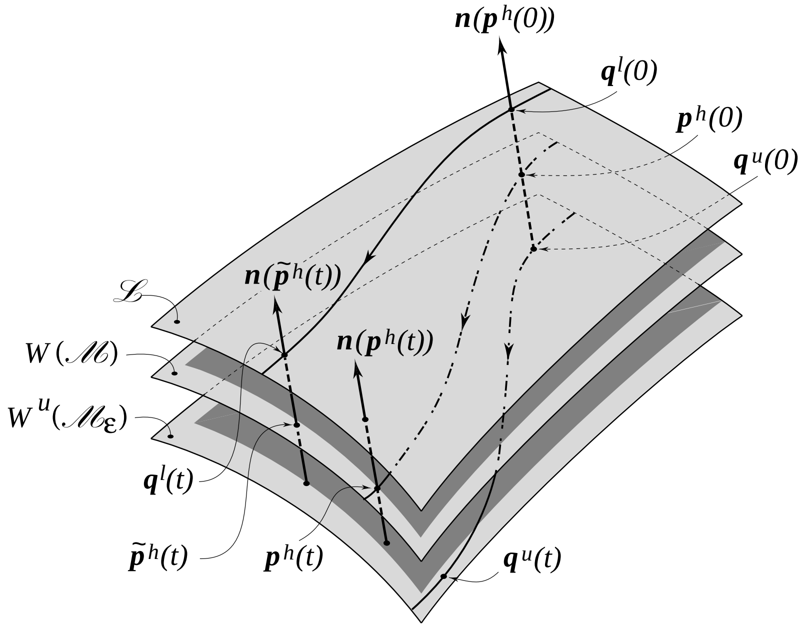


Fig. 6.1. The signed distance between the points at which the normal $\boldsymbol{n}(\boldsymbol{p}^h(t))$ pierces the manifold $\mathscr{L}$ and $W^u(\mathscr{M})_\varepsilon$ is equal to the component of the distance between the points $\boldsymbol{q}^l(t)$ and $\boldsymbol{q}^u(t)$ along the normal $\boldsymbol{n}(\boldsymbol{p}^h(t))$, plus higher order terms, away from folds of the manifold $\mathscr{L}$.

We now show

**Proposition 6.1.** *The signed distance $d^{l,u}(\boldsymbol{p}^h(t))$ between the points at which the normal $\boldsymbol{n}(\boldsymbol{p}^h(t))$ pierces the manifolds $\mathscr{L}$ and $W^u(\mathscr{M}_\varepsilon)$ is equal to*

$$d^{l,u}(\boldsymbol{p}^h(t)) = \frac{\langle \boldsymbol{q}^l(0) - \boldsymbol{q}^u(0), \boldsymbol{n}(\boldsymbol{p}^h(0))\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^h(t))\|} + \mathscr{O}((\varepsilon^\alpha + \varepsilon)^{2-\beta}), \tag{6.6}$$

*where $\beta$ can be taken as small as desired.*

**Proof.** The first term in formula (6.5) can be evaluated as follows. Set $H(X(I), I) = F(I)$; then

$$\boldsymbol{n}(\boldsymbol{p}^h(t)) = \nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]. \tag{6.7}$$

Then

$$\begin{aligned}
&\langle \boldsymbol{p}^l(t) - \boldsymbol{p}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\
&\quad = \langle \boldsymbol{p}^l(t) - \boldsymbol{p}^h(t), \nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\
&\quad = [H(\boldsymbol{p}^l(t)) - F(I(\boldsymbol{p}^l(t)))] - [H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))] \\
&\qquad + \mathcal{O}(\|\boldsymbol{p}^l(t) - \boldsymbol{p}^h(t)\|^2) \\
&\quad = [H(\boldsymbol{p}^l(t)) - F(I(\boldsymbol{p}^l(t)))] - [H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))] + \mathcal{O}((\varepsilon + \varepsilon^\alpha)^2).
\end{aligned}$$

Since $H(\boldsymbol{p}(t)) - F(I(\boldsymbol{p}(t)))$ is a conserved quantity for system (2.4), we must have

$$\begin{aligned}
&[H(\boldsymbol{p}^l(t)) - F(I(\boldsymbol{p}^l(t)))] - [H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))] \\
&\quad = [H(\boldsymbol{p}^l(0)) - F(I(\boldsymbol{p}^l(0)))] - [H(\boldsymbol{p}^h(0)) - F(I(\boldsymbol{p}^h(0)))].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle \boldsymbol{p}^l(t) - \boldsymbol{p}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\
&\quad = [H(\boldsymbol{p}^l(0)) - F(I(\boldsymbol{p}^l(0)))] - [H(\boldsymbol{p}^h(0)) - F(I(\boldsymbol{p}^h(0)))] + \mathcal{O}((\varepsilon + \varepsilon^\alpha)^2) \\
&\quad = \langle \boldsymbol{p}^l(0) - \boldsymbol{p}^h(0), \boldsymbol{n}(\boldsymbol{p}^h(0))\rangle + \mathcal{O}((\varepsilon + \varepsilon^\alpha)^2).
\end{aligned} \tag{6.8}$$

To estimate the second term in formula (6.5), we set up a differential equation for the expression

$$\Delta(t) = \Delta^l(t) - \Delta^h(t) = \langle \boldsymbol{r}^l(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle - \langle \boldsymbol{r}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle. \tag{6.9}$$

This differential equation consists of two parts:

$$\begin{aligned}
\dot{\Delta}^l(t) &= \langle \dot{\boldsymbol{r}}^l(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle + \langle \boldsymbol{r}^l(t), \dot{\boldsymbol{n}}(\boldsymbol{p}^h(t))\rangle, \\
\dot{\Delta}^h(t) &= \langle \dot{\boldsymbol{r}}^h(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle + \langle \boldsymbol{r}^h(t), \dot{\boldsymbol{n}}(\boldsymbol{p}^h(t))\rangle.
\end{aligned}$$

From equation (6.7) we obtain

$$\begin{aligned}
\dot{\boldsymbol{n}}(\boldsymbol{p}^h(t)) &= \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\dot{\boldsymbol{p}}^h(t) \\
&= \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t)).
\end{aligned}$$

Together with equation (6.3), this implies that

$$\begin{aligned}\dot{\Delta}^l(t) &= \langle J\nabla^2 H(\boldsymbol{p}^l(t))\boldsymbol{r}^l(t) + \boldsymbol{g}(\boldsymbol{p}^l(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\ &\quad + \langle \boldsymbol{r}^l(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle \\ &= \langle J\nabla^2 H(\boldsymbol{p}^l(t))\boldsymbol{r}^l(t) + \boldsymbol{g}(\boldsymbol{p}^l(t)), \nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\ &\quad + \langle \boldsymbol{r}^l(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle \\ &= \langle \boldsymbol{g}(\boldsymbol{p}^l(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle - \langle \boldsymbol{r}^l(t), \nabla^2 H(\boldsymbol{p}^l(t))J\nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\ &\quad + \langle \boldsymbol{r}^l(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle, \end{aligned} \tag{6.10a}$$

$$\begin{aligned}\dot{\Delta}^h(t) &= \langle J\nabla^2 H(\boldsymbol{p}^h(t))\boldsymbol{r}^h(t) + \boldsymbol{g}(\boldsymbol{p}^h(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\ &\quad + \langle \boldsymbol{r}^h(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle \\ &= \langle J\nabla^2 H(\boldsymbol{p}^h(t))\boldsymbol{r}^h(t) + \boldsymbol{g}(\boldsymbol{p}^h(t)), \nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\ &\quad + \langle \boldsymbol{r}^h(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle \\ &= \langle \boldsymbol{g}(\boldsymbol{p}^h(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle - \langle \boldsymbol{r}^h(t), \nabla^2 H(\boldsymbol{p}^h(t))J\nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\ &\quad + \langle \boldsymbol{r}^h(t), \nabla^2[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]J\nabla H(\boldsymbol{p}^h(t))\rangle \\ &= \langle \boldsymbol{g}(\boldsymbol{p}^h(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\ &\quad + \langle \boldsymbol{r}^h(t), \nabla^2 H(\boldsymbol{p}^h(t))J\nabla F(I(\boldsymbol{p}^h(t))) - \nabla^2 F(I(\boldsymbol{p}^h(t)))J\nabla H(\boldsymbol{p}^h(t))\rangle. \end{aligned} \tag{6.10b}$$

However, in the last term of equation (6.10b), we have

$$\nabla^2 H(\boldsymbol{p}^h(t))J\nabla F(I(\boldsymbol{p}^h(t))) - \nabla^2 F(I(\boldsymbol{p}^h(t)))J\nabla H(\boldsymbol{p}^h(t)) = 0 \tag{6.11}$$

because

$$\begin{aligned}\nabla^2 H(\boldsymbol{p}^h(t))J\nabla F(I(\boldsymbol{p}^h(t))) &- \nabla^2 F(I(\boldsymbol{p}^h(t)))J\nabla H(\boldsymbol{p}^h(t)) \\ &= \nabla\langle \nabla H(\boldsymbol{p}^h(t)), J\nabla F(I(\boldsymbol{p}^h(t)))\rangle \\ &= \nabla\left[\frac{dF(I(\boldsymbol{p}^h(t)))}{dI}\langle \nabla H(\boldsymbol{p}^h(t)), J\nabla I(\boldsymbol{p}^h(t))\rangle\right],\end{aligned}$$

and the expression $\langle \nabla H(\boldsymbol{p}^h(t)), J\nabla I(\boldsymbol{p}^h(t))\rangle$ vanishes identically since only the first three components of the vector $\nabla H(\boldsymbol{p}^h(t))$ are nonzero, and $J\nabla I(\boldsymbol{p}^h(t)) = (0, 0, 0, 1)$. Therefore, equation (6.10b) simplifies to

$$\dot{\Delta}^h(t) = \langle \boldsymbol{g}(\boldsymbol{p}^h(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle. \tag{6.12}$$

Furthermore, using equation (6.11), we can collect the terms in equation (6.10a) to become

$$\begin{aligned}\dot{\Delta}^l(t) &= \langle \boldsymbol{g}(\boldsymbol{p}^l(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle \\ &\quad + \langle \boldsymbol{r}^l(t), \nabla^2[H(\boldsymbol{p}^h(t)) - H(\boldsymbol{p}^l(t))]J\nabla[H(\boldsymbol{p}^h(t)) - F(I(\boldsymbol{p}^h(t)))]\rangle \\ &= \langle \boldsymbol{g}(\boldsymbol{p}^l(t)), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle + \langle \boldsymbol{r}^l(t), \nabla^2[H(\boldsymbol{p}^h(t)) - H(\boldsymbol{p}^l(t))]J\boldsymbol{n}(\boldsymbol{p}^h(t))\rangle.\end{aligned} \tag{6.13}$$

By using equations (6.12) and (6.13), the differential equation for the expression (6.9) becomes

$$\dot{\Delta}(t) = \langle [\boldsymbol{g}(\boldsymbol{p}^l(t)) - \boldsymbol{g}(\boldsymbol{p}^h(t))] + J\nabla^2[H(\boldsymbol{p}^l(t)) - H(\boldsymbol{p}^h(t))]\boldsymbol{r}^l(t), \boldsymbol{n}(\boldsymbol{p}^h(t))\rangle,$$

which implies that

$$\begin{aligned}\Delta(t) = \Delta(0) + \int_0^t &\langle [\boldsymbol{g}(\boldsymbol{p}^l(s)) - \boldsymbol{g}(\boldsymbol{p}^h(s))] \\ &+ J\nabla^2[H(\boldsymbol{p}^l(s)) - H(\boldsymbol{p}^h(s))]\boldsymbol{r}^l(s), \boldsymbol{n}(\boldsymbol{p}^h(s))\rangle \, ds.\end{aligned}$$

The integrand on the right-hand side of this equation is of the order $\mathcal{O}(\varepsilon^\alpha + \varepsilon)$, therefore we obtain the estimate

$$\Delta(t) = \Delta(0) + \mathcal{O}(\varepsilon^\alpha + \varepsilon). \tag{6.14}$$

Combining equation (6.8), the definition (6.9) and equation (6.14) with equation (6.5), we finally obtain the estimate (6.6). □

For future purposes, it will be more convenient to estimate the distance between the manifolds $\mathcal{L}$ and $W^u(\mathcal{M}_\varepsilon)$ at a point $\tilde{\boldsymbol{p}}^h(t)$, defined as the point on $W(\mathcal{M})$ at which the normal to $W(\mathcal{M})$ passing through $\boldsymbol{q}^l(t)$ intersects $W(\mathcal{M})$ itself. For this we need a little corollary to Proposition 6.1 to modify the result (6.6).

**Proposition 6.2.** *The signed distance $d^{l,u}(\tilde{\boldsymbol{p}}^h(t))$ between the points at which the normal $\boldsymbol{n}(\tilde{\boldsymbol{p}}^h(t))$ pierces the manifolds $\mathcal{L}$ and $W^u(\mathcal{M}_\varepsilon)$ is equal to*

$$d^{l,u}(\tilde{\boldsymbol{p}}^h(t)) = \frac{\langle \boldsymbol{q}^l(0) - \boldsymbol{q}^u(0), \boldsymbol{n}(\boldsymbol{p}^h(0))\rangle}{\|\boldsymbol{n}(\tilde{\boldsymbol{p}}^h(t))\|} + \mathcal{O}((\varepsilon^\alpha + \varepsilon)^{2-\beta}), \tag{6.15}$$

*where $\beta$ can be taken as small as desired.*

**Proof.** This follows simply by the triangle inequality

$$\begin{aligned}\|\tilde{\boldsymbol{p}}^h(t) - \boldsymbol{p}^h(t)\| &= \|\tilde{\boldsymbol{p}}^h(t) - \boldsymbol{q}^l(t) + \boldsymbol{q}^l(t) - \boldsymbol{p}^h(t)\| \\ &\leqq \|\tilde{\boldsymbol{p}}^h(t) - \boldsymbol{q}^l(t)\| + \|\boldsymbol{q}^l(t) - \boldsymbol{p}^h(t)\| \\ &\leqq \mathcal{O}(\varepsilon^\alpha + \varepsilon),\end{aligned}$$

where Gronwall-type estimates are used in the last inequality. This implies that $\boldsymbol{n}(\tilde{\boldsymbol{p}}^h(t)) = \boldsymbol{n}(\boldsymbol{p}^h(t)) + \mathcal{O}(\varepsilon^\alpha + \varepsilon)$, which together with equation (6.6) yields equation (6.15). □

## 7. Distance Estimates Near the Hyperbolic Annulus

In this section, we examine how the distance between the manifolds $\mathscr{L}$ and $W^s(\mathscr{M}_\varepsilon)$ transforms into the distance between $\mathscr{L}$ and $W^u(\mathscr{M}_\varepsilon)$ as $\mathscr{L}$ flies by the annulus $\mathscr{M}_\varepsilon$. (We recall here that the manifold $\mathscr{L}$ is defined as the winding piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$.) The result is given by

**Proposition 7.1.** *In the perturbed problem* (2.1)*, let some trajectory* $\boldsymbol{q}^l(t)$ *enter the neighborhood* $U_\delta(\mathscr{M}_\varepsilon)$ *a distance* $c\varepsilon^\alpha$ *away from the local stable manifold* $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ *of the annulus* $\mathscr{M}_\varepsilon$ *at the time* $t = 0$*, and let this trajectory leave the neighborhood* $U_\delta(\mathscr{M}_\varepsilon)$ *at the time* $t = T$*. Let* $\boldsymbol{n}_{\boldsymbol{q}}(0)$ *be the normal to the unperturbed local stable manifold* $W^s_{\mathrm{loc}}(\mathscr{M})$ *that passes through the point* $\boldsymbol{q}^l(0)$*, and let this normal intersect the local stable manifold* $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ *in the point* $\boldsymbol{q}^s(0)$*. Also, let* $\boldsymbol{n}_{\boldsymbol{q}}(T)$ *be the normal to the unperturbed local unstable manifold* $W^u_{\mathrm{loc}}(\mathscr{M})$ *that passes through the point* $\boldsymbol{q}^l(T)$*, and let this normal intersect the local unstable manifold* $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ *in the point* $\boldsymbol{q}^u(T)$*. Then*

$$\begin{aligned}&\left\langle \boldsymbol{n}_{\boldsymbol{q}}(0), \boldsymbol{q}^l(0) - \boldsymbol{q}^s(0)\right\rangle \\ &\quad = \left\langle \boldsymbol{n}_{\boldsymbol{q}}(T), \boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)\right\rangle + \mathscr{O}\left(\varepsilon^{2\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon^{1+\alpha}\left(\log\frac{1}{\varepsilon}\right)^2\right). \quad (7.1)\end{aligned}$$

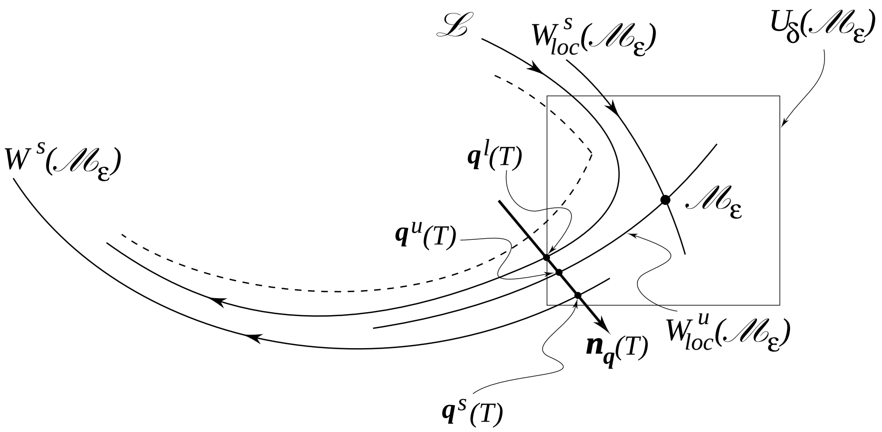


Fig. 7.1. The definition of the exit points $\boldsymbol{q}^l(T)$ and $\boldsymbol{q}^u(T)$. The unperturbed stable and unstable manifolds are represented by dashed lines.

Figure 7.1 gives a sketch of the geometry to which this proposition applies. The proof of Proposition (7.1) is given below in two steps, the first step of which is

**Proposition 7.2.** *In the unperturbed problem* (2.4)*, let some trajectory* $\boldsymbol{p}^l(t)$ *enter some small neighborhood of the annulus* $\mathscr{M}$ *at the time* $t = 0$*, and leave*

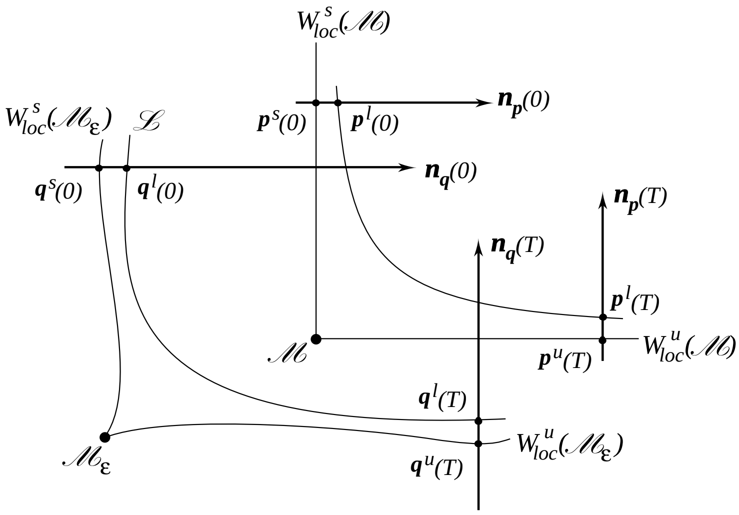


Fig. 7.2. An illustration of Propositions 7.2 and 7.3.

*this neighborhood at the time $t = T$. Let $\boldsymbol{n_p}(0) = \boldsymbol{n}(\boldsymbol{p}^s(0))$ be the normal to the local stable manifold $W^s_{\text{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{p}^l(0)$, and let $\boldsymbol{n_p}(T) = \boldsymbol{n}(\boldsymbol{p}^u(T))$ be the normal to the local unstable manifold $W^u_{\text{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{p}^l(T)$. Then*

$$\begin{aligned}&\left\langle \boldsymbol{n_p}(0), \boldsymbol{p}^l(0) - \boldsymbol{p}^s(0)\right\rangle + \mathscr{O}\left(\|\boldsymbol{p}^l(0) - \boldsymbol{p}^s(0)\|^2\right)\\ &\quad = \left\langle \boldsymbol{n_p}(T), \boldsymbol{p}^l(T) - \boldsymbol{p}^u(T)\right\rangle + \mathscr{O}\left(\|\boldsymbol{p}^l(T) - \boldsymbol{p}^u(T)\|^2\right).\end{aligned} \tag{7.2}$$

**Proof.** The proof of this proposition is almost identical to the first part of the proof of Proposition 6.1, except that the value of the function $H(x, I) - F(I)$, where $F(I) = H(X(I), I)$, must be the same on the stable and unstable manifolds $W^s(\mathscr{M})$ and $W^u(\mathscr{M})$ (see Figure 7.2). □

The second step and main technical result used in proving Proposition 7.1 is contained in

**Proposition 7.3.** *In the perturbed problem (2.1), let some trajectory $\boldsymbol{q}^l(t)$ enter the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ at a distance $c\varepsilon^\alpha$ away from the local stable manifold $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ at the time $t = 0$, and let this trajectory leave the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ at the time $t = T$. Let $\boldsymbol{n_q}(0)$ be the normal to the unperturbed local stable manifold $W^s_{\text{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{q}^l(0)$, and let this normal intersect the local stable manifold $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$ in the point $\boldsymbol{q}^s(0)$. Also, let $\boldsymbol{n_q}(T)$ be the normal to the unperturbed local unstable manifold $W^u_{\text{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{q}^l(T)$, and let this normal intersect the local unstable manifold $W^u_{\text{loc}}(\mathscr{M}_\varepsilon)$ at the point $\boldsymbol{q}^u(T)$. Let $\boldsymbol{p}^l(t)$ be an unperturbed trajectory whose initial point $\boldsymbol{p}^l(0)$ has the $(a, b, I, \psi)$ coordinates identical to the $(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$ coordinates of the point $\boldsymbol{q}^l(0)$, that is, $(c\varepsilon^\alpha, \delta, I^l_\varepsilon(0), \psi^l_\varepsilon(0))$. Let $\boldsymbol{n_p}(0) = \boldsymbol{n}(\boldsymbol{p}^s(0))$ be the normal to the unperturbed local stable manifold*

*$W^s_{\mathrm{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{p}^l(0)$. Let $\boldsymbol{n}_{\boldsymbol{p}}(T) = \boldsymbol{n}(\boldsymbol{p}^u(T))$ be the normal to the unperturbed local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M})$ that passes through the point $\boldsymbol{p}^l(T)$. Then*

$$\begin{aligned}&\left\langle \boldsymbol{n}_{\boldsymbol{q}}(0), \boldsymbol{q}^l(0) - \boldsymbol{q}^s(0)\right\rangle \\ &\quad = \left\langle \boldsymbol{n}_{\boldsymbol{p}}(0), \boldsymbol{p}^l(0) - \boldsymbol{p}^s(0)\right\rangle + \mathscr{O}\left(\varepsilon^{2\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon^{1+\alpha}\left(\log\frac{1}{\varepsilon}\right)^2\right),\end{aligned} \tag{7.3a}$$

$$\begin{aligned}&\left\langle \boldsymbol{n}_{\boldsymbol{q}}(T), \boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)\right\rangle \\ &\quad = \left\langle \boldsymbol{n}_{\boldsymbol{p}}(T), \boldsymbol{p}^l(T) - \boldsymbol{p}^u(T)\right\rangle + \mathscr{O}\left(\varepsilon^{2\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon^{1+\alpha}\left(\log\frac{1}{\varepsilon}\right)^2\right).\end{aligned} \tag{7.3b}$$

**Proof.** From equations (4.1) we compute that

$$a^l(T) = a^l(0)e^{\lambda(c\delta\varepsilon^\alpha, I_\varepsilon(0))T}, \quad b^l(T) = b^l(0)e^{-\lambda(c\delta\varepsilon^\alpha, I_\varepsilon(0))T}.$$

Since $\lambda(c\delta\varepsilon^\alpha, I_\varepsilon(0)) = \lambda(0, I_\varepsilon(0)) + \mathscr{O}(\varepsilon^\alpha)$, we conclude from inequalities (4.5) and (4.6a) that

$$a^l_\varepsilon(T) = a^l(T)\left[1 + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right)\right]. \tag{7.4a}$$

Likewise, we conclude from inequalities (4.5) and (4.6b) that

$$b^l_\varepsilon(T) = b^l(T)\left[1 + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right)\right]. \tag{7.4b}$$

By inequality (4.7), we have

$$|I_\varepsilon(T) - I(T)| = |I_\varepsilon(T) - I_\varepsilon(0)| = \mathscr{O}\left(\varepsilon\log\frac{1}{\varepsilon}\right), \tag{7.5}$$

$$\psi_\varepsilon(T) - \psi(T) = \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right). \tag{7.6}$$

Since the difference between the $(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$ and $(a, b, I, \psi)$ coordinates is $\mathscr{O}(\varepsilon)$, we have the estimate

$$\boldsymbol{q}^l(T) = \mathbf{p}^l(T) + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right),$$

and therefore

$$\boldsymbol{n}_{\boldsymbol{q}}(T) = \boldsymbol{n}_{\boldsymbol{p}}(T) + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right)$$

in any of the three coordinate systems.

Now, the vectors $\boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)$ and $\boldsymbol{p}^l(T) - \boldsymbol{p}^u(T)$ are proportional to the normals $\boldsymbol{n}_{\boldsymbol{q}}(T)$ and $\boldsymbol{n}_{\boldsymbol{p}}(T)$, respectively, so that all their components must be proportional to 1. The easiest component to compute is the $b$ component, which is just equal to $b^l_\varepsilon(T)$ for the vector $\boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)$ and equal to $b^l(T)$ for the vector $\boldsymbol{p}^l(T) - \boldsymbol{p}^u(T)$. By the estimate (7.4b), and since $b^l(T) = c\varepsilon^\alpha$, we have the estimate

$$b^l_\varepsilon(T) = b^l(T) + \mathscr{O}\left(\varepsilon^{2\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon^{1+\alpha}\left(\log\frac{1}{\varepsilon}\right)^2\right).$$

It is in order to establish this estimate that we need our stronger version of the Fenichel normal form and the subsequent Proposition 4.2. This proposition, in turn, implies the crucial estimate (7.4b). The usual Fenichel normal form is not refined enough to provide this estimate, since then the expression in square brackets in (7.4b) would be replaced by the term $\varepsilon^{\text{const.}\delta}$, where $\delta$ is the size of the neighborhood in which the Fenichel normal form is valid and the constant can be either positive or negative.

In what follows, we let the lower case letters $\boldsymbol{p}$ and $\boldsymbol{q}$ denote the positions of unperturbed and perturbed trajectories in the $(x, I, \theta)$ coordinates, and the upper case letters $\boldsymbol{P}$ and $\boldsymbol{Q}$ the positions of these same trajectories in the $(a, b, I, \psi)$ and $(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$ coordinates, respectively. By the discussion in the previous paragraph, we thus have

$$\boldsymbol{Q}^l(T) - \boldsymbol{Q}^u(T) = \boldsymbol{P}^l(T) - \boldsymbol{P}^u(T) + \mathscr{O}\left(\varepsilon^{2\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon^{1+\alpha}\left(\log\frac{1}{\varepsilon}\right)^2\right), \tag{7.7}$$

and so

$$\boldsymbol{Q}^l(T) - \boldsymbol{Q}^u(T) = \mathscr{O}(\varepsilon^\alpha), \quad \boldsymbol{P}^l(T) - \boldsymbol{P}^u(T) = \mathscr{O}(\varepsilon^\alpha). \tag{7.8}$$

Consider the function $N(a, b, I) = N_\varepsilon(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon) = H(x, I) - F(I)$, where $F(I) = H(X(I), I)$. Then, by slight abuse of notation,

$$\begin{aligned}&\left\langle \nabla N_\varepsilon(\boldsymbol{Q}^u(T)),\, \boldsymbol{Q}^l(T) - \boldsymbol{Q}^u(T)\right\rangle + \mathscr{O}\left(\|\boldsymbol{Q}^l(T) - \boldsymbol{Q}^u(T)\|^2\right)\\ &\qquad = \left\langle \boldsymbol{n}_{\boldsymbol{q}}(T), \boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)\right\rangle + \mathscr{O}\left(\|\boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)\|^2\right),\end{aligned}$$

since both expressions are the differentials of the same function in different coordinate systems. By equation (7.8), we obtain

$$\left\langle \nabla N_\varepsilon(\boldsymbol{Q}^u(T)),\, \boldsymbol{Q}^l(T) - \boldsymbol{Q}^u(T)\right\rangle = \left\langle \boldsymbol{n}_{\boldsymbol{q}}(T), \boldsymbol{q}^l(T) - \boldsymbol{q}^u(T)\right\rangle + \mathscr{O}(\varepsilon^{2\alpha}). \tag{7.9}$$

Now

$$\nabla N_\varepsilon(\boldsymbol{Q}^u(T)) = \nabla N(\boldsymbol{P}^u(T)) + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon}\log\frac{1}{\varepsilon} + \varepsilon\left(\log\frac{1}{\varepsilon}\right)^2\right)$$

in any coordinate system, and because of equation (7.7) we obtain

$$\begin{aligned}&\left\langle \nabla N_\varepsilon(\boldsymbol{Q}^u(T)),\, \boldsymbol{Q}^l(T)-\boldsymbol{Q}^u(T)\right\rangle\\ &\qquad = \left\langle \nabla N(\boldsymbol{P}^u(T)),\, \boldsymbol{P}^l(T)-\boldsymbol{P}^u(T)\right\rangle\\ &\qquad\quad + \mathscr{O}\left(\varepsilon^{2\alpha-D\varepsilon}\log\tfrac{1}{\varepsilon}+\varepsilon^{1+\alpha}\left(\log\tfrac{1}{\varepsilon}\right)^2\right).\end{aligned} \tag{7.10}$$

By equation (7.8), we also have

$$\left\langle \nabla N(\boldsymbol{P}^u(T)),\, \boldsymbol{P}^l(T)-\boldsymbol{P}^u(T)\right\rangle = \left\langle \boldsymbol{n}_{\boldsymbol{p}}(T),\, \boldsymbol{p}^l(T)-\boldsymbol{p}^u(T)\right\rangle + \mathscr{O}(\varepsilon^{2\alpha}). \tag{7.11}$$

Combining equations (7.9)–(7.11) finally yields equation (7.3b).

The proof of equation (7.3a) is similar to the last part of this proof. □

## 8. Estimate of the Phase Change Along a Pulse

Recall the definition of the manifold $\mathscr{L}$ as the winding piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$. As discussed in Section 6, away from a small neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$, a perturbed orbit on the manifold $\mathscr{L}$ can be approximated by unperturbed orbits that lie on the homoclinic manifold $W(\mathscr{M})$. Each of these unperturbed orbits is parametrized by one of the solutions (2.6), i.e., $x = x^h(t,I)$, $\theta = \theta^h(t,I)+\theta_0$, but they all have different values of the phase $\theta_0$, in general. In this section, we compute how this phase changes between two consecutive approximating unperturbed homoclinic orbits. This is straightforward in the "slow" case when the frequency $\Omega(X(I),I)$ of the underlying unperturbed periodic orbit on the annulus $\mathscr{M}$ vanishes, but for the general "fast" case considerable care is required for the estimate of the phase change. This estimate is the result of Proposition 8.1.

In order to compute the phase change described in the previous paragraph, we consider an orbit $O^l$ that lies on the manifold $\mathscr{L}$. In particular, let us concentrate on the segment of the orbit $O^l$ along its $j$-th and $(j+1)$-st excursions away from the small neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$. Let the orbit $O^l$ enter the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ through the point $\boldsymbol{q}^l$ from its $j$-th excursion away from $\mathscr{M}_\varepsilon$. Let $\boldsymbol{p}^s$ be the point such that the normal $\boldsymbol{n}(\boldsymbol{p}^s)$ to the unperturbed homoclinic manifold $W(\mathscr{M})$ at the point $\boldsymbol{p}^s$ pierces the manifold $\mathscr{L}$ at the point $\boldsymbol{q}^l$, see Figure 8.1. Moreover, let $\boldsymbol{q}^s$ be the point where the normal $\boldsymbol{n}(\boldsymbol{p}^s)$ intersects the local stable manifold $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$. For the time being, we assume that the points $\boldsymbol{q}^s$ and $\boldsymbol{q}^l$ are $\mathscr{O}(\varepsilon^\alpha)$ close, which we will justify in Section 9. From this assumption it follows that the points $\boldsymbol{p}^s$ and $\boldsymbol{q}^l$ are $\mathscr{O}(\varepsilon+\varepsilon^\alpha)$-close. Note that from their definitions, the points $\boldsymbol{q}^s$ and $\boldsymbol{q}^l$ are connected by the relation

$$\boldsymbol{q}^l = \boldsymbol{q}^s + \frac{\langle \boldsymbol{q}^l-\boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^s)\|}\frac{\boldsymbol{n}(\boldsymbol{p}^s)}{\|\boldsymbol{n}(\boldsymbol{p}^s)\|}. \tag{8.1}$$

Let the unperturbed homoclinic orbit through the point $\boldsymbol{p}^s$ be parametrized by the unperturbed trajectory

$$\boldsymbol{p}^h(t) = \left(x^h(t,I), I, \theta^h(t,I)+\theta_{j-1}(\varepsilon)\right), \tag{8.2}$$

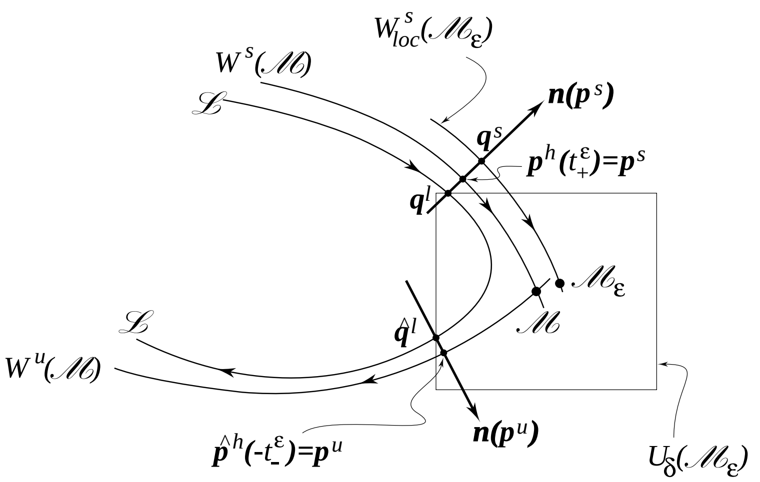


Fig. 8.1. The geometry of Proposition 8.1.

and let this trajectory pass through the point $\boldsymbol{p}^s$ at the time $t = t^\varepsilon_+$, that is,

$$\boldsymbol{p}^h(t^\varepsilon_+) = \boldsymbol{p}^s\,. \tag{8.3}$$

Moreover, let $\boldsymbol{q}^l(t)$ be the trajectory on the orbit $O^l$ that passes through the point $\boldsymbol{q}^l$ at the same time $t = t^\varepsilon_+$, so that $\boldsymbol{q}^l(t^\varepsilon_+) = \boldsymbol{q}^l$. As in Section 6, Gronwall-type estimates ensure that the trajectories $\boldsymbol{q}^l(t)$ and $\boldsymbol{p}^h(t)$ remain $\mathscr{O}(\varepsilon + \varepsilon^\alpha)$-close to each other during their whole flight away from the neighborhood $U_\delta$ all the way back to the boundary of $U_\delta$.

The estimate (4.8) can be used to determine the phase increment of the trajectory $\boldsymbol{q}^l(t)$ during its flight through the neighborhood $U_\delta$. Upon exiting $U_\delta$ again, at the point $\hat{\boldsymbol{q}}^l$ and after a time $T$ say, the normal to the unperturbed homoclinic manifold $W(\mathscr{M})$ that passes through the point $\hat{\boldsymbol{q}}^l$ defines a new point $\boldsymbol{p}^u$ on $W(\mathscr{M})$ (see Figure 8.1), which is $\mathscr{O}(\varepsilon + \varepsilon^\alpha)$-close to $\hat{\boldsymbol{q}}^l$ by the estimates of Section 7. The point $\boldsymbol{p}^u$ in turn defines a new unperturbed trajectory,

$$\hat{\boldsymbol{p}}^h(t) = \Big(x^h(t, \hat{I}(\varepsilon)), \hat{I}(\varepsilon), \theta^h(t, \hat{I}(\varepsilon)) + \theta_j(\varepsilon)\Big)\,,$$

for some $\theta_j(\varepsilon)$. Let $t = -t^\varepsilon_-$ be the time at which the new unperturbed trajectory passes through $\boldsymbol{p}^u$, i.e.,

$$\hat{\boldsymbol{p}}^h(-t^\varepsilon_-) = \boldsymbol{p}^u\,. \tag{8.4}$$

Once again, Gronwall-type estimates ensure that the trajectory $\hat{\boldsymbol{p}}^h(t)$ follows the trajectory $\boldsymbol{q}^l(t)$ on to its whole flight outside the neighborhood $U_\delta$ along its $(j+1)$-st excursion. Since $|\hat{I}(\varepsilon) - I| = \mathscr{O}(\varepsilon \log(1/\varepsilon))$ by formula (4.7), we then see that the segment of the orbit $O^l$ along this excursion can be approximated by the solution $\big(x^h(t, I), I, \theta^h(t, I) + \theta_j(\varepsilon)\big)$, and the relation between $\theta_j(\varepsilon)$ and $\theta_{j-1}(\varepsilon)$ is all we need to compute in order to select the new unperturbed trajectory $\hat{\boldsymbol{p}}^h(t)$. The explicit expression for the phase increment $\theta_j(\varepsilon) - \theta_{j-1}(\varepsilon)$ is provided by

**Proposition 8.1.** *The exit point $\boldsymbol{p}^u$ belongs to the orbit $\big(x^h(t, \hat{I}(\varepsilon)), \theta^h(t, \hat{I}(\varepsilon)) + \theta_j(\varepsilon)\big)$, where*

$$\begin{aligned} \theta_j(\varepsilon) &= \theta_{j-1}(\varepsilon) + \Delta\theta(I) + \frac{1}{\lambda(I)} \Omega(X(I), I) \log \left| \frac{\varsigma(I)}{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s)\rangle} \right| \\ &\quad + \mathscr{O}\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon}\right), \end{aligned} \tag{8.5}$$

*$\varsigma(I)$ is a function that depends on the entries of the Jacobian $D_x^2 H(X(I), I)$,*

$$\varsigma(I) = \frac{2(\lambda(I))^2 |A_2(I)| f_+(I) f_-(I)}{\sqrt{\big[\big(A_2(I)\big)^2 + \big(\lambda(I) - A_0(I)\big)^2\big]\big[\big(A_2(I)\big)^2 + \big(\lambda(I) + A_0(I)\big)^2\big]}} \tag{8.6}$$

*with*

$$A_0(I) = D_{x_1} D_{x_2} H(X(I), I), \quad A_1(I) = D_{x_1}^2 H(X(I), I),$$

$$A_2(I) = D_{x_2}^2 H(X(I), I),$$

$$\begin{aligned} f_+(I) &= \lim_{t \to +\infty} \frac{1}{\lambda(I)} e^{\lambda(I)t} \| D_x H(x^h(t, I), I) \|, \\ f_-(I) &= \lim_{t \to -\infty} \frac{1}{\lambda(I)} e^{-\lambda(I)t} \| D_x H(x^h(t, I), I) \| . \end{aligned} \tag{8.7}$$

Recall that $\lambda(I)$ is the positive eigenvalue of the matrix $J D_x^2 H(X(I), I)$, and that the phase difference $\Delta\theta(I)$ is given by formula (2.8).

A large part of the difficulty in proving this proposition stems from the fact that we must work in two coordinate systems: the $(x, I, \theta)$ coordinates away from the annulus $\mathscr{M}_\varepsilon$ and the $(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$ coordinates near $\mathscr{M}_\varepsilon$. Thus, before the actual proof of Proposition 8.1, we prove two results that address this issue. The first is a technical proposition:

**Proposition 8.2.** *Let $\boldsymbol{q}^s$ and $\boldsymbol{p}^s$ be points on the manifolds $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ and $W^s_{\mathrm{loc}}(\mathscr{M})$ respectively, such that the normal to the unperturbed local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M})$ at $\boldsymbol{p}^s$ pierces the perturbed local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ at $\boldsymbol{q}^s$. Then*

$$\frac{\langle \nabla a_\varepsilon(\boldsymbol{q}^s), \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^s)\|^2} = \frac{1}{\lambda(I) b(\boldsymbol{p}^s)} + \mathscr{O}(\varepsilon).$$

**Proof.** First of all, we have $\nabla a_\varepsilon(\boldsymbol{q}^s) = \nabla a(\boldsymbol{p}^s) + \mathscr{O}(\varepsilon)$. From the expression for $\boldsymbol{n}(\boldsymbol{p}^s)$, viz., expression (3.3), and the fact that $\boldsymbol{p}^s$ is a point on the unperturbed local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M})$, i.e., $a(\boldsymbol{p}^s) = 0$, we have

$$\begin{aligned}
\boldsymbol{n}(\boldsymbol{p}^s) &= \nabla\big(H(\boldsymbol{p}^s) - H(X(I), I)\big) \\
&= \nabla\big(K(a(\boldsymbol{p}^s)\, b(\boldsymbol{p}^s), I) - K(0, I)\big) \\
&= D_{ab}K(0, I)\big[b(\boldsymbol{p}^s)\nabla a(\boldsymbol{p}^s) + a(\boldsymbol{p}^s)\nabla b(\boldsymbol{p}^s)\big] \\
&\quad + D_I K\big(a(\boldsymbol{p}^s)b(\boldsymbol{p}^s), I\big)\nabla I - D_I K(0, I)\nabla I \\
&= \lambda(I) b(\boldsymbol{p}^s)\nabla a(\boldsymbol{p}^s),
\end{aligned} \tag{8.8}$$

where we have used the shorthand notation $I \equiv I(\boldsymbol{p}^s)$. It follows that

$$\nabla a(\boldsymbol{p}^s) = \frac{1}{\lambda(I) b(\boldsymbol{p}^s)} \boldsymbol{n}(\boldsymbol{p}^s), \tag{8.9}$$

so that

$$\frac{\langle \nabla a_\varepsilon(\boldsymbol{q}^s), \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^s)\|^2} = \frac{\langle \nabla a(\boldsymbol{p}^s), \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\|\boldsymbol{n}(\boldsymbol{p}^s)\|^2} + \mathcal{O}(\varepsilon) = \frac{1}{\lambda(I) b(\boldsymbol{p}^s)} + \mathcal{O}(\varepsilon),$$

as desired. □

We remark that had we chosen a point $\boldsymbol{p}^u$ contained in the unperturbed local unstable manifold $W^u_{\text{loc}}(\mathcal{M})$, then the above argument would have yielded $\boldsymbol{n}(\boldsymbol{p}^u) = \lambda(I) a(\boldsymbol{p}^u)\nabla b(\boldsymbol{p}^u)$.

Proposition 8.2 allows us to find an expression for the coordinate $a_\varepsilon(\boldsymbol{q}^l)$ of the point $\boldsymbol{q}^l$ at which the orbit $O^l$ enters the neighborhood $U_\delta(\mathcal{M}_\varepsilon)$. As defined in the second paragraph of this section, let the normal $\boldsymbol{n}(\boldsymbol{p}^s)$ to the unperturbed homoclinic manifold $W(\mathcal{M})$ that passes through the point $\boldsymbol{q}^l$ also pass through the point $\boldsymbol{p}^s$ on $W(\mathcal{M})$, and let this normal pierce the local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$ of the annulus $\mathcal{M}_\varepsilon$ at the point $\boldsymbol{q}^s$. Recall that we have assumed the points $\boldsymbol{q}^l$ and $\boldsymbol{q}^s$ to be at most a distance $\mathcal{O}(\varepsilon^\alpha)$ apart, and so the points $\boldsymbol{q}^l$ and $\boldsymbol{p}^s$ are at most a distance $\mathcal{O}(\varepsilon + \varepsilon^\alpha)$ apart. By using the result of Proposition 8.2, we obtain

**Proposition 8.3.** *The $a_\varepsilon$ coordinate of the point $\boldsymbol{q}^l$ is given by the expression*

$$a_\varepsilon(\boldsymbol{q}^l) = \operatorname{sign} b(\boldsymbol{p}^s) \frac{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\lambda(I)\delta} + \mathcal{O}\left((\varepsilon + \varepsilon^\alpha)^2\right).$$

**Proof.** At the point $\boldsymbol{q}^s$, the coordinate $a_\varepsilon(\boldsymbol{q}^s)$ vanishes. Therefore, since we assumed that the points $\boldsymbol{q}^l$ and $\boldsymbol{q}^s$ are at most a distance $\mathcal{O}(\varepsilon^\alpha)$ apart,

$$a_\varepsilon(\boldsymbol{q}^l) = a_\varepsilon(\boldsymbol{q}^l) - a_\varepsilon(\boldsymbol{q}^s) = \langle \nabla a_\varepsilon(\boldsymbol{q}^s), \boldsymbol{q}^l - \boldsymbol{q}^s\rangle + \mathcal{O}(\varepsilon^{2\alpha}).$$

Using equation (8.1) and Proposition 8.2, we obtain

$$a_\varepsilon(\boldsymbol{q}^l) = \frac{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s)\rangle}{\lambda(I) b(\boldsymbol{p}^s)} + \mathcal{O}\left((\varepsilon + \varepsilon^\alpha)^2\right).$$

If we use $|b(\boldsymbol{p}^s)| = \delta$, the conclusion follows. □

We are now in the position to give the

**Proof of Proposition 8.1.** Recall the parametrization of the unperturbed homoclinic orbit $\boldsymbol{p}^h(t)$, given by equation (8.2), which approximates to $\mathcal{O}(\varepsilon+\varepsilon^\alpha)$ the trajectory $\boldsymbol{q}^l(t)$ during its $j$-th excursion away from the annulus $\mathcal{M}_\varepsilon$. Recall also that the times $t_+^\varepsilon$ and $t_-^\varepsilon$, introduced by formulas (8.3) and (8.4), are defined in terms of the perturbed Fenichel coordinates by the equations

$$\begin{aligned}\delta &= |a_\varepsilon(\hat{\boldsymbol{q}}^l)| = \left|a_\varepsilon\Big(\hat{\boldsymbol{p}}^h(-t_-^\varepsilon) + \mathcal{O}(\varepsilon+\varepsilon^\alpha)\Big)\right| \\ &= |a(x^h(-t_-^\varepsilon, I), I)| + \mathcal{O}(\varepsilon+\varepsilon^\alpha), \\ \delta &= |b_\varepsilon(\boldsymbol{q}^l)| = \left|b_\varepsilon\Big(\boldsymbol{p}^h(+t_+^\varepsilon) + \mathcal{O}(\varepsilon+\varepsilon^\alpha)\Big)\right| \\ &= |b(x^h(+t_+^\varepsilon, I), I)| + \mathcal{O}(\varepsilon+\varepsilon^\alpha)\,.\end{aligned}$$

These equations in turn define two times $t_+^0$ and $t_-^0$ which are $\mathcal{O}(\varepsilon+\varepsilon^\alpha)$-close to $t_+^\varepsilon$ and $t_-^\varepsilon$ by

$$|a(x^h(-t_-^0, I), I)| = \delta, \quad |b(x^h(t_+^0, I), I)| = \delta. \tag{8.10}$$

At the time $t = t_+^\varepsilon$, the angle component of the trajectory $\boldsymbol{p}^h(t)$ is

$$\theta^h(t_+^\varepsilon) = \theta_{j-1}(\varepsilon) + \Omega(X(I), I)t_+^\varepsilon + \int_0^{t_+^\varepsilon} \Big(\Omega(x^h(t, I), I) - \Omega(X(I), I)\Big)\,dt, \tag{8.11}$$

and similarly the angle component of the trajectory $\hat{\mathrm{p}}^h(t)$ at the time $t = -t_-^\varepsilon$ is

$$\begin{aligned}\theta^h(-t_-^\varepsilon) = \theta_j(\varepsilon) - \Omega\Big(X(\hat{I}(\varepsilon)), \hat{I}(\varepsilon)\Big)t_-^\varepsilon - \int_{-t_-^\varepsilon}^0 \Big(\Omega\Big(x^h(t, \hat{I}(\varepsilon)), \hat{I}(\varepsilon)\Big) \\ - \Omega\Big(X(\hat{I}(\varepsilon)), \hat{I}(\varepsilon)\Big)\Big)dt.\end{aligned} \tag{8.12}$$

In order to use the estimate (4.8), we need to find the angle component $\psi_+^h \equiv \psi^h(t_+^\varepsilon) \equiv \psi\left(\boldsymbol{p}^h(t_+^\varepsilon)\right)$ for the point $\boldsymbol{p}^h(t_+^\varepsilon)$ in the Fenichel coordinates. Recall that the Fenichel transformation for the angle component takes the form

$$\psi = \theta + \mathcal{G}(x, I)$$

for some function $\mathcal{G}(x, I)$, given by equation (4.4), which satisfies the identity

$$\mathcal{G}(X(I), I) \equiv 0 \tag{8.13}$$

for all $I$. By the condition (8.13), we have

$$\psi_+^h - \theta^h(t_+^\varepsilon) = -\int_{t_+^\varepsilon}^\infty \Big(\dot{\psi}^h(t) - \dot{\theta}^h(t)\Big)\,dt.$$

Since the trajectory $\boldsymbol{p}^h(t)$ lies on the local stable manifold $W^s_{\mathrm{loc}}(\mathcal{M})$ for $t$ sufficiently large, we have $a(\boldsymbol{p}^h(t), I) = 0$ and $\dot{\psi}^h = \Omega\,(X(I), I)$. It follows that

$$\psi_+^h - \theta^h(t_+^\varepsilon) = -\int_{t_+^\varepsilon}^{\infty} \Big( \Omega(X(I), I) - \Omega(x^h(t, I), I) \Big)\, dt. \tag{8.14}$$

By combining this with equation (8.11), we find

$$\psi_+^h = \theta_{j-1}(\varepsilon) + \Omega(X(I), I)t_+^\varepsilon + \Delta\theta_+(I),$$

where we have defined

$$\Delta\theta_+(I) = \int_0^{+\infty} \Big( \Omega(x^h(t, I), I) - \Omega(X(I), I) \Big)\, dt. \tag{8.15}$$

The result of the previous paragraph, estimate (4.8), and the choice $\boldsymbol{q}^l(t_+^\varepsilon) = \boldsymbol{q}^l$ show that at the time when the trajectory $\boldsymbol{q}^l(t)$ leaves the neighborhood $U_\delta$ again, that is, at $t = T + t_+^\varepsilon$, the phase angle of $\boldsymbol{q}^l(t)$ in the Fenichel coordinates is

$$\begin{aligned} &\psi_+^h + \Omega(X(I), I)T + \mathcal{O}\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon}\right) \\ &\quad = \theta_{j-1}(\varepsilon) + \Omega(X(I), I)(t_+^\varepsilon + T) + \Delta\theta_+(I) + \mathcal{O}\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon}\right) \end{aligned} \tag{8.16}$$

where $T$ is estimated by formula (4.5). Now, by the definition of the exit point $\hat{\mathrm{q}}^l$, this phase angle can also be approximated by the formula $\hat{\psi}_-^h - \theta^h(-t_-^\varepsilon)$ where $\hat{\psi}_-^h \equiv \hat{\psi}^h(-t_-^\varepsilon) \equiv \psi\left(\hat{\mathrm{p}}^h(-t_-^\varepsilon)\right)$. By analogy with formula (8.14), we have

$$\hat{\psi}_-^h - \theta^h(-t_-^\varepsilon) = \int_{-\infty}^{-t_-^\varepsilon} \Big( \Omega\big(X(\hat{I}(\varepsilon)), \hat{I}(\varepsilon)\big) - \Omega\big(x^h(t, \hat{I}(\varepsilon)), \hat{I}(\varepsilon)\big) \Big)\, dt, \tag{8.17}$$

which combined with equation (8.12) and the approximation $|\hat{I}(\varepsilon) - I| = \mathcal{O}(\varepsilon \log(1/\varepsilon))$ by (4.7) yields

$$\hat{\psi}_-^h = \theta_j(\varepsilon) - \Omega(X(I), I)t_-^\varepsilon - \Delta\theta_-(I) + \mathcal{O}\left(\varepsilon \log \frac{1}{\varepsilon}\right), \tag{8.18}$$

where

$$\Delta\theta_-(I) \equiv \int_{-\infty}^{0} \Big( \Omega(x^h(s, I), I) - \Omega(X(I), I) \Big)\, ds. \tag{8.19}$$

Since $\boldsymbol{q}^l(T + t_+^\varepsilon) = \hat{\mathrm{q}}^l$, the two phases (8.16) and (8.18) must be equal, so that

$$\hat{\psi}_-^h = \psi_+^h + \Omega(X(I), I)T + \mathcal{O}\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon}\right).$$

This equation, together with equations (8.16) and (8.18), finally gives the total phase difference as

$$\theta_j(\varepsilon) - \theta_{j-1}(\varepsilon) = \Delta\theta(I) + \omega(I)(t_+^\varepsilon + t_-^\varepsilon + T) + \mathcal{O}\left(\varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon}\right), \tag{8.20}$$

where $\Delta\theta(I) = \Delta\theta_+(I) + \Delta\theta_-(I)$ is the same as that given by formula (2.8), and, as usual, $\omega(I) \equiv \Omega(X(I), I)$.

From the estimate (4.5) we easily compute that the time $T$ of flight through the neighborhood $U_\delta$ of the annulus $\mathscr{M}_\varepsilon$ equals

$$T = \frac{1}{\lambda(I)} \log \left| \frac{\delta}{a_\varepsilon(\boldsymbol{q}^l)} \right| + \mathscr{O}\left( \varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon} \right),$$

and Proposition 8.3 implies that

$$T = \frac{1}{\lambda(I)} \log \left| \frac{\delta^2 \lambda(I)}{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s) \rangle} \right| + \mathscr{O}\left( \varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon} \right).$$

The times $t_+^0$ and $t_-^0$ defined by equation (8.10), which approximate $t_+^\varepsilon$ and $t_-^\varepsilon$ with an error of order $\mathscr{O}(\varepsilon + \varepsilon^\alpha)$, can be expressed in terms of the size $\delta$ of the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$. Thinking of $\delta$ as a function of $t_+^0$ via $\delta = |b(\boldsymbol{p}^h(t_+^0))|$, we have

$$\frac{d\delta}{dt_+^0} = \left( \operatorname{sign} b(\boldsymbol{p}^h(t_+^0)) \right) \dot{b}(\boldsymbol{p}^h(t_+^0)) = -\lambda(0, I)|b(\boldsymbol{p}^h(t_+^0))| = -\lambda(I)\delta,$$

where we have used equation (4.2b) at $a(\boldsymbol{p}^h(t_+^0)) = 0$. Integrating yields

$$t_+^0 = \frac{1}{\lambda(I)} \log \frac{C_+(I)}{\delta},$$

where $C_+(I)$ is the constant of integration which is given by

$$C_+(I) = \lim_{t \to \infty} |b(\boldsymbol{p}^h(t))| e^{\lambda(I)t}. \tag{8.21}$$

An analogous argument for $t_-^0$ yields

$$t_-^0 = \frac{1}{\lambda(I)} \log \frac{C_-(I)}{\delta},$$

where

$$C_-(I) = \lim_{t \to -\infty} |a(\boldsymbol{p}^h(t))| e^{-\lambda(I)t}, \tag{8.22}$$

so that the total phase difference evaluates to

$$\begin{aligned} &\theta_j(\varepsilon) - \theta_{j-1}(\varepsilon) \\ &\quad = \Delta\theta + \frac{\omega(I)}{\lambda(I)} \log \left| \frac{\lambda(I) C_+(I) C_-(I)}{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s) \rangle} \right| + \mathscr{O}\left( \varepsilon^{\alpha - D\varepsilon} + \varepsilon \log \frac{1}{\varepsilon} \right). \end{aligned} \tag{8.23}$$

The expressions (8.21) and (8.22) for the functions $C_+(I)$ and $C_-(I)$ still involve the Fenichel coordinates $b$ and $a$ respectively, and so $C_+(I)$ and $C_-(I)$ cannot be computed explicitly without knowing the explicit form of the functions $a(x, I)$ and $b(x, I)$. We therefore need to determine an expression of $C_+(I)$ and $C_-(I)$ which does not refer to the Fenichel coordinates. This can be done in terms of the linearization of the vector field (2.4a) at $x = X(I)$, as follows. By the L'Hospital rule,

$$
\begin{aligned}
C_+(I) &= \lim_{t\to\infty} \frac{1}{\lambda(I)} \left| \langle D_x b(\boldsymbol{p}^h(t)), J D_x H(x^h(t,I),I)\rangle \right| e^{\lambda(I)t} \\
&= \lim_{t\to\infty} \frac{1}{\lambda(I)} \left| \langle D_x b(X(I),I), J D_x H(x^h(t,I),I)\rangle \right| e^{\lambda(I)t} \\
&= \lim_{t\to\infty} \frac{1}{\lambda(I)} \| D_x H(x^h(t,I),I)\| e^{\lambda(I)t} \\
&\quad \times \left| \left\langle D_x b(X(I),I), \tfrac{J D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \right\rangle \right| \\
&= f_+(I) \left| \left\langle D_x b(X(I),I), \lim_{t\to\infty} \frac{J D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \right\rangle \right|,
\end{aligned} \tag{8.24}
$$

where we have defined, as in equation (8.7),

$$
f_+(I) = \lim_{t\to+\infty} \frac{1}{\lambda(I)} e^{\lambda(I)t} \| D_x H(x^h(t,I),I)\|.
$$

Similarly we have (cf. equation (8.7))

$$
\begin{aligned}
C_-(I) &= \lim_{t\to-\infty} \frac{1}{\lambda(I)} \left| \langle D_x a(X(I),I), J D_x H(x^h(t,I),I)\rangle \right| e^{-\lambda(I)t} \\
&= f_-(I) \left| \left\langle D_x b(X(I),I), \lim_{t\to-\infty} \frac{J D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \right\rangle \right|,
\end{aligned} \tag{8.25}
$$

with

$$
f_-(I) = \lim_{t\to-\infty} \frac{1}{\lambda(I)} e^{-\lambda(I)t} \| D_x H(x^h(t,I),I)\|.
$$

The unit vectors

$$
\boldsymbol{e}_{\pm\lambda} \equiv \mp \lim_{t\to\pm\infty} \frac{J D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \tag{8.26}
$$

span the tangent spaces of the unstable and stable manifolds $W^u(X(I))$ and $W^s(X(I))$ of the equilibrium point $X(I)$, respectively. Since $b(x,I) = 0$ defines the unstable manifold $W^u(X(I))$ in the neighborhood $U_\delta(\mathscr{M})$, the vector $D_x b(X(I),I)$ is normal to $W^u(X(I))$ at $x = X(I)$, and so

$$
D_x b(X(I),I) = \kappa_b \lim_{t\to-\infty} \frac{D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \tag{8.27}
$$

for some proportionality constant $\kappa_b$. Likewise, the vector $D_x a(X(I),I)$ normal to $W^s(X(I))$ at $x = X(I)$ is

$$
D_x a(X(I),I) = \kappa_a \lim_{t\to\infty} \frac{D_x H(x^h(t,I),I)}{\| D_x H(x^h(t,I),I)\|} \tag{8.28}
$$

for some other proportionality constant $\kappa_a$ (see Figure 8.2 for a sketch of the geometry).

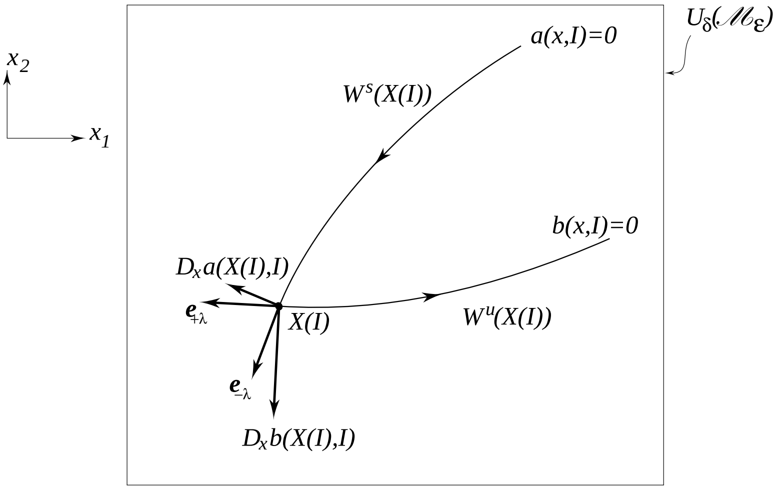


Fig. 8.2. Sketch of the geometry of the vectors normal and tangent to the stable manifold $W^s(X(I))$ and unstable manifold $W^u(X(I))$ at $X(I)$.

The condition that the $x \mapsto (a, b)$ part of the Fenichel transformation be canonical is easily seen to be $|\langle D_x a(X(I), I), J D_x b(X(I), I)\rangle| = 1$. Inserting expressions (8.28) and (8.27) into this condition yields an equation for the constants $\kappa_a$ and $\kappa_b$

$$\begin{aligned} &|\kappa_a \kappa_b| \lim_{t\to\infty} \left| \left\langle \frac{D_x H(x^h(-t, I), I)}{\|D_x H(x^h(-t, I), I)\|}, \frac{J D_x H(x^h(t, I), I)}{\|D_x H(x^h(t, I), I)\|} \right\rangle \right| \\ &\qquad = |\kappa_a \kappa_b|\, |\langle J\boldsymbol{e}_{+\lambda}, \boldsymbol{e}_{-\lambda}\rangle| = |\kappa_a \kappa_b \sigma| = 1, \end{aligned} \tag{8.29}$$

where we have used the definitions (8.26) for the second equality and the definition (3.4) of the signature $\sigma$ for the last equality. Combining equations (8.24), (8.25) with the expressions (8.27), (8.28) and the definitions (8.26) finally yields

$$\begin{aligned} C_+(I) C_-(I) &= |\kappa_a \kappa_b| f_+(I) f_-(I)\, |\langle J\boldsymbol{e}_{+\lambda}, \boldsymbol{e}_{-\lambda}\rangle|^2 \\ &= f_+(I) f_-(I)\, |\langle J\boldsymbol{e}_{+\lambda}, \boldsymbol{e}_{-\lambda}\rangle| = |\sigma| f_+(I) f_-(I), \end{aligned} \tag{8.30}$$

where we have used condition (8.29) in the last equality.

Let us introduce the following notation for the entries of the Hessian of the Hamiltonian $H(x, I)$ at $x = X(I)$,

$$A_0(I) = D_{x_1} D_{x_2} H(X(I), I), \quad A_1(I) = D^2_{x_1} H(X(I), I),$$

$$A_2(I) = D^2_{x_2} H(X(I), I),$$

so that $\lambda(I) = \sqrt{A_0^2(I) - A_1(I) A_2(I)}$. When $A_2(I) \neq 0$, a simple calculation shows that the vectors $\boldsymbol{e}_{\pm\lambda}$ are the eigenvectors of the Hessian $J D^2_x H(X(I), I)$,

$$\boldsymbol{e}_{\pm\lambda} = \frac{1}{\sqrt{(A_2(I))^2 + (\lambda(I) \pm A_0(I))^2}} \begin{pmatrix} A_2(I) \\ -(A_0(I) \pm \lambda(I)) \end{pmatrix}.$$

Using this expression in (8.30) to compute $|\sigma| = |\langle J\boldsymbol{e}_{+\lambda}, \boldsymbol{e}_{-\lambda}\rangle|$ then gives

$$C_+(I)C_-(I) = \frac{2\lambda(I)|A_2(I)|f_+(I)f_-(I)}{\sqrt{\left[\left(A_2(I)\right)^2 + \left(\lambda(I) - A_0(I)\right)^2\right]\left[\left(A_2(I)\right)^2 + \left(\lambda(I) + A_0(I)\right)^2\right]}}. \tag{8.31}$$

When $A_2(I) = 0$, a similar calculation shows that

$$C_+(I)C_-(I) = \frac{2\lambda(I)f_+(I)f_-(I)}{\sqrt{4\,(\lambda(I))^2 + (A_1(I))^2}},$$

which coincides with the limit of (8.31) when $A_2(I) \to 0$, and hence we can take (8.31) as the representative of the general case. Setting

$$\varsigma(I) = \lambda(I)C_+(I)C_-(I)$$

proves the proposition. □

## 9. Proof of the Main Result

In this section we finally tie the results of the previous sections into a proof of Theorem 1. We will obtain this proof in three steps by following an orbit $O^l$ that is contained in the manifold $\mathscr{L}$, the piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ whose properties we have discussed in the previous four sections. In the first of the three steps, we compute the distance between any point on the orbit $O^l$ and the stable manifold $W^s(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ in terms of the $j$-pulse Melnikov function, provided that the orbit $O^l$ does not lie on any of the folds of the manifold $\mathscr{L}$. The second step is Proposition 9.2, which expresses the nonfolding condition of Lemma 1 in terms of the logarithmic derivative of the $j$-pulse Melnikov function. The third step is the actual proof of Theorem 1.

We remark that at every step the exponent $\alpha$ we have used in all previous estimates turns out to be $\alpha = 1$.

The first step is given by

**Proposition 9.1.** *Let $O^l$ be an orbit on the manifold $\mathscr{L}$ and let no part of this orbit lie on a fold of the manifold $\mathscr{L}$. Let $\boldsymbol{q}_j$ be any point on the $j$-th pulse of the orbit $O^l$, and let $\boldsymbol{p}_j$ be the point on the unperturbed homoclinic manifold $W(\mathscr{M})$ such that the normal $\boldsymbol{n}(\boldsymbol{p}_j)$ to $W(\mathscr{M})$ at the point $\boldsymbol{p}_j$ passes through the point $\boldsymbol{q}_j$. Then the signed distance $d^{l,s}(\boldsymbol{p}_j)$ between the point $\boldsymbol{q}_j$ and the manifold $W^s(\mathscr{M}_\varepsilon)$ along the normal $\boldsymbol{n}(\boldsymbol{p}_j)$ equals*

$$d^{l,s}(\boldsymbol{p}_j) = \varepsilon\frac{M_j(\varepsilon, I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_j)\|} + \mathscr{O}\left(\varepsilon^{2-\beta}\right), \tag{9.1}$$

*where $\beta$ can be taken arbitrarily small by decreasing $\varepsilon$. Here the $j$-pulse Melnikov function $M_j(\varepsilon, I, \theta_0, \mu)$ is defined recursively as*

$$M_j(\varepsilon, I, \theta_0, \mu) = \sum_{i=0}^{j-1} M(I, \theta_0 + i\,\Delta\theta + \mathscr{T}_i(\varepsilon, I, \theta_0, \mu), \mu),$$

*with*

$$\mathscr{T}_i(\varepsilon, I, \theta_0, \mu) = \frac{\Omega(X(I), I)}{\lambda(I)} \sum_{r=1}^{i} \log \left| \frac{\varsigma(I)}{\varepsilon M_r(\varepsilon, I, \theta_0, \mu)} \right|,$$

*and $\mathscr{T}_0(\varepsilon, I, \theta_0, \mu) = 0$. The function $\varsigma(I)$ is defined in terms of the entries of the Hessian $J D_x^2 H(X(I), I)$ by equation* (8.6).

**Proof.** The first excursion or pulse of the orbit $O^l$ away from the annulus $\mathscr{M}_\varepsilon$ can be approximated to order $\mathscr{O}(\varepsilon)$ by the solution

$$\boldsymbol{p}_1^h(t) \equiv \mathscr{P}(t) = \left(x^h(t, I), I, \theta^h(t, I) + \theta_0\right).$$

Let $\boldsymbol{q}_1$ be any point on this pulse, and let $\boldsymbol{p}_1$ be the $\mathscr{O}(\varepsilon)$-close point on the unperturbed homoclinic manifold $W(\mathscr{M})$ such that the normal $\boldsymbol{n}(\boldsymbol{p}_1)$ to $W(\mathscr{M})$ at the point $\boldsymbol{p}_1$ passes through the point $\boldsymbol{q}_1$. The standard Melnikov method shows that the signed distance along the normal $\boldsymbol{n}(\boldsymbol{p}_1)$ between the point $\boldsymbol{q}_1$ and the stable manifold $W^s(\mathscr{M}_\varepsilon)$ equals

$$d^{l,s}(\boldsymbol{p}_1) = \varepsilon \frac{M(I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_1)\|} + \mathscr{O}(\varepsilon^2). \tag{9.2}$$

Let us now consider the second pulse of the orbit $O^l$. Lemma 1 and Propositions 7.1 and 6.2 with $\alpha = 1$ show that after its exiting the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$, the distance between the orbit $O^l$ and the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ at any point $\boldsymbol{q}_2$ along the second pulse of $O^l$ equals

$$d^{l,u}(\boldsymbol{p}_2) = \varepsilon \frac{M(I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_2)\|} + \mathscr{O}\left(\varepsilon^{2-\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right), \tag{9.3}$$

where the point $\boldsymbol{p}_2$ is related to the point $\boldsymbol{q}_2$ in the same way as $\boldsymbol{p}_1$ is related to $\boldsymbol{q}_1$, and $\beta$ can be taken as small as we please when we decrease $\varepsilon$.

From equation (9.3) and estimate (4.7) in Proposition 4.2, it follows that the second pulse of the orbit $O^l$ is approximated to $\mathscr{O}(\varepsilon)$ by the solution

$$\boldsymbol{p}_2^h(t) = \Big(x^h(t, I_1(\varepsilon, I, \theta_0, \mu)), I_1(\varepsilon, I, \theta_0, \mu), \theta^h(t, I_1(\varepsilon, I, \theta_0, \mu)) + \theta_1(\varepsilon, I, \theta_0, \mu)\Big),$$

where $|I_1(\varepsilon, I, \theta_0, \mu) - I| = \mathscr{O}(\varepsilon \log(1/\varepsilon))$, and the new phase $\theta_1(\varepsilon, I, \theta_0, \mu)$ is given by formula (8.5), $\theta_1(\varepsilon, I, \theta_0, \mu) \equiv \theta_1(\varepsilon)$, with $\boldsymbol{q}^l = \boldsymbol{q}_1$ and $\boldsymbol{p}^s = \boldsymbol{p}_1$, so that $\boldsymbol{n}(\boldsymbol{p}^s) = \boldsymbol{n}(\boldsymbol{p}_1)$. We thus conclude that the signed distance between the manifolds

$W^u_{\rm loc}(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ along the normal $\boldsymbol{n}(\boldsymbol{p}_2)$ to the unperturbed homoclinic manifold $W(\mathcal{M})$ at the point $\boldsymbol{p}_2$ equals

$$d^{u,s}(\boldsymbol{p}_2) = \varepsilon \frac{M(I, \theta_1(\varepsilon, I, \theta_0, \mu), \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_2)\|} + \mathcal{O}\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right). \tag{9.4}$$

From equations (9.3) and (9.4), it thus follows that the signed distance between the point $\boldsymbol{q}_2$ on the orbit $O^l$ and the manifold $W^s(\mathcal{M}_\varepsilon)$ along the normal $\boldsymbol{n}(\boldsymbol{p}_2)$ equals

$$d^{l,s}(\boldsymbol{p}_2) = \varepsilon \frac{M(I, \theta_0, \mu) + M(I, \theta_1(\varepsilon, I, \theta_0, \mu), \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_2)\|} + \mathcal{O}\left(\varepsilon^{2-\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right). \tag{9.5}$$

Now, from equation (9.2) it follows that if $\boldsymbol{q}_1^s$ is the point where the normal $\boldsymbol{n}(\boldsymbol{p}_1)$ intersects the local stable manifold $W^s_{\rm loc}(\mathcal{M}_\varepsilon)$, we must have

$$\langle \boldsymbol{q}_1 - \boldsymbol{q}_1^s, \boldsymbol{n}(\boldsymbol{p}_1)\rangle = d^{l,s}(\boldsymbol{p}_1)\|\boldsymbol{n}(\boldsymbol{p}_1)\| = \varepsilon M(I, \theta_0, \mu) + \mathcal{O}(\varepsilon^2).$$

Therefore, equation (8.5), with $\boldsymbol{q}^l = \boldsymbol{q}_1$, $\boldsymbol{p}^s = \boldsymbol{p}_1$, and $\boldsymbol{q}^s = \boldsymbol{q}_1^s$, implies that the phase $\theta_1(I, \theta_0, \mu, \varepsilon)$ is given by the expression

$$\theta_1(I, \theta_0, \mu, \varepsilon) = \theta_0 + \Delta\theta(I) + \mathcal{T}_1(\varepsilon, I, \theta_0, \mu) + \mathcal{O}\left(\varepsilon^{1-\beta} + \varepsilon \log \frac{1}{\varepsilon}\right),$$

where

$$\mathcal{T}_1(\varepsilon, I, \theta_0, \mu) = \frac{\Omega(X(I), I)}{\lambda(I)} \log\left(\frac{\varsigma(I)}{\varepsilon M(I, \theta_0, \mu)}\right).$$

We conclude that the distance $d^{l,s}(\boldsymbol{p}_2)$ is given by the quantity

$$d^{l,s}(\boldsymbol{p}_2) = \varepsilon \frac{M_2(\varepsilon, I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_2)\|} + \mathcal{O}\left(\varepsilon^{2-\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right). \tag{9.6}$$

Here the 2-pulse Melnikov function $M_2(\varepsilon, I, \theta_0, \mu)$ is defined as

$$M_2(\varepsilon, I, \theta_0, \mu) = M(I, \theta_0, \mu) + M(I, \theta_0 + \Delta\theta + \mathcal{T}_1(\varepsilon, I, \theta_0, \mu), \mu).$$

We can get rid of the term $\mathcal{O}\left(\varepsilon^2 (\log(1/\varepsilon))^2\right)$ in equation (9.6) by decreasing $\varepsilon$ while keeping $\beta$ fixed.

We proceed by using finite induction along the subsequent pulses of the orbit $O^l$. Let $\boldsymbol{q}_{j-1}$ be a point along the $(j-1)$-st pulse of the orbit $O^l$, with $j > 1$, and assume that its distance from the stable manifold $W^s(\mathcal{M}_\varepsilon)$ is given by an expression analogous to equation (9.5),

$$d^{l,s}(\boldsymbol{p}_{j-1}) = \varepsilon \frac{M_{j-1}(\varepsilon, I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_{j-1})\|} + \mathcal{O}\left(\varepsilon^{2-(j-2)\beta}\right),$$

where the point $\boldsymbol{p}_{j-1}$ on the unperturbed homoclinic manifold $W(\mathcal{M})$ is such that the normal $\boldsymbol{n}(\boldsymbol{p}_j)$ to $W(\mathcal{M})$ passes through the point $\boldsymbol{q}_{j-1}$.

For the $j$-th pulse, we use $\alpha = 1 - (j - 2)\beta$ in Lemma 1, to ensure that the tangent space of the winding manifold $\mathscr{L}$ exits the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ at least as close as order $\mathscr{O}(\varepsilon^{1-(j-1)\beta})$ to the tangent space of the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$. We then refine the estimate of the distance between the orbit $O^l$ in $\mathscr{L}$ and the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ by using Proposition 7.1 with $\alpha = 1$, to ensure that when exiting the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ the leading order of this distance is again $\mathscr{O}(\varepsilon)$. Using Proposition 6.2 with $\alpha = 1$ again shows that the distance between the orbit $O^l$ and the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ at any point $\boldsymbol{q}_j$ along the $j$-th pulse of $O^l$ equals

$$d^{l,u}(\boldsymbol{p}_j) = \varepsilon \frac{M_{j-1}(\varepsilon, I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_j)\|} + \mathscr{O}\left(\varepsilon^{2-(j-1)\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right),$$

where the point $\boldsymbol{p}_j$ is again related to the point $\boldsymbol{q}_j$ by the fact that the normal $\boldsymbol{n}(\boldsymbol{p}_j)$ to $W(\mathscr{M})$ at $\boldsymbol{p}_j$ passes through the point $\boldsymbol{q}_j$, and $\beta$ can be taken as small as we please when we decrease $\varepsilon$. Just as for $j = 2$, we then compute that the signed distance $d^{l,s}(\boldsymbol{p}_j)$ between the point $\boldsymbol{q}_j$ and the manifold $W^s(\mathscr{M}_\varepsilon)$ along the normal $\boldsymbol{n}(\boldsymbol{p}_j)$ is equal to

$$\begin{aligned} d^{l,s}(\boldsymbol{p}_j) &= \varepsilon \frac{M_{j-1}(\varepsilon, I, \theta_0, \mu) + M(I, \theta_{j-1}(\varepsilon, I, \theta_0, \mu), \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_j)\|} \\ &\quad + \mathscr{O}\left(\varepsilon^{2-(j-1)\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right) \\ &= \varepsilon \frac{M_j(\varepsilon, I, \theta_0, \mu)}{\|\boldsymbol{n}(\boldsymbol{p}_j)\|} + \mathscr{O}\left(\varepsilon^{2-(j-1)\beta} + \varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2\right), \end{aligned}$$

where $\beta$ can again be taken arbitrarily small, and the phase angle $\theta_{j-1}(\varepsilon, I, \theta_0, \mu)$ can be computed recursively by using formula (8.5) and the distance $d^{l,s}(\boldsymbol{p}_{j-1})$. Renaming $(j-1)\beta \equiv \beta$ and reducing $\varepsilon$ so that we get rid of the $\mathscr{O}\left(\varepsilon^2 (\log(1/\varepsilon))^2\right)$ term concludes the proof. □

Before we finally prove our main result, Theorem 1, we need to rephrase the nonfolding condition (5.29) in terms of the original $(x, I, \theta)$ coordinates, in particular, in terms of the derivatives of the $j$-pulse Melnikov functions $M_j(\varepsilon, I, \theta_0, \mu)$. This is expressed by

**Proposition 9.2.** *Let the manifold $\mathscr{L}$ be approaching the local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ for the $k$-th time, $k > 0$, and let the trajectory $\boldsymbol{q}^l(t)$ on $\mathscr{L}$ enter the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ at the point $\boldsymbol{q}^l$. Then, at $\boldsymbol{q}^l$, the nonfolding condition* (5.29) *is equivalent to the inequality*

$$\left| \frac{1 - \dfrac{\Omega(X(I), I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \cdots M_k|(\varepsilon, I, \theta_0, \mu)}{1 - \dfrac{\Omega(X(I), I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \cdots M_{k-1}|(\varepsilon, I, \theta_0, \mu)} \right| > B, \tag{9.7}$$

*holding for some constant $B > 0$ independent of $\varepsilon$ and all $\varepsilon$ sufficiently small. When $k = 1$, the denominator of the left-hand side of the inequality* (9.7) *is taken to be* 1.

We remark that for the case of only one angle $\theta_0$ and a homoclinic orbit for the unperturbed $x$-system, it is possible to obtain a simplified condition by using the fact that the Melnikov function is periodic in $\theta_0$, as in the third remark made after Theorem 1. In particular, the inequality (9.7) can be replaced by the simpler inequalities

$$\frac{\Omega(X(I), I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \cdots M_{k-1}|(\varepsilon, I, \theta_0, \mu) \neq 1, \tag{9.8}$$

$$\frac{\Omega(X(I), I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \cdots M_k|(\varepsilon, I, \theta_0, \mu) \neq 1, \tag{9.9}$$

where we assume that $\Omega(X(I), I) \neq 0$.

**Proof of Proposition 9.2.** After the $k$-th pulse, Proposition 9.1 shows that the manifold $\mathscr{L}$ returns to the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ at a distance $\mathscr{O}(\varepsilon)$ close to the local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ at the point $\boldsymbol{q}^l$, whose $a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon$ coordinates are $(A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon), \delta, I_\varepsilon, \psi_\varepsilon)$. Furthermore, a repeated use of Lemma 1 shows that the corresponding tangent spaces are $\mathscr{O}(\varepsilon^{1-\beta})$-close for any small $\beta > 0$, just as long as we take $\varepsilon$ small enough, and the orbit $\boldsymbol{q}^l(t)$ that passes through the point $\boldsymbol{q}^l$ does not lie on a fold of the manifold $\mathscr{L}$. As the orbit $\boldsymbol{q}^l(t)$ flies through the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$, the conclusion of Lemma 1 holds if the inequality (5.29), i.e.,

$$\left| \lambda(0, I_\varepsilon) - \omega(I_\varepsilon) \frac{D_{\psi_\varepsilon} A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)}{|A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)|} \right| > B$$

holds for some constant $B > 0$ independent of $\varepsilon$. Here $\lambda(0, I_\varepsilon) \equiv \lambda(I_\varepsilon)$ is the positive eigenvalue of the matrix $J D^2_x H(X(I_\varepsilon), I_\varepsilon)$, and $\omega(I_\varepsilon) = \Omega(X(I_\varepsilon), I_\varepsilon)$.

In order to prove Proposition 9.2, we need to express all the quantities in inequality (5.29) in terms of the arguments $I$ and $\theta_0$ of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$. This is easy for $\lambda(0, I_\varepsilon)$ and $\omega(I_\varepsilon) = \Omega(X(I_\varepsilon), I_\varepsilon)$, because they do not depend on the angle $\psi_\varepsilon$. Also, for the point $\boldsymbol{q}^l$, the Fenichel coordinate $I_\varepsilon$, introduced in Proposition 4.1, is $\mathscr{O}(\varepsilon)$-close to its $I$-coordinate, and furthermore, $I_\varepsilon$ is $\mathscr{O}(\varepsilon \log(1/\varepsilon))$-close to the argument $I$ of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$ by the inequality (4.7). Computing the logarithmic derivative of the function $A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)$ with respect to $\psi_\varepsilon$ is therefore the main task of this proof.

First, by Proposition 8.3 and the inequality (4.7), we find that

$$A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon) \equiv a_\varepsilon(\boldsymbol{q}^l) = \operatorname{sign} b(\boldsymbol{p}^s) \frac{\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s) \rangle}{\lambda(I)\delta} + \mathscr{O}\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right),$$

where $I$ is now the argument of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$. We recall here that $\boldsymbol{p}^s$ and $\boldsymbol{q}^s$ are the points where the normal $\boldsymbol{n}(\boldsymbol{p}^s)$ to the unperturbed homoclinic manifold $W(\mathscr{M})$ that passes through the point $\boldsymbol{q}^l$ intersects $W(\mathscr{M})$

and the local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M})$, respectively. Furthermore, Proposition 9.1 implies that

$$\langle \boldsymbol{q}^l - \boldsymbol{q}^s, \boldsymbol{n}(\boldsymbol{p}^s)\rangle = \varepsilon M_k(\varepsilon, I, \theta_0, \mu) + \mathscr{O}(\varepsilon^{2-\beta}),$$

and so

$$A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon) = \operatorname{sign} b(\boldsymbol{p}^s)\frac{\varepsilon M_k(\varepsilon, I, \theta_0, \mu)}{\lambda(I)\delta} + \mathscr{O}\left(\varepsilon^{2-\beta} + \varepsilon^2 \log\frac{1}{\varepsilon}\right),$$

where the neglected terms must vary smoothly with all of their arguments including $\varepsilon$ by the results of Sections 4 and 5. Since $D_{\psi_\varepsilon} I = \mathscr{O}(\varepsilon \log(1/\varepsilon))$, we must have

$$\begin{aligned} D_{\psi_\varepsilon} \log |A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)| = {} & D_{\theta_0} \log |M_k(\varepsilon, I, \theta_0, \mu)|\, D_{\psi_\varepsilon}\theta_0 \\ & + \mathscr{O}\left(\varepsilon^{2-2\beta} + \varepsilon^{2-\beta}\log\frac{1}{\varepsilon}\right), \end{aligned} \tag{9.10}$$

where the additional $\beta$ appears in the exponent of the remainder estimate because the differentiation on $\psi_\varepsilon$ involves the tangent space of the manifold $\mathscr{L}$ at the point $\boldsymbol{q}^l$, at which the manifold $\mathscr{L}$ reenters the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ for the $k$-th time, and there this tangent space is only $\mathscr{O}(\varepsilon^{1-\beta})$-close to the corresponding tangent space of local unstable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$. Equation (9.10) shows that in order to complete this proof, we must now compute the derivative $D_{\psi_\varepsilon}\theta_0$.

This can be done as follows. From the proof of Proposition 4.1, we see that

$$\psi_\varepsilon = \theta(\boldsymbol{q}^l) + \mathscr{G}(x(\boldsymbol{q}^l), I(\boldsymbol{q}^l)) + \mathscr{O}(\varepsilon),$$

for some function $\mathscr{G}(x, I)$, whose exact form is unimportant here. Helped by repeated applications of Propositions 8.1 and 9.1, we compute $\theta(\boldsymbol{q}^l)$ as

$$\begin{aligned} \theta(\boldsymbol{q}^l) = {} & \theta_0 + (k-1)\Delta\theta(I) + \mathscr{T}_{k-1}(\varepsilon, I, \theta_0, \mu) + \theta^h(t^0_+(I), I) \\ & + \mathscr{O}\left(\varepsilon^{1-\beta} + \varepsilon\log\frac{1}{\varepsilon}\right), \end{aligned}$$

where the time $t^0_+(I)$ is defined by $|b(x^h(t^0_+(I), I), I)| = \delta$ (see equation (8.10) at the beginning of the proof of Proposition 8.1), and $I$ is the value of the argument of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$. Thus, the angles $\psi_\varepsilon$ and $\theta_0$ are connected by the relation

$$\begin{aligned} \psi_\varepsilon = {} & \theta_0 + (k-1)\Delta\theta(I) + \mathscr{T}_{k-1}(\varepsilon, I, \theta_0, \mu) + \theta^h(t^0_+(I), I) \\ & + \mathscr{G}(x(\boldsymbol{q}^l), I(\boldsymbol{q}^l)) + \mathscr{O}\left(\varepsilon^{1-\beta} + \varepsilon\log\frac{1}{\varepsilon}\right), \end{aligned} \tag{9.11}$$

where the neglected terms again vary smoothly with both angles $\psi_\varepsilon$ and $\theta_0$, as well as with $\varepsilon$, by the results of Sections 4 and 5. Since $D_{\psi_\varepsilon} x(\boldsymbol{q}^l) = \mathscr{O}(\varepsilon)$ and $D_{\psi_\varepsilon} I(\boldsymbol{q}^l) = \mathscr{O}(\varepsilon)$ by the construction of the $(a_\varepsilon, b_\varepsilon, I_\varepsilon, \psi_\varepsilon)$ coordinates in the proof of Proposition 4.1, we obtain by implicit differentiation in the equation (9.11) that

$$D_{\psi_\varepsilon}\theta_0 = \frac{1}{\left(1 + D_{\theta_0}\mathscr{T}_{k-1}(\varepsilon, I, \theta_0, \mu)\right)} + \mathscr{O}\left(\varepsilon^{1-2\beta} + \varepsilon^{1-\beta}\log\frac{1}{\varepsilon}\right).$$

Directly from the definition (3.6) of the angle $\mathscr{T}_j(\varepsilon, I, \theta_0, \mu)$, we can compute its derivative with respect to $\theta_0$:

$$D_{\theta_0}\mathscr{T}_{k-1}(\varepsilon, I, \theta_0, \mu) = -\frac{\omega(I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \ldots M_{k-1}|(\varepsilon, I, \theta_0, \mu),$$

and therefore we finally find

$$\begin{aligned} D_{\psi_\varepsilon}\theta_0 = {} & \frac{1}{1 - \dfrac{\omega(I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \ldots M_{k-1}|(\varepsilon, I, \theta_0, \mu)} \\ & + \mathscr{O}\left(\varepsilon^{1-2\beta} + \varepsilon^{1-\beta}\log\frac{1}{\varepsilon}\right). \end{aligned} \tag{9.12}$$

We now combine equation (9.12), equation (9.10), and a calculation of the derivative $D_{\theta_0} M_k(\varepsilon, I, \theta_0, \mu)$ directly from the definition by formula (3.5), to obtain

$$\begin{aligned} & \lambda(I_\varepsilon) - \omega(I_\varepsilon)\,\frac{D_{\psi_\varepsilon} A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)}{|A_\varepsilon(I_\varepsilon, \psi_\varepsilon, \varepsilon)|} \\ & \quad = \lambda(I)\,\frac{1 - \dfrac{\omega(I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \ldots M_k|(\varepsilon, I, \theta_0, \mu)}{1 - \dfrac{\omega(I)}{\lambda(I)} D_{\theta_0} \log |M_1 M_2 \ldots M_{k-1}|(\varepsilon, I, \theta_0, \mu)} \\ & \quad\quad + \mathscr{O}\left(\varepsilon^{1-2\beta} + \varepsilon^{1-\beta}\log\frac{1}{\varepsilon}\right), \end{aligned}$$

where the number $\beta > 0$ can be taken as small as we please if we keep decreasing $\varepsilon$. This proves Proposition 9.2. □

Finally, we are ready to carry out the

**Proof of Theorem 1.** Consider again an orbit $O^l$ on the manifold $\mathscr{L}$. For $j = 1, \ldots, k-1$, we show recursively using Proposition 9.1 that if the second condition of Theorem 1 is met, then, after completing the $j$-th pulse, the orbit $O^l$ exits the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ along the correct branch of the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$, so that its $(j+1)$-st pulse can again follow an orbit on the unperturbed homoclinic manifold $W(\mathscr{M})$. Furthermore, we show recursively at the same time by using Proposition 9.2 and Lemma 1 that if the nonfolding condition of Theorem 1 is met, then the orbit $O^l$ does not lie on any of the folds of the manifold $\mathscr{L}$. Combined with Proposition 9.1, this shows that the orbit $O^l$ has at least $k$ pulses that follow the unperturbed homoclinic manifold $W(\mathscr{M})$ along excursions away from the annulus $\mathscr{M}_\varepsilon$, and that the distance $d^{l,s}(\boldsymbol{p}_k)$ from any point $\boldsymbol{q}_k$ along the $k$-th pulse of the orbit $O^l$ to the stable manifold $W^s(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ is given by formula (9.1) with $j = k$. In this formula, as always, $\boldsymbol{p}_k$

is the point on the unperturbed homoclinic manifold $W(\mathscr{M})$ such that the normal $\boldsymbol{n}(\boldsymbol{p}_k)$ to $W(\mathscr{M})$ at the point $\boldsymbol{p}_k$ passes through the point $\boldsymbol{q}_k$. By continuity, the same statements are also true for all nearby orbits on the manifold $\mathscr{L}$, which correspond to values of the variables $I$ and $\theta_0$ close to those that correspond to the orbit $O^l$.

Now, let us fix $I = \bar{I}$, $\mu = \bar{\mu}$, and divide the distance $d^{l,s}(\boldsymbol{p}_k)$ by $\varepsilon$, so that

$$\frac{d^{l,s}(\boldsymbol{p}_k)}{\varepsilon} = \frac{M_k(\varepsilon, \bar{I}, \theta_0, \bar{\mu})}{\|\boldsymbol{n}(\boldsymbol{p}_k)\|} + \mathscr{O}(\varepsilon^{1-\beta}). \tag{9.13}$$

By the first condition of our main theorem (Theorem 1), there exists a curve $\bar{\theta}_0(\varepsilon)$ such that

$$M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) = 0.$$

By the third condition of the same theorem, the graph of the $k$-pulse Melnikov function $M_k(\varepsilon, \bar{I}, \theta_0, \bar{\mu})$ as a function of $\theta_0$ intersects the $\theta_0$-axis transversely at $\bar{\theta}_0(\varepsilon)$. Because the graph of the left-hand side of (9.13) is, together with the respective tangent spaces, $\mathscr{O}(\varepsilon^{1-\beta})$-close to the graph of $M_k(\varepsilon, \bar{I}, \theta_0, \bar{\mu})/\|\boldsymbol{n}(\boldsymbol{p}_k)\|$ by the results of Sections 4 and 5, the distance $d^{l,s}(\boldsymbol{p}_k)$ vanishes on a nearby curve $\tilde{\theta}_0(\varepsilon)$ which is $\mathscr{O}(\varepsilon^{1-\beta})$-close to $\bar{\theta}_0(\varepsilon)$.

Now for $\varepsilon$ considered as fixed, a straightforward application of the Implicit Function Theorem ensures the existence of the function $\hat{\theta}_0(\varepsilon, I, \mu)$ in a neighborhood of $I = \bar{I}$ and $\mu = \bar{\mu}$ such that $M_k(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu) = 0$. This, in turn, implies the existence of the two-dimensional surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ which is $\mathscr{O}(\varepsilon^{1-\beta})$-close to the surfaces spanned by the union of unperturbed homoclinic orbits $\mathscr{P}^h(t) = \big(x^h(t, I), I, \theta^h(t, I) + \theta_0\big)$ determined by the sequence of phase angles $\theta_0 = \hat{\theta}_0(\varepsilon, I, \mu) + j\Delta\theta(I) + \mathscr{T}_j(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu)$ for $j = 0, 1, \ldots, k-1$. □

Notice a curious fact about this proof. Since the angle increments $\mathscr{T}_j(\varepsilon, I, \theta_0, \mu)$ grow like $\mathscr{O}(\Omega(X(I), I)\log(1/\varepsilon))$ when $\varepsilon$ decreases, the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$ does not have a limit as $\varepsilon \to 0$ unless $\Omega(X(I), I) = 0$. Nevertheless, the $k$-pulse intersection orbit between the manifolds $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$, which is determined by a simple zero in $\theta_0$ of this function, is better and better approximated by the corresponding segments of the unperturbed homoclinic orbits as $\varepsilon$ becomes smaller and smaller.

## 10. Extensions

In this section, we present some simple but important extensions of Theorem 1. The first extension concerns substituting heteroclinic orbits for homoclinic orbits in Assumption 2. This extension also covers the case when the unperturbed annulus $\mathscr{M}$ is connected to itself by a pair of homoclinic manifolds. The second extension is to higher dimensional $I$ and $\theta$.

First, we consider the extension to heteroclinic manifolds connecting several normally hyperbolic annuli. In this case, we replace Assumption 2 by

**Assumption 3.** *For every $I$ with $I_1 < I < I_2$, and some integer $k$, equation* (2.4a) *possesses $k+1$ hyperbolic equilibria $x = X_j(I)$, $j = 0, \dots, k$, which vary continuously with $I$. The unstable and stable manifolds, $W^u(X_j(I))$ and $W^s(X_{j+1}(I))$, intersect along a heteroclinic orbit, $W_j(I)$, connecting the equilibrium at $x = X_j(I)$ to the equilibrium at $x = X_{j+1}(I)$, for $j = 0, \dots, k$.*

In the full four-dimensional $(x, I, \theta)$-phase space of the system (2.4), the equilibria $x = X_j(I)$ correspond to annuli $\mathscr{M}_j$ filled with periodic orbits $O_j^I$, which are parametrized by the solutions

$$x = X_j(I), \quad I = I, \quad \theta = \Omega(X_j(I), I)t + \theta_0 \equiv \omega_j(I)t + \theta_0. \tag{10.1}$$

The heteroclinic orbits $W_j(I)$ correspond to three-dimensional heteroclinic manifolds $W_j$, parametrized by $t$, $I$, and $\theta_0$ in the solutions

$$x = x_j^h(t, I), \tag{10.2a}$$

$$I = I, \tag{10.2b}$$

$$\theta = \theta_j^h(t, I) + \theta_0 = \int_0^t \Omega(x_j^h(s, I), I)ds + \theta_0. \tag{10.2c}$$

The heteroclinic manifolds $W_j$ can be represented implicitly by the equations

$$H(x, I) - H(X_{j-1}(I), I) = H(x, I) - H(X_j(I), I) = 0. \tag{10.3}$$

Notice that the two expressions in this formula are the same because, by continuity, all the expressions $H(X_j(I), I)$, with $j = 0, \dots, k$, must be equal. The case of one or more homoclinic or heteroclinic cycles is easily included in this notation by letting some of the equilibria coincide, i.e., by letting $X_i(I) = X_j(I)$ for some $i \neq j$.

We define the Melnikov functions $M^{(j)}(I, \theta_0, \mu)$, with $j = 0, \dots, k$ in the usual way by

$$M^{(j)}(I, \theta_0, \mu) = \int_{-\infty}^{\infty} \Big\langle \boldsymbol{n}(\mathscr{P}_j^h(t)), \boldsymbol{g}(\mathscr{P}_j^h(t), \mu, 0) \Big\rangle dt, \tag{10.4}$$

where

$$\mathscr{P}_j^h(t) = (x_j^h(t, I), I, \theta_j^h(t, I) + \theta_0),$$

$$\begin{aligned} &\boldsymbol{n}(\mathscr{P}_j^h(t)) \\ &\quad = \Big(D_x H(x_j^h(t, I), I), D_I H(x_j^h(t, I), I) - D_I H(X_{j-1}(I), I), 0\Big) \qquad (10.5) \\ &\quad = \Big(D_x H(x_j^h(t, I), I), D_I H(x_j^h(t, I), I) - D_I H(X_j(I), I), 0\Big). \end{aligned}$$

By analogy with (3.4), we define the signatures $\sigma_j$ of the normals $\boldsymbol{n}$ to the unperturbed heteroclinic manifolds $W_j$ by

$$\begin{aligned}\sigma_j &= \lim_{t\to+\infty} \frac{\left\langle \boldsymbol{n}(\mathscr{T}_j^h(t)),\, \dot{\mathscr{T}}_{j+1}^h(-t)\right\rangle}{||D_x H(x_j^h(t,I),I)||\; ||D_x H(x_{j+1}^h(-t,I),I)||} \\ &= \lim_{t\to+\infty} \frac{\left\langle D_x H(x_j^h(t,I),I),\, J D_x H(x_{j+1}^h(-t,I),I)\right\rangle}{||D_x H(x_j^h(t,I),I)||\; ||D_x H(x_{j+1}^h(-t,I),I)||},\end{aligned} \tag{10.6}$$

so that $\sigma_j$ is positive if the normal $\boldsymbol{n}$ to $W_j$ points in the direction of the unperturbed flow on the heteroclinic manifold $W_{j+1}(I)$ at a point $(X_j(I), I, \theta)$ in the annulus $\mathscr{M}_j$.

The main difference introduced by the heteroclinic case in the form of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$ lies in the phase jumps $\Delta\theta$, which now depend on the heteroclinic orbit along which they are computed. Specifically, we define the $k$-pulse Melnikov function as

$$M_k(\varepsilon, I, \theta_0, \mu) = \sum_{j=0}^{k-1} M^{(j)}\left(I, \theta_0 + \sum_{i=1}^{j} \Delta\theta_i(I) + \mathscr{T}_j(\varepsilon, I, \theta_0, \mu), \mu\right), \tag{10.7}$$

where

$$\begin{aligned}\Delta\theta_i(I) = &\int_0^{+\infty} \Big(\Omega(x_{i-1}^h(t), I) - \Omega(X_i(I), I)\Big)\, dt \\ &+ \int_{-\infty}^{0} \Big(\Omega(x_i^h(t), I) - \Omega(X_i(I), I)\Big)\, dt,\end{aligned} \tag{10.8}$$

and the sum of the phase jumps is absent if $j = 0$. Here

$$\mathscr{T}_j(\varepsilon, I, \theta_0, \mu) = \sum_{r=1}^{j} \frac{\omega_r(I)}{\lambda_r(I)} \log\left|\frac{\varsigma_r(I)}{\varepsilon M_r(\varepsilon, I, \theta_0, \mu)}\right|, \tag{10.9}$$

$\mathscr{T}_0(\varepsilon, I, \theta_0, \mu) = 0$, $\omega_r(I) = \Omega(X_r(I), I)$, and the functions $\varsigma_r(I)$ are defined in the same way as equation (8.6) at each equilibrium point $X_r(I)$.

Theorem 1 now becomes

**Theorem 2.** *For some integer $k$, some constant $B > 0$ independent of $\varepsilon$, some $I = \bar{I}$, some $\mu = \bar{\mu}$, and all sufficiently small $0 < \varepsilon$ let there exist a function $\theta_0 = \bar{\theta}_0(\varepsilon)$ such that the following conditions are satisfied:*

1. *The $k$-pulse Melnikov function has a simple zero in $\theta_0$, that is, $M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) = 0$, and $|D_{\theta_0} M_k(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu})| > B$.*
2. *$M_i(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}) \not\equiv 0$ for all $i = 1, \ldots, k-1$, $k > 1$, and is positive if the signature $\sigma_i$ for the unperturbed heteroclinic manifold $W_i$ is positive, and negative if $\sigma_i$ is negative.*

3. *For all* $i = 1, \ldots, k-1$, $k > 1$,

$$\left| \frac{1 - \lambda_i(\bar{I}) D_{\theta_0} \log \prod_{j=1}^{i} |M_j|^{\frac{\Omega(X_j(\bar{I}),\bar{I})}{\lambda_j(\bar{I})}} (\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu})}{1 - \lambda_i(\bar{I}) D_{\theta_0} \log \prod_{j=1}^{i-1} |M_j|^{\frac{\Omega(X_j(\bar{I}),\bar{I})}{\lambda_j(\bar{I})}} \left(\varepsilon, \bar{I}, \bar{\theta}_0(\varepsilon), \bar{\mu}\right)} \right| > B, \qquad (10.10)$$

*where* $\pm\lambda_i(I)$ *are the two eigenvalues of the linearization of system* (2.4) *at the equilibrium* $x = X_i(I)$*, and the denominator in* (10.10) *is defined to be* 1 *when* $i = 1$.

*Then for all* $I$ *close to* $\bar{I}$*, all* $\mu$ *close to* $\bar{\mu}$*, and all sufficiently small* $\varepsilon$*, there exists a two-dimensional intersection surface* $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ *along which the stable and unstable manifolds* $W^s(\mathscr{M}_{k,\varepsilon})$ *and* $W^u(\mathscr{M}_{0,\varepsilon})$ *of the perturbed annuli* $\mathscr{M}_{k,\varepsilon}$ *and* $\mathscr{M}_{0,\varepsilon}$ *intersect transversely at an angle of size* $\mathscr{O}(\varepsilon)$*. Moreover, outside of some small neighborhoods of the perturbed annuli* $\mathscr{M}_{i,\varepsilon}$*,* $i = 0, \ldots, k$*, the surface* $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ *is* $\mathscr{O}(\varepsilon)$*-close to the union of surfaces spanned by the orbits* (10.2) *selected by the phase angles*

$$\theta_0 = \hat{\theta}_0(\varepsilon, I, \mu) + \sum_{i=0}^{j} \Delta\theta_j(I) + \mathscr{T}_j(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu),$$

$j = 0, \ldots, k-1$*, where the triple* $(I, \hat{\theta}_0(\varepsilon, I, \mu), \mu)$ *identically satisfies the equation*

$$M_k(\varepsilon, I, \hat{\theta}_0(\varepsilon, I, \mu), \mu) = 0$$

*in some neighborhood of* $I = \bar{I}$ *and* $\mu = \bar{\mu}$*, and* $\hat{\theta}_0(\varepsilon, \bar{I}, \bar{\mu}) = \bar{\theta}_0(\varepsilon)$.

**Proof.** The proof of this theorem is almost identical to the proof of Theorem 1. □

Notice that the case of multiple homoclinic orbits for a single equilibrium point $X(I)$ can be treated by this extension of Theorem 1 with an obvious adaptation of the notation.

We now briefly discuss an extension of our results to higher-dimensional $I$ and $\theta$ variables. Extending Theorems 1 and 2 to the case when $I \in \mathbb{R}^m$ and $\theta \in T^n$, where $T^n$ is the $n$-dimensional torus, is immediate for all positive integers $m$ and $n$, including $m = 0$. We remark, however, that when system (2.1) is Hamiltonian, the annulus $\mathscr{M}_\varepsilon$ is almost everywhere filled with Kolmogorov-Arnold-Moser tori. In this case, a question that arises is whether the stable and unstable manifolds of these tori, as opposed to those of the whole annulus, intersect. In the case of a single action-angle pair $(I, \theta)$ our Theorems 1 and 2, combined with the observation that the Hamiltonian surfaces intersect the surfaces $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ transversely, provide an affirmative answer. In the case when $I$ and $\theta$ are multi-dimensional one needs to compute additional Melnikov functions. We suspect that results similar to the ones presented here can be obtained in this case of multi-dimensional action-angle pairs;

however, the details are beyond the scope of the present paper and are left to future work.

A more interesting case occurs when we can write $\theta = (\theta_1, \theta_2)$, with $\theta_1 \in T^p$ and $\theta_2 \in \mathbb{R}^q$ for some nonnegative integers $p$ and $q$ with $p + q = n$. We can then replace the unperturbed equations (2.4c) for $\theta$ by

$$\dot{\theta}_1 = \Omega_1(x, I) + \tilde{\Omega}_1(x, I, \theta), \tag{10.11a}$$

$$\dot{\theta}_2 = \tilde{\Omega}_2(x, I, \theta). \tag{10.11b}$$

To extend the previous results to this situation we must assume that the equations (10.11a) and (10.11b) can be integrated by quadratures, so that a solution $\theta(t, \theta_0)$ of system (10.11) corresponding to some initial condition $\theta_0$ can be computed explicitly. We must also assume the inequality

$$\left\| \int_{-\infty}^{\infty} \tilde{\Omega}_s\Big(x_i^h(t, I), \theta(t, \theta_0), I\Big) dt \right\| < \infty \tag{10.12}$$

for $s = 1, 2$ and all $j = 1, \ldots, k$. This inequality of course implies that $\tilde{\Omega}_s(X_j(I), \theta, I) = 0$ for $s = 1, 2$ and all $j = 0, \ldots, k$. Notice that the solutions of equations (10.11) define a mapping more general than $\theta_0 \to \theta_0 + \sum_{i=1}^{j} \Delta\theta_j(I)$ in the $\theta$-argument of the $k$-pulse Melnikov function (10.7); the analog of the increments $\Delta\theta_j(I)$ of the “angles” $\theta$ do not simply depend on $I$ only, since the values at the previous mapping, $\theta = \theta^{(j-1)}(\varepsilon, I, \theta_0, \mu)$ say, determine the initial conditions for (10.11) in computing these increments. The $k$-pulse Melnikov function is now defined recursively by

$$M_k(\varepsilon, I, \theta_0, \mu) = \sum_{j=0}^{k-1} M^{(j)}\Big(I, \theta^{(j)}(\varepsilon, I, \theta_0, \mu), \mu\Big), \tag{10.13}$$

where the vector $\theta^{(j)}(\varepsilon, I, \theta_0, \mu)$ is defined implicitly by

$$\begin{aligned} \theta_1^{(j)}(\varepsilon, I, \theta_0, \mu) &= \theta_1^{(j-1)}(\varepsilon, I, \theta_0, \mu) + \Delta\theta_j(I) + \mathscr{T}_j(\varepsilon, I, \theta_0, \mu) \\ &\quad + \int_0^{+\infty} \tilde{\Omega}_1\Big(x_j^h(t, I), \theta(t, \theta^{(j-1)})\Big) dt \\ &\quad + \int_{-\infty}^{0} \tilde{\Omega}_1\Big(x_{j+1}^h(t, I), \theta(t, \theta^{(j)})\Big) dt, \qquad (10.14) \\ \theta_2^{(j)}(\varepsilon, I, \theta_0, \mu) &= \theta_2^{(j-1)}(\varepsilon, I, \theta_0, \mu) + \int_0^{+\infty} \tilde{\Omega}_2\Big(x_j^h(t, I), \theta(t, \theta^{(j-1)})\Big) dt \\ &\quad + \int_{-\infty}^{0} \tilde{\Omega}_2\Big(x_{j+1}^h(t, I), \theta(t, \theta^{(j)})\Big) dt, \end{aligned}$$

for $j = 1, \ldots, k$, with $\Delta\theta_j(I)$ and $\mathscr{T}_j(\varepsilon, I, \theta_0, \mu)$ computed as in (10.8) and (10.9), with $\Omega(x_j^h(t, I), I)$ and $\omega_j(I)$ replaced by $\Omega_1(x_j^h(t, I), I)$ by $\Omega_1(X_j(I), I)$, respectively, and with $\theta^{(0)}(\varepsilon, I, \theta_0, \mu) = \theta_0$. Notice that we have suppressed the arguments of the functions $\theta^{(j)}(\varepsilon, I, \theta_0, \mu)$ on the right-hand side of equations (10.14) for ease of notation.

After these modifications and additional assumptions we can proceed as before and obtain the same results. However, notice that the properties of the $k$-pulse Melnikov functions expressed by formulas (3.10) and (3.11) in the third remark following Theorem 1 depend crucially on the fact that the ordinary Melnikov function $M(I, \theta_0, \mu)$ is periodic in $\theta_0$, i.e., $\theta_0$ lives on $S^1$. Properties (3.10) and (3.11) do not apply, in general, when the angle variables $\theta$ live on a torus $T^m$, $m > 1$. An exception is offered by equation (10.11) because of the condition (10.12). As a consequence of this condition, we have $\dot{\theta}_1 = \Omega_1(X(I), I)$, $\dot{\theta}_2 = 0$ on the unperturbed annulus $\mathscr{M}$. In this case, it is easy to see that the properties (3.10) and (3.11) apply with $\varepsilon_n = \varepsilon \exp(-2n\pi\lambda(I)/\Omega_1(X(I), I))$. An example for this particular extension of Theorem 1 is discussed in the next section.

## 11. Application to an Atmospheric Model

The following example was introduced by E. LORENZ to describe the coupling between wave motions in the atmosphere occurring on fast and slow time scales. After a reduction and rescaling (see [6]), the original five-equation model [46] assumes the form

$$\dot{q} = p - \varepsilon z, \tag{11.1a}$$

$$\dot{p} = -R^2 \sin q, \tag{11.1b}$$

$$\dot{y} = -z, \tag{11.1c}$$

$$\dot{z} = y + R^2 \sin q, \tag{11.1d}$$

with $(q, p, y, z) \in \mathbb{R}^4$. Here $R$ is a reduction parameter which is bounded away from zero. This system has an integral of motion

$$E = \tfrac{1}{2}p^2 - R^2 \cos q + \tfrac{1}{2}\varepsilon(y^2 + z^2). \tag{11.2}$$

The unperturbed system at $\varepsilon = 0$,

$$\dot{q} = p, \tag{11.3a}$$

$$\dot{p} = -R^2 \sin q, \tag{11.3b}$$

$$\dot{y} = -z, \tag{11.3c}$$

$$\dot{z} = y + R^2 \sin q, \tag{11.3d}$$

represents a pendulum acting as external forcing on a harmonic oscillator. The parameter $R$ gives the eigenvalue of the hyperbolic equilibrium point of the pendulum at $(q, p) = (\pm\pi, 0)$. System (11.3) has the integral of motion

$$E_0 = \tfrac{1}{2}p^2 - R^2 \cos q. \tag{11.4}$$

The flow generated by (11.3) does not take place on intersections of the level sets of two integrals of motion, although it is still solvable, i.e., integrable by quadratures. In particular, the plane of the harmonic oscillator $\mathscr{M} = \{q, p, y, z \mid q = \pm\pi, p = 0\}$ is a normally hyperbolic manifold, connected to itself by three-dimensional homoclinic manifolds $W_\pm$, defined implicitly by fixing the value of the constant $E_0$ to be that of the hyperbolic equilibrium point of the pendulum,

$$E_0 = R^2. \tag{11.5}$$

As usual, for the unperturbed system (11.3) the stable and unstable manifolds $W^s_\pm(\mathscr{M})$ and $W^u_\pm(\mathscr{M})$ coincide along the homoclinic manifolds $W_\pm$. However, system (11.3) has the peculiarity that level surfaces of the constant of motion $E_0$ do not intersect the plane $\mathscr{M}$ transversely in the four-dimensional phase space. Thus, the periodic orbits $O^{\rho_0} = \{y, z \mid y^2 + z^2 = \rho_0^2\}$ are not normally hyperbolic in the lower-dimensional space of a constant of motion level set. As a consequence, periodic orbits with different $\rho_0$'s on $\mathscr{M}$ can be connected via heteroclinic excursions.

A parametrization of the manifolds $W_\pm$ can be obtained by integrating the unperturbed system (11.3) using the homoclinic solution for the separatrix of the pendulum component. We have

$$\begin{aligned} W_\pm = \{(q, p, y, z) | q = q^h(t),\ p = p^h(t),& \\ y = y^h(t; \rho_0, \vartheta_0),\ z = z^h(t; \rho_0, \vartheta_0)\},& \end{aligned} \tag{11.6}$$

where

$$q^h(t) = \pm 2 \arcsin\left[\tanh(Rt)\right], \qquad p^h(t) = \pm 2R \operatorname{sech}(Rt), \tag{11.7}$$

$$\begin{aligned} y^h(t; \rho_0, \vartheta_0) &= \rho_0 \cos(t + \vartheta_0) \pm S(t, R)\cos t \pm A(t, R)\sin t, \\ z^h(t; \rho_0, \vartheta_0) &= \rho_0 \sin(t + \vartheta_0) \mp 2R \operatorname{sech}(Rt) \\ &\qquad \pm S(t, R)\sin t \mp A(t, R)\cos t, \end{aligned} \tag{11.8}$$

with $t \in \mathbb{R}$, $\rho_0 \in \mathbb{R}^+$ and $\vartheta_0 \in (-\pi, \pi]$ being three parameters. The functions $S(t, R)$, $A(t, R)$ are

$$\begin{aligned} S(t, R) &\equiv 2R \int_{-\infty}^{t} \operatorname{sech}(Rt')\cos t'\, dt', \\ A(t, R) &\equiv 2R \int_{-\infty}^{t} \operatorname{sech}(Rt')\sin t'\, dt'. \end{aligned} \tag{11.9}$$

Alternatively, for a given $\rho_0$, equations (11.7) and (11.8) can be viewed as a parametrization of the two-dimensional stable and unstable manifolds, $W^s(O^{\rho_0})$ and $W^u(O^{\rho_0})$, of the periodic orbit $O^{\rho_0}$.

The expression (11.8) shows that a given periodic orbit $O^{\rho_0}$ has a one-parameter ($\vartheta_0$) family of heteroclinic connections to other periodic orbits,

$$
\begin{aligned}
y &= \tilde{\rho}_{\pm}(\rho_0, \vartheta_0) \cos\left[t + \tilde{\theta}_{\pm}(\rho_0, \vartheta_0)\right], \\
z &= \tilde{\rho}_{\pm}(\rho_0, \vartheta_0) \sin\left[t + \tilde{\theta}_{\pm}(\rho_0, \vartheta_0)\right]
\end{aligned} \tag{11.10}
$$

in an annulus

$$|\rho_0 - S_\infty| < \rho < |\rho_0 + S_\infty|.$$

Here we denote by $S_\infty$ the value of $S(t, R)$ for $t \to +\infty$, i.e.,

$$S_\infty = 2R \int_{-\infty}^{+\infty} \operatorname{sech}(Rt) \cos t \, dt = 2\pi \operatorname{sech}\left(\frac{\pi}{2R}\right), \tag{11.11}$$

the phase $\tilde{\theta}_{\pm}$ is determined by

$$\tilde{\theta}_{\pm}(\rho_0, \vartheta_0) = \arctan\left(\frac{\rho_0 \sin\vartheta_0}{\rho_0 \cos\vartheta_0 \pm S_\infty}\right), \tag{11.12}$$

and the asymptotic radius is

$$\tilde{\rho}_{\pm}(\rho_0, \vartheta_0) = \left(\rho_0^2 + S_\infty^2 \pm 2\rho_0 S_\infty \cos\vartheta_0\right)^{1/2}. \tag{11.13}$$

Among the heteroclinic connections there are four homoclinic ones, corresponding to the two solutions $\vartheta_0$ of

$$\cos\vartheta_0 = \mp\frac{S_\infty}{2\rho_0}. \tag{11.14}$$

We see that for the periodic orbits within the disk $\mathscr{D}_0(R) \equiv \{\rho_0 | \rho_0 < \frac{1}{2}S_\infty\}$ no homoclinic connection can exist, $\rho_0 = \frac{1}{2}S_\infty$ being the limiting case when only two homoclinic connections are possible. Since the minimum radius of the annulus is $\tilde{\rho}^2(\rho_0, \pi) = (\rho_0 - S_\infty)^2$, periodic orbits *inside* the disk $\mathscr{D}_0(R)$ can only have heteroclinic excursions, which connect to periodic orbits *outside* $\mathscr{D}_0(R)$.

The geometric interpretation of the above parametrization of the homoclinic manifolds $W_\pm$ is particularly simple when viewed in a rotating frame for the $y, z$ space; see Figure 11.1. In this frame (when the frequency of rotation is the same as that of the harmonic oscillator) the center manifold $\mathscr{M}$ is foliated by circles of equilibrium points $O^{\rho_0}$, whose two-dimensional stable and unstable manifolds $W^s_\pm(O^{\rho_0})$, $W^u_\pm(O^{\rho_0})$ are cylinders. The orbits homoclinic (in forward and backward time) to $O^{\rho_0}$ trace the cylinders, and project the circles of equilibrium points $O^{\rho_0}$ back onto the plane $\mathscr{M}$ as a displaced circle of the same radius $\rho_0$ centered at $x = S_\infty$, $z = 0$. Thus, the stable manifold $W^s_+(O^{\rho_0})$ and the unstable manifold $W^u_+(O^{\rho_0})$ intersect *transversely* in the homoclinic manifold $W_+$, and do so along a pair of orbits connecting the equilibrium circle $O^{\rho_0}$ to itself provided that $\rho_0 > \frac{1}{2}S_\infty$. Only at $\rho_0 = \frac{1}{2}S_\infty$, the intersection between the stable manifold $W^s_+(O^{\rho_0})$ and the unstable manifold $W^u_+(O^{\rho_0})$ is nontransverse (a tangency occurs here). For $\rho_0 < \frac{1}{2}S_\infty$, the stable manifold $W^s_+(O^{\rho_0})$ and the unstable manifold $W^u_+(O^{\rho_0})$ miss each other.

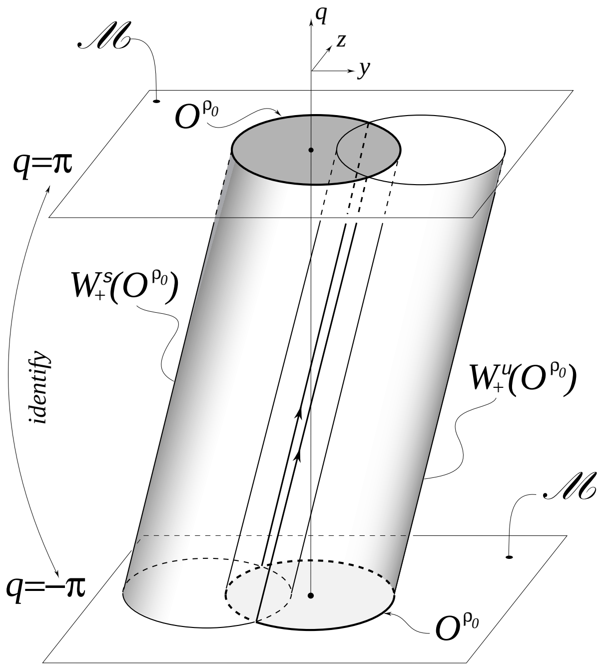


Fig. 11.1. The heteroclinic connections for the rescaled unperturbed problem (11.3) in a rotating frame, for a periodic orbit with $\rho_0 > \frac{1}{2}S_\infty$.

When the perturbation is switched on, the manifold $\mathscr{M}$ deforms into a nearby manifold $\mathscr{M}_\varepsilon$. It can be shown that level surfaces of $E$ now intersect $\mathscr{M}_\varepsilon$ transversely in the four-dimensional phase space and the intersection curves are periodic orbits $O_\varepsilon^{\rho_0}$ close to $O^{\rho_0}$.

The distance along the normal to the unperturbed homoclinic manifolds $W_\pm$ between the unstable manifold $W^u_\pm(\mathscr{M}_\varepsilon)$ and the stable manifold $W^s_\pm(\mathscr{M}_\varepsilon)$ is measured by the Melnikov function, which can be computed explicitly,

$$M_+(\rho_0, \vartheta_0, R) = -S_\infty \left(\rho_0 \cos\vartheta_0 + \tfrac{1}{2}S_\infty\right), \tag{11.15}$$

for the distance between the manifolds $W^s_+(\mathscr{M}_\varepsilon)$ and $W^u_+(\mathscr{M}_\varepsilon)$, and

$$M_-(\rho_0, \vartheta_0, R) = -S_\infty \left(-\rho_0 \cos\vartheta_0 + \tfrac{1}{2}S_\infty\right), \tag{11.16}$$

for the distance between the manifolds $W^s_-(\mathscr{M}_\varepsilon)$ and $W^u_-(\mathscr{M}_\varepsilon)$, respectively. Notice that we have simple zeros of the functions $M_\pm(\rho_0, \vartheta_0, R)$ at the parameter values for homoclinic orbits, as determined by (11.14).

Of the two-parameter ($\rho_0$ and $\vartheta_0$) family of heteroclinic connections (11.7) and (11.8) for the unperturbed problem only the *homoclinic* connections survive after perturbation. This is in accord with the fact that, as soon as $\varepsilon \neq 0$, $E$ rather than $E_0$ is a constant of motion. In particular, the stable and unstable manifolds of the disk $\mathscr{D}_0(R)$ miss each other, as the Melnikov function $M_\pm(\rho_0, \vartheta_0, R)$ are bounded away from zero if $\rho_0 < \frac{1}{2}S_\infty$. However, periodic orbits inside the disk can still be connected by multi-pulse homoclinic orbits, as an application of Theorem 2 will now show.

Let the variable $\theta$ in Theorem 2 be two-dimensional, $\theta = (\theta_1, \theta_2) = (\vartheta, \rho) \in S^1 \times \mathbb{R}$, and satisfy the equations

$$\dot{\vartheta} = 1 + \frac{R^2}{\rho} \sin q \cos \vartheta, \tag{11.17a}$$

$$\dot{\rho} = R^2 \sin q \sin \vartheta, \tag{11.17b}$$

as implied by the last two equations in (11.1) for $y = \rho \cos \vartheta$ and $z = \rho \sin \vartheta$. Let the variable $I$ be absent. By using the asymptotic values $(\vartheta_0, \rho_0)$ of $(\vartheta, \rho)$ for $t \to -\infty$, we can write the $k$-pulse Melnikov function (10.13) in the present case as

$$M_k(\varepsilon, \rho_0, \vartheta_0, R) = \sum_{j=0}^{k-1} M^{(j)}(\rho^{(j)}, \vartheta^{(j)}, R)\,, \tag{11.18}$$

where

$$M^{(j)}(\rho^{(j)}, \vartheta^{(j)}, R) = -S_\infty \left( (-1)^{s_j} \rho^{(j)} \cos \vartheta^{(j)} + \tfrac{1}{2} S_\infty \right), \tag{11.19}$$

$$\rho^{(j)} = \tilde{\rho}\Big(\rho^{(j-1)}, \vartheta^{(j-1)}, s_{j-1}\Big), \quad \vartheta^{(j)} = \mathcal{T}_j + \tilde{\theta}\Big(\rho^{(j-1)}, \vartheta^{(j-1)}, s_{j-1}\Big), \tag{11.20}$$

$$\mathcal{T}_j = \frac{1}{R} \log \left( \frac{32R}{\varepsilon | \sum_{i=0}^{j-1} M^{(i)} |} \right), \tag{11.21}$$

for all $j = 1, \ldots, k-1$, with $(\vartheta^{(0)}, \rho^{(0)}) = (\vartheta_0, \rho_0)$. (In these definitions and some of the following formulas we suppress the arguments $(\varepsilon, \rho_0, \vartheta_0, R)$ of $\mathcal{T}_j$, etc. for ease of notation.) Here we have defined the index $s_j$ to be 0 or 1 according to whether the Melnikov function is computed for the upper or lower separatrix of the pendulum, equations (11.15) and (11.16), respectively. Accordingly, the mapping $\big(\tilde{\theta}, \tilde{\rho}\big)(\cdot, \cdot, s_j)$ is defined as $\big(\tilde{\theta}, \tilde{\rho}\big)(\cdot, \cdot, s_j) = \big(\tilde{\theta}_+, \tilde{\rho}_+\big)(\cdot, \cdot)$ if $s_j = 0$ and $\big(\tilde{\theta}, \tilde{\rho}\big)(\cdot, \cdot, s_j) = \big(\tilde{\theta}_-, \tilde{\rho}_-\big)(\cdot, \cdot)$ if $s_j = 1$.

Theorem 2 now implies

**Proposition 11.1.** *If for some integer $k$, some constant $B > 0$ independent of $\varepsilon$, some $R = \bar{R}$, some $\rho = \bar{\rho}$, all $\varepsilon > 0$ sufficiently small and some function $\vartheta = \bar{\vartheta}(\varepsilon)$ the $k$-pulse Melnikov function* (11.18) *satisfies the three conditions of Theorem 2 for $j = 1, \ldots, k-1$, then the stable and unstable manifolds of the periodic orbit $O_\varepsilon^{\bar{\rho}}$ of* (11.1) *intersect transversely along a homoclinic orbit, which outside of a small neighborhood of the normally hyperbolic invariant manifold $\mathcal{M}_\varepsilon$ is $\mathcal{O}(\varepsilon)$-close to the sequence of heteroclinic orbits selected by $\{\vartheta^{(j)}, \rho^{(j)}\}$, $j = 0, 1 \ldots, k-1$.*

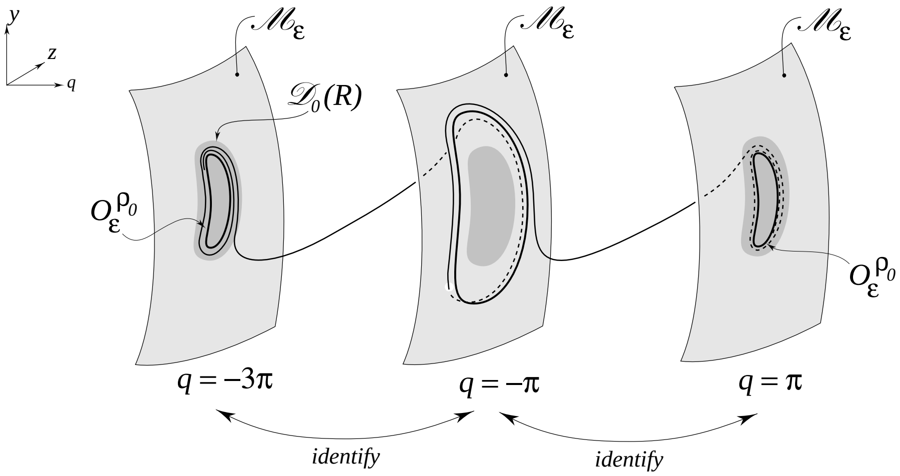


Fig. 11.2. Sketch of a 2-pulse homoclinic orbit as it leaves a periodic orbit $O_\varepsilon^{\rho_0}$ on the perturbed manifold $\mathcal{M}_\varepsilon$ inside the disk $\mathcal{D}_0(R)$, approaches a different periodic orbit flying by the manifold $\mathcal{M}_\varepsilon$, leaves the neighborhood of $\mathcal{M}_\varepsilon$ and follows an unperturbed homoclinic solution back to $O_\varepsilon^{\rho_0}$. The intersection with the manifold $\mathcal{M}_\varepsilon$ is an artifact caused by the suppression of the $p$ coordinate.

In particular, this proposition and expressions (11.18)–(11.21) for $k = 2$ show that a periodic orbit inside the disk $\mathcal{D}_0(R)$ can be connected to itself by a 2-pulse homoclinic orbit when the corresponding radius $\rho_0$ approaches $\frac{1}{2}S_\infty$ from below. This is because the 1-pulse Melnikov function is negative for $\rho_0 < \frac{1}{2}S_\infty$ and the 2-pulse Melnikov function, with $(s_0, s_1) = (0, 1)$, can be written as

$$\begin{aligned} &M_2(\varepsilon, \rho_0, \vartheta_0, R) \\ &\quad = M_+(\rho_0, \vartheta_0, R) + M_-(\tilde{\rho}_+(\rho_0, \vartheta_0), \tilde{\theta}_+(\rho_0, \vartheta_0) + \mathcal{T}_1, R), \\ &\quad = -2S_\infty\ \tilde{\rho}_+(\rho_0, \vartheta_0)\left[\sin\left(\frac{\mathcal{T}_1}{2}\right)\ \sin\left(\tilde{\theta}_+(\rho_0, \vartheta_0) + \frac{\mathcal{T}_1}{2}\right)\right]. \end{aligned} \tag{11.22}$$

Since the variation of the phase delay $\mathcal{T}_1(\varepsilon, \rho_0, \vartheta_0, R)$ between its maximum at $\vartheta_0 = \pi$ and its minimum at $\vartheta_0 = 0$ becomes unbounded as $\rho_0 \uparrow \frac{1}{2}S_\infty$, a nondegenerate zero of the right-hand side of equation (11.22) certainly occurs at some $\vartheta_0$ in $[0, 2\pi)$. Figure 11.2 provides a sketch of a 2-pulse orbit homoclinic to a periodic orbit inside the disk $\mathcal{D}_0(R)$.

We remark that Proposition 11.1 applies indifferently to periodic orbits inside or outside the disk $\mathcal{D}_0(R)$ (where the Melnikov function itself can have simple zeros). System (11.1) is reversible, i.e., if $\boldsymbol{p}(t) = (q(t), p(t), y(t), z(t))$ is a solution of system (11.1), then both

$$(\mathcal{R}_1\boldsymbol{p})(t) = (q(-t), -p(-t), y(-t), -z(-t))$$

and

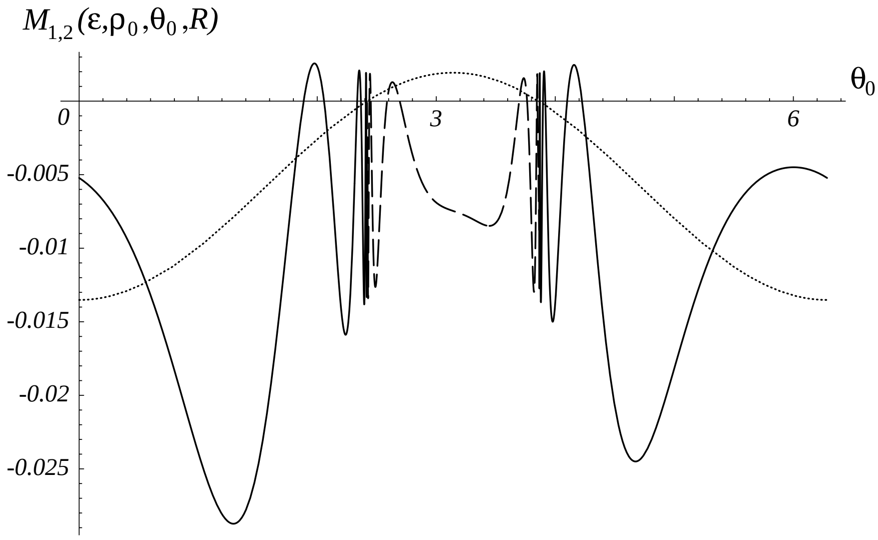


Fig. 11.3. Plot of the 1-pulse Melnikov function $M_1(\varepsilon, \rho_0, \vartheta_0, R) \equiv M_{\mathcal{C}}(\rho_0, \vartheta_0, R)$ (dotted line) and of the 2-pulse Melnikov function $M_2(\varepsilon, \rho_0, \vartheta_0, R)$ (solid and dashed lines) vs. $\vartheta_0 \in [0, 2\pi)$, with $\rho = \frac{2}{3} S_\infty$ and $R = 0.33$. The zero crossings of the solid line determine 2-pulse homoclinic orbits whose $(q, p)$ coordinates make an excursion following first the upper and then the lower separatrix of the pendulum ($+$ and $-$ sign choice, respectively, in equation (11.7)). The zero crossings of the dashed line determine 2-pulse homoclinic orbits whose $(q, p)$ coordinates follow twice the upper separatrix of the pendulum.

$$(\mathscr{R}_2 \boldsymbol{p})(t) = (-q(-t), p(-t), -y(-t), z(-t))$$

are solutions. Accordingly, some of the multi-pulse orbits whose existence is implied by the proposition possess a reversibility symmetry. We can group these orbits in two different classes with respect to their symmetry. For the first class of multi-pulse orbits, there exists a time, which can always be taken to be zero since (11.1) is autonomous, such that, if $p(0) = 0$ *and* $z(0) = 0$, then $\boldsymbol{p}(t)$ is a homoclinic orbit. Orbits in this class are discussed in detail in [6]. Because the Melnikov function $M_+(\rho_0, \vartheta_0, R)$ is negative-definite for periodic orbits inside the disk $\mathscr{D}_0(R)$, this type of symmetry is the only one possible for the 2-pulse homoclinic orbits, 2 being the minimum number of pulses required for the existence of orbits homoclinic to periodic orbits inside $\mathscr{D}_0(R)$.

The second class of multi-pulse orbits is in the fixed set of the second reversibility symmetry, that is, $\boldsymbol{p}(t)$ is such that $q(0) = 0 \bmod \pi$ *and* $y(0) = 0$. The ordinary 1-pulse homoclinic orbits are of this type. Zeros of $M_2(\varepsilon, \rho_0, \vartheta_0, R)$ with $M_+(\rho_0, \vartheta_0, R) > 0$ and $(s_0, s_1) = (0, 0)$ determine 2-pulse homoclinic orbits of the second symmetry type outside $\mathscr{D}_0(R)$. These occur along with 2-pulse homoclinic orbits of the first symmetry type, when $M_+(\rho_0, \vartheta_0, R) < 0$ and $(s_0, s_1) = (0, 1)$. Figure 11.3 shows plots of the 1- and 2-pulse Melnikov functions $M_1(\varepsilon, \rho_0, \vartheta_0, R) \equiv M_+(\rho_0, \vartheta_0, R)$ and $M_2(\varepsilon, \rho_0, \vartheta_0, R)$, as $\vartheta_0$ varies in the period $[0, 2\pi)$, when the periodic orbit determined by $\rho_0$ is outside of the disk $\mathscr{D}_0(R)$, so that the Melnikov function $M_+(\rho_0, \vartheta_0, R)$ defined by equation (11.15) has two zeros in $[0, 2\pi)$. The accumulation of zeros of $M_2(\varepsilon, \rho_0, \vartheta_0, R)$ onto the zeros of $M_1(\varepsilon, \rho_0, \vartheta_0, R)$ mentioned in the fourth remark after Theorem 1 can be clearly seen.

Looking for solutions $R = \bar{R}$, $\rho = \bar{\rho}$, $\vartheta = \bar{\vartheta}(\varepsilon)$ as in Proposition 11.1 shows that a phenomenon similar to that of the disappearance of 1-pulse homoclinic orbits when moving from periodic orbits outside the disk $\mathscr{D}_0(R)$ to orbits within this disk is repeated, with higher-pulse homoclinic orbits of a certain symmetry type, for a sequence of nested disks $\mathscr{D}_1(R, \varepsilon)$, $\mathscr{D}_2(R, \varepsilon)$, etc., whose radii depend on $\varepsilon$ and $R$. Notice that unlike the radius which defines $\mathscr{D}_0(R)$, $\rho_0 = \frac{1}{2}S_\infty$, the radii of $D_k(R, \varepsilon)$, $k > 0$, depend on $\varepsilon$ *and may all vanish* at particular values of $\varepsilon$, $\varepsilon = \varepsilon_n(R)$, $n = 1, 2, \ldots$, with $\varepsilon_n \to 0$ as $n \to \infty$. At these values of $\varepsilon$ the fixed point $(q, p, y, z) = (\pm\pi, 0, 0, 0)$ is connected to itself by a multipulse homoclinic orbit; see [6].

## 12. Application to Orbits Homoclinic to Resonance Bands

In this section, we apply Theorem 1 and its extension Theorem 2 to the phenomenon of orbits homoclinic to resonance bands. Various aspects of this phenomenon were discovered in [22, 23, 32, 34–37, 39, 75]. This phenomenon contains a bounty of homoclinic and heteroclinic orbits, and occurs if, in addition to Assumptions 1 and 2, we also make

**Assumption 4.** *For some $I = I_0$ with $I_1 < I_0 < I_2$, the frequency $\omega(I) = \Omega(X(I), I)$ passes through a simple zero, that is,*

$$\omega(I_0) = 0, \quad \frac{d\omega}{dI}(I_0) \neq 0.$$

This phenomenon also occurs if we make Assumption 4 in conjunction with setting $k = 2$ and $X_0(I) = X_1(I) = X_2(I)$ in Assumption 3. In this second case, we have two not necessarily symmetric manifolds of orbits homoclinic to the annulus $\mathscr{M}$. In what is to follow, for the sake of definiteness we only consider the first of these two situations, with the understanding that the results for the second are almost identical.

Assumption 4 implies that one of the periodic orbits on the annulus $\mathscr{M}$ is really not a periodic orbit but a circle of equilibria. One can easily see from equation (2.8) that pairs of equilibria on this circle that are a distance

$$\Delta\theta(I_0) = \int_{-\infty}^{\infty} \Omega(x^h(s, I_0), I_0)\, ds \tag{12.1}$$

apart, are connected to each other by heteroclinic orbits parametrized by equations (2.6) with $I = I_0$ (see Figure 12.1).

The circle of equilibria that exists on the unperturbed annulus $\mathscr{M}$ breaks up under perturbation into a resonance band that lies on the perturbed annulus $\mathscr{M}_\varepsilon$. This resonance band is described as follows. We first restrict the dynamics of equations (2.1) to the annulus $\mathscr{M}_\varepsilon$ using formula (3.1). Following a standard procedure,

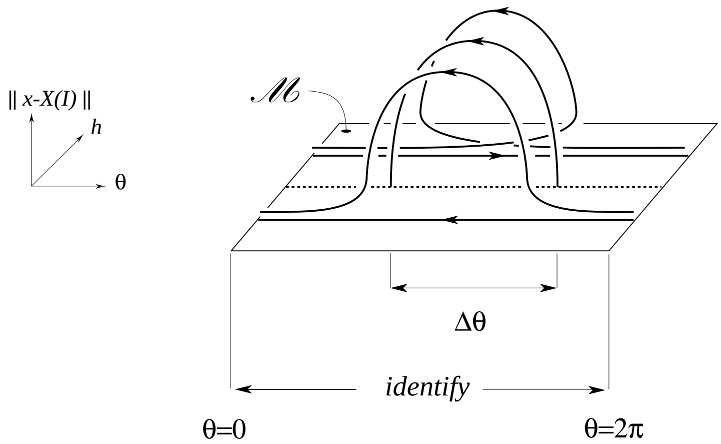


Fig. 12.1. Geometry of manifolds homoclinic to periodic orbits and the circle of equilibria at $I = I_0$. Only one orbit is shown from each such manifold. All the other orbits on the same homoclinic manifold are obtained by translating those shown in the picture along the $\theta$-axis. Orbits on the manifold homoclinic to the circle of equilibria at $I = I_0$ are heteroclinic orbits connecting pairs of points on that circle that are $\Delta\theta$ apart.

described for instance in [19], we "blow up" the region near $I = I_0$ using the transformation $I = I_0 + \sqrt{\varepsilon}\, h$, rescale time using $\tau = \sqrt{\varepsilon} t$, and Taylor expand the restricted equations (2.1) in $\sqrt{\varepsilon}$, to obtain the equations

$$h' = g^I(X(I_0), I_0, \theta, \mu) + \mathscr{O}(\sqrt{\varepsilon}), \quad \theta' = \frac{d\Omega}{dI}(X(I_0), I_0)\ h + \mathscr{O}(\sqrt{\varepsilon}), \quad (12.2)$$

with $' = \frac{d}{d\tau}$. Higher-order terms in these equations can be computed from the Taylor expansion of formula (3.1) in powers of $\sqrt{\varepsilon}$, which can be obtained in terms of algebraic operations and differentiations alone, as shown in [37].

In the limit as $\varepsilon \to 0$, we obtain the *outer system*

$$h' = g^I(X(I_0), I_0, \theta, \mu), \quad \theta' = \frac{d\Omega}{dI}(X(I_0), I_0)\ h, \quad (12.3)$$

which can be derived from the *outer Hamiltonian*

$$\mathscr{H}(h, \theta, \mu) = \frac{1}{2}\frac{d\Omega}{dI}(X(I_0), I_0)\, h^2 + V(\theta, \mu), \quad (12.4)$$

with the potential

$$V(\theta, \mu) = -\int_0^\theta g^I(X(I_0), I_0, s, \mu)ds,$$

via the canonical formulas

$$h' = -D_\theta \mathscr{H}(h, \theta, \mu), \quad \theta' = D_h \mathscr{H}(h, \theta, \mu).$$

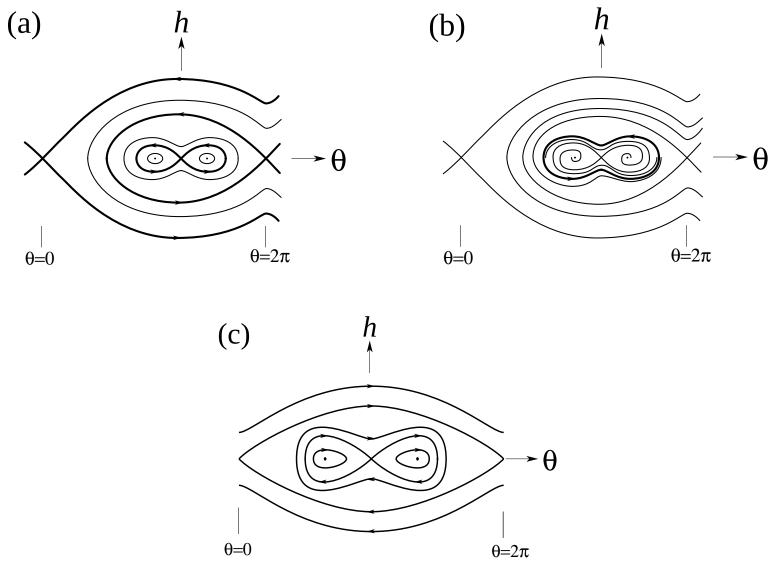


Fig. 12.2. Typical phase portraits of the outer systems (12.2) and (12.3) for $\varepsilon = 0$ and $\varepsilon > 0$, respectively. Figures 12.2a (for $\varepsilon = 0$) and 12.2b (for $\varepsilon = 0$) display the dissipative case, and Figure 12.2c displays the Hamiltonian case (for both $\varepsilon = 0$ and $\varepsilon > 0$). All the points whose $\theta$ coordinates differ by a multiple of $2\pi$ must be identified.

In the case when the perturbed vector field (2.1) is derived from the Hamiltonian (2.2), that is, when equations (2.1) are replaced by equations (2.3), the outer Hamiltonian becomes

$$\mathscr{H}(h, \theta, \mu) = \frac{1}{2}\,\frac{dD_I H}{dI}(X(I_0), I_0)\, h^2 + H_1(X(I_0), I_0, \theta, \mu, 0),$$

and the expression

$$\begin{aligned}\mathscr{H}_\varepsilon(h, \theta, \mu, \varepsilon) &= \frac{\hat{H}|_{\mathscr{M}_\varepsilon}(I_0 + \sqrt{\varepsilon}h, \theta, \mu, \varepsilon) - H(X(I_0), I_0)}{\varepsilon} \\ &= \mathscr{H}(h, \theta, \mu) + \mathscr{O}(\sqrt{\varepsilon}),\end{aligned} \tag{12.5}$$

where $\hat{H}|_{\mathscr{M}_\varepsilon}(I_0 + \sqrt{\varepsilon}h, \theta, \mu, \varepsilon)$ is the restriction of the Hamiltonian (2.2) to the annulus $\mathscr{M}_\varepsilon$ via the formula (3.1), is a conserved quantity which reduces smoothly to the outer Hamiltonian as $\varepsilon \to 0$. System (12.2) can be investigated with the help of system (12.3) by a combination of phase-plane and perturbation techniques, as described for instance in [37] and [36] (see Figure 12.2).

In order to investigate the phase-space structure off of the annulus $\mathscr{M}_\varepsilon$ in the full $(x, I, \theta)$ phase space, in particular, orbits homoclinic or heteroclinic to possible equilibria and periodic orbits of equations (12.2), we set $I = I_0 + \sqrt{\varepsilon}h$ in equations (2.1) and let $\varepsilon \to 0$. In this way, we obtain the *inner system*

$$\dot{x} = JD_x H(x, I_0), \tag{12.6a}$$

$$\dot{h} = 0, \tag{12.6b}$$

$$\dot{\theta} = \Omega\,(x, I_0). \tag{12.6c}$$

In this system, the structure of every slice $h =$ constant is the same as the structure of the slice $I = I_0$ in the unperturbed system (2.4). In particular, the circle of equilibria in the phase space of the unperturbed equations (2.4) at $I = I_0$, $x = X(I_0)$, $0 \leqq \theta \leqq 2\pi$ has been "blown up" to cover a whole cylinder $\mathscr{M}_0$ of equilibria, with $x = X(I_0)$, $0 \leqq \theta \leqq 2\pi$ and arbitrary $h$ in the phase space of the inner equations (12.6). Moreover, pairs of points on this cylinder with equal $h$ coordinates and $\theta$ coordinates a distance $\Delta\theta(I_0)$ apart are connected by heteroclinic orbits given by equations (2.6) with $I = I_0$. The cylinder $\mathscr{M}_0$ of equilibria is thus connected to itself by a three-dimensional manifold of heteroclinic orbits. As in [22, 23, 32, 34–37, 39], we combine the information obtained from systems (12.6) and (12.3) to obtain information about orbits homoclinic to the resonance band on the perturbed annulus $\mathscr{M}_\varepsilon$ at $I = I_0$ in the phase space of system (2.1).

We can use Theorem 1 in the original $(x, I, \theta)$ coordinates to ascertain the existence of a possible surviving $k$-pulse homoclinic intersection surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$. Note that in this case, for any integer $k$, the $k$-pulse Melnikov function at the resonance $I = I_0$ is equal to

$$M_k(I_0, \theta_0, \mu) = \sum_{j=0}^{k-1} M(I_0, \theta_0 + j\Delta\theta(I_0), \mu), \tag{12.7}$$

where the Melnikov function $M(I_0, \theta_0, \mu)$ is given by formula (3.2) with $I = I_0$, and the angle difference $\Delta\theta(I_0)$ is given by formula (12.1). Moreover, one can show along the same lines as in [36] or [22] that in the case when the perturbed vector field (2.1) is derived from the Hamiltonian (2.2), we have

$$\begin{aligned} M_k(I_0, \theta_0, \mu) = {} & H_1(X(I_0), I, \theta_0 - \Delta\theta_-(I_0), \mu, 0) \\ & - H_1(X(I_0), I, \theta_0 + (k-1)\Delta\theta(I_0) + \Delta\theta_+(I_0), \mu, 0). \end{aligned} \tag{12.8}$$

Notice that the $k$-pulse Melnikov function (12.7) in the particular case of orbits homoclinic to resonance bands does not depend on $\varepsilon$. This is because, at most a distance $\mathscr{O}(\sqrt{\varepsilon})$ away from $I = I_0$, the angle increment $\mathscr{T}_k(\varepsilon, I, \theta_0, \mu)$, given by formula (3.6), is of the size $\mathscr{O}\left(\sqrt{\varepsilon}\log(1/\varepsilon)\right)$, and $\mathscr{T}_k(\varepsilon, I_0, \theta_0, \mu) = 0$. It is also clear that the nonfolding condition (3.9) is always fulfilled at $I = I_0$ and also for $I$-values that are $\mathscr{O}(\sqrt{\varepsilon})$ distance away because of Assumption 4, and can therefore be dropped from the hypotheses of Theorem 1. Theorem 1 thus becomes

**Proposition 12.1.** *For some integer $k$, $\theta_0 = \bar{\theta}_0$, and $\mu = \bar{\mu}$ let the following conditions be satisfied:*

1. *The $k$-pulse Melnikov function has a simple zero in $\theta_0$, that is,*

$$M_k(I_0, \bar{\theta}_0, \bar{\mu}) = 0, \quad D_{\theta_0} M_k(I_0, \bar{\theta}_0, \bar{\mu}) \neq 0.$$

2. *$M_i(I_0, \bar{\theta}_0, \bar{\mu}) \neq 0$ for all $i = 1, \ldots, k-1$, $k > 1$, and is positive if the signature $\sigma$, defined by equation* (3.4)*, of the normal* ***n*** *is positive, and negative if $\sigma$ is negative.*

*Then for all $I$ close to $I_0$, and all $\mu$ close to $\bar{\mu}$, there exists a two-dimensional intersection surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ along which the stable and unstable manifolds $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$ of the annulus $\mathscr{M}_\varepsilon$ intersect transversely at an angle of size $\mathscr{O}(\varepsilon)$. In the outer $(x, h, \theta)$ variables, outside of a small neighborhood of the annulus $\mathscr{M}_\varepsilon$, the surface $\Sigma_\varepsilon^\mu(\bar{\theta}_0)$ collapses smoothly onto the union $\Sigma_0^\mu(\bar{\theta}_0)$ of surfaces spanned by the orbits parametrized by formulas* (2.6) *with $I = I_0$, $\theta_0 = \bar{\theta}_0(\mu) + j\Delta\theta(I_0)$, $j = 0, \ldots, k-1$, and with arbitrary $h$. Here $\bar{\theta}_0(\mu)$ is the corresponding simple zero of the $k$-pulse Melnikov function $M_k(I_0, \theta_0, \mu)$. The surface $\Sigma_0^\mu(\bar{\theta}_0)$ takes off from the cylinder $\mathscr{M}_0$ along the line $\theta = \bar{\theta}_0(\mu) - \Delta\theta_-(I_0)$ and eventually lands back on it along the line $\theta = \bar{\theta}_0(\mu) + (k-1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$, where the phase differences $\Delta\theta_-(I_0)$ and $\Delta\theta_+(I_0)$ are defined as*

$$\Delta\theta_+(I_0) = \int_0^\infty \Omega(x^h(s, I_0), I_0)\, ds, \quad \Delta\theta_-(I_0) = \int_{-\infty}^0 \Omega(x^h(s, I_0), I_0)\, ds. \tag{12.9}$$

Recall that the signature $\sigma$ need not be computed if the region enclosed by the unperturbed homoclinic manifold $W(\mathscr{M}_\varepsilon)$ is convex, as explained in the first remark after Theorem 1. Recall also that the definition of the angle differences $\Delta\theta_+(I_0)$ and $\Delta\theta_-(I_0)$ is consistent with the analogous definitions in Section 8 given by formulas (8.15) and (8.19). We call the limiting (nonsmooth) surface $\Sigma_0^\mu(\bar{\theta}_0)$ a *singular homoclinic intersection surface*; see Figure 12.3. We remark that Proposition 12.1 is equivalent to the result obtained in [23]. We combine Proposition 12.1 with the results of [32] to further enlarge the class of orbits homoclinic to resonance bands that can be constructed with the $k$-pulse Melnikov method.

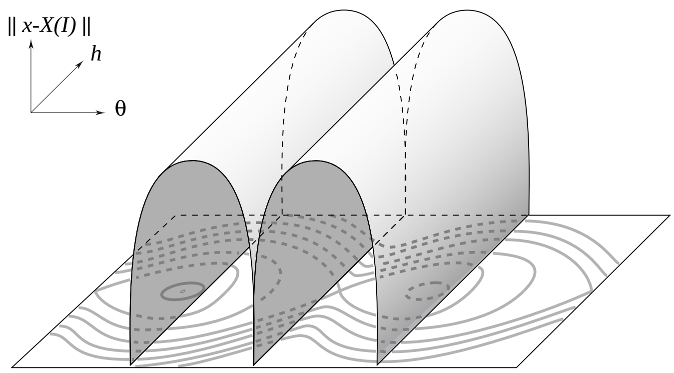


Fig. 12.3. The singular intersection surface with two pulses $\Sigma_0^\mu(\bar{\theta}_0)$ connects equilibria that lie on the line $\theta = \bar{\theta}_0 - \Delta\theta_-(I_0)$ to those that lie on the line $\theta = \bar{\theta}_0 + 2\Delta\theta(I_0) + \Delta\theta_C(I_0)$ on the annulus $\mathscr{M}_\varepsilon$. Gray curves on $\mathscr{M}_\varepsilon$ represent the orbit structure on this annulus under the outer system (12.3).

Gronwall-type estimates, a rescaled version of the estimates given in Proposition 4.2, and KAPLUN's extension theorem [40] now imply

**Proposition 12.2.** *Let $\boldsymbol{q}(t)$ be an orbit on the homoclinic intersection surface $\Sigma_\varepsilon^\mu(\bar\theta_0)$. Let $\Gamma$ be the union of the orbits* (2.6) *with $I = I_0$, $\theta = \bar\theta_0(\mu) + j\Delta\theta(I_0)$, for $j = 0, \ldots, k-1$, and $h = h_0$, the $h$ coordinate of the point $\boldsymbol{q}(0)$. Then there exists a function $\delta(\varepsilon)$ such that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and that, in the $(x, h, \theta)$ coordinates, the orbit $\boldsymbol{q}(t)$ stays $\mathcal{O}(\delta(\varepsilon))$-close to the union $\Gamma$ from the time the orbit $\boldsymbol{q}(t)$ first leaves a small neighborhood of the annulus $\mathcal{M}_\varepsilon$ until the time it last returns to it.*

As we describe below, this estimate immediately allows us to conclude that all the theorems of references [37, 36, 32] remain valid if we substitute a multi-pulse singular homoclinic intersection surface $\Sigma_0^\mu(\bar\theta_0)$ as discussed above for any 1-pulse singular homoclinic intersection surface as discussed in those references. In particular, we can show the existence of orbits homoclinic to resonance bands that have several groups of consecutive pulses interspersed with slow stretches near the annulus $\mathcal{M}_\varepsilon$. We call such groups of pulses *bumps*.

In order to describe multi-bump orbits homoclinic to resonance bands, we must first make two assumptions. The first is

**Assumption 5.** *There exists a positive integer $N$, and $N$ positive integers $K_i$, with $i = 1, \ldots, N$, such that the $K_i$-pulse Melnikov functions $M_{K_i}(I_0, \theta_0, \mu)$ at the resonance have simple zeros at the points $\theta_0 = \bar\theta_{0,i}$, for some $\mu = \bar\mu$. That is, $M_{K_i}(I_0, \bar\theta_{0,i}, \bar\mu) = 0$ and $D_{\theta_0} M_{K_i}(I_0, \bar\theta_{0,i}, \bar\mu) \neq 0$ for each $i = 1, \ldots, N$.*

Under this assumption, Theorem 1 implies that there exist $N$ individual, not necessarily disjoint, $K_i$-pulse homoclinic intersection surfaces $\Sigma_\varepsilon^\mu(\bar\theta_{0,i})$, $i = 1, \ldots, N$, for all $\mu$ close to $\bar\mu$ that limit onto the surfaces $\Sigma_0^\mu(\bar\theta_{0,i})$ as $\varepsilon \to 0$. The second assumption is

**Assumption 6.** *There exist $N-1$ orbit segments $O_i(\bar\mu)$, $i = 2, \ldots, N$, on the annulus $\mathcal{M}_0$ with endpoints $d_i(\bar\mu)$ and $c_i(\bar\mu)$, respectively. The trajectories of the outer system* (12.3) *on the segment $O_i(\bar\mu)$ flow from $d_i(\bar\mu)$ to $c_i(\bar\mu)$ in forward time. Moreover, one of the following two statements holds:*

1. *The line $\theta = \bar\theta_{0,i}(\bar\mu) - \Delta\theta_-(I_0)$ intersects the segment $O_i(\bar\mu)$ transversely at the point $c_i(\bar\mu)$ for $i = 2, \ldots, N$, and the line $\theta = \bar\theta_{0,i}(\bar\mu) + (K_i - 1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$ intersects the segment $O_{i+1}(\bar\mu)$ transversely at the point $d_{i+1}(\bar\mu)$ for $i = 1, \ldots, N-1$.*
2. *The line $\theta = \bar\theta_{0,i}(\bar\mu) - \Delta\theta_-(I_0)$ intersects the segment $O_i(\bar\mu)$ transversely at the point $c_i(\bar\mu)$ for $i = 2, \ldots, N$, and the line $\theta = \bar\theta_{0,i}(\bar\mu) + (K_i - 1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$ intersects the segment $O_{i+1}(\bar\mu)$ transversely at the point $d_{i+1}(\bar\mu)$ for $i = 2, \ldots, N-1$. The $h$ coordinate of the point $d_2(\bar\mu)$ is equal to zero.*

*Finally, for all $i = 2, \ldots, N-1$, the difference in the $h$ coordinates of the two points $c_i(\bar\mu)$ and $d_{i+1}(\bar\mu)$ is equal to zero.*

Note that the difference in the $h$ coordinates of the two points $c_i(\bar\mu)$ and $d_{i+1}(\bar\mu)$ can be computed from equation (12.4), and that the $h$ coordinates of the points $c_i(\bar\mu)$

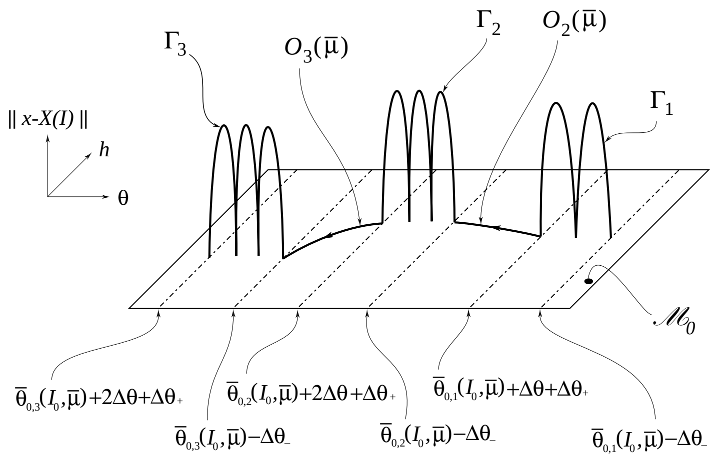


Fig. 12.4. An $N$-transition orbit in the case $N = 3$ (the argument $I_0$ of the functions $\Delta\theta(I_0)$, $\Delta\theta_{\pm}(I_0)$ has been suppressed in the figure). This "orbit" is composed of three heteroclinic orbits $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ of the inner system (12.6), connected by two orbit segments $O_1(\bar{\mu})$ and $O_2(\bar{\mu})$ of the outer system (12.3). The time scales of the inner and outer systems are incompatible.

and $d_i(\bar{\mu})$ cannot be equal to zero when the intersections of the orbit segments $O_i(\bar{\mu})$ with the lines $\theta = \bar{\theta}_{0,i}(I_0, \bar{\mu}) - \Delta\theta_-(I_0)$ and $\theta = \bar{\theta}_{0,i}(I_0, \bar{\mu}) + (K_i - 1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$ are transverse.

Assumption 6 implies that, for each $i = 2, \dots, N-1$, a string of $K_i$ heteroclinic orbits, which we label $\Gamma_i$, contained in the surface $\Sigma_0^{\mu}(\bar{\theta}_{0,i})$ at $\mu = \bar{\mu}$, connects the two intersection points $c_i(\bar{\mu})$ and $d_{i+1}(\bar{\mu})$. Moreover, a string $\Gamma_1$ on the surface $\Sigma_0^{\mu}(\bar{\theta}_{0,1})$ consisting of $K_1$ heteroclinic orbits connects some point $c_1(\bar{\mu})$ on the annulus $\mathscr{M}_0$ to the point $d_2(\bar{\mu})$ on the segment $O_2(\bar{\mu})$, and a string $\Gamma_N$ on the surface $\Sigma_0^{\mu}(\bar{\theta}_{0,N})$ consisting of $K_N$ heteroclinic orbits connects the point $c_N(\bar{\mu})$ on the segment $O_N(\bar{\mu})$ to some point $d_{N+1}(\bar{\mu})$ on the annulus $\mathscr{M}_0$. In this case, we say that there exists an *N-bump singular transition orbit* or a *modified N-bump singular transition orbit*, depending on whether statement 1 or statement 2 in Assumption 6 holds. This $N$-bump singular transition orbit, denoted by $\Gamma$, is a continuous broken curve, which consists of the heteroclinic strings $\Gamma_i$ on the surfaces $\Sigma_0^{\mu}(\bar{\theta}_{0,i})$ for $\mu = \bar{\mu}$ that connect the points $c_i(\bar{\mu})$ and $d_{i+1}(\bar{\mu})$, for each $i = 1, \dots, N$ and the parts of the orbit segments $O_i(\bar{\mu})$ between $d_i(\bar{\mu})$ and $c_i(\bar{\mu})$, for each $i = 2, \dots, N$ (see Figure 12.4). We call the points $c_1(\bar{\mu})$ and $d_{N+1}(\bar{\mu})$ the takeoff and landing points, respectively, of the singular orbit $\Gamma$.

One can easily verify (see [32]) that an $N$-bump singular transition orbit implies the existence of a continuous, two-parameter family of such orbits, the parameters being $h$ and $\mu$. Likewise, a modified $N$-bump singular transition orbit implies the existence of a continuous, one-parameter family of such orbits, the parameter being $\mu$. In other words, a unique $N$-bump modified singular transition orbit $\Gamma(\mu)$ exists at most a distance $\mathscr{O}(\mu - \bar{\mu})$ away from the original $N$-bump modified singular transition orbit $\Gamma$ for any $\mu$ close enough to $\bar{\mu}$.

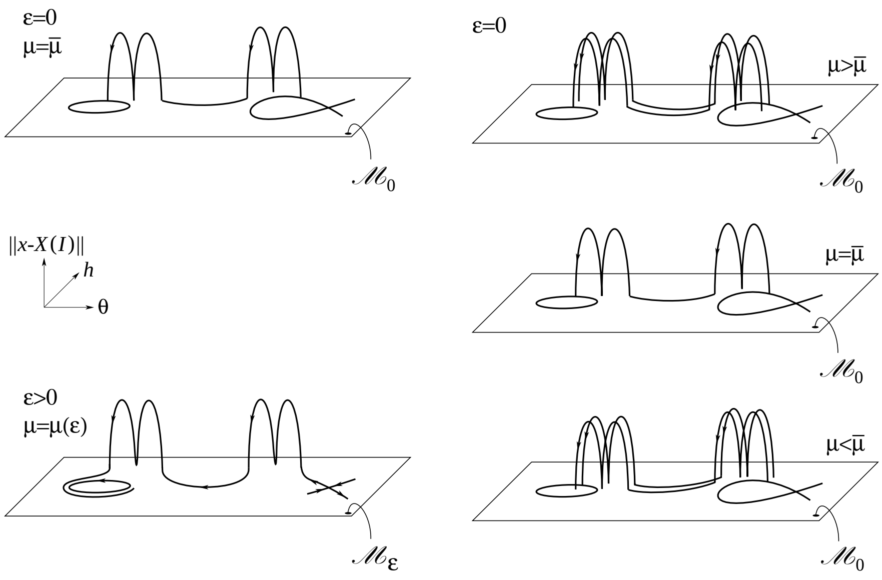


Fig. 12.5. (a) An illustration of Proposition 12.3, showing a two-bump connection between a saddle and an unstable limit cycle on the annulus $\mathscr{M}_\varepsilon$. The first and second bump consist of two pulses. (b) The singular transition orbits $\mathscr{A}_{1,0}(\mu)$ and $\mathscr{A}_{NC1,0}(\mu)$ pass through each other transversely as $\mu$ passes through $\bar{\mu}$.

We are now ready to state four propositions that extend the results of [32] to cover homoclinic and heteroclinic orbits whose bumps are groups of consecutive pulses, rather than single pulses. First, we consider the existence of a heteroclinic connection between either a stable periodic orbit or a saddle of the outer system to either an unstable periodic orbit or another saddle of the outer system.

**Proposition 12.3.** *For $\mu$ near $\mu = \bar{\mu}$, let the curve $O_{1,\varepsilon}(\mu)$ be either a stable periodic orbit for the restricted system* (12.2) *on the annulus $\mathscr{M}_\varepsilon$ or a* (*restricted*) *unstable manifold of a saddle $s_{1,\varepsilon}(\mu)$ for this system, and let the curve $O_{N+1,\varepsilon}(\mu)$ be either an unstable periodic orbit for the restricted system* (12.2) *on $\mathscr{M}_\varepsilon$ or a* (*restricted*) *stable manifold of a saddle $s_{N+1,\varepsilon}(\mu)$ for this system. Let the line $\theta = \bar{\theta}_{0,1}(\mu) - \Delta\theta_-(I_0)$ intersect the curve $O_{1,0}(\mu)$ transversely at the point $c_1(\mu)$, and let the line $\theta = \bar{\theta}_{0,N}(\mu) + (K_N - 1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$ intersect the curve $O_{N+1,0}(\mu)$ transversely at the point $d_{N+1}(\mu)$. At $\mu = \bar{\mu}$, let the points $c_1(\bar{\mu})$ and $d_{N+1}(\bar{\mu})$ be connected by a singular transition orbit $\Gamma$. For $\mu$ near $\mu = \bar{\mu}$, let $\mathscr{A}_{1,0}(\mu)$ be the singular transition orbit whose takeoff point is $c_1(\mu)$, and let $\mathscr{A}_{N+1,0}(\mu)$ be the singular transition orbit whose landing point is $d_{N+1}(\mu)$. Furthermore, assume that the singular transition orbits $\mathscr{A}_{1,0}(\mu)$ and $\mathscr{A}_{N+1,0}(\mu)$ pass through each other (and $\Gamma$) transversely as $\mu$ passes through $\bar{\mu}$. Then, for all small enough $\varepsilon$ and for some $\mu = \mu(\varepsilon)$ with $\mu(0) = \bar{\mu}$, there exists a heteroclinic orbit connecting either the periodic orbit $O_{1,\varepsilon}(\mu(\varepsilon))$ or the saddle $s_{1,\varepsilon}(\mu)$ to either the periodic orbit $O_{N+1,\varepsilon}(\mu(\varepsilon))$ or the saddle $s_{N+1,\varepsilon}(\mu)$. (See Figure 12.5.)*

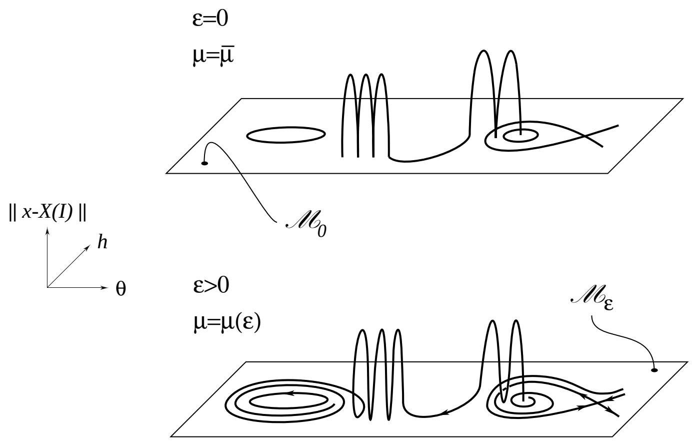


Fig. 12.6. An illustration of Proposition 12.4, showing a two-bump connection between a sink and a stable limit cycle on the annulus $\mathscr{M}_\varepsilon$.

Next, we describe heteroclinic orbits between either two equilibria that are sinks for the restricted system (12.2), or between a sink and a stable periodic orbit for that system; see Figure 12.6.

**Proposition 12.4.** *Let $c_{1,0}(\mu)$ be a center for the outer system* (12.3)*, and at $\mu = \bar{\mu}$ let it be located at*

$$\big(h(c_{1,0}(\bar{\mu})), \theta(c_{1,0}(\bar{\mu}))\big) = \big(0, \bar{\theta}_{0,1}(\bar{\mu}) - \Delta\theta_-(I_0)\big),$$

*with*

$$\frac{d}{d\mu}\left[\theta(c_{1,0}(\mu)) - \bar{\theta}_{0,1}(\mu) + \Delta\theta_-(I_0)\right] \neq 0$$

*at $\mu = \bar{\mu}$. Let the corresponding perturbed equilibrium $c_{1,\varepsilon}(\mu)$ be a sink for the restricted system* (12.2) *for all small enough $\varepsilon$ and all $\mu$ close enough to $\mu = \bar{\mu}$. Let the restricted system* (12.2) *on the annulus $\mathscr{M}_\varepsilon$ possess either another sink $c_\varepsilon(\mu)$ or a stable limit cycle $O_\varepsilon(\mu)$, and denote the associated basin of attraction by $\mathscr{B}_\varepsilon$. Moreover, let there exist a modified singular transition orbit connecting the point $c_{1,0}(\bar{\mu})$ to a point $d_{N+1}(\bar{\mu})$, and assume that the point $d_{N+1}(\bar{\mu})$ lies in a compact domain $\mathscr{R}$ that is all contained in an open region $\mathscr{B}$, the limit as $\varepsilon \to 0$ of the basin of attraction $\mathscr{B}_\varepsilon$. Then, for small $\varepsilon > 0$, there exists a function $\mu = \mu(\varepsilon)$ with $\mu(0) = \bar{\mu}$ and a heteroclinic orbit connecting the equilibrium $c_{1,\varepsilon}(\mu(\varepsilon))$ to either the equilibrium $c_\varepsilon(\mu(\varepsilon))$, or the periodic orbit $O_\varepsilon(\mu(\varepsilon))$.*

Next, we find heteroclinic connections between a saddle or a stable limit cycle on the annulus $\mathscr{M}_\varepsilon$ and a sink or another stable limit cycle on $\mathscr{M}_\varepsilon$, such as the one shown in Figure 12.7.

**Proposition 12.5.** *Let the curve $O_{1,\varepsilon}(\mu)$ be either a stable periodic orbit on the annulus $\mathscr{M}_\varepsilon$, or the* (*restricted*) *unstable manifold of a saddle $s_\varepsilon(\mu)$ on the annulus $\mathscr{M}_\varepsilon$ for all $\mu$ near $\mu = \bar{\mu}$, and all small enough positive $\varepsilon$. Let the restricted system*

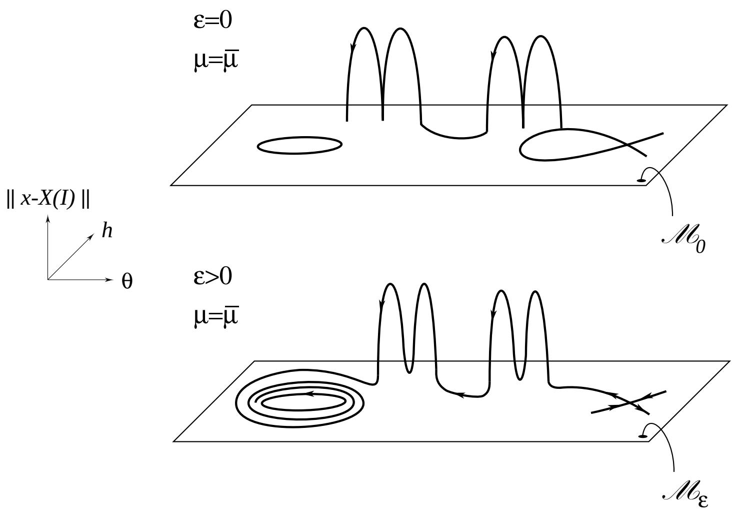


Fig. 12.7. An illustration of Proposition 12.5, showing a two-bump connection between a saddle and a stable limit cycle on the annulus $\mathscr{M}_\varepsilon$.

(12.2) *on the annulus $\mathscr{M}_\varepsilon$ also possess either a sink $c_\varepsilon(\mu)$ or a stable limit cycle $O_\varepsilon(\mu)$, and denote the associated basin of attraction by $\mathscr{B}_\varepsilon$. At $\mu = \bar{\mu}$, let the curve $O_{1,0}(\bar{\mu})$ and the line $\theta = \bar{\theta}_0(\mu) - \Delta\theta_-(I_0)$ intersect transversely at the point $c_{1,0}(\bar{\mu})$. Let the point $c_{1,0}(\bar{\mu})$ be connected via a singular transition orbit $\Gamma$ to the point $d_{N+1,0}(\bar{\mu})$. Moreover, let the point $d_{N+1,0}(\bar{\mu})$ lie in a compact domain $\mathscr{R}$ that is all contained in an open region $\mathscr{B}$, the limit as $\varepsilon \to 0$ of the basin of attraction $\mathscr{B}_\varepsilon$. Then, for all small enough positive $\varepsilon$ and all $\mu$ close enough to $\mu = \bar{\mu}$, there exists a heteroclinic orbit connecting either the periodic orbit $O_{1,\varepsilon}(\mu)$ or the saddle $s_\varepsilon(\mu)$ to either the equilibrium $c_\varepsilon(\mu)$ or the periodic orbit $O_\varepsilon(\mu)$. Moreover, the intersection of the unstable manifolds $W^u(O_{1,\varepsilon}(\mu))$ or $W^u(s_\varepsilon(\mu))$ with the stable manifolds $W^s(c_\varepsilon(\mu))$ or $W^s(O_\varepsilon(\mu))$ is transverse along that heteroclinic orbit.*

We note that, as in [37, 32], we can describe still other heteroclinic connections by inverting the time flow in Propositions 12.4 and 12.5.

Finally, we look at the case when the perturbed vector field (2.1) is derived from the Hamiltonian (2.2), i.e., when equations (2.1) are replaced by equations (2.3). In this case, an easy argument given in [32] shows that the end points of each $N$-bump singular transition orbit must be at the same values of the outer Hamiltonian $\mathscr{H}(h, \theta, \mu)$. Furthermore, if $\Gamma$ is a singular transition orbit that connects a pair of slow-time periodic orbits $O_{1,0}(\bar{\mu})$ and $O_{N+1,0}(\bar{\mu})$, then there exists a pair of two-parameter families,

$$\left\{ O_{1,0}^{\nu}(\mu) \mid \nu_1(\mu) < \nu < \nu_2(\mu), \mu \text{ near } \bar{\mu} \right\},$$

$$\left\{ O_{N+1,0}^{\nu}(\mu) \mid \nu_1(\mu) < \nu < \nu_2(\mu), \mu \text{ near } \bar{\mu} \right\},$$

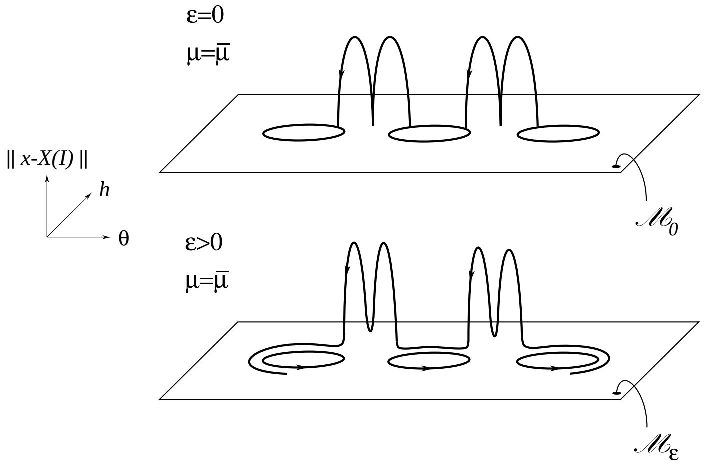


Fig. 12.8. An illustration of Proposition 12.6, showing a three-bump connection between two periodic orbits on the annulus $\mathscr{M}_\varepsilon$ in the Hamiltonian case.

of slow-time periodic orbits near $O_{1,0}(\bar{\mu})$ and $O_{N+1,0}(\bar{\mu})$, such that each pair $O^\nu_{1,0}(\mu)$ and $O^\nu_{N+1,0}(\mu)$ is connected by an $N$-bump singular transition orbit $\Gamma^\nu(\mu)$. Here the parameter $\nu$ is the value of the outer Hamiltonian $\mathscr{H}(h, \theta, \mu)$ along the orbits $O^\nu_{1,0}(\mu)$ and $O^\nu_{N+1,0}(\mu)$.

For small positive $\varepsilon$, we now conclude:

**Proposition 12.6.** *At $\mu = \bar{\mu}$, assume that the line $\theta = \bar{\theta}_{0,1}(\bar{\mu}) - \Delta\theta_-(I_0)$ transversely intersects a periodic orbit $O_{1,0}(\bar{\mu})$ at the point $c_{1,0}(\bar{\mu})$. Assume also that the line $\theta = \bar{\theta}_{0,N}(\bar{\mu}) + (K_N - 1)\Delta\theta(I_0) + \Delta\theta_+(I_0)$ transversely intersects a periodic orbit $O_{N+1,0}(\bar{\mu})$ at the point $d_{N+1,0}(\bar{\mu})$. Furthermore, assume that the points $c_{1,0}(\bar{\mu})$ and $d_{N+1,0}(\bar{\mu})$ are connected to each other by a singular transition orbit $\Gamma$. Then for each $\mu$ near $\bar{\mu}$, there exist a function $\eta(\varepsilon)$, with $\eta(\varepsilon) \to 0$ with $\varepsilon \to 0$, and two continuous families of periodic orbits,*

$$\left\{O^\nu_{1,\varepsilon}(\mu) \mid \nu_1(\mu) + \eta(\varepsilon) < \nu < \nu_2(\mu) - \eta(\varepsilon), \mu \textit{ near } \bar{\mu}\right\},$$

$$\left\{O^\nu_{N+1,\varepsilon}(\mu) \mid \nu_1(\mu) + \eta(\varepsilon) < \nu < \nu_2(\mu) - \eta(\varepsilon), \mu \textit{ near } \bar{\mu}\right\},$$

*such that each pair $O^\nu_{1,\varepsilon}(\mu)$ and $O^\nu_{N+1,\varepsilon}(\mu)$ is connected by an $N$-bump heteroclinic orbit. The value of the parameter $\nu$ along the orbits $O^\nu_{1,\varepsilon}(\mu)$ and $O^\nu_{N+1,\varepsilon}(\mu)$ is equal to the value of the constant $\mathscr{H}_\varepsilon(h, \theta, \mu)$, given by equation* (12.5)*, along these orbits, and also to the value of the outer Hamiltonian $\mathscr{H}(h, \theta, \mu)$ along the orbits $O^\nu_{1,0}(\mu)$ and $O^\nu_{N+1,0}(\mu)$. As $\varepsilon \to 0$, the union of the orbits $O^\nu_{1,\varepsilon}(\mu)$ and $O^\nu_{N+1,\varepsilon}(\mu)$ and the connecting heteroclinic orbit collapses onto the union of the curves $O^\nu_{1,0}(\mu)$ and $O^\nu_{N+1,0}(\mu)$ and the singular transition orbit $\Gamma^\nu(\mu)$. Moreover the intersection of the unstable manifold $W^u\big(O^\nu_{1,\varepsilon}(\mu)\big)$ with the stable manifold $W^s\big(O^\nu_{N+1,\varepsilon}(\mu)\big)$ along the heteroclinic orbit that connects the orbits $O^\nu_{1,\varepsilon}(\mu)$ and*

$O^{\nu}_{N+1,\varepsilon}(\mu)$ *is transverse inside the corresponding level surface of the perturbed Hamiltonian* (2.2). *(See Figure 12.8.)*

The proofs of Propositions 12.3–12.6 proceed with the help of Proposition 12.2 exactly as the proofs of the corresponding theorems in [32].

## 13. An Example of Orbits Homoclinic to Resonance Bands

The example of a Duffing oscillator coupled into an anharmonic oscillator,

$$\dot{p} = \eta^2 q(I - q^2) - \varepsilon\alpha p, \tag{13.1a}$$

$$\dot{q} = p, \tag{13.1b}$$

$$\dot{I} = -\varepsilon I \sin\theta - \varepsilon\beta I - \varepsilon\gamma p^2, \tag{13.1c}$$

$$\dot{\theta} = I - 1 - \tfrac{1}{2}\eta^2 q^2 - \varepsilon\cos\theta, \tag{13.1d}$$

was used in [34–37] to illustrate various types of one-bump and multi-bump orbits homoclinic to resonance bands. In this example $\varepsilon \ll 1$, $\eta$, $\alpha$, $\beta$, and $\gamma$ are positive parameters. System (13.1) has the form (2.1) for nonzero $\alpha$, $\beta$ and $\gamma$, and the form (2.3) with the Hamiltonian

$$\begin{aligned}\hat{H}(p,q,I,\theta) &= H(p,q,I) + \varepsilon H_1(p,q,I,\theta)\\ &= \tfrac{1}{2}I^2 - I + \tfrac{1}{2}p^2 - \tfrac{1}{2}\eta^2 q^2(I - \tfrac{1}{2}q^2) - \varepsilon I\cos\theta,\end{aligned} \tag{13.2}$$

for $\alpha = \beta = \gamma = 0$.

The phase-space structure of system (13.1) with zero $\varepsilon$ is as follows. An unstable invariant annulus, $\mathscr{M}$, with $(p,q) = (0,0)$ and $I$ between any $I_1$ and $I_2$, such that $0 < I_1 < 1 < I_2$, is foliated by periodic orbits $p = q = 0$, $I =$ constant, $\theta = (I-1)t + \theta_0$. The annulus $\mathscr{M}$ is connected to itself by a pair of three-dimensional homoclinic manifolds, $W_+(\mathscr{M})$ and $W_-(\mathscr{M})$, which are parametrized by $t$, $I$ and $\theta_0$ in the homoclinic solutions (see Figure 13.1)

$$p = p^h(t,I) = \mp\sqrt{2}\eta I\ \mathrm{sech}(\eta\sqrt{I}t)\tanh(\eta\sqrt{I}t), \tag{13.3a}$$

$$q = q^h(t,I) = \pm\sqrt{2I}\ \mathrm{sech}(\eta\sqrt{I}t), \tag{13.3b}$$

$$I = I, \tag{13.3c}$$

$$\theta = \theta^h(t,I) + \theta_0 = (I-1)t - \eta\sqrt{I}\tanh(\eta\sqrt{I}t) + \theta_0. \tag{13.3d}$$

The phase difference $\Delta\theta(I)$, defined by equation (2.8), between the end points of any heteroclinic orbit (13.3) is $\Delta\theta(I) = -2\eta\sqrt{I}$, which follows immediately from equation (13.3d).

The frequency $I - 1$ of the periodic orbits on the annulus $\mathscr{M}$ passes through zero transversely at $I = 1$, so that the orbit at $I = 1$ is a circle of equilibria, which

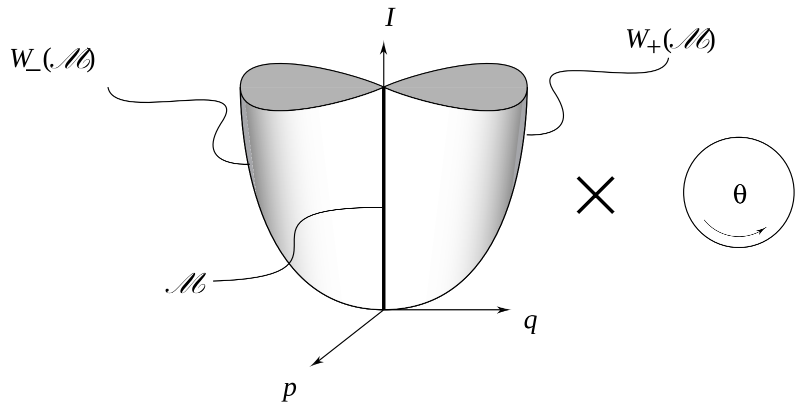


Fig. 13.1. The annulus $\mathscr{M}$ and the homoclinic manifolds $W_C(\mathscr{M})$ and $W_-(\mathscr{M})$ for the example of Section 13.

gives rise to a resonance band for $\varepsilon > 0$. Pairs of points on this circle of equilibria that are $\Delta\theta(1) = -2\eta$ apart are connected by heteroclinic orbits, obtained by inserting the value $I = 1$ into the solutions (13.3).

Since the set $p = q = 0$ is invariant even for nonzero $\varepsilon$, we can take the perturbed annulus $\mathscr{M}_\varepsilon$ to be the same as the annulus $\mathscr{M}$. Thus, we immediately calculate the restricted system (12.2) to be

$$h' = -(1 + \sqrt{\varepsilon}h)\sin\theta - \beta(1 + \sqrt{\varepsilon}), \qquad \theta' = h - \sqrt{\varepsilon}\cos\theta. \tag{13.4}$$

The limiting outer system and the outer Hamiltonian are

$$h' = -\sin\theta - \beta, \qquad \theta' = h, \tag{13.5}$$

$$\mathscr{H}(h, \theta) = \tfrac{1}{2}h^2 - \cos\theta + \beta\theta, \tag{13.6}$$

respectively.

When $0 < \beta < 1$, there are two equilibria on the $h-\theta$ cylinder $\mathscr{M}_0$, a center $c_0$ at $(h, \theta) = (0, -\arcsin\beta)$ and a saddle $s_0$, at $(h, \theta) = (0, -\pi + \arcsin\beta)$. The two branches of the stable and unstable manifolds $\mathscr{W}^s(s_0)$ and $\mathscr{W}^u(s_0)$ to the right of the saddle $s_0$ coincide to form a separatrix that encloses a family of periodic orbits nested around the center. The two branches of the manifolds $\mathscr{W}^s(s_0)$ and $\mathscr{W}^u(s_0)$ to the left of the saddle $s_0$ wind around the cylinder $\mathscr{M}_0$ towards $h = +\infty$ and $h = -\infty$, respectively. For small positive $\sqrt{\varepsilon}$, the saddle $s_0$ persists as a saddle $s_\varepsilon$, the center $c_0$ becomes a sink $c_\varepsilon$, and the separatrix breaks. The top branch of the unstable manifold $\mathscr{W}^u(s_\varepsilon)$ of the perturbed saddle $s_\varepsilon$ falls into the sink $c_\varepsilon$. No periodic orbits are left in this system, and all the points that lie in any compact domain, that is, all contained inside the unperturbed separatrix asymptote to the sink $c_\varepsilon$ (see Figure 13.2).

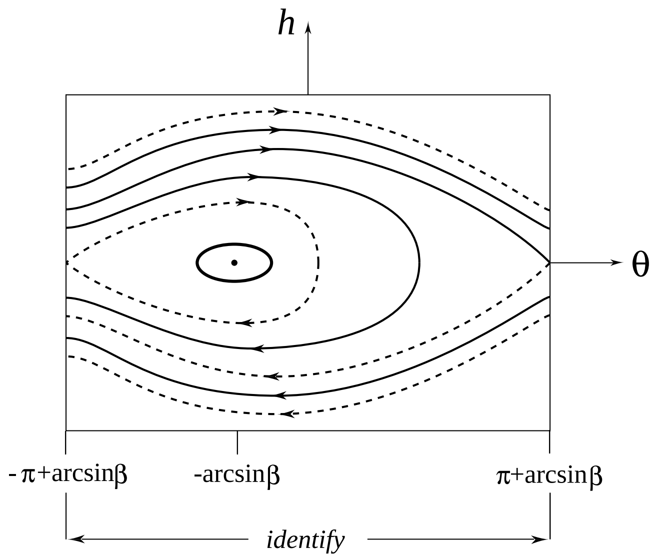


Fig. 13.2. The phase portrait for the outer system in the dissipative case.

The inner system (12.6) for this example is

$$\dot{p} = \eta^2 q(1 - q^2), \tag{13.7a}$$

$$\dot{q} = p, \tag{13.7b}$$

$$\dot{h} = 0, \tag{13.7c}$$

$$\dot{\theta} = -\tfrac{1}{2}\eta^2 q^2. \tag{13.7d}$$

In references [35, 37, 32] we computed that, at the resonance $I = 1$, the Melnikov function $M(I, \theta_0, \eta, \alpha, \beta, \gamma)$ on both homoclinic manifolds $W_+(\mathscr{M})$ and $W_-(\mathscr{M})$ is equal to

$$M(1, \theta_0, \eta, \alpha, \beta, \gamma) = \cos(\theta_0 - \eta) - \cos(\theta_0 + \eta) - \tfrac{4}{3}\alpha\eta + 2\beta\eta + \tfrac{8}{15}\gamma\eta^3. \tag{13.8}$$

Likewise, the $k$-pulse Melnikov function equals

$$\begin{aligned} M_k(1, \theta_0, \eta, \alpha, \beta, \gamma) = \cos(\theta_0 - (2k-1)\eta) - \cos(\theta_0 + \eta) - \tfrac{4}{3}k\alpha\eta \\ + 2k\beta\eta + \tfrac{8}{15}k\gamma\eta^3. \end{aligned} \tag{13.9}$$

If we set $\theta_{k-1} = \theta_0 - (k-1)\eta$, this formula can be rewritten as

$$
\begin{aligned}
&M_k(1, \theta_0, \eta, \alpha, \beta, \gamma) \\
&\quad = M_k(1, \theta_{k-1} + (k-1)\eta, \eta, \alpha, \beta, \gamma) \\
&\quad = \cos(\theta_{k-1} - k\eta) - \cos(\theta_{k-1} + k\eta) - \tfrac{4}{3}k\alpha\eta + 2k\beta\eta + \tfrac{8}{15}k\gamma\eta^3 \\
&\quad = 2\sin k\eta \sin\theta_{k-1} - \tfrac{4}{3}k\alpha\eta + 2k\beta\eta + \tfrac{8}{15}k\gamma\eta^3.
\end{aligned}
\tag{13.10}
$$

If $k\eta$ is not a multiple of $\pi$ and if

$$
\left|\frac{k\eta}{\sin k\eta}\left(\tfrac{2}{3}\alpha - \beta - \tfrac{4}{15}\gamma\eta^2\right)\right| < 1, \tag{13.11}
$$

then the $k$-pulse Melnikov function (13.10) has simple zeros in $\theta_{k-1}$ at some $\theta_{k-1} = \bar{\theta}_{k-1,1}$ and $\theta_{k-1} = \bar{\theta}_{k-1,2} = \pi - \bar{\theta}_{k-1,1}$. If for $i = 1$ or $i = 2$, the values of the $j$-pulse Melnikov function $M_j(1, \bar{\theta}_{0,i}, \eta, \alpha, \beta, \gamma)$, with $\bar{\theta}_{0,i} = \bar{\theta}_{k-1,i} + (k-1)\eta$ and $j = 1, \dots, k-1$, are different from zero, and if $k$, $\eta$, $\alpha$, $\beta$, and $\gamma$ satisfy the inequality (13.11), then the stable and unstable manifolds $W^s(\mathscr{M}_\varepsilon)$ and $W^u(\mathscr{M}_\varepsilon)$ intersect transversely along a symmetric pair of two-dimensional, $k$-pulse homoclinic surfaces, $\Sigma^{\alpha,\beta,\gamma}_{\pm,\varepsilon}(\bar{\theta}_{k-1,i})$. In the phase space of the inner system (13.7), this pair collapses smoothly onto a pair of limiting $k$-pulse surfaces, $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,i})$, parametrized by the expressions (13.3) with $I = 1$, $\theta_0 = \bar{\theta}_{k-1,i} + (k-1-2j)\eta$, where $j = 0, \dots, k-1$, and arbitrary $h$. The sign in each of these expressions is determined by the sign of the corresponding $j$-pulse Melnikov function $M_j(1, \bar{\theta}_{0,i}, \eta, \alpha, \beta, \gamma)$.

We now describe a new example of orbits homoclinic to the saddle $s_\varepsilon$, whose existence follows from Proposition 12.3, provided that we fix $0 < \eta \ll \beta < 1$ and $\tilde{\gamma} = \eta^2\gamma = \mathscr{O}(1)$. If $5\alpha - 2\tilde{\gamma} = 0$, then for any $j = 1, \dots, k$, the $j$-pulse Melnikov function (13.10) becomes equal to the difference in the values of the outer Hamiltonian (13.6) at the endpoints of the singular orbit along which this Melnikov function is calculated. This implies, in particular, that the $k$-pulse singular orbits picked out by the zeros of the Melnikov function must have both their endpoints at the same value of the outer Hamiltonian (13.6). Moreover, since $\eta \ll 1$, for all finite $k$, equation (13.10) becomes approximately $\sin\theta_{k-1} \approx -\beta$, so that $\bar{\theta}_{k-1,1} \approx -\arcsin\beta$ and $\bar{\theta}_{k-1,2} \approx -\pi + \arcsin\beta$, which are the $\theta$ coordinates of the center $c_0$ and the saddle $s_0$, respectively.

From the discussion in the previous paragraph, it easily follows that for $\bar{\theta}_{0,i} = \bar{\theta}_{k-1,i} + (k-1-2j)\eta$, with $i = 1, 2$, the values of the $j$-pulse Melnikov functions $M_j(1, \bar{\theta}_{0,i}, \eta, \alpha, \beta, \gamma)$ are different from zero for all $j = 1, \dots, k-1$, and are in fact of the same sign for all $j$. This sign is negative for $\bar{\theta}_{0,1}$ and positive for $\bar{\theta}_{0,2}$. Therefore, the $k$-pulse homoclinic surfaces $\Sigma^{\alpha,\beta,\gamma}_{\pm,\varepsilon}(\bar{\theta}_{k-1,1})$ and $\Sigma^{\alpha,\beta,\gamma}_{\pm,\varepsilon}(\bar{\theta}_{k-1,2})$ indeed exist in this case for all $k$, and so do the limiting $k$-pulse surfaces, $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,1})$ and $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,2})$. Since the regions enclosed by the two homoclinic manifolds $W_+(\mathscr{M})$ and $W_-(\mathscr{M})$ are both convex, and the normal $\boldsymbol{n} = \big(p, -\eta^2 q(I - q^2), 0, 0\big)$ is easily seen to point out of them, it follows that orbits forming each of the surfaces $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,1})$ are parametrized by expressions

(13.3) with alternating signs, and orbits forming each of the surfaces $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,2})$ are parametrized by expressions (13.3) with the same signs.

From the discussion in the previous two paragraphs, it follows that there exist an integer $n \geqq 1$ and $2n$ symmetric pairs of $k$-pulse singular orbits $\Gamma^{\pm}_{\pm k}$, with $k = 1, \ldots, n$, which lie on the $n$ pairs of limiting intersection surfaces $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,1})$, and connect $2n$ pairs of points on the separatrix on the cylinder $\mathscr{M}_0$. The sign in the subscript of the symbol $\Gamma^{\pm}_{\pm k}$ is the same as the sign of the corresponding surface $\Sigma^{\alpha,\beta,\gamma}_{\pm,0}(\bar{\theta}_{k-1,1})$; the sign in the superscript is the same as the sign of the $h$ coordinate along the singular orbit $\Gamma^{\pm}_{\pm k}$. As mentioned above, the $q$ coordinates along the pulses of the orbit $\Gamma^{\pm}_{\pm k}$ have alternating signs. The equality of the $k$-pulse Melnikov function and the corresponding difference in the values of the outer Hamiltonian implies that for $k > l$, the takeoff point of the singular orbit $\Gamma^{\pm}_{\pm k}$ is to the right of the takeoff point of the singular orbit $\Gamma^{\pm}_{\pm l}$, and the landing point of the singular orbit $\Gamma^{\pm}_{\pm k}$ is to the left of the landing point of the singular orbit $\Gamma^{\pm}_{\pm l}$. Moreover, the takeoff point of any singular orbit $\Gamma^{\pm}_{\pm k}$ is to the right of the landing point of any other singular orbit $\Gamma^{\pm}_{\pm l}$.

We can now form a countable infinity of singular homoclinic orbits as follows. Each such orbit starts along the right-hand branch of the unstable manifold $\mathscr{W}(s_0)$ of the saddle $s_0$ on the annulus $\mathscr{M}_0$. The singular homoclinic orbit then takes off from $\mathscr{M}_0$ along one of the singular $k$-pulse orbits $\Gamma^{+}_{\pm k}$, and lands back on $\mathscr{M}_0$ at a point on the separatrix that connects the saddle $s_0$ to itself. After following the separatrix for a while, the singular homoclinic orbit again takes off along some singular $l$-pulse orbit $\Gamma^{+}_{\pm l}$, and so forth. Eventually, the singular homoclinic orbit lands back on the separatrix and either follows it to the takeoff point of one of the two heteroclinic orbits $\Gamma^{-}_{\pm j}$, where it takes another excursion along one of them before returning to the saddle $s_0$, or else proceeds directly to this saddle.

With each singular homoclinic orbit described in the previous paragraph, we can associate in a one-to-one fashion a sequence $S = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, where $\boldsymbol{\sigma}_1$ is either an empty sequence or else is a string of symbols $\Gamma^{+}_{\pm k}$ with various $k$, and $\boldsymbol{\sigma}_2$ is either empty or equal to one of the symbols $\Gamma^{-}_{\pm k}$. Either $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$ must be nonempty. We denote the singular homoclinic orbit corresponding to the sequence $S$ by $\Gamma^{S}_{0}$. If we denote by $-S$ the sequence obtained from $S$ by interchanging the $+$ and $-$ signs, then the corresponding singular homoclinic orbit $\Gamma^{-S}_{0}$ is the mirror image of the singular orbit $\Gamma^{S}_{0}$ under the transformation $(p, q) \mapsto (-p, -q)$, which preserves equations (13.1).

Proposition 12.3 now implies

**Proposition 13.1.** *Fix* $0 < \eta \ll \beta < 1$ *and* $\tilde{\gamma} = \eta^2\gamma = \mathscr{O}(1)$. *Let the sequence* $S$ *and its corresponding singular homoclinic orbit* $\Gamma^{S}_{0}$ *be as in the preceding two paragraphs. Then, there exists a continuous function* $\alpha_S(\varepsilon)$ *with* $5\alpha_S(0) - 2\tilde{\gamma} = 0$, *such that for small positive* $\varepsilon$ *and* $\alpha = \alpha_S(\varepsilon)$, *there exists an orbit* $\Gamma^{S}_{\varepsilon}$, *homoclinic to the saddle* $s_\varepsilon$. *The orbit* $\Gamma^{S}_{\varepsilon}$ *is* $\mathscr{O}(\delta(\varepsilon))$*-close to the singular orbit* $\Gamma^{S}_{0}$, *where* $\delta(\varepsilon)$ *is a function with* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Moreover,* $\alpha_S(\varepsilon) = \alpha_{-S}(\varepsilon)$, *and the orbit* $\Gamma^{-S}_{\varepsilon}$ *is the mirror image of the orbit* $\Gamma^{S}_{\varepsilon}$ *under the transformation* $(p, q) \mapsto (-p, -q)$.

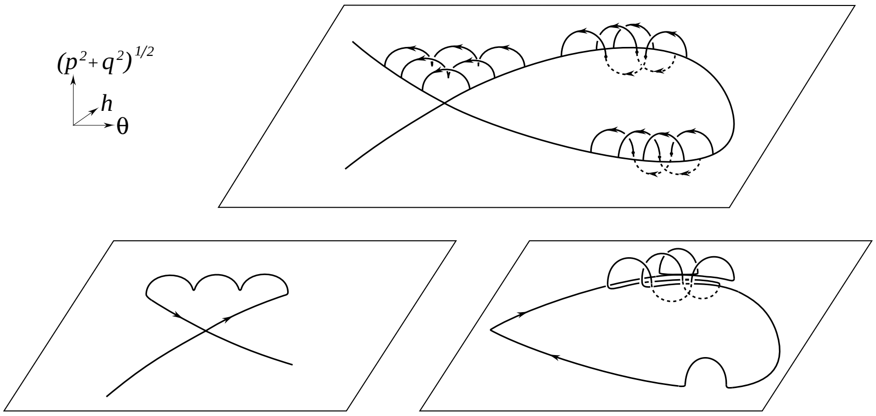


Fig. 13.3. Multi-bump orbits for the sequence constructed using Proposition 13.1.

As noted above, the sign of the $q$ coordinate along the pulses in each $k$-pulse stretch of the orbit $\Gamma_\varepsilon^S$ alternates (see Figure 13.3).

In the same fashion, we can show the existence of $k$-pulse orbits homoclinic to the saddle $s_\varepsilon$ that lie on the surface $\Sigma_{\pm,\varepsilon}^{\alpha,\beta,\gamma}(\bar{\theta}_{k-1,2})$. The $q$ coordinate along all pulses of such an orbit has the same sign. The more complicated counterparts of these orbits consisting of several separate groups of pulses, such as the ones described in Proposition 13.1, do not exist.

Finally, we mention that we can use Propositions 12.4–12.6 to construct Šilnikov orbits [63–65] homoclinic to the spiral-saddle $c_\varepsilon$, heteroclinic connections between the saddle $s_\varepsilon$ and $c_\varepsilon$, as well as orbits homoclinic to periodic orbits in the resonance band in the Hamiltonian case when $\alpha = \beta = \gamma = 0$. All of these homoclinic and heteroclinic orbits consist of several groups of consecutive bumps, separated by long stretches near the annulus $\mathscr{M}_\varepsilon$, and can be associated with symbol sequences.

## 14. Conclusions

The main result of this paper, Theorem 1, and its extensions presented in Section 10, consists of a tool that, for systems of the form (2.1), is comparable in its applicability and efficiency to the celebrated Melnikov method. As such, it is useful both for solving applied problems [6, 7, 38], and for unifying existing perturbation-theoretic approaches to detection of homoclinic and heteroclinic orbits in near-integrable systems. In this latter vein, this paper already develops a unifying approach to the results of references [6, 59, 23, 32]. We also believe that our paper may be used to provide an alternative derivation of the result presented in [21]. However, there are still many unresolved and challenging problems on the way to develop our tool to handle systems even more general than (2.1), thereby achieving the same level of generality that the Melnikov method has reached through years of investigation devoted to it. Moreover, our study raises new problems which further generalizations of the results described in this paper could help solve. Here we mention just some of the obstacles and problems, which we are planning to investigate in future work.

As we have mentioned immediately after Assumption 1, we impose this technical requirement that the unperturbed problem have an analytic right-hand side for the sole purpose of simplifying the presentation. There appear to be no major obstacles to generalizing our results to the case when the Hamiltonian function $H(x, I)$ of the unperturbed system (2.4) has a finite degree of smoothness. This generalization may be achieved by using the Smooth Linearization Theorem for two-dimensional nonlinear saddle-type equilibrium points, described in [11, 61, 4, 67, 68], instead of the analytic theorems described in [49, 60, 62].

On the other hand, while the results of Section 10 show that Theorem 1 is also easily generalized to cover systems with several $I$ and $\theta$ variables, the same is not true for systems in which the vector $x$ has more than two components. There appear to be two major obstacles on the way towards achieving this generalization. The first is that, in general, smooth linearization is not possible for systems with multiple degrees of freedom [11, 61], and thus the results of Sections 4 and 5 cannot be immediately applied to this case. We do believe, however, that the situation should be different for integrable Hamiltonian systems. This is indicated, for example, in the results of [27, 12, 33], but this type of smooth linearization still needs to be formulated in the context of the systems of interest for our theory. Moreover, for multiple degrees of freedom, the situation is further complicated by the possibility of resonances.

The second obstacle is that, when the vector $x$ has more than two components, certain orbits that enter the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the hyperbolic manifold $\mathscr{M}_\varepsilon$ a distance $\mathscr{O}(\varepsilon)$ close to its local stable manifold $W^s_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ may exit this neighborhood a distance $\mathscr{O}(\varepsilon^\alpha)$, with $\alpha \neq 1$, away from the local unstable manifold $W^u_{\mathrm{loc}}(\mathscr{M}_\varepsilon)$ of $\mathscr{M}_\varepsilon$. This property is not limited to complicated nonlinear and nonintegrable systems; it even occurs in simple four-dimensional linear models in which the role of $\mathscr{M}_\varepsilon$ is played by the origin. In the problem of finding a Melnikov function for multi-pulse homoclinic orbits, this property is likely to introduce new singular submanifolds of the manifold $\mathscr{L}$, the piece of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ of the hyperbolic manifold $\mathscr{M}_\varepsilon$ which winds $n$ times in and out of the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of $\mathscr{M}_\varepsilon$ and finally gives rise to an $n$-pulse homoclinic intersection surface. These singular submanifolds will probably have to be described by conditions somewhat analogous to the nonfolding condition (5.29), but these new conditions still need to be derived.

Other generalizations of our result include the case of equations (2.3), in which the perturbed system is also Hamiltonian. In this case, as mentioned in Section 10, the manifold $\mathscr{M}_\varepsilon$ is filled by Kolmogorov-Arnold-Moser tori, and the interesting question is not just whether the stable and unstable manifolds of $\mathscr{M}_\varepsilon$ intersect along multi-pulse homoclinic intersection surfaces, but if this also happens to the stable and unstable manifolds of the individual tori. The answer will be provided by computing additional multi-pulse Melnikov functions in the direction of the $I$ coordinates, as derived in [25, 26, 58]. Standard single-pulse versions of these

Melnikov functions have the rather annoying property that the integrals defining them do not converge absolutely, and one must in fact choose special time sequences to make these integrals converge at all [58]. The time sequences that resolve this non-convergence property are closely related to the geometry of the phase space near the Kolmogorov-Arnold-Moser tori, and the fact that orbits wind densely on these tori. A careful study of this geometry may also be needed while deriving estimates analogous to those in this paper's Sections 7 and 8 for these additional Melnikov functions.

References [6, 48] study the limiting case in which one of the periodic orbits on the two-dimensional unperturbed hyperbolic annulus $\mathscr{M}$ degenerates into a saddle-center equilibrium point. A particular limit of Theorem 1 or its extensions in Section 10 could be developed to cover the general version of this limiting case.

A closely related problem is that of classifying homoclinic orbits as to whether they are "secondary" or "primary," that is, whether or not their existence is a consequence of the existence of some other homoclinic orbits. In particular, in time-periodically modulated planar conservative systems which possess homoclinic tangles [52, 19], all 1-pulse homoclinic orbits are primary, and all multi-pulse homoclinic orbits are secondary. (Here, we do not make any distinction between "secondary," "tertiary," etc., which is sometimes made in the literature.) As stated in the introduction, the existence of these secondary homoclinic orbits follows directly from the topology of the homoclinic tangle, which is a direct consequence of the existence of a single simple zero of the 1-pulse Melnikov function. Some of these secondary homoclinic orbits can be shown to exist directly by using the method of [59] or that developed in this paper. However, it is not clear whether all the secondary homoclinic orbits in such perturbed planar conservative systems can be detected in this way, since the method only provides a sufficient condition for their existence. If the method fails to detect all secondary homoclinic orbits, an interesting question would be just what geometric property distinguishes the ones that are detected from the ones that are not.

A different case of distinction between primary and secondary homoclinic orbits occurs when the dynamics on the manifold $\mathscr{M}_\varepsilon$ is slow, such as in the case of orbits homoclinic to resonance bands discussed in Sections 12 and 13. In this case all multi-pulse orbits, which consist of a single group of fast consecutive pulses, are primary. Multi-bump orbits, which consist of several groups of fast consecutive pulses interspersed with slow segments that slowly travel along the hyperbolic annulus $\mathscr{M}_\varepsilon$, are, on the other hand, secondary. The existence of these secondary orbits can be deduced from the existence of several zeros of the appropriate multi-pulse Melnikov functions, as discussed in Section 12 and [32]. This example also shows that, while secondary homoclinic orbits do come into being only because some primary homoclinic orbits already exist, the actual process of establishing the existence of secondary orbits from that of the primary orbits, developed in [32], need by no means be trivial.

The distinction between primary and secondary homoclinic orbits is much less clear in the model of an atmospheric slow manifold presented in Section 11. In this example, there are occasions in which 2-pulse homoclinic orbits exist without there being any 1-pulse homoclinic orbits nearby, yet, on the other hand, there exist near any $k$-pulse homoclinic orbit cascades of $n$-pulse homoclinic orbits with any $n > k$. It is quite clear that the 2-pulse homoclinic orbits with no nearby 1-pulse homoclinic orbits must be primary, but the answer to the general question of which of the multi-pulse orbits are primary or secondary is as yet unresolved. This question is even more interesting, because, as mentioned at the end of Section 3, a whole class of secondary multi-pulse homoclinic orbits in the model can be obtained by using the Exchange Lemma [31], a technique different from those used in this paper. More generally, in the context of the case when the dynamics on the normally hyperbolic manifold $\mathscr{M}_\varepsilon$ is fast, we notice that the folds of the winding piece $\mathscr{L}$ of the unstable manifold $W^u(\mathscr{M}_\varepsilon)$ certainly have implications on the multiplicity of the transverse intersections with the stable manifold $W^s(\mathscr{M}_\varepsilon)$. Since these intersections define higher-pulse homoclinic orbits, we can expect these folds to play a role on how the multi-pulse homoclinic orbits are organized in the case when the dynamics on the annulus $\mathscr{M}_\varepsilon$ is fast. Exactly what this role is and its relation with the cascade of higher-pulse homoclinic orbits mentioned above have yet to be explored.

Finally, we point out that, apart from its main result, Theorem 1, and its generalizations given in Section 10, an interesting and potentially useful side result of this paper is Lemma 1. This lemma, whose predecessor was developed in [71], addresses the situation in which a manifold $\mathscr{L}$ enters the small neighborhood $U_\delta(\mathscr{M}_\varepsilon)$ of the hyperbolic annulus $\mathscr{M}_\varepsilon$ a distance $\mathscr{O}(\varepsilon)$ close to the local stable manifold $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$, and exits $U_\delta(\mathscr{M}_\varepsilon)$ the same distance from the local unstable manifold $W^u_{\text{loc}}(\mathscr{M}_\varepsilon)$. The manifold $\mathscr{L}$ must have the same dimension as the manifolds $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$ and $W^u_{\text{loc}}(\mathscr{M}_\varepsilon)$. If the tangent spaces of the manifolds $\mathscr{L}$ and $W^s_{\text{loc}}(\mathscr{M}_\varepsilon)$ are also $\mathscr{O}(\varepsilon)$-close when these two manifolds enter the neighborhood $U_\delta(\mathscr{M}_\varepsilon)$, Lemma 1 gives precise conditions for when the tangent spaces of the manifold $\mathscr{L}$ and $W^u_{\text{loc}}(\mathscr{M}_\varepsilon)$ are almost $\mathscr{O}(\varepsilon)$-close as these manifolds exit $U_\delta(\mathscr{M}_\varepsilon)$, and locates folds on $\mathscr{L}$ when these conditions are not met. We have already mentioned in the introduction that, while the tools used for establishing this lemma have much in common with those used for establishing the Exchange Lemma [28–31, 72, 70], the two lemmas describe complementary geometries. We believe that Lemma 1 and its usefulness go beyond the framework of this paper.

In conclusion, in this paper we have developed a new computable criterion that lets us establish the existence of multi-pulse homoclinic and heteroclinic orbits in a large class of near-integrable systems with many degrees of freedom. This criterion consists of finding simple zeros of the $k$-pulse Melnikov function $M_k(\varepsilon, I, \theta_0, \mu)$, and unifies several previously disjoint but related techniques. We have also developed a lemma that describes in detail the behavior of certain manifolds that fly very close to a hyperbolic manifold without intersecting its local stable and unstable manifolds.

*Acknowledgment.* We thank Chris Jones, Nancy Kopell, and Mark Levi for helpful discussions. R. Camassa acknowledges support by the U.S. Department of Energy through its CHAMMP program, its Office of Basic Energy Science program in Applied Mathematical Sciences under contract KC-07-01-01, and the Los Alamos National Laboratory under contract W-7405-ENG-36. G. Kovačič thanks the Theoretical Division and Center for Nonlinear Science at the Los Alamos National Laboratory for their hospitality and support during the summers of 1992 through 1995, and acknowledges support from the U.S. Department of Energy through grant DE-FG02-93ER25154, the National Science Foundation through grants DMS-9403750 and DMS-9502142, and the Alfred P. Sloan Foundation through an Alfred P. Sloan Fellowship. S.-K. TIN acknowledges summer support of 1995 from University of Michigan.

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*(Accepted December 2, 1996)*