Math. Ann. 317, 285–293 (2000)

Mathematische Annalen

A tight closure proof of Fujita's freeness conjecture for very ample line bundles

Karen E. Smith⋆

Received: 7 April 1999

Mathematics Subject Classification (1991): 14C20, 14B15, 13A35

The purpose of this note is to give a simple algebraic proof of the following special case of Fujita's Freeness Conjecture:

Theorem 1. *Let X be a smooth projective algebraic variety of dimension d over a field (of any characteristic), and let L be a very ample line bundle of X. Then $K_X + dL$ is globally generated unless $X = \mathbb{P}^d$ and L is the hyperplane bundle.*

The proof here actually proves a stronger statement than Theorem 1 above. The variety X need not be smooth; F-rationality is sufficient (the definition is recalled in the next section; in characteristic zero it is equivalent to rational singularities). Also, the line bundle L need not be very ample; it is sufficient if L is globally generated and the dimension the complete linear system $|L|$ is greater than d. These generalizations are summarized in Theorem 2. Furthermore, with a little more work, the same ideas prove even stronger statements, which are interesting algebraically, but difficult to interpret geometrically (see Theorem 3).

Fujita's Freeness Conjecture predicts the same conclusion under the much weaker hypothesis that L is only ample. While open in general, for varieties defined over a field of characteristic zero, it is known in dimension four or less [R], [EL], [Ka]. See also [AS] for important progress on the conjecture, and [Ko] for a good survey about it.

In characteristic zero, it is not hard to give a geometric argument of the special case above using the Kodaira Vanishing Theorem. The goal here is to give a simple, quite different proof that is purely algebraic and valid in any characteristic. This argument offers a nice illustration of how tight closure can be used to prove geometric theorems in arbitrary characteristic without the use of the usual tools of desingularization or vanishing theorems.

K.E. Smith
The University of Michigan, Department of Mathematics, East Hall, 525 East University, Ann Arbor, MI 48109-1109, USA

⋆ Supported by the National Science Foundation and the Alfred P. Sloan Foundation

In [S1], a different version of Fujita's Conjecture was considered. There it was shown that $K_X + (d+1)L$ is globally generated (instead of $K_X + dL$) where X is a smooth projective variety of any characteristic of dimension d and L is a globally generated ample line bundle. The proof of Theorem 1 begins by using the same equivalent form of Fujita's Conjecture in terms of local cohomology as in [S1], but more subtle facts about tight closure are needed to reach the sharper conclusion above.

Theorem 1 has a nice application to a seemingly unrelated result. As observed by Ein in [Ei], Theorem 1 implies the finiteness of the Gauss map from X to the appropriate Grassmannian defined by sending a point in X to its tangent plane. A different way of deducing that the dimension of the Gauss image is d is considered in [SSU], which also uses tight closure in a similar way as in this paper.

Thanks to Rob Lazarsfeld for helpful discussions, in particular for pointing me towards [Ei].

The proof of Theorem 1

First some notation and review of facts. Let X be a normal projective algebraic variety of dimension d over a field (of any characteristic), and let L be an ample line bundle on X. Then the section ring $\oplus_{n=0}^{\infty} H^0(X, nL)$ of the pair (X, L) will be denoted S and its unique homogeneous maximal ideal will be denoted m. Recall that S is a normal graded ring of dimension $d+1$ over a field k such that $\text{Proj} S = X$.

The proof uses the following way of interpreting global generation of adjoint linear series in terms of local cohomology.

Proposition A. [S1, 1.1] *With notation as above, the following are equivalent.*

(1) The reflexive sheaf $\mathcal{O}_X(K_X + nL)$ is globally generated;
(2) There exists an integer N such that every element of the local cohomology module $H_m^{d+1}(S)$ of degree less than N has a non-zero multiple of degree $-n$.

Here, $\mathcal{O}_X(K_X)$ denotes the unique reflexive sheaf that agrees with the invertible sheaf of algebraic d-forms on the smooth locus of X and $\mathcal{O}_X(K_X + nL)$ is its tensor product with the n^{th} power of L.

The proof of Theorem 1 will be valid, not only for smooth projective X, but for any projective F-rational variety X. We recall the definition:

Definition. *A local ring of prime characteristic is F-rational if every ideal generated by a system of parameters is tightly closed. A scheme of prime characteristic is F-rational if all its local rings are.*

For algebras essentially of finite type over a field of characteristic zero, one can define a concept of "F-rational type" based on reduction to characteristic p.

The point is that the k-algebra R may be written as a tensor product $k \otimes_A R_A$ where A is a finitely generated $\mathbb{Z}$-algebra contained in k and R_A is a finitely generated A-algebra; then we say that R has F-rational type if on a dense set of $\operatorname{Spec} A$, the closed fibers of the map $\operatorname{Spec} R_A \to \operatorname{Spec} A$ (which are algebras over finite fields of different characteristics) are F-rational. See [S2] for the detailed definition. F-rational type, it turns out, is equivalent to rational singularities [S2, H]. The important fact about F-rational local rings we will use here is the following:

Proposition B [S2]. *If a local ring (R, m) of prime characteristic and dimension $d+1$ is F-rational on its punctured spectrum, then the tight closure of the zero module in the local cohomology module $H_m^{d+1}(R)$ has finite length.*

The proof will also require the following result about tight closure of homogeneous ideals. This is an improvement of a Theorem in [S3]; its proof will appear after the proof of the main theorem.

Theorem C. *Let R be a normal $\mathbb{N}$-graded ring over a perfect field of prime characteristic p, and let I_1 and I_2 be ideals of R generated by homogeneous elements of degrees strictly less than δ and greater than or equal to δ respectively. Let z be an element of R homogeneous of degree δ. Then if $z \in (I_1 + I_2)^*$, then $z \in I_1^* + I_2$.*

Theorem 1 is a special case of the following theorem.

Theorem 2. *Let X be a projective variety of dimension d over a field of arbitrary characteristic. Assume that X is F-rational (type). Let L be a globally generated ample line bundle such that the dimension of the complete linear system $|L|$ associated to L exceeds d. Then $K_X + nL$ is globally generated for all $n \geq d$.*

Because every smooth variety is F-rational (type), and every very ample line bundle (with the exception of the hyperplane bundle on $\mathbb{P}^d$) satisfies the hypothesis of Theorem 2, Theorem 1 follows immediately from Theorem 2.

We now prove Theorem 2. First, a standard argument reduces the problem to the case where the ground field has prime characteristic p (the details are worked out in [S1]). Thus the section ring S may be assumed to be a graded ring of prime characteristic. The point now is really to prove the borderline case, that $K_X + dL$ is globally generated; in any case, the case where $n > d$ is covered by [S1].

Because X is F-rational, its section ring S is F-rational on its punctured spectrum $\operatorname{Spec} S - m$. By Proposition B, this means that the tight closure of the zero module in the local cohomology module $H_m^{d+1}(S)$ has finite length. Thus there exists an integer N such that the tight closure of zero is contained in the submodule of $H_m^{d+1}(S)$ generated by elements of degree N and higher.

To prove the theorem, we use the equivalent formulation **A** above. Let η be a homogeneous element of $H_m^{d+1}(S)$ of degree $-n < min(N, -d-1)$, so that η is not in the tight closure of zero. We need to show that η has a non-zero multiple of degree $-d$. Suppose that this is not the case, that is, suppose that S_{n-d} kills η.

Because L is globally generated, S admits a system of parameters of degree one (if necessary, we can enlarge the ground field, see [S1, last paragraph of Sect. 2]). Let $x_0, x_1, \dots, x_d$ be a system of parameters of degree one. The local cohomology module $H_m^{d+1}(S)$ can be computed as the cokernel of the following map

$$\begin{aligned} S_{x/x_0} \oplus S_{x/x_1} \cdots \oplus S_{x/x_d} &\to S_x \\ \left(\frac{s_0 x_0^t}{x^t}, \frac{s_1 x_1^t}{x^t}, \dots, \frac{s_d x_d^t}{x^t}\right) &\mapsto \frac{\sum_{i=0}^d (-1)^i s_i x_i^t}{x^t} \end{aligned}$$

where x denotes the product $x_0 x_1 \dots x_d$ of the x_i's. (This is the last map in the Cech complex for computing the cohomology of the sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{i=0}^{\infty} \mathcal{O}_X(nL)$ with respect to the affine cover of X given by the $d+1$ open sets U_i where x_i does not vanish.) Thus we represent elements of $H_m^{d+1}(S)$ by fractions $[\frac{z}{x^t}]$, with the square bracket reminding us of the equivalence relation on fractions. In [S2] it is proven that an element $[\frac{z}{x^t}]$ is in the tight closure of the zero module in $H_m^{d+1}(S)$ if and only if the element z is in the tight closure of the ideal $(x_0^t, x_1^t, \dots, x_d^t)$ in S. In particular, the element η above, which is not in the tight closure of zero, can be written $[\frac{z}{x^t}]$, where z is not in $(x_0^t, x_1^t, \dots, x_d^t)^*$. Because the degree of η is $-n$, we see that $-n = \deg z - t(d+1)$.

Now if S_{n-d} kills η, then also the subset consisting of degree $n-d$ monomials in $x_0, x_1, \dots, x_d$ kills η. This means that all elements of the local cohomology module of the form $[\frac{wz}{x^t}]$, where w is in the ideal $(x_0, x_1, \dots, x_d)^{n-d}$ are zero. In particular, all such elements are in the tight closure of zero, so that $(x_0, x_1, \dots, x_d)^{n-d} z \subset (x_0^t, x_1^t, \dots, x_d^t)^*$. That is, $z \in (x_0^t, x_1^t, \dots, x_d^t)^* : (x_0, x_1, \dots, x_d)^{n-d}$.

The colon capturing property of tight closure allows us to manipulate parameters as if they are a regular sequence, up to tight closure; that is, we may formally compute the colon ideal as if the x_i are the variables in a polynomial ring, and the actual colon ideal will be in the tight closure of this formal colon ideal (see [HH, Sect. 7]). In this case, we can use colon capturing to conclude that this colon ideal is contained in

$$\left((x_0^t, x_1^t, \dots, x_d^t) + (x_0, x_1, \dots, x_d)^{(t-1)(d+1)-(n-d)+1}\right)^* .$$

Because the degree of z is $t(d+1) - n$, we see that

$$z \in \left((x_0^t, x_1^t, \dots, x_d^t) + (x_0, x_1, \dots, x_d)^{\deg z}\right)^* .$$

By Theorem C above, in fact,

$$z \in \left(x_0^t, x_1^t, \dots, x_d^t\right)^* + (x_0, x_1, \dots, x_d)^{\deg z} .$$

We write z as $z' + y$ where y and z' are homogeneous of the same degree as z, and y is a k-linear combination of the monomials $x_0^{i_0} x_1^{i_1} \dots x_d^{i_d}$ of degree equal to the degree of z and $z' \in (x_0^t, \dots, x_d^t)^*$. But then

$$\eta = \left[\frac{z'}{x^t}\right] + \left[\frac{y}{x^t}\right]$$

where $[\frac{z'}{x^t}]$ is in the tight closure of zero in $H_m^{d+1}(S)$. But because this tight closure module vanishes in degrees less than N, we see that $[\frac{z'}{x^t}] = 0$, since its degree is $-n < N$. Thus we may assume that

$$\eta = \left[\frac{\sum \lambda_{i_0 i_1 \dots i_d} x_0^{i_0} x_1^{i_1} \dots x_d^{i_d}}{x^t}\right].$$

We are assuming that η is a non-zero element of degree $-n$. Thus at least one of the coefficients λ_I is non-zero. Note that we can also assume that all i_j are strictly less than t, for otherwise, the fraction $\frac{x_0^{i_0} x_1^{i_1} \dots x_d^{i_d}}{x^t}$ represents zero in local cohomology.

Assume that $\lambda = \lambda_{i_0 i_1 \dots i_d}$ is not zero. Let

$$w = x_0^{t-1-i_0} x_1^{t-1-i_1} \dots x_d^{t-1-i_d} s,$$

where s is an arbitrary element of degree one. The degree of w is $(t-1)(d+1) - (\sum i_j) + 1 = (t-1)(d+1) - \deg z + 1 = (t-1)(d+1) - (-n + t(d+1)) + 1 = n - d$. Now multiplying η by w we have

$$w\eta = \left[\frac{\lambda s x^{t-1}}{x^t}\right] = \left[\frac{\lambda s}{x}\right],$$

since each of the other terms in the product is of the form $\left[\frac{\lambda' s x_0^{t-1-i_0+j_0} x_1{}^{t-1-i_1+j_1} \dots x_d^{t-1-i_d+j_d}}{x^t}\right]$ where at least one of the sums $t - 1 - i_k + j_k$ is greater than or equal to t.

Now if S_{n-d} kills η, then the element

$$w\eta = \left[\frac{\lambda s x^{t-1}}{x^t}\right] = \left[\frac{\lambda s}{x}\right],$$

is zero, whence $s \in (x_0, x_1, \dots, x_d)^*$. But since s has degree one, Theorem C above forces $s \in (x_0, \dots, x_d)$.

Now because s is an arbitrary element of degree one, we can obtain a contradiction by choosing s to be not in the linear system spanned by the x_0. This is always possible if the dimension $H^0(X, L)$ exceeds $d + 1$. In particular, if L is very ample, then it is possible (except when $X = \mathbb{P}^d$ and $L = \mathcal{O}(1)$), for otherwise the embedding of X given by the complete linear system $|L|$ would necessarily be an isomorphism $X \to \mathbb{P}^d$. This completes the proof. $\square$

We now prove Theorem C. This Theorem was proven in a slightly different form (but not strong enough for our needs here) in [S3]. We draw heavily from the ideas in that paper, but repeat some arguments for the sake of readability.

Assume that $z \in (I_1 + I_2)^*$ but $z \notin I_1^* + I_2$, and that this example of the failure of our conclusion is choosen so that I_2 has the minimal possible number of generators among all such examples. Let $x_1, \ldots, x_r$ be the generators of I_2, all of which have the same degree as z.

We have equations

$$cz^q - a_q x_1^q \in (I_1 + (x_2, x_3, \ldots, x_r))^{[q]}$$

for all $q = p^e \gg 0$, where $I^{[q]}$ denotes the ideal of R generated by the $q-th$ powers of the generators of I. Here c and a_q may be assumed homogeneous of the same degree. By our minimality assumption, a_q is non-zero for all large q.

As pointed out in [S3], fixing c, there exists a homogeneous $\theta \in E_f = \mathrm{End}_{R^{p^f}}(R)$ for some p^f such that $\theta(c) = 0$ but θ does not kill a_q for all $q > p^f$ unless $a_q = \lambda_q c$ for some $\lambda_q \in K$. (The point is that the decreasing chain of finite dimensional vector spaces

$$\cdots \supset \mathrm{Ann}_{R_d}(\mathrm{Ann}_{E_f}(c)) \supset \mathrm{Ann}_{R_d}(\mathrm{Ann}_{E_{f+1}}(c))$$

of all degree d elements annihilated by all R^{p^f}-linear endomorphisms of R annihilating c must eventually stabilize, and this stable vector space is Kc. Then θ may be taken to be one of the finitely many homogeneous R^{p^f}-module generators for $\mathrm{End}_{R^{p^f}}(R)$.)

Applying θ to the equations above, we have equations

$$\theta(a_q)x_1^q \in (I_1 + (x_2, x_3, \ldots, x_r))^{[q]}$$

for all large q. If a_q is not in the K-span of c in R for infinitely many q, these equations show that $x_1 \in (I_1 + (x_2, x_3, \ldots, x_r))^*$. In this case, $z \in (I_1 + (x_2, x_3, \ldots, x_r))^*$, contrary to our minimality assumption.

Thus it must be that for all large q, we have $a_q = \lambda_q^q c$ for some λ_q^q in K. (Since K is perfect, there is no harm in writing our scalars in the convenient form λ_q^q.) If all, or at least infinitely many, of the λ_q are equal, we get equations of the form

$$c(z - \lambda x_1)^q \in (I_1 + (x_2, x_3, \ldots, x_r))^{[q]}$$

for infinitely many q, which shows that $z - \lambda x_1$ is in $(I_1 + (x_2, x_3, \ldots, x_r))^*$. By our minimality assumption, $z - \lambda x_1$ is in $I_1^* + (x_2, x_3, \ldots, x_r)$, whence z is in $I_1^* + I_2$ and the proof would be complete. Thus we need to show that the

λ_q are equal for infinitely many q. If K is finite, this is immediate; this was the assumption in [S3] which we must now weaken.

Let d be a test element for R not in the K-span of c (such exists because R is normal, and so the test ideal has height two or more). According to the arguments above, we have sets of equations of the form

$$c(z-\lambda_q x_1)^q \in (I_1+(x_2,x_3,\dots,x_r))^{[q]}$$

and

$$d(z-\mu_q x_1)^q \in (I_1+(x_2,x_3,\dots,x_r))^{[q]}$$

for all large q. Multiplying the form by $d\mu_q^q$ and the latter by $c\lambda_q^q$ and subtracting, we get equations

$$cd(\lambda_q-\mu_q)^q z^q \in (I_1+(x_2,x_3,\dots,x_r))^{[q]}.$$

Again, by our minimality assumption, we see that $z\in I_1^*+I_2$, unless $\mu_q=\lambda_q$ for all large q.

But the choice of d was arbitrary, and we can take d to be, for example, c^p, which is clearly not in the K-span of c. This produces equations of the form

$$c^p(z-\lambda_{qp}x_1)^{qp} \in (I_1+(x_2,x_3,\dots,x_r))^{[qp]}.$$

whereas by raising the equations above to the p^{th} power we get

$$c^p(z-\lambda_q x_1)^{qp} \in (I_1+(x_2,x_3,\dots,x_r))^{[qp]}.$$

Subtracting these, we see that either $x_1\in(I_1+(x_2,x_3,\dots,x_r))^*$, whence $z\in(I_1+(x_2,x_3,\dots,x_r))^*$, contrary to minimality, or that $\lambda_{qp}=\lambda_q$. Iterating this process, we see we can find Q such that for all $q\gg Q$, $\lambda_{Qq}=\lambda_Q$. But then the equations

$$c(z-\lambda_{Qq}x_1)^{Qq} \in (I_1+(x_2,x_3,\dots,x_r))^{[Qq]}$$

can be written

$$c(z-\lambda_Q x_1)^{Qq} \in (I_1+(x_2,x_3,\dots,x_r))^{[Qq]}$$

where Q is fixed, for all large q. This shows that

$$z-\lambda_Q x_1 \in (I_1+(x_2,x_3,\dots,x_r))^*.$$

But now by minimality, $z-\lambda_Q x_1\in I_1^*+(x_2,x_3,\dots,x_r)$, whence $z\in I_1^*+I_2$. This completes the proof of Theorem C.

Finally, we point out the following purely algebraic result which is a generalization of Theorem 2.

Theorem 3. *Let S an $\mathbb{N}$-graded domain over a field, F-rational on its punctured spectrum. Assume that S admits a system of parameters $\{x_0, \dots, x_d\}$ of degree one generating an ideal whose tight closure is properly contained in the unique homogeneous maximal ideal m of S. Then every element of the local cohomology module $H_m^{d+1}(S)$ of sufficiently small degree has a non-zero multiple of degree $-d$.*

To prove Theorem 3, we begin with exactly the same observations as in the proof of Theorem 2. Let $\eta = [\frac{z}{x^t}] \in H_m^{d+1}(S)$ be an element of degree $-n$ as in the proof of Theorem 2. If the ideal $(x_0, x_1, \dots, x_d)^{n-d}$ does not annihilate η, then η has a non-zero multiple of degree $-d$ and there is nothing to prove. Otherwise, exactly the same argument shows that we can assume that $z = \sum \lambda_{i_0 i_1 \dots i_d} x_0^{i_0} x_1^{i_1} \dots x_d^{i_d}$, where each exponent i_j is less than t.

Now, since $(x_0, \dots, x_d)^*$ is properly contained in m, we can find a homogeneous element, say w, of positive degree k, which is not in $(x_0, \dots, x_d)^*$. We claim that $(x_0, \dots, x_d)^{n-d-k} w$ does not annilihate η, in which case the proof will be complete, because then η has a non-zero multiple of degree $-d$.

To check the claim, assume on the contrary that $(x_0, \dots, x_d)^{n-d-k} w$ kills η. Then, using the same method as above, we see that $(x_0, \dots, x_d)^{n-d-k} wz \in (x_0^t, \dots, x_d^t)^*$. Using the colon capturing property of tight closure, we see that

$$wz \in \left[(x_0^t, \dots, x_d^t) + (x_0, x_1, \dots, x_d)^{(d+1)(t-1)-(n-d-k)+1}\right]^*.$$

Remembering that $z = \sum \lambda_{i_0 i_1 \cdots d} x_0^{i_0} x_1^{i_1} \dots x_d^{i_d}$, where each monomial in the sum has degree $-n + (d+1)t$, we isolate one term of this sum, say $x_0^{a_0} x_1^{a_1} \dots x_d^{a_d}$, and observe that

$$\begin{aligned}\left(x_0^{a_0} \dots x_d^{a_d}\right) w \in {} & \left[(x_0^t, \dots, x_d^t) + (x_0, \dots, x_d)^{(d(t+1)-n+k)}\right]^* + \\ & \left(\left\{x_0^{b_0} x_1^{b_1} \dots x_d^{b_d} \,\middle|\, \sum b_j = (d+1)t - n, \right.\right. \\ & \qquad \left.\left. \text{with some } b_j \neq a_j \right\}\right) \\ \subset {} & \left[(x_0^t, \dots, x_d^t) + \left(\left\{x_0^{b_0} x_1^{b_1} \dots x_d^{b_d} \,\middle|\, \sum b_j = (d+1)t - n, \right.\right.\right. \\ & \qquad \left.\left.\left. \text{with some } b_j \neq a_j \right\}\right)\right]^*\end{aligned}$$

where $(\{x_0^{b_0} x_1^{b_1} \dots x_d^{b_d} \,|\, \sum b_j = (d+1)t - n, \text{ with some } b_j \neq a_j\})$ denotes the ideal generated by all monomials of degree $(d+1)t - n$ except $x_0^{a_0} \dots x_d^{a_d}$. Now, we again use colon capturing to estimate

$$\begin{aligned}&\left[(x_0^t, \dots, x_d^t) + \left(\left\{x_0^{b_0} x_1^{b_1} \dots x_d^{b_d} \,\middle|\, \sum b_j = (d+1)t - n, \right.\right.\right. \\ &\qquad \left.\left.\left. \text{with some } b_j \neq a_j \right\}\right)\right]^* : (x_0^{a_0} \dots x_d^{a_d}),\end{aligned}$$

which is an ideal containing w. We know that the formal colon is a monomial ideal in $x_0, \ldots, x_d$, so provided the colon ideal is not the unit ideal, the result is certainly contained in $(x_0, \ldots, x_d)^*$. But on the other hand, the formal colon is not the unit ideal, because it it were, then $x_0^{a_0} \ldots x_d^{a_d}$ would be divisible by one of the generators x_j^t or $x_0^{b_0} \ldots x_d^{b_d}$, and it is not. This forces $w \in (x_0, \ldots, x_d)^*$, contrary to the choice of w, completing the proof of the claim, and the proof of Theorem 3.

This raises an interesting question: if S is a section ring, can $(x_0, \ldots, x_d)^* = m$ for some polarized variety (X, L) other than $\mathbb{P}^n, \mathcal{O}(1)$? If L is very ample, we can not have $(x_0, \ldots, x_d)^* = m$ by Theorem C above. It is easy to see that in general, a graded ring can have $(x_0, \ldots, x_d)^* = m$. For example, if $S = k[x, y, z]/(x^2 + y^4 + z^4)$, the tight closure of (y, z) is (x, y, z) in all characteristics. However, this ring can not be a section ring because its Hilbert function is not eventually polynomial.

References

[AS] Angehrn, U. and Siu, Y.T., *Effective freeness and point separation for adjoint bundles,* Invent. Math. **122** (1995), 291–308.

[Ei] Ein, L., *The Ramification divisors for branched coverings of* $\mathbb{P}_k^n$, Math. Ann. **261** (1982), 483–485.

[EL] Ein, L., and Lazarsfeld, R., *Global generation of pluricanonical and adjoint linear series on*, smooth projective threefolds J. American Math Society **6** (1993), 875–903.

[H] Hara, N., *A Frobenius characterization of rational singularities,* Amer. Jour. Math. **120** (1998), 981–996.

[HH1] Hochster, M. and Huneke, C., *Tight closure, invariant theory, and the Briancon-Skoda theorem*, Jour. Amer. Math. Society **3** (1990), 31–116.

[Ka] Kawamata, Y., *On Fujita's freeness conjecture for threefolds and fourfolds,* Math. Ann. **308** (1997), 491–505.

[Ko] Kollár, *Singularities of Pairs, in Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., 62, Part 1,* Amer. Math. Soc. Providence, RI, 1997.

[S1] Smith, Karen E., *Fujita's conjecture in terms of local cohomology,* Jour. of Alg. Geom. **6** (1997), 417–429.

[S2] Smith, Karen E., *F-rational rings have rational singularities,* Amer. Jour. Math. **119 (1)** (1997), 159–180.

[S3] Smith, Karen E., *Tight closure in graded rings*, Journal of Mathematics of Kyoto University **37** (1997), 35–53.

[SSU] Simis, Aron, Smith, Karen E., and Ulrich, Bernd, *An algebraic proof of Zak's theorem on the dimension of the Gauss map* (preprint)