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# The Differentiable Structure of Three Remarkable Diffeomorphism Groups

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## § 1. Introduction

The goal of this paper is to find the "Lie groups" for three well-known Lie algebras: the globally Hamiltonian vector fields, the infinitesimal quantomorphisms and the homogeneous real-valued functions of degree one on the cotangent bundle minus the zero section. In all our considerations the underlying manifold is assumed to be compact without boundary. Before explaining and motivating our results, a very brief review of the "Lie group" structure of diffeomorphism groups is in order ([9, 10, 18]).

If $M$, $N$ are compact, boundaryless, finite-dimensional manifolds, $\tau_N: TN \to N$, the tangent bundle of $N$, an $H^s$-map, $s > (\dim M)/2$, from $M$ to $N$ has by definition all derivatives of order $\leqq s$ in any local chart square integrable. This notion is independent of the charts only for $s > (\dim M)/2$. Then the space $H^s(M, N)$ of all such maps is a Hilbert manifold whose tangent space at every point is the Hilbert space $T_f H^s(M, N) = \{g \in H^s(M, TN) \mid \tau_N \circ g = f\}$. If $E \to M$ is a vector bundle, the same construction works for all $H^s$-sections $H^s(E)$ of $E$, $s > (\dim M)/2$. If $N = \mathbb{R}$, $H^s(M, N)$ will be denoted by $C^s(M, \mathbb{R})$.

Let $\mathscr{D}^{s+1}(M)$ denote the diffeomorphisms of $M$ of Sobolev class $H^{s+1}$, i.e. $\eta \in \mathscr{D}^{s+1}(M)$ if and only if $\eta$ is bijective and $\eta, \eta^{-1}: M \to M$ are of class $H^{s+1}$. $\mathscr{D}^{s+1}(M)$ is a topological group, and since it is open in $H^{s+1}(M, M)$, it is also a Hilbert manifold. Right multiplication $R_\eta: \mathscr{D}^{s+1}(M) \to \mathscr{D}^{s+1}(M)$, $R_\eta(\xi) = \xi \circ \eta$ is $C^\infty$ for each $\eta \in \mathscr{D}^{s+1}(M)$ and if $\eta \in \mathscr{D}^{s+k+1}(M)$, left multiplication $L_\eta: \mathscr{D}^{s+1}(M) \to \mathscr{D}^{s+1}(M)$, $L_\eta(\xi) = \eta \circ \xi$ is $C^k$. The inversion map $\eta \mapsto \eta^{-1}$ in $\mathscr{D}^{s+1}(M)$ is only continuous. The tangent maps of $R_\eta$ and $L_\eta$ at $e$, the identity of $\mathscr{D}^{s+1}(M)$, are given by $T_e R_\eta(X) = X \circ \eta$, $T_e L_\eta(X) = T\eta \circ X$, where $X \in \mathscr{X}^{s+1}(M) = H^{s+1}(TM)$, the set of all $H^{s+1}$-vector fields on $M$. The tangent space at $e$, $T_e \mathscr{D}^{s+1}(M)$ coincides with $\mathscr{X}^{s+1}(M)$, which is a Sobolev space. The bracket $[X, Y]$ of $X, Y \in \mathscr{X}^{s+1}(M)$ is however only of class $H^s$ (one derivative is lost). The usual bracket of vector fields is the Lie algebra bracket of $\mathscr{X}^{s+1}(M)$, i.e. if $\tilde{X}$, $\tilde{Y}$

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denote the *right-invariant* vector fields corresponding to $X$, $Y$ on $\mathscr{D}^{s+1}(M)$, then $[\tilde{X}, \tilde{Y}]_e = [X, Y]$. Moreover, if $X \in \mathscr{X}^{s+1}(M)$, then its flow $\eta_t$ is a $C^1$-one-parameter subgroup of $\mathscr{D}^{s+1}(M)$ and one can define the exponential map $\exp: \mathscr{X}^{s+1}(M) \to \mathscr{D}^{s+1}(M)$ by $X \mapsto \eta_1$. The map exp is continuous, but is not even $C^1$ since it does not cover a neighborhood of the identity. That's how far $\mathscr{D}^{s+1}(M)$ is a "Lie group" and $\mathscr{X}^{s+1}(M)$ its "Lie algebra". Throughout this paper we shall call such a structure a *$H^{s+1}$-Lie group* with a *$H^{s+1}$-Lie algebra.*

Omori [18] defined a differentiable structure on $A^\infty = \bigcap A^s$, where $A^s$ is an $H^s$-Lie group with $H^s$-Lie algebra $\mathscr{A}^s$, as the limit of the topologies of $A^s$ and called $A^\infty$ an ILH (inverse limit of Hilbert) Lie group with ILH-Lie algebra $\mathscr{A}^\infty = \bigcap \mathscr{A}^s$. If $A = \bigcap A^s$, $B = \bigcap B^s$ are ILH-manifolds, a map $f: A \to B$ is $C^k$ if and only if for every index $s$, there exists an index $s'$ such that $f$ has a $C^k$-extension $\bar{f}: A^{s'} \to B^s$. In this sense $\mathscr{D}^\infty(M)$, the group of $C^\infty$-diffeomorphisms of $M$, becomes a Lie group since now left translation and inversion are smooth; the Lie algebra of $\mathscr{D}^\infty(M)$ is $\mathscr{X}^\infty(M)$, the $C^\infty$-vector fields on $M$. Closed Lie subalgebras of ILH-Lie algebras do not necessarily arise from ILH-Lie subgroups. We shall prove that the three Lie algebras considered have closed underlying ILH-Lie subgroups.

Throughout the paper $M$ is compact, finite-dimensional, connected, boundaryless. Define $\mathscr{D}_\chi^{s+1}(M) = \{\eta \in \mathscr{D}^{s+1}(M) \mid \eta^* \chi = \chi\}$, $\mathscr{X}_\chi^{s+1}(M) = \{X \in \mathscr{X}^{s+1}(M) \mid L_X \chi = 0\}$, where $\chi$ is a $p$-form or a vector field and it is always assumed that $s > (\dim M)/2$. Formally $\mathscr{D}_\chi^\infty(M)$ is a Lie group with Lie algebra $\mathscr{X}_\chi^\infty(M)$.

In §2 we consider the following problem. Let $(M, \omega)$ be a compact, symplectic manifold, i.e. $\omega$ is a non-degenerate, closed two-form on $M$. It is known ([9, 18]) that $\mathscr{D}_\omega^\infty(M)$ is an ILH-Lie subgroup of $\mathscr{D}^\infty(M)$ with Lie algebra $\mathscr{X}_\omega^\infty(M)$, the locally Hamiltonian vector fields, and that the commutator algebra $[\mathscr{X}_\omega^\infty(M), \mathscr{X}_\omega^\infty(M)] = \mathscr{H}^\infty(M)$, the globally Hamiltonian vector fields ([2, 3, 8]); recall that a globally Hamiltonian vector field corresponds to the differential of a function under the isomorphism between one-forms and vector fields on $M$ defined by $\omega$. Is there an ILH-Lie subgroup with Lie algebra $\mathscr{H}^\infty(M)$? We prove that this is the case and the group is the commutator subgroup $[\mathscr{D}_\omega^\infty(M)_0, \mathscr{D}_\omega^\infty(M)_0]$ of the identity component $\mathscr{D}_\omega^\infty(M)_0$ of $\mathscr{D}_\omega^\infty(M)$, under some conditions on the first homotopy of $\mathscr{D}_\omega^\infty(M)_0$ which seem always to be satisfied. The proof is based on techniques of Ebin and Marsden [9] and constructs for the $H^{s+1}$-Lie algebra of $H^{s+1}$ globally Hamiltonian vector fields an $H^{s+1}$ closed Lie subgroup of $\mathscr{D}_\omega^{s+1}(M)_0$.

The next two sections represent a first step in a bigger program: the understanding of the Lie group structure of invertible Fourier integral operators of order zero on a compact manifold with canonical relation a canonical transformation. It is believed that remarkable completely integrable wave equations (like KdV, Boussinesq) live on co-adjoint orbits of this group. The formal Lie algebra of this group is formed by the pseudo-differential operators of order one with homogeneous purely imaginary principal symbol. The Lie algebras considered in §3, §4 are exactly these symbols, and thus their corresponding "Lie groups" will be the canonical relations of these Fourier integral operators.

Let $(M, \theta)$ be an exact, regular, compact, contact manifold, i.e. $\theta$ is a one-form such that $\theta \wedge (d\theta)^n$ is a volume element of $M$, $\dim M = 2n+1$, and $M$ is a principal circle bundle over a symplectic manifold $(N, \omega)$ with connection form $\theta$ and curvature $\omega$. In § 3 it is shown that the identity component $\mathscr{D}_\theta^{s+1}(M)_0$ of the quantomorphism group $\mathscr{D}_\theta^{s+1}(M)$ is a principal circle bundle over the $H^{s+1}$-extension of $[\mathscr{D}_\omega^\infty(N)_0, \mathscr{D}_\omega^\infty(N)_0]$. In the course of proving this we reformulate some of Konstant's prequantization theorems [15] in terms of principal circle bundles, thereby shortening the original proofs. We also give a very short direct proof of the fact that $\mathscr{D}_\theta^{s+1}(M)$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}^{s+1}(M)$, a fact shown already in Omori [18] in a different and considerably longer way. If $M$ is the sphere bundle of a Riemannian manifold $V$, the smooth infinitesimal quantomorphisms $\mathscr{X}_\theta^\infty(M)$ are isomorphic to the principal symbols of pseudo-differential operators of order one on $V$ with purely imaginary symbol commuting with the Laplacian of the metric on $V$.

The general principal symbol of a pseudo-differential operator of order one lives on the cotangent bundle minus the zero section. But this manifold is *not* compact, so classical methods of manifolds of maps break down. However, if these symbols are homogeneous, § 4 shows that this Lie algebra is isomorphic to the Lie algebra of infinitesimal contact transformations of the cosphere bundle, which has as underlying ILH-Lie subgroup the group of contact transformations of the cosphere bundle. This result cannot be derived from § 3, since the cosphere bundle has no canonical exact contact structure, but carries a whole family of them, two differing by a factor which is a strictly positive function. This group of contact transformations is shown to be isomorphic to the group of homogeneous canonical diffeomorphisms of the cotangent bundle minus the zero section. In this way, canonical relations of a large class of invertible Fourier integral operators of order zero carry the structure of an ILH-Lie group. The differentiable structure of the group of invertible Fourier integral operators of order zero with canonical relation a canonical transformation will be analyzed in a forthcoming paper.

## § 2. The $H^{s+1}$-Lie Group for the Globally Hamiltonian Vector Fields

It is known that the commutator algebra of $C^\infty$-locally Hamiltonian vector fields $\mathscr{X}_\omega^\infty(M)$ coincides with the $C^\infty$-globally Hamiltonian vector fields $\mathscr{H}^\infty(M)$ ([2, 3, 8]), and that the commutator group of the identity component of $C^\infty$-symplectic diffeomorphisms $\mathscr{D}_\omega^\infty(M)$ is simple (Banyaga [4]). In the spirit of classical Lie theory it is to be expected that the "Lie group" underlying $\mathscr{H}^\infty(M)$ should be the commutator group of the identity component of $\mathscr{D}_\omega^\infty(M)$. This section proves that this is indeed the case in the ILH-sense of Omori [18] by determining explicitly an $H^{s+1}$-Lie group $\mathscr{G}^{s+1}(M)$ for the $H^{s+1}$-globally Hamiltonian vector fields $\mathscr{H}^{s+1}(M)$ on a compact, symplectic manifold $(M, \omega)$. It is *not* true that $\mathscr{H}^{s+1}(M)$ equals $[\mathscr{X}_\omega^{s+1}(M), \mathscr{X}_\omega^{s+1}(M)]$ due to the loss of derivatives. $\mathscr{G}^{s+1}(M)$ will be the kernel of a certain map defined by Calabi [8] and the main technical difficulty is the proof of its $C^\infty$-smoothness.

We start with a few preparatory remarks. Let $\sigma\colon \mathbb{R} \to \Omega^p(M)$ be a time-dependent smooth $p$-form. For each $a, b \in \mathbb{R}$ the definite integral $\int_a^b \sigma$ is a $p$-form defined by $x \mapsto \int_a^b \sigma_t(x)\,dt$, the integration taking place in the fiber at the point $x$. Clearly the integration commutes with the exterior derivative $d$, the interior product $i_X = i(X)$, the Lie derivative $L_X = L(X)$, and pull-backs by smooth maps.

If $\mathscr{D}^{s+1}(M)$, $s > \frac{1}{2}\dim(M)$, denotes the $H^{s+1}$-diffeomorphisms of the compact manifold $M$, an *isotopy* $h$: $[0,1] \to \mathscr{D}^{s+1}(M)$, where $h(0) = e =$ the identity on $M$, is induced by the smooth map $\bar{h}$: $[0,1] \times M \to M$ defined by $\bar{h}(t,x) = h(t)(x)$. A diffeomorphism $\eta \in \mathscr{D}^{s+1}(M)$ is said to be isotopic to $e$, if there exists an isotopy $h$ with $h(1) = \eta$. Two isotopies $h$ and $h'$ are called *homotopic*, if $h(1) = h'(1) = \eta$ and if there exists a smooth homotopy $H: [0,1] \times [0,1] \to \mathscr{D}^{s+1}(M)$ such that, with $H_{s,t} = H(s,t)$, $H_{s,0} = e$, $H_{s,1} = \eta$ for all $s \in [0,1]$ and $H_{0,t} = h(t)$, $H_{1,t} = h'(t)$ for all $t \in [0,1]$. Since the group $\mathscr{D}^{s+1}(M)$ is locally arcwise connected (by smooth arcs), the connected component $\mathscr{D}^{s+1}(M)_0$ of $e$ consists of all diffeomorphisms isotopic to $e$. Let $\widetilde{\mathscr{D}^{s+1}(M)_0}$ denote the universal covering of $\mathscr{D}^{s+1}(M)_0$. An element $\tilde{h} \in \widetilde{\mathscr{D}^{s+1}(M)_0}$ is a pair $(h, [h_t])$ for $[h_t]$ a homotopy class of isotopies from $h \in \mathscr{D}^{s+1}(M)_0$ to $e$.

**Lemma 2.1.** *Let $h_{s,t}$ be a two-parameter family of smooth diffeomorphisms on $M$ such that $h_{0,0} = e$. Define the families of vectorfields $X_{s,t}$ and $Y_{s,t}$ by*

$$X_{s,t} = \frac{\partial h_{s,t}}{\partial t} \circ h_{s,t}^{-1}, \qquad Y_{s,t} = \frac{\partial h_{s,t}}{\partial s} \circ h_{s,t}^{-1}.$$

*Then*

$$[X_{s,t}, Y_{s,t}] = \frac{\partial X_{s,t}}{\partial s} - \frac{\partial Y_{s,t}}{\partial t}. \tag{2.1}$$

The proof is a direct calculation showing that the principal part of both sides is in local charts $DY_{s,t}(X_{s,t}) - DX_{s,t}(Y_{s,t})$, $D$ denoting the Fréchet derivative.

Let $(M, \omega)$ be a compact, symplectic manifold and $\mathscr{D}_\omega^{s+1}(M) = \{\eta \in \mathscr{D}^{s+1}(M) \mid \eta^* \omega = \omega\}$ the group of symplectic $H^{s+1}$-diffeomorphisms. It is known (Ebin-Marsden [9]) that $\mathscr{D}_\omega^{s+1}(M)$ is an $H^{s+1}$-Lie group with Lie algebra $\mathscr{X}_\omega^{s+1}(M) = \{X \in \mathscr{X}^{s+1}(M) \mid L_X \omega = 0\}$. Let $\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$ denote the universal covering of the connected component $\mathscr{D}_\omega^{s+1}(M)_0$ of $e$. Calabi [8] introduced a continuous, surjective group homomorphism $\tilde{S}$: $\widetilde{\mathscr{D}_\omega^{s+1}(M)_0} \to H^1(M, \mathbb{R})$, where $H^1(M, \mathbb{R})$ denotes the first cohomology group of $M$, which we now describe (see also Banyaga [4]).

Let $\tilde{h} = (h, [h_t]) \in \widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$, i.e. $h \in \mathscr{D}_\omega^{s+1}(M)_0$ and $[h_t]$ is a homotopy class of symplectic isotopies from $h$ to $e$. Since $h_t$ is symplectic, the vector field $X_t = \frac{dh_t}{dt} \circ h_t^{-1}$ is locally Hamiltonian, i.e. $i(X_t)\,\omega$ is a closed $H^{s+1}$-one-form on $M$. Thus $A(h_t) = \int_0^1 i(X_t)\,\omega\,dt$ is a closed $H^{s+1}$-one-form defining a cocycle in $H^1(M, \mathbb{R})$. Let $h'_t$ be another isotopy in $\mathscr{D}_\omega^{s+1}(M)_0$ such that $h'_1 = h_1 = h$, $h'_0 = h_0 = e$ and there exists a smoothy homotopy $H$: $[0,1] \times [0,1] \to \mathscr{D}_\omega^{s+1}(M)_0$ with $H_{s,0} = e$, $H_{s,1} = h$, $H_{0,t} = h_t$, $H_{1,t} = h'_t$ for all $s, t \in [0,1]$. Consider the vector fields

$$X_{s,t}=\frac{\partial H_{s,t}}{\partial t}\circ H_{s,t}^{-1}, \qquad Y_{s,t}=\frac{\partial H_{s,t}}{\partial s}\circ H_{s,t}^{-1}$$

and let $A(H_{s,t})=\int_0^1 i(X_{s,t})\,\omega\,dt$. Clearly $A(H_{0,t})=A(h_t)$, $A(H_{1,t})=A(h'_t)$. We are going to prove that

$$A(h'_t)-A(h_t)=d\alpha \qquad \text{where } \alpha=-\int_0^1\int_0^1 \omega(X_{s,t},\ Y_{s,t})\,ds\,dt.$$

Lemma 2.1, $L(X_{s,t})\,\omega=0$, and $Y_{s,1}=Y_{s,0}=0$ give

$$\begin{aligned}\frac{\partial}{\partial s}A(H_{s,t})&=\int_0^1 i\left(\frac{\partial X_{s,t}}{\partial s}\right)\omega\,dt=\int_0^1 i\left(\frac{\partial Y_{s,t}}{\partial t}\right)\omega\,dt+\int_0^1 i([X_{s,t},\ Y_{s,t}])\,\omega\,dt\\ &=\int_0^1 \frac{\partial}{\partial t} i(Y_{s,t})\,\omega\,dt+\int_0^1 L(X_{s,t})\,i(Y_{s,t})\,\omega\,dt-\int_0^1 i(Y_{s,t})\,L(X_{s,t})\,\omega\,dt\\ &=i(Y_{s,1})\,\omega-i(Y_{s,0})\,\omega+\int_0^1 di(X_{s,t})\,i(Y_{s,t})\,\omega\,dt\\ &=-d\int_0^1 \omega(X_{s,t},\ Y_{s,t})\,dt.\end{aligned}$$

Therefore

$$A(h'_t)-A(h_t)=\int_0^1 \frac{\partial}{\partial s}A(H_{s,t})\,ds=-d\int_0^1\int_0^1 \omega(X_{s,t},\ Y_{s,t})\,ds\,dt.$$

This shows that the cohomology class $[A(h_t)]$ does not depend on the choice of $h_t$ in its homotopy class and so we can define the map

$$\tilde{S}\colon \widetilde{\mathscr{D}_\omega^{s+1}(M)_0}\to H^1(M,\mathbb{R}) \qquad \text{by } \tilde{S}(\tilde{h})=[A(h_t)].$$

It is straightforward to see that $\tilde{S}$ is a continous group homomorphism. For surjectivity, if $\beta$ is a closed $H^{s+1}$-one-form, $X$ defined by $i_X\omega=\beta$ is $H^{s+1}$ and $\tilde{S}(\tilde{\phi})=[\beta]$, where $\tilde{\phi}=(\phi_1,\ [\phi_t])\in\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$ and $\phi_t$ is the $H^{s+1}$-flow of $X$.

An equivalent definition of $\tilde{S}$ which will not be used, is the following (Calabi [8]): the value of $\tilde{S}(\tilde{h})$ on a one-cycle $\alpha\colon[0,1]\to M$ is the integral of $\omega$ over the image $\bar{h}([0,1],\ \alpha([0,1]))$ of $\alpha$ by the isotopy $\bar{h}$, i.e.

$$(\tilde{S}(\tilde{h}))(\alpha)=\int_\alpha A(h_t)=\int_{\bar{h}([0,1],\,\alpha([0,1]))}\omega.$$

The proof of this formula is a straightforward verification.

**Proposition 2.1.** *The map* $\tilde{S}\colon\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}\to H^1(M,\mathbb{R})$ *is a* $C^\infty$*-submersion.*

*Proof.* We start out by showing that $\tilde{S}$ has a derivative. Let $h\colon[0,1]\to\mathscr{D}_\omega^{s+1}(M)_0$ be a smooth curve through the identity, i.e. $h(s)=(h_s,\ [h_{s,t}])$, $h_{s,0}=e$, $h_{s,1}=h_s$, $h(0)=\tilde{e}=(h_0,\ [h_{0,t}])$, $h_0=h_{0,t}=e$, and let $\tilde{X}\in T_{\tilde{e}}\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$ be the tangent vector to this curve at $\tilde{e}$, i.e.

$$\tilde{X}=\left.\frac{dh(s)}{ds}\right|_{s=0}=\left.\frac{d}{ds}\right|_{s=0}(h_s,[h_{s,t}])=\left(\left.\frac{dh_s}{ds}\right|_{s=0},\left.\frac{d}{ds}\right|_{s=0}[h_{s,t}]\right)=(X,[X_t]),$$

where

$$X=\left.\frac{dh_s}{ds}\right|_{s=0}\in T_e\mathscr{D}_\omega^{s+1}(M)_0 \quad\text{and}\quad X_t=\left.\frac{dh_t(s)}{ds}\right|_{s=0},$$

for $h_t\colon[0,1]\to\mathscr{D}_\omega^{s+1}(M)_0$, $h_t(s)=h_{s,t}$, $t\in[0,1]$. We have

$$X_0=\left.\frac{dh_0(s)}{ds}\right|_{s=0}=0 \quad\text{and}\quad X_1=\left.\frac{dh_1(s)}{ds}\right|_{s=0}=\left.\frac{dh_s}{ds}\right|_{s=0}=X.$$

If $X_{s,t}=\frac{\partial h_{s,t}}{\partial t}\circ h_{s,t}^{-1}$ we get

$$\begin{aligned}(T_{\tilde{e}}\tilde{S})(\tilde{X})&=\left.\frac{d}{ds}\right|_{s=0}(\tilde{S}\circ h)(s)=\left.\frac{d}{ds}\right|_{s=0}\tilde{S}(h_s,[h_{s,t}])\\&=\left.\frac{d}{ds}\right|_{s=0}\left[\int_0^1 i(X_{s,t})\,\omega\,dt\right]=\left[\int_0^1 i\left(\left.\frac{\partial X_{s,t}}{\partial s}\right|_{s=0}\right)\omega\,dt\right].\end{aligned}$$

But since $h_{0,t}=e$ and $X_{0,t}=0$ it follows that

$$\left.\frac{\partial X_{s,t}}{\partial s}\right|_{s=0}=\left.\frac{\partial^2 h_{s,t}}{\partial s\,\partial t}\right|_{s=0}\circ h_{0,t}^{-1}-TX_{0,t}\circ\left.\frac{\partial h_{s,t}}{\partial s}\right|_{s=0}\circ h_{0,t}^{-1}=\left.\frac{\partial^2 h_{s,t}}{\partial s\,\partial t}\right|_{s=0}.$$

Thus using $h_{s,0}=e$, $\frac{\partial h_{s,0}}{\partial s}=0$ it follows that

$$\begin{aligned}T_{\tilde{e}}\tilde{S}(\tilde{X})&=\left[\int_0^1 i\left(\left.\frac{\partial^2 h_{s,t}}{\partial s\,\partial t}\right|_{s=0}\right)\omega\,dt\right]=\left[\int_0^1\frac{d}{dt}\,i\left(\left.\frac{\partial h_{s,t}}{\partial s}\right|_{s=0}\right)\omega\,dt\right]\\&=\left[i\left(\left.\frac{\partial h_{s,1}}{\partial s}\right|_{s=0}\right)\omega-i\left(\left.\frac{\partial h_{s,0}}{\partial s}\right|_{s=0}\right)\omega\right]=[i(X_1)\,\omega]=[i_X\omega].\end{aligned}$$

The above shows that $\tilde{S}$ is Gateaux differentiable at $\tilde{e}$.

Working in a local chart at $\tilde{e}\in\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$, from the fact that $\tilde{S}$ is a Lie group homomorphism, we conclude that $D\tilde{S}(\eta)=D\tilde{S}(e)\circ R_{\eta^{-1}}$, where $R_{\eta^{-1}}(X)=X\circ\eta^{-1}$, for all $\eta\in\mathscr{D}_\omega^{s+1}(M)_0$ in the chart; this local chart is chosen to be contained in $\mathscr{D}_\omega^{s+1}(M)_0$ and centered at $e$. Since $\eta\mapsto\eta^{-1}$ is continuous, $\eta\mapsto D\tilde{S}(\eta)$ is continuous and so $\tilde{S}$ is $C^1$. Moreover since $T_e\mathscr{D}_\omega^{s+1}(M)_0=\mathscr{X}_\omega^{s+1}(M)$, $T_{\tilde{e}}\tilde{S}$ is clearly surjective. Thus, by right translations, $T_{\tilde{\eta}}\tilde{S}$ is surjective for all $\tilde{\eta}\in\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$. Therefore $\tilde{S}$ is a $C^1$-submersion and hence its kernel is a $C^1$-submanifold of $\widetilde{\mathscr{D}_\omega^{s+1}(M)_0}$, whose tangent space at $\tilde{e}$ is $\operatorname{Ker}(T_{\tilde{e}}\tilde{S})=\{X\in\mathscr{X}_\omega^{s+1}(M)\,|\,[i_X\omega]=0\in H^1(M,\mathbb{R})\}=\mathscr{H}^{s+1}(M)$, i.e. all globally Hamiltonian $H^{s+1}$-vector fields. Clearly $\mathscr{H}^{s+1}(M)$ is a closed subspace of $\mathscr{X}_\omega^{s+1}(M)$ and of finite codimension equal to $\dim(H^1(M,\mathbb{R}))$, since $M$ is compact. Therefore there exists a closed complement $\mathscr{C}^{s+1}$ in $\mathscr{X}_\omega^{s+1}(M)$, i.e.

$$\mathscr{X}_\omega^{s+1}(M)=\mathscr{H}^{s+1}(M)\oplus\mathscr{C}^{s+1}. \tag{2.2}$$

For each $\eta \in \mathscr{D}_{\omega}^{s+1}(M)_0$ put $\mathscr{H}_{\eta}^{s+1} = R_{\eta}\mathscr{H}^{s+1}$, $\mathscr{C}_{\eta}^{s+1} = R_{\eta}\mathscr{C}^{s+1}$

$$\overline{\mathscr{H}}^{s+1} = \bigcup_{\eta} \mathscr{H}_{\eta}^{s+1}, \qquad \overline{\mathscr{C}}^{s+1} = \bigcup_{\eta} \mathscr{C}_{\eta}^{s+1}$$

By the Hodge decomposition with respect to an arbitrary Riemannian metric on $M$, closed $H^{s+1}$-forms split $L^2$-orthogonally in exact $H^{s+1}$-one-forms and harmonic one-forms which are $C^{\infty}$. Equation (2.2) represents this splitting in terms of vectorfields and therefore we conclude that $\mathscr{C}^{s+1} = \mathscr{C}^{\infty}$, i.e. all elements of $\mathscr{C}^{s+1}$, the finite-dimensional complement of $\mathscr{H}^{s+1}(M)$, are $C^{\infty}$-smooth. The following lemma shows that we have a $C^{\infty}$-Whitney sum

$$T\mathscr{D}_{\omega}^{s+1}(M)_0 = \overline{\mathscr{H}}^{s+1} \oplus \overline{\mathscr{C}} \tag{2.3}$$

with $\overline{\mathscr{C}} = \operatorname{Im}(\bar{P})$, $\overline{\mathscr{H}}^{s+1} = \operatorname{Ker}(\bar{P})$, for $\bar{P}_{\eta} = R_{\eta} \circ P \circ R_{\eta^{-1}}$, $P: \mathscr{X}_{\omega}^{s+1}(M) \to \mathscr{C}^{\infty}$ the canonical projection given by (2.2).

**Lemma 2.2** (Ebin-Marsden [9]). *Let $E$ be a vector bundle over $M$ and let $\mathscr{C}$ be a finite-dimensional subspace of the $H^{s+1}$-sections $H^{s+1}(E)$ of $E$, all of whose elements are $C^{\infty}$. Let $P: H^l(E) \to \mathscr{C}$ be a continuous projection, $l \leq s+1$. Then*

(i) $\overline{\mathscr{C}} = \bigcup_{\eta \in \mathscr{D}^{s+1}(M)} R_{\eta}\mathscr{C}$ *is a $C^{\infty}$-subbundle of $H^l(M,E) \mid \mathscr{D}^{s+1}(M)$;*

(ii) $\bar{P}$ *defined by $\bar{P}_{\eta} = R_{\eta} \circ P \circ R_{\eta^{-1}}$ is a $C^{\infty}$-bundle map over the identity;*

(iii) $\overline{\mathscr{C}} = \operatorname{Im}(\bar{P})$, $\operatorname{Ker}(\bar{P})$, $\operatorname{Im}(Id - \bar{P})$, $\operatorname{Ker}(Id - \bar{P})$ *are all $C^{\infty}$-subbundles of $H^l(M,E)$ over $\mathscr{D}^{s+1}(M)$.*

Thus working in a local chart around $\tilde{e}$, we see from (2.3) that $D\tilde{S} \mid \overline{\mathscr{H}}^{s+1} \equiv 0$ and $D\tilde{S}(e): \mathscr{C} \to H^1(M, \mathbb{R})$ is an isomorphism. Consider the trivial bundle $H^1(M,\mathbb{R}) \times \mathscr{D}_{\omega}^{s+1}(M)_0$ over $\mathscr{D}_{\omega}^{s+1}(M)_0$ and define the bundle map $\phi: \overline{\mathscr{C}} \to H^1(M,\mathbb{R}) \times \mathscr{D}_{\omega}^{s+1}(M)_0$, $\phi_{\eta} = D\tilde{S}(\eta) \mid R_{\eta}\mathscr{C}$. Since the $C^{\infty}$-smooth subbundle $\overline{\mathscr{C}}$ is obtained by right-translations of $\mathscr{C}$, the inverse

$$\phi_{\eta}^{-1} = R_{\eta} \circ (DS(\tilde{e}) \mid \mathscr{C})^{-1}: H^1(M,\mathbb{R}) \to R_{\eta}\mathscr{C}$$

is smooth in $\eta$. Thus $\phi$ is the inverse of a $C^{\infty}$-vector bundle isomorphism and hence is $C^{\infty}$-smooth. We showed hence that $\tilde{S}$ is a $C^{\infty}$-smooth submersion.

This proposition implies the following.

**Theorem 2.1.** $\operatorname{Ker}(\tilde{S})$ *is a closed $H^{s+1}$-Lie subgroup of $\widetilde{\mathscr{D}_{\omega}^{s+1}(M)_0}$ with Lie algebra $\mathscr{H}^{s+1}(M)$.*

The kernel of the covering map $\widetilde{\mathscr{D}_{\omega}^{s+1}(M)_0} \to \mathscr{D}_{\omega}^{s+1}(M)_0$ is the first homotopy group $\pi_1(\mathscr{D}_{\omega}^{s+1}(M)_0)$ which is discrete. Let us assume that its image $\Gamma$ under $\tilde{S}$ is closed and hence discrete in $H^1(M,\mathbb{R})$. Then by Theorem 2.1 the quotient map

$$S: \mathscr{D}_{\omega}^{s+1}(M)_0 \to H^1(M,\mathbb{R})/\Gamma$$

is a $C^{\infty}$-smooth submersion. Thus we get the following.

**Theorem 2.2.** *If $\Gamma$ is closed in $H^1(M,\mathbb{R})$, then $\mathscr{G}^{s+1} = \operatorname{Ker}(S)$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}_{\omega}^{s+1}(M)_0$ and of $\mathscr{D}_{\omega}^{s+1}(M)$ with Lie algebra $\mathscr{H}^{s+1}(M)$.*

It is known that $\Gamma$ is discrete if $\dim(M)=2$, if the symplectic form of $M$ is defined by a Kaehler structure, or if the symplectic form on $M$ has integral periods. To our knowledge there is no known example for which $\Gamma$ is not discrete (see also Banyaga [4], [5]).

Banyaga [4] proved that for $s=\infty$, $\operatorname{Ker}(S)=\mathscr{G}^\infty=[\mathscr{D}^\infty_\omega(M)_0, \mathscr{D}^\infty_\omega(M)_0]$ and that this commutator group is simple. He also showed that $\operatorname{Ker}(\tilde{S})=[\widetilde{\mathscr{D}^\infty_\omega(M)_0}, \widetilde{\mathscr{D}^\infty_\omega(M)_0}]$ is a perfect group, i.e. it coincides with its own commutator subgroup. Hence we conclude

**Theorem 2.3.** $[\widetilde{\mathscr{D}^\infty_\omega(M)_0}, \widetilde{\mathscr{D}^\infty_\omega(M)_0}]$ *is a perfect, closed ILH-Lie subgroup of* $\widetilde{\mathscr{D}^\infty_\omega(M)_0}$ *and* $[\mathscr{D}^\infty_\omega(M)_0, \mathscr{D}^\infty_\omega(M)_0]$ *is a simple, closed ILH-Lie subgroup of* $\mathscr{D}^\infty_\omega(M)_0$, *both with Lie algebra* $\mathscr{H}^\infty(M)=[\mathscr{X}^\infty_\omega(M), \mathscr{X}^\infty_\omega(M)]$, *the* $C^\infty$*-globally Hamiltonian vector fields, if* $\Gamma$ *is closed.*

Remark that $\mathscr{G}^{s+1}$ is connected since $\mathscr{G}^\infty=[\mathscr{D}^\infty_\omega(M)_0, \mathscr{D}^\infty_\omega(M)_0]$ is dense in it, and that the covering projection $\widetilde{\mathscr{D}^{s+1}_\omega(M)_0}\to\mathscr{D}^{s+1}_\omega(M)_0$ induces a surjective map $\operatorname{Ker}(\tilde{S})\to\operatorname{Ker}(S)=\mathscr{G}^{s+1}$ with kernel $\pi_1(\mathscr{D}^{s+1}_\omega(M)_0)\cap\operatorname{Ker}(\tilde{S})$. The classical construction of bundle charts for homogeneous spaces (e.g. [11], p. 83) shows that $\tilde{S}$, $S$ are topological locally trivial bundles; these bundle charts are *not* smooth since they involve composition and taking inverses. The homotopy exact sequence for the locally trivial fiber bundle

$$0\to\operatorname{Ker}(\tilde{S})\to\widetilde{\mathscr{D}^{s+1}_\omega(M)_0}\xrightarrow{\tilde{S}}H^1(M,\mathbb{R})\to 0$$

shows that $\pi_1(\operatorname{Ker}(\tilde{S}))=0$ and hence $\operatorname{Ker}(\tilde{S})$ is the universal covering group $\widetilde{\mathscr{G}^{s+1}}$ of $\mathscr{G}^{s+1}$. These results are summarized in the following commutative diagram with all lines and columns exact.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0\to\pi_1(\mathscr{G}^{s+1}) & = & \pi_1(\mathscr{D}^{s+1}_\omega(M)_0)\cap\widetilde{\mathscr{G}^{s+1}} & \hookrightarrow & \pi_1(\mathscr{D}^{s+1}_\omega(M)_0) & \xrightarrow{\tilde{S}} & \Gamma\longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widetilde{\mathscr{G}^{s+1}} & \hookrightarrow & \widetilde{\mathscr{D}^{s+1}_\omega(M)_0} & \xrightarrow{\tilde{S}} & H^1(M,\mathbb{R})\to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{G}^{s+1} & \hookrightarrow & \mathscr{D}^{s+1}_\omega(M)_0 & \xrightarrow{S} & H^1(M,\mathbb{R})/\Gamma\to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

The columns are covering spaces and the lines are topological principal bundles. This diagram for $s=\infty$ appears already in Banyaga [4]. We shall obtain a similar diagram in the next section for the group of quantomorphisms of a compact, regular, exact, contact manifold.

## §3. The Group of Quantomorphisms

In this section we prove that the group of quantomorphisms of a compact, exact, regular, contact manifold (see definitions below) is a smooth principal circle bundle over the $H^{s+1}$-Lie group $\mathscr{G}^{s+1}$ of the quotient manifold constructed in the previous section.

We start by recalling a few standard facts from prequantization ([1, 12, 15, 19, 22]) in the setting of principal circle bundles with connection.

Let $(M, \theta)$ be a compact, exact contact manifold, i.e. $M$ is a smooth $(2n+1)$-dimensional compact manifold with a $C^\infty$-one-form $\theta$ such that $\theta \wedge (d\theta)^n$ is a volume element. The characteristic bundle of $d\theta$

$$R_{d\theta} = \{v \in TM \mid i_v \, d\theta = 0\}$$

is integrable and one-dimensional. At each point, there exists a chart $(x^1, \ldots, x^n, y^1, \ldots, y^n, t)$ such that $\theta = \sum_{i=1}^{n} y^i \, dx^i + dt$. The *Reeb vector field* $E$ is the unique section of the line bundle $R_{d\theta}$ defined by $i_E \theta = 1$, $i_E \, d\theta = 0$; in the charts above, $E = \partial/\partial t$. Let

$$R_\theta = \{v \in TM \mid \theta(v) = 0\}$$

be the characteristic bundle of $\theta$; it is $2n$-dimensional, $TM = R_{d\theta} \oplus R_\theta$, and hence

$$\mathscr{X}^{s+1}(M) = H^{s+1}(R_{d\theta}) \oplus H^{s+1}(R_\theta). \tag{3.1}$$

We have

$$[H^{s+1}(R_\theta), E] \subseteq H^s(R_\theta) \tag{3.2}$$

since for all $Y \in H^{s+1}(R_\theta)$, $\theta([Y, E]) = (d\theta)(Y, E) - Y(\theta(E)) + E(\theta(Y)) = 0$. Note that $H^{s+1}(R_{d\theta}) = \{fE \mid f \in C^{s+1}(M, \mathbb{R})\}$. Since $H^{s+1}(R^*_{d\theta}) = \{\alpha \in H^{s+1}(\Lambda^1) \mid \alpha(X) = 0$ for all $X \in H^{s+1}(R_\theta)\}$, $H^{s+1}(R^*_\theta) = \{\alpha \in H^{s+1}(\Lambda^1) \mid \alpha(X) = 0$ for all $X \in H^{s+1}(R_{d\theta})\}$, it follows

$$H^{s+1}(\Lambda^1) = H^{s+1}(R^*_{d\theta}) \oplus H^{s+1}(R^*_\theta). \tag{3.3}$$

Remark that $X \in H^{s+1}(R_\theta) \mapsto i_X \, d\theta \in H^{s+1}(R^*_\theta)$ is an isomorphism, since $d\theta$ is non-degenerate on $R_\theta$.

The leaves of the foliation $\mathscr{F}$, defined by the integrable line bundle $R_{d\theta}$ are the integral curves of $E$. In all that follows we assume that $\theta$ is *regular* (Boothby-Wang [7], Weinstein [22]), i.e. the flow of $E$ defines a free circle action $\Phi$ on $M$. Then the quotient manifold $N = M/\mathscr{F}$ is automatically smooth and carries a symplectic structure $\omega$, $[\omega] \in H^2(N, \mathbb{Z})$ (the second cohomology group of $N$ with integer coefficients) such that $\pi^* \omega = d\theta$, for $\pi \colon M \to N$ the canonical projection; $\theta$ becomes thus the connection one-form on this principal bundle whose horizontal subbundle is $R_\theta$ and $\omega$ is its curvature form. $M$ is thus the *quantizing manifold* of $N$ whose automorphism group $\mathscr{D}^{s+1}_\theta(M) = \{\eta \in \mathscr{D}^{s+1}(M) \mid \eta^* \theta = \theta\}$,

$s>(\dim M)/2$, is called the *quantomorphism group* of $(M, \theta)$. Its formal Lie algebra of *infinitesimal quantomorphisms* is $\mathscr{X}_\theta^{s+1}(M)=\{X\in\mathscr{X}^{s+1}(M) \mid L_X\theta=0\}$.

Remark that $\eta^*(E)=E$ if and only if $\eta$ is equivariant with respect to the free circle action $\Phi$ and that $\mathscr{D}_\theta^{s+1}(M)\subset\mathscr{D}_E^{s+1}(M)=\{\eta\in\mathscr{D}^{s+1}(M) \mid \eta^*(E)=E\}$, $\mathscr{X}_\theta^{s+1}(M)\subset\mathscr{X}_E^{s+1}(M)=\{X\in\mathscr{X}^{s+1}(M) \mid [X, E]=0\}$. In fact $\mathscr{D}_E^{s+1}(M)$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}^{s+1}(M)$ with $H^{s+1}$-Lie algebra $\mathscr{X}_E^{s+1}(M)$ in view of the following general result of Marsden-Ebin-Fischer [17, p. 167] and Omori [18]. Let $K$ be a compact subgroup of $\mathscr{D}^{s+1}(M)$, i.e. $K$ is a compact Lie group acting $H^{s+1}$-smoothly on $M$; the group $\mathscr{D}_K^{s+1}(M)=\{\eta\in\mathscr{D}^{s+1}(M) \mid \eta\circ\phi=\phi\circ\eta$ for all $\phi\in K\}$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}^{s+1}(M)$ with $H^{s+1}$-Lie algebra $\mathscr{X}_K^{s+1}(M)=\{X\in\mathscr{X}^{s+1}(M) \mid \phi^*(X)=X$ for all $\phi\in K\}$. It will be shown in the proof of Theorem 3.1 that $\mathscr{D}_\theta^{s+1}(M)$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}_E^{s+1}(M)$ and hence also of $\mathscr{D}^{s+1}(M)$.

The assumption of regularity of $\theta$ does *not* follow in general from the condition that $E$ has all orbits closed. This is due to the fact that the period function of a smooth vector field with all integral curves closed is *not* necessarily smooth (e.g. the geodesic flow on the unit sphere bundle of lens spaces, Besse [6], p. 9; see Sullivan [20] for an example of a vector field with all orbits closed but with unbounded period function). However $\theta$ is regular if and only if the Reeb vector field $E$ has all orbits closed and the period function is smooth. Moreover, if $E$ has only periodic orbits, then there exists a smooth circle action on $M$ with the same orbits as $E$ if and only if there exists a Riemannian metric on $M$ with respect to which all orbits of $E$ are geodesics (Wadsley [21]); this circle action however may not be free. But if all geodesics coinciding with the orbits of $E$ are closed and of the *same* length, the circle action is free. See Besse [6] for an extensive study of manifolds *all* of whose geodesics are closed.

**Proposition 3.1** (Kostant [15]). *The following sequence of $H^{s+1}$-Lie algebras is exact*

$$0\to\mathbb{R}\xrightarrow{J}\mathscr{X}_\theta^{s+1}(M)\xrightarrow{P}\mathscr{H}^{s+1}(N)\to 0 \tag{3.4}$$

*Proof.* The map $J(c)=cE$, $c\in\mathbb{R}$ is clearly injective. To define $P$ we proceed in the following way. If $X\in\mathscr{X}_\theta^{s+1}(M)$, then $[X, E]=0$ and hence $X$ projects to an $H^{s+1}$-vector field $Z$ on $N$; $T\pi\circ X=Z\circ\pi$. However if $X=fE+Y$, $Y\in H^{s+1}(R_\theta)$, $[X, E]=0$ is equivalent to $E(f)E=[Y, E]$ and thus by (3.1), (3.2) $E(f)=0$, $[Y, E]=0$. Hence $Y$ also projects to $Z$ on $N$. Remark that $L_X\theta=0$ is equivalent to $df+i_Y d\theta=0$, so that $f\in C^{s+2}(M, \mathbb{R})$ and $dg=i_Z\omega$, for $f=-g\circ\pi\in C^{s+2}(N, \mathbb{R})$. Thus $Z\in\mathscr{H}^{s+1}(N)$ and we define $P(X)=Z$.

If $Z\in\mathscr{H}^{s+1}(N)$, there exists a Hamiltonian $g\in C^{s+2}(N, \mathbb{R})$ for $Z$ and thus a unique $Y\in H^{s+1}(R_\theta)$ such that $i_Y d\theta+df=0$ for $f=-g\circ\pi$. Hence $fE+Y\in\mathscr{X}_\theta^{s+1}(M)$ and $P(fE+Y)=Z$ by definition of $P$, i.e. $P$ is surjective. Since $Z$ determines its Hamiltonians only up to constants, exactness at $\mathscr{X}_\theta^{s+1}(M)$ is also proved. ∎

The goal of this section is to find the group theoretical version of (3.4). For this we need a few more ingredients. If $\gamma$: $[0, 1]\to N$ is a piecewise smooth path and $x\in\pi^{-1}(\gamma(0))$, there exists a unique path $\hat{\gamma}_x$: $[0, 1]\to M$, called the *horizontal lift of* $\gamma$ ([11], p. 290–292) such that $\hat{\gamma}_x(0)=x$, $\pi\circ\hat{\gamma}_x=\gamma$, $\frac{d\hat{\gamma}_x(t)}{dt}$ is a horizontal vector. This operation defines the "horizontal transport along $\gamma$" $H_\gamma$: $\pi^{-1}(\gamma(0))\to\pi^{-1}(\gamma(1))$, $H_\gamma(x)=\hat{\gamma}_x(1)$. Note that if $\gamma$ is closed, there exists a unique $s(\gamma)\in S^1$ such that $H_\gamma=\Phi_{s(\gamma)}$. If one replaces $M$ with Hermitian line bundle over $N$ with invariant connection, $H_\gamma$ becomes the usual parallel transport along $\gamma$. The following proposition is proved by easy direct verification.

**Proposition 3.2.** *Let $\eta\in\mathscr{D}_E^{s+1}(M)$ induce $\phi\in\mathscr{D}^{s+1}(N)$.* The following are equivalent:

(i) $\eta^*\theta=\theta$;

(ii) $(T_x\eta)(R_\theta(x))=R_\theta(\eta(x))$ *for all* $x\in M$;

(iii) $\eta\circ H_\gamma=H_{\phi\circ\gamma}\circ\eta$ (i.e. $\eta\circ\hat{\gamma}_x=(\phi\circ\gamma)^\wedge_{\eta(x)}$ *for all* $x\in M$) *for all piecewise smooth paths* $\gamma$ *in* $N$.

**Proposition 3.3** (Kostant [15]). *The following sequence of groups is exact*

$$0\to S^1\xrightarrow{j}\mathscr{D}_\theta^{s+1}(M)\xrightarrow{p}\mathscr{K}^{s+1}\to 0 \tag{3.5}$$

*where* $\mathscr{K}^{s+1}=\{\phi\in\mathscr{D}_\omega^{s+1}(N)\,|\,H_{\phi\circ\gamma}=H_\gamma$ *for all piecewise smooth closed paths* $\gamma$ *in* $N\}$.

*Proof.* The map $j(s)=\Phi_s$, $s\in S^1$ is injective. Define $p(\eta)\in\mathscr{D}^{s+1}(N)$ to be the diffeomorphism induced by $\eta$ on $N$, i.e. $p(\eta)\circ\pi=\pi\circ\eta$ and remark that since $\pi^*\omega=d\theta$, $p(\eta)\in\mathscr{D}_\omega^{s+1}(N)$. Clearly $p\circ j=e_N$. Let now $p(\eta)=e_N$. Then $\eta$ is of the form $\eta(x)=\Phi((f\circ\pi)(x),x)$ for $f\in H^{s+1}(N,S^1)$. If $\Phi^x\colon S^1\to M$ denotes the map $\Phi^x(s)=\Phi_s(x)$, then since $\Phi$ is a free action, $T_s\Phi^x$ is injective (see e.g. [1], Lemma 4.5.4) and has as image $R_{d\theta}(\Phi_s(x))$, the tangent space at $\Phi_s(x)$ to the orbit. Since for $v_x\in T_xM$

$$(T_x\eta)(v_x)=(T_x\Phi_{(f\circ\pi)(x)})(v_x)+T_{(f\circ\pi)(x)}\Phi^x)(T_{\pi(x)}f(T_x\pi(v_x))),$$

$\eta^*\theta=\theta$ implies

$$(T_{(f\circ\pi)(x)}\Phi^x)(T_{\pi(x)}f(T_x(v_x)))\in R_{d\theta}(\Phi_s(x))\cap R_\theta(\Phi_s(x))=0,$$

i.e. $(T_{\pi(x)}f)(T_x\pi(v_x))=0$ for all $x\in M$, $v_x\in T_xM$ and thus $f=$constant. Hence $\eta=\Phi_f$ and exactness at $\mathscr{D}_\theta^{s+1}(M)$ is proved.

To prove surjectivity of $p$ we proceed in the following way. Let $[z]\in N$ be fixed and $\phi\in\mathscr{D}_\omega^{s+1}(N)$ be given. Let $\eta_{[z]}\colon\pi^{-1}([z])\to\pi^{-1}(\phi([z]))$ be an arbitrary "rotation", i.e. $\eta_{[z]}=\beta^{-1}\circ s\alpha$, where $\alpha\colon\pi^{-1}([z])\to S^1$, $\beta\colon\pi^{-1}(\phi([z]))\to S^1$ are diffeomorphisms given by bundle charts and $s\in S^1$. It is clear that $\eta_{[z]}\circ\Phi_s=\Phi_s\circ\eta_{[z]}$ for all $s\in S^1$. For $x\in M$ consider the smooth path $\gamma$ in $N$ with $\gamma(0)=[z]$, $\gamma(1)=[x]$ and let $\hat{\gamma}$ be its unique horizontal lift with endpoint at $x$. Define then $\eta(x)=(H_{\phi\circ\gamma}\circ\eta_{[z]}\circ H_\gamma^{-1})(x)$. Let now $\rho$ be another path connecting $[z]$ to $[x]$ and let $\sigma=\gamma^{-1}*\rho$ be the closed path based at $[z]$ formed by $\rho$ followed by $\gamma$ in opposite direction. The definitions of $\eta$ with $\gamma$ and $\rho$ coincide if and only if $H_{\phi\circ\sigma}\circ\eta_{[z]}=\eta_{[z]}\circ H_\sigma$, i.e. $H_{\phi\circ\sigma}=H_\sigma$ which is the defining relation of $\mathscr{K}^{s+1}$. It is a routine matter to check that $\eta$ thus defined is indeed of class $H^{s+1}$. Moreover $\pi\circ\eta=\phi\circ\pi$ and $\eta^*\theta=\theta$ since condition (iii) in proposition 3.2 is trivially verified. ∎

The infinitesimal version of (3.5) should be (3.4), but for this we need to know that the identity component $\mathscr{K}_0^{s+1}=\mathscr{G}^{s+1}$ as $H^{s+1}$-Lie groups.

**Theorem 3.1.** *$\mathscr{K}^{s+1}$ has an $H^{s+1}$-Lie group structure making $\mathscr{D}_\theta^{s+1}(M)$ a principal circle bundle over it.*

*Proof.* First we have to show that $\mathscr{D}_\theta^{s+1}(M)$ is an $H^{s+1}$-Lie group. This has been proved already by Omori [18] but we shall give here a much shorter proof based on techniques of Ebin-Marsden [9] (see also the remark at the end of this section).

Let $H_{equ}^{s+1}(R_\theta^*)=\{\alpha\in H^{s+1}(R_\theta^*)\,|\,L_E\alpha=0\}$ be the space of all $\Phi$-equivariant $H^{s+1}$-sections of $R_\theta^*$, and $H_{equ}^{s+1}(\Lambda^2)$ the space of all equivariant $H^{s+1}$-two-forms on $M$. The linear space $\mathscr{A}^s=\{(\alpha, d\alpha)\,|\,\alpha\in H^{s+1}(R_\theta^*)\}$ is clearly closed in $H_{equ}^s(R_\theta^*)\times H_{equ}^s(\Lambda^2)$, so that $(\theta, d\theta)+\mathscr{A}^s$ is a closed affine submanifold of $H^s(\Lambda^1)\times H^s(\Lambda^2)$.

Define

$$\psi_\theta\colon \mathscr{D}_E^{s+1}(M)\to(\theta, d\theta)+\mathscr{A}^s, \qquad \text{by } \psi_\theta(\eta)=(\eta^*\theta, d\eta^*\theta).$$

We show that $\psi_\theta(\mathscr{D}_E^{s+1}(M))\subset(\theta, d\theta)+\mathscr{A}^s$. Since $(\eta^*\theta-\theta)(E)=(\eta^*\theta)(E)-1=\eta^*(\theta(E))-1=0$, it follows $\eta^*\theta-\theta\in H^{s+1}(R_\theta^*)$. Moreover, since $L_E\theta=0$, $L_E(\eta^*\theta-\theta)=\eta^*(L_E\theta)=0$, i.e. $\eta^*\theta-\theta\in H_{equ}^{s+1}(R_\theta^*)$, and $\psi_\theta$ is hence well-defined. It is known (Ebin [10], Ebin-Marsden [9]) that the map $\psi_\theta$ is $C^\infty$ and has derivative

$$D\psi_\theta(e)\colon \mathscr{X}_E^{s+1}(M)\to\mathscr{A}^s, \qquad D\psi_\theta(e)\cdot X=(L_X\theta, dL_X\theta).$$

If $X=fE+Y$, $f\in C^{s+1}(M,\mathbb{R})$, $Y\in H^{s+1}(R_\theta)$, then $E(f)=0$, $[Y,E]=0$ and

$$D\psi_\theta(e)\cdot X=(i_Y d\theta+df, d i_Y d\theta).$$

We prove now that $D\psi_\theta(e)$ is surjective. Take $f=$constant. The relation $(\alpha, d\alpha)=(i_Y d\theta, d i_Y d\theta)$ is satisfied if and only if $\alpha=i_Y d\theta$. Since $\alpha\in H_{equ}^{s+1}(R_\theta^*)$ there exists a unique $Y\in H^{s+1}(R_\theta)$ such that $\alpha=i_Y d\theta$. But $L_E\alpha=0$ implies $0=L_E i_Y d\theta=L_E i_Y d\theta-i_Y L_E d\theta=i_{[E,Y]}d\theta$, i.e. $[E,Y]\in H^s(R_{d\theta})$ so that by (3.1), (3.2) $[Y,E]=0$ and thus $D\psi_\theta(e)$ is surjective.

Since $R_{\eta^{-1}}$, $\eta^*$ are both isomorphism (for fixed $\eta$), it follows that $\psi_\theta$ is a submersion at any $\eta\in\mathscr{D}_E^{s+1}(M)$. Since $\mathscr{D}_\theta^{s+1}(M)=\psi_\theta^{-1}(\theta, d\theta)$, it follows that $\mathscr{D}_\theta^{s+1}(M)$ is a closed $H^{s+1}$-Lie group of $\mathscr{D}_E^{s+1}(M)$ and hence also of $\mathscr{D}^{s+1}(M)$.

It is obvious that $T_e j=J$ and hence $j$ is an immersion. $S^1$ acts freely and properly on $\mathscr{D}_\theta^{s+1}(M)$. Thus the quotient space $\mathscr{D}_\theta^{s+1}(M)/S^1=\mathscr{K}^{s+1}$ has a smooth manifold structure and is the base space of a principal circle bundle. ■

**Theorem 3.2.** *The identity component $\mathscr{K}_0^{s+1}$ equals $\mathscr{G}^{s+1}$ as $H^{s+1}$-Lie group.*

*Proof.* Theorem 3.1 implies that the universal covering $\widetilde{\mathscr{D}_\theta^{s+1}(M)}_0$ of the identity component $\mathscr{D}_\theta^{s+1}(M)_0$ is a principal line bundle over the universal covering $\widetilde{\mathscr{K}_0^{s+1}}$. The map $p$ lifts to $\tilde{p}(\eta, [\eta_t])=(p(\eta), [p(\eta_t)])$. We show that $\widetilde{\mathscr{K}_0^{s+1}}\subset\mathscr{G}^{s+1}=\operatorname{Ker}(\tilde{S})$ (see §2). Let $Z_t=(dp(\eta_t)/dt)\circ p(\eta_t)^{-1}$, $X_t=(d\eta_t/dt)\circ\eta_t^{-1}$. Then it is easy to see that $T\pi\circ X_t=Z_t\circ\pi$. Since $\eta_t^*\theta=\theta$, it follows that $L(X_t)\theta=0$, i.e. if $X_t=Y_t+f_t E$, then $i(Y_t)d\theta=-df_t$. Thus

$$i(X_t)d\theta=i(Y_t)d\theta+f_t(i_E d\theta)=-df_t$$

so that

$$\pi^*(A(p(\eta_t))) = \int_0^1 (i(Y_t)\, d\theta)\, dt = d \int_0^1 (-f_t)\, dt = dg.$$

Since $E(f_t)=0$, it follows $E(g)=0$ and thus $g$ projects to a map $\bar{g}$ on $N$; $\bar{g} \circ \pi = g$. Thus $\pi^*(A(p(\eta_t))) = \pi^*(d\bar{g})$ so that $A(p(\eta_t)) = d\bar{g}$, i.e. $\tilde{S}(p(\eta), [p(\eta_t)]) = 0$. Thus $\widetilde{\mathscr{K}_0^{s+1}} \subset \widetilde{\mathscr{G}^{s+1}}$ as groups. It follows then that $\mathscr{K}_0^{s+1} \subset \mathscr{G}^{s+1}$ as groups.

But $\mathscr{K}_0^{s+1}$ is a manifold whose model space is $\mathscr{X}_\theta^{s+1}(M)/\mathbb{R} = \mathscr{H}^{s+1}$ by Theorem 3.1 and Proposition 3.1, which is also the model space of $\mathscr{G}^{s+1}$ (Theorem 2.2). This means that $\mathscr{K}_0^{s+1}$ is an open subgroup of the connected group $\mathscr{G}^{s+1}$ and thus equal to it. ■

We summarize our results in the following commutative diagram with all lines and columns exact. The lines are smooth principal fiber bundles and the columns covering spaces. Its infinitesimal version is (3.4).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z} & \xhookrightarrow{\tilde{j}} & \pi_1(\mathscr{D}_\theta^{s+1}(M)_0) & \xrightarrow{\tilde{p}} & \pi_1(\mathscr{G}^{s+1}(N)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{R} & \xhookrightarrow{\tilde{j}} & \widetilde{\mathscr{D}_\theta^{s+1}(M)_0} & \xrightarrow{\tilde{p}} & \widetilde{\mathscr{G}^{s+1}(N)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^1 & \xhookrightarrow{j} & \mathscr{D}_\theta^{s+1}(M)_0 & \xrightarrow{p} & \mathscr{G}^{s+1}(N) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*Remark.* Omori [18] has shown that $\mathrm{Con}^{s+1}(M) = \{(\eta, f) \in \mathscr{D}^{s+1}(M) \times C^{s+1}(M, \mathbb{R} \setminus \{0\}) \mid \eta^* \theta = f\theta\}$ is a closed $H^{s+1}$-Lie subgroup of the semidirect product $\mathscr{D}^{s+1}(M) * C^{s+1}(M, \mathbb{R} \setminus \{0\})$ defined by the composition law $(\eta_1, f_1)(\eta_2, f_2) = ((\eta_1 \circ \eta_2), f_2(f_1 \circ \eta_2))$, identity element $(e, 1)$, and inverse $(\eta, f)^{-1} = (\eta^{-1}, 1/(f \circ \eta^{-1}))$, where $C^{s+1}(M, \mathbb{R} \setminus \{0\})$ is regarded as multiplicative group. The Lie algebra of $\mathscr{D}^{s+1}(M) * C^{s+1}(M, \mathbb{R} \setminus \{0\})$ is $\mathscr{X}^{s+1}(M) * C^{s+1}(M, \mathbb{R})$ with bracket

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f))$$

and exponential map

$$\exp(t(X, f)) = \left(F_t, e^{\int_0^t (f \circ F_s)\, ds}\right),$$

where $F_t = \exp(tX)$ is the flow of $X$. The Lie algebra of $\mathrm{Con}^{s+1}(M)$ is $S^{s+1}(M) = \{(Y, g) \in \mathscr{X}^{s+1}(M) * C^{s+1}(M, \mathbb{R}) \mid L_Y \theta = g\theta\}$. Remark that if $(\eta, f) \in \mathrm{Con}^{s+1}(M)$ and $(Y, g) \in S^{s+1}(M)$, the functions $f$ and $g$ are uniquely determined by $\eta$ respectively $Y$, namely $f = (\eta^* \theta) E$, $g = E(\theta(Y))$.

The group $\{\eta \in \mathscr{D}^{s+1}(M) \mid$ there exists an $f \in C^{s+1}(M, \mathbb{R} \setminus \{0\})$ such that $\eta^* \theta = f\theta\}$ is not even a submanifold of $\mathscr{D}^{s+1}(M)$ due to the loss of derivatives (Omori [18], p. 101). However, if $\theta$ is a regular contact one-form, it follows that $\mathscr{D}_\theta^{s+1}(M) * \{1\}$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}^{s+1}(M) * C^{s+1}(M, \mathbb{R} \setminus \{0\})$ (Omori [18], p. 102) which implies that $\mathscr{D}_\theta^{s+1}(M)$ is a closed $H^{s+1}$-Lie subgroup of $\mathscr{D}^{s+1}(M)$.

## § 4. The Group of Homogeneous Symplectic Transformations of $T^*M \setminus \{0\}$

In this section we endow the group of homogeneous symplectic transformations of $T^*M \setminus \{0\}$, $M$ a compact manifold, with an $H^{s+1}$-Lie group structure. This group appears in the study of a class of invertible Fourier integral operators of order zero as their group of canonical relations and has as Lie algebra the principal symbols of pseudo-differential operators of order one with homogeneous, purely imaginary principal symbol. The main problem in finding a Hilbert manifold structure of this group is the *non-compactness* of $T^*M \setminus \{0\}$. This is circumvented by considering the isomorphic group of contact transformations of the cosphere bundle of $M$. Some of the results and techniques of § 3 will be used.

The manifold $P = T^*M \setminus \{0\}$ is *conic*, i.e. the multiplicative group of strictly positive reals $\mathbb{R}_+$ acts smoothly on $P$ by $m(\tau, \alpha_x) = m_\tau(\alpha_x) = \tau\alpha_x$, $\tau > 0$, $\alpha_x \in T_x^*M$, $\alpha_x \neq 0$. This action is free and proper and thus $\pi: P \to Q = P/\mathbb{R}_+$ is a principal fiber bundle over a smooth manifold. $Q$, the *cosphere bundle of* $M$, is compact and odd-dimensional. A function $H: P \to \mathbb{R}$ (vector field $X$, $p$-form $\chi$, smooth map $\eta: P \to P$) is said to be homogeneous of degree $\nu$, $\nu \in \mathbb{R}$, if $m_\tau^* H = \tau^\nu H$ ($m_\tau^* X = \tau^\nu X$, $m_\tau^* \chi = \tau^\nu \chi$, $\eta(\tau\alpha_x) = \tau^\nu \eta(\alpha_x)$) for any $\tau > 0$. Remark that the canonical one ($\theta$) and two-form ($\omega = -d\theta$) of $P$ are homogeneous of degree one and that for any $p$-form $\chi$ homogeneous of degree $\nu$, $d\chi$ is also homogeneous of degree $\nu$.

The next two results are well-known (Hörmander [13], Weinstein [22]).

**Proposition 4.1.** *Let* $\eta: T^*M \setminus \{0\} \to T^*N \setminus \{0\}$ *be a diffeomorphism. The following are equivalent:*

(i) $\eta^* \theta_N = \theta_M$, *where* $\theta_M$, $\theta_N$ *are the canonical one-forms on* $T^*M \setminus \{0\}$ *and* $T^*N \setminus \{0\}$ *respectively;*

(ii) $\eta$ *is symplectic and homogeneous of degree one.*

**Proposition 4.2.** *Let* $\eta: T^*M \to T^*M$ *be a diffeomorphism. The following are equivalent:*

(i) $\eta^* \theta = \theta$;

(ii) $\eta$ *is symplectic and homogeneous of degree one;*

(iii) *there exists a diffeomorphism* $f: M \to M$ *such that* $\eta = T^*f$.

The following is a collection of facts to be used throughout this section.

**Proposition 4.3.** (i) *Let* $H: P = T^*M \setminus \{0\} \to \mathbb{R}$ *be homogeneous of degree* $\nu$. *Then the Hamiltonian vector field* $X_H$ *is homogeneous of degree* $\nu - 1$ *and* $\theta(X_H) = \nu H$.

(ii) *Let* $H_1, H_2: P \to \mathbb{R}$ *be homogeneous of degree* $\nu$, $\mu$. *Then the Poisson bracket* $\{H_1, H_2\}$ *is homogeneous of degree* $\nu + \mu - 1$. *It follows that the homo-*

*geneous functions of degree $\nu$ on $P$ are a Lie algebra for the Poisson bracket if and only if $\nu=1$.*

(iii) *A vector field $X$ on $P$ is homogeneous of degree zero if and only if its flow is homogeneous of degree one, i.e. it commutes with the action $m_\tau$, $F_t \circ m_\tau = m_\tau \circ F_t$ for all $t \in \mathbb{R}$, $\tau > 0$.*

(iv) *$L_X \theta = 0$ if and only if $X$ is globally Hamiltonian, homogeneous of degree zero, with Hamiltonian function $\theta(X)$ homogeneous of degree one.*

*Proof.* (i)

$$\tau\, i(m_\tau^* X_H)\,\omega = m_\tau^*(i_{X_H}\omega) = m_\tau^*(dH) = d(\tau^\nu H) = \tau^\nu dH = \tau^\nu i_{X_H}\omega = \tau\, i(\tau^{\nu-1} X_H)\,\omega,$$

and hence $m_\tau^*(X_H) = \tau^{\nu-1} X_H$. In local coordinates $\theta(X_H) = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} = \nu H$ by Euler's theorem on homogeneous functions.

(ii) and (iii) are trivial verifications.

(iv) If $L_X\theta = 0$, then $F_t^*\theta = \theta$ for $F_t$ the flow of $X$, and hence by proposition 4.1, $F_t$ is symplectic homogeneous of degree one; by (iii) $X$ is homogeneous of degree zero. Since $0 = L_X\theta = i_X d\theta + d i_X \theta$, it follows $i_X\omega = d(\theta(X))$, i.e. $X = X_H$ with $H = \theta(X)$ homogeneous of degree one since $X$ and $\theta$ are homogeneous of degree zero and one respectively. The converse is trivial in view of (i). ■

We shall investigate now the structure of the cosphere bundle $Q$. $Q$ carries no canonical contact structure, but one can construct a whole family of them in the following way. Let $\sigma: Q \to P$ be a global section of the principal fiber bundle $\pi: P \to Q$ and put $\theta_\sigma = \sigma^*\theta$. Such global sections exist in abundance; for example a Riemannian metric on $M$ identifies $T^*M$ with $TM$, $Q$ with the unit sphere bundle and $\sigma$ with the usual inclusion of the sphere bundle in $TM$. The section $\sigma$ is uniquely determined by a smooth function $f_\sigma: P \to \mathbb{R}_+$ defined by $\sigma \circ \pi = f_\sigma e_P$, i.e. $\sigma[\alpha_x] = f_\sigma(\alpha_x)\,\alpha_x$, where $[\alpha_x] = \pi(\alpha_x) \in Q$ denotes the class of $\alpha_x \in P$. Since $f_\sigma(\alpha_x)\,\alpha_x = \sigma[\alpha_x] = \sigma[\tau\alpha_x] = f_\sigma(\tau\alpha_x)\,\tau\alpha_x$, it follows that $f_\sigma$ is homogeneous of degree $-1$. The following formula will be used repeatedly:

$$\pi^*\theta_\sigma = f_\sigma\,\theta. \tag{4.1}$$

Indeed, if $\alpha_x \in P$, $w \in T_{\alpha_x}P$, then

$$\begin{aligned}(\pi^*\theta_\sigma)(\alpha_x)\cdot w &= \theta((\sigma\circ\pi)\,\alpha_x)\cdot T(\sigma\circ\pi)(w) = (\sigma\circ\pi)(\alpha_x)\cdot T(\tau^*\circ\sigma\circ\pi)(w)\\ &= f_\sigma(\alpha_x)\,\alpha_x\cdot(T\tau^*)(w) = f_\sigma(\alpha_x)\,\theta(\alpha_x)\cdot w.\end{aligned}$$

Consider now another global section $\rho: Q \to P$, its function $f_\rho$ and the one-form $\theta_\rho$ on $Q$. Since $\theta_\sigma = \sigma^*\theta = (\sigma\circ\pi\circ\rho)^*\theta = \rho^*\pi^*\theta_\sigma = (f_\sigma\circ\rho)\,\theta_\rho$, we have

$$\theta_\sigma = g_{\sigma\rho}\,\theta_\rho, \qquad g_{\sigma\rho} = f_\sigma\circ\rho, \qquad g_{\sigma\rho}\circ\pi = f_\sigma/f_\rho. \tag{4.2}$$

This relation proves that $\theta_\rho$ is an exact contact form on $Q$ if and only if $\theta_\sigma$ is. But if $\sigma$ is defined via a Riemannian metric, it is well-known that $\theta_\sigma$ is an exact contact form ([1], p. 373) and we proved that *$\theta_\rho$ is an exact contact one-form on $Q$ for any global section $\rho$.*

*Definition* (Kobayashi [14], p. 29). An *$H^{s+1}$-contact transformation* on $Q$ is a diffeomorphism $\phi \in \mathscr{D}^{s+1}(Q)$ such that for any two global sections $\sigma, \rho: Q \to P$, there exists an $H^{s+1}$-function $h_{\sigma\rho}: Q \to \mathbb{R}_+$ satisfying $\phi^* \theta_\sigma = h_{\sigma\rho} \theta_\rho$.

**Proposition 4.4.** *$\phi \in \mathscr{D}^{s+1}(Q)$ is a contact transformation if and only if there exists a global section $\sigma$ and a strictly positive $H^{s+1}$-function $h_\sigma: Q \to \mathbb{R}_+$ such that $\phi^* \theta_\sigma = h_\sigma \theta_\sigma$.*

*Proof.* If $\phi$ is a contact transformation, choose $\sigma = \rho$ and $h_\sigma = h_{\sigma\sigma}$. Conversely, if $\phi^* \theta_\sigma = h_\sigma \theta_\sigma$ for a fixed $\sigma$, then $\phi^* \theta_\rho = h_\rho \theta_\rho$ for any global section $\rho$, because by (4.2) $\phi^* \theta_\rho = \phi^*(g_{\rho\sigma} \theta_\sigma) = (g_{\rho\sigma} \circ \phi)\, \phi^* \theta_\sigma = (g_{\rho\sigma} \circ \phi)\, h_\sigma g_{\sigma\rho} \theta_\rho$, i.e. $h_\rho = (g_{\rho\sigma} \circ \phi)\, h_\sigma g_{\sigma\rho}$. Hence for arbitrary $\sigma$ and $\rho$, $\phi^* \theta_\sigma = h_{\sigma\rho} \theta_\rho$ for $h_{\sigma\rho} = h_\sigma g_{\sigma\rho}$. ∎

*Remarks.* 1. The function $h_\sigma$ is uniquely determined by $\sigma$, namely $h_\sigma = (\phi^* \theta_\sigma)(E_\sigma)$, where $E_\sigma$ is the Reeb vector field on $Q$ determined by $\theta_\sigma$. Therefore the group of $H^{s+1}$-contact transformations on $Q$ is isomorphic to the group

$$\mathrm{Con}_\sigma^{s+1}(Q) = \{(\phi, h) \in \mathscr{D}^{s+1}(Q) * C^{s+1}(Q, \mathbb{R} \setminus \{0\}) \mid \phi^* \theta_\sigma = h \theta_\sigma\},$$

for any fixed but arbitrary global section $\sigma$. For the definition of the semi-direct product $\mathscr{D}^{s+1}(Q) * C^{s+1}(Q, \mathbb{R} \setminus \{0\})$ and its $H^{s+1}$-Lie group structure see the remark at the end of § 3.

2. It is easily seen that $\mathrm{Con}_\sigma^{s+1}(Q)$ and $\mathrm{Con}_\rho^{s+1}(Q)$ are isomorphic as $H^{s+1}$-Lie groups for any two global sections $\sigma, \rho$.

3. The Hamiltonian vector field $X_{1/f_\sigma}$ on $P$ is homogeneous of degree zero (since $f_\sigma$ is homogeneous of degree $-1$) and thus it induces a $\pi$-related vector field $Y$ on $Q$. We claim that $Y = E_\sigma$. To show this, it is enough to prove that $\pi^*(i_Y d\theta_\sigma) = 0$, $\pi^*(i_Y \theta_\sigma) = 1$ since $\pi$ is a surjective submersion. By proposition 4.2(i) we have $\theta(X_{1/f_\sigma}) = 1/f_\sigma$ and hence

$$\pi^*(i_Y \theta_\sigma) = i(X_{1/f_\sigma})(f_\sigma \theta) = f_\sigma \theta(X_{1/f_\sigma}) = 1,$$

$$\begin{aligned}\pi^*(i_Y d\theta_\sigma) &= i(X_{1/f_\sigma})\, d(f_\sigma \theta) = X_{1/f_\sigma}(f_\sigma)\, \theta - \theta(X_{1/f_\sigma})\, df_\sigma + f_\sigma\, i(X_{1/f_\sigma})\, d\theta \\ &= -df_\sigma/f_\sigma - f_\sigma\, d(1/f_\sigma) = 0.\end{aligned}$$

4. Let $j: \sigma(Q) \hookrightarrow P$ be the inclusion. Then $\sigma: (Q, \theta_\sigma) \to (\sigma(Q), j^* \theta)$ is an exact contact diffeomorphism. Since $\sigma \circ \pi = f_\sigma e_P$ is a diffeomorphism, it is easily seen that $df_\sigma$ is nowhere zero and thus $(1/f_\sigma)^{-1}(1) = \sigma(Q)$ is a regular energy surface for the Hamiltonian vector field $X_{1/f_\sigma}$. The Reeb vector field of $j^* \theta$ is $X_{1/f_\sigma} \mid \sigma(Q)$ and its pull-back by $\sigma$ coincides with $E_\sigma$.

The next theorem endows the group of homogeneous symplectic diffeomorphisms $\mathscr{D}_\theta^{s+1}(P) = \{\eta \in \mathscr{D}^{s+1}(P) \mid \eta^* \theta = \theta\}$ with an $H^{s+1}$-Lie group structure and represents the main result of this section.

**Theorem 4.1.** (i) *The map $\Phi: \mathscr{D}_\theta^{s+1}(P) \to \mathrm{Con}_\sigma^{s+1}(Q)$, $\Phi(\eta) = (\phi, h)$, where $\phi$ is uniquely determined by $\phi \circ \pi = \pi \circ \eta$ and $h \circ \pi = (f_\sigma \circ \eta)/f_\sigma$, is a group isomorphism with inverse $\Psi(\phi, h) = (\sigma \circ \phi \circ \pi)/(h \circ \pi)\, f_\sigma$.*

(ii) *The map*

$$\begin{aligned}\Phi': \mathscr{X}_\theta^{s+1}(P) &= \{Y \in \mathscr{X}^{s+1}(P) \mid L_Y \theta = 0\} \to S_\sigma^{s+1}(Q) \\ &= \{(X, k) \in \mathscr{X}^{s+1}(Q) * C^{s+1}(Q, \mathbb{R}) \mid L_X \theta_\sigma = k \theta_\sigma\},\end{aligned}$$

*$\Phi'(X_H)=(X,k)$, where $X$ is uniquely determined by $T\pi\circ X_H=X\circ\pi$ and $k\circ\pi=\{f_\sigma,H\}/f_\sigma$, is a Lie algebra isomorphism with inverse $\Psi'(X,k)=X_H$, for $H=(\theta_\sigma(X)\circ\pi)/f_\sigma$.*

(iii) *Endow $\mathscr{D}_\theta^{s+1}(P)$ and $\mathscr{X}_\theta^{s+1}(P)$ with the $H^{s+1}$-differentiable structures of $\mathrm{Con}_\sigma^{s+1}(Q)$ and $S_\sigma^{s+1}(Q)$ given by the isomorphisms $\Phi$ and $\Phi'$ respectively. Then $T_e\mathscr{D}_\theta^{s+1}(P)=\mathscr{X}_\theta^{s+1}(P)$, $T_e\Phi=\Phi'$, and $T_e\Psi=\Psi'$.*

*Proof.* (i) $\eta\in\mathscr{D}_\theta^{s+1}(P)$ is by Proposition 4.1 symplectic homogeneous of degree one and hence it induces a unique diffeomorphism $\phi$ of $Q$ characterized by $\phi\circ\pi=\pi\circ\eta$. Since $\pi$ is a surjective submersion, $\phi^*\theta_\sigma=h\theta_\sigma$ is equivalent to $\pi^*(\phi^*\theta_\sigma)=\pi^*(h\theta_\sigma)$, a relation easily proved using (4.1) and the definitions of $\phi$ and $h$. Let $\Phi(\eta_1)=(\phi_1,h_1)$, $\Phi(\eta_2)=(\phi_2,h_2)$ and $\Phi(\eta_1\circ\eta_2)=(\phi,h)$. Then $\phi\circ\pi=\pi\circ\eta_1\circ\eta_2=\phi_1\circ\phi_2\circ\pi$,

$$\begin{aligned}h\circ\pi&=(f_\sigma\circ\eta_1\circ\eta_2)/f_\sigma=((f_\sigma\circ\eta_1\circ\eta_2)/(f_\sigma\circ\eta_2))((f_\sigma\circ\eta_2)/f_\sigma)\\&=(h_2\circ\pi)((f_\sigma\circ\eta_1)/f_\sigma)\circ\eta_2=(h_2\circ\pi)(h_1\circ\pi\circ\eta_2)=(h_2(h_1\circ\phi_2))\circ\pi,\end{aligned}$$

and thus $\Phi(\eta_1\circ\eta_2)=(\phi_1,h_1)(\phi_2,h_2)$, i.e. $\Phi$ is a group homomorphism.

If $\Psi(\phi,h)=\eta$, then $\eta^*\theta=(\sigma\circ\phi\circ\pi)^*\theta/f_\sigma(h\circ\pi)=\pi^*(h\theta_\sigma)/f_\sigma(h\circ\pi)=(\pi^*\theta_\sigma)/f_\sigma=\theta$ and $\Psi$ is thus correctly defined.

It is straightforward to verify that $(\Psi\circ\Phi)(\eta)=\eta$ for all $\eta\in\mathscr{D}_\theta^{s+1}(P)$. Conversely, let $(\Phi\circ\Psi)(\phi,h)=\Phi(\eta)=(\psi,k)$. Then $\psi\circ\pi=\pi\circ\eta=\pi\circ((\sigma\circ\phi\circ\pi)/f_\sigma(h\circ\pi))=\pi\circ\sigma\circ\pi\circ\phi=\phi\circ\pi$, i.e. $\psi=\phi$. Since $f_\sigma\circ\sigma\circ\phi\circ\pi=1$ and $f_\sigma$ is homogeneous of degree $-1$, we have $k\circ\pi=(f_\sigma\circ\eta)/f_\sigma=\frac{1}{f_\sigma}\left(f_\sigma\circ\frac{1}{f_\sigma(h\circ\pi)}\sigma\circ\phi\circ\pi\right)=(h\circ\pi)(f_\sigma\circ\sigma\circ\phi\circ\pi)=h\circ\pi$, i.e. $k=h$ and we showed $\Psi=\Phi^{-1}$.

(ii) By Proposition 4.3(iv) $\mathscr{X}_\theta^{s+1}(P)=\{X_H\in\mathscr{X}_{d\theta}^{s+1}(P)\,|\,H$ is homogeneous of degree one$\}$, so that each $X_H\in\mathscr{X}_\theta^{s+1}(P)$ projects to a vector field $X$ on $Q$ characterized by $T\pi\circ X_H=X\circ\pi$. Since $\pi^*(L_X\theta_\sigma)=L_{X_H}(f_\sigma\theta)=\{f_\sigma,H\}\theta+f_\sigma L_{X_H}\theta=(\{f_\sigma,H\}/f_\sigma)f_\sigma\theta=(k\circ\pi)\pi^*\theta_\sigma=\pi^*(k\theta_\sigma)$, i.e. $L_X\theta_\sigma=k\theta_\sigma$, it follows that $\Phi'$ is correctly defined.

Let $\Phi'(X_H)=(X,h)$, $\Phi'(X_K)=(Y,k)$, and $\Phi'([X_H,X_K])=(Z,l)$. Then $Z\circ\pi=T\pi\circ[X_H,X_K]=[X,Y]\circ\pi$ since $X_H$, $X_K$ are $\pi$-related to $X$, $Y$ respectively; hence $Z=[X,Y]$. From the definitions of $h$, $k$, $l$, the relation $[X_H,X_K]=X_{\{K,H\}}$ and the Jacobi identity it follows that

$$\begin{aligned}\pi^*(X(k)-Y(h))&=L_{X_H}(k\circ\pi)-L_{X_K}(h\circ\pi)=L_{X_H}(\{f_\sigma,K\}/f_\sigma)-L_{X_K}(\{f_\sigma,H\}/f_\sigma)\\&=(L_{X_H}\{f_\sigma,K\})/f_\sigma+\{f_\sigma,K\}\{1/f_\sigma,H\}\\&\quad-(L_{X_K}\{f_\sigma,H\})/f_\sigma-\{f_\sigma,H\}\{1/f_\sigma,K\}\\&=-\{H,\{f_\sigma,K\}\}/f_\sigma-\{f_\sigma,K\}\{f_\sigma,H\}/f_\sigma^2\\&\quad+\{K,\{f_\sigma,H\}\}/f_\sigma+\{f_\sigma,H\}\{f_\sigma,K\}/f_\sigma^2\\&=\{f_\sigma,\{K,H\}\}/f_\sigma=l\circ\pi\end{aligned}$$

i.e. $l=X(k)-Y(h)$ and hence $\Phi'([X_H,X_K])=[\Phi'(X_H),\Phi'(X_K)]$.

If $(\Psi'\circ\Phi')(X_H)=\Psi'(X,k)=X_K$, then $K=(\theta_\sigma(X)\circ\pi)/f_\sigma=(\pi^*(i_X\theta_\sigma))/f_\sigma=(i_{X_H}(f_\sigma\theta))/f_\sigma=\theta(X_H)=H$ by Proposition 4.3(i). Conversely, let $(\Phi'\circ\Psi')(X,k)$

$=\Phi'(X_H)=(Y,l)$, where $l\circ\pi=\{f_\sigma,H\}/f_\sigma$ and $H=(\theta_\sigma(X)\circ\pi)/f_\sigma$. Then $k\circ\pi=\pi^*(E_\sigma(\theta_\sigma(X)))=L(X_{1/f_\sigma})(\theta_\sigma(X)\circ\pi)=\{f_\sigma H,1/f_\sigma\}=\{f_\sigma,H\}/f_\sigma=l\circ\pi$ and hence $k=l$. To show that $X=Y$ we associate to $(Y,l)$ the vector field $\Psi'(Y,l)=(\Psi'\circ\Phi')(X_H)=X_H$. By the definition of $\Psi'$ it follows $H=(\theta_\sigma(Y)\circ\pi)/f_\sigma=(\theta_\sigma(X)\circ\pi)/f_\sigma$, i.e. $\theta_\sigma(X-Y)=0$, which is equivalent to $X-Y\in H^{s+1}(R_{\theta_\sigma})$. But since $k=l$ we also have $L_{X-Y}\theta_\sigma=0$, i.e. $X-Y\in H^{s+1}(R_{d\theta_\sigma})$. The relation $H^{s+1}(R_{\theta_\sigma})\cap H^{s+1}(R_{d\theta_\sigma})=0$ (see (3.1)) implies $X=Y$ and hence we proved $\Psi'=(\Phi')^{-1}$.

(iii) Since $T_e\mathscr{D}_\theta^{s+1}(P)=\mathscr{X}_\theta^{s+1}(P)$ as vector spaces and both are isomorphic to $S_\sigma^{s+1}(Q)$ as $H^{s+1}$-Lie algebras, this equality is between $H^{s+1}$-Lie algebras.

Let $X_H\in T_e\mathscr{D}_\theta^{s+1}(P)$ and $\eta_t$ be its flow. Then

$$(T_e\Phi)(X_H)=\left.\frac{d}{dt}\right|_{t=0}\Phi(\eta_t)=\left.\frac{d}{dt}\right|_{t=0}(\phi_t,h_t)=\left(X,\left.\frac{dh_t}{dt}\right|_{t=0}\right),$$

where $\phi_t$ is the flow on $Q$ defined by $\pi\circ\eta_t=\phi_t\circ\pi$; $X$ is the vector field on $Q$ with flow $\phi_t$ and hence $T\pi\circ X_H=X\circ\pi$. Since $h_t\circ\pi=(f_\sigma\circ\eta_t)/f_\sigma$, it follows

$$\left.\frac{dh_t}{dt}\right|_{t=0}\circ\pi=\left.\frac{d}{dt}\right|_{t=0}((f_\sigma\circ\eta_t)/f_\sigma)=df_\sigma(X_H)/f_\sigma=\{f_\sigma,H\}/f_\sigma,$$

and we showed that $T_e\Phi=\Phi'$. Since $\Psi=\Phi^{-1}$, $\Psi'=(\Phi')^{-1}$ it follows that $T_e\Psi=\Psi'$. ∎

*Remarks.* 1. Let $(X,k)\in S_\sigma^{s+1}(Q)$ and let $\phi_t$ be the flow of $X$. Then the flow of $\Psi'(X,k)$ is

$$\eta_t=(\sigma\circ\phi_t\circ\pi)/f_\sigma\, e^{\int_0^t(k\circ\phi_s\circ\pi)\,ds}. \tag{4.3}$$

This follows at once from the commutative diagram

$$\begin{array}{ccc} S_\sigma^{s+1}(Q) & \xrightarrow{\ \Psi'\ } & \mathscr{X}_\theta^{s+1}(P) \\ {\scriptstyle \exp_1}\big\downarrow & & \big\downarrow{\scriptstyle \exp_2} \\ \mathrm{Con}_\sigma^{s+1}(Q) & \xrightarrow{\ \Psi\ } & \mathscr{D}_\theta^{s+1}(P) \end{array}$$

and the formula for $\exp_1$ in the remark at the end of § 3. The flow of $\Psi'(X,k)$ is given by

$$\begin{aligned}\exp_2(t\,\Psi'(X,k))&=\Psi(\exp_1(t(X,k)))=\Psi(\phi_t,e^{\int_0^t(k\circ\phi_s)\,ds})\\ &=(\sigma\circ\phi_t\circ\pi)/f_\sigma\, e^{\int_0^t(k\circ\phi_s\circ\pi)\,ds}.\end{aligned}$$

2. Let $(X,k)\in S_\sigma^{s+1}(Q)$. Then

$$\Psi'(X,k)=Tm_{1/f_\sigma}\circ T\sigma\circ X\circ\pi-(k\circ\pi)(e_P)_{e_P}^l, \tag{4.4}$$

where $m_{1/f_\sigma}: P \to \mathscr{D}^\infty(P)$, $m_{1/f_\sigma}(\alpha_x) = m_{1/f_\sigma(\alpha_x)}$ and $(e_P)^l_{e_P}$ is the vector field "vertical lift" of the identity $e_P$ over itself, i.e.

$$(e_P)^l_{e_P}(\alpha_x) = \frac{d}{dt}\bigg|_{t=0} (\alpha_x + t\alpha_x) \in T_{\alpha_x} P,$$

or in local coordinates, $(e_P)^l_{e_P}(\mathbf{x}, \boldsymbol{\alpha}) = (\mathbf{x}, \boldsymbol{\alpha}, \mathbf{0}, \boldsymbol{\alpha})$.

*Proof.* Let $\eta_t$ be the flow of $\Psi'(X, k)$ given by (4.3); $\eta_t$ is of the form $f_t \psi_t$, where $\psi_t = \sigma \circ \phi_t \circ \pi: P \to P$ and

$$f_t = e^{-\int_0^t (k \circ \phi_s \circ \pi)\, ds} / f_\sigma : P \to \mathbb{R}_+.$$

In local coordinates $\psi_t = (\psi_{1t}, \psi_{2t})$, $f_t \psi_t = (\psi_{1t}, f_t \psi_{2t})$ and hence

$$\frac{d}{dt}\bigg|_{t=0} (f_t \psi_t) = \left( \psi_{10}, f_0 \psi_{20}, \frac{d\psi_{1t}}{dt}\bigg|_{t=0}, f_0 \frac{d\psi_{2t}}{dt}\bigg|_{t=0} + \frac{df_t}{dt}\bigg|_{t=0} \psi_{20} \right).$$

With

$$\frac{d\psi_t}{dt}\bigg|_{t=0} = T\sigma \circ X \circ \pi = ((T\sigma \circ X \circ \pi)_1, (T\sigma \circ X \circ \pi)_2)$$

and

$$\frac{df_t}{dt}\bigg|_{t=0} = -(k \circ \pi)/f_\sigma$$

we get

$$\begin{aligned} \Psi'(X, k)(\mathbf{x}, \boldsymbol{\alpha}) &= \frac{d\eta_t}{dt}\bigg|_{t=0} (\mathbf{x}, \boldsymbol{\alpha}) \\ &= (\mathbf{x}, \boldsymbol{\alpha}, (T\sigma \circ X \circ \pi)_1 (\mathbf{x}, \boldsymbol{\alpha}), (T\sigma \circ X \circ \pi)_2 (\mathbf{x}, \boldsymbol{\alpha})/f_\sigma(\mathbf{x}, \boldsymbol{\alpha}) - (k \circ \pi)(\mathbf{x}, \boldsymbol{\alpha})\, \boldsymbol{\alpha}) \end{aligned}$$

which is easily seen to be the local coordinate expression of the right-hand side of (4.4). ■

3. By Proposition 4.3(iv), $\mathscr{X}^{s+1}_\theta(P)$ is isomorphic to $\mathscr{S}^{s+2}(P) = \{H \in C^{s+2}(P, \mathbb{R}) \mid H \text{ homogeneous of order one}\}$. Thus the ILH-Lie group $\mathscr{D}^\infty_\theta(P)$ of homogeneous canonical transformations of $P$ has its ILH-Lie algebra $\mathscr{X}^\infty_\theta(P)$ isomorphic to $\mathscr{S}^\infty(P)$, the principal symbols of pseudo-differential operators of order one with purely imaginary homogeneous principal symbol. Theorem 4.1 and the two remarks above enable us to pass back and forth between formulas on the cosphere bundle (which defines the ILH-structures) and the usual symbol spaces. The results here will be the starting point in a forthcoming paper in which we investigate the differentiable structure of the group of invertible Fourier integral operators of order zero with canonical relation in $\mathscr{D}^\infty_\theta(P)$.

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**Note added in proof.** We constructed an *explicite chart* for the $H^{s+1}$-Lie group $\mathscr{D}_\theta^{s+1}(P)$. Since this is crucial for the construction of a differentiable structure for the group of invertible Fourier integral operators, this chart construction will be given in the forthcoming paper mentioned above.