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Supersymmetric branes on $AdS_5 \times Y^{p,q}$ and their field theory duals

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ABSTRACT: We systematically study supersymmetric embeddings of D-brane probes of different dimensionality in the $AdS_5 \times Y^{p,q}$ background of type IIB string theory. The main technique employed is the kappa symmetry of the probe's worldvolume theory. In the case of D3-branes, we recover the known three-cycles dual to the dibaryonic operators of the gauge theory and we also find a new family of supersymmetric embeddings. The BPS fluctuations of dibaryons are analyzed and shown to match the gauge theory results. Supersymmetric configurations of D5-branes, representing domain walls, and of spacetime filling D7-branes (which can be used to add flavor) are also found. We also study the baryon vertex and some other embeddings which break supersymmetry but are nevertheless stable.

KEYWORDS: AdS-CFT Correspondence, Supersymmetry and Duality, D-branes, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

The Maldacena conjecture provides a unique window into the strongly coupled physics of gauge theories in terms of a string theory [1, 2]. A crucial ingredient in the AdS/CFT correspondence is the state/operator correspondence. It provides the basis for explicit computations. Calculationally, it is convenient to consider the limit of large 't Hooft coupling where the supergravity approximation is valid. More precisely, chiral operators of the CFT are in correspondence with the modes of supergravity in the dual background. However, much information is contained in the stringy sector of the correspondence and some of it crucially survives the large 't Hooft limit.

For example, as discovered by Witten [3] in discussing the duality in the case of $\mathcal{N}=4$ super Yang-Mills (SYM) theory with gauge group SO($2N$) and its dual $AdS_5\times\mathbb{RP}^5$, the gravity side must contain branes, just to accommodate a class of chiral operators of the gauge theory. The study of branes wrapped in the gravity theory becomes an intrinsic part of the correspondence. It has been extended and understood in a variety of situations. For example, a vertex connecting N fundamental strings –known as the baryon vertex– can be identified with a baryon built out of external quarks, since each string ends on a charge in the fundamental representation of SU(N). Such an object can be constructed by wrapping a D5-brane over the whole five-dimensional compact manifold [3]. Also, domain walls in the field theory side can be understood as D5-branes wrapping 2-cycles of the internal geometry [3, 4]. In quantum field theories that arise from D3-branes placed at conical singularities, an object of particular interest is given by D3-branes wrapped on supersymmetric 3-cycles; these states are dual to dibaryons built from chiral fields charged under two different gauge groups of the resulting quiver theory [3 – 6]. In the absence of a string theory formulation on backgrounds with Ramond-Ramond forms, probe D-branes of various dimensions provide valuable information about the spectrum. More generally, finding particular situations where a semiclassical description captures nontrivial stringy information is an important theme of the AdS/CFT correspondence recently fueled largely by BMN in [7], but having its root in the work of Witten [3] and in considerations of the Wilson loop as a classical string in the supergravity background [8].

Given a Sasaki-Einstein five-dimensional manifold X^5 one can consider placing a stack of N D3-branes at the tip of the (Calabi-Yau) cone over X^5. Taking the Maldacena limit then leads to a duality between string theory on $AdS_5\times X^5$ and a superconformal gauge theory living in the worldvolume of the D3-branes [9]. When the Sasaki-Einstein manifold is the $T^{1,1}$ space –the Calabi-Yau cone being nothing but the conifold– we have the so-called Klebanov-Witten model [10], which is dual to a four-dimensional $\mathcal{N}=1$ superconformal field theory with gauge group SU(N) $\times$ SU(N) coupled to four chiral superfields in the bifundamental representation. Important aspects of this duality, relevant in the context of this article, have been further developed in [4, 5, 11]. The supersymmetry of D-brane probes in the Klebanov-Witten model was studied in full detail in ref. [12].

Recently, a new class of Sasaki-Einstein manifolds $Y^{p,q}$, p and q being two coprime positive integers, has been constructed [13, 14]. The infinite family of spaces $Y^{p,q}$ was shown to be dual to superconformal quiver gauge theories [15, 16]. The study of AdS/CFT

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in these geometries has shed light in many subtle aspects of superconformal field theories in four dimensions. Furthermore, the correspondence successfully passed new tests such as those related to the fact that the central charge of these theories, as well as the R-charges of the fundamental fields, are irrational numbers [17].

In this paper we perform a systematic classification of supersymmetric branes in the $AdS_5 \times Y^{p,q}$ geometry and study their field theoretical interpretation. It is worth reminding that the spectrum of IIB supergravity compactified on $Y^{p,q}$ is not known due to various technical difficulties including the general form of Heun's equation. Therefore, leaving aside the chiral primaries, very little is known about the gravity modes dual to protected operators in the field theory. Our study of supersymmetric objects in the gravity side is a way to obtain information about properties of these operators in the gauge theory side. They comprise interesting physical objects of these theories such as the baryon vertex, domain walls, the introduction of flavor, fat strings, etc. It is very remarkable that we are able to provide precise information about operators with large conformal dimension that grows like N. Moreover, we can also extract information about excitations of these operators.

The main technique we employ to determine the supersymmetric embeddings of D-brane probes in the $AdS_5 \times Y^{p,q}$ background is kappa symmetry [18] and follows the same systematics as in the analysis performed in ref. [12] for the case of the $AdS_5 \times T^{1,1}$ background. Our approach is based on the existence of a matrix Γ_κ which depends on the metric induced on the worldvolume of the probe and characterizes its supersymmetric embeddings. Actually, if ϵ is a Killing spinor of the background, only those embeddings such that $\Gamma_\kappa\, \epsilon = \epsilon$ preserve some supersymmetry [19]. This kappa symmetry condition gives rise to a set of first-order BPS differential equations whose solutions, if they exist, determine the embedding of the probe and the fraction of the original background supersymmetry that it preserves. The configurations found by solving these equations also solve the equations of motion derived from the Dirac-Born-Infeld action of the probe and, actually, we will verify that they saturate a bound for the energy, as it usually happens in the case of worldvolume solitons [20].

The first case we study is that of D3-branes. We are able to find in this case the three-cycles introduced in refs. [15, 16, 21] to describe the different dibaryonic operators of the gauge theory. Moreover, we also find a general class of supersymmetric embeddings of the D3-brane probe characterized by a certain local holomorphicity condition. Contrary to what happens in the case of $T^{1,1}$, globally it turns out that these embeddings, in general, do not define a three-cycle but a submanifold with boundaries. We also study the fluctuations of the D3-brane probe around a dibaryonic configuration and we successfully match the emerging results with those of the corresponding quiver theory. We also find stable non-supersymmetric configurations of D3-branes wrapping a two-dimensional submanifold of $Y^{p,q}$ that define a one dimensional object in the gauge theory that could be interpreted as a *fat* string.

Our analysis continues with the study of D5-brane probes. We find embeddings in which the D5-brane wraps a two-dimensional submanifold and creates a domain wall in AdS_5. When crossing these domain walls the rank of the gauge group factors of the quiver gets shifted [4], this leading to their identification as fractional branes [22]. We also study other stable configurations that break supersymmetry completely but are nevertheless interesting enough. One of these configurations is the baryon vertex, in which the D5-brane wraps the entire $Y^{p,q}$ space. Besides, we also consider the case of D5-branes wrapping a two-dimensional submanifold when a nonvanishing worldvolume flux is present as well as the setting with D5-branes on three-cycles of $Y^{p,q}$.

Finally we turn to the case of D7-brane probes. According to the original proposal of ref. [23], the embeddings in which the D7-branes fill completely the gauge theory directions are specially interesting. These spacetime filling configurations can be used as flavor branes, i.e. as branes whose fluctuations can be identified with the dynamical mesons of the gauge theory (see refs. [24]–[33] for the analysis of the meson spectrum in different theories). Moreover, we show that the configurations in which the D7-brane wraps the entire $Y^{p,q}$ are also supersymmetric.

The organization of the paper is as follows. In section 2 we review some properties of the $Y^{p,q}$ space and the corresponding Calabi-Yau cone that we call $CY^{p,q}$, including the local complex coordinates of the latter. We discuss the corresponding type IIB supergravity solution and present the explicit form of its Killing spinors. We also present the general condition satisfied by supersymmetric embeddings of D-brane probes on this background. Section 3 discusses embeddings of D3-branes on various supersymmetric cycles. We reproduce the three-cycles considered previously in the literature and find a new family of supersymmetric embeddings. Section 3 also contains an analysis of the excitations of wrapped D3-branes and we find perfect agreement with the corresponding field theory results. Section 4 deals with supersymmetric D5-branes which behave as domain walls, while in section 5 we discuss the spacetime filling embeddings of D7-branes. For completeness, we consider other possible embeddings, such as the baryon vertex, in section 6. We conclude and summarize our results in section 7.

2. The $Y^{p,q}$ space and brane probes

Let us consider a solution of IIB supergravity given by a ten-dimensional space whose metric is of the form:

$$ds^2 = ds^2_{AdS_5} + L^2\, ds^2_{Y^{p,q}} \tag{2.1}$$

where $ds^2_{AdS_5}$ is the metric of AdS_5 with radius L and $ds^2_{Y^{p,q}}$ is the metric of the Sasaki-Einstein space $Y^{p,q}$, which can be written as [13, 14]:

$$\begin{aligned} ds^2_{Y^{p,q}} = {} & \frac{1-cy}{6}\,(d\theta^2+\sin^2\theta\, d\phi^2)+\frac{1}{6\,H^2(y)}\,dy^2+\frac{H^2(y)}{6}\,(d\beta-c\cos\theta d\phi)^2 \\ & +\frac{1}{9}\,[d\psi+\cos\theta d\phi+y(d\beta-c\cos\theta d\phi)\,]^2\,, \end{aligned} \tag{2.2}$$

$H(y)$ being given by:

$$H(y)=\sqrt{\frac{a-3y^2+2cy^3}{3(1-cy)}}\,. \tag{2.3}$$

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The metrics $ds^2_{Y^{p,q}}$ are Sasaki-Einstein, which means that the cones $CY^{p,q}$ with metric $dr^2 + r^2 ds^2_{Y^{p,q}}$ are Calabi-Yau manifolds. The metrics in these coordinates neatly display some nice local features of these spaces. Namely, by writing it as

$$ds^2_{Y^{p,q}} = ds^2_4 + \left[\frac{1}{3}d\psi + \sigma\right]^2 , \tag{2.4}$$

it turns out that ds^2_4 is a Kähler-Einstein metric with Kähler form $J_4 = \frac{1}{2}d\sigma$. Notice that this is a local splitting that carries no global information. Indeed, the pair (ds^2_4, J_4) is not in general globally defined. The Killing vector $\frac{\partial}{\partial\psi}$ has constant norm but its orbits do not close (except for certain values of p and q, see below). It defines a foliation of $Y^{p,q}$ whose transverse leaves, as we see, locally have a Kähler-Einstein structure. This aspect will be important in later discussions.

These $Y^{p,q}$ manifolds are topologically $S^2 \times S^3$ and can be regarded as U(1) bundles over manifolds of topology $S^2 \times S^2$. Its isometry group is $\mathrm{SU}(2) \times \mathrm{U}(1)^2$. Notice that the metric (2.2) depends on two constants a and c. The latter, if different from zero, can be set to one by a suitable rescaling of the coordinate y, although it is sometimes convenient to keep the value of c arbitrary in order to be able to recover the $T^{1,1}$ geometry, which corresponds to $c = 0$ [1]. If $c \neq 0$, instead, as we have just said we can set $c = 1$ and the parameter a can be written in terms of two coprime integers p and q (we take $p > q$) as follows:

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\,\sqrt{4p^2 - 3q^2}\,. \tag{2.5}$$

Moreover, the coordinate y ranges between the two smaller roots of the cubic equation

$$\mathcal{Q}(y) \equiv a - 3y^2 + 2cy^3 = 2c\prod_{i=1}^{3}(y - y_i)\,, \tag{2.6}$$

i.e. $y_1 \leq y \leq y_2$ with (for $c = 1$):

$$\begin{aligned} y_1 &= \frac{1}{4p}\left(2p - 3q - \sqrt{4p^2 - 3q^2}\right) < 0\,, \\ y_2 &= \frac{1}{4p}\left(2p + 3q - \sqrt{4p^2 - 3q^2}\right) > 0\,. \end{aligned} \tag{2.7}$$

In order to specify the range of the other variables appearing in the metric, let us introduce the coordinate α by means of the relation:

$$\beta = -(6\alpha + c\psi)\,. \tag{2.8}$$

Then, the coordinates θ, ϕ, ψ and α span the range:

$$0 \leq \theta \leq \pi\,, \qquad 0 < \phi \leq 2\pi\,, \qquad 0 < \psi \leq 2\pi\,, \qquad 0 < \alpha \leq 2\pi\ell\,, \tag{2.9}$$

[1]If $c = 0$, we can set $a = 3$ by rescaling $y \to \xi y$, $\beta \to \xi^{-1}\beta$, and $a \to \xi^2 a$. If we further write $y = \cos\tilde{\theta}$ and $\beta = \tilde{\phi}$, and choose the period of ψ to be 4π, the metric goes to that of $T^{1,1}$.

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where ℓ is (generically an irrational number) given by:

$$\ell = -\frac{q}{4p^2\, y_1\, y_2} = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}\,, \tag{2.10}$$

the metric (2.2) being periodic in these variables. Notice that, whenever $c \neq 0$, the coordinate β is non-periodic: the periodicities of ψ and α are not congruent, unless the manifold is quasi-regular, i.e., there exists a positive integer k such that

$$k^2 = 4p^2 - 3q^2\ . \tag{2.11}$$

For quasi-regular manifolds, ds_4^2 in (2.4) corresponds to a Kähler-Einstein orbifold. Notice that ℓ becomes rational and it is now possible to assign a periodicity to ψ such that β ends up being periodic. If we perform the change of variables (2.8) in (2.2), we get

$$\begin{aligned} ds^2_{Y^{p,q}} =& \frac{1-cy}{6}\,(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{6\,H^2(y)}\,dy^2 + \frac{v(y)}{9}\,(d\psi + \cos\theta d\phi)^2 + \\ &+ w(y)\,[d\alpha + f(y)\,(d\psi + \cos\theta d\phi)\,]^2\,, \end{aligned} \tag{2.12}$$

with $v(y)$, $w(y)$ and $f(y)$ given by

$$v(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}\,, \qquad w(y) = \frac{2(a-y^2)}{1-cy}\,, \qquad f(y) = \frac{ac - 2y + y^2 c}{6(a-y^2)}\ . \tag{2.13}$$

Concerning the AdS_5 space, we will represent it by means of four Minkowski coordinates x^α ($\alpha = 0,1,2,3$) and a radial variable r. In these coordinates, the AdS_5 metric takes the standard form:

$$ds^2_{AdS_5} = \frac{r^2}{L^2}\,dx^2_{1,3} + \frac{L^2}{r^2}\,dr^2\,. \tag{2.14}$$

The ten-dimensional metric (2.1) is then a solution of the equations of motion of type IIB supergravity if, in addition, we have N units of flux of the self-dual Ramond-Ramond five-form $F^{(5)}$. This solution corresponds to the near-horizon region of a stack of N coincident D3-branes extended along the Minkowski coordinates and located at the apex of the $CY^{p,q}$ cone. The explicit expression of $F^{(5)}$ is:

$$g_s\, F^{(5)} = d^4x\, \wedge dh^{-1} + \text{Hodge dual}\,, \tag{2.15}$$

where $h(r)$ is the near-horizon harmonic function, namely:

$$h(r) = \frac{L^4}{r^4}\,. \tag{2.16}$$

The quantization condition of the flux of $F^{(5)}$ determines the constant L in terms of g_s, N, α' and the volume $\text{Vol}(Y^{p,q})$ of the Sasaki-Einstein space:

$$L^4 = \frac{4\pi^4}{\text{Vol}(Y^{p,q})}\, g_s\, N\, (\alpha')^2\,, \tag{2.17}$$

where $\text{Vol}(Y^{p,q})$ can be computed straightforwardly from the metric (2.2), with the result (for $c = 1$):

$$\text{Vol}(Y^{p,q}) = \frac{q^2}{3p^2}\,\frac{2p + \sqrt{4p^2 - 3q^2}}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}\,\pi^3\,. \tag{2.18}$$

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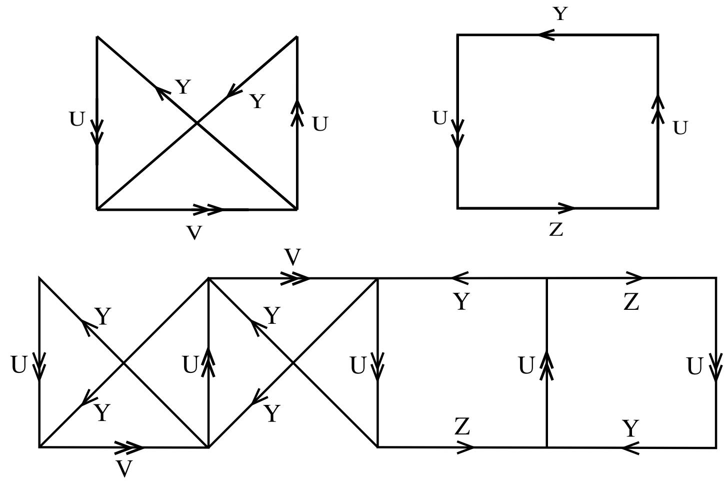

Figure 1: The basic cells σ (upper left) and τ (upper right). $Y^{p,q}$ quivers are built with q σ and $p-q$ τ unit cells. The cubic terms in the superpotential (2.19) come from closed loops of the former and the quartic term arises from the latter. The quiver for $Y^{4,2}$ is given by $\sigma\tilde{\sigma}\tau\tilde{\tau}$ (bottom).

2.1 Quiver theories for $Y^{p,q}$ spaces

The gauge theory dual to IIB on $AdS_5 \times Y^{p,q}$ is by now well understood. Here we quote some of the results that are directly relevant to our discussion. We follow the presentation of ref. [16].

The quivers for $Y^{p,q}$ can be constructed starting with the quiver of $Y^{p,p}$ which is naturally related to the quiver theory obtained from $\mathbb{C}^3/\mathbb{Z}_{2p}$. The gauge group is $\mathrm{SU}(N)^{2p}$ and the superpotential is constructed out of cubic and quartic terms in the four types of bifundamental chiral fields present: two doublets U^α and V^β and two singlets Y and Z of a global SU(2). Namely,

$$W = \sum_{i=1}^{q} \epsilon_{\alpha\beta}(U_i^\alpha V_i^\beta Y_{2i-1} + V_i^\alpha U_{i+1}^\beta Y_{2i}) + \sum_{j=q+1}^{p} \epsilon_{\alpha\beta} Z_j U_{j+1}^\alpha Y_{2j-1} U_j^\beta . \qquad (2.19)$$

Greek indices $\alpha, \beta = 1, 2$ are in SU(2), and Latin subindices i, j refer to the gauge group where the corresponding arrow originates. Equivalently, as explained in [21], the quiver theory for $Y^{p,q}$ can be constructed from two basic cells denoted by σ and τ, and their mirror images with respect to a horizontal axis, $\tilde{\sigma}$ and $\tilde{\tau}$ (see figure 1). Gluing of cells has to respect the orientation of double arrow lines corresponding to the U fields. For example, the quiver $Y^{4,2}$ is given by $\sigma\tilde{\sigma}\tau\tilde{\tau}$. More concrete examples and further discussion can be found in [16, 21].

Here we quote a result of [16] which we will largely reproduce using a study of wrapped branes. The global U(1) symmetries corresponding to the factors appearing in the isometry group of the $Y^{p,q}$ manifold are identified as the R-charge symmetry $\mathrm{U}(1)_R$ and a flavor symmetry $\mathrm{U}(1)_F$. There is also a baryonic $\mathrm{U}(1)_B$ that becomes a gauge symmetry in the gravity dual. The charges of all fields in the quiver with respect to these Abelian symmetries is summarized in table 1.

Field	number	$R-$ charge	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_F$
Y	$p+q$	$\frac{-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2}}{3q^2}$	$p-q$	-1
Z	$p-q$	$\frac{-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2}}{3q^2}$	$p+q$	$+1$
U^α	p	$\frac{2p(2p-\sqrt{4p^2-3q^2})}{3q^2}$	$-p$	0
V^β	q	$\frac{3q-2p+\sqrt{4p^2-3q^2}}{3q}$	q	$+1$

Table 1: Charges for bifundamental chiral fields in the quiver dual to $Y^{p,q}$ [16].

It is worth noting that the above assignment of charges satisfies a number of conditions. For example, the linear anomalies vanish Tr $\mathrm{U}(1)_B =$ Tr $\mathrm{U}(1)_F = 0$, as well as the cubic t' Hooft anomaly Tr $\mathrm{U}(1)_B^3$.

2.2 Complex coordinates on $CY^{p,q}$

We expect some of the supersymmetric embeddings of probes that will be studied in the present paper to be related to the complex coordinates describing $CY^{p,q}$. The relevant coordinates were introduced in [15]. (Here we follow the notation of [34].) The starting point in identifying a good set of complex coordinates is the following set of closed one-forms [15]:

$$\begin{aligned}
\eta^1 &= \frac{1}{\sin\theta}\, d\theta - i d\phi\,, \\
\tilde{\eta}^2 &= -\frac{dy}{H(y)^2} - i(d\beta - c\cos\theta d\phi)\,, \\
\tilde{\eta}^3 &= 3\frac{dr}{r} + i[d\psi + \cos\theta d\phi + y(d\beta - c\cos\theta d\phi)]\,,
\end{aligned} \tag{2.20}$$

in terms of which, the metric of $CY^{p,q}$ can be rewritten as

$$ds^2 = r^2\frac{(1-cy)}{6}\sin^2\!\theta\ |\eta^1|^2 + r^2\frac{H(y)^2}{6}\ |\tilde{\eta}^2|^2 + \frac{r^2}{9}\ |\tilde{\eta}^3|^2\ . \tag{2.21}$$

Unfortunately, $\tilde{\eta}^2$ and $\tilde{\eta}^3$ are not integrable. It is however easy to see that integrable one-forms can be obtained by taking linear combinations of them:

$$\eta^2 = \tilde{\eta}^2 + c\cos\theta\ \eta^1\,, \qquad \eta^3 = \tilde{\eta}^3 + \cos\theta\ \eta^1 + y\ \tilde{\eta}^2\ . \tag{2.22}$$

We can now define $\eta^i = dz_i/z_i$ for $i = 1, 2, 3$, where

$$z_1 = \tan\frac{\theta}{2}\, e^{-i\phi}\,, \qquad z_2 = \frac{(\sin\theta)^c}{f_1(y)}\, e^{-i\beta}\,, \qquad z_3 = r^3\,\frac{\sin\theta}{f_2(y)}\, e^{i\psi}\,, \tag{2.23}$$

with $f_1(y)$ and $f_2(y)$ being given by:

$$f_1(y) = \exp\left(\int \frac{1}{H(y)^2}dy\right)\,, \qquad f_2(y) = \exp\left(\int \frac{y}{H(y)^2}dy\right)\,. \tag{2.24}$$

By using the form of $H(y)$ written in eq. (2.3) it is possible to provide a simpler expression for the functions $f_i(y)$, namely:

$$\begin{aligned}\frac{1}{f_1(y)} &= \sqrt{(y-y_1)^{\frac{1}{y_1}}\,(y_2-y)^{\frac{1}{y_2}}\,(y_3-y)^{\frac{1}{y_3}}}\,,\\ \frac{1}{f_2(y)} &= \sqrt{\mathcal{Q}(y)} = \sqrt{2c}\,\sqrt{(y-y_1)\,(y_2-y)\,(y_3-y)}\,,\end{aligned} \tag{2.25}$$

where $\mathcal{Q}(y)$ has been defined in (2.6), y_1 and y_2 are given in eq. (2.7) and y_3 is the third root of the polynomial $\mathcal{Q}(y)$ which, for $c=1$, is related to $y_{1,2}$ as $y_3 = \frac{3}{2} - y_1 - y_2$. The holomorphic three-form of $CY^{p,q}$ simply reads

$$\Omega = -\frac{1}{18} e^{i\psi} r^3 \sqrt{\frac{\mathcal{Q}(y)}{3}}\,\sin\theta\;\eta^1\wedge\eta^2\wedge\eta^3 = -\frac{1}{18\sqrt{3}}\,\frac{dz_1\wedge dz_2\wedge dz_3}{z_1 z_2}\;. \tag{2.26}$$

Notice that coordinates z_1 and z_2 are local complex coordinates on the transverse leaves of $Y^{p,q}$ (2.4) with Kähler-Einstein metric ds_4^2. They are not globally well-defined as soon as z_2 is periodic in β –which is not a periodic coordinate. Besides, they are meromorphic functions on $CY^{p,q}$ (the function z_1 is singular at $\theta = \pi$ while z_2 has a singularity at $y = y_1$). A set of holomorphic coordinates on $Y^{p,q}$ was constructed in [35].

2.3 Killing spinors for $AdS_5 \times Y^{p,q}$

The $AdS_5 \times Y^{p,q}$ background preserves eight supersymmetries, in agreement with the $\mathcal{N}=1$ superconformal character of the corresponding dual field theory, which has four ordinary supersymmetries and four superconformal ones. In order to verify this statement, and for later use, let us write explicitly the form of the Killing spinors of the background, which are determined by imposing the vanishing of the supersymmetric variations of the dilatino and gravitino. The result of this calculation is greatly simplified in some particular basis of frame one-forms, which we will now specify. In the AdS_5 part of the metric we will choose the natural basis of vielbein one-forms, namely:

$$e^{x^\alpha} = \frac{r}{L}\,dx^\alpha\,, \qquad (\alpha = 0,1,2,3)\,, \qquad\qquad e^r = \frac{L}{r}\,. \tag{2.27}$$

Moreover, in the $Y^{p,q}$ directions we will use the following frame:

$$\begin{aligned} e^1 &= -\frac{L}{\sqrt{6}}\,\frac{1}{H(y)}\,dy\,,\\ e^2 &= -\frac{L}{\sqrt{6}}\,H(y)\,(d\beta - c\cos\theta\,d\phi)\,,\\ e^3 &= \frac{L}{\sqrt{6}}\,\sqrt{1-c\,y}\,d\theta\,,\\ e^4 &= \frac{L}{\sqrt{6}}\,\sqrt{1-c\,y}\,\sin\theta\,d\phi,\\ e^5 &= \frac{L}{3}\,\left(d\psi + y\,d\beta + (1-c\,y)\cos\theta\,d\phi\right)\;. \end{aligned} \tag{2.28}$$

In order to write the expressions of the Killing spinors in a compact form, let us define the matrix Γ_* as:

$$\Gamma_* \equiv i\Gamma_{x^0x^1x^2x^3}\,. \tag{2.29}$$

Then, the Killing spinors ϵ of the $AdS_5 \times Y^{p,q}$ background can be written in terms of a constant spinor η as [2]:

$$\epsilon = e^{-\frac{i}{2}\psi}\, r^{-\frac{\Gamma_*}{2}}\,\Big(1+\frac{\Gamma_r}{2L^2}\,x^\alpha\,\Gamma_{x^\alpha}\,(1+\Gamma_*)\Big)\,\eta\,. \tag{2.30}$$

The spinor η satisfies the projections:

$$\Gamma_{12}\,\eta = -i\eta\,, \qquad\qquad \Gamma_{34}\,\eta = i\eta\,, \tag{2.31}$$

which show that this background preserves eight supersymmetries. Notice that, since the matrix multiplying η in eq. (2.30) commutes with Γ_{12} and Γ_{34}, the spinor ϵ also satisfies the conditions (2.31), i.e.:

$$\Gamma_{12}\,\epsilon = -i\epsilon\,, \qquad\qquad \Gamma_{34}\,\epsilon = i\epsilon\,. \tag{2.32}$$

In eq. (2.30) we are parameterizing the dependence of ϵ on the coordinates of AdS_5 as in ref. [36]. In order to explore this dependence in detail, it is interesting to decompose the constant spinor η according to the different eigenvalues of the matrix Γ_*:

$$\Gamma_*\,\eta_\pm = \pm\eta_\pm\,. \tag{2.33}$$

Using this decomposition we obtain two types of Killing spinors:

$$\begin{aligned} e^{\frac{i}{2}\psi}\,\epsilon_- &= r^{1/2}\,\eta_-\,,\\ e^{\frac{i}{2}\psi}\,\epsilon_+ &= r^{-1/2}\,\eta_+ + \frac{r^{1/2}}{L^2}\,\Gamma_r\,x^\alpha\,\Gamma_{x^\alpha}\,\eta_+\,. \end{aligned} \tag{2.34}$$

The four spinors ϵ_- are independent of the coordinates x^α and $\Gamma_*\epsilon_- = -\epsilon_-$, whereas the ϵ_+'s do depend on the x^α's and are not eigenvectors of Γ_*. The latter correspond to the four superconformal supersymmetries, while the ϵ_-'s correspond to the ordinary ones. Notice also that the only dependence of these spinors on the coordinates of the $Y^{p,q}$ space is through the exponential of the angle ψ in eq. (2.34).

In addition to the Poincare coordinates (x^α, r) used above to represent the AdS_5 metric, it is also convenient to write it in the so-called global coordinates, in which $ds^2_{AdS_5}$ takes the form:

$$ds^2_{AdS_5} = L^2\Big[-\cosh^2\rho\;d\tau^2 + d\rho^2 + \sinh^2\rho\;d\Omega_3^2\Big]\,, \tag{2.35}$$

where $d\Omega_3^2$ is the metric of a unit three-sphere parameterized by three angles $(\alpha^1, \alpha^2, \alpha^3)$:

$$d\Omega_3^2 = (d\alpha^1)^2 + \sin^2\alpha^1\Big((d\alpha^2)^2 + \sin^2\alpha^2\,(d\alpha^3)^2\Big)\,, \tag{2.36}$$

[2] Note that this spinor differs from the one of [12] by a rotation generated by $e^{-\frac{i}{2}\psi\,\Gamma_{34}}$. This rotation accounts for the difference between both frames.

with $0 \leq \alpha^1, \alpha^2 \leq \pi$ and $0 < \alpha^3 \leq 2\pi$. In order to write down the Killing spinors in these coordinates, we will choose the same frame as in eq. (2.28) for the $Y^{p,q}$ part of the metric, while for the AdS_5 directions we will use:

$$
\begin{aligned}
e^{\tau} &= L\cosh\rho\, d\tau\,, \qquad\qquad e^{\rho} = L d\rho\,,\\
e^{\alpha^1} &= L\sinh\rho\, d\alpha^1\,,\\
e^{\alpha^2} &= L\sinh\rho\,\sin\alpha^1\, d\alpha^2\,,\\
e^{\alpha^3} &= L\sinh\rho\,\sin\alpha^1\,\sin\alpha^2\, d\alpha^3\,.
\end{aligned}
\tag{2.37}
$$

If we now define the matrix

$$
\gamma_* \equiv \Gamma_\tau\,\Gamma_\rho\,\Gamma_{\alpha^1\,\alpha^2\,\alpha^3}\,,
\tag{2.38}
$$

then, the Killing spinors in these coordinates can be written as [37]:

$$
\epsilon = e^{-\frac{i}{2}\psi}\, e^{-i\,\frac{\rho}{2}\,\Gamma_\rho\gamma_*}\, e^{-i\,\frac{\tau}{2}\,\Gamma_\tau\gamma_*}\, e^{-\frac{\alpha^1}{2}\,\Gamma_{\alpha^1\rho}}\, e^{-\frac{\alpha^2}{2}\,\Gamma_{\alpha^2\alpha^1}}\, e^{-\frac{\alpha^3}{2}\,\Gamma_{\alpha^3\alpha^2}}\,\eta\,,
\tag{2.39}
$$

where η is a constant spinor that satisfies the same conditions as in eq. (2.31).

2.4 Supersymmetric probes on $AdS_5 \times Y^{p,q}$

In the remainder of this paper we will consider Dp-brane probes moving in the $AdS_5\times Y^{p,q}$ background. If ξ^μ ($\mu = 0,\ldots,p$) are a set of worldvolume coordinates and X^M denote ten-dimensional coordinates, the embedding of the brane probe in the background geometry will be characterized by the set of functions $X^M(\xi^\mu)$, from which the induced metric on the worldvolume is determined as:

$$
g_{\mu\nu} = \partial_\mu X^M\,\partial_\nu X^N\, G_{MN}\,,
\tag{2.40}
$$

where G_{MN} is the ten-dimensional metric. Let $e^{\underline{M}}$ be the frame one-forms of the ten-dimensional metric. These one-forms can be written in terms of the differentials of the coordinates by means of the coefficients $E^{\underline{M}}_N$:

$$
e^{\underline{M}} = E^{\underline{M}}_N\, dX^N\,.
\tag{2.41}
$$

From the $E^{\underline{M}}_N$'s and the embedding functions $X^M(\xi^\mu)$ we define the induced Dirac matrices on the worldvolume as:

$$
\gamma_\mu = \partial_\mu\, X^M\, E^{\underline{N}}_M\,\Gamma_{\underline{N}}\,,
\tag{2.42}
$$

where $\Gamma_{\underline{N}}$ are constant ten-dimensional Dirac matrices.

The supersymmetric embeddings of the brane probes are obtained by imposing the kappa-symmetry condition:

$$
\Gamma_\kappa\,\epsilon = \epsilon\,,
\tag{2.43}
$$

where ϵ is a Killing spinor of the background and Γ_κ is a matrix that depends on the embedding. In order to write the expression of Γ_κ for the type IIB theory it is convenient to decompose the complex spinor ϵ in its real and imaginary parts, ϵ_1 and ϵ_2. These are Majorana-Weyl spinors. They can be subsequently arranged as a two-dimensional vector

$$\epsilon = \epsilon_1 + i\epsilon_2 \;\longleftrightarrow\; \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} . \tag{2.44}$$

The dictionary to go from complex to real spinors is straightforward, namely:

$$\epsilon^* \;\longleftrightarrow\; \tau_3\,\epsilon\,, \qquad i\epsilon^* \;\longleftrightarrow\; \tau_1\,\epsilon\,, \qquad i\epsilon \;\longleftrightarrow\; -\,i\tau_2\,\epsilon\,, \tag{2.45}$$

where the τ_i $(i = 1, 2, 3)$ are nothing but the Pauli matrices. If there are not worldvolume gauge fields on the D-brane, the kappa symmetry matrix of a Dp-brane in the type IIB theory is given by [18]:

$$\Gamma_\kappa = \frac{1}{(p+1)!\sqrt{-g}}\,\epsilon^{\mu_1\cdots\mu_{p+1}}\,(\tau_3)^{\frac{p-3}{2}}\,i\tau_2\,\otimes\,\gamma_{\mu_1\cdots\mu_{p+1}}\,, \tag{2.46}$$

where g is the determinant of the induced metric $g_{\mu\nu}$ and $\gamma_{\mu_1\cdots\mu_{p+1}}$ denotes the antisymmetrized product of the induced Dirac matrices (2.42). To write eq. (2.46) we have assumed that the worldvolume gauge field A is zero. This assumption is consistent with the equations of motion of the probe as far as there are no source terms in the action which could induce a non-vanishing value of A. These source terms must be linear in A and can only appear in the Wess-Zumino term of the probe action, which is responsible for the coupling of the probe to the Ramond-Ramond fields of the background. In the case under study only $F^{(5)}$ is non-zero (see eq. (2.15)) and the only linear term in A is of the form $\int A\wedge F^{(5)}$, which is different from zero only for a D5-brane which captures the flux of $F^{(5)}$. This only happens for the baryon vertex configuration studied in subsection 6.5. In all other cases studied in this paper one can consistently put the worldvolume gauge field to zero. Nevertheless, even if one is not forced to do it, in some cases we can switch on the field A to study how this affects the supersymmetric embeddings. In these cases the expression (2.46) for Γ_κ is no longer valid and we must use the more general formula given in ref. [18].

The kappa symmetry condition $\Gamma_\kappa\,\epsilon = \epsilon$ imposes a new projection to the Killing spinor ϵ which, in general, will not be compatible with those already satisfied by ϵ (see eq. (2.32)). This is so because the new projections involve matrices which do not commute with those appearing in (2.32). The only way of making these two conditions consistent with each other is by requiring the vanishing of the coefficients of those non-commuting matrices, which will give rise to a set of first-order BPS differential equations. By solving these BPS equations we will determine the supersymmetric embeddings of the brane probes we are looking for. Notice also that the kappa symmetry condition must be satisfied *at any point of the probe worldvolume*. It is a local condition whose global meaning, as we will see in a moment, has to be addressed a posteriori. This requirement is not obvious at all since the spinor ϵ depends on the coordinates (see eqs. (2.30) and (2.39)). However this would be guaranteed if we could reduce the $\Gamma_\kappa\,\epsilon = \epsilon$ projection to some algebraic conditions on the constant spinor η of eqs. (2.30) and (2.39). The counting of solutions of the algebraic equations satisfied by η will give us the fraction of supersymmetry preserved by our brane probe.

3. Supersymmetric D3-branes on $AdS_5 \times Y^{p,q}$

Let us now apply the methodology just described to find the supersymmetric configurations of a D3-brane in the $AdS_5 \times Y^{p,q}$ background. The kappa symmetry matrix in this case can be obtained by putting $p = 3$ in the general expression (2.46):

$$\Gamma_\kappa = -\frac{i}{4!\sqrt{-g}}\, \epsilon^{\mu_1\cdots\mu_4}\, \gamma_{\mu_1\cdots\mu_4}\,, \tag{3.1}$$

where we have used the rule (2.45) to write the expression of Γ_κ acting on complex spinors. Given that the $Y^{p,q}$ space is topologically $S^2 \times S^3$, it is natural to consider D3-branes wrapping two- and three-cycles in the Sasaki-Einstein space. A D3-brane wrapping a two-cycle in $Y^{p,q}$ and extended along one of the spatial directions of AdS_5 represents a fat string. We will study such type of configurations in section 6 where we conclude that they are not supersymmetric, although we will find stable non-supersymmetric embeddings of this type.

In this section we will concentrate on the study of supersymmetric configurations of D3-branes wrapping a three-cycle of $Y^{p,q}$. These objects are pointlike from the gauge theory point of view and, on the field theory side, they correspond to dibaryons constructed from the different bifundamental fields. In what follows we will study the kappa symmetry condition for two different sets of worldvolume coordinates, which will correspond to two classes of cycles and dibaryons.

3.1 Singlet supersymmetric three-cycles

Let us use the global coordinates of eq. (2.35) to parameterize the AdS_5 part of the metric and let us consider the following set of worldvolume coordinates:

$$\xi^\mu = (\tau, \theta, \phi, \beta), \tag{3.2}$$

and the following generic ansatz for the embedding:

$$y = y(\theta, \phi, \beta), \qquad \psi(\theta, \phi, \beta). \tag{3.3}$$

The kappa symmetry matrix in this case is:

$$\Gamma_\kappa = -iL\,\frac{\cosh\rho}{\sqrt{-g}}\,\Gamma_\tau\,\gamma_{\theta\phi\beta}\,. \tag{3.4}$$

The induced gamma matrices along the θ, ϕ and β directions can be straightforwardly obtained from (2.42), namely:

$$\begin{aligned}
\frac{1}{L}\gamma_\theta &= \frac{\sqrt{1-c\,y}}{\sqrt{6}}\,\Gamma_3 + \frac{1}{3}\psi_\theta\,\Gamma_5 - \frac{1}{\sqrt{6}\,H}\,y_\theta\,\Gamma_1,\\
\frac{1}{L}\gamma_\phi &= \frac{cH\cos\theta}{\sqrt{6}}\,\Gamma_2 + \frac{\sqrt{1-c\,y}}{\sqrt{6}}\sin\theta\,\Gamma_4 + \frac{1}{3}\left(\psi_\phi + (1-c\,y)\cos\theta\right)\Gamma_5 - \frac{1}{\sqrt{6}\,H}\,y_\phi\,\Gamma_1,\\
\frac{1}{L}\gamma_\beta &= -\frac{H}{\sqrt{6}}\,\Gamma_2 + \frac{1}{3}\left(\psi_\beta + y\right)\Gamma_5 - \frac{1}{\sqrt{6}\,H}\,y_\beta\,\Gamma_1,
\end{aligned} \tag{3.5}$$

where the subscripts in y and ψ denote partial differentiation. By using this result and the projections (2.32) the action of the antisymmetrized product $\gamma_{\theta\phi\beta}$ on the Killing spinor ϵ reads:

$$-\frac{i}{L^3}\,\gamma_{\theta\phi\beta}\,\epsilon = [a_5\,\Gamma_5 + a_1\Gamma_1 + a_3\Gamma_3 + a_{135}\,\Gamma_{135}\,]\epsilon\,, \tag{3.6}$$

where the coefficients on the right-hand side are given by:

$$\begin{aligned}
a_5 &= \frac{1}{18}\Bigg[(y+\psi_\beta)\,[(1-cy)\,\sin\theta + c\,y_\theta\cos\theta] + \\
&\qquad + [\psi_\phi + (1-cy)\,\cos\theta\,]y_\theta - \psi_\theta y_\phi - c\cos\theta\psi_\theta y_\beta\Bigg]\,,\\
a_1 &= -\frac{1-cy}{6\sqrt{6}}\,\sin\theta\left[\frac{y_\beta}{H} - iH\right],\\
a_3 &= -\frac{\sqrt{1-cy}}{6\sqrt{6}}\,[y_\phi + c\cos\theta y_\beta - i\sin\theta y_\theta\,]\,,\\
a_{135} &= \frac{\sqrt{1-cy}}{18}\,\Bigg[\frac{\sin\theta}{H}[\psi_\theta y_\beta - (y+\psi_\beta)\,y_\theta] + H[\psi_\phi + (1+c\psi_\beta)\cos\theta]\, + \\
&\qquad + \frac{i}{H}\Big[(\psi_\phi + (1-cy)\,\cos\theta)y_\beta - (y+\psi_\beta)\,y_\phi\Big] - iH\sin\theta\psi_\theta\Bigg]\,.
\end{aligned} \tag{3.7}$$

As discussed at the end of section 2, in order to implement the kappa symmetry projection we must require the vanishing of the terms in (3.6) which are not compatible with the projection (2.32). Since the matrices Γ_1, Γ_3 and Γ_{135} do not commute with those appearing in the projection (2.32), it follows that we must impose that the corresponding coefficients vanish, i.e.:

$$a_1 = a_3 = a_{135} = 0\,. \tag{3.8}$$

Let us concentrate first on the condition $a_1 = 0$. By looking at its imaginary part:

$$H(y) = 0\,, \tag{3.9}$$

which, in the range of allowed values of y, means:

$$y = y_1\,, \qquad \text{or} \qquad y = y_2\,. \tag{3.10}$$

If $H(y) = 0$, it follows by inspection that $a_1 = a_3 = a_{135} = 0$. Notice that ψ can be an arbitrary function. Moreover, one can check that:

$$\sqrt{-g}\,\big|_{BPS} \;=\; L^4\,\cosh\rho\;\, a_5\big|_{BPS}\,. \tag{3.11}$$

Thus, one has the following equality:

$$\Gamma_\kappa\,\epsilon\big|_{BPS} \;=\; \Gamma_\tau\Gamma_5\,\epsilon\,, \tag{3.12}$$

and, therefore, the condition $\Gamma_\kappa\epsilon = \epsilon$ becomes equivalent to

$$\Gamma_\tau\Gamma_5\,\epsilon = \epsilon\,. \tag{3.13}$$

As it happens in the $T^{1,1}$ case [12], the compatibility of (3.13) with the AdS_5 structure of the spinor implies that the D3-brane must be placed at $\rho = 0$, i.e. at the center of AdS_5. Indeed, as discussed at the end of section 2, we must translate the condition (3.13) into a condition for the constant spinor η of eq. (2.39). Notice that $\Gamma_\tau\Gamma_5$ commutes with all the matrices appearing on the right-hand side of eq. (2.39) except for $\Gamma_\rho\gamma_*$. Since the coefficient of $\Gamma_\rho\gamma_*$ in (2.39) only vanishes for $\rho = 0$, it follows that only at this point the equation $\Gamma_\kappa\,\epsilon = \epsilon$ can be satisfied for every point in the worldvolume and reduces to:

$$\Gamma_\tau\Gamma_5\,\eta = \eta\,. \tag{3.14}$$

Therefore, if we place the D3-brane at the center of the AdS_5 space and wrap it on the three-cycles at $y = y_1$ or $y = y_2$, we obtain a $\frac{1}{8}$ supersymmetric configuration which preserves the Killing spinors of the type (2.39) with η satisfying (2.31) and the additional condition (3.14).

The cycles we have just found have been identified by Martelli and Sparks as those dual to the dibaryonic operators $\det(Y)$ and $\det(Z)$, made out of the bifundamental fields that, as the D3-brane wraps the two-sphere whose isometries are responsible for the global SU(2) group, are singlets under this symmetry [15]. For this reason we will refer to these cycles as singlet (S) cycles. Let us recall how this identification is carried out. First of all, we look at the conformal dimension Δ of the corresponding dual operator. Following the general rule of the AdS/CFT correspondence (and the zero-mode corrections of ref. [5]), $\Delta = LM$, where L is given by (2.17) and M is the mass of the wrapped three-brane. The latter can be computed as $M = T_3\,V_3$, with T_3 being the tension of the D3-brane $(1/T_3 = 8\pi^3(\alpha')^2 g_s)$ and V_3 the volume of the three-cycle. If $g_{\mathcal{C}}$ is the determinant of the spatial part of the induced metric on the three-cycle $\mathcal{C}$, one has:

$$V_3 = \int_{\mathcal{C}} \sqrt{g_{\mathcal{C}}}\;\; d^3\xi\,. \tag{3.15}$$

For the singlet cycles S_i at $y = y_i$ $(i = 1, 2)$ and ψ=constant, the volume V_3 is readily computed, namely:

$$V_3^{\mathrm{S}_i} = \frac{2L^3}{3}\,(1 - cy_i)\,|\,y_i\,|\,(2\pi)^2\,\ell\,. \tag{3.16}$$

Let us define $\lambda_1 = +1$, $\lambda_2 = -1$. Then, if $\Delta_i^{\mathrm{S}} \equiv \Delta^{\mathrm{S}_i}$, one has:

$$\Delta_i^{\mathrm{S}} = \frac{N}{2q^2}\Big[-4p^2 + 3q^2 + 2\lambda_i\,pq + (2p - \lambda_i\,q)\,\sqrt{4p^2 - 3q^2}\,\Big]\,. \tag{3.17}$$

As it should be for a BPS saturated object, the R-charges R_i of the S_i cycles are related to Δ_i^{S} as $R_i = \frac{2}{3}\,\Delta_i^{\mathrm{S}}$. By comparing the values of R_i with those determined in [16] from the gauge theory dual (see table 1) one concludes that, indeed, a D3-brane wrapped at $y = y_1$ $(y = y_2)$ can be identified with the operator $\det(Y)$ $(\det(Z))$ as claimed. Another piece of evidence which supports this claim is the calculation of the baryon number, that can be identified with the third homology class of the three-cycle $\mathcal{C}$ over which the D3-brane is wrapped. This number (in units of N) can be obtained by computing the integral over $\mathcal{C}$

of the pullback of a $(2,1)$ three-form $\Omega_{2,1}$ on $CY^{p,q}$:

$$\mathcal{B}(\mathcal{C}) = \pm i \int_{\mathcal{C}} P[\Omega_{2,1}]_{\mathcal{C}}\,, \tag{3.18}$$

where $P[\cdots]_{\mathcal{C}}$ denotes the pullback to the cycle $\mathcal{C}$ of the form that is inside the brackets. The sign of the right-hand side of (3.18) depends on the orientation of the cycle. The explicit expression of $\Omega_{2,1}$ has been determined in ref. [21]:

$$\Omega_{2,1} = K\Big(\frac{dr}{r} + \frac{i}{L}\,e^5\Big) \wedge \omega\,, \tag{3.19}$$

where e^5 is the one-form of our vielbein (2.28) for the $Y^{p,q}$ space, K is the constant

$$K = \frac{9}{8\pi^2}\,(p^2 - q^2)\,, \tag{3.20}$$

and ω is the two-form:

$$\omega = -\frac{1}{(1-cy)^2\,L^2}\,\Big[e^1 \wedge e^2 + e^3 \wedge e^4\Big]\,. \tag{3.21}$$

Using (θ, ϕ, β) as worldvolume coordinates of the singlet cycles S_i,

$$P[\Omega_{2,1}]_{\mathrm{S}_i} = -i\,\frac{K}{18}\,\frac{y_i}{1-cy_i}\,\sin\theta\, d\theta \wedge d\phi \wedge d\beta\,, \tag{3.22}$$

Then, changing variables from β to α by means of (2.8), and taking into account that $\alpha \in [0, 2\pi\ell]$, one gets:

$$\int_{\mathrm{S}_i} P[\Omega_{2,1}]_{\mathrm{S}_i} = -i\,\frac{8\pi^2}{3}\,\frac{K\ell y_i}{1-cy_i}\,. \tag{3.23}$$

After using the values of y_1 and y_2 displayed in (2.7), we arrive at:

$$\begin{aligned} \mathcal{B}(\mathrm{S}_1) &= -i \int_{\mathrm{S}_1} P[\Omega_{2,1}]_{\mathrm{S}_1} = p - q\,, \\ \mathcal{B}(\mathrm{S}_2) &= i \int_{\mathrm{S}_2} P[\Omega_{2,1}]_{\mathrm{S}_2} = p + q\,. \end{aligned} \tag{3.24}$$

Notice the perfect agreement of $\mathcal{B}(\mathrm{S}_1)$ and $\mathcal{B}(\mathrm{S}_2)$ with the baryon numbers of Y and Z displayed in table 1.

3.2 Doublet supersymmetric three-cycles

Let us now try to find supersymmetric embeddings of D3-branes on three-cycles by using a different set of worldvolume coordinates. As in the previous subsection it is convenient to use the global coordinates (2.35) for the AdS_5 part of the metric and the following set of worldvolume coordinates:

$$\xi^\mu = (\tau, y, \beta, \psi)\ . \tag{3.25}$$

Moreover, we will adopt the ansatz:

$$\theta(y, \beta, \psi)\,, \qquad \phi(y, \beta, \psi)\ . \tag{3.26}$$

The kappa symmetry matrix Γ_κ in this case takes the form:

$$\Gamma_\kappa = -iL\,\frac{\cosh\rho}{\sqrt{-g}}\,\Gamma_\tau\,\gamma_{y\,\beta\,\psi}\,, \tag{3.27}$$

and the induced gamma matrices are:

$$\begin{aligned}
\frac{1}{L}\gamma_y &= -\frac{1}{\sqrt{6}H}\Gamma_1 + \frac{cH\cos\theta}{\sqrt{6}}\phi_y\Gamma_2 + \frac{\sqrt{1-cy}}{\sqrt{6}}\,(\theta_y\Gamma_3 + \phi_y\sin\theta\,\Gamma_4) + \frac{1-c\,y}{3}\cos\theta\phi_y\,\Gamma_5\,,\\
\frac{1}{L}\gamma_\beta &= \frac{H}{\sqrt{6}}\,(-1 + c\cos\theta\,\phi_\beta)\;\Gamma_2 + \frac{\sqrt{1-c\,y}}{\sqrt{6}}\,\theta_\beta\,\Gamma_3 + \frac{\sqrt{1-c\,y}}{\sqrt{6}}\sin\theta\,\phi_\beta\,\Gamma_4\\
&\quad + \frac{1}{3}\Big(\,y + (1-c\,y)\cos\theta\,\phi_\beta\Big)\,\Gamma_5\,, \qquad\qquad (3.28)\\
\frac{1}{L}\gamma_\psi &= \frac{cH\cos\theta}{\sqrt{6}}\,\phi_\psi\,\Gamma_2 + \frac{\sqrt{1-c\,y}}{\sqrt{6}}\,(\theta_\psi\,\Gamma_3 + \sin\theta\,\phi_\psi\,\Gamma_4) + \frac{1}{3}\,(1 + (1-c)\,\cos\theta\phi_\psi)\,\Gamma_5\,.
\end{aligned}$$

By using again the projections (2.32) one easily gets the action of $\gamma_{y\,\beta\,\psi}$ on the Killing spinor

$$-\frac{i}{L^3}\,\gamma_{y\,\beta\,\psi}\,\epsilon = [c_5\,\Gamma_5 + c_1\,\Gamma_1 + c_3\,\Gamma_3 + c_{135}\,\Gamma_{135}]\epsilon\,, \tag{3.29}$$

where the different coefficients appearing on the right-hand side of (3.29) are given by:

$$\begin{aligned}
c_5 &= \frac{1}{18}\Bigg[-1 - \cos\theta(\phi_\psi - c\phi_\beta) + (1-cy)\sin\theta\,\Big[\theta_y(\phi_\beta - y\phi_\psi) - \phi_y(\theta_\beta - y\theta_\psi)\Big]\Bigg]\,,\\
c_1 &= -\frac{1-cy}{6\sqrt{6}}\,\sin\theta\,\Big[\frac{\theta_\beta\phi_\psi - \theta_\psi\phi_\beta}{H} + iH\,(\theta_y\phi_\psi - \theta_\psi\phi_y)\Big]\,,\\
c_3 &= -\frac{\sqrt{1-cy}}{6\sqrt{6}}\Big[\theta_\psi - c\cos\theta\,(\theta_\psi\phi_\beta - \theta_\beta\phi_\psi) + i\sin\theta\phi_\psi\Big]\,,\\
c_{135} &= -\frac{\sqrt{1-cy}}{18}\Bigg[\frac{\sin\theta}{H}(\phi_\beta - y\phi_\psi) + H\Big(\theta_y + \cos\theta\Big[\theta_y(\phi_\psi - c\phi_\beta) - \phi_y(\theta_\psi - c\theta_\beta)\Big]\Big)\\
&\quad + iH\sin\theta\,\phi_y - \frac{i}{H}\Big[\theta_\beta - y\theta_\psi + (1-cy)\cos\theta(\theta_\beta\phi_\psi - \theta_\psi\phi_\beta)\Big]\Bigg]\,. \qquad (3.30)
\end{aligned}$$

Again, we notice that the matrices Γ_1, Γ_3 and Γ_{135} do not commute with the projections (2.32). We must impose:

$$c_1 = c_3 = c_{135} = 0\,. \tag{3.31}$$

From the vanishing of the imaginary part of c_3 we obtain the condition:

$$\sin\theta\,\phi_\psi = 0\,. \tag{3.32}$$

One can solve the condition (3.32) by taking $\sin\theta = 0$, i.e. for $\theta = 0, \pi$. By inspection one easily realizes that c_1, c_3 and c_{135} also vanish for these values of θ and for an arbitrary function $\phi(y,\beta,\psi)$. Therefore, we have the solution

$$\theta = 0, \pi\,, \qquad \phi = \phi(y,\beta,\psi)\,. \tag{3.33}$$

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Another possibility is to take $\phi_\psi = 0$. In this case one readily verifies that c_1 and c_3 vanish if $\theta_\psi = 0$. Thus, let us assume that both ϕ and θ are independent of the angle ψ. From the vanishing of the real and imaginary parts of c_{135} we get two equations for the functions $\theta = \theta(y, \beta)$ and $\phi = \phi(y, \beta)$, namely:

$$\begin{aligned} \theta_y + \frac{\sin\theta}{H^2}\,\phi_\beta + c\cos\theta\,(\phi_y\,\theta_\beta - \theta_y\,\phi_\beta) &= 0\,, \\ \theta_\beta - H^2\,\sin\theta\phi_y &= 0\,. \end{aligned} \tag{3.34}$$

If the BPS equations (3.34) hold, one can verify that the kappa symmetry condition $\Gamma_\kappa\,\epsilon = \epsilon$ reduces, up to a sign, to the projection (3.13) for the Killing spinor. As in the case of the S three-cycles studied in subsection 3.1, by using the explicit expression (2.39) of ϵ in terms of the global coordinates of AdS_5, one concludes that the D3-brane must be placed at $\rho = 0$. The corresponding configuration preserves four supersymmetries.

In the next subsection we will tackle the problem of finding the general solution of the system (3.34). Here we will analyze the trivial solution of this system, namely:

$$\theta = \text{constant}\,, \qquad\qquad \phi = \text{constant}\,. \tag{3.35}$$

This kind of three-cycle was studied in ref. [21] by Herzog, Ejaz and Klebanov (see also [16]), who showed that it corresponds to dibaryons made out of the SU(2) doublet fields U^α. In what follows we will refer to it as doublet (D) cycle. Let us review the arguments leading to this identification. First of all, the volume of the D cycle (3.35) can be computed with the result:

$$V_3^D = \frac{L^3}{3}\,(2\pi)^2\,(y_2 - y_1)\,\ell\,. \tag{3.36}$$

By using the values of y_1 and y_2 (eq. (2.7)), L (eq. (2.17)) and ℓ (eq. (2.10)) we find the following value of the conformal dimension:

$$\Delta^D = N\,\frac{p}{q^2}\Big(2p - \sqrt{4p^2 - 3q^2}\,\Big)\,. \tag{3.37}$$

By comparison with table 1, one can verify that the corresponding R-charge, namely $2/3\,\Delta^D$, is equal to the R-charge of the field U^α multiplied by N. We can check this identification by computing the baryon number. Since, in this case, the pullback of $\Omega_{2,1}$ is:

$$P[\Omega_{2,1}]_{\rm D} = i\,\frac{K}{3(1-cy)^2}\,dy\wedge d\alpha\wedge d\psi \tag{3.38}$$

we get:

$$\mathcal{B}({\rm D}) = -i\int_{\rm D} P[\Omega_{2,1}]_{\rm D} = -p\,, \tag{3.39}$$

which, indeed, coincides with the baryon number of U^α written in table 1.

3.2.1 General integration

Let us now try to integrate in general the first-order differential system (3.34). With this purpose it is more convenient to describe the locus of the D3-brane by means of two functions $y = y(\theta, \phi)$, $\beta = \beta(\theta, \phi)$. Notice that this is equivalent to the description used so far (in which the independent variables were (y, β)), except for the cases in which (θ, ϕ) or (y, β) are constant. The derivatives in these two descriptions are related by simply inverting the Jacobian matrix, i.e.:

$$\begin{pmatrix} y_\theta & y_\phi \\ \beta_\theta & \beta_\phi \end{pmatrix} = \begin{pmatrix} \theta_y & \theta_\beta \\ \phi_y & \phi_\beta \end{pmatrix}^{-1} . \tag{3.40}$$

By using these equations the first-order system (3.34) is equivalent to:

$$\beta_\theta = \frac{y_\phi}{H^2 \sin\theta} , \qquad\qquad \beta_\phi = c\cos\theta - \frac{\sin\theta}{H^2}\, y_\theta . \tag{3.41}$$

These equations can be obtained directly by using θ and ϕ as worldvolume coordinates. Interestingly, in this form the BPS equations can be written as Cauchy-Riemann equations and, thus, they can be integrated in general. This is in agreement with the naive expectation that, at least locally, these equations should determine some kind of holomorphic embeddings. In order to verify this fact, let us introduce new variables u_1 and u_2, related to θ and y as follows:

$$u_1 = \log\left(\tan\frac{\theta}{2}\right), \qquad\qquad u_2 = \log\left(\frac{(\sin\theta)^c}{f_1(y)}\right). \tag{3.42}$$

By comparing the above expressions with the definitions of z_1 and z_2 in eq. (2.23), one gets:

$$u_1 - i\phi = \log z_1 , \qquad\qquad u_2 - i\beta = \log z_2 . \tag{3.43}$$

The relation between u_1 and θ leads to $du_1 = d\theta/\sin\theta$, from which it follows that:

$$\frac{\partial u_2}{\partial u_1} = c\cos\theta - \frac{\sin\theta}{H^2}\, y_\theta , \qquad\qquad \frac{\partial \beta}{\partial u_1} = \sin\theta\, \beta_\theta , \tag{3.44}$$

and it is easy to demonstrate that the BPS equations (3.41) can be written as:

$$\frac{\partial u_2}{\partial u_1} = \frac{\partial \beta}{\partial \phi} , \qquad\qquad \frac{\partial u_2}{\partial \phi} = -\frac{\partial \beta}{\partial u_1} , \tag{3.45}$$

these being the Cauchy-Riemann equations for the variables $u_2 - i\beta = \log z_2$ and $u_1 - i\phi = \log z_1$. Then, the general integral of the BPS equations is

$$\log z_2 = f(\log z_1) , \tag{3.46}$$

where f is an arbitrary (holomorphic) function of $\log z_1$. By exponentiating eq. (3.46) one gets that the general solution of the BPS equations is a function $z_2 = g(z_1)$, in which z_2 is an arbitrary holomorphic function of z_1. This result is analogous to what happened for $T^{1,1}$ [12]. The appearance of a holomorphic function in the local complex coordinates z_1 and z_2 is a consequence of kappa symmetry or, in other words, supersymmetry. But one still has to check that this equation makes sense globally. We will come to this point shortly. The simplest case is that in which $\log z_2$ depends linearly on $\log z_1$, namely

$$\log z_2 = n(\log z_1) + \text{const.} , \tag{3.47}$$

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where n is a constant. By exponentiating this equation we get a relation between z_2 and z_1 of the type:

$$z_2 = \mathcal{C}\, z_1^n\,, \tag{3.48}$$

where $\mathcal{C}$ is a complex constant. If we represent this constant as $\mathcal{C} = Ce^{-i\beta_0}$, the embedding (3.48) reduces to the following real functions $\beta = \beta(\phi)$ and $y = y(\theta)$:

$$\begin{aligned} \beta &= n\phi + \beta_0\,, \\ f_1(y) &= C\,\frac{(\,\sin\theta)^c}{\left(\tan\frac{\theta}{2}\right)^n}\,. \end{aligned} \tag{3.49}$$

This is a nontrivial embedding of a probe D3-brane on $AdS_5 \times Y^{p,q}$. Notice that in the limit $c \to 0$ one recovers the results of [12]. For $c \neq 0$, a key difference arises. As we discussed earlier, z_2 is not globally well-defined in $CY^{p,q}$ due to its dependence on β. As a consequence, eqs. (3.48)–(3.49) describe a kappa-symmetric embedding for the D3-brane on $Y^{p,q}$ but it does not correspond to a wrapped brane. The D3-brane spans a submanifold with boundaries.[3] The only solution corresponding to a probe D3-brane wrapping a three-cycle is $z_1 =$ const. which is the one obtained in the preceding subsection.

In order to remove β while respecting holomorphicity [4], we seem to be forced to let z_3 enter into the game. The reason is simple, any dependence in β disappears if z_2 enters through the product $z_2 z_3$. This would demand embeddings involving the radius that we did not consider. In this respect, it is interesting to point out that this is also the conclusion reached in [35] from a different perspective: there, the complex coordinates corresponding to the generators of the chiral ring are deduced and it turns out that all of them depend on z_1, $z_2 z_3$ and z_3. It would be clearly desirable to understand these generalized wrapped D3-branes in terms of algebraic geometry, following the framework of ref. [11] which, in the case of the conifold, emphasizes the use of global homogeneous coordinates. Unfortunately, the relation between such homogeneous coordinates and the chiral fields of the quiver theory is more complicated in the case of $CY^{p,q}$.

3.3 The calibrating condition

Let us now verify that the BPS equations we have obtained ensure that the three-dimensional submanifolds we have found are calibrated. First, recall that the metric of the $Y^{p,q}$ manifold

[3] In this respect, notice that it might happen that global consistency forces, through boundary conditions, the D3-brane probes to end on other branes.

[4] One might think that a possible caveat to this problem is to choose a different slicing of $Y^{p,q}$ as the one in (2.12), where the metric is written as a U(1) bundle coordinatized by α (the base not being a Kähler-Einstein manifold). The complex coordinates of the slice are

$$\tilde{z}_1 = z_1\,, \qquad \tilde{z}_2 = G(y)\sin\theta e^{i\psi}\,, \tag{3.50}$$

where $G'(y)/G(y) = 3/\sqrt{w(y)}q(y)$. However, a 'holomorphic' ansatz of the form $\tilde{z}_1 = \tilde{z}_2^m$ would be related to an embedding of the form $\phi = \phi(\psi)$ and $\theta = \theta(y)$, which is a particular case of (3.26) albeit it is not kappa symmetric. These complex coordinates $\tilde{z}_1$ and $\tilde{z}_2$ have nothing to do with the complex structure of the Calabi-Yau manifold and, as such, kappa symmetry is not going to lead to a holomorphic embedding in terms of them.

can be written as (2.4),

$$ds^2_{Y^{p,q}} = ds^2_4 + \left[\frac{1}{3}d\psi + \sigma\right]^2 \,,$$

where σ is a one-form given by

$$\sigma = \frac{1}{3}[\,\cos\theta d\phi + y(d\beta - c\cos\theta d\phi)]\,. \tag{3.51}$$

The Kähler form J_4 of the four-dimensional Kähler-Einstein space is just

$$J_4 = \frac{1}{2}\,d\sigma = \frac{1}{L^2}[e^1\wedge e^2 - e^3\wedge e^4]\,, \tag{3.52}$$

where the e^i's are the forms of the vielbein (2.28). From the Sasaki-Einstein space $Y^{p,q}$ we can construct the Calabi-Yau cone $CY^{p,q}$, whose metric is just given by: $ds^2_{CY^{p,q}} = dr^2 + r^2\,ds^2_{Y^{p,q}}$. The Kähler form J of $CY^{p,q}$ is just:

$$J = r^2\,J_4 + \frac{r}{L}\,dr\wedge e^5\,, \tag{3.53}$$

whose explicit expression in terms of the coordinates is:

$$J = -\frac{r^2}{6}\,(1-cy)\,\sin\theta d\theta\wedge d\phi + \frac{1}{3}\,rdr\wedge(d\psi+\cos\theta d\phi) + \frac{1}{6}d(r^2y)\wedge(d\beta - c\cos\theta d\phi)\,. \tag{3.54}$$

Given a three-submanifold in $Y^{p,q}$ one can construct its cone $\mathcal{D}$, which is a four-dimensional submanifold of $CY^{p,q}$. The calibrating condition for a supersymmetric four-submanifold $\mathcal{D}$ of $CY^{p,q}$ is just:

$$P\Big[\frac{1}{2}\,J\wedge J\Big]_{\mathcal{D}} = \mathrm{Vol}(\mathcal{D})\,, \tag{3.55}$$

where $\mathrm{Vol}(\mathcal{D})$ is the volume form of the divisor $\mathcal{D}$. Let us check that the condition (3.55) is indeed satisfied by the cones constructed from our three-submanifolds. In order to verify this fact it is more convenient to describe the embedding by means of functions $y = y(\theta,\phi)$ and $\beta = \beta(\theta,\phi)$. The corresponding BPS equations are the ones written in (3.41). By using them one can verify that the induced volume form for the three-dimensional submanifold is:

$$vol = \frac{1}{18}\Big|\,(1-cy)\sin\theta + c\cos\theta y_\theta + \beta_\theta y_\phi - y_\theta\beta_\phi\Big|_{BPS}\,d\theta\wedge d\phi\wedge d\psi\;. \tag{3.56}$$

By computing the pullback of $J\wedge J$ one can verify that the calibrating condition (3.55) is indeed satisfied for:

$$\mathrm{Vol}(\mathcal{D}) = -r^3\,dr\wedge vol\,, \tag{3.57}$$

which is just the volume form of $\mathcal{D}$ with the metric $ds^2_{CY^{p,q}}$ having a particular orientation. Eq. (3.55) is also satisfied for the cones constructed from the singlet and doublet three-cycles of sections 3.1 and 3.2. This fact is nothing but the expression of the local nature of supersymmetry.

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3.4 Energy bound

The dynamics of the D3-brane probe is governed by the Dirac-Born-Infeld lagrangian that, for the case in which there are not worldvolume gauge fields, reduces to:

$$\mathcal{L} = -\sqrt{-g}\,, \tag{3.58}$$

where we have taken the D3-brane tension equal to one. We have checked that any solution of the first-order equations (3.34) or (3.41) also satisfies the Euler-Lagrange equations derived from the lagrangian density (3.58). Moreover, for the static configurations we are considering here the hamiltonian density $\mathcal{H}$ is, as expected, just $\mathcal{H} = -\mathcal{L}$. We are now going to verify that this energy density satisfies a bound, which is just saturated when the BPS equations (3.34) or (3.41) hold. In what follows we will take θ and ϕ as independent variables. For an arbitrary embedding of a D3-brane described by two functions $\beta = \beta(\theta, \phi)$ and $y = y(\theta, \phi)$ one can show that $\mathcal{H}$ can be written as:

$$\mathcal{H} = \sqrt{\mathcal{Z}^2 + \mathcal{Y}^2 + \mathcal{W}^2}\,, \tag{3.59}$$

where $\mathcal{Z}$, $\mathcal{Y}$ and $\mathcal{W}$ are given by:

$$\begin{aligned}
\mathcal{Z} &= \frac{L^4}{18}\Big[(1-cy)\sin\theta + c\cos\theta\, y_\theta + y_\phi\beta_\theta - y_\theta\beta_\phi\Big]\,,\\
\mathcal{Y} &= \frac{L^4}{18}\sqrt{1-cy}\;H\,\Big[\beta_\phi - c\cos\theta + \frac{\sin\theta}{H^2}\,y_\theta\Big]\,,\\
\mathcal{W} &= \frac{L^4}{18}\sqrt{1-cy}\;H\,\Big[\sin\theta\,\beta_\theta - \frac{y_\phi}{H^2}\Big]\,.
\end{aligned} \tag{3.60}$$

Obviously one has:

$$\mathcal{H} \ge |\,\mathcal{Z}|\,. \tag{3.61}$$

Moreover, since

$$\mathcal{Y}_{|\,BPS} = \mathcal{W}_{|\,BPS} = 0\,, \tag{3.62}$$

the bound saturates when the BPS equations (3.41) are satisfied. Thus, the system of differential equations (3.41) is equivalent to the condition $\mathcal{H} = |\,\mathcal{Z}|$ (actually $\mathcal{Z} \ge 0$ if the BPS equations (3.41) are satisfied). Moreover, for an arbitrary embedding $\mathcal{Z}$ can be written as a total derivative, namely:

$$\mathcal{Z} = \frac{\partial}{\partial\theta}\,\mathcal{Z}^\theta + \frac{\partial}{\partial\phi}\,\mathcal{Z}^\phi\,. \tag{3.63}$$

This result implies that $\mathcal{H}$ is bounded by the integrand of a topological charge. The explicit form of $\mathcal{Z}^\theta$ and $\mathcal{Z}^\phi$ is:

$$\begin{aligned}
\mathcal{Z}^\theta &= -\frac{L^4}{18}\Big[(1-cy)\cos\theta + y\,\beta_\phi\Big]\,,\\
\mathcal{Z}^\phi &= \frac{L^4}{18}\,y\,\beta_\theta\,.
\end{aligned} \tag{3.64}$$

In this way, from the point of view of the D3-branes, the configurations satisfying eq. (3.41) can be regarded as BPS worldvolume solitons.

3.5 BPS fluctuations of dibaryons

In this section we study BPS fluctuations of dibaryon operators in the $Y^{p,q}$ quiver theory. We start with the simplest dibaryon which is singlet under SU(2), say $\det Y$. To construct excited dibaryons we should replace one of the Y factors by any other chiral field transforming in the same representation of the gauge groups. For example, replacing Y by $YU^\alpha V^\beta Y$, we get a new operator of the form

$$\epsilon_1\epsilon^2(YU^\alpha V^\beta Y)Y\cdots Y\,, \tag{3.65}$$

where ϵ_1 and ϵ^2 are abbreviations for the completely anti-symmetric tensors for the respective SU(N) factors of the gauge group. Using the identity

$$\epsilon^{a_1\cdots a_N}\epsilon_{b_1\cdots b_N}=\sum_\sigma(-1)^\sigma\delta^{a_1}_{\sigma(b_1)}\cdots\delta^{a_N}_{\sigma(b_N)}\,, \tag{3.66}$$

the new operator we get can factorize into the original dibaryon and a single-trace operator

$$\mathrm{Tr}(U^\alpha V^\beta Y)\,\det Y\,. \tag{3.67}$$

Indeed for singlet dibaryons, a factorization of this sort always works. This fact seems to imply, at least at weak coupling, that excitation of a singlet dibaryon can be represented as graviton fluctuations in the presence of the original dibaryon.

For the case of dibaryon with SU(2) quantum number the situation is different. Consider, for simplicity, the state with maximum J_3 of the SU(2)

$$\epsilon_1\epsilon^2(U^1\cdots U^1)=\det U^1, \tag{3.68}$$

we can replace one of U^1 factors by $U^1\,\mathcal{O}$, where $\mathcal{O}$ is some operator given by a closed loop in the quiver. As the case of singlet dibaryon, this kind of excitation is factorizable since all SU(2) indices are symmetric. So this kind of operator should be identified with a graviton excitation with wrapped D3-brane in the dual string theory. However if the SU(2) index of the U field is changed in the excitation, i.e. $U^1\to U^2\,\mathcal{O}$, then the resulting operator cannot be written as a product of the original dibaryon and a meson-like operator. Instead it has to be interpreted as a single particle state in AdS. Since the operator also carries the same baryon number, the natural conclusion is that the one-particle state is a BPS excitation of the wrapped D3-brane corresponding to the dibaryon [5].

In order to classify all these BPS excitations of the dibaryon, we have to count all possible inequivalent chiral operators $\mathcal{O}$ that transform in the bifundamental representation of one of the gauge group factors of the theory. In $Y^{p,q}$ quiver gauge theory, these operators correspond to loops in the quiver diagram just like the mesonic chiral operators discussed in [38]. The simplest ones are operators with R-charge 2. They have been thoroughly discussed in [39]. They are given by short loops of length 3 or 4 in the quiver, precisely as those operators entering in the superpotential (2.19). They are single trace operators of the form (in what follows we omit the trace and the SU(2) indices) UVY, VUY or $YUZU$ (see the upper quiver in figure 2). Since they are equivalent in the chiral ring, we can identify

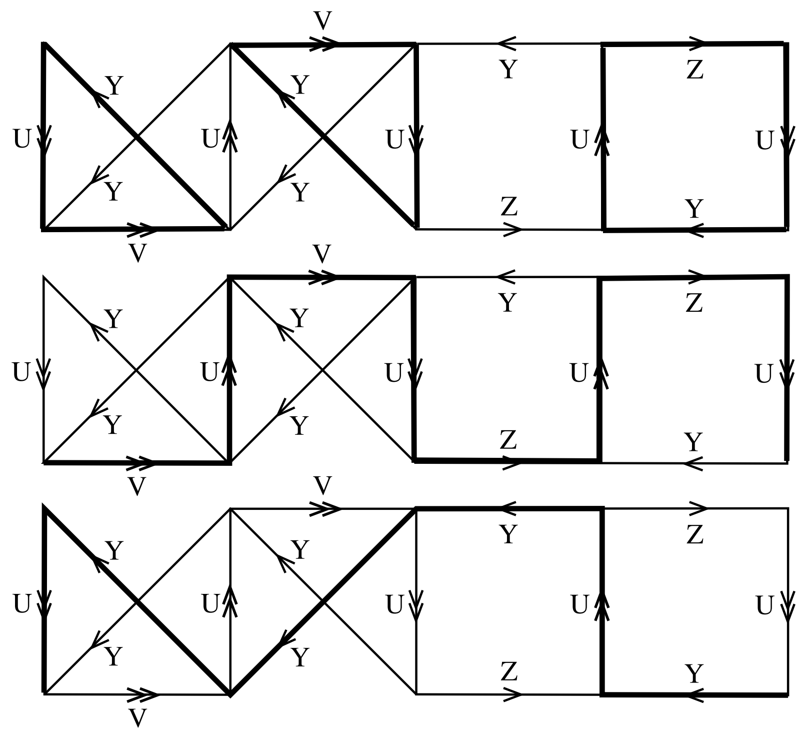

Figure 2: Loops in the $Y^{4,2}$ quiver representing mesonic operators in the chiral ring. There are short loops such as UVY, VUY or $YUZU$ (upper), longest loops as $VUVUZUZU$ (middle) and long loops like $YUYYYU$ (bottom). The representative of each class in the chiral ring is, respectively, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

them as a single operator $\mathcal{O}_1$. It transforms in the spin $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$ representation of the global SU(2). The scalar component vanishes in the chiral ring. Thus, we end up with a spin 1 chiral operator with scaling dimension $\Delta=3$. Its $\mathrm{U}(1)_F$ charge vanishes.

There are also two classes of long loops in the quiver. The first class, whose representative is named $\mathcal{O}_2$, has length $2p$, winds the quiver from the left to the right and is made of p U type operators, q V type operators and $p-q$ Z type operators. For example, in $Y^{4,2}$, a long loop of this class is $VUVUZUZU$ (middle quiver in figure 2). It transforms in the spin $\frac{1}{2}\otimes\cdots\otimes\frac{1}{2}=\frac{p+q}{2}\oplus\cdots$ representation of SU(2). The dots amount to lower dimensional representations that vanish in the chiral ring. The resulting operator, $\mathcal{O}_2$, has spin $\frac{p+q}{2}$. There is another class of long loops which has length $2p-q$, running along the quiver in the opposite direction, build with p Y type operators and $p-q$ U type operators. We name its representative as $\mathcal{O}_3$. In the case of $Y^{4,2}$, it is an operator like $YUYYYU$ (bottom quiver in figure 2). SU(2) indices, again, have to be completely symmetrized, the spin being $\frac{p-q}{2}$. Long loops wind around the quiver and this leads to a nonvanishing value of Q_F [38]. The baryonic charge vanishes for any of these loops. We summarize in table 2 the charge assignments for the three kinds of operators $\mathcal{O}_i$ [38]. We can see that these operators satisfy the BPS condition $\Delta=\frac{3}{2}Q_R$. In fact, they are the building blocks of all other scalar BPS operators. The general BPS excitation corresponds to operators of the

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Operator	Q_R	Q_F	Spin
$\mathcal{O}_1$	2	0	1
$\mathcal{O}_2$	$p+q-\frac{1}{3\ell}$	p	$\frac{p+q}{2}$
$\mathcal{O}_3$	$p-q+\frac{1}{3\ell}$	$-p$	$\frac{p-q}{2}$

Table 2: Charges assignments for the mesonic operators $\mathcal{O}_i$ that generate the chiral ring.

form

$$\mathcal{O} = \prod_{i=1}^{3} \mathcal{O}_i^{n_i} \,. \tag{3.69}$$

It is interesting to notice that the spectrum of fluctuations of a dibaryon must coincide with the mesonic chiral operators in the $Y^{p,q}$ quiver theory. This would provide a nontrivial test of the AdS/CFT correspondence. We show this result explicitly via an analysis of open string fluctuation on wrapped D3-branes.

Now we are interested in describing the excitations of dibaryon operators from the dual string theory. For those excitations that are factorizable, the dual configurations are just the multi-particle states of graviton excitations in the presence of a dibaryon. The correspondence of graviton excitation and mesonic operator were studied in [38, 40]. What we are really interested in are those non-factorizable operators that can be interpreted as open string excitations on the D-brane. This can be analyzed by using the Dirac-Born-Infeld action of the D3-brane. In what follows we will focus on the dibaryon made of U fields, which corresponds to the three-cycle D studied in subsection 3.2 which, for convenience, we will parameterize with the coordinates (y, ψ, α). The analysis of the dibaryon made of V field is similar. For our purpose we will use, as in eq. (2.35), the global coordinate system for the AdS_5 part of the metric and we will take the $Y^{p,q}$ part as written in eq. (2.12). We are interested in the normal modes of oscillation of the wrapped D3-brane around the solution corresponding to some fixed worldline in AdS_5 and some fixed θ and ϕ on the transverse S^2. For such a configuration, the induced metric on the dibaryon is:

$$L^{-2} ds^2_{\text{ind}} = -d\tau^2 + \frac{1}{wv} dy^2 + \frac{v}{9} d\psi^2 + w(d\alpha + f d\psi)^2 \,, \tag{3.70}$$

where the functions $v(y)$, $w(y)$ and $f(y)$ have been defined in eq. (2.13) (in what follows of this subsection we will take $c = 1$).

The fluctuations along the transverse S^2 are the most interesting, since they change the SU(2) quantum numbers and are most readily compared with the chiral primary states in the field theory. Without lost of generality, we consider fluctuations around the north pole of the S^2, i.e. $\theta_0 = 0$. Instead of using coordinates θ and ϕ, it is convenient to go from polar to Cartesian coordinates: $\zeta^1 = \theta \sin\phi$ and $\zeta^2 = \theta \cos\phi$. As a further simplification we perform a shift in the coordinate ψ. The action for the D3-brane is:

$$S = -T_3 \int d^4\xi \sqrt{-\det g} + T_3 \int P[C_{(4)}]\,. \tag{3.71}$$

Let us expand the induced metric g around the static configuration as $g = g_{(0)} + \delta g$, where $g_{(0)}$ is the zeroth order contribution. The corresponding expansion for the action takes the form:

$$S = S_0 - \frac{T_3}{2} \int d^4\xi \sqrt{-\det g_{(0)}}\; \mathrm{Tr}[g_{(0)}^{-1}\,\delta g] + T_3 \int P[C_{(4)}]\,, \tag{3.72}$$

where $S_0 = -T_3 \int d^4\xi \sqrt{-\det g_{(0)}}$. Note that the determinant of the induced metric at zeroth order is a constant: $\sqrt{-\det(g_{(0)})} = \frac{1}{3}L^4$. The five-form field strength is

$$F_5 = (1+*)\, 4\sqrt{\det(G_{Y^{p,q}})}L^4 d\theta \wedge d\phi \wedge dy \wedge d\psi \wedge d\alpha\,. \tag{3.73}$$

Moreover, using that $\sqrt{\det(G_{Y^{p,q}})} = \frac{1-y}{18}\sin\theta$, we can choose the four-form Ramond-Ramond field to be

$$C_4 = \frac{2}{9}(1-y)L^4(\cos\theta - 1)\, d\alpha \wedge dy \wedge d\psi \wedge d\phi\,, \tag{3.74}$$

which is well defined around the north pole of S^2. At quadratic order, the four form C_4 is

$$C_4 = -\sqrt{-\det g_{(0)}}\;\frac{1-y}{3}\epsilon_{ij}\,\zeta^i\, d\zeta^j \wedge d\alpha \wedge dy \wedge d\psi\,. \tag{3.75}$$

The contribution from the Born-Infeld part of the effective action is:

$$\mathrm{Tr}[g_{(0)}^{-1}\,\delta g] = G_{ij}\; g_{(0)}^{\mu\nu}\,(\partial_\mu \zeta^i \partial_\nu \zeta^j) + 2g_{(0)}^{\mu\nu}\; G_{\mu i}\,\partial_\nu \zeta^i, \tag{3.76}$$

where G is the metric of the background, i,j denote the components of G along the $\zeta^{1,2}$ directions and the indices μ,ν refer to the directions of the worldvolume of the cycle. The non-vanishing components of G are:

$$G_{ij} = \frac{1-y}{6}L^2\delta_{ij}, \quad G_{\psi i} = -\frac{1}{2}\left(wf^2 + \frac{v}{9}\right)L^2\epsilon_{ij}\,\zeta^j, \quad G_{\alpha i} = -\frac{wf}{2}L^2\epsilon_{ij}\,\zeta^j\,. \tag{3.77}$$

Using these results one can verify that the effective Lagrangian is proportional to:

$$\sum_i L^2\frac{1-y}{6}g_{(0)}^{\mu\nu}(\partial_\mu\zeta^i\partial_\nu\zeta^i) + 2g_{(0)}^{\mu\nu}\; G_{\mu i}\,\partial_\nu\zeta^i + \frac{2(1-y)}{3}\epsilon_{ij}\,\zeta^i\,\partial_\tau\zeta^j\,. \tag{3.78}$$

The equations of motion for the fluctuation are finally given by

$$\frac{L^2}{6}\,\partial_\mu\Big((1-y)\, g_{(0)}^{\mu\nu}\,\partial_\nu\zeta^i\Big) + 2\partial_\nu(g_{(0)}^{\mu\nu}\, G_{\mu i}) - \frac{2(1-y)}{3}\epsilon_{ij}\,\partial_\tau\zeta^j = 0\,. \tag{3.79}$$

Introducing $\zeta^{\pm} = \zeta^1 \pm i\zeta^2$, the equations of motion reduce to

$$\left(\nabla^2 - \frac{1-y}{6}\partial_\tau^2\right)\zeta^{\pm} \pm i\frac{2(1-y)}{3}\partial_\tau\zeta^{\pm} \pm i\partial_\psi\zeta^{\pm} = 0\,, \tag{3.80}$$

where ∇^2 is the laplacian along the spatial directions of the cycle for the induced metric $g_{(0)}$. The standard strategy to solve this equation is to use separation of variables as

$$\zeta^{\pm} = \exp(-i\omega\tau)\exp\left(i\frac{m}{\ell}\alpha\right)\exp(in\psi)\, Y^{k\pm}_{mn}(y)\,. \tag{3.81}$$

Plugging this ansatz into the equation of motion, we find

$$\begin{aligned}
&\frac{1}{1-y}\frac{d}{dy}\left[(1-y)w(y)v(y)\frac{d}{dy}Y^{k\pm}_{mn}(y)\right] \\
&\quad = \left[\left(\frac{9f^2(y)}{v(y)}+\frac{1}{w(y)}\right)\frac{m^2}{\ell^2}-\frac{18f(y)}{v(y)}\frac{m}{\ell}n+\frac{9}{v(y)}n^2-\omega(\omega\pm 4)\pm\frac{6n}{1-y}\right]Y^{k\pm}_{mn}(y)\,.
\end{aligned} \tag{3.82}$$

The resulting equation has four regular singularities at $y = y_1, y_2, y_3$ and ∞ and is known as Heun's equation (for clarity, in what follows we omit the indices in Y) [41]:

$$\frac{d^2}{dy^2}Y^{\pm}+\left(\sum_{i=1}^{3}\frac{1}{y-y_i}\right)\frac{d}{dy}Y^{\pm}+q(y)Y^{\pm}=0, \tag{3.83}$$

where, in our case

$$\begin{aligned}
q(y) &= \frac{2}{\mathcal{Q}(y)}\left[\mu-\frac{y}{4}\omega(\omega\pm 4)-\frac{1}{2}\sum_{i=1}^{3}\frac{\alpha_i^2\mathcal{Q}'(y_i)}{y-y_i}\right], \\
\mu &= \frac{3}{32}\left(\frac{m}{\ell}+2n\right)\left(\frac{m}{\ell}-6n\right)+\frac{1}{4}\omega(\omega\pm 4)\mp\frac{3n}{2}\,,
\end{aligned} \tag{3.84}$$

with $\mathcal{Q}(y)$ being the function defined in eq. (2.6). Now, given that the R-symmetry is dual to the Reeb Killing vector of $Y^{p,q}$, namely $2\partial/\partial\psi - \frac{1}{3}\partial/\partial\alpha$, we can use the R-charge $Q_R = 2n - m/3\ell$ instead of n as quantum number. The exponents at the regular singularities $y = y_i$ are then given by

$$\alpha_i = \pm\frac{1}{2}\frac{(1-y_i)(m/\ell+3Q_R\, y_i)}{\mathcal{Q}'(y_i)}. \tag{3.85}$$

The exponents at $y=\infty$ are $-\frac{\omega}{2}$ and $\frac{\omega}{2}+2$ for Y^+, while $-\frac{\omega}{2}+2$ and $\frac{\omega}{2}$ for Y^-. We can transform the singularity from $\{y_1, y_2, y_3, \infty\}$ to $\{0, 1, b = \frac{y_1-y_3}{y_1-y_2}, \infty\}$ by introducing a new variable x, defined as:

$$x = \frac{y-y_1}{y_2-y_1}. \tag{3.86}$$

It is also convenient to substitute

$$Y = x^{|\alpha_1|}(1-x)^{|\alpha_2|}(b-x)^{|\alpha_3|}\, h(x)\,, \tag{3.87}$$

which transforms equation (3.83) into the standard form of the Heun's equation

$$\frac{d^2}{dx^2}h(x)+\left(\frac{C}{x}+\frac{D}{x-1}+\frac{E}{x-b}\right)\frac{d}{dx}h(x)+\frac{ABx-k}{x(x-1)(x-b)}h(x)=0. \tag{3.88}$$

Here the Heun's parameters are given by

$$
\begin{aligned}
&A = -\frac{\omega}{2} + \sum_{i=1}^{3} |\alpha_i|\,, \qquad B = \frac{\omega+4}{2} + \sum_{i=1}^{3} |\alpha_i|\,, \\
&C = 1 + 2|\alpha_1|\,, \qquad D = 1 + 2|\alpha_2|\,, \qquad E = 1 + 2|\alpha_3|\,,
\end{aligned}
\tag{3.89}
$$

and

$$
\begin{aligned}
k &= (|\alpha_1| + |\alpha_3|)(|\alpha_1| + |\alpha_3| + 1) - |\alpha_2|^2 \\
&\quad + b\left[(|\alpha_1| + |\alpha_2|)(|\alpha_1| + |\alpha_2| + 1) - |\alpha_3|^2\right] - \tilde{\mu}\,, \\
\tilde{\mu} &= -\frac{1}{y_1 - y_2}\left(\mu - \frac{y_1}{4}\omega(\omega+4)\right) \\
&= \frac{p}{q}\left[\frac{1}{6}(1-y_1)\omega(\omega+4) - \frac{3}{16}Q_R\left(Q_R + \frac{4m}{3\ell}\right) - \frac{1}{2}\left(Q_R + \frac{m}{3\ell}\right)\right], \\
b &= \frac{1}{2}\left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q}\right).
\end{aligned}
\tag{3.90}
$$

We only presented the equation for Y^+; the corresponding equation for Y^- can be obtained by replacing ω with $\omega - 4$ and changing the sign of the last term in (3.90).

Now let us discuss the solutions to this differential equation. For quantum number $Q_R = 2N$ (which implies $m = 0$), we find all α_i equal to $N/2$. If we set $\omega = 3N$, the Heun's parameters A and k both vanish. The corresponding solution $h(x)$ is a constant function. Similarly if $\omega = -3N - 4$, then B and k vanish which also implies a constant $h(x)$. The complete solution of $\zeta^{\pm}$ in these two cases is given by

$$
\begin{aligned}
\zeta_1^{\pm} &= e^{\pm i(-3N\tau + N\psi)} \prod_{i=1}^{3} (y - y_i)^{N/2}\,, \\
\zeta_2^{\pm} &= e^{\pm i((3N+4)\tau + N\psi)} \prod_{i=1}^{3} (y - y_i)^{N/2}\,.
\end{aligned}
\tag{3.91}
$$

These constant solutions represent ground states with fixed quantum numbers and, since they have the lowest possible dimension for a given R-charge, they should be identified with the BPS operators. Indeed, in the solutions (3.91) the energy is quantized in units of $3L^{-1}$, and 3 is precisely the conformal dimension of $\mathcal{O}_1$. This provides a perfect matching of AdS/CFT in this setting.

The situation for quantum numbers $Q_R = N(p \pm q \mp 1/3\ell)$ and $m = \pm N$ is similar to the case we have just discussed. The solutions for $h(x)$ are constant with

$$
\omega = \frac{Np}{2}\left(3 \pm \frac{2p - \sqrt{4p^2 - 3q^2}}{q}\right), \tag{3.92}
$$

and

$$
\omega = -\frac{Np}{2}\left(3 \pm \frac{2p - \sqrt{4p^2 - 3q^2}}{q}\right) - 4. \tag{3.93}
$$

We can see the conformal dimension satisfies $\Delta = \frac{3}{2}Q_R$. So all these solutions are BPS fluctuations which should correspond to the operators $\mathcal{O}_2$ and $\mathcal{O}_3$.

An interesting comment is in order at this point. Notice that the dibaryon excitations should come out with the multiplicities associated to the SU(2) spin (see table 2) of the $\mathcal{O}_i$ operators. However, in order to tackle this problem, we would need to consider at the same time the fluctuation of the D3-brane probes and the zero-mode dynamics corresponding to their collective motion along the sphere with coordinate θ and ϕ (see ref. [5] for a similar discussion in the conifold case). This is an interesting problem that we leave open.

4. Supersymmetric D5-branes in $AdS_5 \times Y^{p,q}$

In this section we will study the supersymmetric configurations of D5-branes in the $AdS_5 \times Y^{p,q}$ background. First of all, notice that in this case Γ_κ acts on the Killing spinors ϵ as:

$$\Gamma_\kappa\,\epsilon = \frac{i}{6!\,\sqrt{-g}}\;\epsilon^{\mu_1\cdots\mu_6}\,\gamma_{\mu_1\cdots\mu_6}\,\epsilon^*\,, \tag{4.1}$$

where we have used the relation (2.45) to translate eq. (2.46). The appearance of the complex conjugation on the right-hand side of eq. (4.1) is crucial in what follows. Indeed, the complex conjugation does not commute with the projections (2.32). Therefore, in order to construct an additional compatible projection involving the $\epsilon \to \epsilon^*$ operation we need to include a product of gamma matrices which anticommutes with both Γ_{12} and Γ_{34}. As in the D3-brane case just analyzed, this compatibility requirement between the $\Gamma_\kappa\,\epsilon = \epsilon$ condition and (2.32) implies a set of differential equations whose solutions, if any, determine the supersymmetric embeddings we are looking for.

We will carry out successfully this program only in the case of a D5-brane extended along a two-dimensional submanifold of $Y^{p,q}$. In analogy with what happens with the conifold [4], one expects that these kinds of configurations represent a domain wall in the gauge theory side such that, when one crosses one of these objects, the gauge groups change and one passes from an $\mathcal{N} = 1$ superconformal field theory to a cascading theory with fractional branes. The supergravity dual of this cascading theory has been obtained in ref. [21]. In the remainder of this section we will find the corresponding configurations of the D5-brane probe. Moreover, in section 6 we will find, based on a different set of worldvolume coordinates, another embedding of this type preserving the same supersymmetry as the one found in the present section and we will analyze the effect of adding flux of the worldvolume gauge fields. In section 6 we will also look at the possibility of having D5-branes wrapped on a three-dimensional submanifold of $Y^{p,q}$. These configurations are not supersymmetric, although we have been able to find stable solutions of the equations of motion. The case in which the D5-brane wraps the entire $Y^{p,q}$ corresponds to the baryon vertex. In this configuration, studied also in section 6, the D5-brane captures the flux of the RR five-form, which acts as a source for the electric worldvolume gauge field. We will conclude in section 6 that this configuration cannot be supersymmetric, in analogy with what happens in the conifold case [12].

4.1 Domain wall solutions

We want to find a configuration in which the D5-brane probe wraps a two-dimensional submanifold of $Y^{p,q}$ and is a codimension one object in AdS_5. Accordingly, let us place the probe at some constant value of one of the Minkowski coordinates (say x^3) and let us extend it along the radial direction. To describe such an embedding we choose the following set of worldvolume coordinates for a D5-brane probe

$$\xi^\mu = (t, x^1, x^2, r, \theta, \phi)\,, \tag{4.2}$$

and we adopt the following ansatz:

$$y = y(\theta,\phi)\,, \qquad \beta = \beta(\theta,\phi)\,, \tag{4.3}$$

with x^3 and ψ constant. The induced Dirac matrices can be computed straightforwardly from eq. (2.42) with the result:

$$\begin{aligned}
\gamma_{x^\mu} &= \frac{r}{L}\,\Gamma_{x^\mu}\,, \qquad \mu = 0,1,2\,,\\
\gamma_r &= \frac{L}{r}\,\Gamma_r\,,\\
\frac{1}{L}\,\gamma_\theta &= -\frac{1}{\sqrt{6}\,H}\,y_\theta\,\Gamma_1 - \frac{H}{\sqrt{6}}\,\beta_\theta\,\Gamma_2 + \frac{\sqrt{1-cy}}{\sqrt{6}}\,\Gamma_3 + \frac{y}{3}\,\beta_\theta\,\Gamma_5\,,\\
\frac{1}{L}\,\gamma_\phi &= -\frac{1}{\sqrt{6}\,H}\,y_\phi\,\Gamma_1 + \frac{H}{\sqrt{6}}\,(c\cos\theta - \beta_\phi)\,\Gamma_2 + \frac{\sqrt{1-cy}}{\sqrt{6}}\,\sin\theta\,\Gamma_4\\
&\quad + \frac{1}{3}[y\,\beta_\phi + (1-cy)\cos\theta]\Gamma_5\,.
\end{aligned} \tag{4.4}$$

From the general expression (4.1) one readily gets that the kappa symmetry matrix acts on the spinor ϵ as:

$$\Gamma_\kappa\,\epsilon = \frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\,\Gamma_{x^0x^1x^2r}\,\gamma_{\theta\phi}\,\epsilon^*\,. \tag{4.5}$$

By using the complex conjugate of the projections (2.32) one gets:

$$\frac{6}{L^2}\,\gamma_{\theta\phi}\,\epsilon^* = [b_I + b_{15}\,\Gamma_{15} + b_{35}\,\Gamma_{35} + b_{13}\,\Gamma_{13}]\epsilon^*\,, \tag{4.6}$$

where the different coefficients are:

$$\begin{aligned}
b_I &= -i\Big[(1-cy)\sin\theta + c\cos\theta\,y_\theta + y_\phi\beta_\theta - y_\theta\beta_\phi\Big]\,,\\
b_{15} &= -\sqrt{\frac{2}{3}}\,\frac{1}{H}\Big[(1-cy)\cos\theta\,y_\theta + y\,(\beta_\phi\,y_\theta - \beta_\theta\,y_\phi)\Big] - i\sqrt{\frac{2}{3}}\,H\,\cos\theta\,\beta_\theta\,,\\
b_{35} &= \sqrt{\frac{2}{3}}\sqrt{1-cy}\Big[(1-cy)\cos\theta + y\beta_\phi\Big] + i\sqrt{\frac{2}{3}}\sqrt{1-cy}\;y\sin\theta\beta_\theta\,,\\
b_{13} &= \sqrt{1-cy}\Big[\frac{y_\phi}{H} - H\beta_\theta\,\sin\theta\Big] + i\sqrt{1-cy}\Big[\frac{\sin\theta}{H}\,y_\theta - H\,(c\cos\theta - \beta_\phi)\Big]\,.
\end{aligned} \tag{4.7}$$

As discussed above, in this case the action of Γ_κ involves the complex conjugation, which does not commute with the projections (2.32). Actually, the only term on the right-hand side of (4.6) which is consistent with (2.32) is the one containing Γ_{13}. Accordingly, we must require:

$$b_I = b_{15} = b_{35} = 0\,. \tag{4.8}$$

From the vanishing of the imaginary part of b_{15} we get:

$$\beta_\theta = 0\,, \tag{4.9}$$

while the vanishing of the real part of b_{15} leads to:

$$\beta_\phi = -\frac{1-cy}{y}\,\cos\theta\,. \tag{4.10}$$

Notice that b_{35} is zero as a consequence of equations (4.9) and (4.10) which, in particular imply that:

$$\beta = \beta(\phi)\,. \tag{4.11}$$

Moreover, by using eq. (4.9), the condition $b_I = 0$ is equivalent to

$$(1-cy)\sin\theta + (c\cos\theta - \beta_\phi)y_\theta = 0\,, \tag{4.12}$$

and plugging the value of β_ϕ from (4.10), one arrives at:

$$y_\theta = -(1-cy)\,y\tan\theta\,. \tag{4.13}$$

In order to implement the kappa symmetry condition at all points of the worldvolume the phase of b_{13} must be constant. This can be achieved by requiring that the real part of b_{13} vanishes, which for $\beta_\theta = 0$ is equivalent to the condition $y_\phi = 0$, i.e.:

$$y = y(\theta)\,. \tag{4.14}$$

The equation (4.13) for $y(\theta)$ is easily integrated, namely:

$$\frac{y}{1-cy} = k\cos\theta\,, \tag{4.15}$$

where k is a constant. Moreover, by separating variables in eq. (4.10), one concludes that:

$$\beta_\phi = m\,, \tag{4.16}$$

where m is a new constant. Plugging (4.15) into eq. (4.10) and using the result (4.16) one concludes that the two constants m and k must be related as:

$$km = -1\,, \tag{4.17}$$

which, in particular implies that k and m cannot vanish. Thus, the embedding of the D5-brane becomes

$$\begin{aligned} \beta &= m\phi + \beta_0\,, \\ y &= -\frac{\cos\theta}{m - c\cos\theta}\,. \end{aligned} \tag{4.18}$$

Notice that the solution (4.18) is symmetric under the change $m \to -m$, $\theta \to \pi - \theta$ and $\phi \to 2\pi - \phi$. Thus, from now on we can assume that $m \geq 0$.

It is now straightforward to verify that the BPS equation are equivalent to impose the following condition on the spinor ϵ:

$$\Gamma_{x^0 x^1 x^2 r 13}\epsilon^* = \sigma\epsilon\,, \tag{4.19}$$

where σ is:

$$\sigma = \mathrm{sign}\Big(\frac{\cos\theta}{y}\Big) = -\mathrm{sign}\Big(m - c\cos\theta\Big)\,. \tag{4.20}$$

Obviously, the only valid solutions are those which correspond to having a constant sign σ along the worldvolume. This always happens for $m/c \geq 1$. In this case the minimal (maximal) value of θ is $\theta = 0$ ($\theta = \pi$) if $|m - c||y_1| > 1$ ($|m - c||y_2| > 1$). Otherwise the angle θ must be restricted to lie in the interval $\theta \in [\theta_1, \theta_2]$, where θ_1 and θ_2 are given by:

$$\theta_i = \arccos\left[\frac{m y_i}{c y_i - 1}\right]\,, \qquad\qquad (i = 1, 2)\,. \tag{4.21}$$

Notice that, similarly to what we obtained in the previous section, eq. (4.18) implies that the configuration we arrived at does not in general correspond to a wrapped brane but to a D5-brane that spans a two-dimensional submanifold with boundaries.

Let us now count the number of supersymmetries preserved by our configuration. In order to do so we must convert eq. (4.19) into an algebraic condition on a constant spinor. With this purpose in mind let us write the general form of ϵ as the sum of the two types of spinors written in eq. (2.34), namely:

$$e^{\frac{i}{2}\psi}\,\epsilon = r^{-\frac{1}{2}}\,\eta_+ + r^{\frac{1}{2}}\left(\frac{\bar{x}^3}{L^2}\,\Gamma_{rx^3}\,\eta_+ + \eta_-\right) + \frac{r^{\frac{1}{2}}}{L^2}\,x^p\,\Gamma_{rx^p}\,\eta_+\,, \tag{4.22}$$

where $\bar{x}^3$ is the constant value of the coordinate x^3 in the embedding and the index p runs over the set $\{0, 1, 2\}$. By substituting eq. (4.22) on both sides of eq. (4.19), one can get the conditions that η_+ and η_- must satisfy. Indeed, let us define the operator $\mathcal{P}$ as follows:

$$\mathcal{P}\,\epsilon \equiv i\sigma e^{i\psi_0}\,\Gamma_{rx^3}\,\Gamma_{13}\,\epsilon^*\,. \tag{4.23}$$

Then, one can check that eq. (4.19) is equivalent to:

$$\begin{aligned} &\mathcal{P}\,\eta_+ = \eta_+\,, \\ &(1 + \mathcal{P})\,\eta_- = -\frac{2\bar{x}^3}{L^2}\,\Gamma_{rx^3}\,\eta_+\,. \end{aligned} \tag{4.24}$$

As $\mathcal{P}^2 = 1$, we can classify the four spinors η_- according to their $\mathcal{P}$-eigenvalue as: $\mathcal{P}\,\eta_-^{(\pm)} = \pm\eta_-^{(\pm)}$. We can now solve the system (4.24) by taking $\eta_+ = 0$ and η_- equal to one of the two spinors $\eta_-^{(-)}$ of negative $\mathcal{P}$-eigenvalue. Moreover, there are other two solutions which correspond to taking a spinor $\eta_-^{(+)}$ of positive $\mathcal{P}$-eigenvalue and a spinor η_+ related to the former as:

$$\eta_+ = \frac{L^2}{\bar{x}^3}\,\Gamma_{rx^3}\;\eta_-^{(+)}\,. \tag{4.25}$$

Notice that, according to the first equation in (4.24), the spinor η_+ must have positive $\mathcal{P}$-eigenvalue, in agreement with eq. (4.25). All together this configuration preserves four supersymmetries, i.e. one half of the supersymmetries of the background, as expected for a domain wall.

4.2 The calibrating condition

For any two-dimensional submanifold L of $Y^{p,q}$ one can construct its three-dimensional cone $\mathcal{L} \subset CY^{p,q}$. The holomorphic $(3,0)$ form Ω of $CY^{p,q}$ can be naturally used to calibrate such submanifolds. Indeed, $\mathcal{L}$ is called a special Lagrangian submanifold of $CY^{p,q}$ if the pullback of Ω to $\mathcal{L}$ is, up to a constant phase, equal to the volume form of $\mathcal{L}$, namely:

$$P[\Omega]_{\mathcal{L}} = e^{i\lambda}\,\mathrm{Vol}\,(\mathcal{L})\,, \tag{4.26}$$

where λ is constant on $\mathcal{L}$. If the cone $\mathcal{L}$ is special Lagrangian, its base L is said to be special Legendrian. It has been argued in ref. [42] that the supersymmetric configurations of a D5-brane extended along a two-dimensional submanifold L of a Sasaki-Einstein space are those for which $\mathcal{L}$ is special Lagrangian. Let us check that this is indeed the case for the embeddings (4.18). First of all, we notice that the expression of Ω written in (2.26) can be recast as:

$$\Omega = e^{i\psi}\, r^2\, \Omega_4\, \wedge [dr + i\,\frac{r}{L}\, e^5]\,, \tag{4.27}$$

where Ω_4 is the two-form:

$$\Omega_4 = \frac{1}{L^2}\,(e^1 + ie^2) \wedge (e^3 - ie^4)\,. \tag{4.28}$$

In eqs. (4.27) and (4.28) $e^1, \ldots, e^5$ are the vielbein one-forms of (2.28). Moreover, the volume form of $\mathcal{L}$ can be written as:

$$\mathrm{Vol}\,(\mathcal{L}) = r^2 dr \wedge \mathrm{Vol}\,(L)\,. \tag{4.29}$$

For our embeddings (4.18) one can check that:

$$\mathrm{Vol}\,(L) = \frac{H}{6}\Big|\frac{\cos\theta}{y}\Big|\,\sqrt{1-cy}\,\Big[1 + (1-cy)\,\frac{y^2}{H^2}\,\tan^2\theta\Big]\, d\theta \wedge d\phi\,. \tag{4.30}$$

It is now straightforward to verify that our embeddings (4.18) satisfy (4.26) with $e^{i\lambda} = -i\sigma e^{i\psi}$, where σ is the constant sign defined in (4.20) (recall that in our ansatz (4.3) the angle ψ is constant). Thus, we conclude that L is special Legendrian, as claimed. Moreover, one can check that:

$$P[J]_{\mathcal{L}} = 0\,. \tag{4.31}$$

4.3 Energy bound

Let us consider a generic embedding $y = y(\theta)$, $\beta = \beta(\phi)$ and let us define the following functions of θ and y

$$\Delta_\theta \equiv -y(1-cy)\tan\theta\,, \qquad \Delta_\phi \equiv -\frac{1-cy}{y}\,\cos\theta\,. \tag{4.32}$$

In terms of these functions the BPS equations (4.10) and (4.13) are simply $y_\theta = \Delta_\theta$ and $\beta_\phi = \Delta_\phi$. We have checked that any solution of this first-order equations also solves the Euler-Lagrange equations derived from the Dirac-Born-Infeld lagrangian (3.58). Moreover, the hamiltonian density $\mathcal{H} = \sqrt{-g}$ satisfies a BPS bound as in (3.61), where $\mathcal{Z}$ is a total derivative. To prove this statement, let us notice that $\mathcal{H}$ can be written as:

$$\mathcal{H} = \frac{r^2}{6}\,\frac{H}{\sqrt{1-cy}}\,\Big|\frac{y}{\cos\theta}\Big|\,\sqrt{\Delta_\phi^2 + (1-cy)\,\frac{\cos^2\theta}{y^2H^2}\,y_\theta^2}\;\times$$
$$\times\sqrt{(c\cos\theta-\beta_\phi)^2 + \frac{\cos^2\theta}{H^2y^2(1-cy)}\,\Delta_\theta^2 + \frac{2y^2}{3H^2}\,(\beta_\phi-\Delta_\phi)^2}\,. \tag{4.33}$$

Let us now rewrite $\mathcal{H}$ as $\mathcal{H} = |\mathcal{Z}| + \mathcal{S}$, where

$$\mathcal{Z} = \frac{r^2}{6}\,\frac{H}{\sqrt{1-cy}}\,\frac{y}{\cos\theta}\,\Big[\frac{\cos^2\theta}{y^2H^2}\,\Delta_\theta\,y_\theta - (c\cos\theta-\beta_\phi)\Delta_\phi\Big]\,. \tag{4.34}$$

One can check that $|\mathcal{Z}|_{|BPS} = \sqrt{-g}_{|BPS}$. Moreover, for arbitrary functions $y = y(\theta)$ and $\beta = \beta(\phi)$, one can verify that $\mathcal{Z}$ is a total derivative, namely:

$$\mathcal{Z} = \frac{\partial}{\partial\theta}\,\mathcal{Z}^\theta + \frac{\partial}{\partial\phi}\,\mathcal{Z}^\phi\,. \tag{4.35}$$

In order to write the explicit expressions of $\mathcal{Z}^\theta$ and $\mathcal{Z}^\phi$, let us define the function $g(y)$ as follows:

$$g(y) \equiv -\int\frac{\sqrt{1-cy}}{H(y)}\,dy\,. \tag{4.36}$$

Then one can verify that eq. (4.35) is satisfied for $\mathcal{Z}^\theta$ and $\mathcal{Z}^\phi$ given by:

$$\begin{aligned}
\mathcal{Z}^\theta &= \frac{r^2}{6}\,\sin\theta\,g(y)\,,\\
\mathcal{Z}^\phi &= \frac{r^2}{6}\Big[-\cos\theta\,g(y)\,\phi + H(y)\,\sqrt{1-cy}\,\,(c\phi\cos\theta-\beta)\Big]\,.
\end{aligned} \tag{4.37}$$

One can prove that $\mathcal{H} \geq |\,\mathcal{Z}|$ is equivalent to:

$$\frac{\cos^2\theta}{y^2(1-cy)}\Big[\Delta_\phi\,\Delta_\theta + (1-cy)\,(c\cos\theta-\beta_\phi)\,y_\theta\Big]^2 +$$
$$\frac{2y^2}{3}\Big[\Delta_\phi^2 + \frac{(1-cy)\cos^2\theta}{y^2H^2}\,y_\theta^2\Big][\beta_\phi-\Delta_\phi]^2 \,\geq\, 0\,, \tag{4.38}$$

which is always satisfied. Moreover, by using that $(c\cos\theta-\beta_\phi)_{|BPS} = \cos\theta/y$, one can prove that this inequality is saturated precisely when the BPS differential equations are satisfied.

5. Supersymmetric D7-branes in $AdS_5\times Y^{p,q}$

For a D7-brane the kappa symmetry matrix (2.46) takes the form:

$$\Gamma_k = -\frac{i}{8!\sqrt{-g}}\epsilon^{\mu_1\ldots\mu_8}\gamma_{\mu_1\ldots\mu_8}\,, \tag{5.1}$$

where, again, we have used the rules of eq. (2.45) to write the expression of Γ_k acting on complex spinors. The D7-branes which fill the four Minkowski spacetime directions and extend along some holographic non-compact direction can be potentially used as flavor branes, i.e. as branes whose fluctuations can be identified with the dynamical mesons of the gauge theory. In this section we will find a family of these configurations which preserve four supersymmetries. In section 6 we will determine another family of supersymmetric spacetime filling configurations of D7-branes and we will also demonstrate that there are embeddings in which the D7-brane wraps the entire $Y^{p,q}$ space and preserve two supersymmetries.

5.1 Spacetime filling D7-brane

Let us choose a system of worldvolume coordinates motivated by the spacetime filling character of the configuration that we are trying to find, namely:

$$\xi = (t, x^1, x^2, x^3, y, \beta, \theta, \phi). \tag{5.2}$$

The ansatz we will adopt for the embedding is:

$$\psi = \psi(\beta, \phi), \qquad r = r(y, \theta). \tag{5.3}$$

In this case the general expression of Γ_κ (eq. (5.1)) reduces to:

$$\Gamma_\kappa = -i\,\frac{r^4}{L^4\sqrt{-g}}\,\Gamma_{x^0\cdots x^3}\,\gamma_{y\beta\theta\phi}\,. \tag{5.4}$$

In order to implement the $\Gamma_\kappa\,\epsilon = \epsilon$ condition we require that the spinor ϵ is an eigenvector of the matrix Γ_* defined in eq. (2.29). Then, according to eq. (2.34), $\Gamma_*\epsilon = -\epsilon$, i.e. ϵ is of the form ϵ_- and, therefore, it satisfies:

$$\Gamma_{x^0\cdots x^3}\,\epsilon_- = i\epsilon_-\,. \tag{5.5}$$

Moreover, as ϵ_- has fixed ten-dimensional chirality, the condition (5.5) implies:

$$\Gamma_{r5}\epsilon_- = -i\epsilon_-\,. \tag{5.6}$$

By using the projection (5.5), one immediately arrives at:

$$\Gamma_\kappa\,\epsilon_- = \frac{r^4}{L^4\sqrt{-g}}\,\gamma_{y\beta\theta\phi}\,\epsilon_-\,. \tag{5.7}$$

The induced gamma matrices appearing on the right-hand side of eq. (5.7) are:

$$\begin{aligned}
\frac{1}{L}\,\gamma_y &= -\frac{1}{\sqrt{6}H}\Gamma_1 + \frac{1}{r}r_y\Gamma_r,\\
\frac{1}{L}\,\gamma_\theta &= \frac{\sqrt{1-c\,y}}{\sqrt{6}}\Gamma_3 + \frac{1}{r}r_\theta\Gamma_r,\\
\frac{1}{L}\,\gamma_\beta &= -\frac{H}{\sqrt{6}}\Gamma_2 + \frac{1}{3}\left(\psi_\beta + y\right)\Gamma_5,\\
\frac{1}{L}\,\gamma_\phi &= \frac{cH\cos\theta}{\sqrt{6}}\Gamma_2 + \frac{\sqrt{1-c\,y}}{\sqrt{6}}\sin\theta\Gamma_4 + \frac{1}{3}\left(\psi_\phi + (1-c\,y)\cos\theta\right)\Gamma_5\,.
\end{aligned} \tag{5.8}$$

After using eqs. (2.32) and (5.6), the action of $\gamma_{y\beta\theta\phi}$ on ϵ can be written as:

$$\frac{1}{L^4}\,\gamma_{y\beta\theta\phi}\,\epsilon_- = [d_I + d_{15}\,\Gamma_{15} + d_{35}\,\Gamma_{35} + d_{13}\Gamma_{13}]\epsilon_-\,, \tag{5.9}$$

where the different coefficients are given by:

$$\begin{aligned}
d_I &= \frac{1-cy}{36}\,\sin\theta + \frac{1-cy}{18}\,\sin\theta\,(y+\psi_\beta)\,\frac{r_y}{r} - \frac{1}{18}\,[(1+c\psi_\beta)\cos\theta + \psi_\phi]\frac{r_\theta}{r}\,,\\
d_{15} &= i\,\frac{1-cy}{6\sqrt{6}}\,H\,\sin\theta\left[\frac{r_y}{r} - \frac{y+\psi_\beta}{3H^2}\right],\\
d_{35} &= -i\,\frac{\sqrt{1-cy}}{6\sqrt{6}}\left[\sin\theta\,\frac{r_\theta}{r} + \frac{1}{3}\,((1+c\psi_\beta)\cos\theta + \psi_\phi)\right],\\
d_{13} &= \frac{\sqrt{1-cy}}{18}\,H\left[\sin\theta\,\frac{y+\psi_\beta}{H^2}\,\frac{r_\theta}{r} + ((1+c\psi_\beta)\,\cos\theta + \psi_\phi)\,\frac{r_y}{r}\right].
\end{aligned} \tag{5.10}$$

As the terms containing the matrices Γ_{15}, Γ_{35} and Γ_{13} give rise to projections which are not compatible with those in eq. (2.32), we have to impose that:

$$d_{15} = d_{35} = d_{13} = 0\,. \tag{5.11}$$

From the vanishing of d_{15} and d_{35} we obtain the following first-order differential equations

$$r_y = \Lambda_y\,, \qquad\qquad r_\theta = \Lambda_\theta\,, \tag{5.12}$$

where we have defined Λ_y and Λ_θ as:

$$\begin{aligned}
\Lambda_y &= \frac{r}{3H^2}(y+\psi_\beta)\,,\\
\Lambda_\theta &= -\frac{r}{3\sin\theta}\,[(1+c\psi_\beta)\,\cos\theta + \psi_\phi]\,.
\end{aligned} \tag{5.13}$$

Notice that the equations (5.12) imply that $d_{13} = 0$. One can also check that $r^4\,d_I = \sqrt{-g}$ if the first-order equations (5.12) hold and, therefore, one has indeed that $\Gamma_\kappa\epsilon_- = \epsilon_-$. Thus, any Killing spinor of the type $\epsilon = \epsilon_-$, with ϵ_- as in eq. (2.34), satisfies the kappa symmetry condition if the BPS equations (5.12) hold. Therefore, these configurations preserve the four ordinary supersymmetries of the background and, as a consequence, they are 1/8 supersymmetric.

5.2 Integration of the first-order equations

Let us now obtain the general solution of the system (5.12). Our first observation is that, according to (5.3), the only dependence on the coordinates β and ϕ appearing in eqs. (5.12) and (5.13) comes from the derivatives of ψ. Therefore, for consistency with the assumed dependence of the functions of the ansatz (5.3), ψ_ϕ and ψ_β must be constants. Thus, let us write:

$$\psi_\phi = n_1\,, \qquad\qquad \psi_\beta = n_2\,, \tag{5.14}$$

which can be trivially integrated, namely:

$$\psi = n_1\,\phi + n_2\,\beta + \text{constant}\,. \tag{5.15}$$

It is now easy to obtain the function $r(\theta, y)$. The equations to integrate are:

$$r_y = \frac{r}{3H^2}\,(y+n_2)\,, \qquad r_\theta = -\frac{r}{3\sin\theta}\Big[(1+cn_2)\cos\theta\, + n_1\Big]\,. \tag{5.16}$$

Let us first integrate the equation for r_θ in (5.16). We get:

$$r(y,\theta) = \frac{A(y)}{\Big[\sin\frac{\theta}{2}\Big]^{\frac{1+n_1+cn_2}{3}}\,\Big[\cos\frac{\theta}{2}\Big]^{\frac{1-n_1+cn_2}{3}}}\,, \tag{5.17}$$

with $A(y)$ a function of y to be determined. Plugging this result in the equation for r_y in (5.16), we get the following equation for A:

$$\frac{1}{A}\,\frac{dA}{dy} = \frac{1}{3}\,\frac{y+n_2}{H^2}\,, \tag{5.18}$$

which can be integrated immediately, namely:

$$A^3(y) = C\Big[f_1(y)\Big]^{n_2}\,f_2(y)\,, \tag{5.19}$$

with C a constant and $f_1(y)$ and $f_2(y)$ being the functions defined in (2.24). Then, we can write $r(y,\theta)$ as:

$$r^3(y,\theta) = C\frac{\Big[f_1(y)\Big]^{n_2}\,f_2(y)}{\Big[\sin\frac{\theta}{2}\Big]^{1+n_1+cn_2}\,\Big[\cos\frac{\theta}{2}\Big]^{1-n_1+cn_2}}\,. \tag{5.20}$$

Several comments concerning the solution displayed in eqs. (5.15) and (5.20) are in order. First of all, after a suitable change of variable it is easy to verify that for $c=0$ one recovers from (5.15) and (5.20) the family of D7-brane embeddings in $AdS_5\times T^{1,1}$ found in ref. [12]. Secondly, the function $r(y,\theta)$ in (5.20) always diverges for some particular values of θ and y, which means that the probe always extends infinitely in the holographic direction. Moreover, for some particular values of n_1 and n_2 there is a minimal value of the coordinate r, which depends on the integration constant C. This fact is important when one tries to use these D7-brane configurations as flavor branes, since this minimal value of r provides us with an energy scale, which is naturally identified with the mass of the dynamical quarks added to the gauge theory. It is also interesting to obtain the form of the solution written in eqs. (5.15) and (5.20) in terms of the complex variables z_i defined in (2.23). After a simple calculation one can verify that this solution can be written as a polynomial equation of the form:

$$z_1^{m_1}\,z_2^{m_2}\,z_3^{m_3} = \text{constant}\,, \tag{5.21}$$

where the m_i's are constants and $m_3\neq 0$.[5] The relation between the m_i's of (5.21) and the n_i's of eqs. (5.15) and (5.20) is:

$$n_1 = \frac{m_1}{m_3}\,, \qquad n_2 = \frac{m_2}{m_3}\,. \tag{5.22}$$

[5]It is natural to expect a condition of the form $f(z_1,z_2,z_3)=0$, where f is a general holomorphic function of its arguments. However, in order to be able to solve the problem analytically we started from a restrictive ansatz (5.3) that, not surprisingly, leads to a particular case of the expected answer.

Notice that when $n_2 = m_2 = 0$ the dependence on β disappears and the configuration is reminiscent of its analog in the conifold case [12]. When $n_2 \neq 0$ the D7-brane winds infinitely the ψ-circle.

5.3 Energy bound

As it happened in the case of D3- and D5-branes, one can verify that any solution of the first-order equations (5.12) also solves the equations of motion. We are now going to check that there exists a bound for the energy which is saturated by the solutions of the first-order equations (5.12). Indeed, let $r(y, \theta)$ and $\psi(\beta, \phi)$ be arbitrary functions. The hamiltonian density $\mathcal{H} = \sqrt{-g}$ in this case can be written as:

$$\mathcal{H} = \frac{r^2}{6} \sin\theta \sqrt{\left(r_\theta^2 + (1-cy)\left[H^2 r_y^2 + \frac{r^2}{6}\right]\right)\left(\Lambda_\theta^2 + (1-cy)\left[H^2 \Lambda_y^2 + \frac{r^2}{6}\right]\right)}\,, \tag{5.23}$$

where Λ_y and Λ_θ are the functions displayed in eq. (5.13). Let us rewrite this function $\mathcal{H}$ as $\mathcal{Z} + \mathcal{S}$, where $\mathcal{Z}$ is given by:

$$\mathcal{Z} = \frac{r^2}{6} \sin\theta \left[r_\theta \Lambda_\theta + (1-cy)\left(H^2 r_y \Lambda_y + \frac{r^2}{6}\right)\right]. \tag{5.24}$$

One can prove that $\mathcal{Z}$ is a total derivative:

$$\mathcal{Z} = \partial_\theta \mathcal{Z}^\theta + \partial_y \mathcal{Z}^y\,, \tag{5.25}$$

where $\mathcal{Z}^\theta$ and $\mathcal{Z}^y$ are:

$$\begin{aligned} \mathcal{Z}^\theta &= -\frac{r^4}{72}\left[\psi_\phi + (1 + c\psi_\beta)\cos\theta\right], \\ \mathcal{Z}^y &= \frac{r^4}{72}(1-cy)(y + \psi_\beta)\sin\theta\,. \end{aligned} \tag{5.26}$$

Moreover, when $\mathcal{Z}$ is given by (5.24), one can demonstrate the bound (3.61). Actually, one can show that the condition $\mathcal{H} \geq |\mathcal{Z}|$ is equivalent to the inequality:

$$(r_\theta - \Lambda_\theta)^2 + H^2 (1-cy)(r_y - \Lambda_y)^2 + \frac{H^2}{r^2}(r_\theta \Lambda_y - r_y \Lambda_\theta)^2 \geq 0\,, \tag{5.27}$$

which is always satisfied and is saturated precisely when the BPS equations (5.12) are satisfied. Notice also that $\mathcal{Z}_{|BPS}$ is positive.

6. Other interesting possibilities

Let us now look at some other configurations of different branes and cycles not considered so far. We first consider D3-branes extended along one of the Minkowski coordinates and along a two-dimensional submanifold of $Y^{p,q}$. These configurations represent "fat" strings from the point of view of the gauge theory. We verify in subsection 6.1 that an embedding of this type breaks completely the supersymmetry, although there exist stable non-supersymmetric fat strings. In subsection 6.2 we find a new configuration of a D5-brane wrapping a two-dimensional submanifold, whereas in subsection 6.3 we add worldvolume flux to the domain wall solutions of section 4. In subsection 6.4 we consider the possibility of having D5-branes wrapping a three-cycle. We show that such embeddings cannot be supersymmetric, even though stable solutions of the equations of motion with these characteristics do exist. In subsection 6.5 we analyze the baryon vertex configuration (a D5-brane wrapping the entire $Y^{p,q}$ space) and we verify that such embedding breaks supersymmetry completely. In subsection 6.6 we explore the existence of spacetime filling supersymmetric configurations of D7-branes by using a set of worldvolume coordinates different from those used in section 5. Finally, in subsection 6.7 we show that a D7-brane can wrap the whole $Y^{p,q}$ space and preserve some fraction of supersymmetry.

6.1 D3-branes on a two-submanifold

Let us take a D3-brane which is extended along one of the spatial directions of the worldvolume of the D3-branes of the background (say x^1) and wraps a two-dimensional cycle. The worldvolume coordinates we will take are

$$\xi^\mu = (x^0, x^1, \theta, \phi)\,, \tag{6.1}$$

and we will look for embeddings with x^2, x^3, r and ψ constant and with

$$y = y(\theta, \phi)\,, \qquad\qquad \beta = \beta(\theta, \phi)\,. \tag{6.2}$$

In this case the kappa symmetry matrix acts on ϵ as:

$$\Gamma_\kappa\,\epsilon = -\frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\,\Gamma_{x^0x^1}\,\gamma_{\theta\phi}\,\epsilon\,. \tag{6.3}$$

The expressions of γ_θ and γ_ϕ are just those given in eq. (4.4). Moreover, $\gamma_{\theta\phi}\,\epsilon$ can be obtained by taking the complex conjugate of eq. (4.6):

$$\frac{6}{L^2}\,\gamma_{\theta\phi}\,\epsilon = [b_I^* + b_{15}^*\,\Gamma_{15} + b_{35}^*\,\Gamma_{35} + b_{13}^*\,\Gamma_{13}]\epsilon\,, \tag{6.4}$$

where the b's are given in eq. (4.7). Since now the complex conjugation does not act on the spinor ϵ, the only possible projection compatible with those of the background is the one originated from the term with the unit matrix in the previous expression. Then, we must require:

$$b_{15} = b_{35} = b_{13} = 0\,. \tag{6.5}$$

The conditions $b_{15} = 0$ and $b_{35} = 0$ are equivalent and give rise to eqs. (4.9) and (4.10), which can be integrated as in eq. (4.18). Moreover, the condition $b_{13} = 0$ leads to the equation:

$$\frac{y}{H^2}\,y_\theta = \cot\theta\,. \tag{6.6}$$

The integration of this equation can be straightforwardly performed in terms of the function $f_2(y)$ defined in eq. (2.24) and can be written as:

$$\frac{1}{\sqrt{a - 3y^2 + 2cy^3}} = k\sin\theta\,, \tag{6.7}$$

with k being a constant of integration, which should be related to the constant m in eq. (4.18). However, the dependence of y on θ written in the last equation does not seem to be compatible with the one of eq. (4.18) (even for $c = 0$). Thus, we conclude that there is no solution for the kappa symmetry condition in this case.

If we forget about the requirement of supersymmetry it is not difficult to find solutions of the Euler-Lagrange equations of motion of the D3-brane probe. Indeed, up to irrelevant global factors, the lagrangian for the D3-brane considered here is the same as the one corresponding to a D5-brane extended along a two-dimensional submanifold of $Y^{p,q}$. Thus, the embeddings written in eq. (4.18) are stable solutions of the equations of motion of the D3-brane which represent a “fat string” from the gauge theory point of view.

6.2 More D5-branes wrapped on a two-cycle

Let us consider a D5-brane wrapped on a two-cycle and let us choose the following set of worldvolume coordinates: $\xi^\mu = (x^0, x^1, x^2, r, \theta, y)$. The embeddings we shall consider have x^3 and ψ constant and $\phi = \phi(\theta, y)$, $\beta = \beta(\theta, y)$. For this case, one has:

$$\Gamma_\kappa\,\epsilon = \frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\,\Gamma_{x^0x^1x^2r}\,\gamma_{\theta y}\,\epsilon^*\,. \tag{6.8}$$

The induced gamma matrices are:

$$\begin{aligned}
\frac{1}{L}\gamma_\theta &= \frac{H}{\sqrt{6}}\Big(-\beta_\theta + c\cos\theta\phi_\theta\Big)\Gamma_2 + \sqrt{\frac{1-cy}{6}}\Big(\Gamma_3 + \sin\theta\phi_\theta\Gamma_4\Big) + \\
&\quad + \frac{1}{3}\Big(y\beta_\theta + (1-cy)\cos\theta\,\phi_\theta\Big)\,\Gamma_5\,,\\
\frac{1}{L}\,\gamma_y &= -\frac{1}{\sqrt{6}H}\,\Gamma_1 + \frac{H}{\sqrt{6}}\,\Big(-\beta_y + c\cos\theta\phi_y\Big)\,\Gamma_2 + \sqrt{\frac{1-cy}{6}}\,\sin\theta\phi_y\,\Gamma_4 + \\
&\quad + \frac{1}{3}\Big(y\beta_y + (1-cy)\cos\theta\phi_y\Big)\,\Gamma_5\,.
\end{aligned} \tag{6.9}$$

Then, one has

$$\frac{6}{L^2}\,\gamma_{\theta y}\,\epsilon^* = \Big(f_I + f_{15}\Gamma_{15} + f_{35}\Gamma_{35} + f_{13}\Gamma_{13}\Big)\,\epsilon^*\,, \tag{6.10}$$

where the different coefficients are given by:

$$\begin{aligned}
f_I &= -i\Big((1-cy)\,\sin\theta\,\phi_y - c\cos\theta\phi_\theta\,+\,\beta_\theta\Big)\,,\\
f_{15} &= \sqrt{\frac{2}{3}}\,\frac{1}{H}\Big(y\beta_\theta + (1-cy)\cos\theta\phi_\theta\Big) + i\sqrt{\frac{2}{3}}\,H\,\cos\theta\,\Big(\beta_y\,\phi_\theta - \beta_\theta\,\phi_y\Big)\,,\\
f_{35} &= \sqrt{\frac{2}{3}}\,\sqrt{1-cy}\Big[\Big(y\beta_y + (1-cy)\cos\theta\phi_y\Big) - i\,y\,\sin\theta\,\Big(\beta_y\,\phi_\theta - \beta_\theta\,\phi_y\Big)\Big]\,,\\
f_{13} &= \sqrt{1-cy}\,\Big[\Big(\frac{1}{H} + H\sin\theta\,(\beta_y\,\phi_\theta - \beta_\theta\,\phi_y)\Big) - i\Big(\frac{\sin\theta}{H}\,\phi_\theta - H(\beta_y - c\cos\theta\phi_y)\Big)\Big]\,.
\end{aligned} \tag{6.11}$$

The BPS conditions in this case are the following:

$$f_I = f_{15} = f_{35} = 0\,. \tag{6.12}$$

From the vanishing of f_I we get the equation:

$$\beta_\theta + (1-cy)\sin\theta\,\phi_y \,-\, c\cos\theta\phi_\theta = 0\,. \tag{6.13}$$

Moreover, the vanishing of f_{15} and f_{35} is equivalent to the equations:

$$\begin{aligned} &y\beta_\theta + (1-cy)\,\cos\theta\phi_\theta = 0\,,\\ &y\beta_y + (1-cy)\cos\theta\,\phi_y = 0\,,\\ &\beta_y\,\phi_\theta - \beta_\theta\,\phi_y = 0\,. \end{aligned} \tag{6.14}$$

Notice that this system of equations is redundant, i.e. the first two equations are equivalent if one uses the last one. Substituting the value of β_θ as given by the first equation in (6.14) into (6.13), one can get a partial differential equation which only involves derivatives of ϕ, namely:

$$\cot\theta\,\phi_\theta - y(1-cy)\,\phi_y = 0\,. \tag{6.15}$$

By using in (6.15) the last equation in (6.14), one gets:

$$\cot\theta\,\beta_\theta - y(1-cy)\,\beta_y = 0\,. \tag{6.16}$$

Eqs. (6.15) and (6.16) can be easily integrated by the method of separation of variables. One gets

$$\begin{aligned} \phi &= A\left[\frac{y}{(1-cy)\cos\theta}\right]^{\alpha} + \phi^0\,,\\ \beta &= \frac{\alpha}{1-\alpha}\,A\left[\frac{y}{(1-cy)\cos\theta}\right]^{\alpha-1} + \beta^0\,, \end{aligned} \tag{6.17}$$

where A, α, ϕ^0 and β^0 are constants of integration and we have used eq. (6.14) to relate the integration constants of ϕ and β. However, in order to implement the condition $\Gamma_\kappa\,\epsilon = \epsilon$, one must require the vanishing of the imaginary part of f_{13}. This only happens if ϕ and β are constant, i.e. when $A = 0$ in the above solution. One can check that this configuration satisfies the equation of motion.

6.3 D5-branes on a two-submanifold with flux

We now analyze the effect of adding flux of the worldvolume gauge field F to the configurations of section 4 [6]. Notice that we now have a non-zero contribution from the Wess-Zumino term of the action, which is of the form:

$$\mathcal{L}_{WZ} = P[C^{(4)}\,]\wedge F\,. \tag{6.18}$$

Let us suppose that we switch on a worldvolume gauge field along the angular directions (θ,ϕ). We will adopt the ansatz:

$$F_{\theta\phi} = q\,K(\theta,\phi)\,, \tag{6.19}$$

[6] A nice discussion of supersymmetric configurations with nonzero gauge field strengths by means of kappa symmetry can be found in ref. [43].

where q is a constant and $K(\theta,\phi)$ a function to be determined. The relevant components of $P[C^{(4)}]$ are

$$P[C^{(4)}]_{x^0x^1x^2r} = h^{-1}\,\frac{\partial x^3}{\partial r}\,, \tag{6.20}$$

where $h = L^4/r^4$. It is clear from the above expression of $\mathcal{L}_{WZ}$ that a nonvanishing value of q induces a dependence of x^3 on r. In what follows we will assume that $x^3 = x^3(r)$, i.e. that x^3 only depends on r. Let us assume that the angular embedding satisfies the same equations as in the case of zero flux. The Lagrangian density in this case is given by:

$$\mathcal{L} = -h^{-\frac{1}{2}}\,\sqrt{1+h^{-1}\,(x')^2}\,\sqrt{g_{\theta\theta}g_{\phi\phi}+q^2\,K^2} + q\,h^{-1}x'K\,, \tag{6.21}$$

where $g_{\theta\theta}$ and $g_{\phi\phi}$ are elements of the induced metric, we have denoted x^3 simply by x and the prime denotes derivative with respect to r. The equation of motion of x is:

$$-\frac{\sqrt{g_{\theta\theta}g_{\phi\phi}+q^2\,K^2}}{\sqrt{1+h^{-1}\,(x')^2}}\,h^{-\frac{3}{2}}\,x' + q\,h^{-1}\,K = \text{constant}\,. \tag{6.22}$$

Taking the constant on the right-hand side of the above equation equal to zero, we get the following solution for x':

$$x'(r) = q\,h^{\frac{1}{2}}\,\frac{K(\theta,\phi)}{\sqrt{g_{\theta\theta}g_{\phi\phi}}}\,. \tag{6.23}$$

Notice that the left-hand side of the above equation depends only on r, whereas the right-hand side can depend on the angles (θ,ϕ). For consistency the dependence of $K(\theta,\phi)$ and $\sqrt{g_{\theta\theta}g_{\phi\phi}}$ on (θ,ϕ) must be the same. Without lost of generality let us take $K(\theta,\phi)$ to be:

$$L^2\,K(\theta,\phi) = \sqrt{g_{\theta\theta}g_{\phi\phi}}\,, \tag{6.24}$$

where the factor L^2 has been introduced for convenience. Using this form of K, the differential equation which determines the dependence of x^3 on r becomes:

$$x'(r) = \frac{q}{r^2}\,, \tag{6.25}$$

which can be immediately integrated, namely:

$$x(r) = \bar{x}^3 - \frac{q}{r}\,. \tag{6.26}$$

Moreover, the expression of K can be obtained by computing the induced metric along the angular directions. It takes the form:

$$K(\theta) = \sigma\,\frac{\sqrt{1-cy}}{6H(y)}\Big[H^2(y)+(1-cy)y^2\,\tan^2\theta\Big]\frac{\cos\theta}{y}\,, \tag{6.27}$$

where $y = y(\theta)$ is the function obtained in section 4 and $\sigma = \text{sign}\Big(\cos\theta/y\Big)$. Actually, notice that K only depends on the angle θ and it is independent of ϕ.

We are now going to verify that the configuration just found is supersymmetric. The expression of Γ_κ in this case has an additional term due to the worldvolume gauge field. Actually, it is straightforward to check that in the present case

$$\Gamma_\kappa\,\epsilon = \frac{i}{\sqrt{-\det(g+F)}}\,\frac{r^3}{L^3}\,\Gamma_{x^0x^1x^2}\Big[\gamma_r\,\gamma_{\theta\phi}\,\epsilon^* \,-\, \gamma_r\,F_{\theta\phi}\,\epsilon\Big]\,. \tag{6.28}$$

Notice that γ_r is given by:

$$\gamma_r = \frac{L}{r}(\Gamma_r + \frac{r^2}{L^2}\,x'\,\Gamma_{x^3})\,. \tag{6.29}$$

For the angular embeddings we are considering it is easy to prove from the results of section 4 that:

$$\gamma_{\theta\phi}\,\epsilon^* = -i\sigma L^2 K(\theta)\,\Gamma_{13}\,\epsilon^*\,. \tag{6.30}$$

By using this result and the value of $F_{\theta\phi}$ (eq. (6.19)), one easily verifies that:

$$\Gamma_\kappa\,\epsilon = -\frac{i}{1+\frac{q^2}{L^4}}\,\Gamma_{x^0x^1x^2r}\Big[i\sigma\Gamma_{13}\,\epsilon^* + \frac{q}{L^2}\,i\sigma\,\Gamma_{rx^3}\,\Gamma_{13}\,\epsilon^* + \frac{q}{L^2}\,\epsilon + \frac{q^2}{L^4}\,\Gamma_{rx^3}\,\epsilon\Big]\,. \tag{6.31}$$

By using the explicit dependence of x on r (eq. (6.26)), one can write the Killing spinor ϵ evaluated on the worldvolume as:

$$e^{\frac{i}{2}\psi}\,\epsilon = r^{-\frac{1}{2}}\Big(1-\frac{q}{L^2}\,\Gamma_{rx^3}\Big)\eta_+ + r^{\frac{1}{2}}\Big(\frac{\bar{x}^3}{L^2}\,\Gamma_{rx^3}\,\eta_+ + \eta_-\Big) \,+\, \frac{r^{\frac{1}{2}}}{L^2}\,x^p\,\Gamma_{rx^p}\,\eta_+\,, \tag{6.32}$$

where the constant spinors $\eta_\pm$ are the ones defined in eq. (2.33). Remarkably, one finds that the condition $\Gamma_\kappa\epsilon = \epsilon$ is verified if η_+ and η_- satisfy the same system (4.24) as is the case of zero flux.

6.4 D5-branes wrapped on a three-cycle

We will now try to find supersymmetric configurations of D5-branes wrapping a three cycle of the $Y^{p,q}$ space. Let us choose the following set of worldvolume coordinates $\xi^\mu = (x^0, x^1, x^2, y, \beta, \psi)$ and consider an embedding with x^3 and r constant, $\theta = \theta(y,\beta)$ and $\phi = \phi(y,\beta)$. In this case:

$$\Gamma_\kappa\,\epsilon = \frac{i}{\sqrt{-g}}\,\frac{r^3}{L^3}\,\Gamma_{x^0x^1x^2}\,\gamma_{y\beta\psi}\,\epsilon^*\,. \tag{6.33}$$

The value of $\gamma_{y\beta\psi}\,\epsilon^*$ can be obtained by taking the complex conjugate of eq. (3.29). As $c_1 = c_3 = 0$ when $\theta_\psi = \phi_\psi = 0$, we can write:

$$\frac{i}{L^3}\,\gamma_{y\beta\psi}\,\epsilon^* = [c_5^*\,\Gamma_5 + c_{135}^*\,\Gamma_{135}]\epsilon^*\,. \tag{6.34}$$

The only possible BPS condition compatible with the projections satisfied by ϵ is $c_5 = 0$, which leads to a projection of the type

$$\Gamma_{x^0x^1x^2}\Gamma_{135}\,\epsilon^* = \lambda\epsilon\,, \tag{6.35}$$

where λ is a phase. Notice that, however, as the spinor ϵ contains a factor $e^{-\frac{i}{2}\psi}$, the two sides of the above equation depend differently on ψ due to the complex conjugation appearing on the left-hand side (λ does not depend on ψ). Thus, these configurations cannot be supersymmetric. We could try to use another set of worldvolume coordinates, in particular one which does not include ψ. After some calculation one can check that there is no consistent solution.

For the ansatz considered above the lagrangian density of the D5-brane is, up to irrelevant factors, the same as the one obtained in subsection 3.2 for a D3-brane wrapping a three-dimensional submanifold of $Y^{p,q}$. Therefore any solution of the first-order equations (3.34) gives rise to an embedding of a D5-brane which solves the equations of motion and saturates an energy bound. This last fact implies that the D5-brane configuration is stable, in spite of the fact that it is not supersymmetric.

6.5 The baryon vertex

If a D5-brane wraps the whole $Y^{p,q}$ space, the flux of the Ramond-Ramond five form $F^{(5)}$ that it captures acts as a source for the electric worldvolume gauge field which, in turn, gives rise to a bundle of fundamental strings emanating from the D5-brane. This is the basic argument of Witten's construction of the baryon vertex [3], which we will explore in detail now. In this case the probe action must include the worldvolume gauge field F in both the Born-Infeld and Wess-Zumino terms. It takes the form:

$$S = -T_5 \int d^6\xi \, \sqrt{-\det(g+F)} + T_5 \int d^6\xi \;\; A \wedge F^{(5)} \,, \tag{6.36}$$

where T_5 is the tension of the D5-brane and A is the one-form potential for F ($F = dA$). In order to analyze the contribution of the Wess-Zumino term in (6.36) let us rewrite the expression (2.15) of $F^{(5)}$ as:

$$F^{(5)} = \frac{L^4}{27} \, (1-cy) \, \sin\theta \, dy \wedge d\beta \wedge d\theta \wedge d\phi \wedge d\psi + \text{Hodge dual} \,, \tag{6.37}$$

where, for simplicity we are taking the string coupling constant g_s equal to one. Let us also choose the following set of worldvolume coordinates:

$$\xi^\mu = (x^0, y, \beta, \theta, \phi, \psi) \,. \tag{6.38}$$

It is clear from the expressions of $F^{(5)}$ in (6.37) and of the Wess-Zumino term in (6.36) that, for consistency, we must turn on the time component of the field A. Actually, we will adopt the following ansatz:

$$r = r(y) \,, \qquad\qquad A_0 = A_0(y) \,. \tag{6.39}$$

The action (6.36) for such a configuration can be written as:

$$S = \frac{T_5 L^4}{108} \, V_4 \, \int dx^0 dy \; \mathcal{L}_{\rm eff} \,, \tag{6.40}$$

where the volume V_4 is:

$$V_4 = 6 \int d\alpha\, d\psi\, d\phi\, d\theta \sin\theta = 96\pi^3\, \ell\,, \tag{6.41}$$

and the effective lagrangian density $\mathcal{L}_{\rm eff}$ is given by:

$$\mathcal{L}_{\rm eff} = (1-cy)\left[-H\sqrt{\frac{r^2}{H^2} + 6\,(r')^2 - 6\,(F_{x^0y})^2} + 4A_0\right]. \tag{6.42}$$

Notice that, for our ansatz (6.39), the electric field is $F_{x^0y} = -\partial_y A_0$. Let us now introduce the displacement field, defined as:

$$D(y) \equiv \frac{\partial \mathcal{L}_{\rm eff}}{\partial F_{x^0y}} = \frac{6(1-cy)HF_{x^0y}}{\sqrt{\frac{r^2}{H^2} + 6\,(r')^2 - 6\,(F_{x^0y})^2}}\,. \tag{6.43}$$

From the equations of motion of the system it is straightforward to determine $D(y)$. Indeed, the variation of S with respect to A_0 gives rise to the Gauss' law:

$$\frac{dD(y)}{dy} = -4(1-cy)\,, \tag{6.44}$$

which can be immediately integrated, namely:

$$D(y) = -4\left(y - \frac{cy^2}{2}\right) + \text{constant}\,. \tag{6.45}$$

By performing a Legendre transform in (6.40) we can obtain the energy of the configuration:

$$E = \frac{T_5 L^4}{108}\, V_4 \int dy\, \mathcal{H}\,, \tag{6.46}$$

where $\mathcal{H}$ is given by:

$$\mathcal{H} = (1-cy)\,H\sqrt{\frac{r^2}{H^2} + 6\,(r')^2 - 6\,(F_{x^0y})^2} + D(y)\,F_{x^0y}\,. \tag{6.47}$$

Moreover, the relation (6.43) between $D(y)$ and F_{x^0y} can be inverted, with the result:

$$F_{x^0y} = \frac{1}{6}\,\frac{\sqrt{\frac{r^2}{H^2} + 6\,(r')^2}}{\sqrt{\frac{D^2}{6} + (1-cy)^2\,H^2}}\, D\,. \tag{6.48}$$

Using the relation (6.48) we can rewrite $\mathcal{H}$ as:

$$\mathcal{H} = \sqrt{\frac{D^2}{6} + (1-cy)^2\,H^2}\,\sqrt{\frac{r^2}{H^2} + 6\,(r')^2}\,, \tag{6.49}$$

where $D(y)$ is the function of the y coordinate displayed in (6.45). The Euler-Lagrange equation derived from $\mathcal{H}$ is a second-order differential equation for the function $r(y)$. This equation is rather involved and we will not attempt to solve it here. In a supersymmetric configuration one expects that there exists a first-order differential equation for $r(y)$ whose solution also solves the equations of motion. This first-order equation has been found in ref. [44] for the $AdS_5 \times S^5$ background. We have not been able to find such first-order equation in this $AdS_5 \times Y^{p,q}$ case. A similar negative result was obtained in [12] for the $AdS_5 \times T^{1,1}$ background. This result is an indication that this baryon vertex configuration is not supersymmetric. Let us check explicitly this fact by analyzing the kappa symmetry condition. The expression of Γ_κ when the worldvolume gauge field is non-zero can be found in ref. [18]. In our case $\Gamma_\kappa \, \epsilon$ reduces to:

$$\Gamma_\kappa \, \epsilon = -\frac{i}{\sqrt{-\det(g+F)}} \left[\frac{r}{L} \, \Gamma_{x^0} \, \gamma_{y\beta\theta\phi\psi} \, \epsilon^* - F_{x^0 y} \, \gamma_{\beta\theta\phi\psi} \, \epsilon \right] . \tag{6.50}$$

The two terms on the right-hand side of (6.50) containing the antisymmetrized products of gamma matrices can be written as:

$$\begin{aligned} \gamma_{y\beta\theta\phi\psi} \, \epsilon^* &= \frac{L^5}{108} \, (1-cy) \sin\theta \Big(\Gamma_5 - \sqrt{6} \, H \, \frac{r'}{r} \, \Gamma_{r15} \Big) \, \epsilon^* \, , \\ \gamma_{\beta\theta\phi\psi} \, \epsilon &= -\frac{L^4}{18\sqrt{6}} \, (1-cy) \, H \, \sin\theta \, \Gamma_{15} \, \epsilon \, . \end{aligned} \tag{6.51}$$

By using this result, we can write $\Gamma_\kappa \, \epsilon$ as:

$$\Gamma_\kappa \, \epsilon = -\frac{i \, L^4 \, (1-cy)}{\sqrt{-\det(g+F)}} \sin\theta \left[\frac{r}{108} \, \Gamma_{x^0} \Gamma_5 \, \epsilon^* + \frac{H}{18\sqrt{6}} \Big(F_{x^0 y} \, \Gamma_{15} \, \epsilon - r' \, \Gamma_{x^0 r 15} \, \epsilon^* \Big) \right] . \tag{6.52}$$

In order to solve the $\Gamma_\kappa \, \epsilon = \epsilon$ equation we shall impose, as in ref. [45], an extra projection such that the contributions of the worldvolume gauge field $F_{x^0 y}$ and of r' in (6.52) cancel each other. This can be achieved by imposing that $\Gamma_{x^0 r} \, \epsilon^* = \epsilon$ and that $F_{x^0 y} = r'$. Notice that the condition $\Gamma_{x^0 r} \, \epsilon^* = \epsilon$ corresponds to having fundamental strings in the radial direction, as expected for a baryon vertex configuration. Moreover, as the spinor ϵ has fixed ten-dimensional chirality, this extra projection implies that $i\Gamma_{x^0} \Gamma_5 \, \epsilon^* = -\epsilon$ which, in turn, is needed to satisfy the $\Gamma_\kappa \, \epsilon = \epsilon$ equation. However, the condition $\Gamma_{x^0 r} \, \epsilon^* = \epsilon$ is incompatible with the conditions (2.32) and, then, it cannot be imposed on the Killing spinors. Thus, as in the analysis of [12], we conclude from this incompatibility argument (which is more general than the particular ansatz we are adopting here) that the baryon vertex configuration breaks completely the supersymmetry of the $AdS_5 \times Y^{p,q}$ background.

6.6 More spacetime filling D7-branes

Let us adopt $\xi^\mu = (x^0, x^1, x^2, x^3, y, \beta, \psi, r)$ as our set of worldvolume coordinates for a D7-brane probe and let us consider a configuration with $\theta = \theta(y,\beta)$ and $\phi = \phi(y,\beta)$. In this case:

$$\Gamma_\kappa = -\frac{i}{\sqrt{-g}} \, \frac{r^4}{L^4} \, \Gamma_{x^0 x^1 x^2 x^3} \, \gamma_{y\beta\psi r} \, . \tag{6.53}$$

Let us take $\epsilon = \epsilon_-$, where $\Gamma_* \epsilon_- = -\epsilon_-$(see eq. (2.34)). As $\gamma_r = \frac{L}{r} \, \Gamma_r$, we can write:

$$\frac{r}{L^4} \, \gamma_{y\beta\psi r} \, \epsilon_- = -[c_5 + c_{135} \, \Gamma_{13}] \epsilon_- \, , \tag{6.54}$$

where the coefficients c_5 and c_{135} are exactly those written in eq. (3.30) for the D (doublet) three-cycles. The BPS condition is just $c_{135} = 0$, which leads to the system of differential equations (3.34). Thus, in this case the D7-brane extends infinitely in the radial direction and wraps a three-dimensional submanifold of the $Y^{p,q}$ space of the type studied in subsection 3.2. These embeddings preserve four supersymmetries.

6.7 D7-branes wrapped on $Y^{p,q}$

Let us take a D7-brane which wraps the entire $Y^{p,q}$ space and is extended along two Minkowski directions. The set of worldvolume coordinates we will use in this case are $\xi^\mu = (x^0, x^1, r, \theta, \phi, y, \beta, \psi)$ and we will assume that x^2 and x^3 are constant. The matrix Γ_κ in this case is:

$$\Gamma_\kappa = -\frac{i}{\sqrt{-g}}\,\gamma_{x^0x^1r\theta\phi y\beta\psi}\,. \tag{6.55}$$

Acting on a spinor ϵ of the background one can prove that

$$\Gamma_\kappa\,\epsilon = i\Gamma_{x^0x^1r5}\,\epsilon\,, \tag{6.56}$$

which can be solved by a spinor $\epsilon_- = r^{\frac{1}{2}}\,e^{-\frac{i}{2}\psi}\,\eta_-$, with η_- satisfying the additional projection $\Gamma_{x^0x^1r5}\,\eta_- = -i\eta_-$. Thus this configuration preserves two supersymmetries.

7. Summary and conclusions

Let us briefly summarize the results of our investigation. Using kappa symmetry as the central tool, we have systematically studied supersymmetric embeddings of branes in the $AdS_5 \times Y^{p,q}$ geometry. Our study focused on three kinds of branes D3, D5 and D7.

D3-branes: this is the case that we studied most exhaustively. For D3-branes wrapping three-cycles in $Y^{p,q}$ we first reproduced all the results present in the literature. In particular, using kappa symmetry, we obtained two kinds of supersymmetric cycles: localized at y_1 and y_2 [15] and localized in the round S^2 [16, 21]. For these branes we found perfect agreement with the field theory results. Moreover, we also found a new class of supersymmetric embeddings of D3-branes in this background. They do not correspond to dibaryonic operators since the D3-brane does not wrap a three-cycle. The field theory interpretation of these new embeddings is not completely clear to us due to various issues with global properties. We believe that they might be a good starting point to find candidates for representatives of the integer part of the third homology group of $Y^{p,q}$, just like the analogous family of cycles found in [5, 12] were representative of the integer part $H_3(T^{1,1}, \mathbb{Z})$. It would be important to understand these wrapped D3-branes in terms of algebraic geometry as well as in terms of operators in the field theory dual, following the framework of ref. [11] which, in the case of the conifold, emphasizes the use of global homogeneous coordinates. It is worth stressing that such global homogeneous coordinates exist in any toric variety [46] but the relation to the field theory operators is less clear in $CY^{p,q}$. We analyzed the spectrum of excitations of a wrapped D3-brane describing an SU(2)-charged dibaryon and found perfect agreement with the field theory expectations. We considered other embeddings and found that a D3-brane wrapping a two-cycle in $Y^{p,q}$ is not a supersymmetric state but, nevertheless, it is stable. In the field theory this configuration describes a fat string.

D5-branes: the embedding that we paid the most attention to is a D5-brane extended along a two-dimensional submanifold in $Y^{p,q}$ and having codimension one in AdS_5. In the field theory this is the kind of brane that represents a domain wall across which the rank of the gauge groups jumps. Alternatively, when we allow the D5-brane to extend infinitely in the holographic direction, we get a configuration dual to a defect conformal field theory of the type analyzed in ref. [47] for the $AdS_5 \times S^5$ background. We showed explicitly that such configuration preserves four supersymmetries and saturates the expected energy bound. For this configuration we also considered turning on a worldvolume flux and found that it can be done in a supersymmetric way. The flux in the worldvolume of the brane provides a bending of the profile of the wall, analogously to what happens in $AdS_5 \times S^5$ [48] and $AdS_5 \times T^{1,1}$ [12]. We showed the consistency of similar embeddings in which the D5-brane wraps a different two-dimensional submanifold in $Y^{p,q}$. We also considered D5-branes wrapping three-cycles. This configuration also looks like a domain wall in the field theory dual but it does not have codimension one in AdS_5 and, although it cannot be supersymmetric, it is stable. Finally, we considered a D5-brane wrapping the whole $Y^{p,q}$, which corresponds to the baryon vertex. We verified that, as in the case of $T^{1,1}$, it is not a supersymmetric configuration.

D7-branes: with the aim of introducing mesons in the corresponding field theory, we considered spacetime filling D7-branes. We explicitly showed that such configurations preserve four supersymmetries and found the precise embedding in terms of the radial coordinate. We found an interpretation of the embedding equation in terms of complex coordinates. We also analyzed other spacetime filling D7-brane embeddings. Finally, we considered a D7-brane that wraps $Y^{p,q}$ and is codimension two in AdS_5. This configuration looks, from the field theory point of view, as a string and preserves two supersymmetries.

We would like to comment on various approximations made in the paper and point out some interesting open problems. We believe that our analysis, though carried out in the case of $Y^{p,q}$ manifolds, is readily adaptable to other Sasaki-Einstein spaces. In particular, the form of the spinor for $L^{a,b,c}$ is essentially the same as in our case, namely $\epsilon^{-i\psi/2}\eta$, where ψ is the coordinate on the U(1) fiber in the canonical presentation of Sasaki-Einstein spaces as a U(1) bundle over a Kähler-Einstein base, and η is a constant spinor satisfying two projections generically written as $\Gamma_{12}\eta = -i\eta$ and $\Gamma_{34}\eta = i\eta$. Note that this structure comes from the Kähler base and is universal.

Part of our analysis of some branes could be made more precise. In particular, it would be interesting to understand the new family of supersymmetric embeddings of D3-branes in terms of algebraic geometry as well as in terms of operators in the field theory. We did not present an analysis of the spectrum of excitations for all of the branes. In particular, we would like to understand the excitations of the spacetime filling D7-branes and the baryon vertex better. We hope that understanding the conformal case will provide the basis for future analysis of deformed theories including the confining ones. For example, as shown by [26] and [29], the spacing of mass eigenvalues for the mesons in the confining case inherits properties of the related conformal theory.

We would like to point out that fully matching the spectrum of wrapped branes with field theory states is largely an ongoing problem. In particular, there are various embeddings that originate from branes of different dimensionality wrapping different cycles which should be distinguishable from the field theory point of view. It would be interesting to understand to what extent the topological data of the space determine the kind of supersymmetric branes that are allowed. Let us finish with a wishful statement. We have found a large spectrum of supersymmetric wrapped branes and also non-supersymmetric but stable branes. In analogy with the situation for strings in flat space and orbifolds one wonders whether there is a sort of holographic K-theory which accounts for all the possible branes in a given background.

We hope to return to these issues in the near future.

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